THE SEXTANT

AND ITS APPLICATIONS;

INCLUDING

THE CORRECTION

OF

OBSERVATIONS FOR INSTRUMENTAL ERRORS,

AND THE

DETERMINATION OF LATITUDE, TIME, AND LONGITUDE
BY VARIOUS METHODS ON LAND AND AT SEA,
WITH EXAMPLES AND TABLES.

BY W. H. SIMMS.

LONDON:
TROUGHTON AND SIMMS, 138 FLEET STREET.
AND
TAYLOR AND FRANCIS, RED LION COURT, FLEET STREET.
1858.

PRINTED BY TAYLOR AND FRANCIS,

RED LION COURT, FLEET STREET.

PREFACE.

The following pages are addressed to those who may be inclined to assist in introducing an improved system of practice with the Sextant, the necessity for which is well known to all who have been in the habit of employing the instrument for nice purposes, whether on land or at sea, and who are therefore aware that this, like other instruments, is subject to certain imperfections, as well in the parts which the operator has the means of adjusting as in those over which he has no such control. The defect most commonly met with was treated by the author in a paper since published in the Memoirs of the Royal Astronomical Society; but as further investigation and experience have shown him the advantage of treating other defects in a similar manner, he has in the First Part of the work now published discussed the subject in a more general form; and although the investigations themselves may appear complicated, the resulting formulæ are so simple as to render their application, with the aid of the examples given, sufficiently easy even to those who are not familiar with the manipulation of algebraical symbols.

The Second Part is devoted exclusively to the applications of the Sextant, the first Chapter treating in succession several methods that may be employed on land in the determination of Latitude, Time, Longitude, Right Ascension, and Declination, when observations are made under the most advantageous circumstances, and when consequently minute accuracy in the calculation will be rewarded with corresponding accuracy in the results. The second Chapter treats of the application of the Sextant to Nautical Astronomy; and in this will be found the several processes which are discussed at length in the first, so modified as to suit the circumstances under which the observations are made. The examples are numerous; and every effort has been made so to select them as to present the greatest variety in the data, in order that the operator may not be at a loss to find something to meet the case he may have to treat in the course of his own experience.

December 1858.

CONTENTS.

PART I.

General Theory of the Sextant.

PART II.

Applications of the Sextant.

CHAPTER I.

CHAPTER II.

Application to Nautical Astronomy.

Tables.

ERRATA.

Page 3 line 5 from top, *for* ∠ B E B *read* ∠ B E *b*.
— 10 — 23 from top, *for* *q* ∠ O G *read* ∠ *q* O G.
— 17 lines 24 and 25 from top, *for* 'image of one object being brought near to the top of the field, the reflected image of the others may appear at the bottom,' *read* 'image of an object being brought near to the top of the field, its reflected image may appear at the bottom.'
— 18. Note: line 2 from bottom, *for* 'coincidence on contact,' *read* 'coincidence or contact.'

THE SEXTANT

AND

ITS APPLICATIONS.

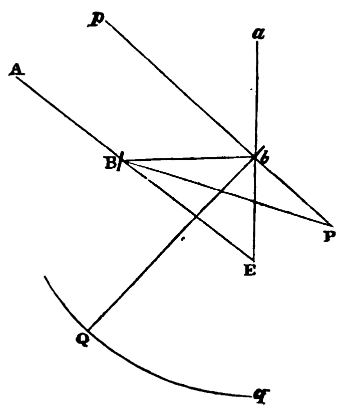
PART I.

GENERAL THEORY OF THE SEXTANT.

As the instrument employed by the navigator and the traveller in the determination of geographical position, the Sextant may perhaps be considered the most important of the many constructions adapted to the measurement of angles; and if the results of observations made with it have not fully satisfied the expectations of those to whose purposes it is peculiarly adapted, this appears to have arisen from a disposition to substitute larger and less manageable instruments, rather than to discuss the causes of such discrepancies as occur in practice with this, and to devise means, either for their removal by mechanical or optical improvement of the instrument itself, or for the elimination of their effects by computation. The Reflecting Circle, simple and repeating, the Transit and Altazimuth—instruments purchased at greater cost, and removed from place to place with greater difficulty and risk—have been extensively employed on scientific expeditions, when a Sextant properly treated would have answered every purpose, and perhaps, considering the disadvantages under which the heavier instruments are used elsewhere than in fixed observatories, would have given results at least equal in accuracy; and the navigator has frequently been led to distrust the results of observation of Lunar Distance, and to place undue confidence in his chronometer, on account of discrepancies in the former, which he is unable to account for, but which are in most instances due to determinable error in his instrument, and can be treated according to fixed and determinable laws. The method of ascertaining the amounts of these errors, and of correcting the observations, being known and reduced to rule, the Sextant will probably occupy its proper place as an astronomical instrument, the occasions on which less portable instruments are made to supersede it will be fewer, and the navigator will see reason to place confidence in the results of his lunar observations and will feel less dependent on his chronometer, of which the not unusual imperfection is the cause of more uneasiness, difficulty, and disaster than is commonly supposed.

1. Before we proceed to consider in detail the possible errors in the parts of a well-constructed Sextant, and their effects upon the observations made, it will be necessary to obtain a clear idea of the conditions requisite to perfection; and although this may be gathered from the exposition of the principles of the Instrument to be found in most treatises on navigation, it will nevertheless be well to present the subject in this place to the reader, in the form which will best prepare him to enter upon the investigations that are to follow.

Suppose two systems of parallel rays, each parallel to the plane of the paper, the one in the direction of A B, the other in that of *a b*. Let rays of the system parallel to *ab* fall upon a plane-reflecting surface *b*, this surface being at right angles with the plane of the paper. Let *b p* in the plane of the paper be perpendicular to this reflecting surface; then the ray *ab* will, after reflexion, follow the direction *b* B in the plane of the paper, the angle *p b* B being equal to the angle *a b p*. Let this reflected ray fall upon a second plane reflecting surface B, this surface likewise being at right angles with the plane of the paper, and B P perpendicular to it. The second reflexion will be in the direction B E, B E being in the plane of the paper, and the angle P B E equal to the angle *b* B P; and the eye being at E, will receive the ray and refer it to the direction A B E. Consequently, if at the same time the direct ray A B fall upon the eye, the objects from which the systems of rays parallel to *a b*, A B respectively emanate will appear to coincide.



Now it will be obvious that if the mirror *b* be made to revolve upon an axis at right angles to the plane of the paper, the original direction *a b* of the ray which, after two reflexions, is seen to coincide with the direct ray, will vary; in other words, the angle between the original directions of the two rays and that between the surfaces of the mirrors are dependent one on the other. To determine the connexion between them, produce A B, $a\,b$ to meet in E, and $p\,b$, B P to meet in P. Then—

angle between mirrors = angle between the normals to them $= \angle\, b\,P\,B$,

and angle between the rays $= \angle\; B\,E\,B$;

and $\because\ \angle a\,b\,B = 2\angle p\,b\,B$, and $\angle b\,B\,E = 2\angle b\,B\,P$,

$\therefore\ \angle a\,b\,B - \angle b\,B\,E = 2(\angle p\,b\,B - \angle b\,B\,P)$.

But $\angle a\,b\,B - \angle b\,B\,E = \angle B\,E\,b$
and $\angle p\,b\,B - \angle b\,B\,P = \angle b\,P\,B$ {because the exterior angle of a triangle is equal to the sum of the interior and opposite angles,

$\therefore\ \angle B\,E\,b = 2\angle b\,P\,B$;

or the angle between the original directions of the rays is the double of that between the mirrors.

If, therefore, with the revolution of the mirror b a radius $b\,Q$ revolve likewise, its extremity Q indicating upon the arc $Q\,q$, which it traces, the angle moved through, this combination of parts will supply the means of ascertaining the angle between the original directions of the rays. For if, in the first instance, we move the mirror b upon its centre until the direct and reflected images of the same object coincide, this object being so far distant that the rays emanating from it may, so far as the operator and his instrument are concerned, be considered parallel, the angle between the original direct rays being in this case nothing, that between the mirrors will be nothing likewise, and the reading of the indicator Q in this position will be that corresponding to the mirrors parallel. The difference between this reading and that obtained in any other position of $b\,Q$ will be the angle between the mirrors; and the double of this will be that between the rays from two objects, which in the second position are seen to coincide.

If the reader will now turn to the instrument itself, he will have no difficulty in identifying b as the index-glass with its attached arm $b\,Q$, B as the horizon-glass, and $Q\,q$ as the divided arc, a telescope being fixed in the direction E B to receive the rays parallel to A B. But with respect to the figures upon the arc, he will remark that each actual half degree is numbered as a degree in order to save him the operation of doubling the reading to obtain the desired angle, with which the indicator or vernier thus supplies him at once.

2. On attentively considering the process which has conducted to the above very simple result, expressing the connexion between the inclination of the mirrors and that of the direct rays from the objects whose images are seen to coincide, we shall perceive that the following conditions are implied:—

(1) That the arc Q q is in one plane, and that the axis of revolution of the index-glass is at right angles to this plane, and passes through the centre of the arc;

(2) That the surfaces of the index- and horizon-glasses are plane, and at right angles to the plane Q q, called the plane of the instrument, and that the interior and exterior surfaces of each of these glasses are parallel one to the other;

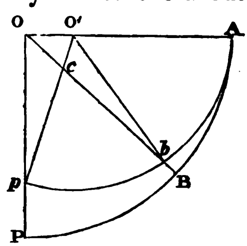
(3) That the optical axis of the telescope at E is parallel to the plane of the instrument.

And he will readily understand that, in the event of any deviation from these, the law will cease to obtain.

We proceed to treat each of these conditions in order.

3. With respect to the arc Q q, that it should be in one plane, and that the axis of revolution should be at right angles to this plane, these conditions can be fulfilled only in the excellence of the workmanship. Very small defects in these particulars will not produce sensible errors in the results; and the operator may easily satisfy himself that such defects do not exist to any important extent. For this purpose he has simply to move the vernier slowly and uniformly from one extremity to the other of the arc, the instrument being held with its face upwards and the clamp released from contact; and should he effect the motion by the application of a uniform pressure, there being no sense of obstruction or evidence of greater friction upon one than another portion of the arc, he may feel convinced that the conditions are sufficiently fulfilled.

4. But we suppose likewise that the axis of revolution passes through the centre of the arc. This is a condition of the greatest importance, and one that is scarcely ever fulfilled. Indeed, a very small deviation produces so great an effect, that it may be doubted whether it is in the power of the maker to fulfil it except by accident; but as the effect follows a certain law which renders it easy to obtain both the relative position of the centres of the arc and axis of revolution, as well as the amount of correction to be applied to any given reading on account of the deviation of one from the other, the difficulty is not one about which he need give himself trouble. To determine the law, let the points O, O′ in the annexed figure represent respectively the centre of the arc A B, and the axis of revolution of the index-bar and vernier O′ A. Let O′ p (= O′ A) be any position whatever of the index, and B the zero division of the arc. Join O B, and with centre O′ and radius O′ A describe the arc A b p. Then O′ b is the position of the index when the reading



is zero, and the reading corresponding to the position $O'p$ of the index is the double of the angle $b\,O\,P$ instead of that of $b\,O'p$, through which the index has actually moved, which angle we wish to obtain.

$O\,b$ and $O'p$ intersecting in c, we have from the triangles $O\,cp$, $O'c\,b$,

$$\angle bO'p + \angle ObO' = \angle bOp + \angle OpO',$$

$$\therefore\ \angle bO'p = \angle bOp + \angle OpO' - \angle ObO'.$$

But

$$\sin OpO' = \frac{OO'}{O'p} \,.\, \sin AOp = \frac{OO'}{O'p} \,.\, \sin (AOB + bOp)\,;$$

and since $O\,O'$ is small compared with $O'p$,

$$\angle OpO' = \frac{O\,O'}{O'p} \cdot \frac{\sin (AOB + bOp)}{\sin 1''},$$

and similarly,

$$\angle ObO' = \frac{OO'}{O'b} \cdot \frac{\sin AOB}{\sin 1''} = \frac{O\,O'}{O'p} \cdot \frac{\sin AOB}{\sin 1''},$$

$$\therefore\ \angle bO'p = \angle bOp + \frac{OO'}{O'p \,.\, \sin 1''}\Big\{\sin (AOB + bOp) - \sin AOB\Big\}.$$

Let $O'p =$ radius of index $= a$, $OO' = b$, $\angle AOB = \alpha$,

$\angle bOp =$ half the uncorrected reading $= \frac{1}{2}\omega$, $\angle bO'p =$ half the corrected reading $= \frac{1}{2}\Omega$.

Then

$$\Omega = \omega + \frac{2b}{a \,.\, \sin 1''}\Big\{\sin (\alpha + \tfrac{1}{2}\omega) - \sin \alpha\Big\}$$

$$= \omega + e \,.\, \{\sin (\alpha + \tfrac{1}{2}\,\omega) - \sin \alpha\} \text{ if } e = \frac{2b}{a \,.\, \sin 1''}.$$

Let ω_0 be the reading corresponding to coincidence of images of a single object, that is, corresponding to $\Omega = 0$,

$$\therefore\ 0 = \omega_0 + e \,.\{\sin (\alpha + \tfrac{1}{2}\,\omega_0) - \sin \alpha\}\,;$$

and subtracting this from the above,

$$\Omega = \omega - \omega_0 - e \,.\, \sin (\alpha + \tfrac{1}{2}\,\omega_0) + e \,.\, \sin (\alpha + \tfrac{1}{2}\,\omega)$$

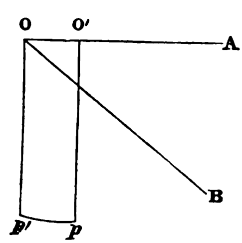
$$= \omega + \epsilon + e \,.\, \sin (\alpha + \tfrac{1}{2}\,\omega) \text{ if } \epsilon = -\omega_0 - e \,.\, \sin (\alpha + \tfrac{1}{2}\,\omega_0),$$

which is the general expression for the true angle Ω when the centre of the divided arc is not a point in the axis of revolution.

5. But in the preceding investigation we have supposed the reading obtained constantly at one point of the index, which, however, is not the case in practice. Attached to the index-bar is a short divided segment, called the vernier, by means of which we subdivide the spaces between the divisions on the arc; and the reading is obtained from that point of the vernier at which

one of its divisions is seen to coincide with one on the arc. The position of this point varies therefore with the minutes and seconds taken from it, and we have now to inquire in what way the results of our investigation are affected by this circumstance.

According to the principle of the vernier, the space between each two divisions on the arc being *m* minutes, that between each two on the vernier is *m* minutes less *m* seconds. Suppose, then, the zero division on the vernier to coincide with any division on the arc, say that representing 50° 10′, the spaces on the arc representing 10′, and those on the vernier 10′−10″. The first division from zero on the vernier, when the centre of this is coincident with that of the arc, will be in a position 10″ short of 50° 20′ on the arc; and consequently, if the vernier be advanced 10″, this first division will coincide with that representing 50° 20′ on the arc. The coincidence, then, of the first division of the vernier with the first on the arc beyond 50° 10′ will correspond to a position of the zero of the vernier representing 50° 10′ 10″ on the arc. Similarly, the coincidence of the second division of the vernier with the second beyond 50° 10′ on the arc, will give for the position of the zero of the vernier 50° 10′ 20″, and so on. With this explanation of the principle of the vernier, the manner in which readings are obtained by its means will be readily understood.



Now, as before, suppose O B to be the radius from the centre to the zero of the arc, $O'p$ any position of the radius to the zero of the vernier, and p' the point on the vernier from which in this position the reading is taken. The reading will clearly be

$$2\angle \mathrm{B\,O}p' - 2\angle p\,\mathrm{O}'p',$$

of which the former part only is affected by the deviation of the centres O, O′ one from the other, commonly called the error of excentricity, the latter $2\,p\,\mathrm{O}'p'$ being, in fact, taken from the arc of the vernier which has O′ for its centre. Hence the correction to be applied to the reading will be that due to the angle $\mathrm{B\,O}p'$, or to the degrees and minutes on the arc corresponding to the position of the coincidence of divisions on the arc and vernier. These degrees and minutes may in general be derived from the reading in a very simple way,—as it will be evident, on consideration of the principle above explained, that for every second taken from the vernier in excess of the degrees and minutes taken from the arc, the position of coincidence will advance one minute, and for every minute in excess it will advance one degree. Thus, suppose the reading to be 54° 27′ 17″. In this case we take 54° 20′ from the arc, and 7′ 17″ from the vernier. The coincidence ought therefore to take place at 54° 20′ + 7° 17′ = 61° 37′ on the arc, supposing each division on the arc to represent 1′, and on the vernier 1′ − 1″. This is true in general; but nevertheless—as it may happen, as a consequence of the error of excentricity, that there are two coincidences within the normal extent of the vernier, or that, there being no coincidence within the normal extent, it occurs beyond it on one side or the other—it will be necessary in such cases to remark the point of coincidence from which the reading is derived, although in other cases this precaution is needless. Hence ω being the reading, and (ω) the degrees and minutes on the arc at the point of coincidence from which the reading is derived, the general expression will be

$$\Omega = \omega + \epsilon + e \, . \sin \{\alpha + \tfrac{1}{2}(\omega)\},$$

where ϵ, e, and α are constants to be determined.

For the purpose of determining the values of these constants, let three known angles, Ω_1, Ω_2, Ω_3 be observed, and let ω_1, ω_2, ω_3 be the readings obtained from the limb of the sextant. Then, supposing these to be affected by no errors save those involving the said constants, we have

$$\Omega_1 = \omega_1 + \epsilon + e.\sin\{\alpha + \tfrac{1}{2}(\omega_1)\}; \quad \Omega_2 = \omega_2 + \epsilon + e.\sin\{\alpha + \tfrac{1}{2}(\omega_2)\};$$
$$\Omega_3 = \omega_3 + \epsilon + e \, . \sin\{\alpha + \tfrac{1}{2}(\omega_3)\}.$$

Subtracting the two last successively from the first,

$$\Omega_1 - \Omega_2 = \omega_1 - \omega_2 + e \, . [\sin\{\alpha + \tfrac{1}{2}(\omega_1)\} - \sin\{\alpha + \tfrac{1}{2}(\omega_2)\}]$$
$$= \omega_1 - \omega_2 + 2e.\sin\tfrac{1}{4}\{(\omega_1) - (\omega_2)\}.\cos\{\alpha + \tfrac{1}{4}[(\omega_1) + (\omega_2)]\},$$
$$\Omega_1 - \Omega_3 = \omega_1 - \omega_3 + 2e.\sin\tfrac{1}{4}\{(\omega_1) - (\omega_3)\}.\cos\{\alpha + \tfrac{1}{4}[(\omega_1) + (\omega_3)]\}.$$

Let $\Omega_1 - \Omega_2 - (\omega_1 - \omega_2) = c_1$, $\Omega_1 - \Omega_3 - (\omega_1 - \omega_3) = c_2$,

$\tfrac{1}{4}\{(\omega_1) - (\omega_2)\} = \beta_1$, $\tfrac{1}{4}\{(\omega_1) + (\omega_2)\} = \beta_2$, $\tfrac{1}{4}\{(\omega_1) - (\omega_3)\} = \gamma_1$, $\tfrac{1}{4}\{(\omega_1) + (\omega_3)\} = \gamma_2$,

then

$$c_1 = 2e \, . \sin\beta_1 \, . \cos(\alpha + \beta_2), \text{ and } c_2 = 2e \, . \sin\gamma_1 \, . \cos(\alpha + \gamma_2),$$
$$\therefore c_1 \, . \sin\gamma_1 \, . \cos(\alpha + \gamma_2) = c_2 \, . \sin\beta_1 \, . \cos(\alpha + \beta_2);$$

and dividing by $\cos\alpha$,

$$c_1.\sin\gamma_1(\cos\gamma_2 - \tan\alpha.\sin\gamma_2) = c_2.\sin\beta_1 \, . (\cos\beta_2 - \tan\alpha.\sin\beta_2),$$
$$\therefore \tan\alpha = \frac{c_2 \, . \sin\beta_1 \, . \cos\beta_2 - c_1 \, . \sin\gamma_1 \, . \cos\gamma_2}{c_2 \, . \sin\beta_1 \, . \sin\beta_2 - c_1 \, . \sin\gamma_1 \, . \sin\gamma_2},$$

whence we may compute $\tan\alpha$, and take α itself from the Tables.

The value of α being thus determined, we may compute that of e from either one of the equations $c_1 = 2e \,.\, \sin \beta_1 \,.\, \cos(\alpha + \beta_2)$, $c_2 = 2e \,.\, \sin \gamma_1 \,.\, \cos(\alpha + \gamma_2)$; and this as well as that of α being known, we may by substitution in any one of the three original equations, determine the value of the remaining constant ϵ.

The angles Ω_1, Ω_2, Ω_3 may be either those between well-known stars computed for the occasion by formulæ which will be found in the course of this work, or two of them may be double altitudes of similar objects on the meridian of a place the latitude of which is known; and they should be such that the readings ω_1, ω_2, ω_3 may be taken, one from about the middle of the arc, and the remaining two from near to each extremity of it. One of these may correspond to actual coincidence of images of the same object, in which case we shall have $\Omega_1 = 0$; or it may be the vertical diameter of the sun observed with a dark glass before the eyepiece of the telescope, the true diameter being taken from the Tables, and the difference of refraction at the upper and lower limb subtracted from it in order to obtain the apparent diameter.

Having determined the values of ϵ, e, and α, we are in a position to compute the correction $\epsilon + e \,.\, \sin\{\alpha + \frac{1}{2}(\omega)\}$ to any reading ω. But it is to be observed that the second term of this correction depends only upon e and α and the variable angle (ω), the two former of which are not liable to vary, save with a variation in the position of the centre of the divided arc with respect to the axis of revolution,—a change by no means probable in a well-constructed instrument, except as the result of a serious accident, or the operations of the maker in effecting a complete repair. Hence, having once obtained the values of e and α, we may tabulate those of $e \,.\, \sin\{\alpha + \frac{1}{2}(\omega)\}$ for every five or ten degrees of (ω), and from the table so constructed take that value which we require for the reading on each particular occasion. But the quantity ϵ depending not only upon e and α, but likewise upon the relative position of the planes of the index- and horizon-glasses in a given position of the index upon the arc, is more liable to change; and its value should be obtained as a preliminary to every important series of observations. This, fortunately, is an operation of no difficulty, since we have simply

$$\epsilon = -\omega_0 - e \,.\, \sin\{\alpha + \tfrac{1}{2}(\omega_0)\},$$

in which ω_0 is the reading corresponding to coincidence of images of a single object, and $e \,.\, \sin\{\alpha + \frac{1}{2}(\omega_0)\}$ the quantity corresponding to (ω_0) in the table of values of $e \,.\, \sin\{\alpha + \frac{1}{2}(\omega)\}$ supposed already constructed. At night, coincidence of the images of a star is easily obtained; but during the day, the only object adapted to the purpose is the sun: and in this case the contact of the limbs of the two images is in general a much better observation than that of complete coincidence.

Let $2r$ be the apparent vertical diameter of the sun,

ω_1, ω'_1 the readings corresponding to opposite coincidences of limbs;

then $$2r = \omega_1 + \epsilon + e \,.\, \sin\{\alpha + \tfrac{1}{2}(\omega_1)\}$$

$$-2r = \omega'_1 + \epsilon + e \,.\, \sin\{\alpha + \tfrac{1}{2}(\omega'_1)\};$$

and by addition,

$$0 = \omega_1 + \omega'_1 + 2\epsilon + e \,.\, \sin\{\alpha + \tfrac{1}{2}(\omega_1)\} + e \,.\, \sin\{\alpha + \tfrac{1}{2}(\omega'_1)\};$$

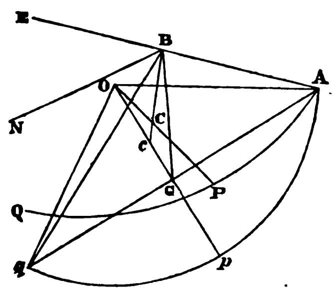
$\therefore \quad \epsilon = -\tfrac{1}{2}(\omega_1 + \omega'_1) -$ half the sum of the quantities in the table corresponding to (ω_1) and (ω'_1).

6. Before we proceed to apply the results of the above investigation to a definite example, it will be proper to consider the remaining conditions relative to the parts of the Sextant, of which the next in order will be those affecting the index- and horizon-glasses.

The surfaces of the index- and horizon-glasses are examined by the maker previously to the operation of silvering those intended to reflect the rays which fall upon them; and the tests applied are in their nature so searching, that any defect can scarcely escape an eye practised in their application. After silvering, it is more difficult to discover imperfections; but on bringing the limb of the reflected image of the sun into contact with that of the direct image of the moon in the middle of the field of the telescope, the angle between the two being not less than 90°, should the former image appear single and sharp, and the contact of limbs remain unbroken when the images are brought into other positions of the field, these new positions being with respect to the first in a line parallel to the plane of the instrument, it may safely be inferred that the defects, if any exist, are unimportant. Should there be indistinctness or appearance of double image about the reflected limb of the sun, or should the contact be broken on changing the position in the manner described, the glasses are imperfect, and the instrument should be supplied with others in their place. Taking for granted, then, that the surfaces of the glasses are plane, and that the interior and exterior surfaces of each are parallel one to the other*, we have next to place them in a position perpendicular to the plane of the instrument.

* We shall consider in its proper place the effect of any deviation from parallelism of the interior and exterior surfaces.

Let O A Q be the plane of the instrument, A P Q being the external edge of the limb supposed circular and to lie in the plane O A Q. O C B the plane of the index-glass in any position, O C being the common intersection of this with the plane O A Q, and B C the edge of the glass.



Let the eye at E see coincident with B the point A of the edge of the limb. Then, the plane of the instrument not being at right angles with the plane of the index-glass, let A O q be supposed at right angles with it, and in this plane and with radius O A describe the arc A p q. The angle P A p will thus represent the deviation of the plane of the index-glass from its correct position.

Let the planes O C B, A O q intersect in O c, let O c be produced to p, and take $\angle p O q = \angle p O A$. Join A q, B q, the former cutting O p in G, and let B N be the normal at B to the plane of the index-glass.

Then in the triangles A O G, q O G we have $AO = Oq$, O G common, and $\angle AOG = q \angle OG$,

$\therefore$ $AG = Gq$, and each of the angles A G O, qGO is a right angle.

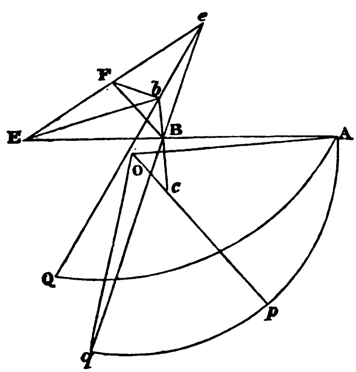
And A q being thus at right angles to O p, the common intersection of the planes A O q, B O c, which are at right angles one to the other, and being situated in one of them, is itself at right angles to the other plane B c O, and therefore to the line G B which lies in it.

And in the triangles A G B, q G B we have $AG = qG$, G B common, and each of the angles A G B, q G B a right angle; $\therefore$ $\angle BAq = \angle BqA$. But A q and B N being each at right angles to the plane OcB, are parallel one to the other, and therefore $\angle NBq$ = alternate $\angle BqA$ and $\angle EBN$ = interior and opposite $\angle BAq$;

$\therefore$ $\angle EBN = \angle NBq$, a result which shows that the point q will be seen reflected at B, and coincident with A seen there direct. Any point Q, therefore, in the edge of the limb will be seen reflected above or below B, except when the angle Q A q vanishes; that is, when the planes A O Q, A O q are coincident, or when the plane of the index-glass and that of the instrument are at right angles one to the other.

This, therefore, furnishes us at once with a test of the relative positions of these two planes; for we have only to place the index about the middle of the arc, and the eye in a position

E behind the centre, looking directly at a point A of the limb where the edge of the index-glass cuts it. If the plane of the index-glass is at right angles to that of the instrument, we shall see reflected at the same point B, the point Q of the limb; but any deviation of the former from this position will cause the reflected limb, where it is cut off by the edge of the index-glass, to appear above or below B. This of course supposes that the limb A Q is circular, and in one plane; consequently, by applying the test in different positions of the index along the arc, we have an additional criterion whereby to judge respecting the figure of the latter. The operation of correcting the position of the index-glass is performed by rubbing away the surfaces of the pins behind it until the condition is fulfilled; and if, on satisfying it for one position of the index, it appears satisfied for others likewise, we may be sure that the limb is sensibly in one plane, and that the index-glass and the centre upon which it moves are nearly perpendicular to this plane. The effect of any small deviation of the index-glass from its correct position we shall investigate in the proper place; but it will be convenient to inquire, in this, what amount of accuracy we may expect after satisfying the above condition to the best of our ability.



In the annexed figure, OAQ (as before) represents the plane of the instrument, A Q the external edge of the limb, O *c* B the plane of the index-glass, and O A*q* a plane perpendicular to it. E is the position of the eye looking towards A, as before.

Draw E F *e* at right angles to the plane O *c* B extended, meeting this plane in F, and take F *e* = F E. Through *e* and B *c* let a plane be supposed to pass and to cut the edge of the limb in Q. Join *e* Q, cutting *c* B, produced, if necessary, in *b*. Join F *b* and E *b*.

Then in triangles E F *b*, *e* F *b* we have E F = F *e*, F *b* common, and rt ∠ E F *b* = rt ∠ *e* F *b*,

$$\therefore \ E b = b e \text{ and } \angle E b F = \angle e b F.$$

The ray Q *b* will therefore be reflected in the direction *b* E, and the point Q of the edge of the limb will be seen at the edge of the index-glass at *b*.

We have now to find the angle B E *b* subtended by the distance between the two broken parts of the limb.

Join e B, F B. Then in triangles E B F, e B F we have, as before, E B $= e$ B and $\angle$ E B F $= \angle e$ B F, and consequently any point in e B produced will be seen reflected at B. But we have already shown that the point seen reflected at B is a point q in the arc A q described with radius O A, and determined by $\angle p$ O $q = \angle p$ O A. Hence e B q is a straight line.

Now in triangles EBb, eBb we have EB$=e$B, E$b$$=eb$, and B$b$ common,

$$\therefore \angle \mathrm{BE}b = \angle \mathrm{B}eb = \angle \mathrm{Q}eq.$$

But

$$\angle \mathrm{Q}eq = \frac{\mathrm{Q}q}{eq \,.\, \sin 1''} \text{ nearly } = \frac{\mathrm{Q}q}{\mathrm{EA} \,.\, \sin 1''} = \frac{\mathrm{OA.QA}q''.\sin 1''.\sin \mathrm{AQ}}{\mathrm{EA}.\sin 1''}$$

$$\text{nearly } = \frac{\mathrm{QA}q''.\sin \mathrm{AQ}}{1 + \frac{\mathrm{EO}}{\mathrm{OA}}} \text{ nearly,—}$$

an expression which, when the angle Q A q is small, and the eye is placed not very far above the plane of the instrument, may be treated as the measure of the angle B E b.

Now our object in practice is to make a given error in the position of the plane of the index-glass produce the greatest effect in the phenomenon through which that error becomes manifest to us; that is, for a given value of Q A q the angle B E b ought to be as great as possible; and this condition will evidently be fulfilled by making A Q$=90°$, and E O as small as we can. But A Q being nearly the double of Ap, may, for our present purpose, be considered as equal to the angle read from the arc in the position O c B of the index; and A and Q being taken as the opposite extremities of the limb, the arc A Q cannot in the Sextant be much greater than 60°, this being its magnitude when the point A, seen directly, is situated behind the zero, and Q, the point seen by reflexion, behind the division numbered 120°. Hence it is clear that, in correcting the position of the plane of the index-glass, the eye should be placed as near to the centre as the parts of the instrument will allow, and that, the point seen directly being that opposite to the zero division of the arc, the zero of the vernier should be placed at 60°, or thereabouts. Let AQ$=60°$, EO$=$ 3 inches, OA$=$ 8 inches, QA$q=1'$, then

$$\mathrm{BE}b = \frac{60'' \times \sin 60°}{1 + \frac{3}{8}} = 38'';$$

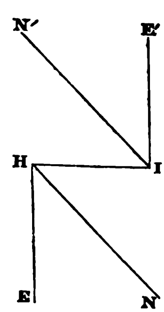
or an error of one minute in the position of the plane of the index-glass will in this case produce a break between the direct and reflected images of the limb, the space between the broken parts subtending at the eye an angle of 38″.

What may be the smallest angle sensible to the eye in an operation of this kind it would be difficult to determine; but in one of a similar character, namely, obtaining index error by reference to the sea horizon without the aid of a telescope, an observer will probably never commit an error exceeding one minute. The line of the edge of the limb, however, is by no means so distinct as that of the horizon at sea in clear weather, and we must moreover allow for some deviation in it from a perfectly plane figure. Suppose $2\frac{1}{2}$ minutes to be the limit of possible error in the observation; then

$$\text{QA}q = \frac{\left\{1 + \frac{\text{EO}}{\text{OA}}\right\} \times \text{BE}b}{\sin \text{AQ}} = \frac{11 \times 2'{\cdot}5}{8 \times \sin 60^\circ} = 4' \text{ nearly.}$$

An error of $2\frac{1}{2}$ minutes, therefore, in the observation of coincidence of the points of the limb seen directly and by reflexion, would in this case introduce one of 4′ in the position of the plane of the index-glass; and this may perhaps be considered the extreme possible limit in a well-constructed instrument. We shall have occasion to refer to this result in the sequel.

7. The plane of the horizon-glass, like that of the index-glass, should be at right angles to the plane of the instrument; but we have no means of testing the position of this independently. The general principle of reflexion, however, leads us at once to a method of examining the position of the planes of the glasses relatively to one another; and as the frame in which the horizon-glass is mounted is provided with an apparatus which enables us to alter the inclination of its plane with respect to that of the instrument, we may thus place it parallel to that of the index-glass in a certain position of the latter, in which case, supposing the axis upon which the index-glass revolves to be perpendicular to the plane of the instrument, the inclination of the planes of the glasses with respect to this plane will be constantly equal; and that of the index-glass being at right angles to it, the plane of the horizon-glass will be so likewise.



Let H E be parallel to the direct ray from a distant object received by the eye along the optical axis of the telescope, H N perpendicular to the plane of the horizon-glass. In the plane E H N make the angle N H I = ∠N H E. Then, when the direct and reflected images of an object coincide, I H will be parallel to the ray from the index-glass upon the horizon-glass which is reflected from the latter along H E. But the ray which falls upon the index-glass will be E′I parallel to H E. If therefore we bisect the angle E′I H by the straight line I N′, I N′ will be perpendicular to the plane of the index-glass.

Since H I meets the parallels HE, IE′, ∴ ∠EHI = ∠E′IH;

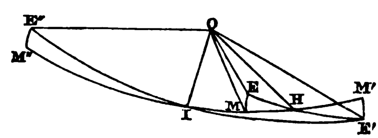
and ∵ ∠NHI = ½ ∠EHI, and ∠N′IH = ½ ∠E′IH;

∴ ∠NHI = ∠N′IH;

and ∴ H N and I N′ are parallel, and consequently the planes of the horizon- and index-glasses, to which these lines respectively are normals, are parallel likewise.

It thus appears that if the direct and reflected images of a distant object, the sun or a star for example, can be made to coincide, the planes of the index- and horizon-glasses, in the position in which the coincidence occurs, are parallel to one another. This is a test which can be immediately applied; and should it appear on trial that, on moving the index, the images approach and recede without coinciding, we have to do no more than to move the proper screw attached to the frame of the horizon-glass until complete coincidence is effected. The planes of the glasses will then be parallel; and since that of the index-glass is supposed not to change its inclination to the plane of the instrument in the course of revolution, this inclination will be constantly the same for both glasses.

8. The last of the conditions to be considered is that affecting the optical axis, which we have supposed to be parallel to the plane of the instrument; and the most convenient way of treating this will be to ascertain the effect of a small deviation from the correct position.



Let O be the centre upon which the index revolves, as well as that of the divided arc; and suppose it likewise the centre of a sphere intersected in M′ H M I M″ by the plane of the instrument. Let O E, meeting the surface of the sphere in E, be parallel to the line of sight in which coincidences are observed, and consequently to the direction of the rays received from one object without reflexion, and of those from a second object, which, after successive reflexions from the index- and horizon-glasses, appear to coincide with the rays from the first. Then if E M be the arc of a great circle formed by a plane perpendicular to that of the instrument, E M will measure the inclination of the line of sight or optical axis.

Let E M $= \eta$, this quantity being supposed small.

O H is perpendicular to the plane of the horizon-glass, and I O to that of the index-glass; so that, ω_0 being the reading which corresponds to coincidence of the direct and reflected images of the same object, and ω to the position of the normals assumed in the figure, we shall have $\angle$ HOI $=\frac{1}{2}(\omega-\omega_0)$. The angle M O H depends upon the construction of the instrument, and may be measured and treated as known. Let $\angle$ MOH $=\beta$.

The rays from the horizon-glass following the direction O E, those falling upon it from the index-glass will be parallel to E′O if, in the plane E O H, we make $\angle$ H O E′ $=\angle$ H O E; and those from the index-glass following the direction E′ O, those which fall from the second object upon this glass will be parallel to O E″ if, in the plane E′O I, we make $\angle$ I O E″ $=\angle$ I O E′; and O E being the direction of the rays from the first object, and O E″ that of those from the second, the true angle between them will be measured by the arc E E″.

Let EE″ $=\Omega$, and describe E′M′, E″M″ arcs of great circles making right angles with the arc M′IM″.

Then from the spherical triangles EHM, E′HM′ we have

$$\sin \text{EM} = \sin \text{EH} . \sin \text{EHM} = \sin \text{E}'\text{H} . \sin \text{E}'\text{HM}' = \sin \text{E}'\text{M}',$$

$$\therefore \text{EM} = \text{E}'\text{M}'.$$

Similarly, from the triangles IE′M′, IE″M″,

$$\text{E}''\text{M}'' = \text{E}'\text{M}' = \therefore \text{EM} = \eta.$$

Also

$$\cos \text{EE}'' = \sin \text{EM} . \sin \text{E}''\text{M}'' + \cos \text{EM} . \cos \text{E}''\text{M}'' . \cos \text{MM}'' *.$$

And

$$\text{MM}'' = \text{IM} + \text{IM}'' = \text{IM} + \text{IM}' = 2\text{IM} + 2\text{MH} = 2\text{IOH} = \omega - \omega_0;$$

$$\therefore \cos \Omega = \sin^2 \eta + \cos^2 \eta . \cos(\omega - \omega_0)$$

$$= \cos(\omega - \omega_0) + \sin^2 \eta \{1 - \cos(\omega - \omega_0)\}$$

$$= \cos(\omega - \omega_0) + 2\eta^2 . \sin^2 1'' . \sin^2 \tfrac{1}{2}(\omega - \omega_0);$$

$$\therefore \Omega = \omega - \omega_0 - \eta^2 . \sin 1'' . \tan \tfrac{1}{2}(\omega - \omega_0);$$

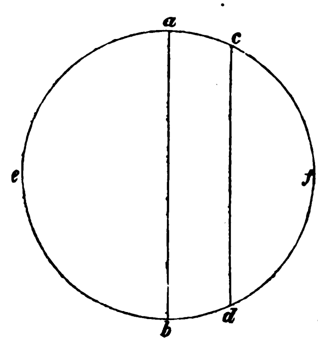
or, ω_0 being small, as it usually is, a deviation η of the optical axis from its correct position will increase the readings by a quantity $\eta^2 . \sin 1'' . \tan \frac{1}{2}\omega$, ω being the degrees and minutes read.

It thus appears that to a coincidence observed in a line of sight parallel to the plane of the instrument, corresponds a reading smaller than that which an observation in a line of sight in any other position will give; from which it follows, that if we so place the telescope that the least reading shall be obtained in the middle of the field or at any point in the line parallel to the plane of the instrument which divides the field into two equal parts, the optical axis itself will be parallel to the plane of the

* This relation will be at once manifest if we produce the arcs M E, M″E″ to meet in the pole of M M″.

instrument. In this case, the images having been made to coincide in the middle of the field, they will appear to overlap when brought into a position on either side of it, the apparent overlap increasing as the square of their distance from the middle line, and, at a given distance from the middle line, varying as the tangent of half the angle between the objects observed. And the optical axis not being parallel to the plane of the instrument, it is evident that the phenomena will be the same with respect to a line not passing through the middle of the field, the images overlapping equally at equal distances upon either side of it: or the telescope may even be so much out of position, that a coincidence being made at one side of the field, the images will continually overlap in their passage towards the other side.

We may express these phenomena in a different manner, and one better adapted to the treatment of the case in practice. Suppose the telescope out of position; and in the annexed diagram let *a c d b* represent the field of view, *c d* being the line parallel to the plane of the instrument on which the least reading is obtained, and *a b*, parallel to it, that which divides the field into two equal parts. Then the direct and reflected images of two objects being made to coincide upon the line *a b*, they will appear to overlap when brought into another position in the direction *e* away from the line *c d*, and to open when moved in the opposite direction towards *c d*, continuing to open until brought upon *c d*, beyond which they will again close, and, should the field be large enough, will at last overlap. Hence, should we, after making the coincidence on the line *a b*, observe an overlapping towards *e* and an opening towards *f*, it will be evident that the line of sight parallel to the plane of the instrument intersects the field (or the field extended) on the side of *a b* towards *f*; and as we wish to make this line identical with *a b*, we must move the telescope with respect to the plane of the instrument, turning it upon an axis parallel to *a b*; and, the inverting eyepiece being employed, it is clear that the end *e* must approach us, and *f* recede from us. The reader has only to refer to the instrument to perceive that this adjustment is effected by means of two screws holding the collar into which the telescope fits; and bearing in mind that the effect of an error varies at a given distance from the true line as the tangent of half the angle between the objects observed,



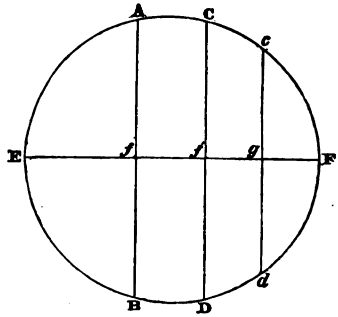
he will see that the above-explained process, reduced to rule, may be briefly expressed in the following terms:—

Make the direct image of one object, and the reflected image of another, coincide in the middle of the field, the angle between the objects being as large as possible. Then if, on being brought away from the middle of the field, the images appear on one side to open and on the other to close, release the screw on the side towards which they close, and tighten that towards which they open, repeating this process until they appear to close immediately on passing from the middle towards either side of the field.

For the purpose of removing any difficulty the operator might experience in determining the middle of the field, the eyepiece of the telescope is furnished with two parallel wires, sufficiently close to enable him to estimate the position of the middle between them, and still sufficiently distant to prevent their interfering with the observation of objects. When the Sextant is prepared for work, these wires are placed parallel to the plane of the instrument, an operation which the observer must perform for himself if the proper position of the eyepiece has not been previously marked by the maker. To perform it, nothing more is requisite than to place the index just so far from the zero position (in which the index- and horizon-glasses are parallel), that, the direct image of one object being brought near to the top of the field, the reflected image of the other may appear at the bottom, or *vice versâ*. Then, one image being brought to the edge of one of the wires, the other image must likewise appear on the same edge of the same wire, and, not being there, the eyepiece must be turned by the hand until this condition is fulfilled. The position once obtained, the eyepiece and its cell should be marked accordingly, in order that the observer may be spared the trouble of repeating the operation on future occasions. This done, and the telescope adjusted, it is evident that all observations should be made in the middle of the field, as nearly as we can estimate it.

The two parallel wires answer likewise another purpose; but to apply them to this it is first necessary to obtain the apparent angle between them. To obtain this angle, turn the eyepiece in its cell until the wires are as nearly perpendicular to the plane of the instrument as we can estimate. Let ω_0 be the reading which corresponds to coincidence of the direct and reflected images of an object, and ω_1 that to the position which places the direct image on the outside of one wire, and the reflected image on the outside of the other. Then $\omega_1 - \omega_0$ will be the angle between the external edges of the two wires. This angle known, we may apply it in the determination of the amount of error in the position of the optical axis, so that, in the event of the adjustment not being quite perfect, we may, without troubling the screws, apply the necessary corrections to angles observed in the middle of the field.

The wires being restored to their correct position parallel to the plane of the instrument, let $2f$ be the distance between their two external edges A B, C D; η the distance of the line $c\,d$ of least reading from the middle of the field. Then E F being at right angles to A B, C D and $c\,d$, we shall have $ff' = 2f$, $fg = \eta + f$, $f'g = \eta - f$. Let ω, ω' be the readings obtained for coincidence of images of two objects on A B, C D respectively. Then, Ω being the true angle,



$$\Omega = \omega - \omega_0 - (\eta + f)^2 . \sin 1'' . \tan \tfrac{1}{2}\,\omega = \omega' - \omega_0 - (\eta - f)^2 . \sin 1'' . \tan \tfrac{1}{2}\,\omega'.$$

And since ω' will differ from ω by a small quantity only, or $\tan \frac{1}{2}\,\omega = \tan \frac{1}{2}\,\omega'$ nearly,

$$\therefore \omega - (\eta + f)^2 . \sin 1'' . \tan \tfrac{1}{2}\,\omega = \omega' - (\eta - f)^2 . \sin 1'' . \tan \tfrac{1}{2}\,\omega \text{ nearly,}$$

or $$\omega - \omega' = 4\,\eta f . \sin 1'' . \tan \tfrac{1}{2}\,\omega,$$

$$\therefore \quad \eta = \frac{\omega - \omega'}{4f . \sin 1'' . \tan \frac{1}{2}\,\omega},$$

from which η is known; and we may thence readily compute the correction due to any reading ω obtained for coincidence made in the middle of the field, or we may take it from the following Table of values of $\eta^2 . \sin 1'' . \tan \frac{1}{2}\,\omega$ computed for various values of η up to 1° and from 0 to 125° of ω*.

* The magnitude of the corrections exhibited in this Table shows the importance of making observations in the middle of the field, a point to which some observers do not attend so carefully as they should. A deviation of 20′, supposing $\eta = 0$, will, at an angle of 100°, produce an error of 8″; and if $\eta = 5'$, one of 12″ or 4″ according to the side upon which the deviation occurs: and it may be here remarked, that the difficulty of keeping the objects in the middle of the field during a time sufficient to enable the operator to effect coincidence on contact is the principle cause of the inferiority of observations made at sea.

ω.	Values of η.												ω.
	5′	10′	15′	20′	25′	30′	35′	40′	45′	50′	55′	60′	
°	″	″	″	″	″	″	″	″	″	″	″	″	°
10	0·0	0·2	0·3	0·6	1·0	1·4	1·9	2·4	3·1	3·8	4·6	5·5	10
20	0·1	0·3	0·7	1·2	1·9	2·8	3·8	4·9	6·2	7·7	9·3	11·1	20
30	0·1	0·5	1·1	1·9	2·9	4·2	5·7	7·5	9·5	11·7	14·1	16·8	30
40	0·2	0·6	1·4	2·5	4·0	5·7	7·8	10·2	12·9	15·9	19·2	22·9	40
50	0·2	0·8	1·8	3·3	5·1	7·3	10·0	13·0	16·5	20·3	24·6	29·3	50
60	0·3	1·0	2·3	4·0	6·3	9·1	12·3	16·1	20·4	25·2	30·5	36·3	60
65	0·3	1·1	2·5	4·4	6·9	10·0	13·6	17·8	22·5	27·8	33·6	40·0	65
70	0·3	1·2	2·7	4·8	7·6	11·0	15·0	19·6	24·7	30·6	37·0	44·0	70
75	0·3	1·3	3·0	5·4	8·4	12·1	16·4	21·4	27·1	33·5	40·5	48·2	75
80	0·4	1·5	3·3	5·9	9·2	13·2	17·9	23·4	29·7	36·6	44·3	52·7	80
85	0·4	1·6	3·6	6·4	10·0	14·4	19·6	25·6	32·4	40·0	48·4	57·6	85
90	0·4	1·7	3·9	7·0	10·9	15·7	21·4	27·9	35·3	43·6	52·8	62·8	90
95	0·5	1·9	4·3	7·6	11·9	17·1	23·3	30·5	38·6	47·6	57·6	68·6	95
100	0·5	2·1	4·7	8·3	13·0	18·7	25·5	33·3	42·1	52·0	62·9	74·9	100
105	0·6	2·3	5·1	9·1	14·2	20·5	27·9	36·4	46·1	56·9	68·8	81·9	105
110	0·6	2·5	5·6	10·0	15·6	22·4	30·5	39·9	50·5	62·3	75·4	89·7	110
115	0·7	2·7	6·2	11·0	17·1	24·7	33·6	43·8	55·5	68·5	82·9	98·6	115
120	0·8	3·0	6·8	12·1	18·9	27·2	37·0	48·4	61·2	75·6	91·4	108·8	120
125	0·8	3·4	7·5	13·4	21·0	30·2	41·1	53·6	67·9	83·8	101·4	120·7	125

It will be evident that the greater the values of ω and f, the more accurately we shall obtain that of η. But f, being half the distance between the wires, must not be so great, that is, the wires must not be so wide apart as to render it difficult to estimate the position of the middle between them, whilst, if they are very close, the resulting value of f will be so small as to be of little service in the determination of η. The eyepiece ought therefore to be furnished with a set of four parallel wires,—two strong wires at a distance from one another, small enough to enable us to estimate the position of the middle between them, and two finer wires upon either side of these and at equal distances from them. The latter may then be exclusively employed in observations made for the purpose of determining the position of the optical axis, as f may thus be made as great as we please within the limits of the field*, without interference with the strong wires, which may be placed in their usual position and be applied solely to their usual object.

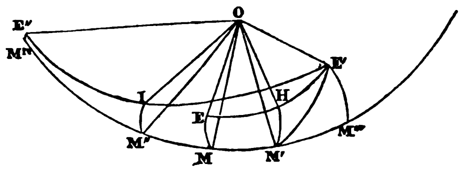
9. We now pass to a more general treatment of the question; and supposing the simultaneous existence of errors in the positions of the optical axis and the planes of the index- and horizon-

* The second term of the expansion of Ω is

$$\tfrac{1}{4} . \tan \tfrac{1}{2}\omega . (\tfrac{1}{3} + \tan^2 \tfrac{1}{2}\omega) . \eta^4 \sin^3 1'' ;$$

and as this does not become sensible in magnitude for any value of η less than 2°, even for the maximum value 125° of ω, we may safely have the parallel wires for this purpose 4° apart.

glasses, we will investigate an expression for the true angle Ω in terms of the angle ω read from the limb after coincidence of images in the middle of the field, and the errors in the positions of these fundamental parts.



Let O be the centre upon which the index revolves as well as that of the divided arc, and suppose it likewise the centre of a sphere intersected in M''', M, M^{iv} by the plane of the instrument. O E is parallel to the optical axis; and the arc E M, at right angles to M''' M M^{iv}, measures the error of its position. Let E M $= \eta$.

O H is parallel to the normal to the plane of the horizon-glass, and O I to that of the index-glass; and the arcs H M', I M'', at right angles to M'''MM^{iv}, measure the errors of position of these planes. Let HM' $= \xi$, IM'' $= \iota$.

In the plane EOH make $\angle$ HOE' $=$ $\angle$ HOE. Then E' O will be parallel to the rays which fall from the index- upon the horizon-glass; and if in the plane E' O I we make $\angle$ I O E'' $=$ $\angle$ I O E', O E'' will be the direction of the rays from the second object as they fall upon the index-glass. The true angle Ω will therefore be measured by the arc E E''.

Let $\angle$ M O M', which depends upon construction and is known, $= \beta$.

Then

$$\cos \mathrm{EH} = \sin \mathrm{EM} \,.\, \sin \mathrm{HM'} + \cos \mathrm{EM} \,.\, \cos \mathrm{HM'} \,.\, \cos \mathrm{MM'}$$
$$= \eta\,\xi.\sin^2 1'' + (1 - \tfrac{1}{2}\,\eta^2 \,.\, \sin^2 1'').(1 - \tfrac{1}{2}\,\xi^2.\sin^2 1'').\cos\beta$$
$$= \cos\beta - \tfrac{1}{2}\big((\eta^2 + \xi^2).\cos\beta - 2\eta\xi\big).\sin^2 1'' = \text{also} \cos \mathrm{E'H},$$
$$\therefore\ \mathrm{EH} = \mathrm{E'H} = \beta + \big((\eta^2 + \xi^2) \,.\, \cos\beta - 2\eta\xi\big) \,.\, \sin 1'' \div 2 \sin\beta.$$

And

$$\sin \mathrm{EM} = \sin \mathrm{HM'} \,.\, \cos \mathrm{EH} + \cos \mathrm{HM'} \,.\, \sin \mathrm{EH} \,.\, \cos \mathrm{E'HM'},$$

or, retaining quantities of the first order only,

$$\eta \,.\, \sin 1'' = \xi \,.\, \cos\beta \,.\, \sin 1'' + \sin\beta \,.\, \cos \mathrm{E'HM'},$$
$$\therefore\ \cos \mathrm{E'HM'} = (\eta - \xi \,.\, \cos\beta) \,.\, \sin 1'' \div \sin\beta \text{ and}$$
$$\mathrm{E'HM'} = 90^\circ - (\eta - \xi \,.\, \cos\beta) \div \sin\beta.$$

Again,

$$\cos \mathrm{E'M'} = \cos \mathrm{HM'} \,.\, \cos \mathrm{E'H} + \sin \mathrm{HM'} \,.\, \sin \mathrm{E'H} \,.\, \cos \mathrm{E'HM'}$$
$$= (1 - \tfrac{1}{2}\,\xi^2 \,.\, \sin^2 1'') \,.\, \big(\cos\beta - \tfrac{1}{2} \,.\, \big((\eta^2 + \xi^2) \,.\, \cos\beta$$
$$- 2\eta\xi\big).\sin^2 1''\big) + \xi \,.\, (\eta - \xi \,.\, \cos\beta) \sin^2 1''$$
$$= \cos\beta - \tfrac{1}{2}\big((\eta^2 + 4\xi^2) \,.\, \cos\beta - 4\eta\xi\big) \,.\, \sin^2 1'',$$
$$\therefore\ \mathrm{E'M'} = \beta + \tfrac{1}{2} \,.\, \big((\eta^2 + 4\xi^2) \,.\, \cos\beta - 4\eta\xi\big) \,.\, \sin 1'' \div \sin\beta.$$

Also

$\sin E'M''' = \cos E'H . \sin HM' - \sin E'H . \cos HM' . \cos E'HM'$,

$E'M'''$ being at right angles to $M'''M$,

$= \xi . \cos\beta . \sin 1'' - (\eta - \xi . \cos\beta) . \sin 1''$

$= (2\xi . \cos\beta - \eta) . \sin 1''$,

$\therefore E'M''' = 2\xi . \cos\beta - \eta$.

From $\Delta E'M'M'''$ we likewise have

$$\cos M'M''' = \cos E'M' \div \cos E'M'''$$
$$= \left(\cos\beta - \tfrac{1}{2} . \left((\eta^2 + 4\xi^2) . \cos\beta - 4\eta\xi\right) . \sin^2 1''\right) \times \left(1 + \tfrac{1}{2}(2\xi . \cos\beta - \eta)^2 . \sin^2 1''\right)$$
$$= \cos\beta + 2\xi . (\eta - \xi . \cos\beta) . \sin^2\beta . \sin^2 1'',$$
$$\therefore M'M''' = \beta - 2\xi . (\eta - \xi . \cos\beta) . \sin\beta . \sin 1''.$$

Let $E''M^{iv}$ be at right angles to the arc $M'''MM^{iv}$. Then, as we had

$E'M''' = 2HM' . \cos MM' - EM$, we shall have

$E''M^{iv} = 2IM'' . \cos M''M''' - E'M'''$.

Let ω_0 be the reading on the limb when OI is in the plane HOM′; ω that for the position assumed in the figure. Then $M'M'' = \frac{1}{2}(\omega - \omega_0)$ and

$M''M''' = \frac{1}{2}(\omega - \omega_0) + \beta - 2\xi . (\eta - \xi . \cos\beta) . \sin\beta . \sin 1''$.

Make $\frac{1}{2}(\omega - \omega_0) + \beta = \theta$,

$\therefore E''M^{iv} = 2\iota . \cos\theta - 2\xi . \cos\beta + \eta$.

Also, as we had

$M'M''' = MM' - 2HM' . (EM - HM' . \cos MM') . \sin MM' . \sin 1''$,

we shall have

$$M''M^{iv} = M''M''' - 2IM'' . (E'M''' - IM'' . \cos M''M''') . \sin M''M''' \sin 1''$$
$$= \theta - 2\xi . (\eta - \xi . \cos\beta) . \sin\beta . \sin 1'' - 2\iota(2\xi . \cos\beta - \eta - \iota . \cos\theta) . \sin\theta . \sin 1''$$
$$= \theta + 2\sin 1'' . (\xi^2 . \sin\beta . \cos\beta + \iota^2 . \sin\theta . \cos\theta - \eta\xi . \sin\beta + \eta\iota . \sin\theta - 2\iota\xi \cos\beta . \sin\theta).$$

And

$$MM'' = \tfrac{1}{2}(\omega - \omega_0) - \beta,$$
$$\therefore MM^{iv} = \omega - \omega_0 + 2\sin 1'' . (\xi^2 . \sin\beta . \cos\beta + \iota^2 . \sin\theta . \cos\theta - \eta\xi . \sin\beta + \eta\iota . \sin\theta - 2\iota\xi . \cos\beta . \sin\theta).$$

Now

$$\cos\Omega = \cos EE'' = \sin EM . \sin E''M^{iv} + \cos EM . \cos E''M^{iv} . \cos MM^{iv}$$
$$= EM . E''M^{iv} . \sin^2 1'' + \left(1 - \tfrac{1}{2}(EM^2 + E''M^{iv2}) . \sin^2 1''\right) . \cos MM^{iv}$$
$$= \cos MM^{iv} - \tfrac{1}{2} . \sin^2 1''\left((EM^2 + E''M^{iv2}) . \cos MM^{iv} - 2\,EM . E''M^{iv}\right)$$
$$\therefore \Omega = MM^{iv} + \tfrac{1}{2} . \sin 1'' . \left((EM^2 + E''M^{iv2}) . \cos MM^{iv} - 2\,EM . E''M^{iv}\right) \div \sin MM^{iv},$$

$$\text{or } \Omega=\begin{cases} \omega-\omega_0+2\sin 1''(\xi^2.\sin\beta.\cos\beta+\iota^2.\sin\theta.\cos\theta-\eta\xi \\ \quad .\sin\beta+\eta\iota.\sin\theta-2\,\iota\xi.\cos\beta.\sin\theta) \\ +\dfrac{\sin 1''}{2\sin(\omega-\omega_0)}\times\begin{pmatrix}(2\eta^2+4\xi^2.\cos^2\beta+4\iota^2.\cos^2\theta-4\eta\xi \\ .\cos\beta+4\eta\iota.\cos\theta-8\,\iota\xi\cos\beta \\ .\cos\theta).\cos(\omega-\omega_0)-2\eta.(2\,\iota \\ .\cos\theta-2\,\xi.\cos\beta+\eta)\end{pmatrix}\end{cases}$$

Collecting the terms, the coefficient of $\eta^2.\sin 1''$ is

$$\frac{\cos(\omega-\omega_0)}{\sin(\omega-\omega_0)}-\frac{1}{\sin(\omega-\omega_0)}=-\tan\tfrac{1}{2}(\omega-\omega_0).$$

The coefficient of $\xi^2.\sin 1''$ is

$$2\sin\beta.\cos\beta+\frac{2\cos^2\beta.\cos(\omega-\omega_0)}{\sin(\omega-\omega_0)}=\frac{2.\cos\beta.\cos(\omega-\omega_0-\beta)}{\sin(\omega-\omega_0)}.$$

The coefficient of $\iota^2.\sin 1''$ is

$$2\sin\theta.\cos\theta+\frac{2.\cos^2\theta.\cos(\omega-\omega_0)}{\sin(\omega-\omega_0)}=\frac{2\cos\theta.\cos(\omega-\omega_0-\theta)}{\sin(\omega-\omega_0)}$$

$$=\frac{2\cos\left(\frac{1}{2}(\omega-\omega_0)+\beta\right).\cos\left(\frac{1}{2}(\omega-\omega_0)-\beta\right)}{\sin(\omega-\omega_0)}.$$

The coefficient of $\eta\xi.\sin 1''$ is

$$-2\sin\beta-\frac{2\cos\beta.\cos(\omega-\omega_0)}{\sin(\omega-\omega_0)}+\frac{2\cos\beta}{\sin(\omega-\omega_0)}=\frac{2\sin\left(\frac{1}{2}(\omega-\omega_0)-\beta\right)}{\cos\frac{1}{2}(\omega-\omega_0)}.$$

The coefficient of $\eta\iota.\sin 1''$ is

$$+2\sin\theta+\frac{2\cos\theta.\cos(\omega-\omega_0)}{\sin(\omega-\omega_0)}-\frac{2\cos\theta}{\sin(\omega-\omega_0)}=\frac{2\sin\beta}{\cos\frac{1}{2}(\omega-\omega_0)}.$$

The coefficient of $\xi\iota.\sin 1''$ is

$$-4\cos\beta.\sin\theta-\frac{4.\cos\beta.\cos\theta.\cos(\omega-\omega_0)}{\sin(\omega-\omega_0)}$$

$$=\frac{-4.\cos\beta.\cos\left(\frac{1}{2}(\omega-\omega_0)-\beta\right)}{\sin(\omega-\omega_0)}.$$

Hence, generally,

$$\Omega=\begin{cases}\omega-\omega_0-\eta^2.\sin 1''.\tan\frac{1}{2}(\omega-\omega_0)+\dfrac{2\xi^2.\sin 1''.\cos\beta.\cos(\omega-\omega_0-\beta)}{\sin(\omega-\omega_0)} \\ +\dfrac{2\iota^2.\sin 1''.\cos\left(\frac{1}{2}(\omega-\omega_0)+\beta\right).\cos\left(\frac{1}{2}(\omega-\omega_0)-\beta\right)}{\sin(\omega-\omega_0)} \\ +\dfrac{2\eta\xi.\sin 1''.\sin\left(\frac{1}{2}(\omega-\omega_0)-\beta\right)}{\cos\frac{1}{2}(\omega-\omega_0)}+\dfrac{2\eta\iota.\sin 1''.\sin\beta}{\cos\frac{1}{2}(\omega-\omega_0)} \\ -\dfrac{4\xi\iota.\sin 1''.\cos\beta.\cos\left(\frac{1}{2}(\omega-\omega_0)-\beta\right)}{\sin(\omega-\omega_0)}.\end{cases}$$

10. From this general expression for the true angle Ω we may derive several important inferences.

1°. From the investigation in section 7, it follows that, when the direct and reflected images of an object coincide, the planes of the index- and horizon-glasses are parallel; and, treating the question in a similar manner, it would be easy to show that, when these planes are equally inclined to that of the instrument, the plane of the index-glass may be made to assume a position in which the direct and reflected images will coincide. These results may, however, be obtained from the general expression for Ω, from which, if we make $\Omega=0$, we shall have

$$0=\omega-\omega_0-\tfrac{1}{2}\cdot\eta^2.(\omega-\omega_0).\sin^2 1''+\frac{2(\xi-\iota)^2.\cos^2\beta}{\omega-\omega_0}$$
$$-2\eta.(\xi-\iota).\sin\beta.\sin 1'',$$

or $$2(\omega-\omega_0)^2-\eta^2.(\omega-\omega_0)^2.\sin^2 1''+4(\xi-\iota)^2.\cos^2\beta-4\eta$$
$$.(\xi-\iota).(\omega-\omega_0).\sin\beta.\sin 1''=0,$$

$$\therefore (\omega-\omega_0)^2.\big(2-\eta^2.\sin^2 1''(1+\tan^2\beta)\big)+\big(2(\xi-\iota).\cos\beta-\eta$$
$$.(\omega-\omega_0).\tan\beta.\sin 1''\big)^2=0.$$

And since the quantity $2-\eta^2.\sin^2 1''(1+\tan^2\beta)$, β not being 90°, must be positive, this equation cannot be satisfied except by $\omega-\omega_0=0$ and $2(\xi-\iota).\cos\beta-\eta.(\omega-\omega_0)\tan\beta.\sin 1''=0$, the latter requiring that $\xi-\iota=0$ likewise, and $\therefore\ \omega=\omega_0$ and $\xi=\iota$.

Hence it follows that, in order that the direct and reflected images of the same object may be made to coincide in some one position of the index, the planes of the index- and horizon-glasses must be equally inclined to the plane of the instrument; and conversely, when the direct and reflected images do coincide, we may conclude that the planes of the index- and horizon-glasses are parallel, and consequently, assuming the axis of revolution of the index to be perpendicular to the plane of the instrument, that the errors ξ and ι are equal in every position.

Now the first step in preparing for a series of observations, is, by adjustment of the horizon-glass, to make the direct and reflected images of an object coincide; and we thus have ξ and ι invariably equal in practice, except in so far as this adjustment may be disturbed in the course of our work. But making $\xi=\iota$, and supposing ω_0 small, as it usually is, the general equation becomes

$$\Omega=\begin{cases}\omega-\omega_0-\eta^2.\sin 1''.\tan\tfrac{1}{2}\omega+\dfrac{2\xi^2.\sin 1''}{\sin\omega}\big(\cos\beta.\cos(\omega-\beta)\\ \quad+\cos(\tfrac{1}{2}\omega+\beta).\cos(\tfrac{1}{2}\omega-\beta)-2\cos\beta.\cos(\tfrac{1}{2}\omega-\beta)\big)\\ \quad+\dfrac{2\eta\xi.\sin 1''}{\cos\tfrac{1}{2}\omega}\big(\sin(\tfrac{1}{2}\omega-\beta)+\sin\beta\big);\end{cases}$$

and this, by substitution of trigonometrical equivalents, may be reduced to

$$\Omega=\omega-\omega_0-\left(\eta-\frac{\xi\,.\cos\left(\frac{1}{4}\,\omega-\beta\right)}{\cos\frac{1}{4}\,\omega}\right)^2.\tan\tfrac{1}{2}\,\omega\,.\sin 1''-2\,\xi^2\,.\tan\tfrac{1}{4}\,\omega\,.\sin 1'',$$

which may therefore be assumed as the formula applicable in practice, ω_0 being the reading corresponding to coincidence of the direct and reflected images of the same object.

2°. In section 8 we supposed the only error to be that affecting the position of the optical axis; and we have there shown that, on this hypothesis, the line of sight upon which we should obtain the minimum reading for coincidence of the direct image of one object with the reflected image of a second would be that parallel to the plane of the instrument. We have now to inquire how this result is affected by the existence of an error ξ in the position of the planes of the index- and horizon-glasses.

The above equation transposed becomes

$$\omega=\Omega+\omega_0+\left(\eta-\frac{\xi.\cos\left(\frac{1}{4}\omega-\beta\right)}{\cos\frac{1}{4}\,\omega}\right)^2.\tan\tfrac{1}{2}\omega.\sin 1''+2\xi^2\,.\tan\tfrac{1}{4}\omega.\sin 1''.$$

Then Ω being a given angle between two objects, and the images being made to coincide in various positions in the field in the manner explained in Section 8, the minimum reading will evidently correspond to the position for which $\eta-\frac{\xi.\cos\left(\frac{1}{4}\omega-\beta\right)}{\cos\frac{1}{4}\,\omega}=0$, the readings increasing upon either side of this position according to the same law as, on the hypothesis assumed in that section, they increased upon either side of the position corresponding to $\eta=0$. Hence in this case the phenomena will be with respect to a position determined by $\eta=\frac{\xi\,.\cos\left(\frac{1}{4}\,\omega-\beta\right)}{\cos\frac{1}{4}\,\omega}$ precisely the same as they were in the former with respect to a line parallel to the plane of the instrument; but since η is now a function of ω, this position will vary with the angle between the objects observed. Let us in effect suppose the telescope adjusted so that for some angle ω not less than 90°, the optical axis shall coincide pretty nearly with the position determined by $\eta=\frac{\xi\,.\cos\left(\frac{1}{4}\,\omega-\beta\right)}{\cos\frac{1}{4}\,\omega}$; that is, let the position of minimum reading for this angle be brought pretty nearly into the middle of the field, in order that the quantities to be treated may not be large. Then Ω being the true angle (greater than 100°) between two objects, and ω_1, ω'_1 the readings obtained for coincidence of images on the outside of each parallel wire, we shall have

$$\Omega_1=\omega_1-\omega_0-\left(\eta-\frac{\xi\,.\cos\left(\frac{1}{4}\,\omega_1-\beta\right)}{\cos\frac{1}{4}\,\omega_1}+f\right)^2.\tan\tfrac{1}{2}\,\omega_1.\sin 1''-2\,\xi^2$$
$$.\tan\tfrac{1}{4}\,\omega_1.\sin 1''$$

$$\Omega_1=\omega_1'-\omega_0-\left(\eta-\frac{\xi.\cos\left(\frac{1}{4}\,\omega_1'-\beta\right)}{\cos\frac{1}{4}\,\omega_1'}-f\right)^2.\tan\tfrac{1}{2}\omega_1'.\sin 1''-2\,\xi^2$$
$$.\tan\tfrac{1}{4}\,\omega'_1\,.\sin 1''$$

$$=\omega_1'-\omega_0-\left(\eta-\frac{\xi\,.\cos\left(\frac{1}{4}\,\omega_1-\beta\right)}{\cos\frac{1}{4}\,\omega_1}-f\right)^2.\tan\tfrac{1}{2}\omega_1\,.\sin 1''-2\,\xi^2$$
$$.\tan\tfrac{1}{4}\,\omega_1\,.\sin 1''\text{ nearly.}$$

And subtracting,

$$0=\omega_1-\omega_1'-4f.\left(\eta-\frac{\xi\,.\cos\left(\frac{1}{4}\,\omega_1-\beta\right)}{\cos\frac{1}{4}\,\omega_1}\right).\tan\tfrac{1}{2}\,\omega_1\,.\sin 1''.$$

Observing in the same way two other objects subtending an angle Ω_2, about 60° suppose, we obtain

$$0=\omega_1-\omega'_1-4f.\left(\eta-\frac{\xi\,.\cos\left(\frac{1}{4}\,\omega_2-\beta\right)}{\cos\frac{1}{4}\,\omega_2}\right).\tan\tfrac{1}{2}\,\omega_2\,.\sin 1''.$$

The former of these two equations gives us

$$\eta-\frac{\xi\,.\cos\left(\frac{1}{4}\,\omega_1-\beta\right)}{\cos\frac{1}{4}\,\omega_1}=\frac{\omega_1-\omega'_1}{4f.\tan\frac{1}{2}\,\omega_1\,.\sin 1''},$$

whilst from the second we have

$$\eta-\frac{\xi\,.\cos\left(\frac{1}{4}\,\omega_2-\beta\right)}{\cos\frac{1}{4}\,\omega_2}=\frac{\omega_2-\omega'_2}{4f.\tan\frac{1}{2}\,\omega_2\,.\sin 1''},$$

and the combination of the two will furnish us with values of η and ξ. So far as regards ξ, however, the determination will not be very exact; for we shall find on computation that the extreme values of $\frac{\cos\left(\frac{1}{4}\,\omega-\beta\right)}{\cos\frac{1}{4}\,\omega}$ throughout the arc of the Sextant differ very little one from the other, and consequently, the coefficients $\frac{\cos\left(\frac{1}{4}\,\omega_1-\beta\right)}{\cos\frac{1}{4}\,\omega_1}$, $\frac{\cos\left(\frac{1}{4}\,\omega_2-\beta\right)}{\cos\frac{1}{4}\,\omega_2}$ being nearly equal, we shall obtain on subtraction,

$$\xi\times\text{a very small quantity}=\frac{1}{4f.\sin 1''}\cdot\left(\frac{\omega_2-\omega'_2}{\tan\frac{1}{2}\,\omega_2}-\frac{\omega_1-\omega'_1}{\tan\frac{1}{2}\,\omega_1}\right);$$

whence it is evident that a small error in any one of the quantities ω_1, ω_1', ω_2, ω'_2 will produce a sensible effect on the resulting value of this element. But the same cause may be viewed in another light; and we shall in fact find that we may, without appreciable error, omit all consideration of the terms in the general equation which depend upon ξ separately. To show this, we must in the first place compute the values of $\frac{\cos(\frac{1}{4}\omega-\beta)}{\cos\frac{1}{4}\omega}=A$.

The angle β depends, as before stated, upon the construction of the instrument, and may, for our present purpose, be consisidered, in any particular sextant, as invariable. Its general value seems to be about 20°, but the operator should of course measure it for himself in the instrument he is about to use. In the sextant from which our examples will be derived its value is 20° nearly; and this has therefore been assumed in the computation of the following Table:—

ω.	A.	ω.	A.
° 0	+0·9397	° 70	+1·0475
10	0·9546	80	1·0642
20	0·9696	90	1·0813
30	0·9847	100	1·0992
40	1·0000	110	1·1177
50	1·0155	120	1·1372
60	+1·0313	125	+1·1473

Now, taking all defects into consideration, we have shown in section 6 that ξ cannot, in a well-constructed instrument, be supposed to exceed 4′; and since the extreme values of A differ only by 0·2076, the difference between the extreme values of $\frac{\xi.\cos(\frac{1}{4}\omega-\beta)}{\cos\frac{1}{4}\omega}$, and therefore the greatest difference between any two positions of minimum reading, supposing ξ to be as great as 4′, will be only 50″. Let us suppose ξ as great as 4′, and the optical axis adjusted to coincide pretty nearly with the line determined by $\eta-4'\times\frac{\cos(\frac{1}{4}.120°-20°)}{\cos\frac{1}{4}.120°}=0$, or $\eta=4'\,33''$. Then, should the adjustment have been performed imperfectly, so that $\eta-4'\times\frac{\cos(\frac{1}{4}.120°-20°)}{\cos\frac{1}{4}.120°}$, instead of being equal to 0, is left equal to a quantity η', the following will be the values of $\eta-4'\times\frac{\cos(\frac{1}{4}\omega-\beta)}{\cos\frac{1}{4}\omega}$ for different values of ω:—

For $\omega=$ 0 it will be $\eta'+4'\times(A_{120}-A_0)=\eta'+47''$
$\omega=$ 20° „ $\eta'+4'\times(A_{120}-A_{20})=\eta'+40$
$\omega=$ 40 „ $\eta'+4'\times(A_{120}-A_{40})=\eta'+33$
$\omega=$ 60 „ $\eta'+4'\times(A_{120}-A_{60})=\eta'+25$
$\omega=$ 80 „ $\eta'+4'\times(A_{120}-A_{80})=\eta'+18$
$\omega=$100 „ $\eta'+4'\times(A_{120}-A_{100})=\eta'+9$
$\omega=$120 „ $\eta'+4'\times(A_{120}-A_{120})=\eta'$
$\omega=$125 „ $\eta'+4'\times(A_{120}-A_{125})=\eta'-2''$.

But supposing η' as great as $30'$, we see from the Table in section 8, that, for $\eta'+$ one minute, the increment to the correction (that is, to the value of $\eta^2 . \tan \frac{1}{2} \omega . \sin 1''$) will, for $\omega = 125°$, be only $1''$, and less for smaller angles. Consequently, an error in the position of the plane of the index-glass not exceeding $4'$ will not produce, with the variation of ω, a variation in the value of $\eta - \frac{\xi . \cos(\frac{1}{4}\omega - \beta)}{\cos \frac{1}{4}\omega}$ which will be at all sensible in the final corrections; and thus, the value of $\eta - \frac{\xi . \cos(\frac{1}{4}\omega - \beta)}{\cos \frac{1}{4}\omega}$ having been determined for $\omega = 120°$ or thereabouts, we may assume it to have the same value in every other position of the index. Hence, substituting η for $\eta - \frac{\xi . \cos(\frac{1}{4}\omega - \beta)}{\cos \frac{1}{4}\omega}$, the general equation becomes

$$\Omega = \omega - \omega_0 - \eta^2 . \tan \tfrac{1}{2}\omega . \sin 1'' - 2\xi^2 . \tan \tfrac{1}{4}\omega . \sin 1''.$$

But supposing ξ as great as $4'$, the value of the term $2\xi^2 . \tan \frac{1}{4}\omega . \sin 1''$ will not exceed $0''\cdot 3$ throughout the possible variation of ω; and as this quantity is quite inappreciable in practice, it is evident that we may safely omit the term, and thus reduce our general equation to

$$\Omega = \omega - \omega_0 - \eta^2 . \tan \tfrac{1}{2}\omega . \sin 1'';$$

and the value of η will be determined from $\eta = \frac{\omega_1 - \omega'_1}{4f . \tan \frac{1}{2}\omega_1 . \sin 1''}$, where ω_1, ω'_1 are the readings obtained for coincidence of images on the outside edges of the two parallel wires whose angular distance one from the other $= 2f$, the angle ω being at all events greater than $90°$, and, if possible, as great as the instrument is capable of measuring.

3°. We have supposed that ξ once made equal to ι, will continue so throughout a series of observations. But, in consequence of instability of mounting, the position of the plane of the horizon-glass with respect to that of the instrument is singularly liable to disturbance; and it is not uncommon, after careful adjustment at the commencement, to find on the termination of work of half an hour's duration, that the direct and reflected images of an object can no longer be made to coincide. It becomes therefore important to inquire to what extent observations are affected by a derangement of this description.

Resuming the general equation as given at the end of section 9, substitute for ξ the values ι and $\iota + \iota'$ successively, and let ω and ω' be the corresponding readings, so that $\omega' - \omega$ shall be the variation occasioned by a disturbance ι' in the position of the plane of the horizon-glass. We shall thus obtain, substituting ω for $\omega - \omega_0$ in the coefficients of the small terms,

$$\omega' - \omega = \sin 1''.\left(2\iota'^2.\frac{\cos\beta.\cos(\omega-\beta)}{\sin\omega}\right.$$

$$\left. + \frac{4\iota\iota'.\cos\beta.\left(\cos(\omega-\beta) - \cos(\frac{1}{2}\omega-\beta)\right)}{\sin\omega} + 2\eta\iota'.\frac{\sin(\frac{1}{2}\omega-\beta)}{\cos\frac{1}{2}\omega}\right).$$

But η being in this case the deviation of the optical axis from the line parallel to the plane of the instrument, we may substitute for it $\eta + \frac{\iota.\cos(\frac{1}{4}\omega-\beta)}{\cos\frac{1}{4}\omega}$, when η will be the deviation of the axis from the position of minimum reading. Making this substitution and reducing, we have

$$\omega' - \omega = 2\iota'^2.\sin 1''.\frac{\cos\beta.\cos(\omega-\beta)}{\sin\omega} - 2\iota\iota'.\sin 1''.\tan\tfrac{1}{4}\omega$$

$$+ 2\eta\iota'.\sin 1''.\frac{\sin(\frac{1}{2}\omega-\beta)}{\cos\frac{1}{2}\omega}.$$

Now ι' being small (generally less than one minute), since in the preceding part of this section we have shown that the maximum value of $2\iota^2.\sin 1''.\tan\frac{\omega}{4}$ is only $0''\cdot 3$, even when ι is as great as $4'$, therefore $2\iota\iota'.\sin 1''.\tan\frac{1}{4}\omega$ will be less than $0''\cdot 3$, and consequently unworthy of notice. Also the term

$$2\iota'^2.\sin 1''.\frac{\cos\beta.\cos(\omega-\beta)}{\sin\omega}$$

diminishes with an increase in the value of ω to $\omega=90°$, after which it increases; but for an angle $\omega=10°$, which is about as small as the smallest observed with a sextant in the course of ordinary practice, this term, when $\iota'=2'$, amounts to $0''\cdot 75$, and at one of $20°$ to $0''\cdot 39$, whilst for $\iota'=1'$ the values will be only $0''\cdot 19$ and $0''\cdot 10$. The last term, $2\eta\iota'.\sin 1''.\frac{\sin(\frac{1}{2}\omega-\beta)}{\cos\frac{1}{2}\omega}$, increases with an increase in ω; and for large angles, when the sum of the first and second terms is practically nothing, this becomes the only term to be considered. Suppose η as great as $15'$, $\iota'=2'$ and $\omega=20$, then this term $=1''\cdot 37$, whilst so rapid is its diminution, that for $\omega=80°$ its value is only $0''\cdot 50$, and for $\iota'=1'$ the values will be $0''\cdot 69$, $0''\cdot 25$. Hence it follows that even though, on the termination of a series of observations, we find that the position of the plane of the horizon-glass has suffered disturbance to an extent exceeding $1'$, but not exceeding $2'$, there will be no occasion for uneasiness respecting the observed angles, except they be either very large or very small—exceeding $90°$ or falling within $10°$,—and respecting the large angles, only when the error of the optical axis is considerable.

Since, however, we have no means of knowing at what point of the series the disturbance may have occurred, it will be proper, in the event of its existence, the circumstances being at the same time such as to render it important, that is, the angles observed being either very large or very small,—it will be proper in this case to reject the observations altogether. In measuring very small angles indeed, the sun's diameter for example, or the phenomena of a solar eclipse, it is evident that we must be particular in the adjustment of the horizon-glass, and must examine the state of this adjustment at the end of short intervals of time.

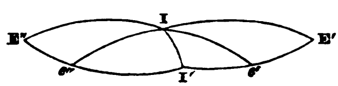
11. We are now prepared to consider the effect of the planes of the internal and external surfaces of the index-glass not being parallel one to the other, as well as of a similar defect in the horizon-glass.

Suppose two lines parallel to the normals to the reflecting and refracting surfaces to meet the surface of a sphere in I, I' respectively, and let the parallel to the emergent ray meet the same surface in E',

$$\angle E'II'=\gamma, \quad II'=\Sigma, \quad E'I=\psi.$$

Then if E'I, E'I' are invariably large compared with II',

$$E'I'=\psi-\Sigma \,.\, \cos\gamma.$$



And the line parallel to the ray which, after reflexion, passes towards the refracting surface meeting the surface of the sphere in e' in the arc E' I', and μ being the index of refraction,

$$\sin e'I'=\frac{1}{\mu}\,.\,\sin E'I'=\frac{1}{\mu}(\sin\psi-\Sigma\,.\,\sin 1''\,.\,\cos\gamma\,.\,\cos\psi).$$

Also $e'I=e'I'+\Sigma\,.\,\cos\gamma=e''I$ if the ray immediately before reflexion meets the surface of the sphere in e'' in the arc $e'I$ produced. But

$$e''I'=e''I+\Sigma\,.\,\cos\gamma= \therefore\, e'I'+2\Sigma\,.\,\cos\gamma.$$

Again, the incident ray meeting the surface of the sphere in E'' in the arc I'e'' produced,

$$\sin E''I'=\mu\,.\,\sin e''I'=\mu\,(\sin e'I'+2\Sigma\,.\,\cos\gamma\,.\,\sin 1''\,.\,\cos e'I')$$

$$=\sin\psi-\Sigma\,.\,\sin 1''.\cos\gamma.\left(\cos\psi-2\mu\sqrt{1-\frac{1}{\mu^2}\,.\,\sin^2\psi}\right)$$

$$=\sin\psi-\Sigma\,.\,\sin 1''\,.\,\cos\gamma\,.\,(\cos\psi-2\sqrt{\mu^2-\sin^2\psi}),$$

$$\therefore\; E''I'=\psi-\Sigma\,.\,\cos\gamma\,.\left(1-\frac{2\sqrt{\mu^2-\sin^2\psi}}{\cos\psi}\right).$$

And

$$E'E''=E''I'+E'I' \text{ nearly } =2\psi-2\Sigma\,.\cos\gamma\,.\left(1-\frac{\sqrt{\mu^2-\sin^2\psi}}{\cos\psi}\right),$$

which is the expression for the angle between the incident and emergent rays.

Now on referring to the figure in section 9, it will be evident that the direction of the emergent ray from the index-glass must be invariably parallel to E′O, in order that, after reflexion from the horizon-glass, it may follow that of OE. And supposing OI′ to represent the normal to the refracting surface, we shall have $\angle E'II'=\gamma$, which, since the angle M″IE′ never differs from a right angle by more than a very small quantity, may be considered constant; and Σ is constant likewise. Also ψ representing the arc E′I, we shall have

$$E'E''=2\psi-2\Sigma\,.\cos\gamma\,.\left(1-\frac{\sqrt{\mu^2-\sin^2\psi}}{\cos\psi}\right);$$

or since $\psi=\frac{1}{2}\,\omega+\beta$ nearly,

$$E'E''=2\psi-2\Sigma\,.\cos\gamma\,.\left(1-\frac{\sqrt{\mu^2-\sin^2(\frac{1}{2}\,\omega+\beta)}}{\cos(\frac{1}{2}\,\omega+\beta)}\right);$$

whereas, had OI, OI′ been coincident, we should have had, to this same position of the index, $E'E''=2\psi$. Hence, since the planes IE′, IE″, EE″ are nearly coincident, we may consider the effect of the deviation IOI′ to be to diminish EE″ by the quantity

$$2\Sigma.\cos\gamma.\left(1-\frac{\sqrt{\mu^2-\sin^2(\frac{1}{2}\,\omega+\beta)}}{\cos(\frac{1}{2}\,\omega+\beta)}\right);$$

and thus, resuming the equation arrived at in the second part of section 10*, assigning to ω_0 its original meaning, viz. the reading of the index when the planes OIM″, OHM′ coincide, and making $2\Sigma\,.\cos\gamma=\sigma$, we shall have

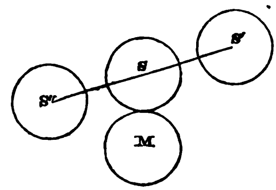
$$\Omega=\omega-\omega_0-\eta^2\,.\tan\tfrac{1}{2}\,\omega\,.\sin 1''-\sigma\,.\left(1-\frac{\sqrt{\mu^2-\sin^2(\frac{1}{2}\omega+\beta)}}{\cos(\frac{1}{2}\,\omega+\beta)}\right),$$

which provides for a defect of the index-glass in respect to the parallelism of the internal and external surfaces.

Suppose, in the next place, a similar defect to exist in the horizon-glass. This will evidently occasion a change in the position of the point E′,—a change depending upon the constant angle EOH, and unaffected by the revolution of the plane OIM″, and consequently by the angle observed. But a small variation

* Σ is supposed so small as not to interfere materially with the conditions which enabled us to reduce the general equation to this simple form.

in the position of E′ producing an equal variation in that of E″, the arc IE″, and consequently EE″, will be augmented or diminished by a constant quantity throughout the variation of ω. Hence the effect of an error in the relative position of the internal and external planes of the horizon-glass will, so far as the reflected image is concerned, be constant; and that it will be the same for the direct image, may be shown in the following manner:— Suppose the limbs of two objects of sensible diameter (the sun and moon for example) to have been brought into contact in the middle of the field without the interposition of the unsilvered portion of the horizon-glass, and let the sun be the object viewed directly. Then the objects being considered to have no relative motion, and the parts of the instrument consequently undisturbed, suppose the interposition of the glass to cause the image of the sun S to appear at S′. Join S′ S and produce it to S″, taking S S″ = S S′. Then it will be evident that, had a second sun appeared at S″ when the first observation was made without the interposition of the glass, it would now, on the interposition, appear at S, the limb being in contact with that of the reflected image of the moon in the middle of the field; and similarly, the glass being interposed and the contact made, the apparent position of an image of the sun for which the contact would occur on the removal of the glass would be S′.



Let the telescope be inverting. Then M being the position of the moon's centre in the heavens, and S′ that of the true sun, if we take S′S = S′S of the preceding figure, and ∠ M S S′ = ∠ M S S′, we shall have for a contact of limbs, the glass being interposed, a reading which would correspond to contact of limbs of objects at M and S instead of at M and S′. But the displacement S S′ and the angle M S S′ are evidently constant; and S S′ being small compared with M S, we have M S′ − M S = − S S′ . cos M S S′ = ∴ a constant; or the effect of the displacement upon the arc measured is invariable.

Hence introducing a general constant c in the place of ω_0, as well as the variable term for eccentricity from section 5, we have

$$\Omega = c + \omega - \eta^2 . \tan \tfrac{1}{2}\omega . \sin 1'' - \sigma . \left(1 - \frac{\sqrt{\mu^2 - \sin^2(\frac{1}{2}\omega + \beta)}}{\cos(\frac{1}{2}\omega + \beta)}\right)$$
$$+ e . \sin\left(\alpha + \tfrac{1}{2}(\omega)\right).$$

Let ω_0 be the reading for coincidence of images of a single object, supposed small,

$$\therefore\ 0=c+\omega_0 \quad -\sigma.\left(1-\frac{\sqrt{\mu^2-\sin^2\beta}}{\cos\beta}\right)+e.\sin\left(\alpha+\tfrac{1}{2}(\omega_0)\right),$$

and

$$\Omega=\omega-\omega_0-\eta^2.\tan\tfrac{1}{2}\omega.\sin 1''+\sigma.\left(\frac{\sqrt{\mu^2-\sin^2(\frac{1}{2}\omega+\beta)}}{\cos\frac{1}{2}\omega+\beta)}\right.$$

$$\left.-\frac{\sqrt{\mu^2-\sin^2\beta}}{\cos\beta}\right)+e.\left(\sin\left(\alpha-\tfrac{1}{2}(\omega)\right)-\sin\left(\alpha+\tfrac{1}{2}(\omega_0)\right)\right)$$

$$=\omega+\epsilon-\eta^2.\tan\tfrac{1}{2}\omega.\sin 1''+\mathrm{B}.\sigma+e.\sin\left(\alpha+\tfrac{1}{2}(\omega)\right),$$

if

$$\epsilon=-\omega_0-e.\sin\left(\alpha+\tfrac{1}{2}(\omega_0)\right)$$

and

$$\mathrm{B}=\frac{\sqrt{\mu^2-\sin^2(\frac{1}{2}\omega+\beta)}}{\cos(\frac{1}{2}\omega+\beta)}-\frac{\sqrt{\mu^2-\sin^2\beta}}{\cos\beta}.$$

12. We have now to determine the values of the constants η, σ, α, e, and ϵ, taking them in the order in which they are here written.

1°. To determine η.

The term $\eta^2.\tan\frac{1}{2}\omega.\sin 1''$ being the only one of the general equation affected by a change in the position with respect to the middle of the field of the telescope in which we observe the coincidence of the direct image of one object with the reflected image of another, it is obvious that we may proceed in the manner already explained in section 10; and ω_1, ω'_1 being the readings obtained for coincidence of images on the outside edges of the two parallel wires whose angular distance one from the other $=2f$, that we shall have $\eta=\dfrac{\omega_1-\omega'_1}{4f.\tan\frac{1}{2}\omega_1.\sin 1''}$; and η being thus determined, we may take from the Table in section 8 the correction $\eta^2.\tan\frac{1}{2}\omega.\sin 1''$ to be applied to any observed angle ω.

The value of η may be expected to remain unchanged through long periods of time; but the process should nevertheless be repeated at intervals.

2°. To determine σ.

The reading ω_0 of the vernier for the coincidence of the direct and reflected images of a single object having been carefully obtained, let us observe in the middle of the field the coincidence of the direct image of one object with the reflected image of another, the true angle between them being Ω_1; and let ω_1 be the reading corrected for the value of η previously determined. We shall then have

$$\Omega_1=\omega_1-\omega_0+\mathrm{B}.\sigma+e.\left(\sin\left(\alpha+\tfrac{1}{2}(\omega_1)\right)-\sin\left(\alpha+\tfrac{1}{2}(\omega_0)\right)\right).$$

Now let the index-glass be removed from its position and inverted, so that the side which was uppermost shall become that resting upon the surface of the index-bar,—a proceeding which will evidently change the sign of σ. Supposing then ω'_0 to be the new reading for coincidence, this not differing much from ω_0, as in a well-constructed instrument it will not, let us observe again the same two objects; and as celestial are better than terrestrial for the purpose, and as these are constantly changing their apparent relative position, let $\Omega_1+\delta\Omega_1$ be the angle between them at the time of the second observation. Then ω'_1 being the reading corrected for η as before, ω'_1 not differing much from ω_1,

$$\Omega_1+\delta\Omega_1=\omega'_1-\omega'_0-\text{B}.\sigma+e.\left(\sin\left(\alpha+\tfrac{1}{2}(\omega_1)\right)-\sin\left(\alpha+\tfrac{1}{2}(\omega_0)\right)\right),$$

whence by subtraction,

$$-\delta\Omega_1=(\omega_1-\omega_0)-(\omega'_1-\omega'_0)+2\text{B}\,.\,\sigma,$$

$$\therefore\ \sigma=\frac{1}{2\text{B}}\cdot\left((\omega'_1-\omega'_0)-(\omega_1-\omega_0)-\delta\Omega_1\right),$$

a formula which, when μ is known, will give us the value of σ affected with the sign with which it must be employed in the first position of the index-glass.

The factor B depends upon the constants μ and β, and the variable angle ω. The value of β seems generally to lie between 18° and 20°, whilst that of μ seldom perhaps differs much from 1·514. With this value of μ, Tables of values of B for 18°, 19°, 20° of β have been constructed, and will be found at the end of this Part, a column being added containing the values of $\frac{d\text{B}}{d\mu}$, which, as we shall presently see, will be serviceable to us in the determination of the true value of μ*.

Now, as in other cases of a similar kind, it is desirable that the factor B which occurs in the denominator of the expression for σ should be as great as possible; and the said Tables show us that this condition will be fulfilled if the angle between the two objects observed is as great as the instrument is capable of measuring.

Suppose Ω_1 to be 120°, or thereabouts, and let us observe other two objects in addition to the first, the arc between the second two subtending an angle of about 100°, and the readings in the two positions of the index-glass being ω_2 and ω'_2. Let B_1 and

* The method of determining the value of β is treated in section 17, pages 42 and 43. In the Sextant, which will furnish us with examples, its exact value has been found to be 19°; but the difference between this and 20°, assumed in page 26, will not sensibly affect the results there obtained.

$\frac{dB_1}{d\mu}$ be the values of B taken from the Table for the angle ω_1, and B_2 and $\frac{dB_2}{d\mu}$ those for the angle ω_2; then making

$$(\omega'_1-\omega'_0)-(\omega_1-\omega_0)-\delta\Omega_1=k_1$$
$$(\omega'_2-\omega'_0)-(\omega_2-\omega_0)-\delta\Omega_2=k_2,$$

we have

$$\sigma=k_1\div2\left(B_1+\frac{dB_1}{d\mu}\cdot\delta\mu\right)=k_2\div2\left(B_2+\frac{dB_2}{d\mu}\cdot\delta\mu\right),$$

and

$$\therefore\ \delta\mu=(k_2\cdot B_1-k_1\cdot B_2)\div\left(k_1\cdot\frac{dB_2}{d\mu}-k_2\cdot\frac{dB_1}{d\mu}\right);$$

and the true value of $\mu=1{\cdot}5140+\delta\mu$ being thus known, we may readily compute that of B for any angle ω, since we shall simply have

value of B for $\mu=1{\cdot}5140+\delta\mu=$ value of B from Table

$+\delta\mu\times$ value of $\frac{dB}{d\mu}$ from Table.

Also σ being computed from $\sigma=k_1\div2\left(B_1+\frac{dB_1}{d\mu}\cdot\delta\mu\right)$, we may construct a table of values of the term $B\cdot\sigma$ for various values of ω, from which the correction to any angle whatever may be at once taken; and the value of σ not being liable to change except with a change of index-glass, this Table will answer as long perhaps as the instrument continues serviceable.

3°. To determine the constants ϵ, e and α.

It will be evident, on referring to section 5, that the process there given in detail will be applicable, ω_1, ω_2, ω_3 being the readings from the limb corrected for the errors η and σ previously ascertained. It is to be remarked, however, that ω_0, and consequently (ω_0), being subject to very small variations only, we may substitute for $e\left(\sin\left(\alpha+\frac{1}{2}(\omega)\right)-\sin\left(\alpha+\frac{1}{2}(\omega_0)\right)\right)$ its equivalent

$2e\cdot\sin\frac{1}{4}\left((\omega)-(\omega_0)\right)\cdot\cos\left(\alpha+\frac{1}{4}\left((\omega)+(\omega_0)\right)\right)$, and tabulate

the values of this instead of those of $e\cdot\sin\left(\alpha+\frac{1}{2}(\omega)\right)$,—a course which has some advantage, inasmuch as the constant correction to ω is thus reduced to the reading ω_0 corresponding to coincidence of the direct and reflected images of the same object. The general equation thus takes the form

$$\Omega=\omega-\omega_0-\eta^2\cdot\tan\tfrac{1}{2}\omega\cdot\sin1''+B\cdot\sigma+2e\cdot\sin\tfrac{1}{4}$$
$$\left((\omega)-(\omega_0)\right)\cdot\cos\left(\alpha+\tfrac{1}{4}\left((\omega)+(\omega_0)\right)\right);$$

or, (ω_0) being small,

$$\Omega=\omega-\omega_0-\eta^2.\tan\tfrac{1}{2}\omega.\sin 1''+\mathrm{B}.\sigma+2e.\sin\tfrac{1}{4}(\omega).\cos\left(\alpha+\tfrac{1}{4}(\omega)\right)$$

Let Ω_1, Ω_2 be two known angles, ω_1, ω_2 the readings from the limb, and let

$$c_1=\Omega_1-(\omega_1-\omega_0)+\eta^2.\tan\tfrac{1}{2}\omega_1.\sin 1''-\mathrm{B}_1.\sigma,$$

$$c_2=\Omega_2-(\omega_2-\omega_0)+\eta^2.\tan\tfrac{1}{2}\omega_2.\sin 1''-\mathrm{B}_2.\sigma,$$

we then have

$$c_1=2e.\sin\tfrac{1}{4}(\omega_1).\cos\left(\alpha+\tfrac{1}{4}(\omega_1)\right),$$

$$c_2=2e.\sin\tfrac{1}{4}(\omega_2).\cos\left(\alpha+\tfrac{1}{4}(\omega_2)\right);$$

and eliminating e and reducing,

$$\tan\alpha=\left(c_1.\sin\tfrac{1}{2}(\omega_2)-c_2.\sin\tfrac{1}{2}(\omega_1)\right)\div\left(c_1.\operatorname{versin}\tfrac{1}{2}(\omega_2)-c_2.\operatorname{versin}\tfrac{1}{2}(\omega_1)\right),$$

whence the value of α may be determined; and that of $2e$ being subsequently derived from

$$c_1=2e.\sin\tfrac{1}{4}(\omega_1).\cos\left(\alpha+\tfrac{1}{4}(\omega_1)\right),$$

or the similar equation depending upon c_2 and (ω_2), we may tabulate the values of

$$2e.\sin\tfrac{1}{4}(\omega).\cos\left(\alpha+\tfrac{1}{4}(\omega)\right).$$

13. We are now supposed to be in possession of three tables applicable to the particular instrument with which we are operating,—the first expressing the values of

$$\eta^2.\tan\tfrac{1}{2}\omega.\sin 1'',$$

the second those of $\mathrm{B}\sigma$, and the third those of

$$2e.\sin\tfrac{1}{4}(\omega).\cos\left(\alpha+\tfrac{1}{4}(\omega)\right);$$

and it is clear that since the elements upon which the quantities in these tables depend are not under ordinary circumstances liable to sensible variation, we may combine the first and second, which have the same argument ω, into one, and from the collective table so constructed, take the correction due to the reading for the errors η and σ on each several occasion. The two separate tables should, however, be preserved on record, in order that in the event of a change in the value of η, we may with greater facility construct a new collective one, the original table of values of $\mathrm{B}.\sigma$ being applicable so long as we retain the same index-glass.

14. To any reading ω taken from the limb, we shall then have to apply three corrections when the fundamental parts only of the instrument have been employed:—

1°. The constant $-\omega_0$, ω_0, being the reading for coincidence of the direct and reflected images of the same object, obtained anew before and after every important series of observations.

2°. The correction from the collective table of the preceding section.

3°. The correction from the table of values of

$$2e \,.\, \sin \tfrac{1}{4}(\omega) \,.\, \cos\left(\alpha + \tfrac{1}{4}(\omega)\right).$$

But the cases in which the fundamental parts only of the Sextant are employed are comparatively few, since, in observing the sun with the moon, or the moon with a star, one of the images necessarily requires darkening by an interposed shade, and in observations of double altitude we generally employ a glass cover to protect the surface of the quicksilver in the trough of the artificial horizon from the disturbing influence of the wind. It becomes therefore important to inquire what new errors the employment of these necessary appendages may introduce into our observations.

15. Belonging to the Sextant are three sets of shades:—

1°. Those which apply to the eye-end of the telescope, and darken both images equally.

2°. Those behind the horizon-glass, which darken the direct image alone.

3°. Those interposed between the index- and horizon-glasses, which darken the reflected image alone.

Defects in the first of these are comparatively unimportant, distinctness of vision being the only condition they are required to fulfil. This will be manifest when we consider that the rays from the direct and reflected images falling upon them in directions parallel one to the other, will, in the event of the internal not being parallel to the external surface of the glass, be displaced equally, and coincidence existing without the shade, will therefore subsist when the shade is interposed, and *vice versâ*.

But of the other shades, those of each set affecting one image alone, it is especially important that the internal and external surfaces should be parallel one to the other, or, not being so, that we should apply a correction to the reading on every occasion on which these are employed; and reasoning again in the way we did in section 11, when treating the effect of a supposed error in that part of the horizon-glass through which we view the direct rays, it is easy to show that the correction will be a constant for each shade.

Let the shades be numbered 1, 2, 3, 4, 5, 6, 7, commencing from that nearest to the index-glass, and terminating with that farthest from the horizon-glass; and let c_1, c_2, c_3, c_4, c_5, c_6, c_7 be the quantities to be applied to the readings obtained with each shade alone, to reduce them to those which we should obtain without the interposition of any shade whatever. Then the correction for any combination, 1, 2 and 5, for example, will be $c_1+c_2+c_5$, and so for the rest.

Now suppose the positive contact of the direct and reflected images of the sun's limb to be observed with a shade before the eyepiece alone, and let ω be the reading. Then $\omega-c_1$ would be the reading on the interposition of shade 1 alone; $\omega-c_1-c_2-c_5$ that obtained on the interposition of shades Nos. 1, 2 and 5, and so on;

$\therefore c_1+c_2+c_5=$ reading obtained with shade before eyepiece alone, minus that obtained on interposition of shades Nos. 1, 2 and 5 alone. Hence, for every combination of shades with which the contact is observed, we have an equation between some of the quantities c_1, c_2, c_3, &c.; and from seven independent equations of this sort, we may evidently determine the values of c_1, c_2, c_3, &c.

The operation of observing the contacts is in practice greatly facilitated by the provision made for elevating or depressing the telescope with respect to the plane of the instrument*, as by this means we are enabled to adjust the images compared to a condition of equal brilliancy. But as the eye seems to judge a contact differently on different occasions, and as a variation in this respect appears to be produced, not only by an inequality in the brilliancy of the two images, but likewise by a variation in the absolute brilliancy when it is the same for both, it will be better to observe both positive and negative contacts, in order that this effect, as well as that of the variation of the sun's apparent vertical diameter, may be eliminated from the mean.

Other equations for the light shades may be derived in the following way, perhaps preferable in general, as being much less wearing than the observation of contacts of the sun's limb.

Select a star whose distance from the moon varies very slowly. View the moon directly with a shade, and observe the distance of the star from the limb. Let the shade be No. 7. Then ω being assumed as the reading without a shade, $\omega-c_7$ will be the reading with it. Suppose we now view the star directly, the moon's reflected image being darkened with shade No. 1; then ω' being assumed as the reading without the shade, $\omega'-c_1$ will be the reading with it. Let $\omega'=\omega+\delta\omega$, the variation $\delta\omega$, which may be assumed as the absolute variation in distance, being

* In indifferently constructed instruments, the elevation or depression of the telescope frequently changes the position of the optical axis materially. The operator should satisfy himself that his instrument is trustworthy in this respect.

computed independently or derived from the observations; then $\omega+\delta\omega-c_1$ will be the second reading. Hence the difference

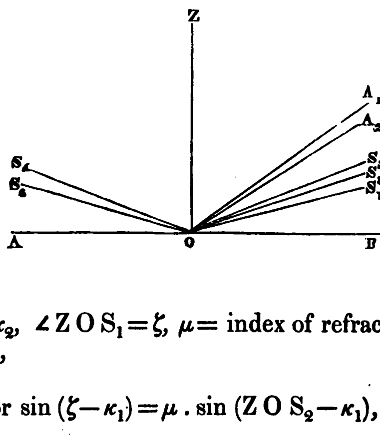
$$\text{first reading} - \text{second reading} = c_1 - c_7 - \delta\omega,$$

$$\therefore c_1 - c_7 = \text{first reading} - \text{second reading} + \delta\omega.$$

The most convenient method of operating in the determination of these constants will be seen from the example.

16. We have lastly to consider the consequence of defects in the glasses which form the cover of the trough of the artificial horizon, presuming the normals to the surfaces of these glasses to be situated in one plane, identical with the plane of observation.

Let A B be the surface of the quicksilver; O Z a vertical line; O A_1 a normal to external surface of the glass upon which the rays from the object observed impinge; O A_2 that to the internal surface of the same glass; S_1 O the direct ray; S_2O the first, and S_3 O the second refracted ray:—



$$\angle ZOA_1 = \kappa_1,\ \angle ZOA_2 = \kappa_2,\ \angle ZOS_1 = \zeta,\ \mu = \text{index of refraction for glass},$$

$$\sin S_1OA_1 = \mu \,.\, \sin S_2OA_1 \text{ or } \sin(\zeta-\kappa_1) = \mu \,.\, \sin(ZOS_2-\kappa_1),$$

$$\sin S_3OA_2 = \mu \,.\, \sin S_2OA_2 \text{ or } \sin(ZOS_3-\kappa_2) = \mu \,.\, \sin(ZOS_2-\kappa_2),$$

$$\therefore \sin(ZOS_3-\kappa_2) = \mu \,.\, \sin(ZOS_2-\kappa_1+\kappa_1-\kappa_2) = \sin(\zeta-\kappa_1) + \mu \,.\, (\kappa_1-\kappa_2) \,.\, \sin 1'' \,.\, \cos(ZOS_2-\kappa_1);$$

or since κ_1 and κ_2 each $=45°$ nearly,

$$\sin(ZOS_3-\kappa_2) = \sin(\zeta-\kappa_1) + (\kappa_1-\kappa_2) \,.\, \sin 1'' \,.\, \sqrt{\mu^2-\sin^2(\zeta-45°)}$$

$$ZOS_3-\kappa_2 = \zeta-\kappa_1+(\kappa_1-\kappa_2) \,.\, \sqrt{\mu^2-\sin^2(\zeta-45°)} \div \cos(\zeta-45°),$$

$$\therefore ZOS_3 = \zeta-(\kappa_1-\kappa_2) \,.\, \left(1-\sqrt{\mu^2-\sin^2(\zeta-45°)} \div \cos(\zeta-45°)\right).$$

Let O S_4 be the ray reflected from A B, $\therefore \angle ZOS_4 = \angle ZOS_3$, And O S_6 being the ray convergent towards the instrument,

$$Z\,O\,S_4 = Z\,O\,S_6 - (\kappa_3 - \kappa_4)\,.\Big(1 - \sqrt{\mu^2 - \sin^2(\zeta - 45^\circ)} \div \cos(\zeta - 45^\circ)\Big) \text{ nearly,}$$

$$\therefore Z\,O\,S_6 = \zeta - \Big(\kappa_1 - \kappa_2 - (\kappa_3 - \kappa_4)\Big).\Big(1 - \sqrt{\mu^2 - \sin^2(\zeta - 45^\circ)} \div \cos(\zeta - 45^\circ)\Big)$$

$$= \zeta - \kappa\,.\Big(1 - \sqrt{\mu^2 - \sin^2(\zeta - 45^\circ)} \div \cos(\zeta - 45^\circ)\Big)$$

if $\kappa = \kappa_1 - \kappa_2 - (\kappa_3 - \kappa_4)$.

And angle measured between the rays S_1 O, O S_6 will be

$$180^\circ - (Z\,O\,S_6 + \zeta),$$

or

$$2(90^\circ - \zeta) + \kappa\,.\Big(1 - \sqrt{\mu^2 - \sin^2(\zeta - 45^\circ)} \div \cos(\zeta - 45^\circ)\Big),$$

instead of $2(90^\circ - \zeta)$, as it would be were the glasses perfect; and we have therefore to correct the reading ω by a quantity,

$$\kappa\,.\Big(1 - \sqrt{\mu^2 - \sin^2(\zeta - 45^\circ)} \div \cos(\zeta - 45^\circ)\Big)$$

$$= \kappa\,.\Big(1 - \sqrt{\mu^2 - \sin^2(45^\circ - \tfrac{1}{2}\omega)} \div \cos(45^\circ - \tfrac{1}{2}\omega)\Big) \text{ nearly.}$$

Assuming then $\mu = 1{\cdot}5140$ as before, the following will be the values of the coefficient of κ for various values of ω from 30° upwards, the sign of the correction depending of course upon that of κ.

ω.		ω.	
30°	0·6502	80°	0·5173
40	0·6041	90	0·5140
50	0·5695	100	0·5173
60	0·5444	110	0·5273
70	0·5273	120	0·5444

To find the value of the constant κ, we have only to reverse the cover, turning the side upon which the rays impinged towards us, and that from which they emerged towards the object, observing the same angle in the two positions. Then since $\kappa = \kappa_1 - \kappa_2 - (\kappa_3 - \kappa_4)$ in the first case, it will $= \kappa_3 - \kappa_4 - (\kappa_1 - \kappa_2)$ in the second, and the sign of the constant being thus changed, the coefficient remaining the same, the correction will be the same number of seconds in the second case as in the first, but will be affected with the contrary sign. Hence the value of κ will be half the difference of the readings divided by the coefficient due to the angle read, and the results will at once show with what sign it should be employed in each position of the cover.

In this treatment of the question, we have supposed the same plane to contain the normals to the several surfaces, as well as the eye and the object observed, conditions seldom probably fulfilled in practice. If, however, we are careful in placing the cover invariably in the same position with respect to the plane of observation, or nearly so, the law we have derived on the above assumption will evidently not lead us into much error, as indeed we may satisfy ourselves by observing, in reversed positions of the cover, angles of various magnitudes between the possible limits.

17. Having now discussed all the errors to which the Sextant and its appendages are liable, we proceed to take an example in an instrument of which nothing is supposed to be known, and for which we wish to construct the requisite tables of corrections.

1°. To determine η.

The distance between the two outer parallel wires having been ascertained by the process explained in section 8 to be

$$123' = 7380'' = 2f,$$

the following observations were obtained:—

α Virginis and α Cygni.

On wire to the right. On wire to the left.

$\omega_1 = 111°14'33''$ (mean of 5 obs.). $\omega'_1 = 111°14'47''$ (mean of 5).

Here $\omega_1 - \omega'_1 = -14''$, $\frac{1}{2}\omega_1 = 55\frac{1}{2}°$ nearly, $4f = 14760''$,

$$\therefore \eta = -\frac{14}{14760 \,.\, \tan 55\frac{1}{2}° \,.\, \sin 1''},$$

and the computation will be as follows:

$\log 4f = 4{\cdot}16909$		$\log(\omega'_1 - \omega_1) =$	$1{\cdot}14613$
$\log\tan\frac{1}{2}\omega_1 = 0{\cdot}16287$			$9{\cdot}01753$
$\log\sin 1'' = 4{\cdot}68557$		Diff. =	$2{\cdot}12860 = \log -\eta$
Sum $= 9{\cdot}01753$			$\therefore \eta = -134'' = -2'\ 14''$*

* We retain the sign of η for the purpose of comparing the result of this observation with that of the following one; otherwise it is manifest that, as the correction varies as the square of η and is always subtractive, the sign of η itself is of no importance.

A second observation was

α Virginis and α Lyræ.

On wire to the right.	On wire to the left.
$\omega_1 = 87° \ 44' \ 29''$ (mean of 3).	$\omega'_1 = 87° \ 44' \ 40''$ (mean of 3).

And treating this in the same way, we shall obtain

$$\eta = -2' \ 41''.$$

Hence we may safely conclude that the quantity represented by η is less than $3'$; and since the correction varies as the square of the error, its amount for $3'$ will very little exceed a third of that due to one of $5'$. But on referring to the Table in section 8 (page 19), we see that the maximum correction for $\eta = 5'$ is only $0''\cdot 8$. Consequently the maximum correction for the value of η now determined, will be less than $0''\cdot 5$, and therefore unworthy of notice.

Before we give in detail the observations made with a view to ascertain the values of σ, e and α, it will be proper to make a few remarks on the determination of ω_0 and β, as well as on the computation of the true apparent distance between two celestial objects.

The quantity ω_0 is the reading on the arc when the direct and reflected images of an object coincide; and the usual method of obtaining it is to observe alternately positive and negative contacts of the limbs of the direct and reflected images of the sun, and, taking half the sum of the mean of the readings with their proper signs, to treat this as the value of ω_0. This course, however, is manifestly improper, inasmuch as the positive readings may be taken from a point on the arc differing considerably from that from which the negative readings are taken, and the error of excentricity will in this case affect the results unequally. If, indeed, we know the values of the constants e and α, we are then in a position to compute the corrections due at the points of the arc from which the readings are taken, and a process of calculation will give us the reading for coincidence itself; but as we want the value of ω_0 in the first place, we must have recourse to some other method of obtaining it. We might indeed observe the coincidence of the direct and reflected images of the sun; but to this there are some objections. The sun is invariably a bad object to observe, its heat affecting the instrument to such an extent as sometimes seriously to disturb the adjustments, and with these the value of ω_0 itself; and, in addition to this, the quantity of light is so great as to render the eye insensible to small distances between the limbs of the images in observing either coincidence or contact. Coincidence of images of a star, however, may be observed with very great nicety; and if we select for the purpose a bright one, and make our observations of coincidence when this first becomes visible in the twilight, we shall find a number of readings rarely exhibiting a difference of more than 3 or 4 seconds between the extremes. This then is the method I am inclined to recommend; and on repeating the operation in the course or at the conclusion of a series of observations, should the sky have darkened in the meantime, a star of inferior magnitude may be taken for the purpose, but still not so small as to occasion any unpleasant exertion of the eye to perceive clearly the coincidence of images. The coincidences should in all cases be made by alternately elevating and depressing the reflected image, so that, the first being obtained by elevating the reflected image, in making the second we should depress it, and so on. The mean of an even number of readings, not fewer than eight, will, I believe, seldom differ 1″ from the truth.

The angle β is that between the optical axis and the normal to the plane of the horizon-glass; and its value in most instruments of modern construction is not far from 20°. A first approximation may be obtained by measuring the sides of the triangle of which the angles are at the middle of the frame of the index-glass, the middle of that of the horizon-glass, and the middle of the collar into which the telescope screws. The half of the second angle, which may be computed from the measured lengths of the sides, will be an approximate value of β.

To determine β more accurately, the operator should remove the index-glass, and then, pushing the index to the extreme end of the arc, should direct the telescope to a distant object and bring the image of it into the middle of the field. The Sextant being held with its plane nearly horizontal, the rays from a second object, which with the first subtends an angle of $180°-2\beta$, will evidently pass the position usually occupied by the index-glass, and, falling directly upon the horizon-glass, will be reflected thence along the optical axis, so that the image of this second object will coincide with the direct image of the first. The operator having placed himself in a position from which two such objects are visible, should now restore the index-glass to its place, and selecting some third object intervening between the first and second, and as nearly as possible in the same plane with these and his eye, should direct the telescope to the middle object, and effect coincidence of the image of this with that of the object to the right reflected from the index- and horizon-glasses. The reading of the vernier corrected for ω_0 alone will be nearly the angle between the two objects. Next, directing the telescope to the object on the extreme left, let him determine in the same way the angle between this and the middle object. The sum of the two angles so obtained will be nearly that between the first and second object, and may be taken as a reasonably good value of $180°-2\beta$; that is, half the supplement of the angle may be employed as the value of β.

To obtain β still more accurately, we may, in the manner to be explained in the second part of this work, determine approximately the latitude of the place, and, with all needful accuracy, the local time. These being known, we may compute the instant at which the double-altitude of some bright star, which culminates sufficiently high, is equal to the angle $180°-2\beta$ ascertained approximately. Then, removing the index-glass as before, and directing the telescope, at a time somewhat earlier than that computed, to the image of the object reflected in the artificial horizon, we must watch the approach of this and the image reflected immediately from the horizon-glass, and note the instant of coincidence of the two. Computing the apparent zenith-distance of the star at the time observed, the result will be a value of β sufficiently accurate for all practical purposes*.

With respect to the computation of true apparent distances between two celestial objects, let A_1, A_2 be their right ascensions, π_1, π_2 their polar distances taken from the Tables. Then if the latitude of the place, and consequently the right ascension of the meridian are unknown, we must observe the altitudes of the objects before and after the time for which we wish to compute their true apparent distance, and thus by interpolation obtain their apparent zenith distances at the time in question†. Let the interpolated zenith distances be ζ'_1, ζ'_2, and ρ_1, ρ_2 the refractions (objects affected by parallax should not be employed in operations such as those which we are now treating): then

* In performing these operations, it will in general be sufficient to remove the index-glass and the clip alone, the plate against which the former rests being so thin that it will not interfere with the passage of the direct rays towards the horizon-glass. If, however, the operator can obtain the value of β before the maker places any part of the apparatus connected with the index-glass in its position, he will thus be saved a considerable amount of trouble, and nothing will then exist to interrupt the rays, as might otherwise be the case should the construction of the instrument not admit of the plate being brought nearly perpendicular to the plane of the horizon-glass.

Should any difficulty be experienced in estimating the position of the middle of the field in the direction in which angles are measured, two parallel wires will be found convenient, these wires being at right angles to the system we have already described in treating the deviation of the optical axis, and forming with the two inner wires of that system a square of which each side may represent about a degree of arc.

† It is necessary to anticipate in this place some portions of the second part of our work; and it must be added that, in observing these altitudes, the readings of the vernier of the Sextant may be assumed to be affected by ω_0 alone, since a small error in the interpolated zenith distance will not sensibly affect the result we have in view.

$\zeta_1=\zeta'_1+\rho_1$, $\zeta_2=\zeta'_2+\rho_2$ will be the true zenith distances. Suppose D to be the true distance between the objects, and D′ the true apparent distance sought, we shall have

$$\cos D=\cos \pi_1 . \cos \pi_2+\sin \pi_1 . \sin \pi_2 . \cos (A_1-A_2),$$

and may compute D by means of either one of the formulæ—

1. $\cos D =\dfrac{\cos \pi_1 . \cos (\pi_2-\theta)}{\cos \theta}$, where

$\tan \theta =\tan \pi_1 . \cos (A_1-A_2)$.

2. $\cos D =\dfrac{\cos \pi_2 . \cos (\pi_1-\theta)}{\cos \theta}$, where

$\tan \theta =\tan \pi_2 . \cos (A_1-A_2)$.

3. $\sin \frac{1}{2} D=\sin \frac{1}{2} (\pi_1+\pi_2) . \sin \theta$, where

$\cos \theta =\dfrac{\sqrt{\sin \pi_1 . \sin \pi_2 . \cos \frac{1}{2} (A_1-A_2)}}{\sin \frac{1}{2} (\pi_1+\pi_2)}$.

4. $\sin \frac{1}{2} D=\sin \frac{1}{2} (\pi_1 \sim \pi_2) . \sec \theta$, where

$\tan \theta =\dfrac{\sqrt{\sin \pi_1 . \sin \pi_2 . \sin \frac{1}{2} (A_1 \sim A_2}}{\sin \frac{1}{2} (\pi_1 \sim \pi_2)}$.

And since we then have

$$\frac{\cos D-\cos \zeta_1 . \cos \zeta_2}{\sin \zeta_1 . \sin \zeta_2}=\frac{\cos D'-\cos \zeta'_1 . \cos \zeta'_2}{\sin \zeta'_1 . \sin \zeta'_2},$$

we obtain in the usual way,

$\sin \frac{1}{2} D'=\sin \frac{1}{2} (\zeta'_1+\zeta'_2) . \cos \theta$, where

$$\sin^2\theta =\frac{\sin \frac{1}{2} (\zeta_1+\zeta_2+D) . \sin \frac{1}{2} (\zeta_1+\zeta_2-D) . \sin \zeta'_1 . \sin \zeta'_2}{\sin \zeta_1 . \sin \zeta_2 . \sin^2 \frac{1}{2} (\zeta'_1+\zeta'_2)},$$

whence D′ may be found.

But if the latitude of the place is known approximately, and the time consequently can be obtained, we may compute the zenith distances ζ_1, ζ_2 in the following way :—

Let ϕ be the colatitude of the place, H_1 the true hour-angle of the first object. Then $\cos \zeta_1=\dfrac{\cos \phi . \cos (\pi_1-\theta)}{\cos \theta}$, where $\tan \theta=\tan \phi . \cos H_1$; and a similar formula will answer for the computation of ζ_2: or, should circumstances require it, we may employ, in the place of this, formulæ similar to Nos. 2, 3 and 4 just given for the computation of D from the data π_1, π_2, A_1-A_2. And having computed ζ_1, ζ_2, and obtaining thence $\zeta'_1=\zeta_1-\rho_1$, $\zeta'_2=\zeta_2-\rho_2$, we may proceed with the computation of D′ as before.

Another method of computing D′ is the following :—

Having obtained ζ_1, ζ_2, ρ_1, ρ_2 in the manner just explained, we have, making π'_1, π'_2, A'_1, A'_2 the apparent polar distances and right ascensions of the objects,

$$\pi'_1=\pi_1-\rho_1\times\frac{\sin\pi_1\,.\cos\phi-\cos\pi_1\,.\sin\phi\,.\cos H_1}{\sin\zeta_1},$$

$$\pi'_2=\pi_2-\rho_2\times\frac{\sin\pi_2\,.\cos\phi-\cos\pi_2\,.\sin\phi\,.\cos H_2}{\sin\zeta_2},$$

$$A'_1=A_1+\rho_1\times\frac{\sin\phi\,.\sin H_1}{\sin\pi_1\,.\sin\zeta_1},\quad A'_2=A_2+\rho_2\times\frac{\sin\phi\,.\sin H_2}{\sin\pi_2\,.\sin\zeta_2},$$

hour-angles to the east of the meridian being reckoned negative; and we may thus calculate the quantities π'_1, π'_2, A'_1, A'_2, and thence determine D' by means of any one of the four formulæ above given for the determination of D, substituting therein π'_1, π'_2, A'_1, A'_2, D' in the place of the same letters without accents.

Example. Two objects having been watched near the meridian, and their greatest apparent altitudes observed, one of the objects being on the northern, the other on the southern meridian, the mean of the two results which these gave for the latitude of the place, the readings having been corrected by the quantities ω_0, $\eta^2.\tan\frac{1}{2}\omega.\sin 1''$ only, was 50° 36′ N., which we therefore employ for the present as an approximate latitude.

Observed altitudes of two equatorial stars, one to the east, the other to the west of the meridian, the readings having been corrected only as before, and the approximate latitude employed in the reduction, gave the right ascension of the meridian $16^h\,30^m\,9^s$, when a watch, going mean time nearly, indicated $8^h\,44^m\,27^s$. The date being the 18th of July, 1858, we wish to know the true apparent distance between Arcturus and α Lyræ when the same watch showed $8^h\,30^m\,19^s$.

The difference between the mean times $=14^m\,8^s=14^m\,10^s$ sidereal time;

	h	m	s
∴ Right ascension of meridian at $8^h\,30^m\,19^s$ by watch =	16	15	59
Right ascension of Arcturus =	14	9	13
H_1 = +	2	6	·46 = 31° 42′*

Also $\phi=90°-50°\,36'=39°\,24'$; and from the Tables $\pi_1=70°\,5'$.

$\log\tan\phi=9{\cdot}91456$	$\log\cos\phi=9{\cdot}88803$
$\log\cos H_1=9{\cdot}92983$	$\log\cos\theta=9{\cdot}91364$
Sum $=9{\cdot}84439=\log\tan\theta$	Difference $=9{\cdot}97439$
$\theta=34°\,57'$	$\log\cos(\pi_1-\theta)=9{\cdot}91266$
$\pi_1-\theta=35°\,8'$	Sum $=9{\cdot}88705=\log\cos\zeta_1$
	$\zeta_1=39°\,33'$

* In computing the true zenith distances for the purpose of obtaining refraction only, it is unnecessary to take account of seconds of arc.

and the Barometer being at 30·12 ins., Thermometer 61°, we find from the Tables,

$$\log \rho_1 = 1{\cdot}6740, \text{ or } \rho_1 = 47''{\cdot}2, \text{ and } \therefore \zeta'_1 = 39^\circ\, 32'\, 12''{\cdot}8.$$

Similarly, for α Lyræ we find

$$\zeta_2 = 26^\circ\, 42',\ \log \rho_2 = 1{\cdot}4590,$$

or

$$\rho_2 = 28''{\cdot}8 \text{ and } \zeta'_2 = 26^\circ\, 41'\, 13''{\cdot}2.$$

To compute the effect of ρ_1 on the polar distance and right ascension, we have

$\log\cos\pi_1 = 9{\cdot}5323$	$\log\sin\pi_1 = 9{\cdot}9732$
$\log\sin\phi = 9{\cdot}8026$	$\log\cos\phi = 9{\cdot}8880$
$\log\cos H_1 = 9{\cdot}9298$	Sum $= 9{\cdot}8612 = $ log of $0{\cdot}7265$
Sum $= 9{\cdot}2647$	$=$ log of $0{\cdot}1840$
$\log\rho_1 = 1{\cdot}6740$	Difference $= +0{\cdot}5425$
$\log 0{\cdot}5425 = 9{\cdot}7344$	$\pi_1 = 70^\circ\, 4'\, 42''{\cdot}4$
Sum $= 1{\cdot}4084$	$40''{\cdot}2 =$ effect in N.P.D.
$\log\sin\zeta_1 = 9{\cdot}8040$	$\pi'_1 = 70^\circ\, 4'\, \ 2''{\cdot}2$
Difference $= 1{\cdot}6044 =$ log of $40''{\cdot}2$	

$\log\sin\phi = 9{\cdot}8026$	$\log\sin\pi_1 = 9{\cdot}9732$
$\log\sin H_1 = 9{\cdot}7206$	$\log\sin\zeta_1 = 9{\cdot}8040$
$\log\rho_1 = 1{\cdot}6740$	Sum $= 9{\cdot}7772$
Sum $= 1{\cdot}1972$	
$9{\cdot}7772$	
Difference $= 1{\cdot}4200 =$ log of $26''{\cdot}3 =$ effect in right ascension.	

Similarly, for α Lyræ we shall find

$$\pi'_2 = 51^\circ\, 20'\, 28''{\cdot}5, \text{ and effect in right ascension} = -29''{\cdot}1.$$

$$\therefore\ A'_1 = 14^h\ 9^m\ 13^s{\cdot}32 + 26''{\cdot}3$$
$$A'_2 = 18\ \ 32\ \ 10{\cdot}75 - 29{\cdot}1$$
$$\therefore\ A'_1 - A'_2 = -\ 4\ \ 22\ \ 57{\cdot}43 + 55{\cdot}4 = -65^\circ\, 43'\, 26''{\cdot}1.$$

Next to compute D′, we have

$\log\tan\pi'_2 = 0{\cdot}0969265$	$\log\cos\pi'_2 = 9{\cdot}7956580$
$\log\cos(A'_1 - A'_2) = 9{\cdot}6139832$	$\log\cos\theta = 9{\cdot}9491039$
Sum $= \log\tan\theta = 9{\cdot}7109097$	Difference $= 9{\cdot}8465541$
$\theta = 27^\circ\, 12'\, 1''{\cdot}1$	$\log\cos(\pi_1 - \theta) = 9{\cdot}8650656$
$\pi_1 - \theta = 42^\circ\, 52'\, 1''{\cdot}1$	Sum $= 9{\cdot}7116197 = \log\cos D'$,

$$\therefore\ D' = 59^\circ\, 1'\, 3''.$$

2°. To determine σ.

The observations made with this object, and which answer likewise for the determination of e and α, were the following:—

July 14, 1858.—$\omega_0 = +0' 28''$. Bar. 29·87 ins. Therm. 63°. Arcturus and α Cygni. Mean of 10 observed distances 80° 44′ 52″ at mean of times $17^h\,57^m\,48^s$ sidereal.

July 17, 1858*.—$\omega_0 = +1' 4''$. Bar. 30·02 ins. Therm. 60°.

	h m s		° ′ ″	
Arcturus and α Lyræ	at 16 17 25,	obs. dist.	59 2 1	mean of 10 obs.
Polaris, Antares	16 44 8,	"	117 0 34	"
α Lyræ, α Aquilæ	17 4 57,	"	34 11 44	"

July 18, 1858.—Index-glass inverted. $\omega_0 = +1' 5''$. Bar. 30·12 ins. Therm. 61°.

	h m s		° ′ ″	
Arcturus, α Lyræ	at 16 15 59,	obs. dist.	59 2 0	mean of 10 obs.
Arcturus, α Cygni	16 55 59,	"	80 45 36	" 12 "
α Lyræ, α Aquilæ	17 22 13,	"	34 11 49	" 10 "

July 19, 1858.—$\omega_0 = +1' 8''$. Bar. 30·07 ins. Therm. 60°.

	h m s		° ′ ″	
Polaris, Antares	at 16 41 15,	obs. dist.	117 0 40	mean of 16 obs.

To commence then with the largest angle: if we compute the true apparent distance between Polaris and Antares on the 17th of July at $16^h\,44^m\,8^s$ sidereal time, we shall find it 116° 59′ 44″. Hence, remembering that $\eta = 0$, taking the coefficient of σ from the Table for $\beta = 19°$, and denoting the correction for excentricity by c_1,

$$116°\,59'\,44'' = 117°\ 0'\,34'' - 1'\,4'' + 3.78\,\sigma + c_1$$
$$= 116°\,59'\,30'' \qquad + 3{\cdot}78\,\sigma + c_1,$$
$$\therefore +14'' = 3{\cdot}78\,\sigma + c_1.$$

Treating in the same manner the observation of the 19th of July with the index-glass in the inverted position,

$$116°\,59'\,45'' = 117°\,0'\,40'' - 1'\,8'' - 3{\cdot}78\,\sigma + c_1,$$
$$\therefore\ 13'' = -3{\cdot}78\,\sigma + c_1.$$

Subtracting this equation from the former,

$$1'' = 7{\cdot}56\,\sigma \text{ and } \sigma = \pm 0''{\cdot}13,$$

the former sign applying to the first, the latter to the second position of the index-glass.

But we wish to derive the best value of σ from the results furnished by all the observations; and we therefore treat them all as we have done those of Polaris and Antares, presuming throughout that $\mu = 1{\cdot}514$, since, σ being so very small, it would be useless to attempt to derive from them a value of $\delta\mu$, which

* Between the 14th and 17th the horizon-glass was dismounted and replaced, an operation which changed the value of ω_0.

must be very small likewise. Hence arranging the equations in pairs,

$$\left.\begin{aligned} 14'' &= +3{\cdot}78\,\sigma + c_1 \\ 13 &= -3{\cdot}78\,\sigma + c_1 \end{aligned}\right\}\text{Polaris, Antares} \ldots\ldots \quad 1$$

$$\left.\begin{aligned} 13 &= +0{\cdot}88\,\sigma + c_2 \\ 15 &= -0{\cdot}88\,\sigma + c_2 \end{aligned}\right\}\text{Arcturus, } \alpha \text{ Cygni} \ldots\ldots \quad 2$$

$$\left.\begin{aligned} 7 &= +0{\cdot}43\,\sigma + c_3 \\ 8 &= -0{\cdot}43\,\sigma + c_3 \end{aligned}\right\}\text{Arcturus, } \alpha \text{ Lyræ} \ldots\ldots \quad 3$$

$$\left.\begin{aligned} 5 &= +0{\cdot}16\,\sigma + c_4 \\ 4 &= -0{\cdot}16\,\sigma + c_4 \end{aligned}\right\}\alpha \text{ Lyræ, } \alpha \text{ Aquilæ} \ldots\ldots \quad 4$$

Subtracting the second of each pair from the first,

$$\begin{aligned} +1'' &= 7{\cdot}56\,\sigma \\ -2 &= 1{\cdot}76\,\sigma \\ -1 &= 0{\cdot}86\,\sigma \\ +1 &= 0{\cdot}32\,\sigma. \end{aligned}$$

Multiplying each equation by the coefficient of σ in it, and adding the results, our final equation becomes

$$61{\cdot}09 \quad \sigma = +3''{\cdot}50,$$

and $\therefore\ \sigma = +0\ {\cdot}057$ for the first, and $-0''{\cdot}057$ for the second position of the index-glass.

Assuming then that $\sigma = 0''{\cdot}057$, we see, from the Table at the end, that the correction due for this to a reading of 125° will be only $6{\cdot}2 \times 0''{\cdot}057 = 0''{\cdot}35$; and as this is the maximum and is less than half a second, we may for all practical purposes assume $\sigma = 0$.

3°. To determine α and $2e$.

Adding the second to the first of each of the above pairs of equations and dividing by 2, we have

$$13''{\cdot}5 = c_1;\ 14'' = c_2;\ 7''{\cdot}5 = c_3;\ 4''{\cdot}5 = c_4,$$

which are evidently the quantities c_1, c_2, c_3, c_4 of the third part of section 12, (ω_1) being 117° 37′, $(\omega_2) = 85°\ 53'$, $(\omega_3) = 61°\ 0'$, and $(\omega_4) = 35°\ 52'$. Hence from the first and third, one being derived from near the extremity and the other from about the middle of the arc, we have

$$\tan\alpha = \frac{c_1 \,.\, \sin\frac{1}{2}(\omega_3) - c_3 \,.\, \sin\frac{1}{2}(\omega_1)}{c_1 \,.\, \text{versin}\,\frac{1}{2}(\omega_3) - c_3 \,.\, \text{versin}\,\frac{1}{2}(\omega_1)} = \frac{6{\cdot}8517 - 6{\cdot}4165}{1{\cdot}8680 - 3{\cdot}6165} = -\frac{0{\cdot}4352}{1{\cdot}7485},$$

$$\therefore \log - \tan\alpha = 9{\cdot}39602 \text{ and } \alpha = -14°\ 25'.$$

Also from $c_1 = 2e \,.\, \sin\frac{1}{2}(\omega_1) \,.\, \cos\left(\alpha + \frac{1}{4}(\omega_1)\right)$ we obtain $\log 2e = 1{\cdot}45417$, results which would be sufficiently exact for ordinary purposes.

To obtain the best values of α and $2e$ from the combination of our four pairs of observations, it would be necessary that the approximate values should differ very little from these, so that the approximate being α' and $2e'$, and $\alpha'+\delta\alpha$, $2e'+e'$ the best, the results of the substitution of the latter for α and $2e$ in the equations

$$c_1=2e.\sin\tfrac{1}{4}(\omega_1).\cos\left(\alpha+\tfrac{1}{4}(\omega_1)\right),\ c_2=2e.\sin\tfrac{1}{4}(\omega_2).\cos\left(\alpha+\tfrac{1}{4}(\omega_2)\right),\ \text{\&c.}$$

might, on making $\sin\delta\alpha=x$, be assumed as

$$c_1'=e'.\sin\tfrac{1}{4}(\omega_1).\cos\left(\alpha'+\tfrac{1}{4}(\omega_1)\right)-x.2e'.\sin\tfrac{1}{4}(\omega_1).\sin\left(\alpha'+\tfrac{1}{4}(\omega_1)\right),\ \text{\&c.},$$

in which case the latter equations, on being treated according to the usual method, would furnish two final equations for the determination of x or $\sin\delta\alpha$ and e'. But in the example before us it will be remarked that, assuming $-14°\,25'$ to be within a few degrees of the value of α, each of the angles

$$\alpha+\tfrac{1}{4}(\omega_1),\ \alpha+\tfrac{1}{4}(\omega_2),\ \text{\&c.}$$

is small, and its cosine not only nearly equal to unity, but likewise so little affected by a variation of even 10° in α as to make no sensible difference in the amount of any correction

$$2e.\sin\tfrac{1}{4}(\omega).\cos\left(\alpha+\tfrac{1}{4}(\omega)\right).$$

Under these circumstances, therefore, we cannot expect to obtain a value of α sufficiently good for the application of the above method, although it would be otherwise were some of the angles

$$\alpha+\tfrac{1}{4}(\omega_1),\ \alpha+\tfrac{1}{4}(\omega_2),\ \text{\&c.}$$

nearly equal to 90°; and the consideration that an error of even some degrees in the assumed value of α will not sensibly affect the value of the correction at any point of the arc, which thus depends almost entirely upon that of $2e$, suggests that it will be sufficient to assume the value already obtained for α as the truth, and, substituting this in the equations of condition, apply the results to the determination of $2e$ alone. We thus obtain

$$2e.\sin\tfrac{1}{4}(\omega_1).\cos\left(\alpha+\tfrac{1}{4}(\omega_1)\right)=\cdot4742\times2e=13''\cdot5.$$
$$2e.\sin\tfrac{1}{4}(\omega_2).\cos\left(\alpha+\tfrac{1}{4}(\omega_2)\right)=\cdot3632\times2e=14\ \cdot0.$$
$$2e.\sin\tfrac{1}{4}(\omega_3).\cos\left(\alpha+\tfrac{1}{4}(\omega_3)\right)=\cdot2630\times2e=\ 7\ \cdot5.$$
$$2e.\sin\tfrac{1}{4}(\omega_4).\cos\left(\alpha+\tfrac{1}{4}(\omega_4)\right)=\cdot1552\times2e=\ 4\ \cdot5.$$

And multiplying each of these by its coefficient of $2e$, and adding the results, the final equation becomes

$$\cdot4501\times2e=14\cdot1583,$$

and $$\therefore\log 2e=1\cdot49770.$$

Assuming this as the value of $\log 2e$ and $-14°\,25'$ as that of α, we proceed to form with these data our table of values of

$$2e.\sin\tfrac{1}{4}(\omega).\cos\left(\alpha+\tfrac{1}{4}(\omega)\right)=\text{E}.$$

(ω).	E.	(ω).	E.
°	″	°	″
10	+1·3	70	+ 9·4
20	2·7	80	10·7
30	4·1	90	11·9
40	5·4	100	13·1
50	6·8	110	14·1
60	+8·1	120	+15·1

This, as the effects of the errors η and σ are insensible, is the only table which we have to employ in correcting the readings from the limb of the Sextant with which we are now dealing; and although the instrument was supposed to be previously untried, and the above observations were made on this hypothesis for the purpose of furnishing an example, it will be proper to remark, in order to show the amount of confidence to be placed in our results, that a table similar to the above, for the same instrument, was formed in March 1857, the objects observed and the angles between them being on that occasion different from those we have now employed. On the first occasion the angles observed were seven in number, and included—

102° 56′ between Capella and Arcturus.
78 5 between Capella and α Hydræ.
72 50 between α Hydræ and α Ursæ Majoris.
67 51 between Procyon and α Ursæ Majoris.
45 1 between Procyon and Aldebaran.
30 41 between Capella and Aldebaran.
22 48 between Procyon and Pollux.

And the resulting Table of corrections was:—

(ω).	E.	(ω).	E.
°	″	°	″
10	+1·5	70	+10·2
20	3·0	80	11·5
30	4·5	90	12·7
40	6·0	100	13·8
50	7·5	110	14·8
60	+8·9	120	+15·8

And on comparing this with the former it will be noticed that the maximum difference is only 0″·8, a fact which not only tends to confirm our results, but which likewise proves the permanency of the constants upon which they depend, the instrument having, between the first and second of the above dates, made a voyage round the world, not to mention sundry journeys by land, in the course of some of which it was unavoidably exposed to much motion. It must not, however, be expected that the quantity to be found in our Table is the exact amount necessary to reduce a particular observed angle to equality with the result of computation. Differences up to 3″, and perhaps above this, will sometimes occur, inasmuch as we have to consider not only the probable error of observation, but that likewise of division, that arising from possible flexure of the frame of the instrument affecting differently angles measured in different planes, and those of the tables of right ascensions, declinations and refraction, affecting the values of the angles computed.

Our example having now included the treatment of errors in the several fundamental parts of the instrument, we proceed

4°. To determine the corrections to be applied on the interposition of shades.

The observations and the resulting equations were—

July 7, 1858.

		′	″
With shade on eyepiece	Mean of positive readings for 14 contacts of sun's limb. .	+32	2·4*
	Mean of negative readings for the same number	−30	54·6
	Half sum . . . . . .=		+33·9
With shades Nos. 4 & 5	Mean of positive readings for 14 contacts	+32	2·5*
	Mean of negative readings for the same number	−30	55·7
	Half sum . . . . . .=		+33·4

Hence $+33''{\cdot}4+c_4+c_5=+33''{\cdot}9,$

$\therefore c_4+c_5=+\ 0\ {\cdot}5$ . . . . (1)

		′	″
With shades Nos. 2, 3, 5	Mean of positive readings for 4 contacts	+32	1·3
	Mean of negative readings for the same number	−30	57·0
	Half sum . . . . . .=		+32·2

$\therefore c_2+c_3+c_5=+1''{\cdot}7.$

July 14, 1858.

Twelve contacts on each side with shade on eyepiece, and as many with shades Nos. 2, 3 and 5, gave

$$c_2+c_3+c_5=+1''{\cdot}5.$$

* These observations were made in the following order. Firstly, a pair of contacts with shade on eyepiece, succeeded by a pair with shades 4 and 5; and so on alternately. We thus become aware of any change in the value of ω_0 in consequence of disturbances produced by the heat of the sun.

Hence, taking the mean of this and the result of July 7,

$$c_2+c_3+c_5=+1''\cdot 6 \ldots\ldots\ldots\ldots \quad (2)$$

And proceeding in the some way on the 15th of July and succeeding days, the following equations were obtained:—

$$c_4+c_6+c_7=+1''\cdot 7, \quad \ldots\ldots \quad (3)$$

$$c_1+c_2+c_5=+0''\cdot 8; \quad \ldots\ldots \quad (4)$$

and as a test of these and the former results,

$$c_1+c_2+c_6+c_7=+2''\cdot 3.$$

Now adding equations (3) and (4), and subtracting (1) from the sum, we obtain

$$c_1+c_2+c_6+c_7=+2''\cdot 0,$$

a result differing only $0''\cdot 3$ from that given by the test.

July 25, 1858.

The apparent distance of the moon's limb from α Aquilæ was observed, the moon being viewed directly, and with shades Nos. 6 and 7 alternately; this arrangement being adopted in order that the mean of the times of observation with No. 6 should be very nearly the same as that of the times with No. 7, the correction to be applied to the mean of one set of observed distances to reduce it to the distance corresponding to the mean of the times of the other set thus becoming a small fraction of the whole variation between the first and last observation of each.

Mean of ten observations with No. 6 gave

At $9^h\ 55^m\ 25^s$, angle from Sextant $33°\ 5'\ 45''\cdot 3$;

the time here given being the mean of ten observed times from the Watch, and the angle the mean of the ten corresponding readings.

Mean of ten observations with No. 7 gave

At $9^h\ 55^m\ 11^s$, angle from Sextant $33°\ 5'\ 41''\cdot 3$.

The interval between the means of the times is 14 seconds; and on comparing the first with the last observation of the set with No. 6, it was found that the variation of the angle in 14 seconds amounted to $0''\cdot 3$, the angle increasing with the time; and the same result was obtained from the set with No. 7. Hence, to determine the angle which would have been shown with No. 7 at $9^h\ 55^m\ 25^s$, we shall have to add $0''\cdot 3$ to the angle with the same shade at $9^h\ 55^m\ 11^s$, and we obtain therefore for shade No. 7,

At $9^h\ 55^m\ 25^s$, angle from Sextant $33°\ 5'\ 41''\cdot 6$.

Let x be the angle at this time without a shade, then

$$x = 33° \; 5' \; 45''{\cdot}3 + c_6 = 33° \; 5' \; 41''{\cdot}6 + c_7,$$

$$\therefore \; c_6 - c_7 = -3''{\cdot}7.$$

Four other similar sets having been taken with the same shades, the mean of the five results was

$$c_6 - c_7 = -3''{\cdot}4. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Again, returning to the sun, a light shade before the eyepiece combined with Nos. 2 and 6 gave

$$c_2 + c_6 = -1''{\cdot}3. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

And the seventh equation, resulting from observations of the sun likewise, was

$$c_4 + c_5 + c_6 = -2''{\cdot}4. \quad . \quad . \quad . \quad . \quad . \quad (7)$$

Finally, a proof equation from observations with shades 2, 4 and 5, and 4 and 5,

$$1' \; 11''{\cdot}2 + c_2 + c_4 + c_5 = 1' \; 12''{\cdot}7 + c_4 + c_5,$$

or

$$c_2 = +1''{\cdot}5.$$

We have now to solve the equations numbered from (1) to (7).

From (1) and (7) we obtain $c_6 = -2''{\cdot}4 - (c_4 + c_5) = -2''{\cdot}9$; and substituting this value of c_6 in (5) and (6) successively,

$$c_7 = +0''{\cdot}5,$$

$$c_2 = +1{\cdot}6,$$

the latter of which differs only a tenth of a second from the value of c_2 given by the proof.

Substituting the values of c_6 and c_7 in (3),

$$c_4 = +4''{\cdot}1;$$

and from this and (1),

$$c_5 = -3''{\cdot}6.$$

Substituting the values of c_2 and c_5 in (4), we obtain

$$c_1 = +2''{\cdot}8;$$

and the same values substituted in (2) give

$$c_3 = +3''{\cdot}6.$$

Hence arranging the results,

$$\begin{aligned} c_1 &= +\overset{''}{2}{\cdot}8 \\ c_2 &= +1{\cdot}6 \\ c_3 &= +3{\cdot}6 \\ c_4 &= +4{\cdot}1 \\ c_5 &= -3{\cdot}6 \\ c_6 &= -2{\cdot}9 \\ c_7 &= +0{\cdot}5 \end{aligned}$$

And from these we see that, although in every combination of shades in which the maker had an opportunity of adjusting them in the ordinary way, the sum of the corrections is very small; nevertheless, in treating them separately, the correction is in some instances important, and will make a difference of many seconds of time in the longitude deduced from an observed lunar distance.

5°. In determining the errors introduced by the glass cover of the artificial horizon, the most convenient course of proceeding will be to observe a number of circum-meridian double altitudes of a star, reversing the cover after every observation. The several angles read being reduced to meridional zenith distance*, it is obvious, from what has been said in section 16, that those resulting from the observations in the first position of the cover being affected with an error $+\frac{1}{2}k$, the remainder will be affected with one $-\frac{1}{2}.k$; and the difference between the mean of the first and that of the second will therefore at once give k, which may be considered the amount of correction due to the mean between the greatest and least angles read in the course of the observations. Let ω be this angle, K the coefficient corresponding to it in the Table in section 16; then

$$\mathrm{K}.\kappa=k, \text{ or } \kappa=k\div\mathrm{K};$$

and κ being thus known, we may, with the help of the Table referred to, compute the corrections due to various angles between the limits of possible observation, these corrections being positive in one position of the cover and negative in the other.

Example. One end of the cover being marked N, and this letter affixed to an observation signifying that the end so marked was on that occasion placed towards the observer, thirty circum-meridian double altitudes of α Ursæ Majoris were observed, and these reduced gave

$$\text{Meridian Z. D. from 15 marked N, } 55^\circ\ 39'\ 19''\cdot7,$$

$$\text{From the remaining 15, } 55\ 39\ 20\cdot7;$$

and the mean between the extreme readings was 68° 24′.

Here

$$\omega=68^\circ\ 24',\ \mathrm{K}=0\cdot5299,\ k=1'',$$

$$\therefore\ \kappa=1''\div0\cdot5299=1''\cdot89;$$

and as the zenith distance for the position N was too small, or he angle observed too great, the corrections to the latter will in this position be negative, in the other positive, or for the marked end towards the observer $\kappa=-1''\cdot89$.

* The method of performing this operation will be found treated in Part II. Section 4.

Similar observations of Canopus gave $\kappa = +0''.55$, and from others of Antares was obtained $\kappa = +2''.62$; and the mean between the three values being thus $+0''.43$, and consequently the maximum correction within the limits of possible observation only $0''.28$, the cover was considered perfect, and accordingly in the examples in Part II., against the letters A H in the columns of corrections to the angle will be found invariably 0.

18. We have now treated in detail all the errors which would affect observations with the Sextant were the instrument perfectly inflexible and stable in all its parts, so that a change in the direction of the telescope or of the plane of the limb with respect to the direction of gravity would not produce any variation in the relative position of the planes of the index- and horizon-glasses, or of these with respect to the zero division of the arc in a given position of the zero of the vernier. This perfection, however, is in all probability unattainable in practice; and in most instances we shall find that the value of ω_0 will sensibly vary with a change in the position of the object from which it is determined. The amount of variation will differ with the instrument, and must be ascertained by experiment. In the Sextant which furnishes our examples, no perceptible difference has been discovered when the telescope is elevated towards an object in the heavens between 10° and 60° from the horizon, be the plane of the limb inclined as it may to the vertical; and as in all, or almost all, observations of distance one of the two objects is situated within these limits, the instrument may so far be considered perfect. But on depressing the telescope towards an object reflected on the surface of the mercury in the artificial horizon a change is at once perceived; and for any depression of between 10° and 60°, the value of ω_0 has been found to be constantly 11″ less than that obtained in the positions of elevation. This result, derived from an observation which can be made immediately and repeated as often as we please, has been confirmed by the results of circum-meridian observations of objects on the northern and southern meridians, the latitudes derived from double altitudes on the northern side being constantly, throughout the limits of possible observation, about this amount less in the northern hemisphere and greater in the southern than those derived from observations of the same class in the opposite direction. Thus the mean of the latitudes derived from three sets of observations of Capella to the north, at a double altitude or observed angle of 102°, the readings having been corrected for all the errors heretofore treated, being 6° 53′ 28″·9 N., that derived from similar observations of α Ursæ Majoris on the same side at an observed angle of 61° was 6° 53′ 28″·3 N.; whilst those derived from Antares and Canopus to the south,

the former at an angle of 114°, and the latter at one of 63°, were respectively 6° 53′ 39″·9 N. and 6° 53′ 40″·2. The mean of the former being 6° 53′ 28″·6, and of the latter 6° 53′ 40″·1, the difference 11″·5 is the amount by which we may consider the double altitude corrected for errors already discussed in each case too small; and another set of observations on both parts of the meridian, made long subsequently in a high northern latitude, giving a difference of 11″·3, the mean of this and the former, amounting to +11″·4, will be treated as a correction to be applied to observations of double altitude alone,—a result, as before stated, confirmed by observations for ω_0 of an object reflected in the artificial horizon. This last correction we shall distinguish by the letter F.

The amount of the correction F, it may be again remarked, is peculiar to the instrument; and it is possible that in some sextants it will not, as in this, be applicable to observations by reflexion alone. Should there be in any case sensible differences in the values of ω_0 derived in different positions of the telescope or plane of the limb, it will be prudent, when making observations for very nice purposes, to obtain a special value from one of the objects observed, and in that position of the instrument in which the observations are made.

19. A set of corrections once obtained, the Sextant will be found to possess great advantages over the Reflecting Circle. The latter, on account of its weight, cannot be employed conveniently except with a stand, which not only adds materially to the apparatus to be moved from place to place, but is ill-adapted to the peculiar way in which instruments of this description should be used. Directing the telescope to one of the two objects observed, we have to retain this in the middle of the field, and by motion of the instrument round the optical axis, to shoot the reflected image of the second object rapidly from one side to the other of the first; and this motion it is difficult to effect except with the hand unimpeded by any stiffness in the support of the instrument itself. After a little practice, the operator will be perfectly successful in giving this motion; and with respect to objects approaching towards or receding from one another, the index being set a little in advance, he will be able to perform it so immediately before and after coincidence as, with a chronometer beating half-seconds, to be at no loss to estimate the tenth of a second at which the coincidence occurred; that is, provided the relative motion is sufficiently rapid to render the variation of distance in a tenth of a second appreciable. Nor is the advantage above named the only one. The mean of the three readings obtained from the vernier of the Reflecting Circle will be nearly independent of the effect of excen-

tricity; but these three readings requiring some time to obtain, it will be impossible to multiply observations to the extent to which we can do this with the Sextant, and to which it is desirable to multiply them on account of the optical power of the instrument being in general considerably inferior to the circular power—the vernier, if the divisions are fine, indicating distinctly angles which are scarcely sensible with the telescope. The correction for excentricity, moreover, is only one of several; and the remainder are as necessary in the Reflecting Circle as they are in the Sextant. By employing the former we should therefore dispense only with the determination of two constants and the computation of a single table, a work involving very little labour, and having to be performed perhaps only once in the course of practice with an instrument, and entail upon ourselves a large amount of trouble on every occasion of making observations, to derive from them, after all, an inferior result.

It only remains, in concluding the first part of this work, to direct the reader's attention to the suggestions given at the end of the second chapter of the succeeding part, some of which are as applicable to operations on shore as to those at sea.

Tables of values of

$$B=\frac{\sqrt{\mu^2-\sin^2(\frac{1}{2}\omega+\beta)}}{\cos(\frac{1}{2}\omega+\beta)}-\frac{\sqrt{\mu^2-\sin^2\beta}}{\cos\beta},$$

and of $\frac{dB}{d\mu}=\frac{\mu}{\cos(\frac{1}{2}\omega+\beta)\sqrt{\mu^2-\sin^2(\frac{1}{2}\omega+\beta)}}-\frac{\mu}{\cos\beta\sqrt{\mu^2-\sin^2\beta}}$,

for $\beta=18°$, 19°, and 20° (see page 33).

$\beta=18°$.

ω.	B.	$\frac{dB}{d\mu}$.	ω.	B.	$\frac{dB}{d\mu}$.	ω.	B.	$\frac{dB}{d\mu}$.
0°	0·00	0·00	85°	0·96	1·41	115°	3·09	4·12
5	0·01	0·02	88	1·06	1·55	116	3·25	4·31
10	0·03	0·05	90	1·14	1·65	116½	3·33	4·41
15	0·05	0·08	92	1·22	1·76	117	3·41	4·51
20	0·07	0·12	94	1·31	1·88	117½	3·50	4·62
25	0·10	0·16	96	1·41	2·01	118	3·59	4·73
30	0·13	0·20	98	1·52	2·15	118½	3·69	4·85
35	0·16	0·26	100	1·64	2·30	119	3·79	4·97
40	0·20	0·32	102	1·77	2·47	119½	3·89	5·10
45	0·24	0·38	104	1·91	2·66	120	4·00	5·23
50	0·29	0·46	105	1·99	2·75	120½	4·11	5·36
54	0·34	0·53	106	2·07	2·86	121	4·23	5·51
58	0·39	0·60	107	2·16	2·97	121½	4·35	5·65
62	0·44	0·69	108	2·25	3·09	122	4·48	5·81
66	0·51	0·78	109	2·35	3·21	122½	4·62	5·97
70	0·58	0·88	110	2·46	3·34	123	4·76	6·14
73	0·64	0·97	111	2·57	3·48	123½	4·91	6·32
76	0·71	1·06	112	2·69	3·62	124	5·06	6·51
79	0·78	1·17	113	2·81	3·78	124½	5·23	6·71
82	0·87	1·28	114	2·95	3·94	125	5·40	6·91

$\beta=19°$.

ω.	B.	$\frac{dB}{d\mu}$.	ω.	B.	$\frac{dB}{d\mu}$.	ω.	B.	$\frac{dB}{d\mu}$.
0°	0·00	0·00	85°	1·02	1·49	115°	3·41	4·51
5	0·02	0·03	88	1·13	1·64	116	3·59	4·73
10	0·03	0·05	90	1·22	1·75	116½	3·68	4·84
15	0·05	0·09	92	1·31	1·87	117	3·78	4·96
20	0·08	0·12	94	1·41	2·00	117½	3·89	5·09
25	0·10	0·17	96	1·51	2·14	118	4·00	5·22
30	0·13	0·22	98	1·63	2·29	118½	4·11	5·36
35	0·17	0·27	100	1·76	2·46	119	4·23	5·50
40	0·21	0·33	102	1·91	2·65	119½	4·35	5·65
45	0·25	0·40	104	2·07	2·85	120	4·48	5·80
50	0·31	0·48	105	2·16	2·96	120½	4·61	5·96
54	0·35	0·55	106	2·25	3·08	121	4·75	6·13
58	0·41	0·63	107	2·35	3·20	121½	4·90	6·31
62	0·47	0·72	108	2·45	3·33	122	5·06	6·50
66	0·54	0·82	109	2·56	3·47	122½	5·22	6·70
70	0·61	0·93	110	2·68	3·61	123	5·40	6·90
73	0·68	1·02	111	2·81	3·77	123½	5·58	7·12
76	0·75	1·12	112	2·94	3·93	124	5·77	7·35
79	0·83	1·23	113	3·09	4·11	124½	5·98	7·60
82	0·92	1·36	114	3·24	4·30	125	6·19	7·85

$\beta=20°$.

ω.	B.	$\frac{dB}{d\mu}$.	ω.	B.	$\frac{dB}{d\mu}$.	ω.	B.	$\frac{dB}{d\mu}$.
0°	0·00	0·00	85°	1·09	1·58	115°	3·78	4·95
5	0·02	0·03	88	1·21	1·74	116	3·99	5·21
10	0·04	0·06	90	1·30	1·86	116½	4·10	5·35
15	0·06	0·09	92	1·40	1·99	117	4·22	5·49
20	0·08	0·13	94	1·51	2·13	117½	4·34	5·64
25	0·11	0·18	96	1·63	2·28	118	4·47	5·79
30	0·14	0·23	98	1·76	2·45	118½	4·61	5·95
35	0·18	0·28	100	1·90	2·64	119	4·75	6·12
40	0·22	0·35	102	2·06	2·84	119½	4·90	6·30
45	0·27	0·42	104	2·24	3·07	120	5·05	6·49
50	0·32	0·51	105	2·34	3·19	120½	5·22	6·69
54	0·37	0·58	106	2·45	3·32	121	5·39	6·89
58	0·43	0·67	107	2·56	3·46	121½	5·57	7·11
62	0·50	0·76	108	2·67	3·60	122	5·77	7·34
66	0·57	0·86	109	2·80	3·76	122½	5·97	7·59
70	0·65	0·98	110	2·94	3·93	123	6·19	7·84
73	0·72	1·08	111	3·08	4·10	123½	6·42	8·12
76	0·80	1·19	112	3·24	4·29	124	6·66	8·41
79	0·88	1·30	113	3·40	4·50	124½	6·92	8·72
82	0·98	1.44	114	3·58	4·72	125	7·20	9·05

To extend the above Tables to cases in which β is less than 18°,

Let $(B)^{17}_{\omega}$, $(B')^{18}_{0}$ signify respectively the values of B and $\frac{dB}{d\mu}$ for an angle ω, supposing β to be 17°. Then

$$(B)^{17}_{\omega}=(B)^{18}_{\omega-2^\circ}+0\cdot005;\quad (B')^{17}_{\omega}=(B')^{18}_{\omega-2^\circ}+0\cdot008;$$

$$(B)^{16}_{\omega}=(B)^{18}_{\omega-4^\circ}+0\cdot010;\quad (B')^{16}_{\omega}=(B')^{18}_{\omega-4^\circ}+0\cdot016;$$

$$(B)^{15}_{\omega}=(B)^{18}_{\omega-6^\circ}+0\cdot014;\quad (B')^{15}_{\omega}=(B')^{18}_{\omega-6^\circ}+0\cdot023;$$

$$(B)^{14}_{\omega}=(B)^{18}_{\omega-8^\circ}+0\cdot018;\quad (B')^{14}_{\omega}=(B')^{18}_{\omega-8^\circ}+0\cdot030.$$

NOTE.

When the internal and external surfaces of the index-glass are not parallel one to the other, it is evident that of an incident ray some portion of the light will be reflected immediately from the external surface, whilst the remainder, being refracted towards the internal surface, will be thence reflected, and emerge eventually in a direction different from that pursued by the former. A consequence of a defective index-glass is, therefore, the production of two images of an object, that resulting from reflexion at the silvered surface being brighter than the other when the angle of incidence is small, and consequently to be distinguished without difficulty. But this image continually diminishing in brilliancy with augmentation of the angle whilst under the same circumstances the other increases, the two approach rapidly towards equality in this respect, and, the index being at about 115° on the arc, are frequently so nearly alike that one may possibly be mistaken for the other. An index-glass so defective as to produce two images separated by a sensible angle should not be employed, although one producing only a perceptible elongation of the image may be safely treated according to the method detailed in sections 11 and 12, the coincidence observed on every occasion being that of the direct image of one object with the middle of the elongated image of the other produced by reflexion; and the only modification resulting from this course will be that the value of σ derived from observations in reversed positions of the glass will be that of $\Sigma \,.\, \cos\gamma$ instead of $2\Sigma \,.\, \cos\gamma$, a difference of no practical consequence, since the correction to an angle will now be $B\Sigma\cos\gamma$ and not $2B\,\Sigma\cos\gamma$, as it would if the single image emanating from the silvered surface were that invariably observed. Pursuing the investigation of section 11, we obtain for the angular separation of the images the expression

$$\frac{2\Sigma\,.\,\sqrt{\mu^2-\sin^2(\frac{1}{2}\omega+\beta)}}{\cos(\frac{1}{2}\omega+\beta)}\,.\,\sqrt{\cos^2\gamma+\tfrac{1}{4}\,.\,\sin^2\gamma\,.\,\cos^2(\tfrac{1}{2}\omega+\beta)},$$

which, when $\gamma=0$ or $180°$, becomes

$$\frac{2\Sigma\,.\,\sqrt{\mu^2-\sin^2(\frac{1}{2}\omega+\beta)}}{\cos(\frac{1}{2}\omega+\beta)}=p\,.\,\Sigma\,;$$

and when $\gamma=90°$ or $270°$,

$$\Sigma\,.\,\sqrt{\mu^2-\sin^2(\tfrac{1}{2}\omega+\beta)}=q\,.\,\Sigma,$$

so that if $r\Sigma=$ total angular separation, we have

$$r^2=p^2.\cos^2\gamma+q^2.\sin^2\gamma.$$

Now as the greatest value of q between $\omega=0$ and $125°$ can never exceed 1·5, and as the value of this quantity is very little affected by a variation in that of β, whereas the value of p may be as great as 20 when ω is large, and in this case only will vary considerably with a variation of β, the greatest value of p moreover occurring with the smallest of q, it is evident that it will not be necessary to consider the dependence of the value of r upon that of β, except in the case in which the term $q^2\,.\,\sin^2\gamma$ is small compared with $p^2\,.\,\cos^2\gamma$, or that in which γ is not nearly equal to $90°$, p and its variations being at the same time considerable, so that

$$r=p\,.\,\cos\gamma \text{ nearly.}$$

But on computing the values of p, we find that whereas at $\omega=125°$ the value is 17·6 if $\beta=20°$, it amounts only to 10·6 if $\beta=15°$, so that the ratio 176 : 106 represents very nearly the advantage gained at $\omega=125°$ by the reduction of β from $20°$ to $15°$, an index-glass which with $\beta=20°$ would produce a double image with an angular separation of $17''\cdot6$ producing with $\beta=15°$ a separation of only $10''\cdot6$, and for smaller separations in the same proportion. The angle between the optical axis and the perpendicular to the plane of the horizon-glass ought therefore, in the construction of the instrument, to be made as small as a due regard to the free action of the several parts will permit, a conclusion otherwise evident on inspection of the tabulated values of the coefficient B.

PART II.

APPLICATIONS OF THE SEXTANT.

CHAPTER I.

GENERAL TREATMENT OF THE SUBJECT.

THE Sextant may be employed in every operation that involves the measurement of an angle, the objects subtending the angle to be measured being so far distant that the direct rays from each which fall upon the eye may, for all practical purposes, be considered parallel to those which fall from the same object upon the index-glass—or, in other words, that the distance between the eye and the index-glass shall not subtend at either object any sensible angle. This condition is seldom fulfilled except in the case of objects in the heavens; and accordingly it is to the observation of these that the Sextant is especially applicable.

The purposes for which we observe celestial objects are generally the following:—

1°. The determination of the geographical latitude of a place.

2°. The determination of local time, the latitude of the place being known.

3°. The determination of the longitude of a place, the latitude and time being known.

And we propose to treat each of these in order, commencing with the determination of latitude.

1. Geographical latitude is the angle between the plane of the equator and the vertical line or direction of gravity at the place; and its complement, called the colatitude, is equal to the true meridian zenith distance of a celestial object $\pm$ the polar distance of the same object, the upper or lower sign being taken according as the object is at upper or lower transit, and the zenith distance, if measured towards the pole which we assume as the origin of polar distance, being reckoned positive, if away from it, negative. The difference between this result and 90° will be the latitude reckoned towards the assumed pole when

the result is less than 90°, and away from it if greater. But as the angle between the image of an object reflected in the artificial horizon and its direct image is the double of its apparent altitude, if in the case of an object on the meridian we observe with the Sextant the angle between these two images and divide this by 2, the result will be the apparent meridian altitude of the object; and subtracting this from 90°, we shall obtain the apparent meridian zenith distance. This again being corrected for the effects of refraction and parallax, will give us the true meridian zenith distance; and as the polar distance may be taken from the Tables, we shall thus be in possession of all the elements requisite for the calculation of latitude.

As we are supposed in the first instance not to know the time, and consequently when to look for the object on the meridian, it will be necessary to watch it closely as it increases its altitude, and to obtain the reading at the moment when the angle between the two images attains a maximum for upper, or a minimum for lower transit. Let Ω be this reading corrected for instrumental errors,

$$\therefore \tfrac{1}{2} . \Omega = \text{apparent meridian altitude,}$$

$$90^\circ - \tfrac{1}{2} . \Omega = \text{apparent meridian zenith distance,}$$

$$\text{true meridian Z. D.} = Z = 90^\circ - \tfrac{1}{2} . \Omega + \text{refraction} - \text{parallax,}$$

Z being reckoned + or − according as it is measured towards or away from the pole which we assume as the origin of the polar distance π. Then

$$\text{colatitude} = Z \pm \pi,$$

the upper or lower sign being taken according as the object is at upper or lower transit.

Example. July 26, 1858. Barometer 29·92 inch. Thermometer 59°. Horizon cover N.

α Ophiuchi on Southern Meridian.

$-\omega_0$	$= -1'\ 9''\cdot 0$*	ω	$= 104^\circ\ 10'\ 37''\ (\omega) = 104^\circ\ 47'$
Correction for η & σ	$= 0\ \ 0\cdot 0$*	——	$-\ 0\ 43\cdot 3$
E	$= +0\ 14\cdot 3$†	Ω	$= 104\ \ 9\ 53\cdot 7$
F	$= +0\ 11\cdot 4$‡	$\tfrac{1}{2} . \Omega$	$= 52\ \ 4\ 56\cdot 9$
AH	$= 0\ \ 0\cdot 0$§	$90^\circ - \tfrac{1}{2}\Omega$	$= 37\ 55\ \ 3\cdot 1$
Sum	$= -0\ 43\cdot 3$	Refraction	$= 44\cdot 5$
		Z	$= -\ 37\ 55\ 47\cdot 6$
		π	$= 77\ 20\ \ 3\cdot 5$
		Colatitude	$= 39\ 24\ 15\cdot 9$
		Latitude	$= 50\ 35\ 44\cdot 1$ N.

* Part I. Section 17, subdivisions 1 and 2.

† Taken from the Table constructed in 1857, given in subdivision 3 of Section 17, Part. I.

‡ Part I. Section 18. § Part I. Section 17, subdivision 5.

In determining latitude by the above method, we should be limited to a single observation of an object; and for the purpose therefore of obtaining a mean of a number of results, in which error of observation would probably not appear, we should be compelled to observe a number of objects, or the same object on the meridian a number of times. But this process would cause our observations to extend over a considerable period, whereas in cases in which the Sextant is employed, we generally want to obtain a good result in the course of a short period, sometimes immediately. A single observation, however, of the above description will furnish us with a latitude sufficiently exact to employ in obtaining time; and this known, we shall be able to extend our observations for the purpose of obtaining the other element with greater accuracy. We have therefore—

2. To determine the local sidereal time, the latitude of the place being known either exactly or approximately.

Let A and π represent the tabular right ascension and polar distance of an object; ζ its true zenith distance at observed time t; ϕ the true colatitude of the place; and let T be the true sidereal time or right ascension of the meridian when the watch or clock showed t.

Making $H = T - A$, we have

$$\cos\zeta = \cos\phi \,.\, \cos\pi + \sin\phi \,.\, \sin\pi \,.\, \cos H;$$

and from this, putting θ for $\frac{1}{2} \,.\, (\zeta + \phi + \pi)$, we obtain

$$\left.\begin{aligned} \sin\tfrac{1}{2}H &= \pm\sqrt{\frac{\sin(\theta-\phi) \,.\, \sin(\theta-\pi)}{\sin\phi \,.\, \sin\pi}} \\ \tan\tfrac{1}{2}H &= \pm\sqrt{\frac{\sin(\theta-\phi) \,.\, \sin(\theta-\pi)}{\sin\theta \,.\, \sin(\theta-\zeta)}} \end{aligned}\right\}\ \text{H being reckoned + or − according as the object is west or east of the meridian,}$$

either of which formulæ may be employed in determining H when the true values of ζ, ϕ, and π are known, the second, however, being generally preferable to the first; and H determined, we have at once $T = A + H$.

The above expressions are perfectly general, and apply, whatever be the values of π and H. There are, however, particular values of these quantities, which, supposing a small error to exist in ζ, consequent on one in the observed altitude, or in the computed refraction or parallax, will cause such an error to produce a minimum effect on the calculated value of H; and objects as nearly as possible in the corresponding position in the heavens should therefore be employed in the determination of time.

To ascertain what are these values, let us suppose the true zenith distance, instead of being ζ, to be $\zeta+\delta\zeta$; the true hour-angle will then be $H+\delta H$, where

$$\delta H=\frac{dH}{d\zeta}\cdot\delta\zeta=\frac{\delta\zeta\cdot\sin\zeta}{\sin\phi\cdot\sin\pi\cdot\sin H}=\frac{\delta\zeta}{\sin\phi\cdot\sin *\text{'s azimuth}},$$

which, ϕ being given, is evidently a minimum when the sine of the azimuth is a maximum, or when the azimuth is as near 90° as possible. Now, for objects whose declination is not greater than the latitude, and of the same name with it, the azimuth may be 90°; and when it is (that is, when they cross the great circle which passes through the zenith at right angles to the meridian, and is called the prime vertical), they are most favourably situated for observation for time*; and with respect to other objects whose declinations do not fulfil the above conditions, it is evident that the greater the azimuth the smaller will be the effect of an error of observation on the computed time; and such objects, when their declinations are in name opposite to that of the latitude, should therefore be observed as far from the meridian as possible consistently with their having sufficient altitude to render the results of the tables of refraction reliable; and the above expression for δH shows us, moreover, that an object should never be observed for time when its azimuth is small, that is when near the meridian if far distant from the pole, and that an object near the pole itself should not be observed for this purpose under any circumstances.

It is to be remarked, likewise, that the expression for δH changes sign with a change in the sign of H, thus showing us that the effect of permanent small instrumental error will not appear in the mean of the results of two observations, one of an object west of the meridian, the other of an object east of it, the latter at the time of the second observation having about the same azimuth east as the former had west at the time of the first, or *vice versâ*.

Let us in the next place consider the effect of an error in the assumed colatitude ϕ.

In this case

$$\delta H=\frac{dH}{d\phi}\cdot\delta\phi=\frac{\cos\phi\cdot\sin\pi\cdot\cos H-\sin\phi\cdot\cos\pi}{\sin\phi\cdot\sin\pi\cdot\sin H}\cdot\delta\phi;$$

* In Baily's Astronomical Tables and Formulæ, page 153, and in other similar works, will be found a Table whereby we can determine at sight the altitude at which an object of given declination may be most advantageously observed for time in a given latitude.

and as the numerator of this expression vanishes when

$$\cos H = \tan \phi \,.\, \cot \pi,$$

or when the azimuth is 90°, it follows that a small error in the assumed latitude produces no effect on the time obtained from an altitude observed on the prime vertical; and this is therefore on every account the most favourable position for observation for the purpose in question.

With respect to an object which does not pass the prime vertical because its declination is greater than the latitude, if we put the coefficient of $\delta\phi$ into the form

$$\cot \phi \,.\, \cot H - \cot \pi \,.\, \text{cosec } H,$$

we shall readily obtain as the condition of minimum,

$$\cos H = \cot \phi \,.\, \tan \pi,$$

showing in this case that the minimum effect of an error $\delta\phi$ is produced when the vertical arc passing through the object becomes a tangent to the parallel of declination, that is, when the azimuth of the object is a maximum. And in the case of an object whose declination is of a name different from that of the latitude, since $\cot \pi$ will be negative, and $\cot H$, $\text{cosec } H$ continually diminish as H increases, such an object should be observed as far as possible from the meridian, subject to the same condition as that established in the case of an error in zenith distance.

We likewise remark that, as in the case of an error in ζ, so in that of one in ϕ, the expression for δH changes sign with a change in the sign of H, thus showing that the effect of a small error in the latitude will not appear in the mean of the results of two observations upon opposite sides of the meridian, the two objects at the times observed having about the same polar distance and hour-angle.

From what has preceded, we may conclude generally,—

1°. Whether the latitude be known exactly or approximately, all objects observed for time should be situated as nearly as possible on the prime vertical.

2°. When the latitude is known approximately only, a single observation for time should be that of an object very close to the prime vertical; or, if two observations are obtained, the objects observed should have about the same polar distance, and be situated at about equal distances from the meridian upon opposite sides of it. In the former case the single result, in the latter the mean of the two results will be independent of any small error in the assumed value of the latitude.

Example.—July 26, 1858. Bar. 29·92 ins. Therm. 59°. Horizon cover N. Assumed latitude 50° 35′ 44″ N.

Arcturus west of Meridian.

	Time by watch.	Reading.	(ω).	E‡.
	h m s	° ′ ″	° ′	′ ″
	9 25 18†	76 1 0	77 0	+0 11·4
	26 43	75 34 14	79 44	+0 11·5
	27 40	75 16 38	81 48	+0 11·7
Means	9 26 33·7	75 37 17	E	=+0 11·5
		−0 46	$-\omega_0$	=−1 9·0
Ω	=	75 36 31	for η and σ	= 0 0·0
½.Ω	=	37 48 16	F	=+0 11·4
90°−½.Ω	=	52 11 44	AH	= 0 0·0
Refraction	=	+1 13	Sum	=−0 46§
ζ	=	52 12 57		
φ	=	39 24 16		
π	=	70 4 42		
Sum	=	161 41 55		

θ	= 80 50 58	log sin	=9·9944376		
θ−ζ	= 28 38 1	log sin	=9·6805230		
θ−φ	= 41 26 42	log sin			=9·8207929
θ−π	= 10 46 16	log sin			=9·2715768
		Sums	9·6749606		9·0923697
					9·6749606
				Difference	=9·4174091
				log tan ½H = Half	=9·7087046
				½H	=27° 4′ 56″
				H	=54 9 52
					h m s
					=+3 36 39·5
				RA of *	= 14 9 13·2
				Sidereal time T	= 17 45 52·7

The watch employed was losing 10 seconds from mean time in the course of 24 hours ‖; and therefore, should we wish to find the local sidereal time at any other time indicated by the

† These times were taken with an ordinary pocket-watch; and tenths of seconds could not be observed.

‡ In some instruments we shall find E vary materially with a few degrees of variation in (ω); that is, with a few minutes in the reading.

§ It is unnecessary to notice tenths of seconds of arc in the sums.

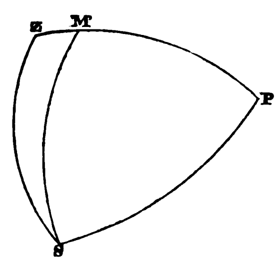
‖ Chronometers going sidereal time will be found much more convenient than those going mean time in all astronomical observations, whether on shore or at sea.

watch, we shall have to correct the interval shown by this for retardation at the above rate; and having converted the corrected interval into sidereal time, apply this to $17^h\ 45^m\ 52^s{\cdot}7$.

Another method of obtaining time is suggested by the equation $\cos\zeta = \cos\phi . \cos\pi + \sin\phi . \sin\pi . \cos H$; from which we see that for equal values of H with opposite signs, the value of $\cos\zeta$, and therefore of ζ, is the same. Hence it follows that a celestial object which does not change its right ascension and polar distance, will at equal distances upon opposite sides of the meridian have equal altitudes; and if therefore we observe it on its approach towards the meridian, noting the time when the Sextant gives a certain reading for the angle between the direct image in the heavens and the reflected image in the artificial horizon, and again, after its passage over the meridian, note a second time when the reading is the same as the first, the mean of the two times shown by the watch will be that indicated at the instant of transit, and the right ascension of the object will be the corresponding local sidereal time. This method is seldom had recourse to, since when the interval is small, and the object consequently near the meridian, its apparent motion is too slow to admit of nice observation of the time of coincidence of images; and when the object is sufficiently far from the meridian to render this part of the operation trustworthy, the interval between the observations must be so considerable that the observer would in the mean time obtain more complete and satisfactory results in another way.

It will be evident that we cannot, in the determination either of latitude or time, have recourse to objects whose right ascension and polar distance change rapidly, without an approximate knowledge of the longitude of the place of observation, or, which amounts to the same thing, of the time at the place for which the Tables are constructed. This known, we may employ the sun, planets, moon, and circumpolar stars, bearing in mind, however, that the tabular places of the moon are much less to be trusted than those of other objects.

3. Having now obtained the local sidereal time, we are in possession of all that is requisite to enable us to extend our observations for latitude, and we will therefore next consider generally the method of deriving this element from an observed altitude of an object at a given angle from the meridian.



In the annexed figure, let Z represent the zenith; P the pole; S the position of a celestial object; and SM an arc at right angles to PZ.

PS = polar distance = π, $\angle$ZPS = hour-angle = H,
ZS = true zenith distance = ζ, PZ = colatitude = ϕ.

$$PM = Z_1,\quad ZM = Z_2,\ \text{and}\ \angle PZS = Z.$$

From ΔPMS we have $\tan Z_1 = \tan\pi . \cos H$; whence Z_1 is known.

From ΔPZS we have $\sin Z = \dfrac{\sin\pi . \sin H}{\sin\zeta}$; whence Z is known*.

From ΔZSM we have $\tan Z_2 = \tan\zeta . \cos Z$; whence Z_2 is known*.

$$\phi = Z_1 + Z_2.$$

Although this method of proceeding is perfectly general, applying to an object in any position of the heavens, nevertheless, as in the case of observations for determination of time, so in this there are particular positions more favourable than others for the purpose in view; and what these are we have therefore to ascertain.

We have already found

$$\delta H = \frac{\cos\phi . \sin\pi . \cos H - \sin\phi . \cos\pi}{\sin\phi . \sin\pi . \sin H} . \delta\phi;$$

and we consequently have

$$\delta\phi = \frac{\sin\phi . \sin\pi . \sin H}{\cos\phi . \sin\pi . \cos H - \sin\phi . \cos\pi} . \delta H,$$

an expression from which we infer that the values of the elements which make δH a minimum when $\delta\phi$ is given, make the latter a maximum when δH is given, and *vice versâ*; and the positions most favourable for observation for time, on the supposition of a small error in the assumed latitude, are thus the most unfavourable for observation for latitude in the event of the existence of a small error in the assumed time. All objects observed for latitude should therefore be near the meridian when their polar distance is considerable; but a circumpolar star may be advantageously employed at any hour-angle whatever.

Again, from the equation

$$\cos\zeta = \cos\phi . \cos\pi + \sin\phi . \sin\pi . \cos H,$$

we derive

$$\delta\phi = \frac{\sin\zeta}{\sin\phi . \cos\pi - \cos\phi . \sin\pi . \cos H} . \delta\zeta;$$

* It is not necessary to take the value of Z itself from the Tables, since all that we require in the computation of Z_2 is the cosine of this angle, which may be taken at once from the proper column in the line opposite to that in which we find sin Z, or interpolating between two lines if requisite.

and since

$$\sin^2\zeta = 1 - (\cos\phi \,.\, \cos\pi + \sin\phi \,.\, \sin\pi \,.\, \cos H)^2$$
$$= (\sin\phi \,.\, \cos\pi - \cos\phi \,.\, \sin\pi \,.\, \cos H)^2 + \sin^2\pi \,.\, \sin^2 H,$$

the coefficient of $\delta\zeta$ becomes

$$\sqrt{1 + \left(\frac{\sin\pi \,.\, \sin H}{\cos\phi \,.\, \sin\pi \,.\, \cos H - \sin\phi \,.\, \cos\pi}\right)^2},$$

which for a given value of π is a minimum when $H = 0$, or when the object is on the meridian, and for a given value of H diminishes with a diminution in π, that is, as the object is nearer to the pole. On every account, therefore, the azimuth of objects observed for latitude should be as small as possible.

Example. July 29, 1858. Bar. 29·98 inches. Therm. 59°. No cover over horizon.

Polaris.

	Sidereal time.	Reading.	(ω).	E.
	h m s	° ′ ″	° ′	′ ″
	18 2 47·6	100 22 45	103 5	+0 14·1
	3 53·8	23 36	103 56	14·2
	5 2·0	24 31	104 51	14·3
	6 21·2	25 25	105 45	14·4
	7 29·4	26 23	106 43	14·5
	9 5·6	27 28	107 48	14·6
	10 8·8	28 18	108 38	14·7
	11 15·0	29 9	109 29	14·7
Means	18 7 0·43	100 25 56·9	E	= +0 14·4
		0 44·2	$-\omega_0$	= −1 10·0
	Ω	= 100 25 12·7	for η & σ	= 0 0·0
	½Ω	= 50 12 36·4	F	= +0 11·4
	90° − ½ . Ω	= 39 47 23·6	AH	= 0 0·0
	Refraction	= + 0 47·6	Sum	= −0 44·2
	ζ	= 39 48 11·2		

Now the right ascension of Polaris, corrected for nutation at upper transit at Greenwich on the 29th of July, was $1^h\ 7^m\ 32^s{\cdot}74$; and the sidereal time at the place, known to be about five minutes west of Greenwich, being $18^h\ 7^m$, the sidereal time at Greenwich at the moment of observation was about $18^h\ 12^m$; that is, about seven hours before the transit of the star. Interpolating for this interval between the right ascensions at upper transit on the 28th and 29th, we have

		h m s	
RA of Polaris at time of observation		= 1 7 32·50	
RA of meridian		= 18 7 0·43	
Hour-angle H		= 7 0 32·07	= − 105° 8′ 1″·1
And in the same way we obtain π =			1 26 53·7
log tan π = 8·4028134	log sin π = 8·4026746		log tan ζ = 9·9207809
log − cos H = 9·4167592	log sin H = 9·9846710		log cos Z* = 9·9996844
log − tan Z_1 = 7·8195726	Sum = 8·3873456		log tan Z_2 = 9·9204653
	log sin ζ = 9·8062827		Z_2 = 39° 46′ 57″·5
	log sin Z = 8·5810629		− Z_1 = 0 22 41·4
			ϕ = 39 24 16·1

4. But when an object is near the meridian, there is a method of proceeding which much facilitates the calculation of latitude, and enables us therefore to multiply our observations without fear of being overwhelmed with the subsequent labour of computation. This method is called "Reduction to the meridian," and depends upon the expansion of ζ into a series of powers of sin H, H being in this case small. Thus we have

$$\zeta=\phi-\pi+\frac{\sin\phi\,.\,\sin\pi}{\sin(\phi-\pi)}\,.\,\frac{2\sin^2\frac{1}{2}H}{\sin 1''}-\left(\frac{\sin\phi\,.\,\sin\pi}{\sin(\phi-\pi)}\right)^2.\cot(\phi-\pi)\,.\,\frac{2\sin^4\frac{1}{2}H}{\sin 1''};$$

and making $\phi-\pi=Z=$ true meridian ZD, and transposing,

$$Z=\zeta-\frac{\sin\phi.\sin\pi}{\sin Z}\,.\,\frac{2.\sin^2\frac{1}{2}H}{\sin 1''}+\left(\frac{\sin\phi.\sin\pi}{\sin Z}\right)^2.\cot Z.\frac{2.\sin^4\frac{1}{2}H}{\sin 1''}.$$

Now in the coefficients of the small quantities on the second side of this equation, it will be sufficient to substitute an approximate value of Z, which may be either the smallest of a number of values of ζ if the observations have been continuous as the object approached towards and receded from the meridian, or the result of the substitution of an approximate value of ϕ in $\phi-\pi$ if the observations were interrupted; and these coefficients are evidently independent of the hour-angle H. Hence denoting the coefficients by Q_1 and Q_2, if we have a number of true circum-meridian zenith distances ζ_1, ζ_2, &c. ζ_n, at hour-angles H_1, H_2, &c. H_n, these will give us

* Cos Z will evidently be positive or negative according as the object is situated on the polar or equatorial side of the prime vertical at the time of observation.

$$Z=\zeta_1-Q_1 \cdot \frac{2 \cdot \sin^2 \frac{1}{2} H_1}{\sin 1''} + Q_2 \cdot \frac{2 \cdot \sin^4 \frac{1}{2} H_1}{\sin 1''},$$

$$Z=\zeta_2-Q_1 \cdot \frac{2 \cdot \sin^2 \frac{1}{2} H_2}{\sin 1''} + Q_2 \cdot \frac{2 \cdot \sin^4 \frac{1}{2} H_2}{\sin 1''},$$

&c. = &c.

$$Z=\zeta_n-Q_1 \cdot \frac{2 \sin^2 \frac{1}{2} H_n}{\sin 1''} + Q_2 \cdot \frac{2 \cdot \sin^4 \frac{1}{2} H_n}{\sin 1''};$$

and taking the mean,

$$Z=\frac{1}{n} \cdot \Sigma\zeta - Q_1 \cdot \frac{1}{n} \cdot \Sigma \frac{2 \cdot \sin^2 \frac{1}{2} H}{\sin 1''} + Q_2 \cdot \frac{1}{n} \cdot \Sigma \frac{2 \cdot \sin^4 \frac{1}{2} H}{\sin 1''},$$

a result which will probably be much more accurate than that obtained from the observation of one double altitude.

To facilitate the application of this method, tables of values of the quantities $\frac{2 \sin^2 \frac{1}{2} H}{\sin 1''} \cdot \frac{2 \sin^4 \frac{1}{2} H}{\sin 1''}$ have been constructed for every second of time of H within such limits as it would be proper to have recourse to it; and these Tables will be found in Baily's Astronomical Tables and Formulæ, page 154, and in other works of a similar description. The process will be made perfectly clear by the following example, in which it will be unnecessary to give in detail either the determination of the hour-angles from the observed times, or that of the true zenith distances from the readings, these operations having been already sufficiently explained and illustrated in the preceding sections.

July 26, 1858. Bar. 29·92 inches. Therm. 59°.

α Ophiuchi on Southern Meridian.

H.	ζ.	$\frac{2 \sin^2 \frac{1}{2} H}{\sin 1''}$.	$\frac{2 \sin^4 \frac{1}{2} H}{\sin 1''}$.
m s	° ′ ″	″	″
−12 17·0	38 0 47·9	296·2	0·21
8 51·4	37 58 17·7	154·0	0·06
6 17·0	37 57 4·3	77·5	0·01
− 1 23·2	37 55 50·3	3·8	0·00
+ 1 42·3	37 55 47·8	5·7	0·00
4 25·7	37 56 26·8	38·5	0·00
7 53·3	37 57 51·8	122·2	0·04
+ 8 50·5	37 58 27·3	153·5	0·06
Means	37 57 34·2	106·43	0·05

For the computation of Q_1 and Q_2, assuming $Z=-37° 56'$ and $\phi=Z+\pi=39° 24'$,

log sin ϕ	= 9·80259	log Q_1	= 0·00320	$\frac{1}{n}.\Sigma\zeta=$	37° 57′ 34″·2
log sin π	= 9·98930	log Q_1^2	= 0·00640		
Sum	= 9·79189	log cot Z	= 0·10823		
log sin Z	= 9·78869	log Q_2	= 0·11463		
log Q_1	= 0·00320	log 0·05	= 8·77815		
log 106 . 43	= 2·02706	log 2nd correct.	= 8·89278 —	nat. no. =	+ 0 0 ·08
log 1st correct.	= 2·03026 —	natural no.	= 107″·22	=	− 1 47 ·22
				Z =	−37 55 47·1
				π =	77 20 3·5
				ϕ =	39 24 16·4

Should the object observed change its polar distance, as in the case of the sun, then Z in any one of the above equations will represent the meridian zenith distance, on the supposition that the polar distance at the time of observation is that at the time of transit. Hence if h be the interval to meridian passage expressed in minutes of time, and $\Delta\pi$ the variation of π in one minute,

$$\text{true meridian ZD} = Z - h \,.\, \Delta\pi,$$

and the correction to the value of Z derived from all the observations on the supposition of the polar distance remaining unchanged, will be

$$-\frac{1}{n}\,.\,\Sigma\, h\,.\,\Delta\pi,$$

or we shall have to subtract from the value of Z given by the above process, the product of the mean of the intervals, with their proper signs, and the variation of the polar distance in one minute, the mean of the intervals being expressed in minutes.

5. To find the latitude and time by means of two observed altitudes, either of two different objects, or of the same object, an interval elapsing between the observations, and the latitude being known to lie between given limits within some minutes one of the other.

Let ϕ, ϕ' be the given limits within which the colatitude lies; T_1, T_2 the times of observation by the watch, T being the interval T_2-T_1 corrected for rate; ζ_1, ζ_2 the true zenith distances of the observed objects at T_1, T_2; π_1 and π_2 their polar distances.

Then, assuming that ϕ is the true colatitude, we may, by the process given in section 2, determine the time from each of the observations. Let us do so; and H_1, H_2 being the resulting hour-angles, let t_1, t_2 be the corresponding times. Then, if $\phi+\delta\phi$ be the true colatitude,

$$\text{true time at } T_1=t_1+\frac{\cos\phi.\sin\pi_1.\cos H_1-\sin\phi.\cos\pi_1}{\sin\phi.\sin\pi_1.\sin H_1}.\delta\phi=t_1+m_1.\delta\phi;$$

$$\text{at } T_2=t_2+\frac{\cos\phi.\sin\pi_2.\cos H_2-\sin\phi.\cos\pi_2}{\sin\phi.\sin\pi_2.\sin H_2}.\delta\phi=t_2+m_2.\delta\phi;$$

and by subtraction,

$$\text{Interval}=T=t_2-t_1+(m_2-m_1).\delta\phi,$$

$$\therefore \delta\phi=\big(T-(t_2-t_1)\big)\div(m_2-m_1),$$

$\therefore$ true colatitude $\phi+\delta\phi$ is known, as are likewise the true times $t_1+m_1.\delta\phi$, $t_2+m_2.\delta\phi$ at T_1, T_2 respectively.

Instead of computing m_1, m_2 by the above expressions, it will be found more convenient to compute again in the ordinary way, with colatitude ϕ', the times t'_1, t'_2. Then if n be the number of seconds of arc in the difference $\phi'-\phi$, we shall have

$$m_1=\frac{1}{n}(t'_1-t_1),\ m_2=\frac{1}{n}(t'_2-t_2).$$

The expressions for m_1, m_2 are useful however as showing us the cases in which m_2-m_1 will be small, and in which therefore the results of the process will be comparatively inaccurate. These are evidently,

1°. When $\pi_1=\pi_2$ and $\sin H_1=\sin H_2$ nearly, or when the same object is observed twice, the interval between the observations being small when it is near the meridian, or even considerable when it is distant from the meridian, and upon the same side of it on both occasions.

2°. When the numerators of the expressions are both small, that is, when the objects are situated near the prime vertical, either upon the same or opposite sides of the zenith;

and in other cases likewise in which the quantities nearly fulfil the condition $m_2=m_1$. But it is evident that if we want the latitude particularly, and are comparatively indifferent about time, we may with advantage observe the same object twice upon opposite sides of the meridian, and not very far away from it; whereas, if we are anxious about both elements, the best combination will be that of an object near the meridian on one side, with one near the prime vertical on the other, or the latter with a circumpolar object at any hour-angle whatever. We remark, moreover, that it is not absolutely necessary that the true latitude should lie between the assumed limits; it may lie a little beyond either one or the other.

Example.—July 14, 1858. Bar. 29·87 inches. Therm. 63°. Assumed limits of latitude 50° 30′ N. and 50° 40′ N., or $\phi=39° 20'$, $\phi'=39° 30'$.

β Ursæ Minoris.

At $9^h\ 47^m\ 19^s{\cdot}6$ by watch, $\zeta=28°\ 22'\ 25''$.

Arcturus.

At $10\ 19\ 2{\cdot}3$ $\zeta=52\ 39\ 52$

The first observation reduced with colatitude	39° 20′ gives	t_1	=	$17^h\ 18^m\ 16^s{\cdot}0$
" "	39 30	t'_1	=	17 15 14·9
The second observation, similarly reduced, gives		t_2	=	17 48 41·4
		t'_2	=	17 48 57·8

The watch was losing 10 seconds daily from mean time.

Interval by watch	= $31^m\ 42^s{\cdot}7$			
Correction for Rate	= +0·2			
True mean interval	= 31 42·9			
Acceleration	= +5·2			
T	= 31 48·1			
t_2-t_1	= 30 25·4			
Difference	= 1 22·7	$=82^s{\cdot}7$	log	= 1·91751
$t'_1-t_1=-181^s{\cdot}1$ log	= 2.25792			
$n=$ 600·0 log	= 2·77815			
Diff. = log $-m_1$	= 9·47977	$m_1=-0{\cdot}30184$		
$t'_2-t_2=+$ 16·4 log	= 1·21484			
log n	= 2·77815			
Difference = log m_2	= 8·43669	$m_2=+0{\cdot}02733$		
		$m_2-m_1=+0{\cdot}32917$	log	= 9·51742

$\log\delta\phi$	= 2·40009	Difference	= $\log\delta\phi=2{\cdot}40009=\log 251''$
$\log m_2$	= 8·43669	$\delta\phi$	= $+0°\ 4'\ 11''$
$\log m_2\,.\,\delta\phi$	= 0·83678	ϕ	= 39 20 0
$m_2\,.\,\delta\phi$	= $+\ 0^m\ 6^s{\cdot}9$	true colatitude	= 39 24 11
t_2	= $17^h 48\ 41{\cdot}4$	latitude	= 50 35 49
Sum	= 17 48 48·3	= true sidereal time when the watch indicated $10^h\ 19^m\ 2^s{\cdot}3$.	

Should the resulting value of the latitude in any case fall much without the assumed limits, we may assume new limits which shall include it and repeat the process.

6. Of the two elements of geographical position, latitude may, as we have seen, be determined with considerable accuracy by means of observations sufficiently simple in themselves as well as

easy of reduction, and we have now to treat the second element, longitude.

Difference of longitude is measured by difference of time, either mean or sidereal, so that a phenomenon being visible at the same instant from two places, and spectators at those places observing each the local time of its occurrence, the difference between the times observed will be the difference of longitude.

The moon in the course of its revolution in its orbit is continually changing its apparent position with respect to other celestial objects, and were this apparent position unaffected by either parallax or refraction, we should have at every instant of time, in the angular distance between this and another object, a phenomenon such as we desire, by means of observation of which difference of longitude could be at once ascertained. But although the introduction of parallax and refraction complicates the problem, the phenomena presented by the moon's motion are nevertheless available for our purpose; for we can ascertain the effect of these complicating causes, and thus, having given the apparent distance of the moon from the sun, a planet, or star as seen by a spectator on the surface, can determine what would be its apparent distance at the same instant to a supposed spectator at the centre of the earth. The process of calculation which this determination involves being performed upon observations made at two places, we have two local times corresponding to two distances between the moon and another celestial object, as the phenomena would appear to an observer at the centre; and the observation being repeated at one of the places at intervals of time, and the successive observed distances reduced to the centre in the same manner, we may by interpolation ascertain the local time at this place corresponding to the reduced distance resulting from the observation made at the other. The difference between the local time thus interpolated for one place and the time observed at the other, being that of the local times of occurrence of the same phenomenon, will be the difference of longitude.

But the laws which regulate the moon's motion, as well as the constants involved in the formulæ expressing those laws, being tolerably well known, we are able to predict what would be the apparent position of that object to a supposed spectator at the centre of the earth at any given time of a given meridian; and in the 'Nautical Almanac' we find in tabular form, not only the geocentric right ascensions and declinations of the moon for every hour of mean time of the meridian of Greenwich, but likewise the geocentric distances of certain objects from the moon's centre for every third hour of time of the same meridian. These Tables, therefore, if they were absolutely exact, would render observations at two places unnecessary, since, having observed a lunar distance at the place the longitude of which we wish to determine, and having thence calculated the geocentric distance at the time of observation, we might find from the Tables the Greenwich mean time at which the moon and object observed were separated by the resulting angle; and this being reduced to sidereal time, the difference between the Greenwich sidereal time and the local sidereal time observed would be the longitude of the place of observation east or west of Greenwich. The Tables are indeed sufficiently exact to enable us to obtain in this way a result not far from the truth; but if we wish to turn our observations to the best account, we shall bear in mind that the moon's tabular places are liable to future correction, the amount of which may either be ascertained on the publication of the results of observations made at fixed observatories, or determined more accurately at some more remote date by careful comparison of such observations in great numbers.

Now the circumstances of the operator who observes a lunar distance are not always the same. He may either know nothing of the longitude of the place of observation, and may wish to determine it, or he may know the longitude approximately and wish to ascertain it with greater accuracy; or, again, he may wish to obtain a tolerably good result immediately and be indifferent about correcting this subsequently for errors in the tables, or he may desire his result to remain a perpetual record, and therefore to express it in a form in which it will be possible for those afterwards interested in it to make the proper corrections on account of errors which may at some future time be discovered in the data employed by himself in the reduction. We will suppose the case which includes all, and take that of an operator who does not know his longitude within an hour*, but wishes to obtain at once a result sufficiently accurate for immediate purposes, and at the same time such as may be corrected subsequently, and rendered as trustworthy as a result of an observation of this class can well be.

We suppose the colatitude ϕ and sidereal time or right ascension of meridian T obtained. The observation for longitude will then consist of the following parts:—

1°. From five to ten observed distances of the moon's limb from the limb of the sun or the centre of a planet or star at intervals of from 1 to 2 minutes.

2°. Three altitudes of the moon at such intervals as will

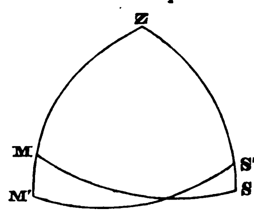
* It is scarcely possible to conceive circumstances under which a traveller conveying a Sextant would not know his longitude within an hour, or even the half of it.

enable us to obtain by interpolation the apparent zenith distance of this object at the mean of the times of the observed distances.

3°. Three similar observations of the second object.

4°. The barometer and thermometer at the time of observation.

To determine from these a result in the first place sufficiently accurate for immediate purposes, we suppose the earth's figure spherical, and the centre situated in the line which represents the direction of gravity at the place, at a distance equal to that of the centre of the spheroid; and as a consequence, we assume parallax and refraction to take effect each in a vertical plane.



Let Z be the zenith; M the position of the moon's centre as seen from the centre of the earth at the mean of the times of the observed distances; and M', in the arc Z M produced, its apparent position as seen from the place of observation.

Similarly, let S and S' be the two corresponding positions of the centre of the second object. Then

M' S' = apparent distance of the centres of the two objects at the mean of the observed times,
= corrected mean of observed distances between limbs $\pm$ apparent radius of moon $\pm$ apparent radius of second object . . . . . $=u_2$
Z M' = apparent zenith distance of moon's centre . . $=\zeta_2$
Z M = ζ_2 − parallax + refraction . . . . $=\zeta_1$
Z S' = apparent zenith distance of second object . . $=z_2$
Z S = z_2 − parallax + refraction . . . . $=z_1$
M S = geocentric distance sought . . . . $=u_1$

and proceeding as we did in section 17 of Part I., making

$$\theta_2=\tfrac{1}{2}\,.\,(u_2+\zeta_2+z_2) \text{ and } \sin^2\theta_1=\frac{\sin\theta_2.\sin(\theta_2-u_2).\sin\zeta_1.\sin z_1}{\sin\zeta_2.\sin z_2.\sin^2\frac{1}{2}(\zeta_1+z_1)},$$

we shall have

$$\sin\tfrac{1}{2}u_1=\sin\tfrac{1}{2}(\zeta_1+z_1)\,.\cos\theta_1\,;$$

and thus, the values of u_2, ζ_2, z_2, ζ_1, z_1 being known, we may compute that of u_1.

But of the quantities u_2, ζ_2, z_2, ζ_1, z_1, two, viz. ζ_2 and z_2, may be at once derived from the observations of altitude of the objects; and if

ρ = radius to place ÷ earth's equatorial radius*,

p = equatorial horizontal parallax of moon at time of observation,

P = equatorial horizontal parallax of second object†,

we may compute ζ_1 and z_1 by means of the formulæ

$$\zeta_1 = \zeta_2 + \text{refraction} - \rho \,.\, p\,\ddagger . \sin(\zeta_2 + \text{refraction}),$$

$$z_1 = z_2 + \text{refraction} - \quad \text{P} \,.\, \sin(z_2 + \text{refraction}).$$

To determine u_2, let

r = horizontal radius of moon at time of observation,

R = radius of second object.

Then,

$$\text{apparent radius of moon} = r \,.\, \frac{\sin(\zeta_2 + \text{refr.})\,\ddagger}{\sin \zeta_1} - \text{effect of refr.},$$

apparent radius of second object = R − effect of refraction,

where the effect of refraction in the first case is

(refraction at centre − refraction at upper limb) × $\cos^2$ Z M′ S′,

and in the second,

(refraction at centre − refraction at upper limb) × $\cos^2$ Z S′ M′.

These last corrections are invariably small and quite unnecessary when the moon and sun (if this be the second object) are within 40° of the zenith; and little will be gained by applying them even for much greater zenith distances, except when we are pursuing the more accurate method of reduction which we are about to treat. In cases, however, in which we wish to apply them, we may employ the approximate value of u_2, which will be derived from the assumption that r and R are the apparent radii

* If e be the excentricity of the earth's meridians $=0{\cdot}081697$, and ϕ' the geocentric colatitude of the place, we shall have $\cot\phi' = (1-e^2) \,.\, \cot\phi$, and $\rho = \sqrt{1-e^2} \div \sqrt{1-e^2 \,.\, \sin^2\phi'}$.

† We write 'equatorial horizontal parallax' for the sake of uniformity; but in fact the horizontal parallax of all objects except the moon is the same practically for every point on the earth's surface.

‡ ρp, which is called the horizontal parallax at the place, may be found at once by subtracting from p the correction for latitude given in most collections of tables. And similarly, a little further on, the augmentation to the moon's horizontal radius may be taken from the Tables and applied to r,—this, however, not including the effect of refraction.

of the objects, and with this and the values of ζ_2, z_2 compute the angles Z M′ S′, Z S′ M′ from

$$\sin^2 \tfrac{1}{2} \, ZM'S' = \frac{\sin \frac{1}{2}(z_2 - \zeta_2 + u_2) \, . \, \sin \frac{1}{2}(z_2 + \zeta_2 - u_2)}{\sin \zeta_2 \, . \, \sin u_2},$$

$$\sin^2 \tfrac{1}{2} \, ZS'M' = \frac{\sin \frac{1}{2}(\zeta_2 + z_2 - u_2) \, . \, \sin \frac{1}{2}(\zeta_2 - z_2 + u_2)}{\sin z_2 \, . \, \sin u_2},$$

taking the difference of refraction at centre and limb from the Tables.

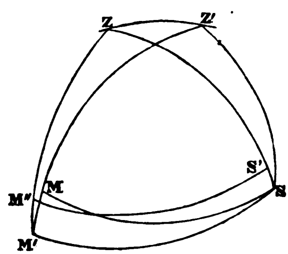
Let (r) and (R) be the apparent radii thus determined; Ω the mean of the readings corrected for instrumental errors; then

$$u_2 = \Omega \pm (r) \pm (R),$$

and the five quantities necessary to the determination of u_1 being thus known, we may compute this by the formula already given.

Let M be the Greenwich mean time at which the tabular geocentric distance is u_1*, and S the corresponding Greenwich sidereal time. Then longitude of place of observation $=$ S $-$ T, reckoned positive or negative according as it is west or east.

7. But we propose, in the next place, to obtain a more accurate result than the above process will give, and at the same time, by the introduction of terms expressing the effects of tabular errors, to render it more complete.



Let the earth's figure be now treated as spheroidal, Z′ being the point where the radius to the place produced meets the sphere, and Z the zenith, as before. Let M be the position of the moon's centre as seen from the centre of the earth, and M′ in the arc Z′M produced its place as affected by parallax alone. Then M′ M″ in the arc Z M′ being the effect of refraction, M″ will be the moon's apparent position at the place of observation.

Similarly, let S and S′ be the true and apparent places of the centre of the second object. These we may suppose situated in the same vertical arc Z S, the effect of parallax upon any object except the moon being so small as to render insensible the difference between the result derived on this hypothesis and that which we should obtain by treating it as occurring in the arc Z′ S. Then M″ S′ being the apparent distance between the centres of the objects, M′ S will be the distance corrected for

* For the method of obtaining M, refer to the explanation given at the end of the 'Nautical Almanac' under the head 'pages xiii to xviii of each month.'

the effect of refraction alone upon the moon, and that of parallax and refraction combined upon the second object; and M S will be the true geocentric distance sought.

The notation we shall employ in the treatment of the question will be as follows:—

ϕ = geographical colatitude of the place of observation.
ϕ' = geocentric colatitude.
ρ = radius to place ÷ earth's equatorial radius.
T = RA of meridian at time of observation.
l = approximate longitude, determined by the process last described, if not previously known.
T + l = assumed Greenwich sidereal time; and this converted into mean time giving t to the nearest minute,
t will be the assumed Greenwich mean time, for which we take from the Tables the following quantities:—
α = moon's geocentric right ascension.
π = moon's geocentric polar distance (π and ϕ being referred to the same pole).
p = moon's equatorial horizontal parallax.
r = moon's horizontal semidiameter.
A, Π, P, R the same elements for the second object.

For the remaining quantities,—

$$h=T-\alpha; \qquad H=T-A,$$
$$Z'M=\zeta, \quad Z'M'=\zeta'; \quad ZM'=\zeta_1, \quad ZM''=\zeta_2,$$
$$Z'S=z \quad ; \quad ZS=z_1, \quad ZS'=z_2,$$
$$M''S'=u_2, \quad M'S=u_1, \quad MS=u.$$

Now from the triangles Z M′ S, Z M″ S′, we have,

$$\text{making } \theta_2=\tfrac{1}{2}\,.\,(u_2+\zeta_2+z_2) \text{ and } \sin^2\theta_1=\frac{\sin\theta_2\,.\,\sin(\theta_2-u_2)\,.\sin\zeta_1\,.\sin z_1}{\sin\zeta_2\,.\,\sin z_2\,.\,\sin^2\frac{1}{2}(\zeta_1+z_1)},$$

$$\sin\tfrac{1}{2}u_1=\sin\tfrac{1}{2}(\zeta_1+z_1)\,.\cos\theta_1;$$

and similarly from the triangles Z′ M′ S, Z′ M S,

$$\text{making } \theta'=\tfrac{1}{2}\,.\,(u_1+\zeta'+z) \text{ and } \sin^2\theta=\frac{\sin\theta'\,.\sin(\theta'-u_1)\,.\sin\zeta}{\sin\zeta'\,.\sin^2\frac{1}{2}(\zeta+z)},$$

$$\sin\tfrac{1}{2}u=\sin\tfrac{1}{2}(\zeta+z)\,.\cos\theta,$$

and thus, the quantities ζ, ζ', ζ_1, ζ_2, z, z_1, z_2, u_2 being known, we may compute successively u_1 and u.

But for the computation of ζ we have (see section 17 of Part I.)

$$\tan\theta=\tan\phi'\,.\cos h, \qquad \cos\zeta=\frac{\cos\phi'\,.\cos(\pi-\theta)}{\cos\theta},$$

and $\zeta' = \zeta + \rho p . \sin\zeta + \frac{1}{2} . \rho^2 p^2 . \sin 2\zeta . \sin 1'' + \frac{1}{3} . \rho^3 p^3 . \sin 3\zeta . \sin^2 1'' +$ &c.

$= \zeta + \rho p . \sin\zeta + (p)$*.

We might in the next place compute Z M from the equation

$$\cos \mathrm{Z\,M} = \cos\phi . \cos\pi + \sin\phi . \sin\pi . \cos h,$$

and then from the triangles Z Z′ M, Z Z′ M′ derive Z M′ $= \zeta_1$, just as we derived u from the triangles M Z′ S, M′ Z′ S′. But the equation

$$\frac{\cos u_1 - \cos\zeta_1 . \cos z_1}{\sin\zeta_1 . \sin z_1} = \frac{\cos u_2 - \cos\zeta_2 . \cos z_2}{\sin\zeta_2 . \sin z_2},$$

through which we derive the formula for the computation of u_1, being in fact equivalent to

$$u_1 = u_2 + \mathrm{Q} . (\zeta_2 - \zeta_1) + \mathrm{Q}' . (z_2 - z_1) + \text{ \&c.},$$

and the effects of refraction $\zeta_2 - \zeta_1$, and refraction and parallax $z_2 - z_1$, being not only small at any altitude at which it would be proper to make observations of this class, but likewise not sensibly changed in value in consequence of small variations in the values of ζ_1 and z_1, we see that small errors in the values of ζ_1 and z_1, as they will affect those of the coefficients Q, Q′ alone, will produce no sensible error in the result. Hence if in treating the quantities which enter into the equation between u_1 and u_2 we suppose $\zeta_1 = \mathrm{ZM}' = \mathrm{ZM} + \mathrm{MM}'$, our result will not be affected to any appreciable extent.

Let $p' = \rho p . \sin\zeta + (p) = \zeta' - \zeta$,

$$\therefore \cos(\zeta_1 - p') = \frac{\cos\phi . \cos(\pi - \theta)}{\cos\theta}, \text{ where } \tan\theta = \tan\phi . \cos h\dagger,$$

from which we may compute $\zeta_1 - p'$, and thence derive successively ζ_1 and $\zeta_2 = \zeta_1 -$ refraction.

* Appended to this Part will be found a Table of values of (p) for various values of p and ζ.

† Another method of proceeding will be to compute the angle ZZ′M from $\sin \mathrm{ZZ'M} = \frac{\sin\pi . \sin h}{\sin\zeta}$, and thence $\zeta_1 = \zeta' \pm (\phi' - \phi) . \cos \mathrm{ZZ'M}$, the sign depending upon the quadrant in which the object is situated; but when the moon is very near the zenith, the best way will be to obtain ζ_1 by the more general process through the triangles ZZ′M, ZZ′M′, already mentioned. When the second object is the sun or a planet, and this happens to be near the zenith, we must treat it in the same manner as we do the moon; but the better course in this case will be to wait until it has fallen a few degrees.

Similarly, for the computation of z, z_1, z_2, we have

$$\cos z = \frac{\cos\phi' \,.\, \cos(\Pi-\theta)}{\cos\theta}, \text{ where } \tan\theta = \tan\phi' \,.\, \cos \mathrm{H},$$

$$\cos z_1 = \frac{\cos\phi \,.\, \cos(\Pi-\theta)}{\cos\theta}, \text{ where } \tan\theta = \tan\phi \,.\, \cos \mathrm{H},$$

$$z_2 = z_1 + \rho \mathrm{P} \,.\, \sin z_1 - \text{refraction}.$$

And the quantities ζ, ζ', ζ_1, ζ_2, z, z_1, z_2 are thus determined.

Also $$u_2 = \Omega \pm (r) \pm (\mathrm{R}),$$

where $$(r) = r \,.\, \frac{\sin\zeta'}{\sin\zeta} - \text{effect of refraction},$$

$$(\mathrm{R}) = \mathrm{R} \quad - \text{effect of refraction},$$

the effects of refraction upon the radii being computed in the manner already explained; and all the quantities which enter into the equations being now known, we may compute u_1 and u successively, and by means of the latter determine from the Tables the Greenwich mean time.

Let us now consider how the value of u will be affected by the employment of an erroneous value of the moon's equatorial horizontal parallax.

u being derived from u_1 through the equation

$$\frac{\cos u - \cos\zeta \,.\, \cos z}{\sin\zeta} = \frac{\cos u_1 - \cos\zeta' \,.\, \cos z}{\sin\zeta'}$$

or $$\cos u \,.\, \sin\zeta' = \cos u_1 \,.\, \sin\zeta + \cos z \,.\, \sin(\zeta' - \zeta),$$

we will suppose the moon's equatorial horizontal parallax instead of being p to be $p + \delta p$. Then, since $\rho = 1$ nearly, the arc ζ' would, by the employment of the correct value of the parallax, be increased by the quantity $\delta p \,.\, \sin\zeta'$; and to find the effect of this increment on the value of u, we may differentiate the above equation, assuming the two variable quantities to be u and ζ'. Hence

$$-\sin u \,.\, \sin\zeta' \,.\, \frac{du}{d\zeta'} + \cos u \,.\, \cos\zeta' = \cos z \,.\, \cos(\zeta' - \zeta) = \cos z \text{ nearly},$$

and $$\frac{du}{d\zeta'} = \frac{\cos u \,.\, \cos\zeta' - \cos z}{\sin u \,.\, \sin\zeta'};$$

this, multiplied by $\delta\zeta' = \delta p \,.\, \sin\zeta'$, gives

$$\delta u = \frac{\cos u \,.\, \cos\zeta' - \cos z}{\sin u} \,.\, \delta p.$$

Again, should the moon's horizontal semidiameter instead of being r be $r+\delta r$, the increment to u on this account will be $\pm\delta r$, the sign depending upon that of the apparent lunar radius, as applied to the observed distance in the expression for u_2.

Suppose, in the next place, that there is an error in the assumed Greenwich mean time t, this giving rise of course to errors in the RA and PD of the moon derived from the Tables, and consequently to others in the computed zenith distances, of which however it will be necessary to consider only ζ and ζ', since it has been already shown that a small error in ζ_1 affecting ζ_2 to an equal extent will not materially affect the value of u_1.

Let $\alpha+\Delta\alpha$, $\pi+\Delta\pi$ be the moon's tabular RA and PD at $t+1$ minute; then if $\delta\zeta$ be the increment to ζ,

$$\delta\zeta=\frac{1}{\sin\zeta}.\Big((\cos\phi'.\sin\pi-\sin\phi'.\cos\pi.\cos h).\Delta\pi-\sin\phi'.\sin\pi.\sin h.\Delta\alpha\Big)^{*},$$

and the new value of ζ' will be $\zeta'+\delta\zeta.(1+\rho p.\cos\zeta.\sin 1'')$. Now we have already $\frac{du}{d\zeta'}=\frac{\cos u.\cos\zeta'-\cos z}{\sin u.\sin\zeta'}$, and we shall similarly find $\frac{du}{d\zeta}=-\frac{\cos u_1.\cos\zeta-\cos z}{\sin u_1.\sin\zeta}$, which, when entering into the coefficient of a small quantity, may be considered the same as $-\frac{du}{d\zeta'}$;

$$\therefore \delta u=\frac{du}{d\zeta}.\delta\zeta+\frac{du}{d\zeta'}.\delta\zeta'=\frac{du}{d\zeta'}.\rho p.\cos\zeta.\sin 1''.\delta\zeta$$

$$=\frac{\cos u.\cos\zeta'-\cos z}{\sin u}.p.\cot\zeta'.\sin 1''.\delta\zeta,$$

which is the expression for the difference arising from an error of one minute. Let M be the Greenwich mean time at which the tabular distance is u, and n the number of seconds of time in $M-t$. Then, finally,

$$\delta u=\frac{\cos u.\cos\zeta'-\cos z}{60.\sin u}.np.\cot\zeta'.\sin 1''.\delta\zeta.$$

Hence collecting the terms, the expression for the corrected distance will be

$$u+\frac{\cos u.\cos\zeta'-\cos z}{\sin u}.\delta p\pm\delta r+\frac{\cos u.\cos\zeta'-\cos z}{60.\sin u}.np.\cot\zeta'.\sin 1''.\delta\zeta.$$

* Instead of computing this, we may if we please calculate ζ by the ordinary formula, assuming the moon's RA and PD to be $a+\Delta a,\pi+\Delta\pi$; and the difference between this value of ζ and that already obtained with RA and PD, a and π, will be $\delta\zeta$.

But U representing tabular distance, we shall have, generally,

$$\cos U = \cos\pi \,.\, \cos\Pi + \sin\pi \,.\, \sin\Pi \,.\, \cos(\alpha - A);$$

and if the tabular right ascensions and polar distances are incorrect, the true values of these elements being $\alpha + \delta\alpha$, $A + \delta A$, $\pi + \delta\pi$, $\Pi + \delta\Pi$, the true distance will be

$$U + \frac{\sin\pi \,.\, \sin\Pi \,.\, \sin(\alpha - A)}{\sin U} \,.\, (\delta\alpha - \delta A)$$
$$+ \frac{\sin\pi \,.\, \cos\Pi - \cos\pi \,.\, \sin\Pi \,.\, \cos(\alpha - A)}{\sin U} \,.\, \delta\pi$$
$$+ \frac{\cos\pi \,.\, \sin\Pi - \sin\pi \,.\, \cos\Pi \,.\, \cos(\alpha - A)}{\sin U} \,.\, \delta\Pi.$$

Hence M being the Greenwich mean time at which the tabular distance is u, and Δu the tabular variation of the distance in one mean second, the true distance at $M + \tau$ seconds will be

$$u + \tau \,.\, \Delta u + \frac{\sin\pi \,.\, \sin\Pi \,.\, \sin(\alpha - A)}{\sin u} \,.\, (\delta\alpha - \delta A)$$
$$+ \frac{\sin\pi \,.\, \cos\Pi - \cos\pi \,.\, \sin\Pi \,.\, \cos(\alpha - A)}{\sin u} \,.\, \delta\pi$$
$$+ \frac{\cos\pi \,.\, \sin\Pi - \sin\pi \,.\, \cos\Pi \,.\, \cos(\alpha - A)}{\sin u} \,.\, \delta\Pi.$$

Equating this with the expression which depends on the observation, and making

$$\frac{\sin\pi \,.\, \sin\Pi \,.\, \sin(A - \alpha)}{\Delta u \,.\, \sin u} = s_1, \quad \frac{\cos\pi \,.\, \sin\Pi \,.\, \cos(A - \alpha) - \sin\pi \,.\, \cos\Pi}{\Delta u \,.\, \sin u} = s_2,$$

$$\frac{\sin\pi \,.\, \cos\Pi \,.\, \cos(A - \alpha) - \cos\pi \,.\, \sin\Pi}{\Delta u \,.\, \sin u} = s_3, \quad \frac{\cos u \,.\, \cos\zeta' - \cos z}{\Delta u \,.\, \sin u} = s_4,$$

we have

$$\tau = s_1 \,.\, (\delta\alpha - \delta A) + s_2 \,.\, \delta\pi + s_3 \,.\, \delta\Pi + s_4 \,.\, \delta p \pm \frac{1}{\Delta u} \,.\, \delta r$$
$$+ \frac{1}{60} \,.\, s_4 \,.\, np \,.\, \cot\zeta' \,.\, \sin 1'' \,.\, \delta\zeta,$$

and corrected Greenwich mean time $= M + \tau$.

Let S be the Greenwich sidereal time corresponding to Greenwich mean time,

$$M + \frac{1}{60} \,.\, s_4 \,.\, n \,.\, p \,.\, \cot\zeta' \,.\, \sin 1'' \,.\, \delta\zeta,$$

corrected Greenwich sidereal time

$$=\mathrm{S}+s_1\,.\,(\delta\alpha-\delta\mathrm{A})+s_2\,.\,\delta\pi+s_3\,.\,\delta\Pi+s_4\,.\,\delta p\pm\frac{1}{\Delta u}\,.\,\delta r\,*,$$

and longitude in time

$$=\mathrm{S}-\mathrm{T}+s_1\,.\,(\delta\alpha-\delta\mathrm{A})+s_2\,.\,\delta\pi+s_3\,.\,\delta\Pi+s_4\,.\,\delta p\pm\frac{1}{\Delta u}\,.\,\delta r,$$

the corrections $\delta\alpha$, δA, $\delta\pi$, &c. being expressed in seconds of arc.

In the above investigation we have omitted to notice the effect of the tabular errors $-\delta\alpha$, $-\delta\Pi$, &c. upon the computed zenith distances, these being so small in general as in this respect to be unworthy of notice. Even should the moon's tabular place be 15″ in error, and should this lie entirely in the arc Z′M, the moon itself being close to the point Z′, the effect of such an error on the parallax can scarcely exceed 0″·25; and in the extreme case in which the second object is situated in the same vertical plane with the moon, the computed value of u can be in error only to an equal amount. It is therefore clear that we may safely neglect this, as we may likewise, and for similar reasons, the effect of such small error as is likely to exist in the latitude after its determination by any of the processes already described.

Before we take an example, it will be well to collect the formulæ into a convenient shape, leaving the example itself to indicate the parts of the calculation in which it will be necessary to proceed to seven places of decimals, and those in which five and four respectively will suffice.

Approximate Method. Altitudes of both objects observed.

Given:
- T the sidereal time at the place.
- Ω the mean of the readings corrected for instrumental errors.
- ζ_2 the apparent zenith distance of moon's centre at T.
- z_2 the apparent zenith distance of centre of second object at T.
- p the moon's equatorial horizontal parallax.
- P the horizontal parallax of second object.
- r the moon's horizontal radius.
- R the horizontal radius of second object.
- ρ the radius to place (earth's equatorial radius $=1$).

To find the true distance u, and thence the longitude of the place.

1. $\zeta_1=\zeta_2+\text{refraction}-\rho p\,.\,\sin\,(\zeta_2+\text{refraction})$.
2. $z_1=z_2+\text{refraction}-\mathrm{P}\,.\,\sin\,(z_2+\text{refraction})$.
3. $(r)=r\,.\,\dfrac{\sin\,(\zeta_2+\text{refraction})}{\sin\zeta_1}$.
4. $u_2=\Omega\pm(r)\pm\mathrm{R}$.
5. $\theta_2=\frac{1}{2}\,.\,(u_2+\zeta_2+z_2)$.

* In treating these small corrections it is unnecessary to introduce the factor for converting mean into sidereal time.

6. $\sin^2\theta_1 = \dfrac{\sin\theta_2 \,.\, \sin(\theta_2 - u_2) \,.\, \sin\zeta_1 \,.\, \sin z_1}{\sin\zeta_2 \,.\, \sin z_2 \,.\, \sin^2\frac{1}{2}(\zeta_1 + z_1)}$.

7. $\sin\frac{1}{2}u_1 = \sin\frac{1}{2}(\zeta_1 + z_1) \,.\, \cos\theta_1$.

8. M = Greenwich mean time of geocentric distance u_1.

9. S = Greenwich sidereal time corresponding to M.

10. Longitude in time = S − T.

Final Method. Altitudes not observed.

Given:

ϕ the geographical, ϕ' the geocentric latitude; ρ the radius to place.
T the local sidereal time; t the approximate Greenwich mean time.
a, π the moon's geocentric RA and PD at t; Δa, $\Delta\pi$ their variations in one minute.
p, r the moon's equatorial horizontal parallax and radius at t.
A, Π, P, R the geocentric RA, PD, HP and R[s] of second object at t.
Ω the mean of the readings corrected for instrumental errors.

To find true distance u, and longitude of place.

1. h	$= \mathrm{T} - a$	H	$= \mathrm{T} - \mathrm{A}$.
2. $\tan\theta$	$= \tan\phi' \,.\, \cos h$	$\cos\zeta$	$= \dfrac{\cos\phi' \,.\, \cos(\pi - \theta)}{\cos\theta}$.
3. p'	$= \rho p \,.\, \sin\zeta + (p)$	ζ'	$= \zeta + p'$.
4. $\tan\theta$	$= \tan\phi \,.\, \cos h$	$\cos(\zeta_1 - p')$	$= \dfrac{\cos\phi \,.\, \cos(\pi - \theta)}{\cos\theta}$.
5. ζ_2	$= \zeta_1 -$ refraction.		
6. $\tan\theta$	$= \tan\phi' \,.\, \cos \mathrm{H}$	$\cos z$	$= \dfrac{\cos\phi' \,.\, \cos(\Pi - \theta)}{\cos\theta}$.
7. $\tan\theta$	$= \tan\phi \,.\, \cos \mathrm{H}$	$\cos z_1$	$= \dfrac{\cos\phi \,.\, \cos(\Pi - \theta)}{\cos\theta}$.
8. z_2	$= z_1 + \rho\mathrm{P} \,.\, \sin z_1 -$ refraction.		
9. v_1	$= \Omega \pm r \pm \mathrm{R}$	v_2	$= \frac{1}{2}(v_1 + \zeta_2 + z_2)$.
10. $\sin^2\frac{1}{2}\mathrm{M}'$	$= \dfrac{\sin(v_2 - v_1) \,.\, \sin(v_2 - \zeta_2)}{\sin v_1 \,.\, \sin\zeta_2}$	$\sin^2\frac{1}{2}s'$	$= \dfrac{\sin(v_2 - v_1) \,.\, \sin(v_2 - z_2)}{\sin v_1 \,.\, \sin z_2}$.

11. $(r) = r \,.\, \dfrac{\sin\zeta'}{\sin\zeta}$ − (refraction at moon's centre − refraction at limb) × $\cos^2\mathrm{M}'$.

12. (R) = R − (refraction at sun's centre − refraction at limb) × $\cos^2\mathrm{S}'$.

13. $u_2 = \Omega \pm (r) \pm (\mathrm{R})$.

14. $\theta_2 = \frac{1}{2}(u_2 + \zeta_2 + z_2)$ $\qquad \sin^2\theta_1 = \dfrac{\sin\theta_2 \,.\, \sin(\theta_2 - u_2) \,.\, \sin\zeta_1 \,.\, \sin z_1}{\sin\zeta_2 \,.\, \sin z_2 \,.\, \sin^2\frac{1}{2}(\zeta_1 + z_1)}$.

15. $\sin\frac{1}{2}u_1 = \sin\frac{1}{2}(\zeta_1 + z_1) \,.\, \cos\theta_1$.

16. $\theta' = \frac{1}{2}(u_1 + \zeta' + z)$ $\qquad \sin^2\theta = \dfrac{\sin\theta' \,.\, \sin(\theta' - u_1) \,.\, \sin\zeta}{\sin\zeta' \,.\, \sin^2\frac{1}{2}(\zeta + z)}$.

17. $\sin\frac{1}{2}u = \sin\frac{1}{2}(\zeta + z) \,.\, \cos\theta$.

18. M = GMT of tabular geocentric distance u; n the number of seconds in $\mathrm{M} - t$.

19. Δu = tabular variation of u in one mean second.

20. $\delta\zeta = \dfrac{1}{\sin\zeta} \,.\, \big((\cos\phi' \,.\, \sin\pi - \sin\phi' \,.\, \cos\pi \,.\, \cos h) \,.\, \Delta\pi - \sin\phi' \,.\, \sin\pi \,.\, \sin h \,.\, \Delta a\big)$.

21. $s_1 = \dfrac{\sin\pi \,.\, \sin\Pi \,.\, \sin(A-\alpha)}{\Delta u \,.\, \sin u}$. $s_2 = \dfrac{\cos\pi \,.\, \sin\Pi \,.\, \cos(A-\alpha) - \sin\pi \,.\, \cos\Pi}{\Delta u \,.\, \sin u}$

22. $s_3 = \dfrac{\sin\pi \,.\, \cos\Pi \,.\, \cos(A-\alpha) - \cos\pi \,.\, \sin\Pi}{\Delta u \,.\, \sin u}$ $s_4 = \dfrac{\cos u \,.\, \cos\zeta' - \cos z}{\Delta u \,.\, \sin u}$.

23. $\text{True GMT} = M + s_1 \,.\, (\delta\alpha - \delta A) + s_2 \,.\, \delta\pi + s_3 \,.\, \delta\Pi + s_4 \,.\, \delta p \pm \dfrac{1}{\Delta u} \,.\, \delta r + \dfrac{s_4 n p}{60} \,.\, \cot\zeta' \,.\, \sin 1'' \,.\, \delta\zeta$.

24. $S = \text{Greenwich sidereal time to } M + \dfrac{s_4 n p}{60} \,.\, \cot\zeta' \,.\, \sin 1'' \,.\, \delta\zeta$ mean time.

25. $\text{Longitude in time} = S - T + s_1 .(\delta\alpha - \delta A) + s_2 .\delta\pi + s_3 .\delta\Pi + s_4 .\delta p \pm \dfrac{1}{\Delta u} \,.\, \delta r$.

The corrections $\delta\alpha$, δA, $\delta\pi$, $\delta\Pi$, δp, δr to the tabular elements being expressed in seconds of arc, and the terms representing the corrections to the longitude in seconds of time.

It will be obvious that when, knowing the longitude approximately, we employ the first or approximate method of reduction, we may, instead of observing the altitudes of the objects, compute their zenith distances by the ordinary formulæ.

Example. July 31, 1858. Bar. 30·23 in. Therm. 64°. $\omega_0 = +1' 6''$. Latitude 50° 36′,

Sun and Moon, the former direct, the shade employed being No. 5.

Local sidereal time.	Reading.	(ω).	E.
h m s	° ′ ″	° ′	′ ″
4 27 21	106 8 37	114 37	+0 15·3
4 28 50	7 58	113 58	15·2
4 30 9	7 30	113 30	15·2
4 31 55	6 40	112 40	15·1
4 33 28	6 1	112 1	15·0
Means 4 30 20·6	106 7 21·2	E =	+0 15·2
	−0 54·4	$-\omega_0$ =	−1 6·0
	Ω = 106 6 26·8	η and σ =	0 0·0
		c_s =	−0 3·6
		Sum =	−0 54·4

Observed altitudes gave—

Apparent ZD of moon's upper limb at $4^h\ 30^m\ 20^s{\cdot}6 = 55°\ 10'\ 41''$

„ sun's lower „ $= 59\ 39\ 56$

The longitude of the place was known to be between $0^h\ 0^m$ and $0^h\ 30^m$ west of Greenwich. Suppose we assume it at $0^h\ 15^m$ west, which will give the approximate Greenwich sidereal time $4^h\ 45^m\ 20^s{\cdot}6$, and mean time $31^d\ 20^h\ 7^m$. For this latter we take from the Tables,

Moon's horizontal semidiameter	$= 15'\ 45'' = r$
Moon's equatorial horizontal parallax	$= 57\ 41 = p$
Sun's semidiameter	$= 15\ 48 = R$
Sun's horizontal parallax	$= 0\ 8 = P$

and are now prepared to proceed with the calculation.

The rigorous method of proceeding would be to correct the interpolated zenith distances of the limbs for refraction and parallax, and applying to the results the true semidiameters, to determine thus the true zenith distances ζ_1, z_1 of the centres. The apparent zenith distances of the centres being then derived from these by the formulæ

$$\zeta_2 = \zeta_1 + \rho p \, . \sin \zeta_1 + (p) - \text{refraction}$$
$$z_2 = z_1 + \text{P} \, . \sin z_1 \qquad - \text{refraction.}$$

But in an operation of an approximate kind, no advantage would be gained by this exactness, as at most it could only make a difference of a second or two in the zenith distances, which would produce no sensible effect on the result. To adhere then to the course adapted to our formulæ, we have

Moon's horizontal radius	= 15′ 45″	Apparent ZD of upper limb	= 55° 10′ 41″
Augmentation at 55° ZD	= 9		= 15 54
Moon's apparent radius	= 15 54	ζ_2	= 55 26 35
		Refraction	= 1 23
		Sum	= 55 27 58 log sin = 9·91582

log p = 3·53920
log ρ = 9·99914*
Sum = 3·45416

47 26 Nat No. = 47′ 26″

ζ_1 = 54 40 32 log sin = 9·91163
log sin (ζ_2 + refraction) = 9·91582
Difference = 0·00419
log r = 2·97543
log (r) = 2·97962
(r) = 15′ 54″

Apparent ZD of Sun's lower limb	= 59° 39′ 56″
Apparent radius	= 15 48
z_2	= 59 24 8
Refraction	= 1 36
Sum	= 59 25 44 log sin = 9·93500

log P = 0·90309
Sum = 0·83809

0 7 Nat No. = 0′ 7″

z_1	= 59° 25′ 37″	Ω	= 106° 6′ 27″
ζ_1	= 54 40 32	(r)	= 15 54
Sum	= 114 6 9	R	= 15 48
$\frac{1}{2}(\zeta_1 + z_1)$	= 57 3 4·5	u_2	= 106 38 9

* For geographical latitude 50° 36′, and may either be taken from the Tables or computed by the formula already given.

Brought forward

$\frac{1}{2}(\zeta_1+z_1)$	$= 57°\ 3'\ 4''.5$	u_2	$= 106°\ 38'\ 9''$
		ζ_2	$= 55\ 26\ 35$
		z_2	$= 59\ 24\ 8$
		Sum	$= 221\ 28\ 52$
		θ_2	$= 110\ 44\ 26$
		$\theta_2 - u_2 =$	$4\ 6\ 17$

log sin θ_2	$= 9.9709015$	log sin $\frac{1}{2}(\zeta_1+z_1)$	$= 9.9238434$
log sin $(\theta_2 - u_2)$	$= 8.8547896$	Double	$= 9.8476868$
log sin ζ_1	$= 9.9116321$	log sin ζ_2	$= 9.9156967$
log sin z_1	$= 9.9349937$	log sin z_2	$= 9.9348830$
Sums	$= 8.6723169$		9.6982665
	9.6982665		
log sin² θ_1	$= 8.9740504$	log sin $\frac{1}{2}(\zeta_1+z_1)$	$= 9.9238434$
log sin θ_1	$= 9.4870252$	log cos θ_1	$= 9.9785162$
		log sin $\frac{1}{2} u_1$	$= 9.9023596$
		$\frac{1}{2} u_1$	$= 53°\ 0'\ 7''$
		u_1	$= 106\ 0\ 14$

Tabular distance July, 31ᵈ 18ʰ	$= 107\ 0\ 27$	PL = 2855*
Difference	$= 1\ 0\ 13$	PL = 4756
PL of	$1^h\ 56^m\ 11^s$ = Diff.	= 1901
	$31^d\ 18\ 0\ 0$	
Sum	$= 31\ 19\ 56\ 11$	
Correction for 2nd difference	$= +0\ 4$	
M	$= 31\ 19\ 56\ 15$	

Sidereal time at Greenwich mean noon on the 31st	8	35	16.28
19 hours mean in sidereal time	19	3	7.27
56 minutes		56	9.20
15 seconds			15.04
S =	4	34	48
T =	4	30	21
Longitude in time =	+0	4	27

We might now, if we wished, amend the calculation by the same method, taking from the Tables the moon's equatorial horizontal parallax and semidiameter for 19ʰ 56ᵐ G.M.T. instead of 20ʰ 7ᵐ first assumed; but the difference in the result would be very trifling, the variation of the parallax during the interval

* The particulars of this process are given in the explanation appended to the 'Nautical Almanac' of every year.

being only $0''.3$, and that of the radius practically nothing. We propose, therefore, as a second example, to treat the same observation according to the more accurate method, taking the following quantities from the Tables for 19^h 56^m G. M. T.*.

$\alpha = 1^h\ 18^m\ 23^s.15$†	$\Delta\alpha = +30''.75$		
$\pi = 77°\ 49'\ 4''$	$\Delta\pi = -14''.59$		
$p = \quad 57'\ 40''.6$	$r = 15'\ 45''.2$		
$A = 8^h\ 44^m\ 35^s.35$,	$\Pi = 71°\ 54'\ 9''$,	$P = 8''.5$,	$R = 15'\ 47''.9$.

And with respect to the remaining quantities:—

$\phi = 39°\ 24'\ 17''$,	$\phi' = 39°\ 35'\ 35''$,	$\log\rho = 9.9991354$‡.
$T = 4^h\ 30^m\ 20^s.6$,	$\Omega = 106°\ 6'\ 26''.8$	

1.

$T =$	$4^h\ 30^m\ 20^s.60$		$4^h\ 30^m\ 20^s.60$
$\alpha =$	$1\ \ 18\ \ 23\ .15$	$A =$	$8\ \ 44\ \ 35\ .35$
$h =$	$+3\ \ 11\ \ 57\ .45 = +47°\ 59'\ 22''$	$H =$	$-4\ \ 14\ \ 14\ .75 = -63°\ 33'\ 41''$

2 and 3.

$\log\tan\phi' = 9.91754$	$\log\cos\phi' = 9.88682$		$\log\rho = 9.99914$
$\log\cos h = 9.82560$	$\log\cos\theta = 9.94196$		$\log p = 3.53915$
$\log\tan\theta = 9.74314$	Difference $= 9.94486$	$\zeta = 54°\ 34'\ 53''$	$\log\sin = 9.91113$
$\theta = 28°\ 57'\ 56''$	$\log\cos(\pi-\theta) = 9.81823$		Sum $= 3.44942$
$\pi = 77\ \ 49\ \ 4$	$\log\cos\zeta = 9.76309$		Nat. N. $= 46'\ 54''.6$
$\pi-\theta = 48\ \ 51\ \ 8$			$(p) = 27''.5$
		$47'\ 22''.1\ p'$	$= 47'\ 22''.1$
		$\zeta' = 55°\ 22'\ 15''.1$	

---

* In the next chapter the reader will find an example of reduction in the case of a star observed with the moon, the zenith distances being either observed or computed.

† These are interpolated from the Tables with first and second differences. A Table for facilitating this operation will be found appended to this Part; but it is to be remarked that such extreme accuracy is by no means necessary. The parallax of the moon, and the elements of the sun's place have in this instance been computed in the same manner.

‡ Should the operator prefer computing ϕ' and ρ to taking them from tables, he may employ the formulæ of the note at page 79. Thus

$\log(1+e) = 0.0341058$	$\log e = 8.9122061$
$\log(1-e) = 9.9629860$	$\log\sin\phi' = 9.8043648$
Sum $= 9.9970918$	$\log e.\sin\phi' = 8.7165709 = \log .052068$
$\log\cot\phi = 0.0853674$	$\log(1+e.\sin\phi') = 0.0220438$
$\log\cot\phi' = 0.0824592$	$\log(1-e.\sin\phi') = 9.9767772$
$\phi' = 39°\ 35'\ 35''$	Sum $= 9.9988210$
	$\log(1-e^2) = 9.9970918$
	$\log\rho^2 = 9.9982708$
	$\log\rho = 9.9991354$

4, 5.

log tan ϕ	= 9·91463	log cos ϕ	= 9·88800		
log cos h	= 9·82560	log cos θ	= 9·94265		
log tan θ	= 9·74023	Difference	= 9·94535		
θ	= 28° 48′ 12″	log cos $(\pi-\theta)$	= 9·81682		
π	= 77 49 4	log cos (ζ_1-p')	= 9·76217	ζ_1-p'	= 54° 40′ 3″
$\pi-\theta$	= 49 0 52			p'	= 47 22
				ζ_1	= 55 27 25
				Refraction	= 1 22·7
				ζ_2	= 55 26 2·3

6

log tan ϕ'	= 9·91754	log cos ϕ'	= 9·88682
log cos H	= 9·64859	log cos θ	= 9·97238
log tan θ	= 9·56613	Difference	= 9·91444
θ	= 20° 12′ 57″	log cos $(\Pi-\theta)$	= 9·79237
Π	= 71 54 9	log cos z	= 9·70681
$\Pi-\theta$	= 51 41 12	z	= 59° 23′ 44″

7, 8.

log tan ϕ	= 9·91463	log cos ϕ	= 9·88800		
log cos H	= 9·64859	log cos θ	= 9·97274		
log tan θ	= 9·56322	Difference	= 9·91526		
θ	= 20° 5′ 29″	log cos $(\Pi-\theta)$	= 9·79117	log ρ	= 9·99914
Π	= 71 54 9	log cos z_1	= 9·70643	log P	= 0·92942
$\Pi-\theta$	= 51 48 40	z_1	= 59° 25′ 30″	log sin	= 9·93499
				Sum	= 0·86355
				Nat. No.	= + 7″·3
				Refraction	= − 96″·3
		−	1 29·0	Sum	= − 89″·0
		z_2	= 59 24 1·0		

9, 10.

Ω	= 106° 6′				
$r+\mathrm{R}$	= 32				
v_1	= 106 38				
ζ_2	= 55 26				
z_2	= 59 24				
Sum	= 221 28	log sin (v_2-v_1)	= 8·8543		
v_2	= 110 44	log sin v_1	= 9·9814		
v_2-v_1	= 4 6	Difference	= 8·8729	———	8·8729
$v_2-\zeta_2$	= 55 18	log sin $(v_2-\zeta_2)$	= 9·9149	log sin (v_2-z_2)	= 9·8925
v_2-z_2	= 51 20	Sum	= 8·7878	Sum	= 8·7654
		log sin ζ_2	= 9·9156	log sin z_2	= 9·9347
		log sin² ½ M′	= 8·8722	log sin² ½ s'	= 8·8307
		log sin ½ M′	= 9·4361	log sin ½ s'	= 9·4154

$\frac{1}{2}$ M′	= 15° 50′	$\frac{1}{2}$ *s*′	= 15° 5′
M′	= 31 40	*s*′	= 30 10

11, 12.

The difference between the refraction for limb and centre of moon . . . . =0″·8

„ of sun . . . . =1″·0

log *r*	= 2·97552	log cos M′	= 9·9300	log cos S′	= 9·9368
log sin ζ′	= 9·91532	Double	= 9·8600	Double	= 9·8736
Sum	= 2·89084	log 0·8	= 9·9031	log 1	= 0·0000
log sin ζ	= 9·91113	Sum	= 9·7631	Sum	= 9·8736
Difference	= 2·97971	Nat. No.	= 0·6	Nat. No.	= 0″·7
Nat. No.	= 15′ 54″·4			R	= 15 47·9
	− 0 ·6			(R)	= 15 47·2
(*r*)	= 15 53 ·8	z_1	= 59° 25′ 30″		
(R)	= 15 47 ·2	ζ_1	= 55 27 25		

13, &c.

Ω	= 106° 6 26 ·8	$\frac{1}{2}(\zeta_1+z_1)$	= 57 26 27·5	log sin	9·9257439
u_2	= 106 38 7 ·8	log sin ζ_1	= 9·9157692	Double	9·8514878
ζ_2	= 55 26 2 ·3			log sin	9·9156493
z_2	= 59 24 1 ·0			log sin	9·9348743
Sum	= 221 28 11 ·1	log sin z_1	= 9·9349850		
θ_2	= 110 44 5 ·6	log sin	= 9·9709178		
θ_2-u_2	= 4 5 57 ·8	log sin	= 8·8542259		
		Sums	= 8·6758979		9·7020114
			9·7020114		
		log sin² θ_1	= 8·9738865	log sin $\frac{1}{2}(\zeta_1+z_1)$	= 9·9257439
		log sin θ_1	= 9·4869433	log cos θ_1	= 9·9785247
				log sin $\frac{1}{2}u_1$	= 9·9042686

$\frac{1}{2}u_1$	= 53 20 17 ·5		
u_1	= 106 40 35 ·0		
ζ′	= 55 22 15 ·1	ζ	= 54° 34′ 53″·0
z	= 59 23 44 ·0		59 23 44 ·0
Sum	= 221 26 34 ·1	$\frac{1}{2}(\zeta+z)$	= 56 59 18 ·5
θ′	= 110 43 17 ·1		
$\theta'-u_1$	= 4 2 42 ·1		

log sin θ′	= 9·9709564
log sin $(\theta'-u_1)$	= 8·8484381
log sin ζ	= 9·9111254
Sum	= 8·7305199
log sin ζ′	= 9·9153193
Difference	= 8·8152006
Half	= 9·4076003

Brought forward

Half	=9·4076003		
log sin $\frac{1}{2}(\zeta+z)$	=9·9235347		9·9235347
log sin θ	=9·4840656	log cos θ	=9·9788217
		log sin $\frac{1}{2}u$	= 9·9023564
		$\frac{1}{2}u$	= 53° 0′ 5″·0
		u	= 106 0 10

Tabular distance at $31^d\ 18^h$ G.M.T. = 107 0 27 PL =28555

Difference = 1 0 17 PL =47508

PL of $1^h\ 56^m\ 21^s$ = Difference = 18953

31^d 18 0 0

Sum = 31 19 56 21

Correction for second difference = + 0 4

M	=	31 19 56 25
t	=	19 56 0
n	=	+25

19, &c.

$\Delta u = -0''\cdot 5181$	log =9·7144	log $-\frac{1}{\Delta u}$	=0·2856
	log sin u =9·9828	$\frac{1}{\Delta u}$	= −1·930
log $-\Delta u$. sin u =	Sum =9·6972		

We might now obtain $\delta\zeta$ by the formula of line 20; but as an example of this will be found in the next chapter, we will here compute $\zeta+\delta\zeta$ by the ordinary process,

$$\pi+\Delta\pi=77^\circ\ 48'\ 49'',\ h+\delta h=h-\Delta\alpha=47^\circ\ 58'\ 51''.$$

log tan ϕ'	=9·91754	log cos ϕ'	=9·88682		
log cos $(h+\delta h)$	=9·82567	log cos θ	=9·94195		
log tan θ	=9·74321	Difference	=9·94487		
θ	=28° 58′ 10″	log cos $(\pi+\Delta\pi-\theta)$	=9·81830		
$\pi+\Delta\pi$	=77 48 49	log cos $(\zeta+\delta\zeta)$	=9·76317	$\zeta+\delta\zeta$	=54° 34′ 25″
$\pi+\Delta\pi-\theta$	=48 50 39			ζ	=54 34 53
				$\delta\zeta$	= −28

21, &c. A$-\alpha=h-$H$=111^\circ\ 33'$.

log sin π	=9·9901	log cos π	=9·3244	log sin π	=9·9901
log sin Π	=9·9780		9·9780	log cos Π	=9·4924
log sin (A$-\alpha$)	=9·9685	Sum	=9·3024	Sum	=9·4825
Sum	=9·9366				

Brought forward

log sin (A − α)	= 9·9685	Sum	= 9·3024	Sum	= 9·4825		
Sum	= 9·9366	log − cos (A − α)	= 9·5651	Nat. No.	= + ·3038		
log − Δu . sin u	= 9·6972	Sum	= 8·8675	Nat. No.	= − ·0737		
log − s_1	= 0·2394			Difference	= − ·3775	log	= 9·5769
s_1	= − 1·736					log − Δu . sin u	= 9·6972
						log s_2	= 9·8797
						s_2	= + 0·758

log sin π . cos Π	= 9·4825						
log − cos (A − α)	= 9·5651						
Sum	= 9·0476	Nat. No.	= − ·1126				
log cos π . sin Π	= 9·3024	cos π . sin Π	= + ·2007				
		Difference	= − ·3123	log	= 9·4946		
				log − Δu . sin u	= 9·6972		
				log s_3	= 9·7974	s_3	= + 0·627

log − cos u	= 9·4404						
log cos ζ'	= 9·7546						
Sum	= 9·1950	Nat. No.	= − ·1567				
		Nat. cos z	= ·5091				
		Difference	= − ·6658	log	= 9·8233		
				log − Δu . sin u	= 9·6972		
				log s_4	= 0·1261	s_4	= + 1·337
				log n	= 1.3979		
				log p	= 3·5392		
				log cot ζ'	= 9·8393		
				log sin 1″	= 4·6856		
				log − $\delta\zeta$	= 1·4472		
				log $\frac{1}{60}$	= 8·2218		
				Sum	= 9·2571	Nat. No.	= − 0^s·18

a correction to M, which in the result of a process of this description it is unnecessary to notice. S will therefore in this case be the Greenwich sidereal time corresponding to mean time M: or

S = 4^h 34^m 58^s

T = 4 30 21

Longitude in time = + 4 37 − 1·736 × ($\delta\alpha - \delta A$)

$+ 0{\cdot}758 \times \delta\pi + 0{\cdot}627 \times \delta\Pi + 1{\cdot}337 \times \delta p - 1{\cdot}930 \times \delta r$.

8. There is yet another method of treating lunar distance, the principle of which may be briefly explained.

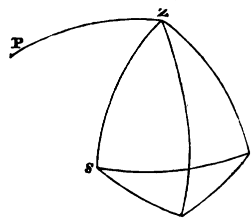
The distance between the two objects as affected by parallax being expressed in terms of the tabular elements at an assumed

Greenwich mean time, and of the corrections due to these elements, as well as to the assumed time and the latitude of the place, we may calculate the several parts, and, having done this, equate the whole with the distance derived from observation, correcting the latter for refraction alone, and adding terms representing the quantities to be applied on account of errors in the tabular semi-diameters. But reduction by this method is extremely laborious, and the results will be practically the same as those obtained by the above method in every case in which the Sextant can be employed to any purpose,—although, in cases in which the distance is so small as it is on the occasion of an occultation or eclipse for example, it will be necessary to have recourse to it or to some other equally exact process. The reader who wishes to make himself acquainted with this method may consult the Appendix to the 'Nautical Almanac' for 1854, where he will find it treated at length in a paper by Professor Challis; and should he be disposed to apply it to the observation which we have treated by the other, he will obtain the result,

$$\text{Longitude} = +4^{m}\ 37^{s} - 1\cdot754 \times (\delta\alpha - \delta A) + 0\cdot765 \times \delta\pi + 0\cdot630 \times \delta\Pi + 1\cdot350 \times \delta p - 1\cdot930 \times \delta r + 0\cdot008 \times \delta\phi,$$

which, as will be seen, is practically identical with the above, the last term showing us also how small an effect is produced by an error in the assumed latitude.

9. In addition to the subjects treated in the foregoing pages, the Sextant is sometimes applied to the determination of the right ascension and polar distance of a celestial object, for example a comet. In this case it will be necessary to observe the distances between the object and two well known stars, the three forming as nearly as possible an equilateral triangle, and to remark at the same time the position of the former with respect to the arc joining the two latter,—whether above or below it, to the right or left. The observations being corrected for instrumental errors, the process of reduction will be sufficiently simple.



Let Z be the zenith; M the apparent position of the object of which we wish to ascertain the right ascension and polar distance; and S, S′ the apparent positions of the stars with which it is observed.

By the ordinary formulæ we may compute the apparent zenith distances ZS, ZS′, the azimuths PZS, PZS′, and the apparent distance SS′; and the apparent distances SM, S′M are derived from observation.

1°. From ΔZSS′, having the three sides given, we may compute the angle ZSS′.

2°. From ΔMSS′, we may similarly compute the angle MSS′.

3°. $\therefore \angle ZSM = \angle ZSS' \pm \angle MSS'$ is known.

4°. From ΔZSM, having given two sides ZS and SM, and the included angle ZSM, we may compute ZM, which is the apparent zenith distance of M, and the angle SZM, which is its azimuth from the vertical ZS. Correcting the former for refraction, and applying the latter with its proper sign to the azimuth PZS of S, we know the zenith distance and azimuth of M affected by parallax alone.

5°. Treating the zenith distance and azimuth just found by the ordinary formulæ, we may determine the right ascension and polar distance of M affected by parallax alone.

10. It will not be out of place, in conclusion, to present in order the formulæ by means of which the origin of the elements of position is transferred from the pole to the zenith, and *vice versâ*, these being frequently employed in the course of practice with the Sextant.

1°. Given the colatitude ϕ of the place, and the hour-angle H and polar distance π of the object, to compute the true zenith distance ζ and azimuth Z, the latter measured from the polar meridian, and reckoned positive or negative according as it is west or east.

For ζ:

1. $\tan\theta = \tan\phi \,.\, \cos H$, $\cos\zeta = \dfrac{\cos\phi \,.\, \cos(\pi-\theta)}{\cos\theta}$.*

To be preferred when ϕ is less than π, and ζ not small.

2. $\tan\theta = \tan\pi \,.\, \cos H$, $\cos\zeta = \dfrac{\cos\pi \,.\, \cos(\phi-\theta)}{\cos\theta}$.

To be preferred when π is less than ϕ, and ζ not small.

3. $\tan\theta = \dfrac{\sqrt{\sin\pi \,.\, \sin\phi} \,.\, \sin\frac{1}{2}H}{\sin\frac{1}{2}(\pi-\phi)}$, $\sin\frac{1}{2}\zeta = \dfrac{\sin\frac{1}{2}(\pi-\phi)}{\cos\theta}$.

To be preferred when ζ is small.

For Z:

1. $\sin Z = \dfrac{\sin\pi \,.\, \sin H}{\sin\zeta}$.

May be employed when Z is either small or nearly 180°.

2. $\theta = \frac{1}{2}(\phi+\pi+\zeta)$, $\tan\frac{1}{2}Z = \sqrt{\dfrac{\sin(\theta-\phi) \,.\, \sin(\theta-\zeta)}{\sin\theta \,.\, \sin(\theta-\pi)}}$.

To be employed when Z is nearly 90°, or when under any

* This is the formula invariably given in the treatment of lunar distances; but it must be understood that the others ought to be employed on occasions on which the conditions render them preferable.

circumstances there is a doubt as to whether we should take the angle from the Tables or its supplement as the true value.

2°. Given the colatitude ϕ of the place, and the true zenith distance ζ and azimuth Z of an object, to compute the true polar distance and hour-angle.

For π.

1. $\tan\theta = \tan\phi \,.\, \cos Z$, $\cos\pi = \dfrac{\cos\phi \,.\, \cos(\zeta - \theta)}{\cos\theta}$

To be preferred when ϕ is less than ζ.

2. $\tan\theta = \tan\zeta \,.\, \cos Z$, $\cos\pi = \dfrac{\cos\zeta \,.\, \cos(\phi - \theta)}{\cos\theta}$.

To be preferred when ζ is less than ϕ.

3. $\tan\theta = \dfrac{\sqrt{\sin\phi \,.\, \sin\zeta \,.\, \sin\frac{1}{2}Z}}{\sin\frac{1}{2}(\phi - \zeta)}$, $\sin\frac{1}{2}\pi = \dfrac{\sin\frac{1}{2}(\phi - \zeta)}{\cos\theta}$.

For H.

1. $\sin H = \dfrac{\sin\zeta \,.\, \sin Z}{\sin\pi}$.

May be employed when H is either small or nearly 180°.

2. $\theta = \frac{1}{2}(\phi + \pi + \zeta)$, $\tan\frac{1}{2}H = \sqrt{\dfrac{\sin(\theta - \phi) \,.\, \sin(\theta - \pi)}{\sin\theta \,.\, \sin(\theta - \zeta)}}$.

To be employed when H is nearly 90°, or when under any circumstances there is a doubt as to whether we should take the angle from the Tables or its supplement as the true value.

CHAPTER II.

APPLICATION TO NAUTICAL ASTRONOMY.

OBSERVATIONS made at sea for the same purposes as those made on land differ from these only in the degree of accuracy attainable, being inferior on several accounts. In the first place, the ship's motion disturbs and embarrasses the observer; and in the second, it being impossible to have recourse to the reflecting surface of the mercury in the artificial horizon, we are compelled to refer to the sea-horizon, and, observing in a vertical plane the apparent elevation of the object above this line often itself ill-defined, the angle must be corrected for dip, the amount of correction to be applied depending upon the height of the observer above the surface, and upon other causes the effect of which it is difficult to appreciate. All that can be done is to apply the correction due to what may be considered the observer's mean height above the surface, that is, the height at which his eye would be above a surface of perfectly still water, taking no account of the effects of variations in the atmosphere from an assumed mean state. Some general remarks on the precautions to be taken with a view to secure the greatest possible amount of accuracy will be found at the end of the Chapter. At present we shall assume that an apparent altitude of a celestial object is its apparent elevation above the line of sea-horizon less the dip, and proceed to notice the several classes of observation in order, commencing with that usually employed in the determination of latitude, and called a meridian altitude.

1. When we know the exact time of meridian passage of a celestial object, we may observe its apparent elevation above the sea-horizon at that instant, and, correcting the observation for dip, refraction, and parallax, we shall have a true meridian altitude, from which, as shown in section 1 of the preceding Chapter, we may at once determine the latitude. If, however, the object be the sun or moon, it is the elevation of the limb and not that of the centre that is observed; and in this case therefore we have to apply the semidiameter to our corrected angle, when the result will be the true altitude of this point. Let Ω be the apparent elevation, that is the angle read, corrected for instrumental errors. Then

true altitude of centre

$$= \Omega - \text{dip} - \text{refraction} + \text{parallax} \pm \text{semidiameter},$$

and the true zenith distance of centre being Z,

$Z = 90° - \Omega + \text{dip} + \text{refraction} - \text{parallax} \mp \text{semidiameter}.$

Also Z being reckoned + or − according as it is measured towards or away from the pole which we assume as the origin of the polar distance Π of the object, we shall have

$$\text{colatitude} = \phi = Z \pm \Pi,$$

the upper or lower sign being taken according as the object is at upper or lower transit, and

$$\text{latitude} = 90° - \text{colatitude}$$

reckoned towards the assumed pole when ϕ is less than 90°, and away from it if greater.

But when we do not know the exact time of meridian passage, we may, if the ship is stationary, observe the greatest apparent elevation, and this will of course, in the case of objects which do not change their polar distance, be the apparent elevation on the meridian. When, however, the ship is in motion, or the object observed is changing its polar distance, it may happen that its apparent elevation a little before or after meridian passage will be greater than that at the instant of meridian passage itself; and it becomes therefore desirable to inquire what effect these causes will produce upon the latitude determined on the assumption that the maximum elevation is that which occurs on the meridian.

Suppose a ship's course to make an angle α with the polar meridian, this angle being reckoned positive towards the east, and her rate of sailing to be m knots or nautical miles per hour,

$$\therefore \text{ variation of colatitude in one mean minute} = -\frac{m}{60} \cdot \cos\alpha,$$

effect of ship's motion upon the hour-angle in the same time

$$= +\frac{m}{60} \cdot \frac{\sin\alpha}{\sin\phi}.$$

Let Φ be the colatitude, A and Π the right ascension and polar distance of the object at instant of meridian passage; δA, $\delta\Pi$ the variations in minutes of arc of A and Π in one mean minute of time. Then T representing the mean time at meridian passage, and t any number of minutes from it,

Colatitude at $T + t$ will be $\Phi - \dfrac{t \cdot m \cdot \cos\alpha}{60}$

Right ascension of meridian $A + 15\,\mu\, t + \dfrac{t\, m \cdot \sin\alpha}{60 \sin\Phi}$, μ being the factor for converting mean to sidereal time $= 1{\cdot}00274$.

RA of object at $T+t$ will be $A+t.\delta A$.

Polar distance $\Pi+t.\delta\Pi$.

Hour-angle $t.\left(15\mu-\delta A+\frac{m.\sin\alpha}{60\sin\Phi}\right)$.

Now ζ, ϕ, π, H representing generally zenith distance, colatitude, polar distance and hour-angle, we have

$$\cos\zeta=\cos\phi.\cos\pi+\sin\phi.\sin\pi.\cos H=\cos(\phi-\pi)-2\sin\phi.\sin\pi.\sin^2\tfrac{1}{2}H$$
$$=\cos(\phi-\pi)-\tfrac{1}{2}.\sin\phi.\sin\pi.H^2.\sin^2 1' \text{ when H is small,}$$

$$\text{or } \zeta=\phi-\pi+\frac{\sin\phi.\sin\pi}{2\sin(\phi-\pi)}.H^2.\sin 1'.$$

Substituting in this for ϕ, π and H their values $\Phi-\frac{tm.\cos\alpha}{60}$, $\Pi+t.\delta\Pi$, &c., and ζ now representing the true zenith distance within a few minutes of meridian passage,

$$\zeta=\Phi-\Pi-t.\left(\delta\Pi+\frac{m.\cos\alpha}{60}\right)$$
$$+\tfrac{1}{2}.t^2.\frac{\sin\Phi.\sin\Pi}{\sin(\Phi-\Pi)}.\left(15\mu-\delta A+\frac{m.\sin\alpha}{60\sin\Phi}\right)^2.\sin 1'.$$
$$=\Phi-\Pi-at+\tfrac{1}{2}bt^2 \text{ suppose.}$$

When ζ is a minimum, $\frac{d\zeta}{dt}=0$, or $-a+bt=0$ and $t=\frac{a}{b}$; and the minimum value of ζ being ζ', we shall have

$$\zeta'=\Phi-\Pi-\frac{a^2}{b}+\tfrac{1}{2}.\frac{a^2}{b}=Z-\tfrac{1}{2}.\frac{a^2}{b},$$

if $Z=\Phi-\Pi$ be the zenith distance on the meridian.

Hence

$$t=\frac{\delta\Pi+\frac{m.\cos\alpha}{60}}{\frac{\sin\Phi.\sin\Pi}{\sin(\Phi-\Pi)}.\left(15\mu-\delta A+\frac{m.\sin\alpha}{60.\sin\Phi}\right)^2.\sin 1'}$$

$$Z-\zeta'=\tfrac{1}{2}.\frac{\left(\delta\Pi+\frac{m.\cos\alpha}{60}\right)^2}{\frac{\sin\Phi.\sin\Pi}{\sin(\Phi-\Pi)}.\left(15\mu-\delta A+\frac{m.\sin\alpha}{60.\sin\Phi}\right)^2.\sin 1'}.$$

Now for all celestial objects whatever δA is very small compared with 15μ, and so likewise at any practicable rate of sailing is $\frac{m.\sin\alpha}{60.\sin\Phi}$, unless Φ be very small, that is, unless the latitude be not far from 90°. On these considerations therefore we may reduce the above expressions to

$$t=\frac{1}{60.(15\mu)^2.\sin 1'}\cdot\frac{\sin(\Phi-\Pi)}{\sin\Phi.\sin\Pi}.(60\,\delta\Pi+m.\cos\alpha),$$

$$Z-\zeta'=\frac{1}{2.60^2(15\mu)^2.\sin 1'}\cdot\frac{\sin(\Phi-\Pi)}{\sin\Phi.\sin\Pi}.(60\,\delta\Pi+m.\cos\alpha)^2$$

and

$$\Phi=\zeta'+\Pi+\frac{1}{2.60^2.(15\mu)^2.\sin 1'}\cdot\frac{\sin(\Phi-\Pi)}{\sin\Phi.\sin\Pi}(60\,\delta\Pi+m.\cos\alpha)^2$$

is a formula by means of which we may compute the colatitude at the instant of meridian passage, it being sufficient to substitute its approximate value only in the coefficient of $(60\,\delta\,\Pi+m.\cos\alpha)^2$.

Should it be more convenient to us to obtain at once the colatitude at the time $T+t$ of observation, we shall have, representing this by ϕ, and the polar distance of the object at the same time by π,

$$\Phi=\phi+\frac{t\,m.\cos\alpha}{60},\qquad \Pi=\pi-t.\delta\,\Pi,$$

$$\therefore\ \phi=\zeta'+\pi-\frac{1}{60}.t.(60\,\delta\Pi+m.\cos\alpha)$$

$$+\frac{1}{2.60^2.(15\mu)^2.\sin 1'}\cdot\frac{\sin(\Phi-\Pi)}{\sin\Phi.\sin\Pi}.(60\,\delta\,\Pi+m.\cos\alpha)^2;$$

and substituting for t its value, and making

$$k=\frac{1}{2.60^2.(15\mu)^2.\sin 1''},\text{ or }\log k=7{\cdot}32438,$$

$$\phi=\zeta'+\pi-k.\frac{\sin(\Phi-\Pi)}{\sin\Phi.\sin\Pi}.(60\,\delta\,\Pi+m.\cos\alpha)^2,$$

$$=\zeta'+\pi-k.\frac{\sin\zeta'}{\sin\pi.\sin(\zeta'+\pi)}.(60\,\delta\,\pi+m.\cos\alpha)^2,$$

and time of meridian passage =

$$T+t-k'.\frac{\sin\zeta'}{\sin\pi.\sin(\zeta'+\pi)}.(60\,\delta\,\pi+m.\cos\alpha),$$

where $k'=120\,k$, or $\log k'=9{\cdot}40356$.

It will be remarked that, according to the ordinary process, we should treat $\zeta'+\pi$ as the colatitude at the time of observation, considering this time identical with that of transit.

For a fixed star we have $\delta\Pi = 0$, and the expressions become

$$\phi = \zeta' + \Pi + k \cdot \frac{\sin(\Phi - \Pi)}{\sin\Phi . \sin\Pi} \cdot (m . \cos\alpha)^2,$$

$$\Phi = \zeta' + \Pi - k \cdot \frac{\sin(\Phi - \Pi)}{\sin\Phi . \sin\Pi} \cdot (m . \cos\alpha)^2.$$

But the colatitude we derive by the ordinary process being $\zeta' + \Pi$, it follows that this is less than the true colatitude at meridian passage, and greater than that at the moment of observation by the same quantity $k \cdot \frac{\sin(\Phi - \Pi)}{\sin\Phi . \sin\Pi} \cdot (m . \cos\alpha)^2$. Hence the latitude derived in the ordinary way from an observed maximum altitude of a star is that of the ship at the mean between the time of observation and that of the star's meridian passage, the interval t being in this case $k' \cdot \frac{\sin(\Phi - \Pi)}{\sin\Phi . \sin\Pi} . m . \cos\alpha$.

The correction $k \cdot \frac{\sin(\Phi - \Pi)}{\sin\Phi . \sin\Pi} \cdot (m . \cos\alpha)^2$, and the interval $t = k' \cdot \frac{\sin(\Phi - \Pi)}{\sin\Phi . \sin\Pi} . m . \cos\alpha$ are evidently greatest in magnitude when $\cos\alpha = \pm 1$, or $\alpha = 0$ or $180°$, and in this case they are reduced to $\frac{k . m^2 . \sin(\Phi - \Pi)}{\sin\Phi . \sin\Pi}$ and $\pm \frac{k' m . \sin(\Phi - \Pi)}{\sin\Phi . \sin\Pi}$. If $\Phi = \Pi$, or the object observed be in the zenith, the correction and interval each $= 0$; but when Φ is greater than Π, then since $\frac{\sin(\Phi - \Pi)}{\sin\Phi . \sin\Pi} = \cot\Pi - \cot\Phi$ is positive and, Φ being given, increases in magnitude with a diminution in the value of Π, the error we commit in assuming the minimum zenith distance to correspond to meridian passage increases as we approach the pole. If, again, Φ is less than Π, and $\cot\Pi - \cot\Phi$ consequently negative, this increases its negative value with an increase in the value of Π, and in this case the error therefore increases as we recede from the pole from which we reckon Π. Hence generally, the effect of our erroneous assumption increases from the zenith towards the poles.

To find at what distances from the poles in a given latitude the error will amount to half a minute of arc, the value of α being 0 or 180°, or the ship's course being due north or south, we have

$$k m^2 . (\cot\Pi - \cot\Phi) = \pm 0{\cdot}5, \text{ or } \cot\Pi = \cot\Phi \pm \frac{0{\cdot}5}{k m^2}.$$

Suppose the rate of sailing to be 10 knots, or $m = 10$, we then have

$$\cot\Pi = \cot\Phi \pm 2{\cdot}3691.$$

Hence if $\Phi=90°$, or the ship be near the equator,

$$\cot \Pi = \pm 2{\cdot}3691, \text{ and } \Pi=22° \ 53' \text{ or } 157° \ 7',$$

showing us that in this case the effect of the ordinary erroneous assumption upon the latitude derived from the observation of objects within 22° 53′ of either pole will exceed half a minute, this effect increasing as the object is nearer to the pole, and diminishing in the opposite direction.

If $\Phi=80°$, or the latitude be about 10°, we shall have

$$\cot \Pi = \cot 80° \pm 2{\cdot}3691 = 0{\cdot}1763 \pm 2{\cdot}3691 = +2{\cdot}5454, \text{ or } -2{\cdot}1928,$$

giving $\Pi=21° \ 27'$ or 155° 29′, and showing that the error will exceed half a minute in the case of objects within 21° 27′ of the upper, and 24° 31′ of the lower pole.

Similarly for $\Phi=70°$, the limits of Π will be 20° 6′ from the upper, and 26° 30′ from the lower pole; for $\Phi=60°$ they will be 18° 45′ and 29° 10′; for $\Phi=50°$, 17° 19′ and 33° 10′; for $\Phi=40°$, 15° 40′ and 40° 20′, the second position in the last three cases being below the horizon. For $\Phi=30°$ they will be 13° 7′ and 65° 12′; for $\Phi=20°$, or latitude 70°, they will be 11° 4′ and 110° 44′; whilst for latitudes greater than this, the course being nearly north or south, and the rate of sailing at all considerable, the limits will be such that the results of observations of objects throughout a very large portion of the sky at upper transit will be subject to a sensible amount of error.

Dealing, however, with cases of ordinary occurrence, the above results prove to us that the latitude being between 60° N. and 60° S., there is no star of any magnitude excepting α and β Ursæ Minoris which we may not treat in the ordinary way, assuming the greatest apparent altitude to correspond to meridian passage, without introducing sensible error into our results, provided the rate of sailing do not much exceed 10 knots the hour. Observations of the two stars excepted are usually treated in a different manner, which we shall discuss in the proper place; but, as an example of the calculation, we will determine the difference between the true meridian and minimum zenith distances of β Ursæ Minoris, supposing the latitude about 1° N., the ship's course N.N.E. true, and rate of sailing 11 knots.

Resuming the equation

$$Z-\zeta'=k \cdot \frac{\sin(\Phi-\Pi)}{\sin \Phi . \sin \Pi} \cdot (m \cdot \cos \alpha)^2, \text{ we have}$$

$$\Phi=89°, \quad \Pi=15° \ 16', \quad \alpha=22° \ 30', \quad m=11$$

$\log m$	$=1{\cdot}04139$
$\log \cos \alpha$	$=9{\cdot}96562$
Sum	$=1{\cdot}00701$
Double	$=2{\cdot}01402$

Brought forward

Double	$=2\cdot01402$		
$\log k$	$=7\cdot32438$	$\log\sin\Phi$	$=9\cdot99993$
$\log\sin(\Phi-\Pi)$	$=9\cdot98226$	$\log\sin\Pi$	$=9\cdot42047$
Sum	$=9\cdot32066$	Sum	$=9\cdot42040$
	$9\cdot42040$		
Difference	$=9\cdot90026=\log(Z-\zeta')$ or $Z-\zeta'=0'\cdot7948=0'\,48''$		

or the meridian zenith distance was 48″ in excess of the corrected minimum. If we substitute the same quantities in the equation $t=k'.\dfrac{\sin(\Phi-\Pi)}{\sin\Phi.\sin\Pi}.m.\cos\alpha$, we find $t=9^{m}\cdot39=9^{m}\,23^{s}$; or the star continued to rise $9^{m}\,23^{s}$ after it had passed the meridian, and attained an altitude 48″ greater than that which it had at the instant of transit. The result of the ordinary process would therefore in this instance place the ship 48″ north of her position at time of transit, and as much south of that which she occupied at the moment of observation.

On looking through the limits of polar distance within which corrections to latitude derived in the ordinary way from observations of maximum altitude are necessary, we observe that, except for very high latitudes, they do not include any possible position of the sun or moon. It is to be remembered, however, that these objects change their polar distances; and in considering the case with them we must therefore resume the equations in their more general form.

Let $n=\dfrac{\sin(\Phi-\Pi)}{\sin\Phi.\sin\Pi}$. Then

$$Z-\zeta'=k.n.(60\,\delta\Pi+m.\cos\alpha)^2,\quad t=k'.n.(60\,\delta\Pi+m.\cos\alpha),$$

which, for given values of Φ and Π, are evidently greatest in magnitude when $\alpha=0$ or $180°$, according as $\delta\Pi$ is positive or negative. Now in the case both of sun and moon, $\delta\Pi$ is a maximum when Π is about $90°$; and its general value is then $\frac{1}{60}$ for the sun and $\frac{16}{60}$ for the moon. In the former case the value of $60\,\delta\Pi$ being 1 only, is small compared with any value of m which will make that of $Z-\zeta'$ at all important, and the sun may therefore be classed with the fixed stars; and we may assume without appreciable error that between the latitudes 60° N. and 60° S., the maximum altitude corresponds to meridian passage.

But that it is otherwise with the moon a single example will suffice to show.

Suppose $\Pi=90°$, $\delta\Pi=+\frac{16}{60}$, $m=10$, $\alpha=0$.

Then $n=-\cot\Phi$.

$$Z-\zeta'=-k.(16+10)^2.\cot\Phi,\quad t=-k'(16+10).\cot\Phi;$$

and when $\Phi=40°$, or latitude$=50°$, the values of these quantities become

$$Z-\zeta'=-1'{\cdot}70=-1'\,42'',\quad t=-7^{m}{\cdot}847=-7^{m}\,51^{s},$$

and will be greater in higher latitudes.

Now Z being reckoned away from the pole will be negative; and it follows therefore that in this case the moon will attain her maximum altitude $7^{m}\,43^{s}$ before her meridian passage, and that altitude will exceed the meridian altitude by $1'\,42''$.

For the moon then our formula will be—

ζ' = true least ZD of centre $=90°-\Omega+$dip$+$refraction$-$parallax$\mp$semidiameter,

$T+t$ = local mean time of observation,

π = polar distance at $T+t$,

$$\phi=\text{colatitude at } T+t=\zeta'+\pi-k\,.\,\frac{\sin\zeta'}{\sin\pi\,.\,\sin(\zeta'+\pi)}\,.\,(60\,\delta\pi+m\,.\cos\alpha)^2,$$

$$t=k'.\frac{\sin\zeta'}{\sin\pi\,.\,\sin(\zeta'+\pi)}\,.\,(60\,\delta\pi+m\,.\cos\alpha),$$

T = time of meridian passage = time of observation $-t$,

$$\Phi=\text{colatitude at } T=\zeta'+\pi-t\,.\,\delta\pi+k\,.\,\frac{\sin\zeta'}{\sin\pi\,.\,\sin(\zeta'+\pi)}\,.\,(60\delta\pi+m\,.\cos\alpha)^2,$$

and true latitude at either time $=90° \sim$ colatitude*;

whilst for all other objects not circumpolar, the ship being between 60° N. and 60° S., and the rate of sailing not much exceeding ten knots the hour, we shall have

$$\zeta'=Z,\ \pi=\Pi,\ \text{and}\ \Phi=\phi=Z\pm\Pi.$$

We will now take examples of determination of latitude.

1°. By an observed maximum altitude of a star.

2°. By a similar observation of the sun.

3°. By a like observation of the moon.

1°. On the 30th of June, 1857, the reading from the limb of the Sextant for the greatest apparent altitude on the southern meridian of α Centauri above the sea-horizon was 26° 3′ 0″, that for coincidence of images being +0′ 17″. The barometer being 30·10 inches, thermometer 82°†, and the elevation of the observer above the level of the sea 17 feet, it is required to find the latitude at the time of observation.

* A little consideration and a few practical trials will enable the navigator, without the necessity of going through the calculation on every occasion, to form a correct opinion as to whether in any particular case it will be necessary to correct the latitude determined in the ordinary way from an observed maximum altitude of the moon.

† The barometer and thermometer are observed for the purpose of enabling us to determine the amount of refraction, which, in our examples, is stated apart from that of the dip. In the treatment of ordinary observations of altitude, however, it will in general be sufficient to employ the correction for the two collectively, taking it from the Tables constructed for the purpose, to be found in most works on nautical astronomy.

	′	″
$-\omega_0$ =	−0	17*
Correction for η & σ =	0	0*
E =	+0	4*
Sum =	−0	13
Dip =	+4	5
Refraction =	+1	52
Sum =	+5	57

	°	′	″
ω =	26	3	0
		−0	13
Ω =	26	2	47
$90^\circ - \Omega$ =	63	57	13
		+5	57
Z =	−64	3	10†
Π =	150	14	51
Φ =	86	11	41
Latitude =	3	48	19 N.

2°. On the 19th of March, 1858, the reading for the greatest apparent altitude of the sun's lower limb on the northern meridian with Shade No. 4, was 55° 14′ 0″, that for coincidence of images being + 0′ 17″. The barometer being 30·10 inches, thermometer 82°, and the elevation of the observer above the level of the sea 11 feet, it is required to find the latitude at the time of observation, on the supposition that the approximate longitude was 76° west from Greenwich.

Here it will be remarked that we introduce a new datum,—the approximate longitude; and this, or the approximate Greenwich mean time, will be necessary in every case of observation of the sun or moon, in order that we may be able to take from the Tables the polar distance of the object at the time. Thus, in this instance, the time being apparent noon at the place, will be about $0^h\ 7^m$ local mean time, mean noon preceding apparent noon on the day in question by about seven minutes; and adding to this the longitude = 76° W. = $5^h\ 4^m$, the Greenwich mean time at the instant of observation was about $19^d\ 5^h\ 11^m$; and for this we have to find the polar distance of the sun's centre, which in the present case we shall measure from the south pole.

		°	′	″	
Sun's declination at Greenwich mean noon on the 19th . .	=	0	33	8	S.
Difference for $5^h\ 11^m = 5{\cdot}2 \times 59''{\cdot}26$ nearly	=		5	8	
Sun's declination at time of observation	=	0	28	0	S.
Sun's south polar distance at same time	=	89	32	0	

* Consult Section 17 of Part I. It must be remembered that each of these corrections, although small in the instrument which furnishes us with our examples, amounts in some Sextants to several minutes of arc. On account of their defective character we do not usually take notice of seconds of arc in observations for latitude at sea; but it is as well to do so on occasions on which we wish our result to represent the nearest minute.

† Reckoned negative because measured on the southern meridian, whereas we assume the north pole as the origin of the polar distance Π. See Section 1. Part II. Chap. 1.

In the example we have taken we have then

	′	″
$-\omega_0=-$	0	17
Correction for η & $\sigma=$	0	0
$E=+$	0	9
$c_4=+$	0	4
Sum $=-$	0	4
Dip $=+$	3	17
Refraction $=+$	0	38
Parallax $=-$	0	5
Semidiameter $=-$	16	5
Sum $=-$	12	15

	°	′	″
ω =	55	14	0
——		−0	4
Ω =	55	13	56
$90^\circ-\Omega$ =	34	46	4
——		−12	15
Z =	−34	33	49
Π =	89	32	0
Φ =	54	58	11
Latitude =	35	1	49 S.

3°. On the 1st of June, 1857, the reading for the greatest apparent altitude of the moon's upper limb, without shade, on the southern meridian, was 39° 20′ 0″, that for coincidence of images being + 0′ 17″. The barometer standing at 29·85 inches, thermometer at 57°, and the elevation of the observer above the level of the sea being 17 feet, it is required to find the latitude at the time of observation, on the supposition that the approximate Greenwich mean time was $8^h 28^m$, the ship's course due south, and rate of sailing $3\frac{1}{2}$ knots.

	′	″
$-\omega_0=-$	0	17
Correction for η & $\sigma=$	0	0
$E=+$	0	6
Sum $=-$	0	11
Dip $=+$	4	5
Refraction $=+$	1	10
Parallax $=-$	42	8*
Semidiameter $=+$	14	53
Sum $=-$	22	0

	°	′	″
ω =	39	20	0
——		− 0	11
Ω =	39	19	49
$90^\circ-\Omega$ =	50	40	11
——		−22	0
ζ' =	−50	18	11
π =	93	1	40
$\zeta'+\pi$ =	42	43	29

Now $60\,\delta\pi=+14{\cdot}2$, $m=3{\cdot}5$, $\cos\alpha=-1$, $60\,\delta\pi+m\,.\cos\alpha=+10{\cdot}7$.

---

* The detail of the computation of parallax is as follows:—

Tabular equatorial horizontal parallax at $8^h\ 28^m = 54'\ 30''$
Reduction for place (approximate latitude 47°) $= -\ 5$
Horizontal parallax at place $= 54\ 25 = 3265''$

log 3265 $= 3{\cdot}51388$
log sin $(90^\circ-\Omega+\text{dip}+\text{refraction}) = 9{\cdot}88896$
Sum = log parallax $= 3{\cdot}40284 = \log 2528''$

log 10·7	=1·02938		
double	=2·05876		
log k	=7·32438	log sin π	=9·99939
log − sin ζ'	=9·88617	log sin $(\zeta'+\pi)$	=9·83154
Sum	=9·26931	Sum	=9·83093
	9·83093		
Difference	=9·43838=log 0′·2744=log 16″		

$\zeta'+\pi=42^\circ\ 43'\ 29''$
Correction = − 16
Difference = 42 43 45 = ϕ = colatitude at time of observation.
Latitude at time of observation = 47 16 15 N.

If we compute t we shall find it equal to $-3^m{\cdot}0774 = -3^m\ 5^s$, showing that the moon passed the meridian $3^m\ 5^s$ after it attained its greatest altitude and commenced to fall. Also

	° ′ ″
Colatitude of ship at time of transit $=\zeta'+\pi-t.\delta\pi-16''$	=42 43 57
Latitude at same time	=47 16 3 N.

the ordinary process giving $42^\circ\ 43'\ 29''$ as the colatitude, $47^\circ\ 16'\ 31''$ as the latitude at the time of observation, which we treat as identical with that of transit.

2. The observation next requiring notice is that for Time, and to this we may apply rules and treatment the same as those established with reference to the same case on land.

Let ζ be the true zenith distance of an object situated as nearly as possible on the prime vertical at an observed time t, the local sidereal time at which we wish to find. Let this be T; and let π and A be the polar distance and right ascension of the object at the instant of observation, the former measured from the pole to which we refer the colatitude ϕ. Make $\text{H}=\text{T}-\text{A}$, and $\theta=\frac{1}{2}(\zeta+\phi+\pi)$. Then

$$\left.\begin{aligned}\sin\tfrac{1}{2}\text{H}&=\sqrt{\frac{\sin(\theta-\phi)\,.\,\sin(\theta-\pi)}{\sin\phi\,.\,\sin\pi}}\\ \tan\tfrac{1}{2}\text{H}&=\sqrt{\frac{\sin(\theta-\phi)\,.\,\sin(\theta-\pi)}{\sin\theta\,.\,\sin(\theta-\zeta)}}\end{aligned}\right\}\ \begin{array}{l}\text{H being reckoned}\\ +\text{ or }-\text{ according}\\ \text{as the object is to}\\ \text{the west or east of}\\ \text{the meridian;}\end{array}$$

either of which formulæ may be employed in determining H when the true values of ζ, ϕ and π are known, the second, however, being preferable in general to the first. H being determined, we have

$$\text{T}=\text{A}+\text{H}.$$

Local mean time may either be obtained from sidereal time by means of the proper Tables, or it may be derived from an observed altitude of the sun; the hour-angle in this case furnishing us with apparent time, which differs from mean by the equation of time, the latter quantity with its variations being given in the Almanac.

Example 1. On the 24th of June, 1857, at $8^h\ 32^m\ 38^s$ by watch, the observed altitude of Regulus to the west of the meridian was 43° 17′ 20″, reading for coincidence of images being +0′ 17″, barometer 30·13 inches, thermometer 81°, and elevation of the observer above level of sea 17 feet. Supposing the latitude of the ship at the instant to be 10° 27′ N., and approximate longitude $26\frac{1}{2}$° W., it is required to find the local sidereal and mean times.

		′	″
$-\omega_0$	=	−0	17
Correction for η and σ	=	0	0
E	=	+0	8†
Sum	=	−0	9
Dip	=	+4	5
Refraction	=	+0	58
Sum	=	+5	3

log sin $(\theta-\phi)$	= 9·57900	log sin θ	= 9·99069
log sin $(\theta-\pi)$	= 9·61783	log sin $(\theta-\zeta)$	= 9·91359
Sum	= 9·19683		9·90428
	9·90428		
Difference	= 9·29255		
Half	= 9·64628 = log tan ½H		
½H	= 23° 53′ 14″		

		h	m	s
H	= +47 46 28 = +	3	11	6
Right ascension of *	=	10	0	46
Local sidereal time	=	13	11	52

		°	′	″
ω	=	43	17	20
		−	0	9
Ω	=	43	17	11
$90°-\Omega$	=	46	42	49
		+	5	3
ζ	=	46	47	52
ϕ	=	79	33	0
π	=	77	20	8
Sum	=	203	41	0
θ	=	101	50	30
$\theta-\zeta$	=	55	2	38
$\theta-\phi$	=	22	17	30
$\theta-\pi$	=	24	30	22

† Had we taken three observations for the purpose of employing the mean in our calculation, we should have corrected each angle separately, and Ω would thus have been the mean of the three corrected readings. The correction E is in this case taken from the Table for (ω)=43° 10′ +7° 20′=50° 30′, whereas, had a second angle 43° 0′ been read, the value of E for this would have been that corresponding to (ω)=43° 0′, which in some Sextants would differ materially from that due to 50° 30′. In the case of observations for time at sea, as in that of observations for latitude, it is neither usual nor necessary in general to take notice of seconds of arc, the minute being the smallest part recognized; but the corrections due to the angles read from the Sextant employed by the author are so very small, that, to render his examples complete, he is compelled to introduce seconds upon every occasion.

This observation having been made in the evening, it will be evident that the preceding transit of the first point of Aries, or of 0^h sidereal time, occurred on the 23rd of June. Hence, to convert the above into mean time, we have

		d	h	m	s
Mean time of transit of first point of Aries at Greenwich		23	17	50	40·09
Difference for longitude 26½ W. $= -\frac{26 \cdot 5}{360} \times 3^m\ 55^s \cdot 91$	=				$-17 \cdot 37$ *
Mean time of transit of first point of Aries at place	=	23	17	50	22·72
13 hours sidereal time in mean time			12	57	52·22
11 minutes				10	58·20
52 seconds					51·86
Local mean time = Sum	=	24	7	0	5·00

The watch employed on this occasion was losing $7^s \cdot 5$ from mean time in the course of 24 mean hours; and therefore, should we wish to find the local sidereal time at any other time indicated by the watch, we shall have to correct the observed interval for retardation at the above rate, and converting the corrected interval into sidereal time, apply this, together with the difference of longitude, to $13^h\ 11^m\ 52^s$. Thus, should we wish to know the local sidereal time at $9^h\ 36^m\ 45^s$ by watch, the ship having made $0°\ 6' = 0^m\ 24^s$ to the westward during the interval between our observations, we shall have

	h	m	s		h	m	s	
Interval by watch	= 1	4	7					
Correction for rate			+0·3					
True mean interval	1	4	7·3	=	1	4	17·8	sidereal.
Correction for difference of longitude W.							−24·0	
Corrected interval					1	3	53·8	
First local sidereal time					13	11	52	
Local sidereal time at second place at $9^h\ 36^m\ 45^s$ by watch				=	14	15	44·8	

Example 2. On the 1st of December, 1857, at $16^h\ 32^m\ 36^s$ by watch, and approximate Greenwich mean time $1^d\ 16^h\ 54^m$, the observed altitude of the sun's lower limb to the east of the meridian, with shade No. 4, was 55° 22′ 0″, the reading for coincidence of images being +0′ 17″, barometer 30·20 inches, thermometer 58°, and elevation of the observer above the level of the sea 17 feet. Supposing the latitude of the ship at the instant to be 41° 26′ S., it is required to find the local mean and sidereal time.

* Since 26½° = $1^h\ 46^m$, this correction may be obtained at once by subtracting from $1^h\ 46^m$ its equivalent in mean time = $1^h\ 45^m\ 42^s \cdot 63$. The subject of the conversion of mean time into sidereal and *vice versâ* is treated annually in the explanation appended to the 'Nautical Almanac.'

		′	″
$-\omega_0$	$=-$	0	17
for η and σ	$=$	0	0
E	$=+$	0	8
c_4	$=+$	0	4
Sum	$=-$	0	5
Dip	$=+$	4	5
Refraction	$=+$	0	40
Sun's parallax	$=-$	0	5
Semidiameter	$=-$	16	16
Sum	$=-$	11	36

		°	′	″
ω	$=$	55	22	0
			-0	5
Ω	$=$	55	21	55
$90^\circ-\Omega$	$=$	34	38	5
			-11	36
ζ	$=$	34	26	29
At $16^h\ 54^m$ G. M. T. ϕ	$=$	48	34	0
At $16^h\ 54^m$ G. M. T. π	$=$	68	1	40 S.
Sum	$=$	151	2	9
θ	$=$	75	31	5
$\theta-\zeta$	$=$	41	4	36
$\theta-\phi$	$=$	26	57	5
$\theta-\pi$	$=$	7	29	25

log sin $(\theta-\phi)$	$=9{\cdot}65632$	log sin θ	$=9{\cdot}98597$
log sin $(\theta-\pi)$	$=9{\cdot}11514$	log sin $(\theta-\zeta)$	$=9{\cdot}81761$
Sum	$=8{\cdot}77146$		$9{\cdot}80358$
	$9{\cdot}80358$		
Difference	$=8{\cdot}96788$		
Half	$=9{\cdot}48394=\log\tan\frac{1}{2}$H		
$\frac{1}{2}$H	$=-16^\circ\ 56'\ 55''$		
H	$=-33\ 53\ 50$		$= -2^h\ 15^m\ 35^s$
			$24\ \ 0\ \ 0$
Local apparent time			$=1^d\ 21\ 44\ 25$
Equation of time at $16^h\ 54^m$ G. M. T.			$=\ \ -10\ 25$
Local mean time			$=1\ 21\ 34\ \ 0$

To find the local sidereal time, we have

		h	m	s
Sun's right ascension at $16^h\ 54^m$ G. M. T.	$=$	16	33	33
H	$=$	-2	15	35
Local sidereal time	$=$	14	17	58

3. The approximate Greenwich mean time being known, and the approximate longitude likewise, to determine the latitude by means of an observed altitude of an object near the meridian, or by one of a circumpolar star at any hour-angle whatever.

Let M be the approximate Greenwich mean time, S the corresponding Greenwich sidereal time, and l the approximate longitude, + if west, − if east. Then $S-l$ will be the approximate local sidereal time.

Let A and π be the right ascension and polar distance of the object at Greenwich mean time M; ζ the true zenith distance of the object deduced from the observed altitude after correction of the latter for instrumental error, dip, refraction, &c.*

* As a sufficient number of examples of the process of deriving true zenith distance from observed altitude have already been given, it will be unnecessary to repeat it.

∴ approximate hour-angle of object $=S-l-A=H$ suppose; and by the method of section 3 of Part II. Chap. I., we shall have, for the computation of the colatitude ϕ,

$$\tan z_1 = \tan \pi . \cos H\ ;\ \sin Z = \frac{\sin \pi . \sin H}{\sin \zeta}\ ;\ \tan z_2 = \tan \zeta . \cos Z\ ;$$

$$\phi = z_1 + z_2\ ;\ \text{latitude} = 90^\circ - \phi\ ;$$

z_1 being greater or less than 90° according as π is greater or less than the same, and cos Z being positive or negative according as the object is at the time of observation near the polar or equatorial meridian.

This method will be found useful in a case which frequently occurs at sea, viz. a good altitude of the sun or other object being obtained a little before or after meridian passage, clouds interfering with the observation of the greatest apparent altitude; and as a small error in the hour-angle, when the azimuth of the object is small, produces a very small effect upon the calculated colatitude, the result of the process may be depended upon in every case in which the assumed Greenwich time and longitude are not very far from the truth, the object itself not being close to the zenith.

Example 1. April 6, 1858. Greenwich mean time by watch $=4^h\ 11^m$; assumed longitude $=58^\circ$ W. $= +3^h\ 52^m$; corrected zenith distance of sun's centre $=59^\circ\ 50'$ towards northern meridian.

		h	m	s
Sidereal time at Greenwich mean noon on the 6th April	=	0	57	56
4 hours mean time in sidereal	=	4	0	39
11 minutes	=		11	2
Sum = S	=	5	9	37
l	=	+3	52	0
Difference = S − l	=	1	17	37
Sun's right ascension at $4^h\ 11^m$ Greenwich mean time	=	1	1	2
Difference = H	=		+16	35 = +4° 9′
Sun's south polar distance at $4^h\ 11^m$ G. M. T.	=			96 31

And the detail of the computation will be—

log − tan π = 0·94222	log sin π = 9·99718	log tan ζ = 0·23565
log cos H = 9·99886	log sin H = 8·85955	log − cos Z* = 9·99850
log − tan z_1 = 0·94108	Sum = 8·85673	log tan − z_2 = 0·23415
z_1 = 96° 32′	log sin ζ = 9·93680	
z_2 = − 59 45	log sin Z = 8·91993	
ϕ = 36 47		

Latitude = 53 13 S. at time of observation.

* Refer to Note to Section 3. Part II. Chap. I.

In obtaining the hour-angle H in this example, we have employed the process which is applicable to all objects alike; but in the case of the sun we may obtain it by another, in which it will be unnecessary to convert the assumed Greenwich mean time into sidereal, or to interpolate the right ascension from the Tables. Thus—

	h m s	
Assumed Greenwich mean time	4 11 0	
Assumed longitude west	3 52 0	
Local mean time	0 19 0	
Equation of time at $4^h\ 11^m$ to be subtracted from mean	2 25	
Local apparent time = Sun's hour-angle	= +16 35	= +4° 9′

The same result as before.

Example 2. May 28, 1857. Greenwich mean time by watch $=9^h\ 45^m\ 17^s$; assumed longitude $=11^\circ\ 3'$ W. $= +0^h\ 44^m\ 12^s$; corrected zenith distance of Polaris $=42^\circ\ 39'$ towards northern meridian.

	h m s	
Sidereal time at Greenwich mean noon on the 28th May	4 23 54	
9 hours mean time in sidereal	9 1 29	
45 minutes	45 7	
17 seconds	17	
Sum = S	14 10 47	
l	+44 12	
Difference = S − l	13 26 35	
✱'s right ascension = A	1 6 19	
Difference = H	= −11 39 44	= −174° 56′
✱'s north polar distance		= 1 26

log tan π = 8·39832	log sin π = 8·39818	log tan ζ = 9·96433
log − cos H = 9·99830	log − sin H = 8·94603	log cos Z = 0·00000
log tan − z_1 = 8·39662	Sum = 7·34421	log tan z_2 = 9·96433
	log sin ζ = 9·83092	z_2 = 42° 39′
	log − sin Z = 7·51329	z_1 = −1 26
		ϕ = 41 13
		Latitude = 48 47 N.

4. To find the latitude and time by means of two observed altitudes either of two different objects or the same object, an interval elapsing between the observations, and the latitude being known to lie between given limits not very far one from the other.

The method of section 5 of Chapter I. will be applicable, on our introducing into the formulæ the proper corrections for the motion of the ship during the interval between the observations.

Let Λ, Λ' be the given limits within which the latitude is supposed to lie; T_1, T_2 the times of observation by watch, T being the interval T_2-T_1 corrected for rate. Then if t_1 be the time obtained from the first observation reduced with latitude Λ, t'_1 that obtained with latitude Λ', n the number of minutes of arc in $\Lambda'-\Lambda$, and m the number to be added to Λ to obtain the true latitude at T_1,

$$\left.\begin{array}{l}\text{True time at } T_1 \text{ in the first}\\ \quad\text{position of the ship}\end{array}\right\}=t_1+\frac{m}{n}.(t'_1-t_1).$$

Let a be the increment to the latitude, b that to the west longitude during the interval T, a being expressed in minutes of arc, b in seconds of time. Then from the first observation we obtain,

$$\left.\begin{array}{l}\text{True time at } T_1 \text{ in the second}\\ \quad\text{position of the ship}\end{array}\right\}=t_1+\frac{m}{n}.(t'_1-t_1)-b$$

$$\left.\begin{array}{l}\text{True time at } T_2 \text{ in the same}\\ \quad\text{position of the ship}\end{array}\right\}=t_1+\frac{m}{n}.(t'_1-t_1)-b+\mathrm{T}.$$

But $\Lambda+m$ being the latitude at T_1, and a the increment produced by the motion of the ship during the interval T,

$$\text{True latitude at } T_2=\Lambda+m+a,$$

$$\left.\begin{array}{l}\text{True time at } T_2 \text{ from the}\\ \quad\text{second observation}\end{array}\right\}=t_2+\frac{m+a}{n}.(t'_2-t_2).$$

Equating this with the result of the first observation, and transposing,

$$\frac{m}{n}\{t'_2-t_2-(t'_1-t_1)\}=\mathrm{T}-b-(t_2-t_1)-\frac{a}{n}.(t'_2-t_2);$$

or making $\mathrm{T}-b-(t_2-t_1)=c$,

$$m=\frac{cn-a(t'_2-t_2)}{t'_2-t_2-(t'_1-t_1)};$$

and m being computed by means of this formula, we obtain at once the true latitnde $\Lambda+m$, and the true time,

$$t_1+\frac{m}{n}(t'_1-t_1) \text{ at } T_1.$$

If the two observations follow so closely that the motion of the ship during the interval is practically nothing, we have $a=0$ and $b=0$, and

$$m=\frac{n.\{\mathrm{T}-(t_2-t_1)\}}{(t'_2-t'_1)-(t_2-t_1)};$$

or the number of minutes to be added to the smaller of the assumed latitudes will be obtained by multiplying the number in the difference between the assumed latitudes by the seconds of time in the excess of the interval given by the watch over that given on reduction of the observations with the smaller latitude and dividing the product by the number of seconds by which the interval obtained with the greater latitude exceeds that obtained with the smaller.

The remarks made in section 5 of Part I. on the application of this method may be repeated here.

When we wish to obtain latitude particularly, and are indifferent about time, we may with advantage observe the same object twice upon opposite sides of the meridian, and not very far away from it, whereas, if we are anxious about both elements, the best combination will be that of an object near the meridian, or one side with a second object near the prime vertical on the other, or the latter with a circumpolar object at any hour-angle whatever; but in no case should both objects be near the prime vertical, either on the same or on opposite sides of the meridian. And, furthermore, should the resulting latitude in any case fall much without the assumed limits, we must assume new limits which shall include it, and repeat the process.

The true time should be obtained from that of the two objects which gives the least difference between the times which result from the reduction of the observation with the two assumed latitudes, in order that the effeet of seconds of arc or decimals omitted from the value of m may be as small as possible.

Should the sun be the object observed on both occasions, it will be more convenient to employ mean time than sidereal in the reductions; but should an observation of the sun be combined with one of the moon, or the objects observed be planets or stars, it will be proper to use sidereal time.

Example 1. A ship's position being supposed to lie between latitudes 55° 50′ and 56° 10′, the sun was observed twice, and the observations gave the following results:—

	h	m	s	
Time by watch at first observation	1	13	21	$=T_1$
Time computed with latitude 55° 50′ . . .	=21	23	34	$=t_1$
,, with latitude 56 10 . . .	=21	26	22	$=t'_1$
Time by watch at second observation	1	59	14	$=T_2$
Time computed with latitude 55 50 . . .	=22	7	53	$=t_2$
,, with latitude 56 10 . . .	=22	12	7	$=t'_2$

and the ship during the interval had increased her latitude 3′·1* and had gone to the westward 3′·7* = 15 seconds of time nearly, the watch losing 7 seconds daily from mean time.

Here $n=20$, $a=+3\cdot1$, $b=+15$,

T	= 0h 45m 53s	t'_2-t_2	= 4m 14s = 254s	log	= 2·40483
$-b$	= − 0 15	t'_1-t_1	= 2 48 = 168	log + a	= 0·49136
$-(t_2-t_1)$	= −44 19	Differ.	= 86	Sum	= 2·89619
c	= + 1 19 = +79s			Nat. No.	= +787
cn	= +1580				
$a(t'_2-t_2)$	= + 787				
Difference	= + 793	log	= 2·89927		
		log 86	= 1·93450	Λ	= 55° 50′
		log m	= 0·96477	m	= +9
		log 168	= 2·22531	Sum	= 55 59 = latitude at T_1
		Sum	= 3·19008		h m s
		log n	= 1·30103	t_1	= 21 23 34
		Difference	= 1·88905 = log		+1 17
			True mean time at T_1	=	21 24 51

Example 2. Aug. 25, 1857. $\Lambda=51°\,40'$ S, $\Lambda'=51°\,50'$ S.

At $T_1=9^h\,30^m\,51^s$, corrected ZD of α Virginis $=56°\,0'$

$T_2=9\ 35\ 0$, „ Canopus $=72\ 58$

And on reduction with Λ and Λ′, we shall find

$$t_1=16^h\,24^m\,59^s \qquad t'_1=16^h\,24^m\,22^s$$
$$t_2=16\ 26\ 4 \qquad t'_2=16\ 29\ 31 \quad \text{sidereal.}$$

Hence $T=4^m\,9^s$ mean $=4^m\,10^s$ sidereal time, an interval so short that we may assume

$a=0$ and $b=0$; $t_2-t_1=65^s$; $t_2'-t_1'=309^s$; and

$$m=\frac{10\times(250-65)}{309-65}=\frac{1850}{244}=8' \text{ nearly,}$$

$$\therefore \text{ true latitude } =51°\,40'+8'=51°\,48' \text{ S};$$

and selecting α Virginis for the determination of time because t'_1-t_1 is less than t'_2-t_2,

true sidereal time at $T_1=16^h\,24^m\,59^s-\frac{8}{10}\times37^s=16^h\,24^m\,29^s$.

* It will be found convenient to employ minutes and decimals.

The importance of this selection will be apparent, if we derive the time in the same manner from Canopus, assuming $m = 8'$ exactly. For $16^h\,26^m\,4^s + \frac{8}{10} \times 207^s = 16^h\,28^m\,50^s$; and subtracting the interval $4^m\,10^s$ from this, we should have $16^h\,24^m\,40^s$ for the sidereal time at T_1, a result differing 11^s from the former. But computing accurately,

			h	m	s	
log 1850	= 3·26717					
log 244	= 2·38739					
log m	= 0·87978					
log 37	= 1·56820					
Sum	= 2·44798					
log n	= 1·00000	t_1 =	16	24	59	
Difference	= 1·44798 = log				28	
	True sidereal time at T_1	=	16	24	31	by α Virginis.
log m	= 0·87978					
log 207	= 2·31597					
Sum	= 3·19575					
log n	= 1·00000	t_2 =	16	26	4	
Difference	= 2·19575 = log			2	37	
	True sidereal time at T_2	=	16	28	41	by Canopus.

and the interval between the former and this is $4^m\,10^s$, as it should be. Thus the omission of the decimals in the value of m produces a difference of 2 seconds only in the time derived from the observation of α Virginis, whereas in that derived from the observation of Canopus it causes one of 9 seconds.

The process just explained is in principle that known to navigators as 'Sumner's Method;' and there is none in the whole range of Nautical Astronomy of more frequent utility. It enables the operator, without entering upon calculations with which he is not familiar, to obtain latitude and time whenever two celestial objects in favourable positions are visible at the same moment, or when the same object, having been once observed and subsequently obscured, reappears after a favourable interval. When we require latitude and time for reduction of an observed lunar distance, it enables us to obtain them from a combination of altitudes of two objects so near to the proper instant that we are comparatively independent of errors in reckoning course and distance, whether arising from causes appertaining to the vessel herself, or to currents in the ocean; and when the navigator wishes to direct his attention to these currents, he will find a repetition of the process at intervals of half an hour or an hour give results that will be serviceable to himself, and add materially to the accurate knowledge already possessed on this interesting and important subject*. And again, should the sun and moon be visible at the same time, and be favourably situated for the application of the method, and should the navigator sight a point of land the latitude and longitude of which he wishes to ascertain, he may at convenient times obtain his own positions, and to these refer that of the point in question by observations with his azimuth compass, or, still better, by observed angles between the sun's limb and the point, the instrument employed being the Sextant.

Thus let Ω be the corrected angle observed, ζ the sun's apparent zenith distance, Z its azimuth at the time. Then the difference between the azimuth of the sun and that of the object being θ, we shall have $\cos\theta = \cos(\Omega + \text{sun's radius}) \div \sin\zeta$, and hence deriving θ, and Z from the formula $\sin Z = \frac{\sin\pi \cdot \sin\zeta}{\sin H}$, the sum or difference of θ and Z will give the true bearing of the point at the time of observation more accurately than it could be observed with the compass, and independent of magnetic variation. Two positions of the ship being given, together with the true azimuths of the point from those positions, the determination of its latitude and longitude is a simple problem of trigonometry; but the introduction of a third or even fourth position and azimuth is desirable with a view to obtain a more accurate result. On such occasions then will the above method be found serviceable, and the scientific navigator will certainly spare no pains in making himself familiar with it by constant practice. One unaccustomed to the manipulation of the positive and negative signs may probably experience some little difficulty at first; but this will be soon overcome; and as a check on the accuracy of his work, he may reduce the observations independently with the latitudes resulting from the operation, when, if all be correct, the times, making proper allowance for the elapsed interval and the motion of the ship, will be the same.

5. We now proceed to the determination of Greenwich mean time by means of an observed distance between the moon and some other celestial object,—an operation which ought to be performed at sea at least three times in the course of every month, should favourable opportunities present themselves. Unfortunately, however, it is rarely performed under any circumstances; and the navigator, dependent entirely upon his chronometer, is doubtless frequently involved in difficulties, and on many occa-

* It must be borne in mind that the simultaneous visibility of two objects favourably situated is in this case indispensable. The accuracy of the result derived from two observations of the same object will depend upon that of the reckoning for the interval; and an unknown current affecting this, will affect the other likewise.

sions meets with accidents in consequence. A chronometer that will maintain anything like a uniform rate at sea is not to be had at command; and it is by no means uncommon to find variations of many seconds daily from the assumed rate, in some instances steadily increasing in one direction, in others dependent on temperature, and so positive and negative alternately. The navigator who wishes to be constantly in possession of reasonably accurate knowledge of his position, and to obtain the means of ascertaining subsequently with greater accuracy what that position was at a given time, will therefore have recourse to observations of lunar distance, and, in reducing them, will give attention to the various circumstances affecting the result, whether that employed immediately in the ordinary course of his operations, or that to be arrived at eventually, which may be important not only to himself but to the world at large. There are probably many dangers scattered over the ocean, the positions of which are by no means exactly known; and although the navigator, after following on two or three occasions a track at the time supposed to pass near to their places as indicated on the chart, should he perceive no evidence of their existence, may be inclined to entertain doubt respecting it, we must remember that they may nevertheless exist, and that the determination of their positions having been in the first instance erroneous, in consequence perhaps of undue dependence upon a chronometer, or indifferent observation and reduction of lunar distance, many succeeding navigators, committing similar errors, may have followed tracks which have not approached either the indicated or actual places. The interest and utility moreover of much of the information that may be collected at sea must depend to a great extent upon accurate knowledge of position; and on all accounts, therefore, it is extremely desirable that the subject should receive from the navigator the attention which its importance demands.

But a good Sextant, in the first place, is essential to anything like success in lunar observations; and even though the instrument be in every respect excellent, it is equally essential that the corrections for eccentricity and other errors be obtained in the manner explained in the first part of this work. The Sextant which was there taken for the purpose of furnishing examples of the processes discussed, is one upon which extraordinary pains were bestowed; and nevertheless the reader will perceive that at one point of the arc a correction of 16″ is due to the reading, and that the interposition of one of the shades produces an error of 4″, so that the angle resulting from an observation with this instrument might, if corrected only for index error in the ordinary way, be incorrect to the extent of 20″; and the time obtained by means of a lunar distance might therefore be in error to an amount exceeding 40^s from this cause alone. But there are perhaps few Sextants in which the error which varies with the reading does not amount at its maximum to $40''$; in many it exceeds $1'$; and instances are to be met with in which it amounts to $5'$. The importance of this system of correction is therefore evident; for an error of $5'$ in the angle would produce one of about ten minutes of time in the longitude. It is true that could we obtain a pair of observations, one of an object to the west, and the second of an object to the east of the moon, and so situated that the mean of the readings in the one case should exactly equal the mean in the other, the same shade being employed on both occasions; and furthermore, should the proportional logarithm of the variation of distance for the one be equal to that for the other, the mean of the two results derived from these might be independent of instrumental errors: but such a combination of circumstances is so rare that we might wait a year or more for its occurrence with favourable weather likewise; and as in many instruments a difference of even a few degrees between the means of the readings will produce one of several seconds in the corrections due, it would not be proper to treat the errors with indifference in expectation of such a contingency. Presuming, therefore, that the operator is in possession of a good instrument, and that he has obtained a table of corrections for it in all its parts, the next thing to be considered is the best mode of obtaining observations.

As a comfortable position is essential to anything like accuracy in observation, the operator should be furnished with a chair, with a back inclined at an angle of about 45° to the vertical, and provided with arms sufficiently elevated for the support of the elbows when the telescope of the Sextant is directed to any object between 30° and 60° from the horizon. With such a convenience there will be comparatively little difficulty in observing lunar distance even with a considerable amount of motion.

In the next place, much will depend upon the power of the telescope employed. With respect to this, I have seldom found in the course of my own experience that a power exceeding 9 can be used at sea with any advantage; and on occasions of much motion, or of observation of the moon and sun when the former is faint on account of its proximity to the latter, a power of 5 has proved more satisfactory. In very fine weather a power of 12 or 13 may be tried; but although with this the instrument is perfectly manageable on land, it is seldom so at sea.

Thirdly, the operator should, if possible, make all the requisite observations himself, and not depend upon others, either for noting time or for observation of altitude. A watch which can be relied upon for an hour should be carefully compared with the ship's chronometer immediately before the commencement of operations, and then, for the purpose of noting time, fixed in some convenient position on deck; and it should be compared again with the chronometer immediately on the conclusion of operations*. The first step should be the observation of altitudes of two objects favourably situated for the application of the method explained in section 4; these will give us latitude and local time. The next step should be the observation of the distance, the number of observations taken varying from 5 to 10 according to the state of the weather, 5 being sufficient when it is favourable; but in no case should the intervals between the observations be long, inasmuch as the moon's apparent motion towards or away from an object not being uniform, the introduction of long intervals would vitiate the mean†. The last step should be a repetition of the first, either upon the same objects or others equally favourably situated; these will give us latitude and time again, and we may thus, by interpolation between these and the results of the first step, obtain the latitude and time at the mean of the times of the observed distances, and thence compute the zenith distances of the moon and the object observed with it. Should a second, or a second and third observer be employed to observe altitude, they should not await any signal from the observer of distance, but on noticing that he is recording an observation, should each observe an altitude as speedily as possible, taking the times from the watch employed by the observer of distance, and recording each his own observation.

The observer should not omit to record the readings of the barometer and external thermometer, as these are essential for the computation of refraction. And after what has been said in the first and second parts of this work, it is perhaps needless to remind him of the necessity of obtaining the value of ω_0 by coincidences observed at the commencement and termination of his operations.

With respect to the reduction: should the navigator be anxious only to take his vessel in safety from one point of land to another, and therefore be indifferent to a few seconds of time in his determinations of longitude, the approximate method will answer

* The operator should learn to count seconds; and in observing, he should make the count 0 coincide with the instant of observation, and continue counting 1, 2, 3, &c. until he looks at the dial of the watch. The number counted being deducted from the time then indicated by the hands, the result will be the time to be recorded.

† I have generally found it possible to observe at intervals of about $1\frac{1}{2}$ minute.

his purpose, the zenith distances of the objects being either derived from observations of altitude, or computed from the latitude and time obtained in the manner already indicated. Hence, referring to Section 6 of Chapter I. of this Part*, and remembering that, as observations are supposed to have been continued at intervals from the time of leaving land, the Greenwich mean time ought to be known approximately, the formulæ will be

No. 1. Approximate Method. Altitudes of both objects observed.

Given:
- τ the mean of the times of observation of distance.
- Ω the mean of the readings corrected for instrumental errors.
- t the approximate Greenwich mean time at τ.
- ζ_2 the apparent zenith distance of moon's centre at τ.
- z_2 the apparent zenith distance of centre of second object at τ.
- p the moon's horizontal parallax at the place†.
- P the horizontal parallax of the second object.
- r the moon's augmented semidiameter ‡.
- R the tabular semidiameter of the second object.

To find the error of the watch compared with Greenwich mean time.

1. $\zeta_1 = \zeta_2 + \text{refraction} - p \,.\, \sin(\zeta_2 + \text{refraction})$.
2. $z_1 = z_2 + \text{refraction} - \text{P} \,.\, \sin(z_2 + \text{refraction})$.
3. $u_2 = \Omega \pm r \pm \text{R}$.
4. $\theta_2 = \frac{1}{2}(u_2 + \zeta_2 + z_2)$.
5. $\sin^2\theta_1 = \dfrac{\sin\theta_2 \,.\, \sin(\theta_2 - u_2) \,.\, \sin\zeta_1 \,.\, \sin z_1}{\sin\zeta_2 \,.\, \sin z_2 \,.\, \sin^2\frac{1}{2}(\zeta_1 + z_1)}$.
6. $\sin\frac{1}{2}u_1 = \sin\frac{1}{2}(\zeta_1 + z_1) \,.\, \cos\theta_1$.
7. M = Greenwich mean time of geocentric distance u_1.
8. $\tau - \text{M}$ = Error of watch.

No. 2. Approximate Method. Altitudes of objects not observed.

Given:
- Notation the same as in No. 1, with the following additions.
- T the local sidereal time at τ.
- ϕ the geographical colatitude at τ.
- a, π the moon's geocentric RA and PD at t.
- A, Π, the geocentric RA and PD of the second object.

1. $h = \text{T} - a$ $\qquad$ $\text{H} = \text{T} - \text{A}$.
2. $\tan\theta = \tan\phi \,.\, \cos h$ $\qquad$ $\cos\zeta_1 = \dfrac{\cos\phi \,.\, \cos(\pi - \theta)}{\cos\theta}$.
3. $p' = p \,.\, \sin\zeta_1 + (p)$§ $\qquad$ $\zeta_2 = \zeta_1 + p' - \text{refraction}$.

---

* The reader should peruse Section 6 of Chapter I., as many particulars of importance are noticed there, which, to avoid repetition, are assumed in this Part as understood.

† Obtained by subtracting the correction for latitude from the equatorial horizontal parallax at assumed Greenwich mean time t.

‡ Obtained by adding the correction for altitude to the horizontal semidiameter.

§ Taken from Table I. annexed to this Part.

4. $\tan\theta = \tan\phi \,.\, \cos H$ $\qquad \cos z_1 = \frac{\cos\phi \,.\, \cos(\Pi - \theta)}{\cos\theta}$.

5. $z_2 \quad = z_1 + P \,.\, \sin z_1 - \text{refraction}.$

And the remaining quantities $u_2, \theta_2, \theta_1, u_1$, M and $\tau - M$ will be obtained in this case exactly as they were in No. 1.

Also S being the Greenwich sidereal time corresponding to M, S−T will be the longitude.

But if the navigator desires, for the several reasons already noticed as well as for others, to obtain a result as accurate as possible for immediate purposes, and to possess in a compendious form the means likewise of correcting this result at some future period, he will have recourse to the second method of proceeding given in Part I., the formula of which, as it will be the same in the two cases, it is unnecessary to repeat. The Greenwich mean time to be immediately adopted will in this case be

$$M + \frac{s_4 np}{60} \,.\, \cot\zeta' \,.\, \sin 1'' \,.\, \delta\zeta :$$

and in all cases, whether the method of reduction be the approximate or the more accurate, the error of the chronometer obtained should be employed with the proper daily correction for rate in the ordinary determination of local time and longitude between one set of observations and another. In the first instance it will be necessary to adopt the rate given by the maker; but this may be amended as observations are multiplied, and as the interval between the first and last sets increases, the sums of the errors in the times divided by the number of days becoming thus smaller and smaller, and at last inappreciable. Instances, however, have been known in which the variation of rate was so considerable as to render it necessary to adopt for the time that given by the results of observations separated by comparatively short intervals.

Example 1. Method 1. June 1, 1857. $\omega_0 = +0' \, 17''$. Bar. 29·85 inches. Therm. 57°. Latitude 47° 11′ N.

Moon and α Leonis, with Shade No. 2.

	Time by watch.	Reading.			
	h m s	° ′ ″		′ ″	
τ	=9 37 32	39 40 23	E*	=+0 7	Means of 6 obs.
Assumed error	=−10 0	−0 8	$-\omega_0$	=−0 17	
t	=9 48†	Ω=39 40 15	η and σ=	0 0	
			c_2	=+0 2	
			Sum	=−0 8	

* Mean value of E. Refer to examples in Chap. I. † To the nearest minute.

ζ_2 interpolated from observation	$=53°\ 49'\ 0''$		
Refraction	$=\ \ +1\ 18$		
Sum	$=53\ 50\ 18$		
Moon's equat. hor. par. at t $=54'\ 28''$		P = 0	
Correction for latitude $=\ -6$			
p $=54\ 22$		z_2 interpolated	$=57°\ 26'\ 0''$
$\log p$ $=3.51348$		Refraction	$=\ +1\ 29$
long sin (ζ_2+refraction) $=9.90707$		P . sin (z_2+refr.)	$=\ 0\ 0$
Sum $=3.42055=\log$	$43\ 54$	z_1	$=57\ 27\ 29$
ζ_1	$=53\ 6\ 24$	——	$53\ 6\ 24$
Moon's hor. ½ diam. at t $=14'\ 52''.5$		Sum	$=110\ 33\ 53$
Augmentation for ζ_2 $=\ +8\ .5$		$\frac{1}{2}(\zeta_1+z_1)$	$=55\ 16\ 57$
(r) $=15\ 1$		$\log\sin\frac{1}{2}(\zeta_1+z_1)$	$=9.9148560$
Ω $=39°\ 40\ 15$		Double	$=9.8297120$
u_2 $=39\ 55\ 16$	$\log\sin\zeta_1=9.9029567$		
ζ_2 $=53\ 49\ 0$		log sin	$=9.9069446$
z_2 $=57\ 26\ 0$		log sin	$=9.9257069$
Sum $=151\ 10\ 16$	$\log\sin z_1=9.9258265$		
θ_2 $=75\ 35\ 8$	log sin $=9.9861088$		
θ_2-u_2 $=35\ 39\ 52$	log sin $=9.7656962$		
	Sums $=9.5805882$		9.6623635
	9.6623635		

Difference $=\log\sin^2\theta_1=9.9182247$ $\log\sin\frac{1}{2}(\zeta_1+z_1)=9.9148560$

$\log\sin\theta_1=9.9591124$ $\log\cos\theta_1=9.6172956$

Sum $=\log\sin\frac{1}{2}u_1=9.5321516$

$\frac{1}{2}u_1 = 19°\ 54'\ 32''$

$u_1 = 39\ 49\ 4$

Tabular distance at $1^d\ 9^h$ G. M. T. $=39\ 25\ 17$ PL $=3037$*

Difference $=\ 23\ 47$ PL $=8790$

PL of $0^h\ 47^m\ 52^s$ $=5753$

$1^d\ 9\ 0\ 0$

Sum $=1\ 9\ 47\ 52$

Correction for 2nd difference $=\ -1$

M $=1\ 9\ 47\ 51$

$\tau=\ 9\ 37\ 32$

Error of watch $=\ -10\ 19$

---

* The process for the determination of M from u_1 is fully explained in the 'Nautical Almanac' for every year, in that for 1857 at pages 536, 537.

It will be remarked, that amongst the particulars at the head of this example we have given the latitude 47° 11′ N. This element, however, is not required to any degree of exactness, inasmuch as we employ it only in obtaining from the Tables the correction to the equatorial horizontal parallax of the moon.

Example 2. Method 2.

Resuming the observation of Example 1, and supposing that instead of the quantities ζ_2 and z_2 being given we have only

$$\text{latitude} = 47^\circ\ 11'\ \text{N and } \phi = 42^\circ\ 49',$$

and local sidereal time at $\tau = 13^h\ 42^m\ 5^s$,

the calculation will be as follows:—

a = moon's geocentric RA at $t = 12^h\ 27^m\ 36^s$ $\quad h = 1^h\ 14^m\ 29^s = 18^\circ\ 37'$
π = moon's geocentric NPD at $t = 93^\circ\ 20'$
A = RA of α Leonis $= 10^h\ 0^m\ 46^s$ $\quad$ H = 3 41 19 = 55 20
Π = NPD of α Leonis $= 77^\circ\ 20'$*

log tan ϕ = 9·96687	log cos ϕ = 9·86542		
log cos h = 9·97666	log cos θ = 9·87589		
log tan θ = 9·94353	Difference = 9·98953		
θ = 41° 17′	log cos $(\pi - \theta)$ = 9·78886	log p	= 3·51348
π = 93 20	log cos ζ_1 = 9·77839	log sin ζ_1	= 9·90296
$\pi - \theta$ = 52 3	ζ_1 = 53° 6′ 0″*	Sum	= 3·41644
	p' = 43 54	Nat. No.	= 2608·″8
	Sum = 53 49 54	(p)	= 25·0
	Refraction = −1 18	p'	= 2634″ = 43′ 54″
	ζ_2 = 53 48 36		

log tan ϕ = 9·96687	log cos ϕ = 9·86542		
log cos H = 9·75496	log cos θ = 9·94678		
log tan θ = 9·72183	Difference = 9·91864		
θ = 27° 47′	log cos $(\Pi - \theta)$ = 9·81210		
Π = 77 20	log cos z_1 = 9·73074	z_1	= 57° 27′ 0″
Π − θ = 49 33		Refraction	= −1 29
		z_2	= 57 25 31

* It would be useless in nautical practice to take these quantities from the Tables to a greater degree of accuracy than we have here done.

And computing with these elements in the same manner as before, we shall find $u_1 = 39° \; 49' \; 3''$, a result differing $1''$ from the former.

Example 3. Method 3.

Treating the same observation again, we shall have

$$\phi' = 43° \; 0' \qquad \log \rho = 9{\cdot}99920^*$$
$$\Delta\alpha = +26''{\cdot}4 \qquad \Delta\pi = +14''{\cdot}2,$$

and the remaining quantities will be as before, except that p will now represent the moon's equatorial horizontal parallax $= 54' \; 28''$.

$\log \tan \phi' = 9{\cdot}96966$	$\log \cos \phi' = 9{\cdot}86413$	$\log \sin \zeta = 9{\cdot}90199$	
$\log \cos h = 9{\cdot}97666$	$\log \cos \theta = 9{\cdot}87467$	$\log p = 3{\cdot}51428$	
$\log \tan \theta = 9{\cdot}94632$	Difference $= 9{\cdot}98946$	$\log \rho = 9{\cdot}99920$	
$\theta = 41° \; 28'$	$\log \cos(\pi - \theta) = 9{\cdot}79063$	Sum $= 3{\cdot}41547$	$= \log \; 43' \; 23''$
$\pi = 93 \; 20$	$\log \cos \zeta = 9{\cdot}78009$	$\zeta = 52° \; 56' \; 0''$	$(p) = +25$
$\pi - \theta = 51 \; 52$		$p' \; 43 \; 48$	
		$\zeta' = 53 \; 39 \; 48$	$\log \sin = 9{\cdot}90609$

$\log \tan \phi = 9{\cdot}96687$	$\log \cos \phi = 9{\cdot}86542$	From line 11 of formula, the effect of refraction upon the radius being inappreciable	$\log r = 2{\cdot}95061$
$\log \cos h = 9{\cdot}97666$	$\log \cos \theta = 9{\cdot}87589$		Sum $= 2{\cdot}85670$
$\log \tan \theta = 9{\cdot}94353$	Difference $= 9{\cdot}98953$		$\log \sin \zeta = 9{\cdot}90199$
$\theta = 41° \; 17'$	$\log \cos(\pi - \theta) = 9{\cdot}78886$		$\log (r) = 2{\cdot}95471$
$\pi = 93 \; 20$	$\log \cos(\zeta_1 - p') = 9{\cdot}77839$	$\zeta_1 - p' = 53° \; 6' \; 0''$	
$\pi - \theta = 52 \; 3$		$p' = 43 \; 48$	
		$\zeta_1 = 53 \; 49 \; 48$	
		Refraction $= 1 \; 18$	
		$\zeta_2 = 53 \; 48 \; 30$	

$\log \tan \phi' = 9{\cdot}96966$	$\log \cos \phi' = 9{\cdot}86413$
$\log \cos H = 9{\cdot}75496$	$\log \cos \theta = 9{\cdot}94617$
$\log \tan \theta = 9{\cdot}72462$	Difference $= 9{\cdot}91796$
$\theta = 27° \; 57'$	$\log \cos(\Pi - \theta) = 9{\cdot}81358$
$\Pi = 77 \; 20$	$\log \cos z = 9{\cdot}73154$ or $z = 57° \; 23'$
$\Pi - \theta = 49 \; 23$	

* Values of $\phi' - \phi$ and $\log \rho$ may be found in most collections of Tables, or they may be computed by the formula of the Note at the bottom of page 79.

And as z_1 will be the same as in the last example, we shall have

		z_1	$=57^\circ\ 27'\ 0''$		
		z_2	$=57\ 25\ 31$		
Ω	$=39^\circ\ 40'\ 15''$	$\frac{1}{2}(\zeta_1+z_1)$	$=55\ 38\ 24$	log sin	$=9{\cdot}9167211$
(r)	$=\ \ 15\ \ 1$			Double	$=9{\cdot}8334422$
u_2	$=39\ 55\ 16$	log sin ζ_1	$=9{\cdot}9070185$		
ζ_2	$=53\ 48\ 30$			log sin	$=9{\cdot}9068984$
z_2	$=57\ 25\ 31$			log sin	$=9{\cdot}9256678$
Sum	$=151\ \ 9\ 17$	log sin z_1	$=9{\cdot}9257875$		
θ_2	$=75\ 34\ 38{\cdot}5$	log sin	$=9{\cdot}9860928$		
θ_2-u_2	$=35\ 39\ 22{\cdot}5$	log sin	$=9{\cdot}7656097$		
		Sums	$9{\cdot}5845085$		$9{\cdot}6660084$
			$9{\cdot}6660084$		

Difference = log sin² θ_1 $=9{\cdot}9185001$ log sin $\frac{1}{2}(\zeta_1+z_1)=9{\cdot}9167211$

log sin θ_1 $=9{\cdot}9592501$ log cos θ_1 $=9{\cdot}6166298$

Sum = log sin $\frac{1}{2}u_1$ $=9{\cdot}5333509$

$\frac{1}{2}u_1$	$=19^\circ\ 57'\ 59''$		
u_1	$=39\ 55\ 58$		
ζ'	$=53\ 39\ 48$	ζ	$=52^\circ\ 56'\ 0''$
z	$=57\ 23\ \ 0$		$57\ 23\ \ 0$
Sum	$=150\ 58\ 46$	$\frac{1}{2}(\zeta+z)$	$=55\ \ 9\ 30$
θ'	$=75\ 29\ 23$	log sin	$=9{\cdot}9859214$
$\theta'-u_1$	$=35\ 33\ 25$	log sin	$=9{\cdot}7645585$
		log sin ζ	$=9{\cdot}9019674$
		Sum	$=9{\cdot}6524473$
		log sin ζ'	$=9{\cdot}9060921$
		Difference	$=9{\cdot}7463552$
		Half	$=9{\cdot}8731776$

log sin $\frac{1}{2}(\zeta+z)=9{\cdot}9142024$ ———— $9{\cdot}9142024$

Difference = log sin $\theta=9{\cdot}9589752$ log cos $\theta=9{\cdot}6179565$

log sin $\frac{1}{2}u=9{\cdot}5321589$

$\frac{1}{2}u=19^\circ\ 54'\ 33''{\cdot}6$

$u=39\ 49\ \ 7$

whence, proceeding as before;

$\Delta u\ =+\ 0''{\cdot}4969$

$A-\alpha=-2^h\ 26^m\ 50^s=-36^\circ\ 43'$

$M=9^h\ 47^m\ 57^s$

$t=9\ \ 48\ \ \ 0$

$n=\ \ \ \ \ \ -3$

(For the remainder of the calculation the angles to minutes and four places of decimals in the logarithms will suffice.)

log cos ϕ'	=9·8641	log sin ϕ'	=9·8338		9·8338
log sin π	=9·9993	log − cos π	=8·7645	log sin π	=9·9993
Sum	=9·8634	log cos h	=9·9767	log sin h	=9·5042
Nat. No.	= ·7301	Sum	=8·5750	log $\Delta\alpha$	=1·4216
	−·0376—	Nat. No.	− ·0376	Sum	=0·7589=log 5″·740
Differ.	=+·7677	log	=9·8852		
		log $\Delta\pi$	=1·1523		
		Sum	=1·0375=log 10″·903		
			5 ·740		
		Difference=	5 ·163	log	=0·7129
				log sin ζ	=9·9020
				log $\delta\zeta$	=0·8109

(The computation of this quantity is never necessary when we know the approximate Greenwich mean time within a minute, or more in some cases. But in an example it is of course expedient to include it; and as in consequence of accident the navigator may not know his time within a few minutes, he may in some instances find it convenient to include in his result the term which involves $\delta\zeta$, and thus save himself the trouble of first treating his observation by the approximate method No. 1 as a preliminary to its more accurate treatment.)

log sin π	=9·9993	log − cos π	=8·7645	log sin π	=9·9993
log sin Π	=9·9893		9·9893	log cos Π	=9·3410
log − sin (A − α)	=9·7766	log cos (A − α)	=9·9040	Sum	=9·3403
Sum	=9·7652	Sum	=8·6578	Nat. No.	= −·219
	9·5027	Nat. No.	= −·045	——	−·045
Diff. = log − s_1	=0·2625	log Δu	=9·6963	Difference=	−·264 log=9·4216
s_1	= −1·83	log sin u	=9·8064		9·5027
		Sum	=9·5027	Difference = log − s_2	=9·9189
				s_2	= −0·83

log sin π . cos Π	=9·3403	log − cos π . sin Π	=8·7538		
log cos (A − α)	=9·9039	Nat. No.	= −·057		
Sum	=9·2442	——Nat. No.	= +·176		
		Difference=	+·233	log	=9·3674
				log Δu . sin u	=9·5027
				log s_3	=9·8647 s_3 = +0·73

log cos u	$=9{\cdot}8854$		log Δu	$=9{\cdot}6963$
log cos ζ'	$=9{\cdot}7727$		log coefficient of δr	$=0{\cdot}3037$
Sum	$=9{\cdot}6581=$ log	$0{\cdot}455$	Nat. No.	$=2{\cdot}01$
	Nat. cos $z=$	$0{\cdot}539$		
	Difference $=-$	$\cdot 084$	log $=8{\cdot}9243$	

log Δu . sin u	$=9{\cdot}5027$	
log $-s_4$	$=9{\cdot}4216$	$s_4=-0{\cdot}26$
log $-n$	$=0{\cdot}4771$	
log p	$=3{\cdot}5143$	
log cot ζ'	$=9{\cdot}8666$	
log sin $1''$	$=4{\cdot}6856$	
log $\delta\zeta$	$=0{\cdot}8109$	
Sum	$=8{\cdot}7761$	
log 60	$=1{\cdot}7782$	
Difference	$=6{\cdot}9979=$ log $0^s{\cdot}000995$	

And as this last quantity is inappreciable, we shall have

$$\text{G.M.T.} \quad = \overset{h}{9}\ \overset{m}{47}\ \overset{s}{57} - 1{\cdot}83 \times (\delta\alpha - \delta A) - 0{\cdot}83 \times \delta\pi + 0{\cdot}73 \times \delta\Pi - 0{\cdot}26 \times \delta p + 2{\cdot}01 \times \delta r$$

$$\tau \quad = 9\ 37\ 32$$

$$\text{Error of Watch} = -10\ 25 + 1{\cdot}83 \times (\delta\alpha - \delta A) + 0{\cdot}83 \times \delta\pi - 0{\cdot}73 \times \delta\Pi + 0{\cdot}26 \times \delta p - 2{\cdot}01 \times \delta r.$$

Converting 9 47 57 mean into sidereal time,

$$S \quad = 14\ 29\ 13$$

$$T \quad = 13\ 42\ \ 5$$

$$\text{Longitude} \quad = +47\ \ 8 - 1{\cdot}83 \times (\delta\alpha - \delta A) - 0{\cdot}83 \times \delta\pi + 0{\cdot}73 \times \delta\Pi - 0{\cdot}26 \times \delta p + 2{\cdot}01 \times \delta r.$$

The chronometer error, then, derived from this observation, to be adopted whilst at sea, is $-10^m\ 25^s$, differing, it will be remarked, from the result of the reduction by the first method by 6 seconds, and from that by the second method by about 8 seconds. But the differences between the results of this and the other methods will sometimes be considerably greater, as in the instance of a second observation on the same night of another object with the moon, in which they amounted to 20 and 19 seconds respectively; and it is to be remembered that as these differences depend upon the magnitude of $\phi'-\phi$, they will in general be greater in middle than in either very high or very low latitudes, the moon in the latter case not being very near the zenith.

As regards the errors of the Tables, these, as already remarked, are to be obtained at some subsequent date from the results of observations made at fixed observatories. In the present instance, according to those made at Greenwich, $\delta\alpha = -0^s{\cdot}40 = -6''$ and $\delta\pi = -8''$; but the values of these quantities, as well as of δp and δr, will eventually be ascertained with all the accuracy that can result from comparison of similar observations made at different places in considerable numbers. Our present object being simply to furnish an example, it will be sufficient to assume the above to be the values of $\delta\alpha$ and $\delta\pi$, and that δp and δr each $=0$; and as the star is well known, we may suppose likewise, without risk of serious error, that δA and $\delta \Pi$ are inappreciable. Hence the results,

$$\text{Error of Watch} = -10^m\ 25^s - 1{\cdot}83 \times 6 - 0{\cdot}83 \times 8 = -10^m\ 43^s,$$

$$\text{Longitude} = +47\quad 8 + 1{\cdot}83 \times 6 + 0{\cdot}83 \times 8 = +47\quad 26$$

$$= 11^\circ\ 51'\ 30''\ \text{W.},$$

are probably not very far from the truth. The general result, however, being recorded, we shall at some future date be able to determine more accurately what was the error of the chronometer on the 1st of June at $9^h\ 48^m$ Greenwich mean time, and from this and the results of subsequent observations similarly reduced ascertain its rate during each interval. We shall thus be prepared to amend the results of daily determination of longitude, and, should we have any particular reason for wishing to know the position of the ship at a given time, shall have the means of obtaining it with a degree of accuracy unattainable in general by any other course of proceeding. It should be understood, however, that it is always desirable that a second object be observed with the moon, situated, if possible, upon the side opposite to that of the first; but even one observation, made with a Sextant for which the operator has obtained a table of corrections, and reduced according to the most accurate method, is of far greater value than the mean of many made in the ordinary way, and reduced either according to the approximate method, or by means of the Tables in common use among navigators.

6. It remains only, in conclusion, to make a few general remarks on the Sextant, and the mode of handling it.

1°. A double-frame Sextant of from 7 to 8 inches radius will be found very convenient as regards size and weight*; and gold

* There is probably less liability to flexure in the double-frame than in the solid-frame Sextant; and in a well-constructed instrument of the former description, the maximum effect has been found not to exceed 3″. In instruments of the other description, the amount of error seems to depend to a great extent upon the mode in which the handle is connected with the frame; and it is to be feared that in many instances it is even greater than in the Sextant which furnished the examples in Part I., in which the maxi-

is perhaps the best material for receiving the divisions. The instrument should be furnished with a good supply of eyepieces of various power,—and with a supplementary handle to screw into the ordinary handle at right angles to it, for convenience in observing angles between objects in planes nearly horizontal.

2°. For every necessary observation at sea, except those of lunar distance and altitudes of stars of inferior brilliancy, a pocket-sextant will be found sufficient and more convenient than the larger instrument. The latter should be reserved for the more delicate operations, and not exposed to spray except in cases of emergency. Moisture condensing from the atmosphere will generally evaporate from the surfaces of the glasses without involving the necessity of wiping them, an operation which, as it is very likely to disturb their positions and sure to injure their polish, should be performed as seldom as possible.

3°. The Sextant should never be held except by the handle; but in cases in which the frame interferes with this mode of handling in removing it from or replacing it in the box, that part only of the frame should be touched which is situated immediately about the centre of gravity.

4°. A good light is indispensable for reading the vernier by night. This may be easily provided on land in a sheltered position convenient to the observer; and at sea, a signal lantern with a good reflector, if lashed in a convenient place with the axis of the reflector or lens inclined a little downward, will answer every purpose. A light of the same description, directed upon the surface of the horizon-glass by an attendant standing behind the observer, may be had recourse to when the light in the sky is not sufficient to render the wires of the telescope distinctly visible.

5°. In observing angles between objects not situated in the same vertical plane, the operator should be seated in a chair of the description mentioned in the last section.

6°. Altitudes of the sun at sea may be observed with any power under 9, depending upon the state of the weather; but for those of the moon and stars, a power not exceeding 5 should be employed; and the horizon not being very well seen, a power of $2\frac{1}{2}$ to $3\frac{1}{2}$ with a large field will be best. Altitudes of stars can be most satisfactorily observed when the objects first become visible in the twilight, or immediately before they disappear in the dawn; but should it be necessary to make observations of this description by moonlight, the operator should be particularly

mum was $11''\cdot4$. From some experiments recently tried, however, there is reason to believe that an improved construction will reduce the amount materially.

careful that he does not take the shadow of a distant cloud for the line of the horizon, the two being, under such circumstances, extremely difficult to distinguish*.

7°. When observing an angle between two objects in relative motion, it will be better, should the weather be very fine, to set the index a little in advance and to await the instant of coincidence of images. This is always practicable on shore; but as at sea an inopportune movement of the ship may disturb the observer at the critical time, it is generally necessary to trust to manipulation of the tangent-screw to effect the coincidence at the moment at which it can be most safely determined, that is, when the images can be retained in the middle of the field during an interval sufficient for the performance of the operation.

8°. In observing lunar distances, the operator may direct the telescope to either of the two objects, preference depending upon the position in which it will be necessary to hold the instrument, which should invariably be that in which it can be most conveniently managed. If the glasses are in order, the reflected image of a star of the third magnitude will be sufficiently distinct, and may readily be brought into contact with the moon's limb, the direct image of the latter being darkened by the proper shade.

9°. In reading the vernier, the vision should, if possible, be made to embrace four divisions—those most nearly coinciding with divisions on the arc, and the two outside them likewise,—as comparison of the relative departure of the latter from the neighbouring divisions will materially assist the operator in estimating the second to be adopted in the angles.

* With respect to the eye-pieces, these should be made to adapt both to the long and short telescope, by which means we obtain a greater variety of power; and the inverting eye-piece of lowest power applied to the short telescope will be found to answer well for observation of altitude of stars.

It is to be remarked that the error of collimation not infrequently varies considerably with the eye-piece employed,—a circumstance which suggests the propriety of having the diaphragm fixed within the tube of the telescope, or, each eye-piece carrying its own diaphragm as now, the application of means for adjustment independent of that which takes effect upon the whole length of the tube.

Table of values of $(p) = \frac{1}{2}p^2 . \sin 2\zeta . \sin 1'' + \frac{1}{3}p^3 . \sin 3\zeta . \sin^2 1''$ for various values of p and ζ.

ζ	Values of p.							
	54′	55′	56′	57′	58′	59′	60′	61′
0	″	″	″	″	″	″	″	″
1	0·9	0·9	1·0	1·0	1·0	1·1	1·1	1·2
2	1·8	1·9	1·9	2·0	2·1	2·2	2·2	2·3
3	2·7	2·8	2·9	3·0	3·1	3·2	3·3	3·5
4	3·6	3·7	3·9	4·0	4·2	4·3	4·5	4·6
5	4·5	4·7	4·8	5·0	5·2	5·4	5·6	5·7
6	5·4	5·6	5·8	6·0	6·2	6·4	6·6	6·9
7	6·3	6·5	6·8	7·0	7·2	7·5	7·7	8·0
8	7·1	7·4	7·7	8·0	8·2	8·5	8·8	9·1
9	8·0	8·3	8·6	8·9	9·2	9·6	9·9	10·2
10	8·8	9·2	9·5	9·9	10·2	10·6	10·9	11·3
11	9·7	10·0	10·4	10·8	11·2	11·6	12·0	12·4
12	10·5	10·9	11·3	11·7	12·1	12·6	13·0	13·4
13	11·3	11·8	12·2	12·6	13·1	13·5	14·0	14·5
14	12·1	12·6	13·1	13·5	14·0	14·5	15·0	15·5
15	12·9	13·4	13·9	14·4	14·9	15·4	16·0	16·5
16	13·7	14·2	14·7	15·3	15·8	16·4	16·9	17·5
17	14·4	15·0	15·5	16·1	16·7	17·3	17·9	18·5
18	15·2	15·8	16·3	16·9	17·5	18·1	18·8	19·4
19	15·9	16·5	17·1	17·7	18·4	19·0	19·7	20·3
20	16·6	17·2	17·9	18·5	19·2	19·8	20·5	21·2
21	17·3	17·9	18·6	19·3	19·9	20·6	21·4	22·1
22	17·9	18·6	19·3	20·0	20·7	21·4	22·2	22·9
23	18·6	19·3	20·0	20·7	21·4	22·2	22·9	23·7
24	19·2	19·9	20·6	21·4	22·1	22·9	23·7	24·5
25	19·8	20·5	21·3	22·0	22·2	23·6	24·4	25·2
26	20·3	21·1	21·9	22·7	23·5	24·3	25·1	26·0
27	20·9	21·6	22·4	23·2	24·1	24·9	25·8	26·7
28	21·4	22·2	23·0	23·8	24·7	25·5	26·4	27·3
29	21·9	22·7	23·5	24·4	25·2	26·1	27·0	27·9
30	22·3	23·1	24·0	24·9	25·8	26·7	27·6	28·5
31	22·7	23·6	24·5	25·3	26·3	27·2	28·1	29·1
32	23·1	24·0	24·9	25·8	26·7	27·7	28·6	29·6
33	23·5	24·4	25·3	26·2	27·2	28·1	29·1	30·1
34	23·9	24·8	25·7	26·6	27·5	28·5	29·5	30·5
35	24·2	25·1	26·0	26·9	27·9	28·9	29·9	30·9
36	24·5	25·4	26·3	27·3	28·2	29·2	30·2	31·3
37	24·7	25·6	26·6	27·5	28·5	29·5	30·5	31·6
38	24·9	25·9	26·8	27·8	28·8	29·8	30·8	31·9
39	25·1	26·1	27·0	28·0	29·0	30·0	31·1	32·1
40	25·3	26·2	27·2	28·2	29·2	30·2	31·3	32·3

TABLE (continued).

ζ	Values of p.							
	54′	55′	56′	57′	58′	59′	60′	61′
°	″	″	″	″	″	″	″	″
41	25·4	26·4	27·4	28·3	29·4	30·4	31·4	32·5
42	25·5	26·5	27·5	28·5	29·5	30·5	31·5	32·6
43	25·6	26·6	27·5	28·5	29·6	30·6	31·6	32·7
44	25·6	26·6	27·6	28·6	29·6	30·6	31·7	32·7
45	25·6	26·6	27·6	28·6	29·6	30·6	31·7	32·7
46	25·6	26·6	27·6	28·6	29·6	30·6	31·6	32·7
47	25·6	26·5	27·5	28·5	29·5	30·5	31·6	32·6
48	25·5	26·4	27·4	28·4	29·4	30·4	31·5	32·5
49	25·4	26·3	27·3	28·3	29·3	30·3	31·3	32·4
50	25·3	26·1	27·1	28·1	29·1	30·1	31·1	32·2
51	25·0	26·0	26·9	27·9	28·9	29·9	30·9	31·9
52	24·8	25·7	26·7	27·6	28·6	29·6	30·6	31·7
53	24·6	25·5	26·4	27·4	28·3	29·3	30·3	31·4
54	24·3	25·2	26·1	27·1	28·1	29·0	30·0	31·0
55	24·0	24·9	25·8	26·7	27·7	28·6	29·6	30·6
56	23·7	24·5	25·4	26·4	27·3	28·2	29·2	30·2
57	23·3	24·2	25·1	26·0	26·9	27·8	28·8	29·7
58	22·9	23·8	24·6	25·5	26·4	27·3	28·3	29·2
59	22·5	23·3	24·2	25·1	25·9	26·8	27·8	28·7
60	22·0	22·9	23·7	24·6	25·4	26·3	27·2	28·1
61	21·6	22·4	23·2	24·0	24·9	25·7	26·6	27·5
62	21·1	21·9	22·7	23·5	24·3	25·1	26·0	26·9
63	20·6	21·3	22·1	22·9	23·7	24·5	25·4	26·2
64	20·0	20·7	21·5	22·3	23·1	23·9	24·7	25·5
65	19·4	20·2	20·9	21·6	22·4	23·2	24·0	24·8
66	18·8	19·5	20·3	21·0	21·7	22·5	23·2	24·0
67	18·2	18·9	19·6	20·3	21·0	21·7	22·5	23·2
68	17·6	18·2	18·9	19·6	20·3	21·0	21·7	22·4
69	16·9	17·5	18·2	18·8	19·5	20·2	20·9	21·6
70	16·2	16·8	17·4	18·1	18·7	19·4	20·0	20·7
71	15·5	16·1	16·7	17·3	17·9	18·5	19·1	19·8
72	14·8	15·4	15·9	16·5	17·1	17·7	18·3	18·9
73	14·1	14·6	15·1	15·7	16·2	16·8	17·3	17·9
74	13·3	13·8	14·3	14·8	15·3	15·9	16·4	17·0
75	12·5	13·0	13·5	14·0	14·5	14·9	15·5	16·0
76	11·8	12·2	12·6	13·1	13·5	14·0	14·5	15·0
77	11·0	11·4	11·8	12·2	12·6	13·1	13·5	13·9
78	10·1	10·5	10·9	11·3	11·7	12·1	12·5	12·9
79	9·3	9·7	10·0	10·4	10·7	11·1	11·5	11·8
80	8·5	8·8	9·1	9·4	9·8	10·1	10·4	10·8
81	7·6	7·9	8·2	8·5	8·8	9·0	9·4	9·7

If u_0, u_1, u_2 be the Moon's RA or PD at hours H_0, H_1, H_2, the RA or PD at H_1 hours $+ n$ minutes will be $u_1 + B_1 . \Delta u_0 + B_2 . \Delta^2 u_0$; and variation between $\overset{h}{H_1} + \overset{m}{n}$ and $\overset{h}{H_1} + (\overset{m}{n+1}) = \frac{1}{60} . \Delta u_0 + b . \Delta^2 u_0$.

n.	log B_1.	log B_2.	log *b*.	*n*.	log B_1.	log B_2.	log *b*.
1	8·22185	7·92800	7·94201	31	9·71321	9·59307	8·23257
2	8·52288	8·23609	7·95558	32	9·72700	9·61161	8·23958
3	8·69897	8·41913	7·96874	33	9·74036	9·62966	8·24647
4	8·82391	8·55091	7·98152	34	9·75333	9·64727	8·25326
5	8·92082	8·65455	7·99393	35	9·76592	9·66446	8·25994
6	9·00000	8·74036	8·00599	36	9·77815	9·68124	8·26652
7	9·06695	8·81384	8·01773	37	9·79005	9·69764	8·27300
8	9·12494	8·87827	8·02916	38	9·80163	9·71368	8·27939
9	9·17609	8·93576	8·04029	39	9·81291	9·72937	8·28568
10	9·22185	8·98777	8·05116	40	9·82391	9·74473	8·29189
11	9·26324	9·03532	8·06175	41	9·83463	9·75977	8·29800
12	9·30103	9·07918	8·07209	42	9·84510	9·77452	8·30404
13	9·33579	9·11993	8·08219	43	9·85532	9·78897	8·30998
14	9·36798	9·15803	8·09206	44	9·86530	9·80315	8·31585
15	9·39794	9·19382	8·10171	45	9·87506	9·81707	8·32164
16	9·42597	9·22760	8·11115	46	9·88461	9·83073	8·32736
17	9·45230	9·25961	8·12039	47	9·89395	9·84415	8·33300
18	9·47712	9·29003	8·12944	48	9·90309	9·85733	8·33857
19	9·50060	9·31905	8·13830	49	9·91204	9·87029	8·34406
20	9·52288	9·34679	8·14699	50	9·92082	9·88303	8·34949
21	9·54407	9·37337	8·15550	51	9·92942	9·89556	8·35486
22	9·56427	9·39890	8·16386	52	9·93785	9·90789	8·36015
23	9·58358	9·42347	8·17205	53	9·94612	9·92002	8·36538
24	9·60206	9·44716	8·18009	54	9·95424	9·93197	8·37055
25	9·61979	9·47003	8·18799	55	9·96221	9·94373	8·37567
26	9·63682	9·49214	8·19575	56	9·97004	9·95531	8·38071
27	9·65321	9·51355	8·20337	57	9·97772	9·96673	8·38571
28	9·66901	9·53431	8·21085	58	9·98528	9·97798	8·39064
29	9·68425	9·55446	8·21821	59	9·99270	9·98907	8·39552
30	9·69897	9·57403	8·22545				

THE END.

Printed by Taylor and Francis, Red Lion Court, Fleet Street.