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SECOND EDITION

The Principles of Surveying

by J. CLENDINNING
O.B.E., B.Sc.(Eng.)
Surveyor-General, Gold Coast, 1926–1938

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PREFACE

This is the second of two volumes in which an attempt has been made to give, in as limited a compass as possible, a good grounding in the elements of those parts of the theory and practice of plane surveying that are used by engineers in ordinary civil engineering work and are required by students taking the examination in surveying for the Associate Membership of the Institution of Civil Engineers. It deals with the main principles and practice of surveying, while the first volume deals with the principles and use of the main instruments commonly used in civil engineering practice.

In his ordinary work, the civil engineer is not concerned with geodesy or the higher branches of surveying, and accordingly the field covered in these two books is confined to the elements of plane surveying. For this reason, they are to be regarded as forming a text-book of an elementary or intermediate standard which can, if necessary, be used as an introduction to books of a more advanced kind.

As the ground covered is elementary, and is all part of generally accepted and well-established practice, all that is claimed for these two volumes is that the material has been carefully selected for the object in view and every effort has been made to compress the treatment into the minimum space consistent with a sound exposition of rudimentary principles and practice. At the same time, special attention has been paid to matters such as the reduction of angular observations, the calculation of bearings from the observed angles, and computations relating to rectangular co-ordinates, which, although very simple in themselves, sometimes cause trouble to beginners; and every effort has therefore been made to make the treatment of these subjects as clear and easy as possible. Many examples are worked out in the text and most chapters end with a series of questions, many of them taken from examination papers set by the Institution of Civil Engineers, which will enable the student to test and consolidate his knowledge.

Special thanks are due to Professor C. A. Hart, D.Sc., of University College, London (now Vice-Chancellor of the University of Roorkee,

India), for kindly providing me with the field notes and plan of the survey used as an example of a simple chain survey in Chapter III; to the Institution of Civil Engineers for permission to use questions taken from the examination papers for Associate Membership; to Messrs. Hilger and Watts, Ltd., for permitting the use of the illustration of a station pointer on p. 230, which is taken from a catalogue of Messrs. E. R. Watts & Son, Ltd.; and to Messrs. W. F. Stanley & Co., Ltd., for permission to use the illustration, taken from their catalogue, of an Amsler planimeter which appears on p. 244.

J. CLENDINNING.

ANGMERING-ON-SEA,
SUSSEX.
1st June, 1950.

PREFACE TO THE SECOND EDITION

In this edition, apart from some minor amendments and additions in the main text, principally in the Sections on Air Survey, and the addition of some supplementary notes in the Appendix on matters connected with Plane Surveying which it seemed desirable to include, the principal change has been the addition of six chapters on Field Astronomy to cover that part of the syllabus of the examination in surveying of the Institution of Civil Engineers which includes Field Astronomy, as this subject was not included in the syllabus when the First Edition was written. Consequently, the book has now been divided into two parts, Part I dealing with Plane Surveying and Part II with Field Astronomy.

In Field Astronomy the syllabus only calls for a knowledge of " Field Astronomy as required for the determination of azimuth " and consequently it would appear that candidates for the examination are not required to know anything about observations for latitude and longitude. However, although azimuth may be the observation with which the ordinary civil engineer is mostly concerned, it does sometimes happen, particulary in unmapped country, that he may have to determine his own local time, latitude, and longitude, as a knowledge of approximate values of certain of these elements is required for computing the results in some of the methods used in finding azimuth. Also, any treatise on Field Astronomy would hardly be complete without some account of observations for latitude and longitude. Accordingly, I have added short descriptions of the simpler methods available for determining these quantities but candidates for the A.M.I.C.E. examination can use their own discretion about omitting the sections on latitude and longitude observations at the end of Chapter XVIII, in which these methods are described.

In the worked out examples in Field Astronomy the data used have been taken from the 1959 edition of the *Star Almanac for Land Surveyors*, which was the latest edition available when the chapters in question were written. There are small changes in these data from year to year so that the values given in later editions of the *Almanac* will not be the same as those used in the examples. However, the student should be able, if he so desires, to re-work the examples using the data in the edition of the *Star Almanac* which happens to refer to the particular year with which he is concerned, and he is strongly advised to do this.

In conclusion, I should like to express my thanks to the Institution of Civil Engineers for permission to reproduce some of the questions set in recent papers at the examinations for Associate Membership; to the Controller of H.M. Stationery Office, for permission to use data taken from the *Star Almanac for Land Surveyors*; and to Brigadier K. M. Papworth, O.B.E., M.C., and Colonel D. R. Crone, C.I.E., O.B.E., for reading through the greater part of the typescript of Part II and for their very helpful criticism and suggestions, and also to the former for checking most of the worked out examples in Chapters XVII and XVIII.

J. CLENDINNING.

ANGMERING-ON-SEA,
SUSSEX.
July, 1959.

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PART I

Plane Surveying

CHAPTER I

INTRODUCTION

Surveying as practised by civil engineers usually has one or other of two objects in view. The first is to produce a plan or map which can be used as a basis for planning some kind of works or development. The second is to lay out pegs or marks on the ground in such a way that a foreman or workman can construct on the ground, correctly and efficiently and without waste, the structure or works which the engineer has shown on his plans.

There are several branches or divisions of surveying and these are:

1. Geodetic surveying.
2. Topographical surveying.
3. Cadastral surveying.
4. Engineering surveying.
5. Mining surveying.
6. Hydrographical surveying.

Geodetic surveying may be taken to consist of surveys covering such a large area that the spherical or spheroidal shape of the earth has to be taken into account if serious error is not to be incurred. It is the most accurate of all forms of survey, and its main object is generally to provide points, very accurately fixed, which can be used as fixed points whose positions and elevations can be accepted without question when " tying " other surveys of lesser accuracy to them. Sometimes, however, geodetic surveys are executed for purely scientific purposes, the main one being to determine the exact size and shape of the earth.

Topographical surveys are surveys made for producing maps or plans showing the main physical features on the ground, i.e. towns, villages, roads, railways, rivers, lakes, woods and forests, etc., and also (by means of contours or form lines) the " vertical relief ", or heights, hollows, hills and mountains. If the scale of the resulting map or plan is about 1/10,000 or smaller (i.e. one unit of measurement on the plan represents 10,000 similar units on the ground), it is generally called a *map*. If the scale is greater than 1/10,000, it is usually called a *plan*. The well-known one-inch map of Great Britain (1/63,360) is a topographical map, and the old 1/2500 and the new 1/1250 sheets are plans.

Cadastral surveys are surveys made for producing plans showing property boundaries or plans on which areas necessary for the assessment of property or land taxes may be computed.

Engineering surveys are surveys made specifically for engineering purposes.

Mining surveys are surveys of mining works and workings, surface and underground, or other surveys made specifically for mining purposes.

Hydrographical surveys are surveys of water areas, particularly the sea, made for the purpose of showing the depth of the water at different points, the nature of the bottom, currents, the shore line or lines where the edge of the water merges with dry land or earth, lighthouses, beacons, buoys, etc., and everything of importance to navigation or needed in connection with engineering operations involving work under water. Admiralty charts are hydrographic plans.

When surveys cover only a limited area, they can be regarded as having been made on a plane surface, so that ordinary theorems in plane geometry and plane trigonometry are applicable, but when very extensive areas, covering, say, 1000 square miles or more, are involved, it is necessary to take the spheroidal shape of the earth into account. Nearly all surveys made for purely engineering purposes cover only a limited area and can therefore be treated as plane surveys. Consequently, in this book we shall concern ourselves only with the methods of ordinary plane surveying.

1. General Principles of Surveying.

The main principles to be observed in surveying are to "work from the whole to the part", and to use methods which are accurate enough for the object in view but which, since increased accuracy means greater labour and cost, are no more accurate than the necessity of the case demands. These principles are best exemplified in the case of a large national survey.

Here the first thing that is done is the establishment of a number of fairly widely separated points fixed with the most refined apparatus and methods. Next, the wide gaps between these *primary* points are filled in with a number of *secondary* points at much closer intervals than the primary points, and surveyed by methods which are rigorous and accurate, but not so rigorous or so accurate as those used in fixing the positions of the primary points. This still leaves rather wide gaps between fixed points, so a number of *tertiary* points are fixed to fill in the gaps, the positions of these tertiary points not being so accurately surveyed as are those of the secondary and primary points. The result is a network of points, fairly thickly spaced, which can be used by the ordinary surveyor who is engaged in surveying the detail on the ground as fixed points whose positions he can accept and use to control his own work, his work not being nearly so accurate as any that has gone before. In this way, the work has proceeded from the whole to the part, and each stage is of no greater accuracy than is necessary for the purpose for which it was designed.

It is to be noted that, if the final object of the survey is merely to produce a map or plan, the accuracy of the last stage of the work, i.e. the survey of the detail, need only be such that the errors in this stage are too small to be plotted, but the accuracy of the fixings of the original primary, secondary, and tertiary points will need to be greater than this, that of the fixings of the primary points being very much greater. The reason for this is that all survey work, even the most refined, is subject to error, and errors are very quickly propagated and generally very much magnified as the work proceeds. Hence, if the primary work were not of the utmost possible accuracy, and the secondary work only slightly less so, very small errors at the beginning would soon become very large errors as the work was extended over a large area.

In the above example, the primary points control the secondary, the secondary the tertiary, and the tertiary the detail survey. Errors in the primary can lead to large errors in the secondary, and so on.

Much the same principle is observed even in simple surveys. Thus, in a chain survey of a small estate, lines are first run round the perimeter, with a number of clear cross lines between, or else the outer perimeter is surveyed with a theodolite traverse, and lines are then run across the interior. These lines are fairly accurately measured and are the first to be plotted to see that they all fit in properly. Minor chain lines, which may be of lesser accuracy, are then run between the main lines until the area is split up into convenient blocks for the survey of the detail. We shall see later on how this process works in practice.

In none of these cases do we start with the survey of detail and build up from block to block or from detail to tertiary points and then to primary points: in all cases, the points first laid down are the most accurately surveyed and serve as a skeleton on which to hang the later work, all of which is adjusted to them.

2. Methods used in Surveying.

Nearly every operation in surveying is based ultimately on fixing on a horizontal plane the position of one or more points with relation to the position of one or more others, or/and determining the elevation or vertical height of one or more points above a definite horizontal *datum plane*, which is very often taken as Mean Sea Level.

There are four main methods used in fixing the position of a point on the horizontal plane.

1. By triangulation from two points whose positions are already fixed and known.
2. By bearing and distance from a single fixed point.
3. By offset from a chain line.
4. By resection.

In fig. 1.1, A and B are two points whose positions are known. This means that we know (or can compute) the distance between the two points, and the direction of one from the other. C is a point

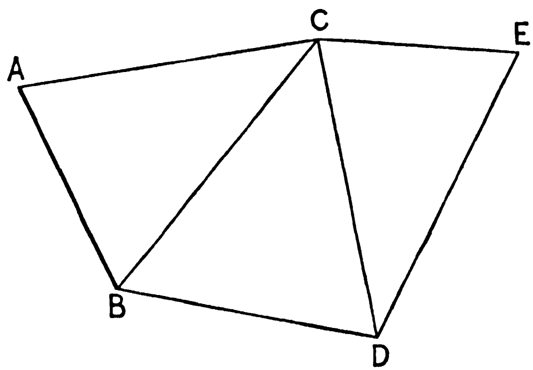


Fig. 1.1

whose position is required. If now (*a*) two of the angles of the triangle ABC are observed, or (*b*) the distances AC and BC are measured, the size and shape of the triangle can be fully determined, either by drawing or by computation, and hence the position of C with relation to both A and B can be found.

When the position of C is fixed, we know the direction and length of the side BC, and hence, from this side, using similar methods to those already employed, we can fix a fourth point D, and after it a fifth point E, and so on. This is the principle of the process known as *triangulation*, which is much used in survey work.

In practical triangulation involving angular observations, the three angles of every triangle are measured wherever possible, as this not only acts as a check, but it also serves to add considerably to accuracy. If the angles BAC and ABC only in fig. 1.1 are observed, the point C is said to be fixed by *intersection*.

In fixing a point by bearing and distance, we measure the distance AB, fig. 1.2, where A is the fixed point and B the point to be fixed, and also measure the bearing or direction of the line AB.

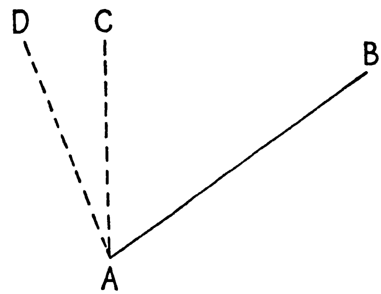


Fig. 1.2

For certain purposes, when very great accuracy is not needed, we can measure bearings or directions directly by means of a magnetic compass. This will give the bearing or direction of the line with reference to a fixed direction known as *magnetic north*, which is shown as the line AC in the figure. For more accurate work, the bearing must be obtained by sextant or theodolite by observing the angle DAB between a fixed point, say D in fig. 1.2, whose bearing or direction from A is known, and the point B. When this angle is known, we can compute the bearing of AB and this, combined with the measured distance, enables us to fix the position of B.

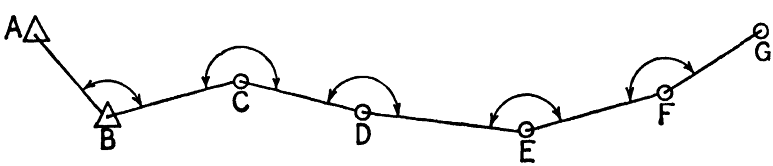


Fig. 1.3

This principle is used in *traversing*, a process also extensively used in surveying. A traverse consists of a series of zigzag lines whose bearings and distances are measured. Thus, in fig. 1.3, starting from the fixed point B we measure the distances BC, CD, DE, EF and FG, and also either the bearings of BC, CD, DE, EF, FG or else the angles ABC, BCD, CDE, DEF, EFG, where in the latter case the bearing or direction of the point A from B is known, and the bearing of BC is obtained by calculation from it and the measured angle ABC. Then, knowing the bearing and distance BC, we can fix the position of C, and after that, knowing the bearing and distance CD, we can fix the position of D, and so on, the bearing of CD, if not observed directly from compass observations, being obtained from the bearing of BC, which we have already found, and the observed angle BCD.

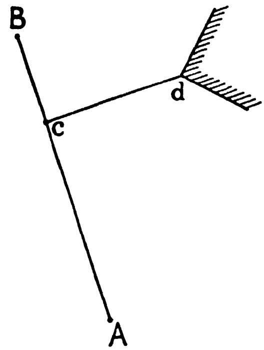


Fig. 1.4

The term *offset* in surveying is applied to a line laid out at right angles to a chain line to fix some point of detail. In fig. 1.4, AB is part of a chain line and **c** is a point on it whose distance from A, the beginning of the line, is noted and recorded. **d** is a point of detail whose position is to be plotted. The point **c** is chosen so that the line **cd** is perpendicular to the line AB, and the distance **cd** is measured. When the line AB is plotted on paper, and the line **cd** laid out the correct distance from the plotted position of **c**, so that **dc** is perpendicular to AB, the position of the point **d** is at once plotted. It will be seen that an offset is really a special case of fixing by bearing and distance.

Offsets are mainly used in the survey of detail in chain surveying and their length is generally limited to something less than 100 ft. Offsets much longer than this are very seldom used, except perhaps for fixing the positions of spot heights in connection with contouring.

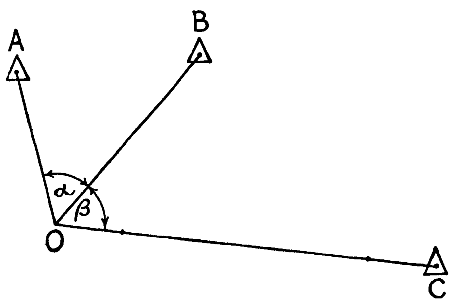


Fig. 1.5

It sometimes happens that the fixed points from which it is desired to fix the position of a new point are inaccessible, or are in positions which are inconvenient to use as observing points, but, provided all four points do not lie on or near a single circle, it is possible to fix a point by angular observations taken at it to three fixed points. Thus, in fig. 1.5, A, B and C are three points whose exact positions are known or are plotted on a plan. Then the position of the point O can be fixed if the angles AOB and BOC are measured. Alternatively, if the work is being done by plane-table, there is a method which, by suitable pointings of the alidade or sight rule in the directions of A, B and C, enables the position of O to be determined graphically in the field. This method of fixing the position of a point by observations to three fixed points is known as *resection*, but it breaks down if a circle drawn through the three fixed points passes through, or near, the point to be fixed.

If prismatic compass fixings only are required, the point O in fig. 1.5 can be fixed from two fixed points only by observing magnetic bearings to them from the point to be fixed. In addition, a fixing can be obtained from two fixed points only if a subsidiary station is chosen suitably placed with regard to the station to be fixed, and angular observations at each of these stations are taken to the other one and to the two fixed points.*

In determining differences of elevation between points, several methods are available. These are:

1. By observing vertical angles between points when the lengths of the lines joining them are known.
2. By ordinary spirit levelling.
3. By readings on a barometer or aneroid.
4. By readings with the hypsometer or boiling-point thermometer.

Of these methods, (1) and (2) are, in general, more accurate than (3) and (4), good spirit levelling being the most accurate of all.

3. Errors in Surveying.

All survey operations are subject to errors of observation, but certain types of error are more serious than others.

The first type of error is a *gross error* or mistake. This means a serious mistake in reading an instrument: for instance, reading 130° instead of 150° when reading the circle of a theodolite, or booking a reading of 80 on a chain when it is really 60. Every care must be taken to avoid making mistakes of this kind, since the results may naturally be very serious.

The second kind of error is a *constant error* which has the same value and sign for every single observation. For example, an index error in a sextant will affect every angle measured with that sextant by the same amount.

Sometimes constant errors cancel out. Suppose the first graduation on a level staff is marked 1 ft. instead of zero. Then every single

* See *The Principles and Use of Surveying Instruments*, pp. 104–5 and 175–7.

sight taken on the staff will be one foot longer than it should be, and hence the apparent reading will be one foot too high. But, since a level is used to measure differences of elevation, and these differences are obtained by subtracting one staff reading from another, the error will cancel out and the true difference of elevation will be obtained. Nevertheless, constant errors are to be avoided as much as possible.

Systematic error is an error which has always the same sign, not necessarily always the same magnitude, at every observation. Thus, a chain may be uniformly stretched so that the error in apparent length of any part of it is proportional to the length. If a line is measured with this chain, the apparent length will be too short by an amount equal to the length of the line multiplied by the amount of the error per unit length.

If we knew the amounts and signs of constant or systematic errors we could allow for them by applying calculated corrections. This is often done, but sometimes, although systematic error is suspected, neither its magnitude nor its sign is known, and consequently no correction is possible.

Accidental errors of observation are the small errors of observation that vary in magnitude *and in sign* with every single observation. Their occurrence depends on the laws of chance and, their magnitudes and signs being unknown, their effects cannot be calculated and allowed for. Small errors are more likely to occur than large ones. The small errors in reading a levelling staff due to "shimmer" in the atmosphere or to temperature changes, small errors in reading an angle, etc., are of this type.

It should be noted that, in the case of systematic error, the total error in a measurement which is dependent on the sum of a series of repeated readings of the same quantity, is directly proportional to the total measurement, but, in the case of accidental errors of observation, the total error is proportional to the square root of the total measurement, or rather to the square root of the number of repetitions of readings. Thus, if there is a systematic error of k units per unit length in the reading of a chain, the total error from this cause in the length L of a line measured with that chain will be $k \times L$. On the other hand, errors of ordinary levelling tend on the whole to be of the accidental type, so that the total error in the measurement of a difference of elevation between two points L units of length apart will be $K\sqrt{L}$, where K is the "probable" accidental error per unit length of line. Hence, since the effects of systematic and constant errors tend to be propagated according to a linear law, and the effects of accidental errors according to a square root law, it is more important to reduce or eliminate constant and systematic errors than it is to reduce or eliminate the small purely accidental errors.

In many cases, but not in all, constant and systematic errors can be reduced or eliminated by using suitable methods of observation. Thus, errors of vertical collimation in a theodolite may be entirely eliminated by observing angles "face right" and "face left" and taking the mean of the two sets of readings. Similarly, errors of horizontal collimation in levelling may be eliminated by keeping 'backsights" and "foresights" equal in length.

CHAPTER II

CHAIN SURVEYING

When the country concerned is not too wooded or broken, complete surveys of small areas can be made by the use of a chain and linen tape alone. For extensive surveys, or in broken or very wooded country, the chain must be supplemented by a compass for the measurement of bearings, or, for more accurate work, by a sextant or theodolite for the measurement of angles. Even when part of the work is done by compass or theodolite, however, the methods of chain surveying are often used for the survey and filling in of detail.

The chain and linen box tape can also be used for much setting-out work, even for the setting out of railway curves, although in this last case it is usual and better to set out with theodolite and steel band.

Before we proceed to describe the methods of fixing detail from chain lines and making a survey by chain, it is necessary to consider the ranging out of straight lines, setting out angles, particularly right angles, and the methods of working round obstacles and across gaps, such as rivers and lakes. It will be assumed that the student is familiar with the method of using the chain as described in Chapter II of *Principles and Use of Surveying Instruments.*

1. Ranging Straight Lines.

Ranging a line means establishing a set of intermediate points on a straight line between two points already fixed on the ground.

The simplest case occurs when the two points are intervisible and a start is made from one end of the line. Let A and B, fig. 2.1, be the

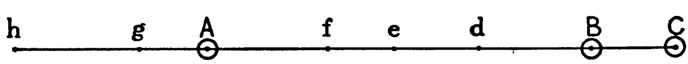


Fig. 2.1

two points between which it is necessary to establish a number of intermediate points. Having set up vertical ranging poles at A and B, move a short distance behind the point B, the point from which it is proposed to work, to a point C, so that on looking towards A the ranging poles at A and B appear to be in a straight line with the eye. Standing at C, get an assistant to hold a pole near some intermediate point **d**, and, using suitable signals or shouts, get him to move his pole right or left of the line until A, **d** and B all appear to be in a straight line. If required, points **g** and **h** beyond A can be lined in in a similar manner.

The assistant should be made to stand to one side of the line during these operations, so that his body does not obstruct the sight to the distant point. He must also hold his ranging pole vertical by supporting it loosely between the forefinger and thumb so that it tends to hang vertical under its own weight.

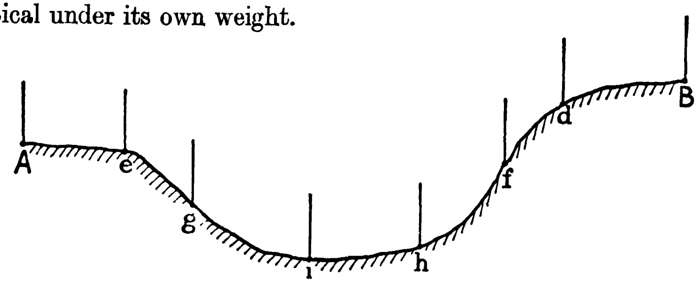


Fig. 2.2

Sometimes the whole of the intermediate line cannot be seen from the ends. In fig. 2.2 there is a gulley between A and B, and neither point can be seen from points inside the gulley. Establish points **d** and **e** on the line between A and B on the edges of the gulley and mark them by ranging poles. Then move to **d** and from behind **d** line in a pole at **f** on the straight line between **d** and **e** and possibly another pole at **g** between **f** and **e**. If necessary, move to **f** and put in intermediate poles at **h** and **i**. In this way, the line can easily be laid out over the gulley.

Another case arises when the ends of the line are not intervisible. This problem can easily be solved when a theodolite is available by making a survey of the relative positions of the two ends and calculating a bearing between them which can be laid out on the ground. In heavy bush country, where heavy clearing is involved, this is the easiest method, even if lines have to be specially cut for the legs of a preliminary traverse. For many purposes, however, such an elaborate procedure is not necessary, and a line can be established by ranging pole alone or by ranging pole and chain.

In fig. 2.3 a hill intervenes between A and B, so that these points are not intervisible. If a point C on the hill can be chosen such that it views A and B, it can be ranged in by a line ranger or by the method now to be described. Intermediate points between A and C and between C and B can then be ranged in in the ordinary way.

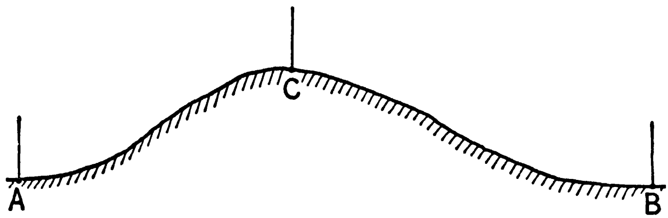


Fig. 2.3

Choose the point C which, as closely as can be judged by eye, is on, or very close to, the line AB and line in a point D between A and C. On going to D, fig. 2.4, stand behind the pole there and look in the direction of B. In all probability it will be seen that D, C and B are not in a straight line. From D line in a point E between D and

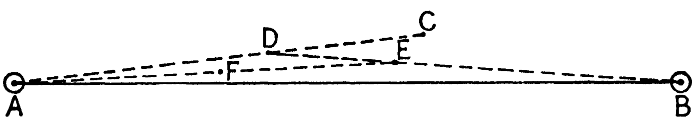


Fig. 2.4

B. Proceed to E and from a point behind it see if E, D and A are on line. If not, line in the point F on the line EA. Proceeding in this way, keep moving the poles closer and closer to AB until, after a few trials, they are seen to lie on it.

If, as in fig. 2.5, it is not possible to choose a point between A and B from which both points can be seen, estimate the direction of

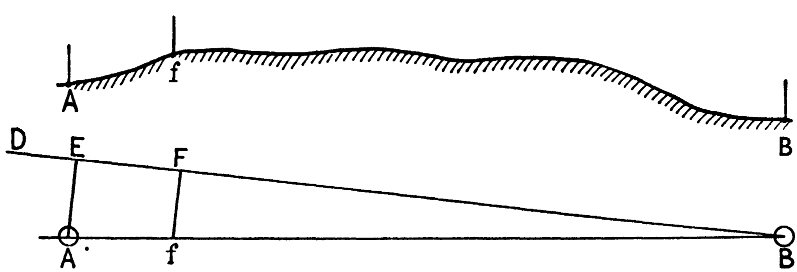


Fig. 2.5

A from B as closely as possible, and range and chain a straight line BD in that direction, leaving intermediate numbered pegs along BD at the end of each chain length. From A lay out a line AE perpendicular to BD at E (p. 14) and measure the length of AE and the chainage of E. Then, points such as f can be found by drawing a perpendicular to BE at F and laying out Ff such that $Ff = EA \cdot \frac{BF}{BE}$.

2. To Lay Out a Right Angle from a Point on a Chain Line.

The operations to be considered in the next few pages are all based on simple geometrical propositions, and are the equivalent on the ground of easy problems in geometrical drawing.

Let AB, fig. 2.6, be a chain line, and let it be required to lay out a line from a point C on AB at right angles to AB.

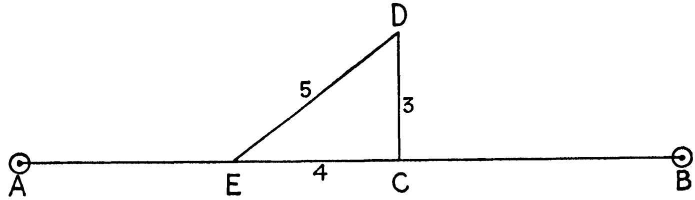


Fig. 2.6

If an optical square is available, stand at C and, viewing a ranging pole at A or B directly through the square, signal to a man holding a ranging pole at D until the image of the latter in the instrument appears to coincide with the pole seen directly through the aperture in the square.

If no optical square is available, the work can be done with the chain alone, although neither of the two methods to be described is as convenient as using an optical square. The principle of the first method depends on the well-known fact that, in a right-angled triangle, the sum of the squares on the two sides containing the right angle is equal to the square on the hypotenuse. Thus, with sides of 3, 4 and 5, we have $3^2 + 4^2 = 5^2$, the sides of lengths 3 and 4 containing the right angle, and the side of length 5 being the hypotenuse.

Standing at C, lay out a point E in line with A so that the distance CE measures 4 units, and put arrows in the ground at E and C. Fastening one end of the chain at C, or getting an assistant to hold it there, get an assistant to hold the graduation 8 at E. Hold the chain at graduation 3 and move to the side of the line towards which the right angle is to be laid out until the two lengths of the chain are taut at some point D. Place an arrow at D. D is then a point on a line at right angles at C to the line AB, as the sides of the triangle ECD are equal to 5, 4 and 3, as shown in the diagram.

Instead of taking short lengths of 5, 4 and 3, it is generally better to use a multiple of these figures. Thus, EC could be made 40 links, DC 30 links, and DE 50 links.

Another method is to lay out two equal distances CE and CF, each about 40 links long, on either side of C and both on the line AB

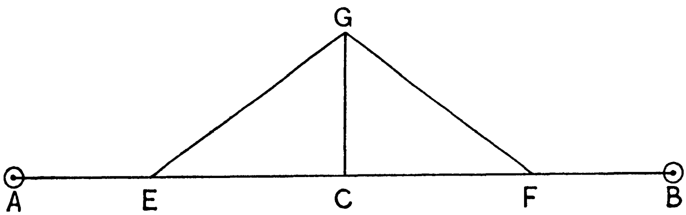


Fig. 2.7

(fig. 2.7). Then, if the ends of the chain are held at E and F, an arrow is held at the centre of the chain, and the two lengths pulled equally taut, the arrow will be at a point G such that GC is at right angles to AB.

Any of the above described methods is good enough for ordinary chain survey work, but cases sometimes occur where the work will not be sufficiently accurate unless it is done with a theodolite. In this case, set up the instrument at C, sight on A or B, and lay off an angle of 90° on the horizontal circle. The line of collimation will then be perpendicular to AB.

3. To drop a Perpendicular from a Given Point to a Given Straight Line.

Let it be required to lay out a line perpendicular to the line AB from a given point C (fig. 2.8).

Fasten the end of the chain at C, or get a chainman to hold it there, and choose some convenient length CD to form one side of an isosceles triangle CDE. Get an assistant to hold the graduation mark on the chain at D and, standing at G a short distance behind A, signal to him to move the tightened chain until D appears to be in line with AB and get him to put in an arrow. Have a similar arrow put in at E, where CE = CD and E is on the line AB. Measure the distance

DE and put in an arrow at F on the line AB such that DF = $\frac{1}{2}$DE. The line FC is then at right angles to AB and passes through C.

If the point C is inaccessible, as in fig. 2.9, choose suitable points D and E on the line AB and from D and E lay out lines DF and EG

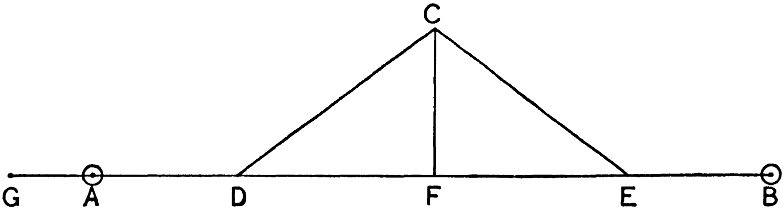


Fig. 2.8

perpendicular to EC and DC respectively. Then line out a point H so that it lies on the intersection of the lines DF and EG. Finally, from H lay out HK perpendicular to AB. The line KH when produced should then pass through C, as should be verified by standing behind a pole at K and seeing if K, H and C are on one straight line.

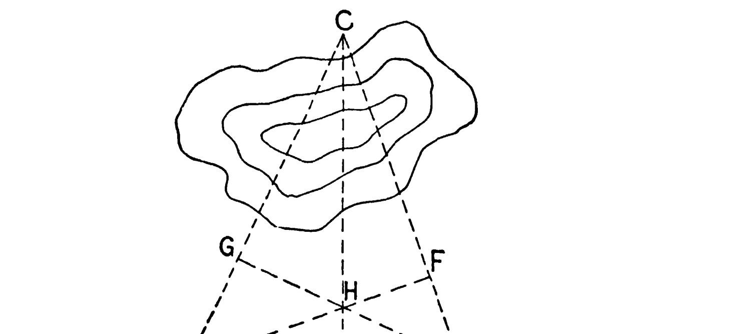


Fig. 2.9

4. To run a Line through a Given Point Parallel to a Chain Line.

Let AB, fig. 2.10, be the chain line and C the point through which it is desired to lay out a line parallel to AB.

From C lay out CD perpendicular to AB and measure the length of CD. Choose a point E on the line AB as far as possible from D and at E erect a perpendicular EF to AB equal in length to CD. The points C and F will then be on a straight line parallel to AB.

An alternative method is to choose points D and E, fig. 2.11, suitably placed on the line AB and measure the distance CD, leaving arrows at every chain length near the point which appears to be the

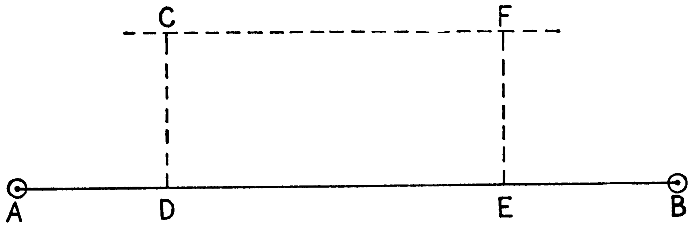


Fig. 2.10

centre of the line. Having obtained the length of CD, use the arrows to put a mark at F so that CF = FD. Measure the distance EF and prolong the line to G so that FG = EF. Then the line CG is parallel to AB.

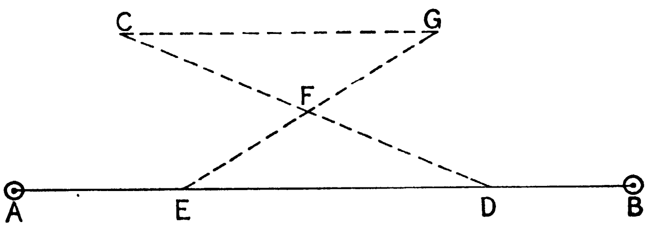


Fig. 2.11

In the above case the point C is supposed to be accessible. If it is inaccessible, as in fig. 2.12, establish the foot D of the perpendicular CD on the line AB by the method described on p. 15 and obtain the distance DC by one of the methods described on pp. 17–18. A point F on the line CF can now be obtained by laying out EF perpendicular to AB and making EF equal in length to DC.

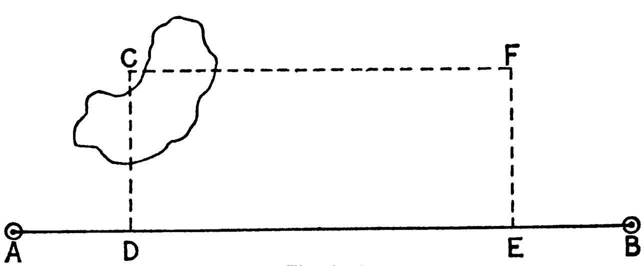


Fig. 2.12

5. Obstacles in Chaining.

Cases often occur in the field where the distance between two points is required, but direct chaining from one point to the other is impossible because of some sort of obstacle.

There are two main cases to be considered: (1) obstacles which obstruct chaining but not ranging, (2) obstacles which obstruct both chaining and ranging, and, of those which come under (1), we may distinguish between (*a*) obstacles round which chaining is possible, and (*b*) obstacles round which it is not possible to chain.

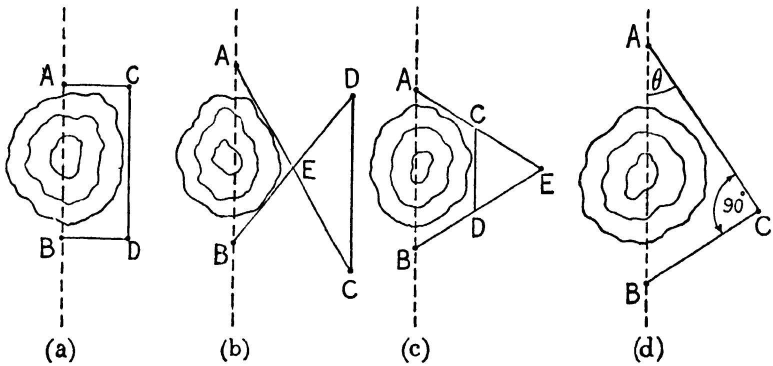


Fig. 2.13

Case 1 (*a*).—In fig. 2.13*a* the line crosses a lake between A and B and the distance AB is required.

At A and B range out lines AC and BD perpendicular to the chain line and make AC equal to BD in length. Then the line CD can be chained and will be parallel and equal in length to AB.

Other alternative methods are shown in figs. 2.13*b*, *c* and *d*.

In fig. 2.13*b*, E is the middle point of a chained line AC chosen so that the lines EA and EB clear the obstacle. Produce BE to D and make ED = BE. Then DC is equal and parallel to AB.

In fig. 2.13*c*, C is the middle point of AE and D the middle point of BE. Then AB = 2 × distance CD.

In fig. 2.13*d*, a perpendicular BC is dropped on a line AC and AC and BC are measured. Calculate θ from $\tan\theta = BC/AC$, and AB from $AB = AC \sec\theta = BC \operatorname{cosec}\theta$.

Case 1 (*b*).—Here a wide river prevents chaining round the obstacle.

In fig. 2.14*a* (p. 18), take a point C on line AB produced and erect perpendiculars to AB at C and B. Take point D on the perpendicular from C and line in E so as to be on the perpendicular from B and on the line DA. Measure BC, BE and CD. Then

$$\frac{AB}{BE} = \frac{BC}{CD - BE}, \text{ or } AB = \frac{BE \times BC}{CD - BE}.$$

In fig. 2.14*b*, lay out and measure BC perpendicular to BA and mark the middle point E. At C lay out line CD perpendicular to BC, and find point D on this perpendicular such that D, E and A are all in a straight line. Measure CD, which is equal to BA.

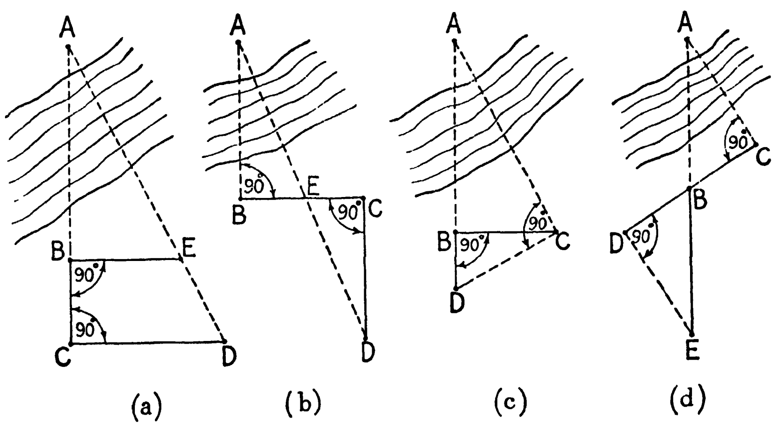


Fig. 2.14

In fig. 2.14*c*, set out BC perpendicular to BA, and at C set out line CD at right angles to AC. Choose point D on this line so that D is in line with B and A. Measure BD and BC.

Then $$\frac{AB}{BC} = \frac{BC}{BD} \text{ and so } AB = \frac{BC^2}{BD}.$$

In fig. 2.14*d*, choose a line AC which makes an angle of about 30° with AB, and from B drop line BC perpendicular to AC. Measure CB and prolong CB to D so that BD = CB. At D lay out line DE at right angles to CBD and find the point E on this line which is in line with B and A. Measure BE, which is equal to BA.

Case 2.—Obstacles which prevent both chaining and ranging. In fig. 2.15, a building interferes with the direct ranging as well as with the chaining of the line AB.

(1). In fig. 2.15*a*, set out lines AC and BD perpendicular at A and B respectively to AB, and make AC = BD. From C set out points

E and F on line CD, ahead of D and clear of the obstacle. At E and F set out lines EG and FH, equal in length to CA or DB and at right angles to CDEF. Measure DE. Then GH is a continuation of AB and BG = DE.

(2). In fig. 2.15*b*, choose a suitable point C, measure AC and BC and lay out points D and E on lines BC and AC, so that

$$\frac{\mathrm{CE}}{\mathrm{CA}} = \frac{\mathrm{CD}}{\mathrm{CB}} = k, \text{ say.}$$

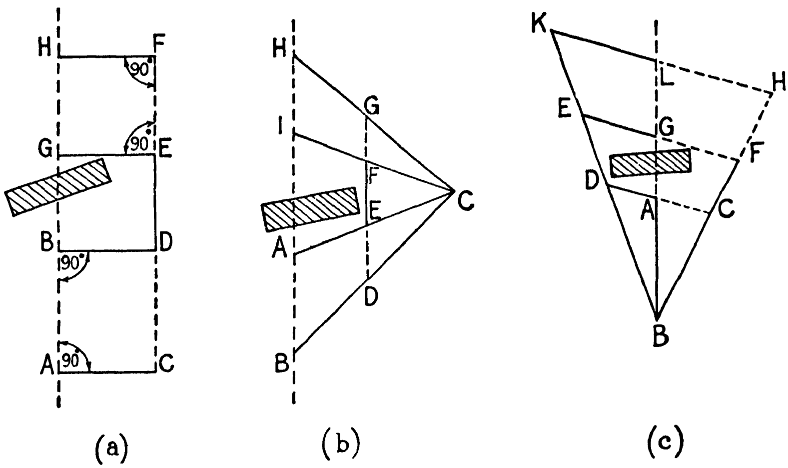


Fig. 2.15

Set out points F and G in line with D and E and measure CF and CG. Produce CF and CG to I and H, making

$$\frac{\mathrm{CI}}{\mathrm{CF}} = \frac{\mathrm{CH}}{\mathrm{CG}} = \frac{\mathrm{CA}}{\mathrm{CE}} = \frac{1}{k}.$$

Measure EF. Then H and I are on BA produced and

$$\mathrm{AI} = \mathrm{EF} \times \frac{\mathrm{CA}}{\mathrm{CE}} = \frac{1}{k}\,\mathrm{EF}.$$

(3). In fig. 2.15*c*, lay out line BE and measure BD and BE. Let $\mathrm{BD} = k \times \mathrm{BE}$. Lay out line BF and line in point C at the intersection of BF and DA produced. Measure BC and make $\mathrm{BF} = \frac{1}{k} \times \mathrm{BC}$. Measure DA and on line EF put in point G so that $\mathrm{EG} = \frac{1}{k} \times \mathrm{DA}$.

G is then a point on BA produced. Similarly, find another point L on BA produced. Measure BA.

Then $$AG = BA\left(\frac{1}{k} - 1\right)$$

and GL is a continuation of BA.

6. The Chain Traverse.

An irregular boundary or a winding road or stream may sometimes be traversed by chain alone, although a compass or theodolite traverse is usually preferable in cases where there is no alternative to a traverse of some kind.

The method will be understood from figs. 2.16 and 2.17, the bends in the lines being fixed by small triangles, of which the lengths of all three sides are measured.

It will be seen from either of the above examples that a good deal of room is required at the bends in order to get in the extra tie lines or triangles. Hence, the method is not always possible, even when

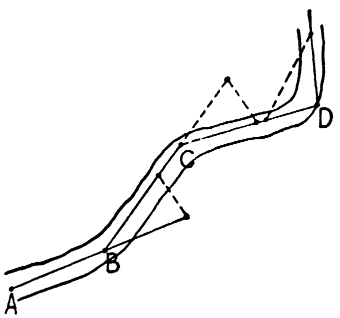


Fig. 2.16

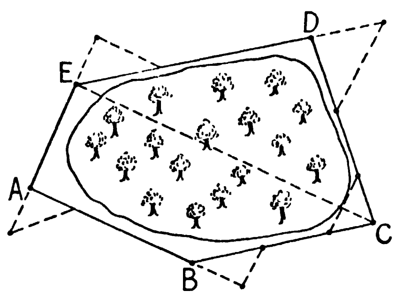


Fig. 2.17

it is otherwise allowable, and the most satisfactory method of survey in such cases is normally an instrumental traverse, either using a compass to measure the directions of the lines, or a theodolite or sextant to measure the angles at the bends. The survey of a closed figure like that shown in fig. 2.17 can be strengthened by one or more tie lines across it, such as the line EC. The method is not sound in practice if any other is available, because the directions of long legs are determined by measurements of triangles with short sides, so that any small linear error in the measurement of the sides of a triangle will be magnified in the swing of the end of the leg whose direction is determined by the triangle.

CHAPTER III

CHAIN SURVEYING: SURVEY AND PLOTTING OF DETAIL

In the previous chapter we have considered the ranging and setting out of survey lines and some problems in field geometry which arise in chain surveying. In this chapter we shall consider the survey and plotting of detail, the keeping of field books, and the methods of making a complete survey by chain alone.

1. Survey of Detail.

The survey of detail from a chain line is carried out mainly by right-angled offsets or by small *offset* triangles, the latter being small triangles of which two sides are measured from two points on the chain line, as ends of the base. The important point in this kind of work is to run the chain lines as close as possible to the detail to be surveyed, so that offsets are as short, and offset triangles as small, as possible. For accurate work, and when important detail has to be surveyed, the lengths of offsets, and the size of the offset triangles, should be strictly limited and care should be taken to see that offsets are set out as closely as possible at right angles to the chain line. When less important or ill-defined detail has to be surveyed, longer offsets and larger offset triangles may be used, and less care taken in setting out the offsets at right angles to the chain line. In addition, detail, such as walls and hedges, that is sloping fairly sharply away from the chain line requires more careful survey than detail that is running more or less parallel to the line. The first rule of chain survey being to *work from the whole to the part*, this leads to the second rule, which is to try and *keep the chain lines as close to and as nearly parallel as possible to the general run of the detail.*

When a point of detail is on a line whose direction is otherwise established, the point can be fixed by noting the chainage of the point where the line, produced if necessary, cuts the chain line, and then measuring the distance along the line from this point to the point to be fixed.

Up to very recently, the Ordnance Survey, in the revision of the large-scale plans, made extensive use of a system of graphical intersection from points already fixed. Thus, in fig. 3.1, P is the point of intersection of two fences which it is desired to fix, PE being an existing fence, which is shown on the plan. The surveyor determines a point E on the fence which he sees is in line with the church spire A and the corner of a building B, both of these points being easily identified both on the ground and on the plan which is being revised. The intersection of the line AB and the line of the fence fixes the point E, and a measurement of the length of EP along the fence on which

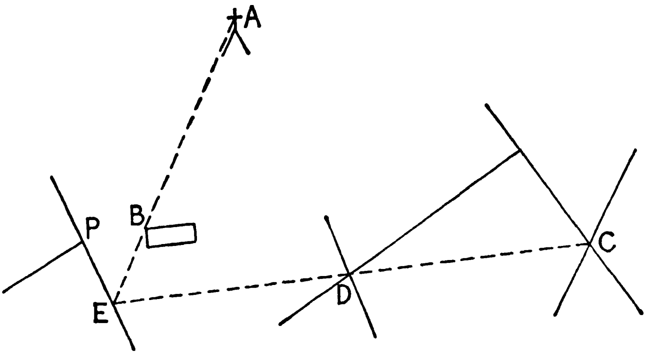


Fig. 3.1

these points lie fixes the point P. The same method can sometimes be used in a new survey when a number of points have been established by other means. In the particular case illustrated, the fixing of E could, if desired, be checked or strengthened if the two fence intersections D and C which are in line with E on the ground are also in line with it on the plan, as the intersection of the lines AB and CD would give an independent fixing for E.

As a preliminary example of the survey of detail from a chain line consider the building ABCDEFGHA in fig. 3.2 which is to be fixed from the chain line XY. If it is an important building, we are fairly safe in assuming that the corners will be right angles.

The face ABCDEF could be fixed by right-angled offsets from the chain line to the corners A, B, C, E and F, and the corner D by the distances CD and ED, the plotting of B, C, E and F being checked, if necessary, by the lengths AB, BC, EF. The corners A and B are rather far from the chain line, so that, instead of fixing them by right-angled offsets, it would be better to fix them by the offset triangles **Aab** and **Bbc**. The corner G can also be fixed by the right-angled offset jG, but, as it is rather far away from the line and is an important point governing the survey of the back of the building, it would be better to fix it by the offset triangle Gjk. The corner H could most easily be fixed by noting the chainage of the point **a**, where the line HA produced meets the chain line, and then measuring the distance AH from **A**.

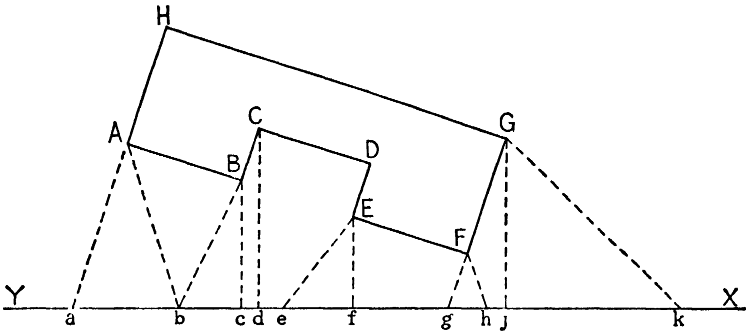


Fig. 3.2

2. Booking of Detail.

The great principle to be kept in view when keeping the field notes of any survey is to follow a definite system and to keep the notes in such a way that they can be followed, and the work plotted by a draughtsman who is totally unfamiliar with the ground and has no means of referring doubtful points to the surveyor.

The form of the notebook generally used for chain survey work is an oblong book about 7 in. by 5 in., with two parallel lines about ½ in. apart running up the centre of the page parallel to the longer side. The chainages are entered between these lines and particulars of the detail, with the relevant offset or other measurements, are entered on either side of them. In all cases, the detail is booked running up the page, from bottom to top, in the direction of the chainage.

Fig. 3.3 (p. 24) shows the field notes relating to the survey of part of the line CD in the survey shown on the plate opposite. The page opens with a reference at the bottom to the page on which the survey of the first part of the line appears. The first chainage point is 615 with offsets of 1 ft. to the right to a tree and 23 ft. to the edge of a grass verge. Other offsets to the verge occur at chainages 693, 697, 703 and 705, and at 709 it cuts across the chain line at an angle. Note that here the points where the verge crosses the chain line are shown opposite each other on the central lines of the field book, even although the actual cut is at an angle, the chainage being written between the cuts. The same thing will be noted at chainage 724 and where a fence crosses the chain line at 752. A corner of the fence comes on the right at chainage 718, with an offset of 9, and there are other offsets to it on the right at 729, 731 and 749 and to the left at 756, 768, 778, etc.

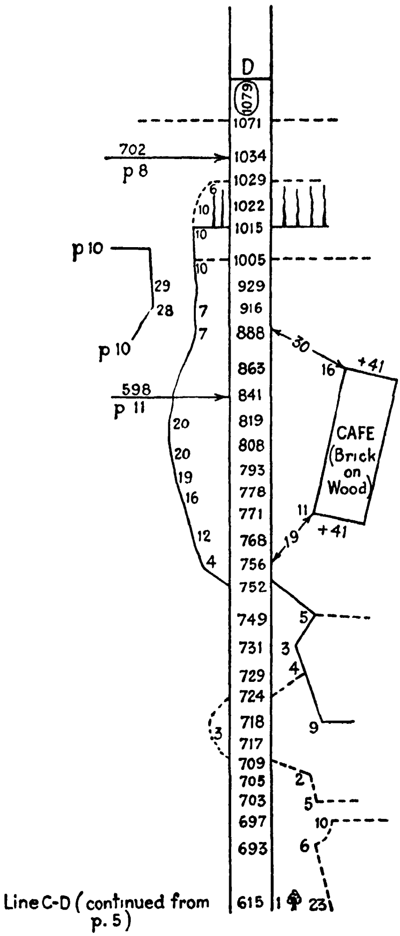


Fig. 3.3

A building on the right starts at chainage 771 with an offset of 11, and the fixing is strengthened and checked by a measurement of 19 from chainage 756, so that the points 756–771 form the base of a small right-angled offset triangle. Similarly, the other end of the building is fixed by a right-angled offset triangle with base 863–888 and hypotenuse 30 from 888. The building is rectangular, and the lengths of the ends, 41 ft., are written alongside these ends with a plus sign before each figure. Such a measurement is known as a *plus measurement*, the plus indicating that the figure is a measurement additional to any that has gone before it. On the other hand, when two or more offsets are taken from the same chainage point, the full measurement from the chain line to each point is recorded. Thus, the offset to the verge at 615 is 23 ft. from the chainage line, not 23 ft. from the tree that is 1 ft. from the chainage line; and at chainage 916 the outer fence is 28 ft. from the chainage line, not 28 ft. from the inner fence. Occasionally, however, it is convenient to record a measurement on an offset or line from a point on the offset not on the chainage line. In this case, a plus sign should be put before the additional measurement to show that it is an additional measurement and is not taken from the chainage line.

Two chain lines run off to the left at the chainages 841 and 1034, and these are indicated by an arrow and a horizontal line, with the length of the chain line written above the horizontal line, and the number of the page in the field book on which the survey of it is to be found written below it. This system of writing the length of a line over the page number is a useful one for identifying different lines. There is an outer fence to the left with offsets of 28 and 29 at chainages 916 and 929, and other details of this fence are to be found on p. 10 of the field book.

The total length of the line is 1079, and this is written vertically between, and parallel to, the ruled lines and ringed as the last entry for the line.

In keeping the chainage book, no attempt is made to draw rigidly to scale, although it will be found to be of assistance in maintaining clarity and neatness in note-keeping if the notes are kept very roughly to scale as far as can be judged by eye.

3. Plotting Detail: Offset Scale.

Detail is plotted from a chain line by means of a small offset scale. This is a small scale about 2 in. in length, graduated exactly similarly to the main scale, and with both ends cut exactly at right angles to

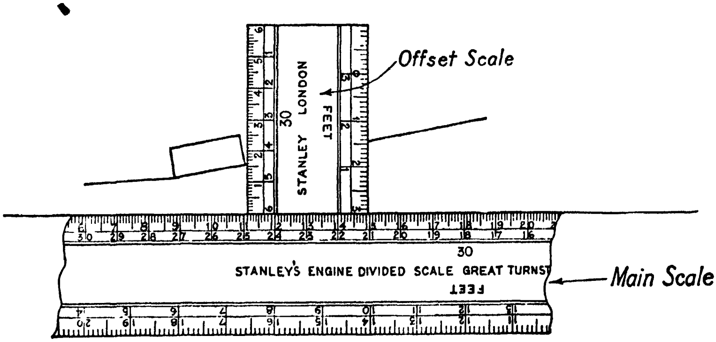


Fig. 3.4

the graduated edges. The main scale is laid on the plan against the line from which the detail is to be plotted, and weights are put on the ends of the scale to hold it in position. The offset scale is placed alongside the main scale as shown in fig. 3.4, with one end against the graduation on the main scale corresponding to the chainage of the point to be plotted. The offset measurement is then easily plotted from the offset scale.

4. Survey and Booking of Complete Surveys.

The method of making and booking a complete survey will be understood from the plan opposite p. 23, and from figs. 3.5 and 3.6. The first stage is to fix a framework for the main chainage lines. In the

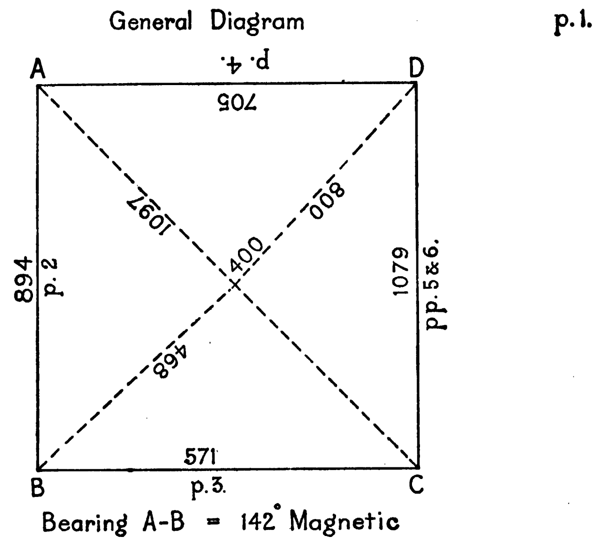


Fig. 3.5

simple example given in the plan this consists of the four outer lines AB, BC, CD and DA, the lengths of which are given in the *General Diagram* shown in fig. 3.5. The points C and A being intervisible, the line CA was measured, and this, with the measured lengths of AB and BC, enables the triangle ABC to be plotted on AB as base, the direction of AB being fixed by a given magnetic bearing. Hence, the point C is fixed. The lengths of the sides CD and DA were also measured; this enables the triangle CDA to be plotted from the base CA, so fixing the point D.

As a check, the lines joining B and D to the chainage point 400 on the line CA were also measured, so that their plotted lengths could be compared with their measured lengths. All of these lines are shown in the General Diagram which is drawn at the beginning of the field book. On this diagram it will be noted that, as before, the different lines are identified by their lengths and field-book page numbers, except that the line CA, and the lines joining points B and D to the chainage point 400 on it, have no page numbers because no detail was surveyed from them, and their lengths were entered direct on the General Diagram as shown as soon as they were measured.

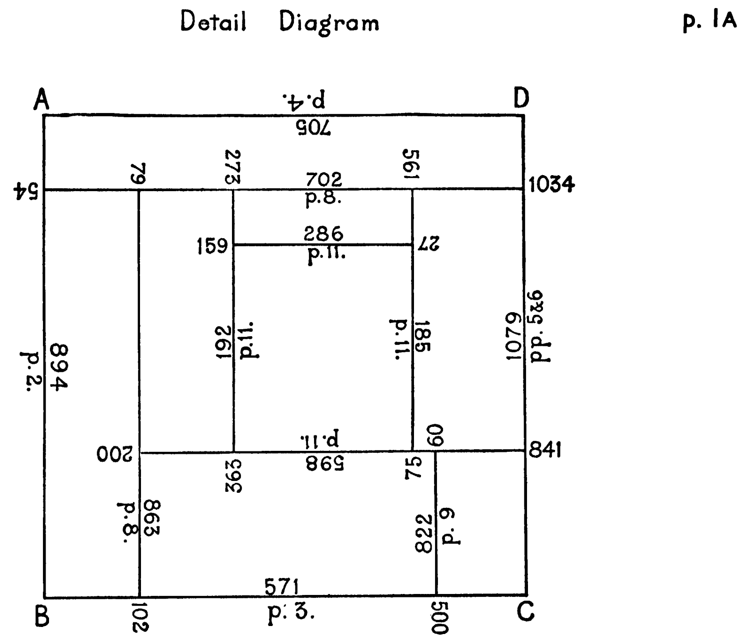


Fig. 3.6

Note here that it is not necessary that the central pole at chainage 400 on line CA should be on this line. If some other point O near this point had been taken, and the lengths AO, BO, CO and DO measured, the figure could have been plotted from the triangles AOB, BOC, COD, with the measured length of AD as check.

The *Detail Diagram*, or diagram illustrating the layout of the detail lines, is shown in fig. 3.6. This diagram follows immediately after the General Diagram. The first line fixed is one of length 702, which starts at chainage 54 on line AB and ends at chainage 1034 on line CD. This line holds one end of each of three others at chainages 79, 273 and 561. The line from chainage 79 is fixed at its other end by being tied to line BC at chainage 102, and the ends of the other two lines are tied to a line joining chainage 841 on line CD to a point at chainage 200 on the line of length 863 which joins BC at chainage 102. In every case, there is a check, as every line can be plotted from the points where it meets other lines, and, after plotting, the plotted length should agree with the length measured on the ground and recorded on the diagram. During plotting, these lines are first drawn in faintly in pencil, and then, if everything fits, they are inked in in red or light blue and the detail plotted from them line by line.

The detail diagram is followed in the book by the notes relating to the various detail lines, which are booked in the manner already described, each line being booked on a separate page or pages.

5. Drawing the Plan.

The plan is usually drawn on best-quality Whatman paper, a ruled margin of about $1\frac{1}{2}$ in. to 2 in. being left all round. If unmounted drawing paper is used, it is well, because of the way in which paper expands and contracts with changes in atmospheric conditions, to damp and stretch it before pasting it round the edges to the drawing board with good cornflour paste.

The most commonly used scales for large-scale plans are 1 chain (links or feet) to the inch and 2 chains to the inch, but scales of 3, 4, 5, 6, 7, 8, 9 and 10 chains (links or feet) to the inch are also used. Boxes, containing wooden or ivory scales for all these scales, together with a set of offset scales, can be obtained from instrument makers.

The other instruments needed are a drawing board, weights, steel straight edge, compasses with lengthening bar and pencil and ink points, beam compasses, pencil and ink bows, dividers, drawing pens, proportional compasses, set squares, T-square, parallel ruler, large and small brass protractors, French curves, and a set of railway curves.

Since paper is very liable to shrink and expand, a scale should be drawn at the very beginning at the bottom of the plan. The survey should be plotted facing north so that the left- and right-hand edges are, as far as possible, parallel to the direction representing true north, and north should be at the top of the sheet. The directions of true and magnetic north should be drawn in when these are known.

After having been pencilled in, the work should be penned in in dense black with the best-quality Indian or Chinese ink. Colour may be used, if desired, to show water (light blue), roads (light brown), buildings (grey), or different classes of land or cultivation.

Some of the conventional signs that are commonly used for large-scale plans are shown in fig. 3.7.

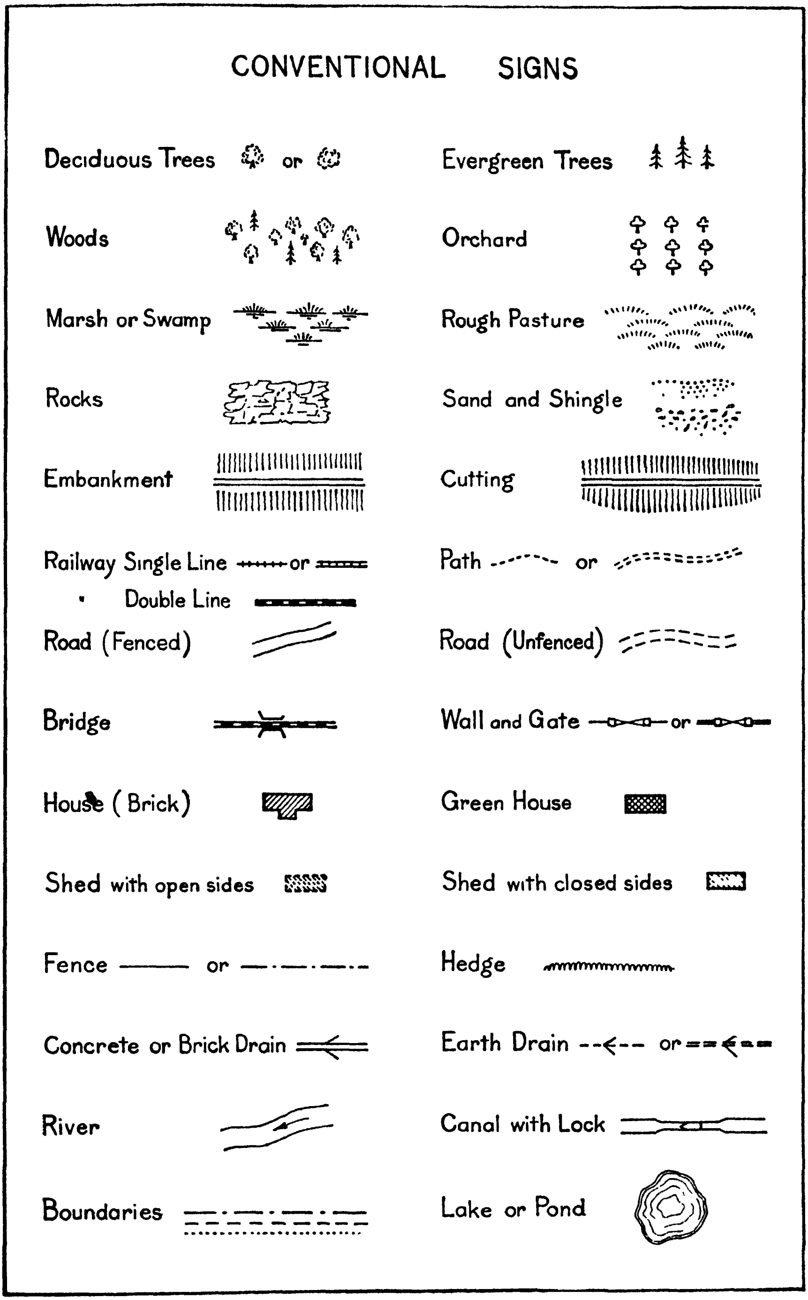

CONVENTIONAL SIGNS
Deciduous Trees or
Evergreen Trees
Woods
Orchard
Marsh or Swamp
Rough Pasture
Rocks
Sand and Shingle
Embankment
Cutting
Railway Single Line or
Double Line
Path or
Road (Fenced)
Road (Unfenced)
Bridge
Wall and Gate or
House (Brick)
Green House
Shed with open sides
Shed with closed sides
Fence or
Hedge
Concrete or Brick Drain
Earth Drain or
River
Canal with Lock
Boundaries
Lake or Pond

Fig. 3.7

QUESTIONS ON CHAPTERS II AND III

1. It is required to find the perpendicular distance from a point to a chain line a short distance away, the foot of the perpendicular on the chain line being inaccessible. Describe how you would find this distance by chain survey methods.
2. A perpendicular is to be laid out from a point C on a chain line AB, and a point D is taken on the chain line 24 ft. from C, and between C and A. It being assumed that a point E on the perpendicular will be found by a right-angled triangle, of which the sides CD and CE containing the right angle will measure 24 ft. and 32 ft. respectively, what must be the length of the side DE?
3. Describe three methods of finding the distance across a lake 300 yd. across, by means of chain methods only.
4. Describe how you would lay out a straight line between two points A and B by chain survey methods. The point B is in open country, but is inaccessible, and a wide and long belt of forest lies between A and B, so that these points are not intervisible. It is assumed that cutting in the forest is permissible and is necessary, as it is not possible to work round it.
5. How would you obtain the distance AB in the last example before setting out the actual line?
6. Show by dimensional sketches how you would lay out with a chain lines from a given point on a given line to make angles of 45°, 60° and 66° 22′ with the given line.
7. Describe with a sketch how you would lay out a hexagonal figure from a base 100 ft. long.
8. How would you lay out a similar figure of 300-ft. side if it were to enclose a lake covering the centre?
9. Name three commonly used methods of fixing detail with reference to a chain line.

CHAPTER IV

ANGLES AND BEARINGS

ANGLES

1. Angular Units.

The sexagesimal method of reckoning angles is general throughout the British Empire, although sometimes, mainly for rough work, angles are reckoned in degrees and decimals of a degree.

In the sexagesimal system there are 360 degrees in a complete circle, or 4 right angles, each degree being divided into 60 minutes, and each minute into 60 seconds. Hence, there are $60 \times 60 = 3600$ seconds in a degree and $60 \times 60 \times 360 = 1{,}296{,}000$ seconds in a complete circle of 4 right angles.

In theoretical work in mathematics, angles are expressed in radians or decimals of a radian, and there are 2π radians in the complete circle, where $\pi = 3{\cdot}1415926536$. Hence,

$$360^\circ = 2\pi \text{ radians},$$

$$1^\circ = \frac{2\pi}{360} = 0{\cdot}01745329 \text{ radian},$$

$$1' = \frac{2\pi}{360 \times 60} = 0{\cdot}000290888 \text{ radian},$$

$$1'' = \frac{2\pi}{360 \times 60 \times 60} = 0{\cdot}0000048481 \text{ radian},$$

and

$$\begin{aligned} 1 \text{ radian} &= 57{\cdot}2957795 \text{ degrees}, \\ &= 3437{\cdot}74677 \text{ minutes}, \\ &= 206264{\cdot}806 \text{ seconds}. \end{aligned}$$

In all ordinary survey work a radian is taken as 206,265 seconds.

In the case of small angles up to about 10 minutes of arc, the radian measure of an angle is very little different from the sine of the angle. Hence, if θ is a small angle expressed in minutes or seconds, and r is the value of the angle in radians, we may write

$$r = \theta' \sin 1', \qquad \text{or} \qquad r = \frac{\theta'}{3437{\cdot}75};$$

and

$$r = \theta'' \sin 1'', \qquad \text{or} \qquad r = \frac{\theta''}{206{,}265},$$

where θ' is the value of the angle expressed in minutes and θ'' is its value expressed in seconds ($\log \sin 1' = \bar{4}{\cdot}463\,7261$, $\log \sin 1'' = \bar{6}{\cdot}685\,5749$).

Similarly, we can write

$$\theta' = \frac{r}{\sin 1'}, \qquad \text{or} \qquad \theta' = 3437{\cdot}75r,$$

and

$$\theta'' = \frac{r}{\sin 1''}, \qquad \text{or} \qquad \theta'' = 206{,}265r.$$

This explains why the factors $\sin 1'$ and $\sin 1''$ or $1/\sin 1'$ and $1/\sin 1''$ enter into many formulæ in higher surveying. Note, however, that these factors must only be used when *small* angles are involved; when large angles are involved, we write

$$r = \theta^\circ \text{ arc}^\circ, \qquad \theta^\circ = \frac{r}{\text{arc}^\circ},$$

$$r = \theta' \text{ arc}', \qquad \theta' = \frac{r}{\text{arc}'},$$

$$r = \theta'' \text{ arc}'', \qquad \theta'' = \frac{r}{\text{arc}''},$$

where arc°, arc′, arc″ are the number of radians in one degree, one minute, and one second respectively.

In Continental practice, angles are often reckoned in *grades* and decimals of a grade, there being 400 grades in 4 right angles, or 100 grades to a right angle. As this unit is practically never used in British engineering or surveying work, we shall not consider it further.

2. Methods of Reckoning Angles.

The horizontal circles of most theodolites are graduated and numbered clockwise from 0° through 90°, 180°, 270° to 360° (or 0°), as in fig. 4.1, but some are numbered from 0° through 90° to 180° on either side of a diameter passing through the zero mark, as in fig. 4.2.

The first system is called the *whole-circle* system, and is more generally useful for ordinary survey work, but, in theodolites to be used for railway and road work, which consists largely of layirg out

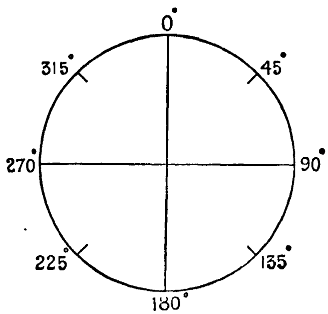


Fig. 4.1

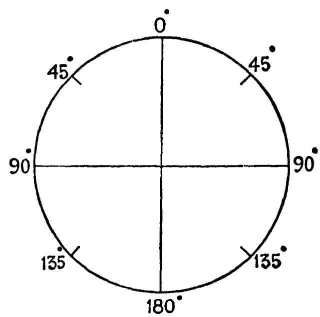


Fig. 4.2

curves by means of small deflection angles, the second system, called the *half-circle* system, is the more convenient. In the whole-circle system, there need never be any doubt about how an angle is to be reckoned, but with the half-circle system it is important to note whether an angle is to be reckoned left or right of the zero mark, and this may lead to error unless great care is taken both in reading and booking.

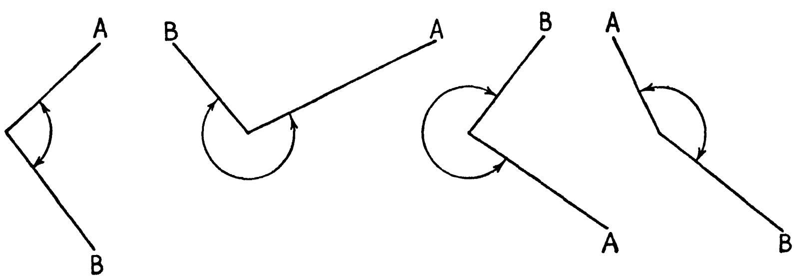


Fig. 4.3

In fig. 4.3 A is the first point sighted, or the point from which the angle is to be measured, and B is the second point sighted, or the point to which the angle is to be measured. Then, with the whole-circle system of numbering and measurement, the angle recorded in each case is the one which is marked by the circular arc. If the theodolite is set and clamped to read zero when the telescope is pointed at station

A, and if the lower circle is kept clamped and the upper one is then unclamped and re-clamped after the telescope has been set to sight B, the reading on the graduated arc will be the angle between A and B reckoned clockwise from A. If the instrument is set and clamped with the circle in *any* position when the telescope is pointed to A and the reading taken, and the reading again taken when the telescope is pointed at B, the observed angle is the first reading subtracted from the second. Thus:

Angle No.	1	2	3	4
Reading to B	165° 34′	337° 17′	36° 12′	123° 38′
Reading to A	23 13	36 29	131 54	315 22
Angle measured clockwise from A.	142 21	300 48	264 18	168 16

In the last two cases, the reading to B is less than the reading to A; in such an event 360° must be added to the reading to B, and the reading to A subtracted from the sum. Hence we have the rule:

To obtain the angle measured clockwise from A, subtract the reading to A from the reading to B. If the reading to B is less than the reading to A, add 360° *to the former and subtract the reading to A from the sum.*

It will be noted that, with this system, it is very important to see that readings are booked correctly against the stations to which they are taken. Thus, if the reading to A were booked as being taken to B and that to B booked as being taken to A, the deduced angle would be the angle measured clockwise from B to A, which is 360° minus the correct angle. This point is particularly important in traverse work where the angles are often very close to 180°, and so an error is not immediately apparent.

If the deflection method of reckoning angles is used in conjunction with a theodolite graduated on the half-circle system, the angles must be booked left (L) or right (R) according as the instrument is rotated to the left or to the right to bring the line of sight from the first station to the second. In this case, the instrument may be set to read zero when the telescope is pointed to the first station. When the upper circle is unclamped and the line of sight directed to the second station, the reading on the circle will be the deflection angle, measured either anticlockwise (Left) or clockwise (Right) from the first station.

An alternative method of reckoning deflection angles, which is the one usually used in traverse and in railway work, is to reckon and measure the deflection angles from the *forward direction* of the line from which the angle is being observed, i.e. from the direction of a continuation beyond the instrument of the line joining the first point sighted, or the rear station, to the instrument station. Here the instrument is first set to read 0° when the telescope is pointed to the rear station. The telescope is transited; then, the lower plate of the theodolite remaining clamped, the clamp of the upper plate is loosened and the telescope sighted on the second, or forward, station; the upper plate is then clamped. The reading on the circle will now be the deflection angle, right or left, from the forward direction of the line joining the first station to the instrument station.

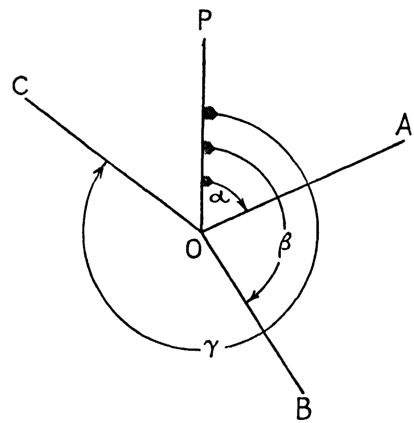


Fig. 4.4

BEARINGS

The *bearing* of an object is the angle between some fixed direction and the direction of the object. Thus, in fig. 4.4, if O is the position of the observer and OP the fixed direction from which bearings are reckoned, the bearing of the point A is the angle POA.

Usually bearings are reckoned clockwise from 0° through 90°, 180° and 270° to 360° from the fixed direction, so that in fig. 4.4 the angles marked α, β and γ are the bearings from O to A, B and C respectively. Bearings reckoned in this way are called *whole-circle bearings.*

3. Azimuths.

If the fixed direction is the true geographical north from the point of observation, bearings are called *azimuths.* Hence, in fig. 4.4., if OP is the direction of the true geographical north from O, i.e. the northward direction of the geographical meridian through O, the bearings α, β and γ become the azimuths of A, B and C from O. Azimuths are generally reckoned from 0° to 360° clockwise from north, but sometimes, and particularly in astronomical and geodetic work, they are reckoned from 0° at geographical south clockwise through 90°, 180°, 270° to 360°. In British practice, however, and now even in nearly all geodetic work, azimuths are generally reckoned clockwise from north.

4. Magnetic Bearings.

If the fixed direction is the direction of magnetic north, the bearings are called *magnetic bearings*. Hence, α, β and γ in fig. 4.4 are magnetic bearings if OP is the direction of magnetic north.

5. Grid Bearings.

There is another kind of bearing which has come much into use during the last few years. If the sheets of the "New Popular" and the new Seventh Editions (National Grid) of the Ordnance Survey one-inch map, or sheets of the new 1/1250 plans of towns, are examined, it will be found that they are divided into squares by a series of straight horizontal and vertical lines. These lines are parallel to those on adjoining sheets, although they are numbered differently. The vertical lines are not true north and south lines unless on one particular line which corresponds with a meridian that is roughly the central meridian of the country. The whole network forms what is called a *grid* and the northward direction of the vertical lines is called *grid north*. *Grid bearings* are bearings referred to the direction of grid north as given by the vertical lines. At any point the difference between the directions of true north and grid north is a small angle called the *convergence*, and this varies according to the position of the point east or west of the central meridian, being greater in magnitude the farther away the point is from this meridian. The amount of the convergence can be calculated, and we then have:

$$\text{azimuth} = \text{grid bearing} \ \frac{\text{plus}}{\text{minus}} \ \text{convergence}$$

according as the point is east or west of the central meridian. In the case of the Seventh Edition of the Ordnance Survey one-inch map, there is a small table at the bottom of the sheet which gives, for the centre and for each of the four corners of the map, the angles which the directions of true and magnetic north make with the vertical sheet edges.

6. Reduced Bearings.

In computing, and in work with the magnetic compass, it is often convenient to use what are called *reduced bearings*. A reduced bearing is the angle between the main vertical line marking the direction to which bearings are referred and the given line, measured from 0° to 90° the shortest way, east or west and north or south of the point, to that line. Thus, in fig. 4.5, in which the circle is divided into four quadrants numbered I, II, III and IV, the reduced bearings of the lines OA, OB, OC and OD are indicated by the Greek letters α, β, γ and δ, and are all reckoned the shortest way, east or west, from the line SN. If the bearings are given on the whole-circle system, it can easily be seen from the figure that we have the following rules for obtaining reduced bearings:

If the whole-circle bearing lies in the first quadrant, i.e. between 0° *and* 90°, *the reduced bearing is the same as the whole-circle bearing.*

If the whole-circle bearing lies in the second quadrant, i.e. between 90° *and* 180°, *the reduced bearing is* 180° *minus whole-circle bearing.*

If the whole-circle bearing lies in the third quadrant, i.e. between 180° *and* 270°, *the reduced bearing is whole-circle bearing minus* 180°.

If the whole-circle bearing lies in the fourth quadrant, i.e. between 270° *and* 360°, *the reduced bearing is* 360° *minus whole-circle bearing.*

It should be noted that a reduced bearing never exceeds 90° in value, and, when bearings are derived and expressed in the first place as whole-circle bearings, and reduced bearings are used only as a convenience in computing, a reduced bearing need take no account of the quadrant in which the line lies. If, however, it is desired to specify the quadrant in which a reduced bearing lies, this is done by putting the letter N or S before the figures giving the actual bearing, according as to whether the latter is measured from the direction of north or south, and then inserting after the figures the letters E or W to show whether the bearing lies east or west of the north and south line. Thus, the bearings α, β, γ and δ in fig. 4.5 would be written as NαE, SβE, SγW and NδW respectively. Magnetic compasses are often graduated on the quadrantal system, with the letters, N, E, S and W marked on the card or rim, and accordingly magnetic bearings are commonly booked and expressed in terms of reduced bearings, with the proper distinguishing letters before and after them to specify the quadrant.

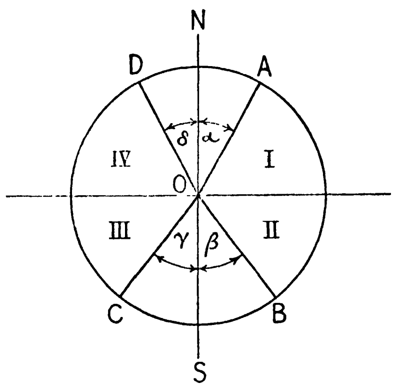


Fig. 4.5

The rules given above are so simple that it is hardly worth while attempting to memorize them, as any given case can easily be worked out from first principles. As practice is gained, the computation becomes almost automatic without conscious effort. The importance of reduced bearings in computing lies in the fact that most mathematical tables only tabulate the values of the trigonometrical functions and their logarithms in terms of angles lying between 0° and 90°. Accordingly, when whole-circle bearings are used, it is usually necessary to convert them into reduced bearings before entering the tables.

7. Calculation of Bearings from Included Angles.

In fig. 4.6, OR is the reference line from which bearings are reckoned. An angle AOB is measured from station A, the first station observed, to station B, the second station observed, and the bearing of the line OA is known. To find the bearing of the line OB.

Let both angles and bearings be reckoned on the whole-circle system and let the bearing ROA be denoted by α, and the measured angle AOB be denoted by θ. The senses in which the angles and bearings

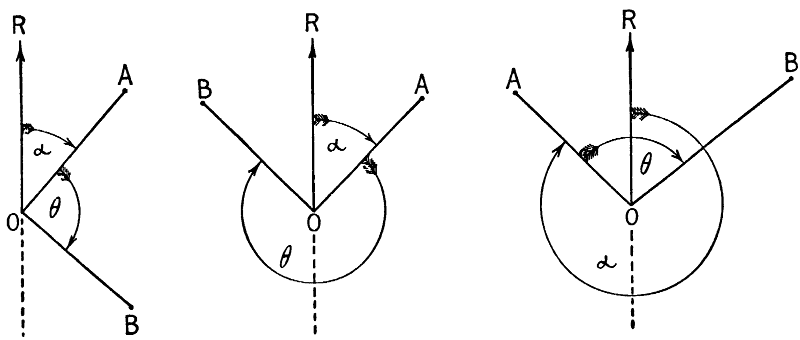


Fig. 4.6

are reckoned are shown in the figure by arrows. It can now easily be seen that, in the first two cases, the bearing of OB (=angle ROB) reckoned clockwise from R is $\alpha + \theta$. In the third case, OB lies in the first quadrant so that its value is between 0° and 90°, while OA lies in the fourth quadrant so that its value lies between 270° and 360°. Consequently, when θ is added to α we have $\theta + \alpha = (\alpha + \text{AOR}) + \text{ROB} = 360° + \text{ROB}$, or $\text{ROB} = \theta + \alpha - 360°$. But ROB, reckoned clockwise from OR, is the bearing of OB. Hence, we have the following simple rule for the calculation of bearings from included angles, both reckoned on the whole-circle system:

Add the observed angle to the bearing of the first point observed and subtract 360° *if the sum is greater than* 360°. *The result is the bearing of the second point observed.*

Examples:

Bearing of OA	..	53°	153°	171°	236°	236°	316°	89°
Angle AOB	..	37	68	215	71	164	191	76
Bearing of OB	..	90	221	26	307	40	147	165

If angles are measured as deflection angles from the forward direction of the line joining the first station observed to the instrument station, but bearings are reckoned on the whole-circle system, we can easily deduce the following rule:

To/From the forward bearing of the first line add/subtract the deflection angle according as it is right/left of the forward direction of the first line. Subtract 360° *if the result is greater than* 360°. *Add* 360° *to the bearing of the first line if the deflection angle is left and greater in magnitude than the bearing of the first line.*

8. Back Bearings.

In fig. 4.7, AR is the reference direction from which bearings are reckoned. Then the bearing of the line AB is the angle marked α. At B draw BR′ parallel to AR. At B bearings are reckoned clockwise

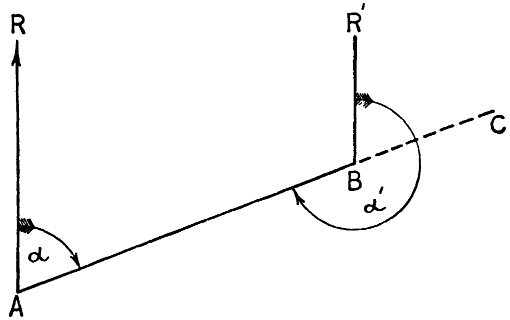


Fig. 4.7

from BR′, and the bearing of the line BA is the angle marked α', which, it will easily be seen, is $180° + \alpha$. If the direction AB is taken as the *forward direction* of the line and the bearing in that direction as the *forward bearing*, the bearing in the *back* or *reverse direction* BA differs from the forward bearing by 180°, and is known as the *back* or *reverse bearing* of the line as viewed from station A. It will thus be seen that the back bearing of AB at station A is the forward bearing of BA at station B.

By drawing diagrams for each case, the student can verify the following rules:

If forward bearing is in quadrants I or II, back bearing = forward bearing + 180°.

If forward bearing is in quadrants III or IV, back bearing = forward bearing − 180°.

These rules also follow from the rules for working out bearings from angles, because, since BA is the direction of AB turned clockwise through 180°, the bearing of BA can be obtained by adding the angle of reversal (180°) to the bearing of AB.

Examples:

Forward bearing ..	67°	131°	216°	331°	348°	12°
	180	180	−180	−180	−180	180
Back bearing ..	247	311	36	151	168	192

9. Carrying Bearings Forward.

We are now in a position to consider the computation of the bearings of a series of connected lines in which the forward bearing of the first

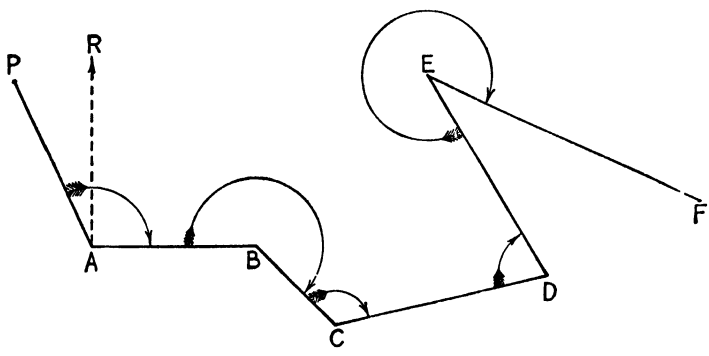


Fig. 4.8

line is known, or can be observed, and the angle between each pair of lines has been measured. This is a practical problem which arises in the computation of every theodolite traverse.

Fig. 4.8 shows a traverse starting at station A and ending at station F. AP is a line whose bearing is known, and the angle PAB is measured so as to obtain the bearing of AB, the first traverse leg. The angles at B, C, D and E, each measured whole-circle clockwise from the rear station, are also observed. AR is the standard direction from which all bearings are reckoned. The procedure, which depends on finding the back bearing of each line and then adding directly to it the observed angle at the station, can best be illustrated by a numerical example.

Let us therefore assume that the bearing of AP is 334° 10′ 46″ and the angles, each measured to the nearest second, are:

$$A = 115^\circ\ 10'\ 18''; \quad B = 224^\circ\ 36'\ 41''; \quad C = 123^\circ\ 22'\ 04''; \quad D = 72^\circ\ 46'\ 51''; \quad E = 326^\circ\ 54'\ 56''.$$

The work can then be set out as follows:

AP =	334°	10′	46″
A =	115	10	18
	449	21	04
	360		
AB =	89	21	04
	180		
BA =	269	21	04
B =	224	36	41
	493	57	45
	360		
BC =	133	57	45
	180		
CB =	313	57	45

CB =	313°	57′	45″
C =	123	22	04
	437	19	49
	360		
CD =	77	19	49
	180		
DC =	257	19	49
D =	72	46	51
DE =	330	06	40
	180		
ED =	150	06	40
E =	326	54	56
	477	01	36
	360		
EF =	117	01	36

Here AB means the bearing of the line AB in the direction A to B and BA the back bearing, or the bearing in the direction B to A, and *A*, *B*, *C*, *D* and *E* are the observed angles at the stations measured clockwise from the back station as shown. The work has also been set out here in full, but, after a very little practice, the computer will no longer find it necessary to write down 360° or 180° when these quantities have to be subtracted or added, as this part of the work can be done mentally, and the forward and back bearings written down straight away.

If it is assumed that the positions of the points B, C, D, E and F have not to be fixed, so that it is not necessary to measure the lengths of the legs AB, BC, CD, DE and EF, the traverse becomes a *bearing traverse.* Such a traverse is sometimes useful in order to carry a bearing, and a bearing alone, from a point A to a distant point F, and the points A and F are not intervisible.

An alternative method of carrying bearings forward from observed angles is to use angles of deflection, measured right or left of the first line of the angle *as viewed from the forward direction of that line,* i.e. the direction in which work is proceeding. To calculate these angles of deflection and determine their " sense ", i.e. their direction right or left, use the following rule, which can easily be verified from the figure:

When the whole-circle angle is greater than 180°, *the deflection angle is Right and is obtained by subtracting* 180° *from the whole-circle angle. When the whole-circle angle is less than* 180°, *the deflection angle is Left and is obtained by subtracting the whole-circle angle from* 180°.

Then, having obtained the deflection angles, use the rule given on p. 39 to get the bearing of the forward (second) line.

Thus, in the last example, denoting the deflection angles by dashes above the letter:

$$\begin{array}{rrrrl} A = & 115^\circ & 10' & 18'' & \\ & 180 & & & \\ \hline A' = & 64 & 49 & 42 & \text{L.} \end{array} \qquad \begin{array}{rrrrl} B = & 224^\circ & 36' & 41'' & \\ & 180 & & & \\ \hline B' = & 44 & 36 & 41 & \text{R.} \end{array} \qquad \begin{array}{rrrrl} C = & 123^\circ & 22' & 04'' & \\ & 180 & & & \\ \hline C' = & 56 & 37 & 56 & \text{L.} \end{array}$$

$$\begin{array}{rrrrl} D = & 72' & 46' & 51'' & \\ & 180 & & & \\ \hline D' = & 107 & 13 & 09 & \text{L.} \end{array} \qquad \begin{array}{rrrrl} E = & 326^\circ & 54' & 56'' & \\ & 180 & & & \\ \hline E' = & 146 & 54 & 56 & \text{R.} \end{array}$$

Also, if we are to apply the same rule at A as we apply at B, C, D and E, we must start with the initial bearing in the direction PA, which is 154° 10′ 46″. Accordingly, we write:

$$\begin{array}{rrrrl} \text{PA} = & 154^\circ & 10' & 46'' & \\ A' = & 64 & 49 & 42 & \text{L.} \\ \hline \text{AB} = & 89 & 21 & 04 & \\ B' = & 44 & 36 & 41 & \text{R.} \\ \hline \text{BC} = & 133 & 57 & 45 & \\ C' = & 56 & 37 & 56 & \text{L.} \\ \hline \text{CD} = & 77 & 19 & 49 & \end{array} \qquad \begin{array}{rrrrl} \text{CD} = & 77^\circ & 19' & 49'' & \\ & 360 & & & \\ \hline & 437 & 19 & 49 & \\ D' = & 107 & 13 & 09 & \text{L.} \\ \hline \text{DE} = & 330 & 06 & 40 & \\ E' = & 146 & 54 & 56 & \text{R.} \\ \hline & 477 & 01 & 36 & \\ & 360 & & & \\ \hline \text{EF} = & 117 & 01 & 36 & \end{array}$$

There is not a great deal to choose between these two methods as regards the actual amount of work involved, but, on the whole, probably the first method is the quicker and, if anything, it is the simpler to remember and to derive from first principles. Moreover, the second method has the disadvantage of all deflection angles in that it is so easy to make a mistake about the direction of a deflection. One good plan is to work by one method and check by the other.

A quick check on the minutes and seconds, and also on the units in the degrees, may be obtained by adding all the angles and the initial bearing together. The result will be the bearing of the final line plus some multiple of 180°. Thus, in the above example:

AP	=	334°	10′	46″
A	=	115	10	18
B	=	224	36	41
C	=	123	22	04
D	=	72	46	51
E	=	326	54	56
Sum	=	1197	01	36
6 × 180	=	1080		
EF	=	117	01	36

This method, therefore, is not by itself a check against a wrong addition or subtraction of 180°, but it is useful nevertheless as a check on the minutes and seconds, and on the odd degrees, and in survey work it is always well to have independent checks on all computations.

10. Check on the Angles of a Closed Figure.

When a closed figure is involved, the angles may be checked by adding them together. In fig. 4.9*a* (p. 44) the survey was made in the direction of the large arrow, so that the angles measured were the interior angles, yielding the following results:

A	B	C	D	E	F
121°	92°	146°	122°	134°	105°

The sum of these is equal to 720°, which is equal to $n \times 180° - 360°$, where n is the number of sides in the figure, in this case 6. In fact, we have the rule:

In any closed figure of n sides the sum of the internal angles is equal to $n \times 180° - 360°$.

This rule can easily be proved by joining the corners of the figure to some arbitrarily chosen internal pole O. This gives n triangles, or one triangle to each side, and the sum of the angles in each triangle is 180°. Hence, the sum for the n triangles is $n \times 180°$. But this includes the four right angles at O, which therefore have to be deducted.

In practice, the actual sum will seldom or never be exactly the same as the theoretical sum, but the difference should be very small, and will represent an accumulation of the ordinary small errors of observation.

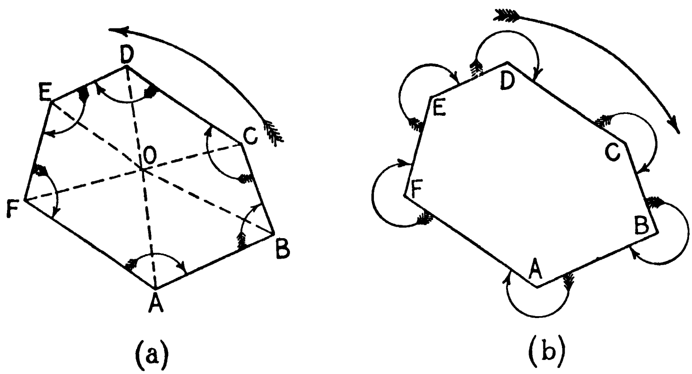


Fig. 4.9

If the survey had been made in the direction of the large arrow shown in fig. 4.9*b*, the external angles of the figure would have been measured as indicated, and it can easily be verified that *the theoretical sum is now* $n \times 180° + 360°$.

QUESTIONS ON CHAPTER IV

1. Convert the following whole-circle angles into angles on the half-circle system:

   36° 12′; 76° 14′; 237° 10′; 296° 14′ 51″; 136° 12′ 18″; 260° 48′ 19″.

2. The following are the readings on a theodolite graduated on the whole-circle system:

Station A:	0° 18′;	96° 18′ 02″;	261° 19′ 12″;	325° 14′ 10″.
Station B:	64° 11′;	212° 17′ 28″;	334° 41′ 18″;	351° 27′ 22″.
Station C:	168° 12′;	337° 46′ 52″;	358° 26′ 07″;	72° 18′ 16″.
Station D:	235° 05′;	18° 36′ 14″;	14° 12′ 48″;	151° 19′ 36″.
Station E:	331° 58′;	78° 41′ 15″;	112° 16′ 19″;	231° 14′ 47″.

   Write down the whole-circle angles actually measured, all reckoned from the R.O., Station A, each set of angles being taken from a different instrument station.

3. The following whole-circle readings were taken on a theodolite:

   Station B: 136° 12′; 0° 15′; 321° 17′; 221° 54′.

   Station A: 221° 18′; 248° 53′; 93° 42′; 46° 16′.

   What are the whole-circle angles reckoned (1) clockwise from Station A, (2) clockwise from Station B?

4. Write down the back or reverse bearings corresponding to the following forward bearings:

   16° 14′ 42″; 234° 17′ 28″; 196° 36′ 13″; 328° 16′ 54″; 93° 56′ 27″; 291° 06′ 21″; 358° 12′ 16″.

5. Write down the reduced bearings corresponding to the whole-circle forward bearings in Question 4, specifying by the letters N, S, E, and W, the quadrant in which the reduced bearing lies.

6. The bearing of the line OA and the included angle, measured whole-circle clockwise from OA, are as follows:

   Bearing of OA: 318° 16′; 18° 26′; 168° 11′; 238° 54′.

   Included angle AOB: 331° 14′; 194° 23′; 212° 14′; 116° 12′.

   In each case write down the bearing of OB.

7. Convert the following bearings, measured right or left from OA, into whole-circle bearings referred to OA:

   18° 31′ R; 36° 14′ L; 122° 12′ L; 131° 13′ R; 110° 12′ L; 164° 48′ R.

8. Convert the following observed reduced magnetic bearings into whole-circle bearings:

   S. 34½° W.; N. 27¼° E.; N. 72¼° W.; S. 86½° E.; S. 22½° E.; N. 48¼° W.; S. 38½° W.; N. 78¾° E.; S. 64° 14′ W.; N. 81° 56′ W.

9. The bearing of a line AP is 312° 18′ 33″ and the following whole-circle angles were observed at stations A, B, C and D:

   PAB = 164° 14′ 18″ BCD = 246° 13′ 22″

   ABC = 210° 17′ 17″ CDE = 64° 42′ 55″

   What is the bearing of the line DE?

10. Convert the above angles into deflection angles measured left or right from the forward direction of the first leg, and then calculate the whole-circle bearings.

11. The following were the whole-circle interior angles observed in a closed figure of 6 sides. Do these angles close the figure? If not, by how much do they fail to close?

    110° 16′ 36″; 122° 41′ 16″; 191° 38′ 10″; 36° 12′ 54″; 131° 55′ 12″; 127° 14′ 22″.

CHAPTER V

RECTANGULAR CO-ORDINATES

Let O in fig. 5.1 be a fixed point, OX a fixed direction passing through O, OY a direction passing through O at right angles to OX, and P some point whose position with relation to O we wish to define. From P draw lines PN and PM perpendicular to OX and OY respectively. As O and the lines OX and OY are fixed, the lengths of the lines MP and NP, which are equal respectively to ON and OM, define

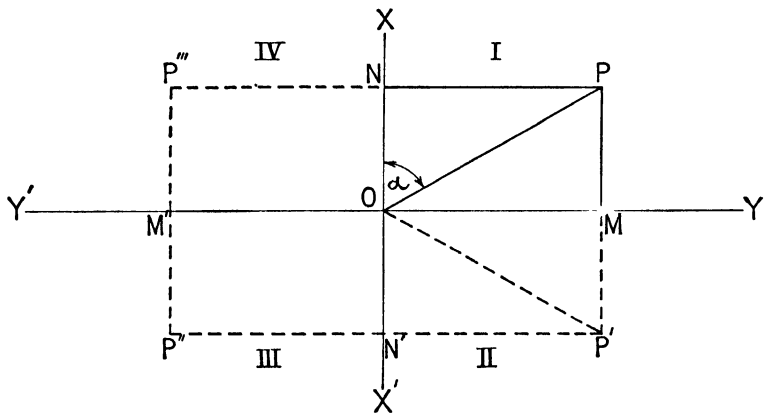


Fig. 5.1

the position of P with reference to O. MP and NP, or ON and OM, are then called the *rectangular co-ordinates* of P and are usually written x and y. Thus, if P is defined as being 13,256 ft. north of OY and 27,450 ft. east of OX, we write:

$$x = 13{,}256; \quad y = 27{,}450,$$

and these quantities clearly define the position of P. The line OX is called the *axis of X* and the line OY the *axis of Y*.

The reader already familiar with the elements of co-ordinate geometry will here recall that, in ordinary mathematical work, the horizontal axis is usually taken as the axis of X and the vertical axis as the axis of Y. In survey work, however, it is more usual to take the axes as we have taken them above.

Now consider the point P′, which lies in the second quadrant so that ON′ = ON and N′P′ = OM = NP. The co-ordinates of the point are x = ON′ and y = OM. Hence, unless we distinguish co-ordinates in the second quadrant in some way from those in the first, the co-ordinates of P′, as written, would be exactly the same as those of P. Similarly, those of P″ in the third quadrant and of P‴ in the fourth would be equal to those of P. Accordingly, in order to clarify matters, we specify that all x co-ordinates lying above (or to the north of) the line YOY′ must be taken as positive, and those lying below (or to the south of) it as negative, while all y co-ordinates lying to the right (or to the east) of the line XOX′ must be taken as positive, and those lying to the left (or to the west) of it must be taken as negative. We may then draw up the following table:

Point in Quadrant	x co-ordinate	y co-ordinate
I	+	+
II	−	+
III	−	−
IV	+	−

In this way, supposing, as before, that in all cases the numerical value of the x co-ordinate was 13,256 ft. and of the y co-ordinate 27,450 ft., we would write the co-ordinates of the points P, P′, P″ and P‴ as

P	$x = +13{,}256$	$y = +27{,}450$
P′	$x = -13{,}256$	$y = +27{,}450$
P″	$x = -13{,}256$	$y = -27{,}450$
P‴	$x = +13{,}256$	$y = -27{,}450$

In actual practice in survey work we usually try to choose O, the *origin of co-ordinates*, so that it lies to the south and west of the area under survey, so that all co-ordinates, both x and y, are positive, although the *differences* of co-ordinates of different points may be either positive or negative according to the way one point lies with reference to the other.

In writing the co-ordinates of a point, the usual method is to write the x first and the y second, with a comma between. Thus: −13,256, +27,450, meaning $x = -13{,}256$, $y = +27{,}450$. In mathematical work a round bracket is usually put round the symbols standing for co-ordinates defining the position of a point, and this is also sometimes done when writing numerical values, e.g. (x, y), (—13,256, +27,450).

Another way of specifying the position of P in fig. 5.1 would be to use the angle XOP and the distance OP. If the direction of OX is taken as the fixed direction from which all angles are measured, the angle XOP is the bearing of P from O. Hence, this method is known in survey work as the *bearing and distance* method. The positions of the points P′, P″ and P‴ are then defined by the angles XOP′, XOP″ and XOP‴, the bearings of the points measured clockwise from OX, and the lengths OP′, OP″ and OP‴.

1. Relation between Bearings and Distances and Rectangular Co-ordinates.

Denoting the angle (or bearing) XOP by α and the distance OP by l, we see at once that

$$x = \text{ON} = \text{OP} \cos \alpha = l \cos \alpha,$$
$$y = \text{OM} = \text{NP} = \text{OP} \sin \alpha = l \sin \alpha.$$

If the point is at P′ in the second quadrant, the cosine of the angle XOP′ is negative, and the sine positive, while x is negative and y positive. Hence, we have as before

$$x = \text{ON}' = l \cos \alpha,$$
$$y = \text{OM} = \text{N}'\text{P}' = l \sin \alpha,$$

where, owing to the negative sign of $\cos \alpha$, x is negative, as it should be. From this it follows that the signs of the rectangular co-ordinates depend on the quadrant in which the bearing lies, and we can consequently construct the following table of signs:

Point in Quadrant	Bearing between	x	y
I	0° and 90°	+	+
II	90° and 180°	—	+
III	180° and 270°	—	—
IV	270° and 360°	+	—

2. Calculation of Co-ordinates from Bearing and Distance.

Most tables of logarithmic sines and cosines tabulate values only in terms of angles from 0° to 90°. In this case it is necessary to work out the reduced bearing (p. 36) and use this bearing for entering the tables, but the signs of the co-ordinates will depend on the quadrant in which the real bearing falls, as tabulated above. Thus, let OP = 5316·7 and the bearing of OP be 233° 43′. The reduced bearing for obtaining the cosine and sine will be $\alpha' = 233° 43' - 180° = 53° 43'$, but, as the real bearing is in the third quadrant, both x and y will be negative. The computation may be arranged as follows, assuming that logarithms to six places, and not a calculating machine, are used:

$$\begin{array}{ll} \log\cos\alpha' = \bar{1}{\cdot}772\,159 & \log\sin\alpha' = \bar{1}{\cdot}906\,389 \\ \log l = 3{\cdot}725\,642 & \log l = 3{\cdot}725\,642 \\ \hline \log x = 3{\cdot}497\,801 & \log y = 3{\cdot}632\,031 \\ x = -3146{\cdot}3 & y = -4285{\cdot}8 \end{array}$$

In practical computing it is better to set out the computation as follows:

$$\begin{array}{lll} \log x = 3{\cdot}497\,801 & & x = -3146{\cdot}3 \\ \log\cos\alpha' = \bar{1}{\cdot}772\,159 & & \\ \log l = 3{\cdot}725\,642 & & \\ \log\sin\alpha' = \bar{1}{\cdot}906\,389 & & \\ \log y = 3{\cdot}632\,031 & & y = -4285{\cdot}8 \end{array}$$

Here $\log l$ has only been written down once, instead of twice, thus not only saving a little writing but also running less risk of a copying error. Log x is obtained by adding $\log l$ to $\log\cos\alpha'$ and $\log y$ by adding $\log\sin\alpha'$ to $\log l$.

If a calculating machine is used instead of logarithms, the work might be set out as follows:

$$\begin{array}{ll} \cos\alpha = 0{\cdot}591\,779 & \sin\alpha = 0{\cdot}806\,100 \\ l = 5316{\cdot}7 & l = 5316{\cdot}7 \\ \hline x = l\cos\alpha = -3146{\cdot}3 & y = l\sin\alpha = -4285{\cdot}8 \end{array}$$

or, better still,

$$\begin{array}{rl} x = & -3146{\cdot}3 \\ \cos\alpha = & 0{\cdot}591\,779 \\ l = & 5316{\cdot}7 \\ \sin\alpha = & 0{\cdot}806\,100 \\ y = & -4285{\cdot}8 \end{array}$$

Here the natural cosine and natural sine take the place of the logarithmic cosine and logarithmic sine, and the length l is written instead of $\log l$. The natural cosine and natural sine are set in turn on the table of the machine, or both together if the machine is a double one, the handle turned to represent a multiplication by 5316·7, and the answers read on the register.

3. Calculation of Bearing and Distance from Co-ordinates.

If we are given the co-ordinates, the bearing and distance can also easily be computed, for, either from fig. 5.1 (p. 46) or from the relations $x = l \cos \alpha$, $y = l \sin \alpha$, we have

$$\tan \alpha = \frac{y}{x} \text{ and } l = x \sec \alpha = y \operatorname{cosec} \alpha.$$

The quadrant in which α lies depends on the signs of x and y, for which purpose the table on p. 48 may be consulted. The length l may be calculated from either of the expressions $l = x \sec \alpha$ or $l = y \operatorname{cosec} \alpha$, but in practice it is well to calculate from both, using one result as a check on the other.

Example.—Let $x = -4281{\cdot}6$; $y = +3614{\cdot}2$. Here, x is minus and y plus so that the bearing lies in the second quadrant. Using five-figure logarithms.

$$\begin{aligned} \log y &= 3{\cdot}558\,01 \\ \log x &= 3{\cdot}631\,61 \\ \hline \log \tan \alpha &= \bar{1}{\cdot}926\,40 \end{aligned}$$

Giving α' (the reduced bearing) $= 40° \; 10'{\cdot}1$. Hence, as the true bearing is in the second quadrant,

$$\alpha = 180° - 40° \; 10'{\cdot}1 = 139° \; 49'{\cdot}9.$$

Again,

$$\begin{aligned} \log x &= 3{\cdot}631\,61 & \qquad \log y &= 3{\cdot}558\,01 \\ \log \cos \alpha' &= \bar{1}{\cdot}883\,18 & \log \sin \alpha' &= \bar{1}{\cdot}809\,58 \\ \hline \log l &= 3{\cdot}748\,43 & \log l &= 3{\cdot}748\,43 \end{aligned}$$

the log l being obtained by subtracting the log cos α' or the log sin α' from the log x or the log y. Consequently, we have

$$l = 5603{\cdot}1; \quad \alpha = 139° \; 49'{\cdot}9.$$

If one co-ordinate is very much less in magnitude than the other, the reduced bearing will lie close to either 0° or 90°. In this case the log sine or the log cosine will be changing very rapidly in comparison with the log cosine or the log sine, and accordingly, if we are working with long lines and using six- or seven-figure logarithms, there may be small differences between the values of l obtained from the log sine and the log cosine.

In such an event, the value derived from the quantity which is changing most slowly should be accepted. Thus, working with co-ordinates to two decimal places and using seven-figure logarithms, let

$$x = +18{,}342{\cdot}16; \quad y = -394{\cdot}13.$$

Then,

$$\begin{aligned} \log y &= 2{\cdot}595\,6395 \\ \log x &= 4{\cdot}263\,4505 \\ \hline \log \tan \alpha' &= \bar{2}{\cdot}332\,1890 \end{aligned}$$

$$\alpha' = 1^\circ\, 13'{\cdot}859; \quad \alpha = 358^\circ\, 46'{\cdot}141.$$

$$\begin{aligned} \log x &= 4{\cdot}263\,4505 & \qquad \log y &= 2{\cdot}595\,6395 \\ \log \cos \alpha' &= \bar{1}{\cdot}999\,8998 & \log \sin \alpha' &= \bar{2}{\cdot}332\,0913 \\ \hline \log l &= 4{\cdot}263\,5507 & \log l &= 4{\cdot}263\,5482 \end{aligned}$$

$$\therefore\ l = 18{,}346{\cdot}40 \quad \text{or} \quad l = 18{,}346{\cdot}29.$$

Here we would accept the value 18,346·40 since the log cosine is changing much more slowly than the log sine. It should be noted, however, that in interpolating for α' and log sin α' from Chambers's Tables, which were used and which only tabulate the logarithms of the trigonometrical functions at intervals of 1′ of arc, we have interpolated linearly from the tables and have not taken into account the progressive variations in the differences between tabulated values. If we had employed special formulæ of interpolation involving the " second and third differences ", or the rules given in Section 5, p. 53, for finding the value of the small arc from the log tangent, and that of the log sine of a small angle, the two results would have agreed or would have been very much closer to one another.

4. The Negative Characteristic when Computing with Logarithms.

In survey work, as in many other forms of computing, the negative characteristic of the logarithm of a trigonometrical function, or of a logarithm dependent on it, is often not written with a bar over the number. Instead, we write a whole number, such as 9 or 19, it being understood that this number, which makes the whole logarithm positive, means that it must be added to -10, -20, etc. Thus, $\bar{1}{\cdot}883\,18$ would be written as 9·883 18, which really means the same thing as $-10 + 9 + 0{\cdot}883\,18$ or $-1 + 0{\cdot}883\,18$, and $\bar{3}{\cdot}774\,31$ would be written as 7·774 31, meaning $-10 + 7 + 0{\cdot}774\,31$. In tables of trigonometrical functions, a positive characteristic is always given, even when the actual characteristic is negative. Thus, the log sines of 12° and 0° 03′ are given as 9·317 88 and 6·940 85, meaning $\bar{1}{\cdot}317\,88$ and $\bar{4}{\cdot}940\,85$ respectively.

Sometimes the logarithm is written with the positive characteristic and followed by a -10, -20, etc. Thus, 9·317 88 $-$ 10, 18·476 28 $-$ 20, meaning $\bar{1}$·317 88 and $\bar{2}$·476 28.

With this convention, the calculation of $\log l$ in the first of the last two examples would be written:

$$\begin{aligned} \log x &= 3{\cdot}631\,61 \\ \log \cos \alpha' &= \underline{9{\cdot}883\,18} \\ \log l &= 3{\cdot}748\,43 \end{aligned} \qquad \begin{aligned} \log y &= 3{\cdot}558\,01 \\ \log \sin \alpha' &= \underline{9{\cdot}809\,58} \\ \log l &= 3{\cdot}748\,43 \end{aligned}$$

In this case of subtraction, 10 is added mentally to the characteristic of the upper figure, as the omitted -10 in the lower $\log \cos \alpha'$, $\log \sin \alpha'$ line becomes $+10$ when the sign is reversed on subtraction. The first subtraction, in fact, may be written $3{\cdot}631\,61 - (-10 + 9{\cdot}883\,18) = 13{\cdot}631\,61 - 9{\cdot}883\,18$. If the two logarithms had to be added instead of subtracted, 10 would have to be subtracted from the sum of the characteristics. Thus, the sum of $\log x$ and $\log \cos \alpha'$ would be $3{\cdot}631\,61 + (-10 + 9{\cdot}883\,18) = -10 + 13{\cdot}514\,79 = 3{\cdot}514\,79$.

Again, suppose that it is desired to find $z = l \sin \alpha \cos \beta$, where $l = 3187{\cdot}4$, $\alpha = 46° \, 51'$, $\beta = 63° \, 15'$, we have:

$$\begin{aligned} \log l &= 3{\cdot}503\,44 \\ \log \sin \alpha &= 9{\cdot}863\,06 \\ \log \cos \beta &= \underline{9{\cdot}653\,31} \\ \log z &= 3{\cdot}019\,81 \qquad \therefore z = 1046{\cdot}7. \end{aligned}$$

In this example, the apparent sum is really 23·019 81, but, as there is a -10 to be added in the case of both $\log \sin \alpha$ and $\log \cos \beta$, 20 has to be subtracted from 23·019 81, leaving 3·019 81.

This method of dealing with negative characteristics may be a little confusing at first to those not used to it, but, keeping in mind the negative 10, or multiple of 10, for which a positive number has been substituted, and by a careful study of the examples in the following pages, the matter should become clear.

5. Logarithmic Trigonometrical Functions of Angles close to 0° and 90°.

When angles are very small, values of the logarithmic sine and tangent are changing very rapidly and unevenly, and, unless the tables are given at very close intervals, ordinary linear interpolation will not give true values. Similarly, the logarithmic cosine and tangent of angles very close to 90° change very rapidly. Different rules for such cases are available, but the following, which are those given in the

Explanation of *Chambers's Seven-Figure Mathematical Tables*, are convenient and easy to use:

$$\log\sin\alpha = \log\alpha'' + 4{\cdot}685\,5749 - \tfrac{1}{3}(\log\sec\alpha - 10) \quad . \quad (1)$$
$$\log\tan\alpha = \log\alpha'' + 4{\cdot}685\,5749 + \tfrac{2}{3}(\log\sec\alpha - 10) \quad . \quad (2)$$
$$\log\alpha'' = \log\sin\alpha + \bar{5}{\cdot}314\,4251 + \tfrac{1}{3}(\log\sec\alpha - 10) \quad . \quad (3)$$
$$\log\alpha'' = \log\tan\alpha + \bar{5}{\cdot}314\,4251 - \tfrac{2}{3}(\log\sec\alpha - 10) \quad . \quad (4)$$

α'' is the value of the angle α expressed in seconds. In (3) and (4) the first thing to do is to find an approximate value of the angle by ordinary straight interpolation from the tables, and from this approximate value obtain the value of $\log\sec\alpha$. The characteristics of the logarithmic sine and tangent are the positive values in each case.

Thus, to find the angle corresponding to $\log\tan\alpha' = 8{\cdot}332\,1890$ and its log sine which entered into the example on p. 51, an approximation by direct interpolation gives $\alpha' = 1°\ 13'{\cdot}86$, and so $\log\sec\alpha = 10{\cdot}000\,1002$.

$$\begin{aligned}
\log\tan\alpha' &= 8{\cdot}332\,1890 \\
&\ \underline{\bar{5}{\cdot}314\,4251} \\
&\ 3{\cdot}646\,6141 \\
\tfrac{2}{3}(\log\sec\alpha - 10) &= \underline{0{\cdot}000\,0668} \\
\log\alpha'' &= 3{\cdot}646\,5473 \\
\therefore\ \alpha'' &= 4431''{\cdot}464. \\
\therefore\ \alpha' &= 1°\ 13'\ 51''{\cdot}464,
\end{aligned}$$

and to find $\log\sin\alpha'$,

$$\begin{aligned}
\log\alpha'' &= 3{\cdot}646\,5473 \text{ (from } \log\alpha'' \text{ above)} \\
&\ \underline{4{\cdot}685\,5749} \\
&\ 8{\cdot}332\,1222 \\
\tfrac{1}{3}(\log\sec\alpha - 10) &= \underline{0{\cdot}000\,0334} \\
\log\sin\alpha' &= 8{\cdot}332\,0888.
\end{aligned}$$

If this value of $\log\sin\alpha'$ is substituted for the value of $\log\sin\alpha'$ in the example on p. 51, the value obtained for $\log l$ is $4{\cdot}263\,5507$, which agrees exactly with the value originally obtained from $\log x$ and the log cosine.

When angles are close to 90°, the log cosine and log cotangent can be found by subtracting the angle from 90°, and then finding the log sine or log tangent of the resulting small angle.

6. Differences of Co-ordinates.

Although the positions of different points are defined in terms of rectangular co-ordinates referred to a single origin and a single vertical and horizontal axis, most survey calculations involve differences of co-ordinates rather than the co-ordinates themselves.

In fig. 5.2 we have two points A and B plotted in relation to an origin O and rectangular axes OX and OY. From A and B drop perpendiculars AC, AE and BD, BF on the axes OX and OY respectively. Produce CA to meet BF at G. Then, since AG and BG are

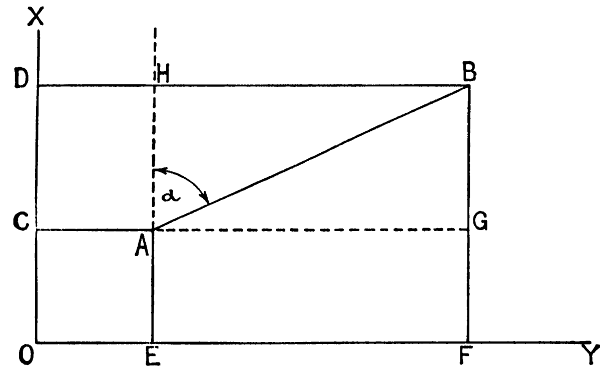


Fig. 5.2

perpendicular to OX and OY, the angle BGA is a right angle. Also, the rectangular co-ordinates of A are OC $= x_1$ and OE $= y_1$, and of B, OD $= x_2$ and OF $= y_2$. Hence, in the right-angled triangle AGB,

$$\text{GB} = \text{CD} = \text{OD} - \text{OC} = x_2 - x_1 = \Delta x,$$
$$\text{AG} = \text{EF} = \text{OF} - \text{OE} = y_2 - y_1 = \Delta y,$$

where the symbols Δx, Δy denote finite differences in x and y.

If we produce EA to H, we have $\angle$ABG $= \angle$HAB. But AH is parallel to OX, and the direction of OX is the fixed direction from which bearings are reckoned. Consequently, $\angle$HAB $= \alpha$ is the bearing of the line AB, and we have

$$\text{GB} = \Delta x = \text{AB} \cos \alpha = l \cos \alpha,$$
$$\text{AG} = \Delta y = \text{AB} \sin \alpha = l \sin \alpha,$$

where l is the length of AB.

These expressions are exactly similar in form to those found on p. 48. In fact, for the calculation of Δx and Δy we may consider the original origin to be transferred to A and then, using axes through A parallel to the original axes, treat the line AB exactly as we treated OP in fig. 5.1 (p. 46).

From these formulæ it follows that

$$\begin{aligned} x_2 &= x_1 + \Delta x \\ & x_1 + l \cos \alpha; \\ y_2 &= y_1 + \Delta y \\ & y_1 + l \sin \alpha. \end{aligned}$$

The reverse formulæ, the calculation of bearing and distance from given co-ordinates, follow exactly as before, for we have

$$\tan \alpha = \frac{AG}{BG} = \frac{EF}{CD} = \frac{\Delta y}{\Delta x},$$

and

$$\begin{aligned} l = AB = GB \sec \alpha = CD \sec \alpha &= (x_2 - x_1) \sec \alpha = \Delta x \sec \alpha \\ = AG \operatorname{cosec} \alpha = EF \operatorname{cosec} \alpha &= (y_2 - y_1) \operatorname{cosec} \alpha = \Delta y \operatorname{cosec} \alpha. \end{aligned}$$

As before, we must pay proper attention to signs to determine the quadrant in which AB lies, using the same rules as before but substituting Δx and Δy for x and y. All the formulæ for co-ordinate differences hold, of course, for all the four quadrants in which the line AB may lie, provided the proper signs are given to the trigonometrical functions or to Δx and Δy.

Example 1.—Given that the co-ordinates of a point A are $x = 10{,}481{\cdot}6$, $y = 26{,}384{\cdot}2$, and that the length and bearing of the line AB are 3784·6 and 296° 34′, find the co-ordinates of the point B.

The bearing is in the fourth quadrant so that Δx is positive and Δy negative. Also the reduced bearing is $\alpha' = 360° - 296° \, 34' = 63° \, 26'$. Writing down the computation as before, but with Δx and Δy substituted for x and y,

$\log \Delta x$	= 3·228 56	$\Delta x = +1692{\cdot}6$
$\log \cos \alpha'$	= 9·650 54	
$\log l$	= 3·578 02	
$\log \sin \alpha'$	= 9·951 54	
$\log \Delta y$	= 3·529 56	$\Delta y = -3385{\cdot}0$

		x	y
Co-ordinates of A	=	10,481·6	26,384·2
Δx and Δy		+ 1,692·6	− 3,385·0
Co-ordinates of B	=	12,174·2	22,999·2

Example 2.—Given that the co-ordinates of the points A and B are $x_1 = 43{,}223{\cdot}62$, $y_1 = 28{,}461{\cdot}13$, and $x_2 = 37{,}334{\cdot}18$, $y_2 = 36{,}141{\cdot}84$ respectively, find the bearing and length of the line AB.

		x			y
Co-ordinates of B	=	37,334·18			36,141·84
Co-ordinates of A	=	43,223·62			28,461·13
Δx	=	− 5,889·44	Δy	=	+ 7,680·71

The bearing is in the second quadrant since Δx is minus and Δy plus. Using six-figure logarithms,

$$\begin{aligned} \log \Delta y &= 3{\cdot}885\ 401 \\ \log \Delta x &= 3{\cdot}770\ 074 \\ \log \tan \alpha' &= 0{\cdot}115\ 327 \\ \therefore\ \alpha' &= 52^\circ\ 31'\ 10''{\cdot}5, \end{aligned}$$

and, as the true bearing is in the second quadrant,

$$\alpha = 180^\circ - \alpha' = 127^\circ\ 28'\ 49''{\cdot}5.$$

Again,

$$\begin{aligned} \log \Delta x &= 3{\cdot}770\ 074 & \log \Delta y &= 3{\cdot}885\ 401 \\ \log \cos \alpha' &= 9{\cdot}784\ 253 & \log \sin \alpha' &= 9{\cdot}899\ 581 \\ \log l &= 3{\cdot}985\ 821 & \log l &= 3{\cdot}985\ 820 \end{aligned}$$

$$\therefore\ l = 9678{\cdot}78; \quad \alpha = 127^\circ\ 28'\ 49''{\cdot}5.$$

This calculation of a bearing from co-ordinates is one that is often needed in survey work. For instance, a straight line has to be cut through dense forest between two fixed points, or else two fixed points have to be joined by a tunnel. In both cases, the co-ordinates of the fixed points are first found by triangulation or traverse and the bearing between the points calculated as above. Then, starting at one point with a line of known bearing, the angle necessary to align the theodolite on the computed bearing can easily be calculated and the instrument set to that angle.

7. Total Co-ordinates: Latitudes and Departures.

Let A, B, C, D, E (fig. 5.3) be five points whose rectangular co-ordinates referred to origin O and rectangular axes OX and OY are (x_1, y_1), (x_2, y_2), (x_3, y_3), (x_4, y_4), (x_5, y_5). From A, B, C, D and E draw perpendiculars AE, BF, CG, DH and EI on axis OX and AJ, BK, CL, DM and EN on axis OY. The co-ordinates of A are therefore $x_1 = \text{OE}$, $y_1 = \text{OJ}$; of B, $x_2 = \text{OF}$, $y_2 = \text{OK}$; of C, $x_3 = \text{OG}$, $y_3 = \text{OL}$; and so on.

Let $x_2 - x_1 = \Delta x_1$, $y_2 - y_1 = \Delta y_1$.

Then $x_2 = \text{OF} = \text{OE} - \text{EF}$. But $\text{EF} = \Delta x_1$, which, since x_2 is less than x_1, is negative. Thus, allowing for the sign of Δx_1, we can write $x_2 = x_1 + \Delta x_1$. Similarly, $y_2 = \text{OK} = \text{OJ} + \text{JK} = y_1 + \Delta y_1$.

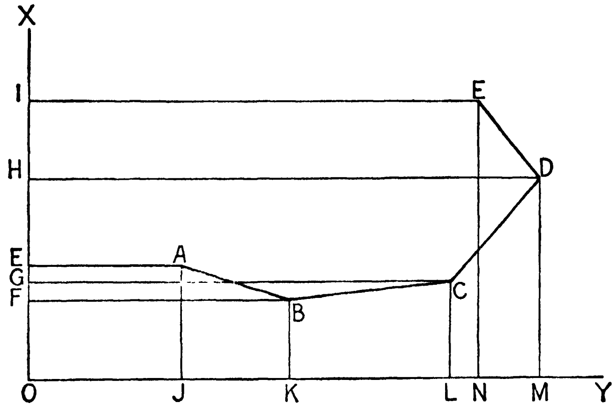


Fig. 5.3

Also, it will be seen that $x_5 = \text{OI} = \text{OH} + \text{HI} = x_4 + \Delta x_4$ and $y_5 = \text{OM} - \text{MN}$, which, since Δy_4 is negative, may be written $y_5 = y_4 + \Delta y_4$. In general, we may therefore write

$$x_n = x_1 + \Delta x_1 + \Delta x_2 + \Delta x_3 + \ldots + \Delta x_{n-2} + \Delta x_{n-1},$$
$$y_n = y_1 + \Delta y_1 + \Delta y_2 + \Delta y_3 + \ldots + \Delta y_{n-2} + \Delta y_{n-1},$$

the proper signs being given in each case to the differences Δx_1, Δx_2, Δx_3, Δx_4, . . . and Δy_1, Δy_2, Δy_3, Δy_4 . . . in accordance with the bearings of the lines and the particular quadrants in which they lie.

The differences of co-ordinates Δx_1, Δx_2, Δx_3, . . . and Δy_1, Δy_2, Δy_3, . . . are often, particularly in traverse work, called the *latitudes* and *departures*, and the co-ordinates of a point referred to the origin the *total latitude* and *total departure*. Latitudes and departures are also sometimes called *northings* or *southings* and *eastings* or *westings*, according to the direction in which they run, and the total latitude and total departure the *total northing* or *total southing* and *total easting* or *total westing*, as the case may be. Very often, however, the word "total" is omitted from these expressions and the terms "latitude" and "departure", or "northing", "southing", "easting" and "westing" applied to describe the actual co-ordinates of a point.

If $l_1, l_2, l_3, \ldots l_{n-2}, l_{n-1}$ are the lengths of the legs AB, BC, CD, etc., and $\alpha_1, \alpha_2, \alpha_3, \ldots \alpha_{n-2}, \alpha_{n-1}$ are the *forward* bearings of these legs reckoned clockwise from directions parallel to the direction of the axis OX, then

$$\Delta x_1 = l_1 \cos \alpha_1; \ \Delta x_2 = l_2 \cos \alpha_2; \ \Delta x_3 = l_3 \cos \alpha_3; \ \ldots$$
$$\Delta y_1 = l_1 \sin \alpha_1; \ \Delta y_2 = l_2 \sin \alpha_2; \ \Delta y_3 = l_3 \sin \alpha_3; \ \ldots$$

and we have

$$x_n = x_1 + l_1 \cos \alpha_1 + l_2 \cos \alpha_2 + l_3 \cos \alpha_3 + \ldots + l_{n-2} \cos \alpha_{n-2} + l_{n-1} \cos \alpha_{n-1},$$
$$y_n = y_1 + l_1 \sin \alpha_1 + l_2 \sin \alpha_2 + l_3 \sin \alpha_3 + \ldots + l_{n-2} \sin \alpha_{n-2} + l_{n-1} \sin \alpha_{n-1}.$$

The last two expressions comprise the fundamental formulæ used in computing traverses, and in Chap. VII we shall give a numerical example of a traverse computation and of the way in which it is best arranged.

8. Calculation of Areas from Co-ordinates.

In fig. 5.4 ABCDEFA is a closed traverse of which A is the most westerly station. Through A draw the line **x**A**x**′ parallel to the axis of X, and from B, C, D, E and F draw perpendiculars B**b**, C**c**, D**d**, E**e** and F**f** to **x**A**x**′. Then it can be seen that

area of ABCDEFA = area B**bc**C + area C**cd**D + area D**de**E + area E**ef**F − area A**f**F − area A**b**B.

But the area of any figure such as D**de**E = $\frac{1}{2}$(**de**)(**d**D + **e**E). Now $\frac{1}{2}$(**d**D + **e**E) = **qp**, where **p** is the middle point of the leg DE and **pq** is the perpendicular from **p** on **x**A**x**′. This quantity represents the *departure* of the point **p** with reference to A and is called the *longitude* of the leg DE. Denoting it by L_4, let L_1, L_2, L_3, L_5, L_6 be the longitudes of the legs AB, BC, CD, EF, FA. The quantity **de**, taken with its proper sign, is the latitude of the leg DE, which we shall call Δx_4. Accordingly,

$$\text{area DdeE} = \Delta x_4 \times L_4,$$

and, if Δx_1, Δx_2, Δx_3, Δx_5, Δx_6 denote the latitudes of the other legs,

$$\text{area ABCDEFA} = \Delta x_2 L_2 + \Delta x_3 L_3 + \Delta x_4 L_4 + \Delta x_5 L_5 - \Delta x_6 L_6 - \Delta x_1 L_1.$$

Here it will be noticed that the Δx's of all the plus terms are positive since they are measured upwards from one point to the next, while the Δx's of the negative terms are negative since they are measured downwards from one point to the next. Hence, figures which have positive latitudes are positive, and those which have negative latitudes are negative, so that the total area is the algebraic sum of the products of the latitudes and longitudes.

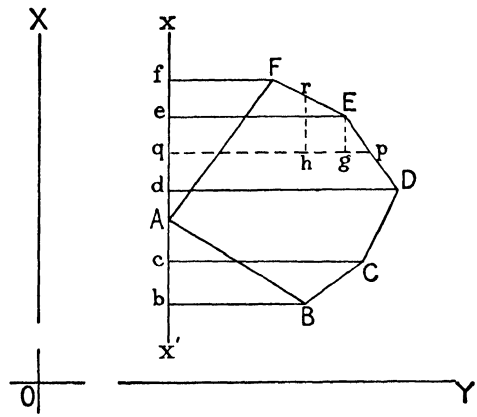


Fig. 5.4

In practice, instead of taking the mean of the two departures of the ends of the leg as the longitude of the leg, it is more convenient simply to take the sums of the two departures and call this the *double longitude*. Then we have the rule:

Twice area of figure equals algebraic sum of the latitudes each multiplied by its corresponding double longitude.

Also, to obtain the double longitudes we have the rule:

The double longitude of either of the lines meeting at the most westerly point is the departure of that line. That of any other line is the algebraic sum of the double longitude of the previous line, plus the departure of that line, plus the departure of the line itself.

Thus, if **r** is the middle point of EF and **rh** and **Eg** are perpendiculars from **r** and E on **qp**, the double longitude of line EF in fig. 5.4 is 2(**qp** — **pg** — **gh**), i.e. double longitude of DE + departure of DE + departure of EF.

Example.

Line	Latitude, ft.	Departure, ft.	Double longitude	Latitude × double longitude	
				+	−
AB	−548	643	643		352,364
BC	327	464	1750	572,250	
CD	484	291	2505	1,212,420	
DE	431	−174	2622	1,130,082	
EF	181	−586	1862	337,022	
FA	−875	−638	638		558,250

3,251,774   −910,614
− 910,614

Algebraic sum = twice area = 2,341,160.

Area = 1,170,580 sq. ft.
= 26·873 acres.

Note the check on the working of the double longitudes, as here the sum of the double longitude of the line before the closing line plus the departure of that line plus the departure of the closing line should equal in magnitude the departure of the latter.

Another rule for determining areas from co-ordinates which the reader may be interested to derive for himself is:

Twice the area of a closed figure equals the algebraic sum of the products formed by multiplying the total latitude (or x co-ordinates) of each station by the algebraic sum of the departures of the two lines which adjoin the station.

In order to avoid unnecessarily large figures, the total latitudes are best taken from one station, which is thus used as a local origin. This method is useful for checking a computation carried out by the method of double longitudes. Thus, the following example is a check on the example given above.

Example.

Line	Latitude	Departure	Station	Total latitude	Sum of adjoining departures	Products +	Products −
AB	−548	643	A	0			
BC	327	464	B	−548	+1107		606,636
CD	484	291	C	−221	+ 755		166,855
DE	431	−174	D	+263	+ 117	30,771	
EF	181	−586	E	+694	− 760		527,440
FA	−875	−638	F	+875	−1224		1,071,000

+ 30,771 − 2,371,931
+ 30,771

Algebraic sum = twice area = 2,341,160

Area = 1,170,580 sq. ft.
= 26·873 acres.

QUESTIONS ON CHAPTER V

1. Given that the co-ordinates of a point A are

$$x \text{ (northing)} = +\ 10{,}342{\cdot}1;\quad y \text{ (easting)} = -8369{\cdot}6;$$

and that the distance and whole-circle bearing to point B are $l = 3621{\cdot}1$ and $\alpha = 74° \ 18' \ 30''$, find the co-ordinates of B.

2. Given that the co-ordinates of A are as above, but $l = 3621{\cdot}1$ and $\alpha = 221° \ 16' \ 20''$, find the co-ordinates of B.

3. Given that the co-ordinates of two points A and B are

	x	y
A	41,612·6	31,178·4
B	42,196·3	24,579·6

find the bearing and distance from A to B.

4. Given that the co-ordinates of three points A, B and C are

	x	y
A	26,141·6	10,862·1
B	19,327·8	13,995·5
C	21,743·6	12,061·7

find the bearings and distances AB, AC, BC.

5. Given the following distances and bearings of the lines AB and AC, find the bearing and distance of BC.

	l	α
AB	4316·1	241° 18′ 10″
AC	2197·6	319° 10′ 50″

6. The following are the latitudes and departures of a closed traverse:

	Latitude	Departure
AB	+ 411	+ 563
BC	+ 83	+ 897
CD	−1398	+ 236
DE	− 397	−1058
EA	+1301	− 638

Find the area of the figure enclosed by the traverse correct to the second decimal place of an acre.

CHAPTER VI

TRIANGULATION

1. Introduction.

In Chapter II we have described simple cases where, by means of a triangle of which the lengths of all the sides are known or are measured, a third point can be fixed from two others whose positions are already fixed. In this chapter we shall consider the application of the same principle, but on a very much larger scale, and with triangles of which the length of one side is known and at least two of the three angles are measured.

The object of the triangulation now to be considered is to establish over a certain area a number of points whose positions are accurately fixed, and from which a detail survey can be carried out, or else to fix the relative positions of two or more widely separated points. The area to be covered may be very large, as in the case of a national survey, or it may be comparatively small, as when triangulation is used as the main framework on which to hang the survey of a town or large estate. Corresponding to the area involved, the average length of side will vary within certain limits. Thus, in the case of a national survey where the area may be very large, the lengths of sides of the main triangles will be as long as visibility and the nature of the ground permit—say between 20 and 80 miles: in the case of a small survey, such as that of a town, the lengths of the sides of the triangles will be very short—say half a mile to two miles.

In national surveys, triangulation may be arranged in chains or in networks. In the former case, simple figures of triangulation, such as single triangles or simple braced quadrilaterals, are extended one after another until long, but relatively narrow, chains of triangulation are built up, the area concerned being split up into rough rectangles each side of which consists of a chain of triangulation. Fig. 6.1 shows part of such a scheme. In this figure, the chains **a**, **b**, **c** and **d** consist of braced quadrilaterals and **e**, **f** and **g** of series of single triangles, all linked together. The areas between the main chains are *broken down* by chains of secondary and minor triangulation, or else by numerous theodolite traverses. This breaking-down process is not shown in the diagram.

A small network of triangulation is shown in fig. 6.2 (p. 65), the arrangement of the triangles being, of course, controlled by the nature of the local topography. The trigonometrical framework of the Ordnance Survey of Great Britain consists of a close network; in the United States, this framework consists of chains of triangulation.

When the area concerned is comparatively small, as in most surveys that are likely to be met with in ordinary engineering practice, a

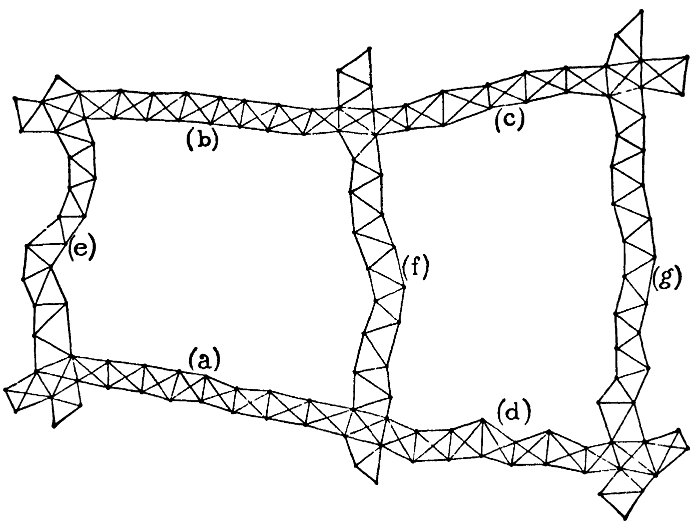


Fig. 6.1

network is a more suitable arrangement and is more usual than closed circuits of chains of triangulation. If, however, the work is only required for the fixation of points a considerable distance apart, as in a survey for fixing points near the ends of a proposed tunnel which are required for the computation of a bearing, a network is not needed and a single chain would be used.

All the triangulation in a scheme is not always of the same order of accuracy, but is generally divided into *primary*, *secondary* and *tertiary*, or *first-order*, *second-order* and *third-order* work. The primary work is the most accurate and forms a skeleton on which the secondary and tertiary work is hung, the secondary work being more accurate than the tertiary and serving to control it. In addition, the lengths of the sides of the triangles vary with the order of accuracy, those of the primary triangulation in a national survey being from 20 to 60 miles in length, or even more; of the secondary from 10 to 20 miles, and of the tertiary from 1 or 2 to 10 miles. Single triangles of the first order therefore cover a fairly considerable area, leaving large gaps to be filled in with the secondary work. The secondary triangulation breaks down the primary, but still leaves points too inconveniently far apart for the survey of detail. Consequently, the gaps left by the secondary work are filled in with the tertiary triangles. The real object of the secondary and tertiary work is therefore to enable points close enough for detail survey to be established. Sometimes, however, if the work is needed for something other than as a control for the detail survey, the secondary and tertiary work is omitted altogether.

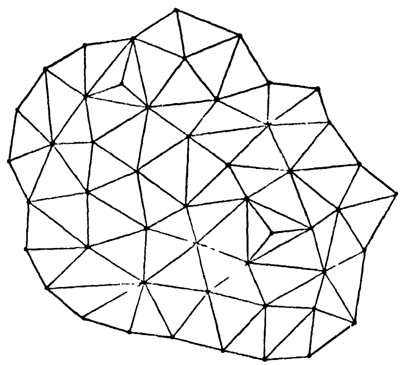


Fig. 6.2

There is no great difference in the instruments or methods used in surveying the different orders of triangulation. Often all three are measured with the same instrument, or type of instrument, but more observations are taken at the primary points than at the secondary, and more at the secondary than at the tertiary points. Sometimes smaller theodolites are used for the observation of the minor triangles.

The necessity for the points between which observations have to be taken being intervisible usually results in the primary and secondary points being on high hills—often inconveniently high for the topographer or detail surveyor. Sometimes, in order to ensure intervisibility, special towers of anything up to about 120 ft. in height, from and to which observations can be taken, have to be used.

In order to start a triangulation, it is necessary to have one side of a triangle whose length and azimuth or bearing are known or measured, and one point, usually one end of the same line, whose position has been fixed in terms of latitude and longitude, or other suitable system of co-ordinates. If the side is measured, it is known as a *base line*. Since, however, the measurement of the length of a base line demands a stretch of fairly level country over which the two ends of the line are intervisible, it would be very difficult to get a suitable line as long as a side of primary triangulation. Moreover, most of the primary points would be on hills, and precise linear measurements up or down the slopes of these hills would be virtually impossible. Accordingly, it is usual to select a line considerably shorter than the sides of the primary triangulation and, by means of carefully selected and very carefully observed triangles, gradually increasing in size, to " extend " the base until the length of one side of the triangulation can be computed.

Thus, in fig. 6.3, **ab** is the measured base, and all the angles of the triangles **abc**, **abd**, **acd** and **bcd** are observed. This enables the length of **cd** to be computed in (in this case) four different ways. Similarly,

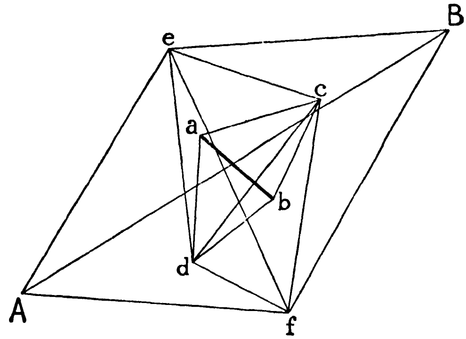


Fig. 6.3

all the angles of the triangles **cdf**, **cfe**, **cde** and **dfe** are observed, so giving the length **ef**. This process is repeated with the figure AeBf, and this enables the length of AB, a side of the main triangulation, to be computed from **ef**. Such a figure as this, which enables a long length to be computed from a rather short measured base, is known as a *base extension*. With small schemes of triangulation, in which the triangles have relatively short sides, a base extension is normally not needed because the triangles are small, with short sides, and the length of one of them can easily be measured directly. The scheme shown in the diagram, it should be noted, is only one example of a great number of other possible schemes.

Up to the time of the invention of *Invar*, a metal with a very low coefficient of thermal expansion, the measurement of a base line was an exceedingly tedious, slow, and costly operation, with the consequence that the length of a base line seldom exceeded six miles. The application of Invar, in the form of wires or long bands, has revolutionized the measurement of bases, so that to-day lines between

10 and 20 miles in length are not uncommon, and can be measured in very much less time than much shorter lines could be measured with the older apparatus. In addition, it is now possible, and usual, to introduce check bases at intervals of about 120 to 200 miles, whereas up to fairly recent years base lines at such close intervals could seldom or never be measured. The comparison of the length of a check base, as computed through the triangulation from the original base, with the length as determined by direct measurement gives an indication of the accuracy of the work. In modern first-order work, the difference between the computed and observed length of a base line situated about 150 miles from the original base line should not exceed 1/20,000 of the length of the check base, and in most cases in practice it is considerably less.

After the base line has been measured, the latitude and longitude of one end of the line and its azimuth are determined by astronomical observations of the utmost precision, unless at least one station can be connected easily to a point and line where values of these quantities are already available. It is also necessary to determine the elevation of the base line above mean sea-level. This is done by careful levelling from sea-level or from a bench mark whose elevation above mean sea-level has already been determined.

The observation of the angles of the triangulation, both horizontal and vertical, follows the initial astronomical and levelling operations. Nowadays, on long lines it is usual to employ luminous signals, either heliograph by day or special electric lamps by night, on which to sight, these signals being accurately centred over the station mark and accurately pointed at the station from which the observations are being taken. The signals and observing instrument are set on special high observation towers if this is necessary to secure intervisibility.

On short lines, such as are used in secondary and tertiary work, *stopped-down* heliographs or electric lamps may be used as signals, but, as an alternative, non-luminous signals, in the form of large tripods or quadripods or posts with crossed sighting vanes, are often employed.

In first-order work, angles are measured with theodolites of the Wild or geodetic Tavistock type, or similar instruments, which enable the horizontal circle to be read direct to a single second of arc and by estimation to tenths of a second. A single observation consists of two sights, one circle left and the other circle right with change of swing with each sight, and about 16 observations to a single station are taken, each observation commencing on a different zero or initial setting on the horizontal circle when the telescope is pointed at the first of the two stations between which the angle is being observed. In secondary work, the number of observations to a station is reduced to about eight, and in tertiary work to about two or four. Observations of vertical angles consist of about the same number of measurements, but of course there is here no question of change of zero, although there is the usual change of circle and swing.

Observation of the angles marks the completion of the field work and the computations are now begun. These involve an adjustment of the angles of the triangles to make them satisfy certain geometrical conditions, the solution of the triangles, and finally the calculation of the co-ordinates and elevations of the stations.

Having thus given a general outline of the procedure involved in carrying out a triangulation scheme, we shall now proceed to describe in some detail the different stages of the work. These stages are:

1. Reconnaissance of scheme, including selection of site of base line.
2. Marking of stations and erection of signals.
3. Base-line measurement.
4. Determination of co-ordinates of one end of base line and of the azimuth or bearing of the line.
5. Measurement of angles.
6. Computation of results.

2. Reconnaissance of Scheme.

Before any observations on a triangulation scheme can be commenced, it is necessary first to reconnoitre the ground and select suitable stations. If maps are available, a provisional scheme can be drawn up on them in the office, and the possibilities and details of this scheme verified later on the ground. The scheme must admit of the following:

1. Suitable shaped triangles, i.e. triangles which have no very acute or very obtuse angles. The ideal is as nearly equilateral as possible.
2. Clear intervisibility between stations.
3. Avoidance of grazing rays, i.e. rays which come within a few feet of the ground at any point on their length, or of rays which come very close to large solid objects such as cliff faces, large buildings, etc.
4. Accessibility of stations.
5. Suitability of points not only for providing strong figures but also as regards their use for subsequent breaking-down operations.
6. A suitable site for a base line and its extension, and also possibly suitable sites for check base lines.

In drawing up a scheme on paper, regard must be paid to the effects of curvature of the earth and of terrestrial refraction on the intervisibility of distant points. In fig. 6.4, AA′ represents the surface of the sea, or a surface parallel to it, on a spherical earth in which the distance AA′ is D. B is a point at height h_1 above A, and B′ a point where a line of sight from B to the horizon at C will meet the vertical at A′. Let d be the distance of C from B and h_2 the height of B′ above A′. Owing to the curvature of the earth and the bending

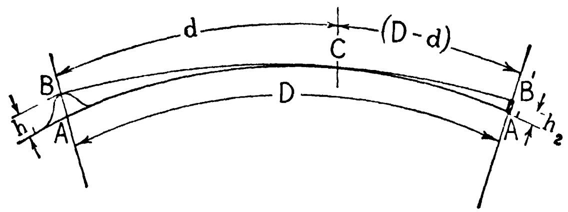


Fig. 6.4

of rays of light by refraction, the line BCB′ will be curved and not straight. Then it can be shown that d and h_1 are connected by the relation

$$h_1 = (1 - 2k)\frac{d^2}{2R} \text{ or } d^2 = \frac{2Rh_1}{1 - 2k},$$

where R is the radius of the earth and k is a constant known as the *coefficient of refraction*. The value of k varies slightly according to the time of day, being slightly different in day-time from what it is at night, and according to whether the sight is over land or sea. Its mean value may be taken at 0·07 and, using this value and a mean value for R, the expressions become

$$h_1 = 0{\cdot}574d^2 \text{ or } d = 1{\cdot}32\sqrt{h_1},$$

where h_1 is expressed in feet and d in miles.

Similarly, with h_2 expressed in feet and D and d in miles,

$$h_2 = 0{\cdot}574(D - d)^2,$$

so that d may be calculated from the second of the first two expressions, and the value so obtained substituted in the last expression to give h_2. This height will be the minimum height at A′ which can be seen from B supposing that the surface ACA′ is the surface of the sea or a level land surface parallel to it.

Now suppose that, as in fig. 6.5, instead of the horizon at C there is a hill there of height H at distance d from A, and we wish to know if a line of sight from B to B′ would clear the hill.

Let h be the height above C of a point which would lie exactly on the line of sight from B to B′. Then h will be given by

$$h = \frac{h_2 d}{D} + \frac{h_1(D-d)}{D} - Kd(D-d)\operatorname{cosec}^2 z,$$

where z is the zenith distance of B′ from B and K is the constant $5280(1 - 2k)/(2R) = 0{\cdot}574$ approximately. If H is greater than h, the line will not clear the hill at C, but if H is less than h, the points B and B′ will be intervisible.

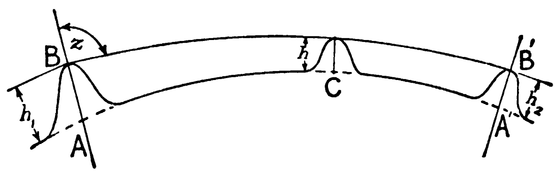


Fig. 6.5

In this formula z, which is the angle of elevation or depression measured from the zenith at B, is usually very close to 90°, so that in practically all cases that are met with in practice we can take $\operatorname{cosec} z = 1$ with sufficient accuracy.

If we substitute H for h in the last formula we can use it to solve for h_2, the height which a signal at B′ must have in order to be visible from B. In this case, however, an amount of at least 10 ft. should be added to H in order to avoid the possibility of a grazing ray at C, and also to allow for possible vagaries in the value of K.

Having drawn up on paper what appears to be a feasible scheme, a party proceeds to the field to see that all the proposed rays are clear and the terminals intervisible. If necessary, heliographs or lamps are used to verify the intervisibility of the end points of very long lines. As lines are verified, they are plotted on a *Trig. Diagram*, usually mounted on a plane-table, and at the same time compass bearings are taken at each station and rays drawn to all distant points that look as if they might be possible trig. points. If no map is available, the reconnaissance party will have to make its own diagram by using compass bearings or plane-table shots to get approximate fixings for all points, thus building up from a roughly measured base a graphical triangulation in which rough elevations of selected stations will be obtained by aneroid or Indian clinometer.

At each station visited, panorama sketches in different directions are made with compass bearings to the principal features figured on each sketch. A careful description of every station is compiled. This description, besides giving all data necessary for finding the point, must give such information as how the point is approached; means of transport used to reach it; details regarding local food supplies for labourers; nearest water supply, its amount, quality, and situation; nearest place where sand and stone for concrete may be obtained; local fuel supplies, etc. In fact, everything that any surveyor visiting the point later would require to know to reach the point, identify it, obtain materials for station building, and maintain himself and a party of labourers.

In Canada in recent years a special technique for trigonometrical reconnaissance from aeroplanes has been worked out, and a good deal of work done by this means.

3. Selection of Base-line Site.

The essential points to be kept in view when selecting a site for a base line are:

1. A reasonably long stretch of moderately flat country suitable for measurements with long metal bands.
2. Intervisibility of ends of base.
3. Ends elevated sufficiently to avoid a grazing ray along the base, but with approach slopes not too steep for accurate linear measurements to be made along them.
4. Good extension figure with well-shaped triangles from base to one side of the main triangles.

The length of the base line will depend on the length of side of the triangulation and the nature of the ground, but it is advisable to make the base as long as possible so as to reach the main figures in the fewest possible steps. Up to fairly recent years, bases varied from about a tenth to a half of the average length of side of the main triangles—say 3 to 10 miles long—but, as the use of Invar wires or bands has rendered the measurement of a base line a comparatively easy affair, the modern tendency is to use long lines of anything up to 15 or even 20 miles when the ground permits.

In the case of small local schemes of triangulation, where the triangles have very short sides, the base is generally one side of a main triangle, so that no base extension, in the ordinary sense of the word, is necessary.

4. Station Preparation and Signal Building.

After the preliminary reconnaissance has been completed, or even while it is in progress, the work of station preparation and signal building commences. This means putting in permanent station ground marks, clearing rays of bush and obstacles, and, when necessary, erecting special signals or observing towers. The best form of station mark is a concrete pillar about four feet high on a good solid foundation, flat on top so that an instrument can be set directly on it, the exact point to be taken as centre being a centre punch on a brass bolt let into the top surface of the pillar. In small local schemes, the station mark is a low concrete pillar about 1 ft. high and 9 in. square on top, set on a foundation about 15 in. square and from 2 to 3 ft. deep.

Modern practice tends more and more towards the use of nothing but luminous signals—heliograph or lamp—and special lamps, either electrical or acetylene, can be obtained for the purpose. Several makers now make special sets of traversing equipment for use with the "three-tripod system" of observing; this equipment is well adapted for small schemes where errors due to bad centring of signal or theodolite are much more serious than they are in the case of major triangulation schemes with long-sided triangles. In this apparatus, the upper part of the theodolite, including lower and upper plates and vertical circle, is removable from the tribrach, the latter being made to take either the theodolite or special targets which are very accurately centred and which can be used as sighting marks. A number of spare tripods, tribrachs and targets are provided. Consequently, when observations are complete at one station, the observer removes the upper part of the theodolite and sets a target in the tribrach, after which he proceeds to the next station to be observed, and sets the theodolite on the tribrach there after removing the target to which he observed from the last station. In this way, errors of centring are reduced to a minimum. The targets supplied with the equipment are fitted with electrical bulbs for night observations.

The acetylene and electrical lamps for use on long lines show powerful lights which, on clear nights, can be seen for distances up to 60 or 70 miles. Heliographs also are visible over similar distances when atmospheric and solar conditions are favourable.

When luminous signals are not employed, some form of opaque signal must be erected. For rather long sights, this usually takes the form of a large quadripod or tripod 10–20 ft. high, the upper part of which is covered with boarding or cloth down to about 6 or 7 ft. from the ground. On top of the main structure is a wooden post carrying a flag, or a red and a white flag, the object of which is to enable the signal, through the waving of the flags in the breeze, to be picked up fairly easily in the field of view of the telescope. Sights are taken to the pointed apex of the signal, which is set accurately over the station mark; when observations are being taken from the station, the instrument is set over the station mark under the centre of the structure. When the latter is being erected, care has to be taken to see that the legs are so placed as not to interfere with sights to any of the distant stations.

The targets supplied for the three-tripod system of observing can also be used for daylight sights with triangles with short sides. If this type of equipment is not available, ordinary ranging poles, accurately plumbed over the station mark and stayed with wire (not rope or string which alters in length with changes in atmospheric humidity), may be used as signals, red and white flags being attached to the top of each pole to make it easier to pick it up in the telescope. In conditions of good visibility, these will be visible up to distances of two or three miles. If longer sights are involved, say up to about ten miles, a more solid signal, consisting of a solid wooden post 8–12 ft. high and with cross pieces as shown in fig. 6.6 makes a good signal.

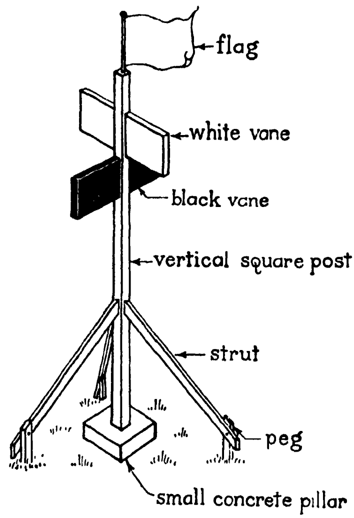


Fig. 6.6

It is impossible to lay too much emphasis on the fact that in all triangulation schemes it is most important to see that signals are truly vertical and accurately centred over the station mark, as nothing will reduce the accuracy of the work more than carelessness with regard to this point. This holds particularly with respect to triangles with short sides, and for this reason by far the best type of signal for local schemes is the type used in conjunction with the three-tripod system of observing.

In flat or wooded country it often happens that special high observing towers are necessary for long sights. Illustrations of these will be found in most textbooks on geodesy. Each tower consists of an inner tower supporting the theodolite and an outer one carrying a platform for the observer. These two towers are entirely independent of one another and are separately stayed, so that vibrations caused by the movement of the observer are not transmitted to the theodolite. If luminous signals are not used with the tower, the top of the latter must be pointed and opaque, so that the cross hair of the theodolite can be centred on it from a long distance away, the centre of the tower being, of course, carefully centred over the station mark.

The Ordnance Survey, in the new triangulation of Great Britain, are using a portable type of tower, called the "Bilby tower", after its designer, J. S. Bilby, of the United States Coast and Geodetic Survey; it can be dismantled and moved from place to place. This tower, which is in varying heights up to about 120 ft., consists of the usual inner tower for the theodolite and an outer independent tower for the observer; it is made up of light metal rods and angle irons which can be bolted together or unbolted as required. Bilby towers have been extensively used in the United States, where they originated, and in Canada.

5. Base-line Measurement.

The apparatus now used in the measurement of geodetic base lines has been described in Chapter II of *Principles and Use of Surveying Instruments*. It consists of a long band of Invar supported "in catenary" by means of weights suspended by cords over pulleys carried

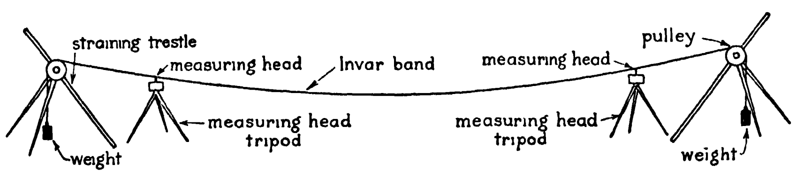


Fig. 6.7

by straining trestles in the manner shown in fig. 6.7, the end marks being set over special measuring heads. The measurement consists in observing at every set-up of the band the small differences in length between the marks on the ends of the band and index marks on the measuring heads.

Sometimes tension is applied at one end of the band by a spring balance supported on a straining pole as shown in Chap. II, fig. 2.16, of *Principles and Use of Surveying Instruments*, the other end of the band being held steady by means of a special steadying pole or lever.

In order to determine slope correction and correction for height above sea-level, spirit levels are either taken along the line or else angles of slope are observed from measuring head to measuring head. At least one of the points on the triangulation, usually one end of the base, is connected to a bench mark whose height above sea-level is known, but if no such height exists and it is impossible to run a special line of levels to the sea, a rough value may be obtained by careful barometric levelling. If heights and slopes along the base are determined by spirit levelling, the line of levels is taken over the pegs over which the measuring heads are set, and the height of each measuring head over the peg measured. If heights and slopes are determined from angles of slope, a small telescopic clinometer, which fits on the spike on each measuring head, is used to sight a special target placed on the measuring head at the other end of the span.

During measurement, readings are taken at every set-up of the band on two thermometers held near the ends of the band.

Before measurement commences, pegs are put in at every point where a measuring head will be placed. If possible, the base is made a whole number of band lengths long so as to avoid a short bay at one end. If a short bay is unavoidable, its length is measured with a special graduated tape kept for the purpose and used either along the ground or in catenary. Pegs to mark the positions of measuring heads must be put in carefully as otherwise the head may come outside the limits of the scales used for the measurement of end differences.

Steel bands are often employed instead of Invar for the measurement of bases not intended for geodetic work, measurements being made either along the ground surface or else in catenary. If they are made along the ground, the earth is levelled off so that the band lies properly flat throughout its length: otherwise, pegs are put in on grade between the end points at, say, 10-ft. intervals, and these pegs become supports for the band. The end marks are on small sheets of stout zinc nailed on top of pegs. On these sheets the line of the base is marked out by a longitudinal line laid out with the aid of a theodolite, and a fine scratch at right angles to this line, and near the centre of the metal sheet, takes the place of the fiducial mark on a measuring head. Tension in this case is applied by spring balance at one end of the band.

For catenary measurements, if no measuring heads are available, the marks are the intersection of two crossed lines on a round-headed brass nail carried on the centre of a stout wooden peg about 42 in. high. Each peg must be very steady and is therefore best held fixed by three stout wooden struts arranged like the legs of a tripod and nailed to the peg at one end and fixed in the ground at the other. Tension is applied through a spring balance at one end of the band, the other end being held steady by a steadying pole stuck lightly into the ground at its lower end. A steadying pole is also used with the spring balance to enable a steady pull to be maintained.

When a base line, whether for major or minor work, is laid out, all pegs and marks indicating where the end of the band will come, or over which a measuring head will be placed, must, of course, be properly and accurately lined in by theodolite. For convenience in measurement and computation, a long line is divided into a number of sections, each section representing the equivalent of a day or two's work, and it is advisable to mark the end of each section by a small concrete pillar carrying an accurately placed centre mark.

6. Measurement and Booking of End Differences.

The measurement of end differences means the measurement of the small differences between the end marks on the band and the fixed marks on the measuring heads, zinc plates or pegs. In base measurement, no attempt is made to hold one end of the band against the fixed mark, so that differences have to be measured at each end of the band. The algebraic sum of all these differences over the whole length of the line, plus the number of spans multiplied by the nominal length of the straight line joining the end marks, plus the length of the odd length bay, is the apparent length of the line.

End differences are measured either by means of a rule held against or beside the band and the fixed mark, or else by means of graduations on the band itself, a magnifying glass fixed to the measuring head or held in the hand being used to view the graduations clearly. If the band is a metre band, the scales are graduated in millimetres and readings estimated to tenths of a millimetre. If the length of the band is a whole number of feet, the scales will generally be graduated in hundredths of a foot and readings estimated to thousandths.

Great care must be taken to book signs correctly and the rule to be followed is:

If the end mark of the band falls between the measuring heads, the end difference is booked as a positive quantity; if the end mark of the band lies between the measuring head and the straining trestle, the difference is booked as a negative quantity.

In many bands intended for base measurement, the scales at the ends both run in the same direction—that of the length of the tape looking at it from the zero or rear end. In this case the zero mark must always lie behind the fixed mark, between that mark and the straining trestle. Consequently, the rear reading will be negative. The mark at the forward end will lie between two measuring heads and will therefore be positive, so that here the apparent distance between the fixed marks is nominal length of span, plus forward reading, minus back reading. Hence, if the back reading is numerically greater than the forward reading, the span between the fixed marks is less than

Span No.	Band No.	Readings in mm.		Difference	Mean	Temp. °F.	Height of measuring head
		Forward	Rear				
16	A.41	+17·9	−12·3	+5·6		63·8	$h_1 = 3{\cdot}48$
		+20·4	−14·6	+5·8	+5·68	61·7	$h_2 = 4{\cdot}01$
		+14·8	− 9·1	+5·7		62·75	
		+16·3	−10·7	+5·6			
	A.64	+12·6	− 6·9	+5·7		63·1	
		+13·8	− 7·9	+5·9	+5·78	62·6	
		+10·6	− 4·8	+5·8	+5·73	62·85	
		+11·0	− 5·3	+5·7			

the nominal span; if the forward reading is numerically greater than the back reading, the span between the fixed marks is greater than the nominal span.

Some bands have scales which are graduated in both directions outwards from the zero and end marks. In this case readings on the outer scales are positive, and on the inner scales negative, and the correction to be applied to the nominal span is the algebraic sum of the two readings.

In all base-line measurements, even in short ones measured for minor work only, the band is moved slightly after the first pair of readings, and several other pairs taken on different parts of the scales, the direction of movement of the band being reversed after each pair. If tension is applied by weight, the pulley of the straining trestle is rotated through an angle each time the band is moved, so that the cord attaching band to weight works on a different part of the circumference for each pair of readings. In addition, in base lines for first-order triangulation, sets of readings are taken with at least two bands, and, after the line has been completely measured in one direction, it is remeasured in the opposite direction. Normally, one measurement is made in each direction, but there is no great difficulty in making four measurements, two of which are in opposite directions to the other two.

The table on p. 77 shows an example of the booking of end differences when using a 50-metre tape. The mean of the end differences being +5·73 mm., the observed length of the span is 50·00573 metres.

7. Field Standardizations.

The field bands used in base measurement are standardized before and after measurement, and, if the base is long and measurement is going to take more than a day or two, it is well to standardize at the beginning of each day's work. For this purpose, one or more bands, which have themselves been standardized and had their coefficients of expansion determined at the National Physical Laboratory, or similar Institution, are kept as field reference bands and used for nothing else but field standardizations.

A field standardization is carried out by measuring the distance between a pair of measuring heads or other marks with the standard bands and then measuring the same distance with the field bands. The differences in the computed lengths, after all corrections have been applied, will give the correction to be applied to the field bands. At least two or three times the number of measurements used in ordinary work should be made with each band when standardizing, as it is very important that a standardization should be of the utmost accuracy if the effects of a systematic error are to be avoided.

Example.—A short base consisting of two measuring heads was measured with a couple of standardized bands, and the mean corrected length of this base was found to be 100·1213 ft. The same base was then measured with an ordinary field band, and the corrected result, assuming the band to be 100 ft. long, was 100·1106 ft. What was the correct length of the band?

The base as measured by the 100-ft. field band was 100·1213 − 100·1106 = 0·0107 ft. too short, so the band must have been that much too long. Hence, the correct length of the band is 100 + 0·0107 = 100·0107 ft.

It should be noted in passing that bands should always be standardized with their own straining equipment, and the conditions of field standardization should be as near as possible to the conditions of ordinary field work.

8. Corrections to be applied to Measured Lengths.

There are a number of corrections to be applied to all measurements made with Invar or steel bands. In base-line measurement these are:

1. Correction for standardization.
2. Correction for temperature.
3. Correction for slope.
4. Correction for change of pull.
5. Correction for index error of spring balance if tension is applied by spring balance.
6. Correction for sag, or *catenary correction.*
7. Correction for height above sea-level.

There is also a very small additional correction to the standard bands when the locality in which they were standardized is entirely different from that in which they were used. This correction is due to the variation of gravity with latitude and height above sea-level, which affects the true value of the pull when this is applied by weights.

In ordinary measurements with steel bands, apart from base measurement, some of the above corrections are too small to be worth applying, but others may be applicable. All depends on the degree of accuracy desired.

1. *Correction for Standardization.*

This is a correction to the observed length for the difference between the nominal length of the band and its true length. It is best expressed as a percentage and applied to the total length measured, using the rule:

A line measured with a band that is too long/short will be too short/long and the correction is additive/subtractive.

Example.—A band whose true length is 100·0114 ft. was used to measure a line of which the apparent length was 6,012·2372 ft. Find the true length of the line.

The band used is 0·0114 per cent too long, and consequently the measured length was too short, so that the correction to be applied is additive. Hence,

$$\text{true length of line} = 6{,}012{\cdot}2372 + 60{\cdot}12 \times 0{\cdot}0114$$
$$= 6{,}012{\cdot}2372 + 0{\cdot}6854 = 6{,}012{\cdot}9226 \text{ ft.}$$

The correction for standardization is often combined with the temperature correction as in the two examples at the end of the next section.

2. *Correction for Temperature.*

When a band has been standardized, its length is known when it is at a certain defined temperature and under a stated pull. If the temperature or pull is altered, the length of the band will alter. In practice, the band is very seldom used at the temperature at which it has been standardized, because the temperature of the air is constantly changing with time and place.

Let T_α be the temperature for which the band has been standardized, T_β the temperature at which it has been used, and l_α its length at temperature T_α. Then its length at temperature T_β will be given by

$$l_\beta = l_\alpha + l_\alpha k(T_\beta - T_\alpha),$$

where k is a constant, known as the *coefficient of expansion*, which gives the amount by which unit length alters with unit change of temperature. The second term on the right in this expression is the temperature correction.

The value of k varies greatly with Invar bands and even in the same band it changes slowly with time, so that an Invar band should have its coefficient of expansion determined shortly before or shortly after an important base measurement. With Invar, the coefficient is usually positive, but it may actually be negative, in which case the band decreases in length with increase in temperature; the numerical value, however, seldom exceeds 0·000 0005 per degree Fahrenheit. The coefficient of expansion of ordinary steel bands is always positive, and, if not specially determined or given, may be assumed to be 0·000 0062 per degree Fahrenheit.

Temperature correction may either be applied to each measured bay, as is generally done in base-line work, or it may be applied to a line as a whole, taking the mean temperature observed during the measurement of the line. As noted above, this correction and the one for standardization may be combined to form a single correction. This is done by working out the temperature at which the band is of standard length and then using this temperature as the one from which all differences in temperature are to be reckoned when calculating temperature correction, as in the two following examples:

Example 1.—The length of a steel band was found to be 99·9846 ft. at 62° F. Find the temperature at which it is of nominal length (100 ft.), assuming the coefficient of expansion to be 0·000 0062 per 1° F.

The tape is 0·0154 ft. too short at 62° F., and the increase in length for an increase in temperature of $T_\beta - T_\alpha$ will be 0·000 0062 × 99·9846 × $(T_\beta - T_\alpha)$, which, with sufficient approximation, may be written 0·000 0062 × 100 × $(T_\beta - T_\alpha)$. This must be equal to 0·0154,

$$\therefore\ 0{\cdot}000\,0062 \times 100 \times (T_\beta - T_\alpha) = 0{\cdot}0154,$$

giving $$(T_\beta - T_\alpha) = 24{\cdot}8^\circ \text{ F.},$$

hence the tape is of nominal length at 62 + 24·8 = 86·8° F.

Example 2.—A line 8326·14 ft. long was measured with the tape referred to in the last Example, the mean temperature of measurement being 71·4° F. Find the true length of the line.

Here $$(T_\beta - T_\alpha) = 71{\cdot}4 - 86{\cdot}8 = -15{\cdot}4^\circ \text{ F.}$$

and change of length in a line 8326 ft. long at this temperature

$$= 15{\cdot}4 \times 8326 \times 0{\cdot}000\,0062 = 0{\cdot}79 \text{ ft.}$$

But at this temperature the band will be too short, and hence the measured length of line will be too long. Hence, the true length is

$$8326{\cdot}14 - 0{\cdot}79 = 8325{\cdot}35 \text{ ft.}$$

3. *Correction for Slope.*

Since a plan represents a truly horizontal surface, and considerations of mathematical convenience make it essential to reduce all measured distances to the corresponding distances on a horizontal plane, a correction for slope of line must be applied to every individual bay and at every change of slope. The formulæ to be used will depend on whether angles of slope are measured directly or whether slope correction is to be determined from observed differences of elevation.

(i) *Slope Correction from Angles of Slope.*—Let l be the measured length of the line and θ the observed angle of slope. Then the horizontal projection of l will be given by

$$\begin{aligned} l_h &= l \cos \theta \\ &= l - l(1 - \cos \theta). \end{aligned}$$

Hence, the correction to be applied to l is $l(1 - \cos \theta)$ and *this correction will always be subtractive.*

The quantity $(1 - \cos \theta)$ is the versine of the angle θ and can easily be found from tables of natural versines such as are given in *Chambers's Seven-figure Mathematical Tables.*

Thus, if $\theta = 3^\circ\ 43'$ and $l = 100{\cdot}1032$, the natural versine of $3^\circ\ 43' = 0{\cdot}002103$. Consequently, the correction is 100·1 × 0·002103 = 0·2105, so that the corrected length is 99·8927.

Slope correction is also tabulated in some books of tables specially prepared for surveyors and in some textbooks. Also, it is better to correct lines by means of a small correction rather than compute corrected values by the formula $l_h = l \cos \theta$; the latter method involves a multiplication of large figures, whereas the correction is a small one, involving few figures, so that it can often be worked out by slide rule. Moreover, it will be noticed from the example that, in the multiplication by the versine, the decimal place in the length need not ordinarily be used.

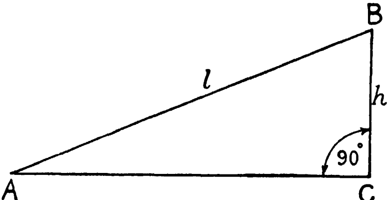


Fig. 6.8

(ii) *Slope Correction from Differences of Elevation.*—If differences of elevation are measured, and not angles of slope, let h be the difference in elevation between the two ends of the line. Then, in fig. 6.8,

$$AC = \sqrt{l^2 - h^2} = l\sqrt{1 - \frac{h^2}{l^2}}.$$

Expanding the expression under the radical sign by means of the binomial theorem, we have

$$AC = l\left(1 - \tfrac{1}{2}\frac{h^2}{l^2} - \tfrac{1}{8}\frac{h^4}{l^4} \ldots\right),$$

and the correction to be *subtracted* from the measured length is

$$l - l\left(1 - \tfrac{1}{2}\frac{h^2}{l^2} - \tfrac{1}{8}\frac{h^4}{l^4} - \ldots\right)$$

$$= \tfrac{1}{2}\frac{h^2}{l} + \tfrac{1}{8}\frac{h^4}{l^3} + \ldots$$

Example.—Let $h = 3{\cdot}25$, $l = 100{\cdot}1645$. Correction $= 0{\cdot}052\,73 + 0{\cdot}000\,01 = 0{\cdot}0527$. Consequently,

corrected length $= 100{\cdot}1645 - 0{\cdot}0527 = 100{\cdot}1118$.

From this example it will be seen that the second term in the formula can usually be neglected unless the difference in height of the two ends is very considerable.

When a band is used in catenary there is also another small correction for slope which will be considered with the correction for sag. This is due to a deformation in the shape of the curve when the catenary is on a slope. If ordinary sag correction is applicable, it should, of course, be applied before correcting for slope.

4. *Correction for Change of Pull.*

This correction is only applicable when the band is used at a different pull to that at which it was standardized. Let P_s be the pull used in standardization and P the pull used during measurement. Let a be the sectional area of the band in square inches and E be Young's modulus of elasticity in pounds per square inch. Then, from the ordinary laws of elasticity, the correction is

$$C_p = l \times \frac{(P - P_s)}{aE}.$$

This correction must be added to the standardized length of the band when $(P - P_s)$ is positive, and subtracted from it when it is negative.

For ordinary steel bands E may be taken at 28,500,000 lb. per square inch and for Invar bands it may be taken at 22,000,000 lb. per square inch.

Example.—Let the length of a steel band standardized on the flat at a tension of 15 lb. be 100·1018 ft., and let it be used at a tension of 20 lb., the width and thickness being 0·25 in. and 0·02 in. Find the correction for the change of pull.

$$C_p = 100{\cdot}1 \times \frac{5}{0{\cdot}25 \times 0{\cdot}02 \times 28{,}500{,}000} = 0{\cdot}0035.$$

Hence, length of tape under 20 lb. pull = 100·1018 + 0·0035 = 100·1053 ft.

A change of pull would, of course, also affect the sag correction, so that a new sag correction must be computed from

$$S = S_s \times \frac{P_s^2}{P^2},$$

where S is the sag correction at pull P and S_s the sag correction at pull P_s.

If a standard band has been standardized with weights at a place of very different latitude and elevation from those of the place where it is to be used, a small correction for alteration of pull must be introduced to allow for the variation of gravity with latitude and height of station. This alteration in pull is given by

$$P - P_s = 0{\cdot}00529P_s(\sin^2 \phi - \sin^2 \phi_s) - 0{\cdot}000\,00006P_s(h - h_s),$$

where ϕ_s and ϕ are the latitudes of the place where the band was standardized and where it was used, and h_s and h the corresponding elevations in feet above sea-level. In this formula, the second term on the right involving the difference in elevation is negligible in all normal cases. If the band was standardized with a spring balance, the correction does not apply. In both cases, however, the weight of the band is affected slightly, but, since this weight and the weights applying tension are similarly affected, this can only make a difference to sag correction (and that to a usually negligible extent) when pull is applied by spring balance. The alteration in weight may be calculated by substituting w and w_s for P and P_s in the formula given above.

5. *Correction for Index Error of Spring Balance.*

If pull is applied by spring balance, and the same balance was not used in standardizing, a correction must be applied to the pull to allow for the fact that the balance is used in a horizontal position, whereas the probability is that it was graduated when used in a vertical position.

Let I be the reading when the balance is suspended in its normal position with hook downwards and no weight attached, I' the reading when the balance is suspended vertically by the hook, and W the total weight of the balance. Then,

$$\text{index error for the horizontal position} = \tfrac{1}{2}(W - I - I').$$

This correction is additive to the ordinary readings when it is positive, I being taken as negative when the index is below zero.

6. *Correction for Sag.*

The correction for sag allows for the difference between the length of the chord and that of the curved arc when the band is hanging freely under the applied tension and its own weight, the value of the correction being given by

$$S = \frac{1}{24} \frac{w^2 l^3}{P^2},$$

where w is the weight of the band *per unit length*, l is the length of the band between end marks, and P is the applied pull. A proof of this expression will be found in the Appendix, p. 254.

Example.—Let $l = 100$ ft., $w = 15$ oz. per 100 ft., and $P = 10$ lb. Then

$$S = \frac{1}{24}\left(\frac{15}{16 \times 100}\right)^2 \frac{100^3}{10^2} = 0{\cdot}0366 \text{ ft.}$$

Hence the length of the chord joining the end marks is $100 - 0{\cdot}0366 = 99{\cdot}9634$ ft.

If the band is used on a slope, the sag correction will depend on whether the measured pull is applied at the upper or lower ends of the band. If it is applied at the upper end, the sag correction becomes

$$S_s = S \cos^2 \theta \left(1 + \frac{wl}{P} \sin \theta\right),$$

where S_s is the sag correction on the slope θ and S the sag correction when both ends of the band are at the same level.

If the measured pull is applied at the lower end, we have

$$S_s = S \cos^2 \theta \left(1 - \frac{wl}{P} \sin \theta\right).$$

In both these cases, the second term is small, so that, for a steel band, we may take

$$S_s = S \cos^2 \theta.$$

Sometimes very long bands are used with one or more intermediate supports dividing the total span into two or more equal spans. If there are n equal spans, the total sag correction is n times the correction for one span, so that, if l is the total length of the band, the length of one span is l/n, and the total sag correction for the whole band is

$$S = \frac{1}{24} n \times \frac{w^2}{P^2}\left(\frac{l}{n}\right)^3 = \frac{1}{24}\frac{w^2 l^3}{n^2 P^2}.$$

Sag correction, of course, is always subtractive if the band has been standardized on the flat, and the chord length between end marks is required; it is additive if the band has been standardized in catenary, and the length on the flat is needed.

7. *Correction for Height above Sea-level.*

In fig. 6.9 (p. 86), DC is a line on the surface of mean sea-level on a spherical earth of radius $R = \text{OD}$, O being the centre of the earth. AB is a parallel line on a surface at height h above DC, so that A is vertically above D and B is on the vertical at C. It is obvious from the figure that AB is longer than DC, and a little consideration will show that, provided the work is accurate enough for the difference between AB and DC to be appreciable, confusion would result if all measured distances were not reduced to a common surface. Accordingly, it is customary to reduce the measured lengths of base lines of all important triangulation schemes to their equivalent lengths at mean sea-level (M.S.L.). Once this is done, all lengths computed from the base through the triangulation will likewise be reduced automatically to mean sea-level.

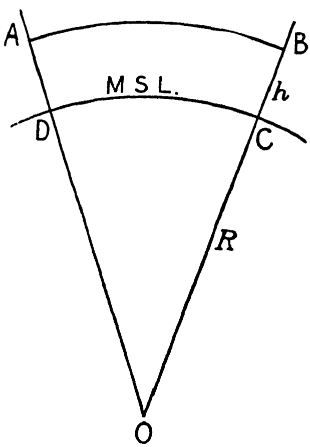


Fig. 6.9

If AB $= l$ and DC $= l - \delta l$, we have from the diagram,

$$\frac{l - \delta l}{l} = \frac{R}{R + h}.$$

$$\therefore \quad (l - \delta l)(R + h) = Rl,$$

or $$lh - R\delta l - h\delta l = 0.$$

$$\therefore \quad \delta l = \frac{lh}{R + h}.$$

δl is the amount which has to be subtracted from AB to give DC, and hence it is the correction required. It is subtractive for all heights above mean sea-level and additive for all heights below it.

Sea-level correction is seldom applied to every measured bay, but the mean height for a section of one or two miles is taken and a correction based on this mean height applied to that section as a whole. It is also necessary to apply the correction to precise traverses when these traverses are measured at an appreciable height above sea-level and form part of a network intended for framework purposes. In such cases, the correction is applied to the total latitudes and total departures of the individual sections into which the traverse is divided, the latitudes and departures of intermediate points being corrected proportionately and automatically during the course of the adjustment between the section terminals.

9. Accuracy of Base-line Measurement.

The measurement of a geodetic base line is one of the most accurate operations that a surveyor is expected to carry out. Thus, if a line is measured twice with Invar bands, once in one direction and once in the other, the discrepancy between the two measures will normally not exceed about 1/2,000,000. Since, however, the discrepancy does not take into account such unknown factors as small constant or systematic errors of standardization, constant errors of pull, temperature, etc., that do not become manifest in the difference between the measures, the real "probable error" will seldom be much less than 1/500,000.

With steel bands used without special base-measuring apparatus either along the ground or in catenary, using the methods described on pp. 75–76, the difference between the forward and back measures should not exceed about 1/100,000 and the "probable error" of the base may be assumed to be of the order of about 1/50,000. If special base-measuring apparatus is used, the results should be somewhat better than this, especially if the work is done at night when temperatures are steadier than during the day.

10. Field Determination of Sag Correction.

If the weight per unit length w of the band is not known, and no scales and weights are available for determining it, the sag correction can easily be found, with sufficient accuracy for work in which a steel band is used, from observations of the actual sag at the centre of a horizontal span.

Let y be the depth of the sag at the middle of the span below a horizontal line joining the end marks. Then it can easily be shown that

$$y = \frac{wl^2}{8P} \text{ or } w = \frac{8Py}{l^2}.$$

Inserting this in the formula for sag correction, we have

$$S = \frac{1}{24}\left(\frac{8Py}{l^2}\right)^2 \frac{l^3}{P^2} = \frac{8}{3}\frac{y^2}{l},$$

a formula which, be it noted, is independent of w and P. Hence, a value deduced by this method is not affected by uncertainties or errors in respect of assumed values of these quantities.

The sensitiveness of this method depends on the fact that an error in the determination of sag will be very considerably more than the resulting error in the sag correction. In fact, if we differentiate the last expression with respect to y we see that the error δS corresponding to an error of δy in y is given by

$$\delta S = \frac{16}{3} \cdot \frac{y}{l} \delta y.$$

Hence, if $y = 1$ ft., $l = 100$ ft., and $\delta y = 0{\cdot}005$ ft., we have $\delta S = 0{\cdot}0003$ ft.

The sag can be measured with the aid of a level and a graduated scale supported vertically beside the band at the centre of its length, the level being used to ensure that the line joining the end marks of the band is truly horizontal, and also to determine the exact point on the graduated scale where this line intersects it. Alternatively, vertical angles can be observed by theodolite to the ends of the band and to the centre point of the span, and the corresponding distances measured from the horizontal axis of the instrument. This will enable the differences in height below or above the horizontal plane through the horizontal axis to be calculated, and from these differences the sag can be obtained by subtraction.

11. Measurement of Horizontal Angles.

There are two main ways of measuring the horizontal angles of a triangulation scheme. One is the method of *repetitions* and the other is the method of *directions*, the latter being the one most commonly used.

In the method of repetitions, a multiple of the angle is measured and the observed angle is divided by the number of repetitions, the procedure in observing being described on pp. 110–111 of *Principles and Use of Surveying Instruments*. This method is more suitable for the observation of very small angles, as in subtense methods of determining distance, than for ordinary trigonometrical observing.

In the method of directions, observations are commenced with a pointing to one selected station known as the *Referring Object*, or *R.O.*, and the instrument is swung round about its vertical axis to intersect in turn the other stations visible from the station of observation, the reading on the horizontal circle being taken at each pointing. Thus, in fig. 6.10, the instrument is set and clamped to sight R.O. and the readings on both verniers or both micrometers taken. With the lower plate kept clamped in this position, the upper plate and telescope are swung round to sight in turn on A, B, C and D, the readings on both of the verniers or micrometers being taken at each pointing. After D has been sighted and the horizontal circle readings taken, the swing is continued until the theodolite again points to R.O., new readings on the verniers being taken. If there were no errors of observation, the new readings to R.O. would be the same as the previous ones, but in practice there usually is a small discrepancy which is adjusted equally among the angles.

Thus, if the second reading on R.O. were less than the original reading by 5″, 1″ would be added to the reading to A, 2″ to the reading to B, 3″ to the reading to C, 4″ to the reading to D, and 5″ to the second reading to R.O. Alternatively, 1″ would be added to each of the deduced angles (R.O.)PA, APB, BPC, CPD and DP(R.O.).

One complete *round* having been thus observed on one face and one swing, the telescope is transited so as to change face, and the instrument again set to read on R.O., D, C, B, A and R.O. in turn. This completes one set of observations, and the mean of the two measures of each angle is accepted as a single value of that angle.

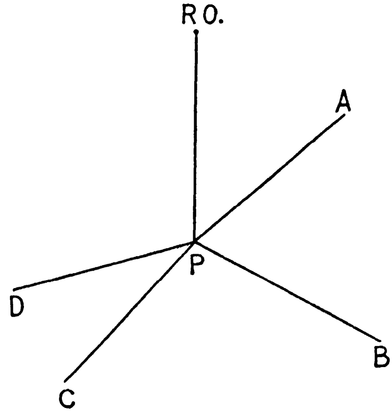


Fig. 6.10

In ordinary work a number of similar observations are made on different *zeroes*. This means that the operations are repeated with different initial settings of the lower circle when the telescope is directed to R.O. If there are to be n zeroes, the difference between successive settings with an instrument with two verniers or micrometers would be $180°/n$. Thus, with 8 zeroes, the initial settings would be such that the horizontal circle readings when the telescope was pointed to R.O. would be 0°, 22° 30′, 45°, 67° 30′, 90°, 112° 30′, 135° and 157° 30′ for each set of face-right and face-left observations. Sometimes the number of zeroes is doubled by changing zero between face-right and face-left observations. Change of zero is, of course, made by setting the upper circle to the reading of the desired zero and then, with upper circle clamped at that reading, loosening the clamp of the lower circle and swinging the instrument to sight R.O., the final setting being made by clamping the lower circle and using the lower-circle tangent screw.

In first-order geodetic triangulation the horizontal angles are generally measured at night on 16 zeroes, with change of face and swing on each zero, using an instrument reading direct to single seconds and by estimation to a tenth of a second. Secondary and tertiary work is generally measured by day on smaller instruments, with 8 zeroes on secondary work and 4 on tertiary.

The object of changing swing, face and zero is to eliminate the effects of errors of backlash and friction in moving parts; of vertical collimation; of eccentric mounting of verniers and graduated arc; and systematic errors of graduation of the circle.

Sometimes the sight back to R.O. at the end of a swing is not made. Instead, face is changed after the last station is observed, and all the stations observed in the reverse direction, starting from the last station and closing back on R.O., any difference between the first and last readings on the R.O. being equally distributed among all the angles.

Again, it often happens that, before an entire set of observations can be completed, the R.O. becomes invisible, although other stations can be seen. In that case, another point may be used as a temporary R.O., and observations taken to any others that are visible, additional observations to the station used as the original R.O. being made later.

An example of booking horizontal angles at a trigonometrical station is given opposite. The full degrees, minutes and seconds are only entered for the readings on the "A" micrometer, and seconds only for the "B" micrometer. The closing error on R.O., and the adjustments to the different directions, are entered above the means and taken into account when working out the angle measured from R.O. Individual angles may then be obtained by subtraction from the angles from R.O., and summed and meaned on separate sheets of foolscap.

12. Accuracy of Angular Measures in Triangulation.

In triangulation, the angular error of closure of the triangles, i.e. the difference between the sum of the observed angles of the triangle and their theoretical sum of 180° plus spherical excess (if any), affords a very good indication of the accuracy of measurement of the horizontal angles. For example, the United States Coast and Geodetic Survey lays down that the average error of triangular closure of first-order triangulation must not exceed 1″ and the maximum must not exceed 3″. For second-order work, the corresponding figures are 3″ and 8″ respectively, and for third-order work they are 6″ and 12″. In the case of small schemes measured with an ordinary vernier theodolite

SPECIMEN PAGE FROM ANGLE BOOK

Date. *26.3.48* Station: *Inverton T.P. 138* Observer : *John Smith*

Instrument. *C. T. & S. No. 415.* Value of *1* Divn. of Level *3″.* Weather *Clear. Light S.W. Wind.* Height of Inst. *4·73*

Station	Face	Horizontal angles				Vertical angles							Remarks
		"A" Micro	"B" Micro	Mean	Angle from R.O.	"C" Micro	"D" Micro	Mean	Level O	Level E	Corr.	Corrected Vertical angle	
		° ′ ″	″	″	° ′ ″	° ′ ″	″	″			″	° ′ ″	
Two Tree Hill 142	R	*0 0 31*	*33*	*32*		*179 15 45*	*49*	*47*	*14·2*	*10·4*	*+5·7*	*0 44 18·7*	R.O.
				−1									
Abigon 144		*41 15 26*	*25*	*25·5*	*41 14 52·5*	*178 23 22*	*25*	*23·5*	*14·8*	*11·2*	*+5·4*	*1 36 41·9*	
				−2									
Cape Mount 145		*136 46 18*	*21*	*19·5*	*136 45 45·5*	*178 57 17*	*23*	*20*	*15·2*	*11·7*	*+5·3*	*1 02 45·3*	
				−3									
Laviston 139		*210 33 52*	*55*	*53·5*	*210 33 18·5*	*177 46 37*	*45*	*41*	*15·0*	*11·6*	*+5·1*	*2 13 24·1*	
				−4									
Solgon 141		*241 54 41*	*44*	*42·5*	*241 54 06·5*	*181 16 51*	*58*	*54·5*	*14·7*	*11 0*	*+5 6*	*−1 16 48·9*	
				−5									
Two Tree Hill 142		*0 0 36*	*38*	*37*									R.O.
Two Tree Hill 142	L	*180 0 23*	*26*	*24·5*		*0 44 33*	*38*	*35·5*	*8·6*	*11·3*	*−4·0*	*0 44 31·5*	R.O.
				−0·5									
Solgon 141		*61 54 27*	*24*	*25 5*	*241 54 00·5*	*358 43 16*	*19*	*17·5*	*9·1*	*12·0*	*−4·4*	*−1 16 46 9*	
				−1·0									
Laviston 139		*30 33 42*	*39*	*40·5*	*210 33 15 0*	*2 13 36*	*31*	*33·5*	*9·3*	*12·7*	*−5·1*	*2 13 28·4*	
				−1·5									
Cape Mount 145		*316 46 04*	*08*	*06·0*	*136 45 40·0*	*1 02 45*	*41*	*43*	*9·2*	*12·5*	*−5 0*	*1 02 38·0*	
				−2·0									
Abigon 144		*221 15 17*	*14*	*15·5*	*41 14 49·0*	*1 37 00*	*36′ 57″*	*36′ 58·5″*	*8·9*	*11·8*	*−4·4*	*1 36 54·1*	
				−2 5									
Two Tree Hill 142		*180 00 26*	*28*	*27·0*									R.O.

reading direct to 30″, and with, say, angles observed on four zeroes, the average closing of a triangle should be somewhere about 10″, with no closure greater than 30″ allowed.

13. Measurement of Vertical Angles.

Vertical angles are often not measured at the same time as the horizontal ones, as they are not so important as the latter, and it is advisable to make the utmost of good observing weather when the horizontal angles are being observed. In addition, on long lines, it is best to observe vertical angles only at certain times of the day when refraction is at its minimum and varying least. These times are usually between about 10 a.m. and 3 p.m. Vertical angles are observed singly or in rounds, face right and face left, but, of course, there can be no change of zero with such observations.

The level bubble on the vernier frame is read at every observation immediately after the station is sighted and before the verniers are read, and, if necessary, a correction applied to the reduced angle. This correction is given by the expression

$$\text{correction} = \frac{(O - E)}{2} d,$$

where O is the reading at the object glass end of the level, E the reading at the eyepiece end, and d is the value of one division of the level tube in seconds. If the tube is graduated continuously from the eye end towards the objective end, or vice versa, and G is the middle graduation, the correction is given by

$$\text{correction} = \frac{(O + E - 2G)}{2} d.$$

In order to obtain vertical heights from the vertical angles, it is necessary to measure the heights of the horizontal axis of the theodolite and of the signals above the station marks.

The method of booking vertical angles is shown on p. 91. The O and E readings are inserted in the 10th and 11th columns of the page, and the level correction in the 12th column, these corrections being applied directly to each observed value so that the 13th column gives the corrected vertical angle.

14. Computation of Triangulation.

The computations of a network of triangulation consists of the following:

1. Computation of length of base line.
2. Abstract of angles and station adjustments.
3. Adjustment of figures.
4. Solution of triangles.
5. Computation of astronomical observations (if any).
6. Computation of co-ordinates.
7. Computation of vertical heights.

1. *Computation of Length of Base Line.*

This consists in computing and applying to the observed lengths the different corrections described on pp. 79–86. As it is very important that there should be no mistake in the computed length of the base line, it is desirable to have two separate and entirely independent computations made by two different computers, each working direct from the field books.

2. *Abstract of Angles and Station Adjustments.*

This entails abstracting the values of the observed angles from the field books on to special sheets of paper, and taking out the arithmetical means after "closing the horizon" (pp. 88–90). In first- and second-order work it often happens that different combinations of angles have been observed, and in such cases it is usual to obtain the "most probable values" for the different angles by means of a *station adjustment* carried out by a special mathematical process known as the *method of least squares.* Adjustments of this kind are beyond the scope of this book and, in any case, in all minor work it is usual, when more than one value of an angle is available, simply to take the mean of all the available values. If the angles have been observed in straight rounds by directions from a single R.O., no question of a station adjustment arises.

3. *Adjustment of Figures.*

In first- and second-order triangulation the work generally consists of a series of somewhat complicated figures in which a point can be fixed by more than a single triangle. Such figures are normally adjusted by use of the method of least squares, which, besides giving the most probable values of the angles, makes the figure geometrically consistent, so that the same result will be obtained no matter from what triangles a point is fixed, and only a single value will be obtained for each angle.

The need for some adjustment of the kind can be seen from fig. 6.11, which represents a braced quadrilateral ABCD in which all the angles have been observed. If the observations were without error, the sum of the angles in each triangle would add up exactly to 180°, or to 180° plus the spherical excess (p. 96) in the case of a very large triangle, but, as there are always some small residual errors of observation present, the observed angles will not add up to 180° exactly. They can easily be adjusted to do so, however, but this does not necessarily mean that the figure is geometrically consistent from a computational point of view.

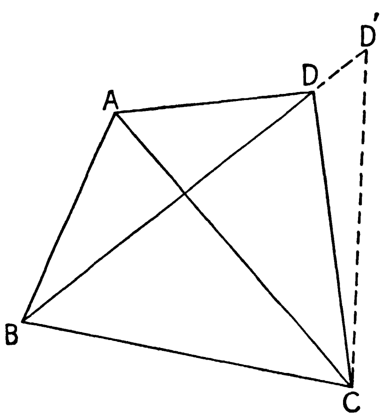


Fig. 6.11

Thus, in the figure, if the point D′ is taken on BD produced, and D′ and C are joined, the angles of the triangles ADB and BD′C will add up to 180°, and, as angle BDC = BD′C + DCD′, so will those of every other triangle in the figure. If we used the angles of the triangle BD′C and the side BC to obtain the other sides, we would obtain the length BD′. On the other hand, if we used the angles of the triangle ABC and the side BC to obtain AB, and then used this side and the angles of the triangle ABD to obtain the other sides of this triangle, we would get BD as the length of the side opposite A. Thus, although the angles of the different triangles will all satisfy the conditions of closure of the triangles, they do not necessarily close the sides. Hence, in addition to the angular conditions which have to be satisfied to make the figure geometrically consistent, we need a *side condition* as well. This is one sort of problem which can be solved by the method of least squares.

In minor work, such as that with which the ordinary engineer has to deal, a rigid least-squares adjustment, which demands some knowledge of the method of least squares, is hardly necessary, but approximate methods of adjustment depending on the use of simple empirical formulæ, such as the one described in the Appendix, pages 263–268, are available. As a purely arbitrary substitute, a simple system of averaging results is sometimes used as follows: First of all, before the triangles are solved, their angles are adjusted to add up to 180°. This is done, for each single triangle, by distributing the amount by which the angles fail to close equally among the three angles, which means that one-third of the closing error is added to, or subtracted from, each angle. Thus:

Angle	Observed angle	Correction	Adjusted angle
A	60° 48′ 13″	−6″	60° 48′ 07″
B	73 31 55	−6	73 31 49
C	45 40 10	−6	45 40 04
Sum	180 00 18	−18	180 00 00

Here, the sum of the angles exceeds the theoretical sum by 18″, and this quantity has to be distributed equally among the angles. Hence, 6″ must be deducted from each angle, and when this is done, the adjusted angles add up to 180° exactly; it is these adjusted angles which are to be used in the solution of the triangle.

After the angles have been adjusted and the triangles solved, it will generally be possible, in the absence of a least-square adjustment, to derive several slightly different values from different triangles for certain computed sides. As the work proceeds, these values can be meaned and co-ordinates computed. Again, different values of the co-ordinates will be found from different points; these values can be meaned to give finally accepted co-ordinates, and from these final values, fresh and final values for the lengths and bearings of the sides and diagonals can be recomputed. These recomputed values will then be those which are finally accepted.

4. *Solution of Triangles.*

In ordinary triangulation the triangles which have to be solved have one side and the angles given, so that the formula to be adopted is the ordinary sine formula

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C},$$

or $b = a \sin B \operatorname{cosec} A$; $c = a \sin C \operatorname{cosec} A$, where a is the given side. As both b and c are needed, the logarithmic computation is best arranged as in the example that follows.

Example.—Let $a = 6325{\cdot}56$ and let the observed angles be $A = 57°\ 15'\ 12''$, $B = 49°\ 51'\ 57''$, $C = 72°\ 52'\ 41''$. Find b and c.

Here the sum of the angles is 10″ less than the theoretical sum of 180°, so that 10/3 = 3⅓″ have to be added to each angle. Consequently, the adjusted angles to be used in the solution are as set out in the last column of the following table:

Angle	Observed angle	Correction	Adjusted angle
A	57° 15′ 12″	+ 3·4″	57° 15′ 15·4″
B	49 51 57	+ 3·3	49 52 00·3
C	72 52 41	+ 3·3	72 52 44·3
Sum	179 59 50	+10·0	180 00 00

The actual logarithmic computation is set out as follows:

log b	= 3·759 667	b = 5749·99
log sin B	= 9·883 405	
log cosec A	= 0·075 163	
log a	= 3·801 099	
log sin C	= 9·980 315	
log c	= 3·856 577	c = 7187·48

In this calculation, the log sines of B and C, log cosec A and log a are written down in the order shown; the addition of the upper three lines gives log b and the addition of the lower three lines gives log c. This is better than the following common arrangement which involves writing down log a and log sin A twice.

log sin B	= 9·883 405	log sin C	= 9·980 315
log a	= 3·801 099	log a	= 3·801 099
	3·684 504		3·781 414
log sin A	= 9·924 837	log sin A	= 9·924 837
log b	= 3·759 667	log c	= 3·856 577

In general, when the whole of the work is computed by logarithms, it is not necessary to look out or tabulate the true values of b and c as the logarithms only are required, and, should the true values be needed at any time, they can easily be found from the tabulated logarithmic values. The log cosecant is, of course, found by subtracting the log sine from zero, which is the logarithm of unity.

It may be added that the above description of the solution of triangles applies strictly only to surveys which can be treated as plane surveys. In geodetic work, the curvature of the earth has to be taken into account, and in this case the triangles are no longer plane triangles, but spherical or spheroidal triangles. Here the theoretical sum of the three angles of a triangle is not 180° but 180° plus a small quantity known as the *spherical excess*. This quantity can be found (in seconds) by dividing the area of the triangle by the square of the radius of the earth multiplied by sin 1″, and it amounts to 1″ roughly for every 76 square miles of area. It can, however, be proved mathematically that, for all triangles which have sides that can be sighted over, it is sufficiently accurate to treat a spherical triangle as a plane triangle if each angle is reduced by one-third of the spherical excess, so as to reduce the angles to plane angles and their sum to 180°.

5. *Computation of Astronomical Observations.*

The data required at the beginning of a systematic trigonometrical survey are the azimuth or bearing of one line, generally the base line, and the co-ordinates of one point, generally one end of the base. If these are not available from other sources, they can be obtained by means of astronomical observations. These observations are described in Chap. XVIII. If astronomical observations are not possible, and if no points exist on the ground from which initial bearings and co-ordinates can be obtained, the magnetic bearing of one line can be observed by compass and all bearings based on it, and co-ordinates on assumed co-ordinates of one point. Magnetic bearings are neither so accurate nor satisfactory as bearings based on an azimuth properly determined with a theodolite.

6. *Computation of Co-ordinates.*

In geodetic work, co-ordinates are often computed in the first place in terms of geographical co-ordinates, i.e. latitude and longitude, and later in terms of special rectangular co-ordinates in which allowance is made for the curvature of the earth. Small local surveys, however, such as this book is concerned with, are computed in terms of simple, plane, rectangular co-ordinates similar to those described in Chapter V. Accordingly, after the triangles are solved, the bearings of the various lines are deduced from the adjusted angles, and, the lengths of these lines being known from the solution of the triangles, plane rectangular co-ordinates are computed in the ordinary way by the rules already given.

7. *Calculation of Vertical Heights: Trigonometrical Levelling.*

No matter how accurately vertical angles are observed, or how carefully the results are computed, elevations determined by the observations of vertical angles are never so accurate as elevations determined by spirit levelling. There are cases, however, where a height is required and where it would be very difficult, if not impossible, to establish one by ordinary spirit levelling. Such cases occur more particularly on steep mountains or hills, and in these cases it often happens that a height is particularly needed by a plane-tabler or topographer engaged on small-scale topographical work who depends on being able to sight his fixed points from considerable distances away. Extreme accuracy for this class of work is not necessary, and also, as a general rule, accurately fixed heights of high hills are not needed for most engineering purposes, since the tendency is for engineering schemes and works, such as roads, railways, pipe lines, etc., to keep to the low-lying plains where ordinary levelling is comparatively easy and is the best method to use. On the other hand, trigonometrical levelling will enable a large stretch of country to be covered much more rapidly than would a network of lines of spirit levels.

If the earth were perfectly flat and rays of light were not bent in their passage through the atmosphere, the computation of vertical heights would be very simple, since they could be calculated from the simple formula $h = d \tan \theta$, where h is the difference in height between a horizontal plane through the horizontal axis of the instrument and the point to which observations are taken, d is the horizontal distance from the instrument to the point whose elevation is required, and θ is the observed vertical angle. Unfortunately, except for comparatively rough work or for very short sights, the curvature of the earth and the effects of vertical refraction are very appreciable and must be taken into account. Thus, as regards curvature, a level line parallel to the earth's surface, such as the line ACA′ in fig. 6.4 (p. 69), falls away from a plane tangential to the earth at any point by about 8 in. in a distance of a mile, and this difference increases with the square of the distance. In ordinary spirit levelling the effect is not appreciable because of the very short sights, and, moreover, it can be almost entirely eliminated by keeping backsights and foresights very approximately equal in length.

The full solution for the determination of differences of elevation by trigonometrical levelling when long lines are involved is not difficult, but the problem is one in geodetic rather than in plane surveying, and the solution will be found in most books on geodesy or advanced surveying. Where only small surveys are involved, and lines do not exceed about 10 miles in length, the following formula may be used with sufficient accuracy for most practical purposes:

$$h_2 - h_1 = d \tan \theta + \frac{d^2}{2R}(1 - 2k),$$

where the second term is the curvature and refraction term, and is the same as that given on p. 69 in connection with the determination of the intervisibility of points. If h_2, h_1 and d are in feet, this may be written

$$h_2 - h_1 = d \tan \theta + 0{\cdot}574 \left(\frac{d}{5280}\right)^2$$
$$= d \tan \theta + 0{\cdot}000\,000\,0206 d^2.$$

In nearly all cases in practice, the height of the ground mark at the observing end is known, and it is the height of the ground station at the other point, not the height of signal, which is required. Hence, the height of instrument and of signal above ground marks must be measured and taken into account as in the following example.

Example.—The elevation of the ground station at point B is 571·3 ft. and the height of the instrument above the ground mark is 4·1 ft. The angle of elevation observed to point A is 4° 25′ and log d, as obtained from the solution of the triangles, is 3·856 577. Height of signal at A above ground mark is 10·5 ft. Find the elevation of the ground mark at A.

$\log d$	$= 3{\cdot}8566$	$\log d$	$= 3{\cdot}857$
$\log \tan \theta$	$= 8{\cdot}8878$		
$\log d \tan \theta$	$= 2{\cdot}7444$	$2 \log d$	$= 7{\cdot}714$
		$\log (206 \times 10^{-10})$	$= \bar{8}{\cdot}313$
$d \tan \theta$	$= +555{\cdot}2$	$\log \{(206 \times 10^{-10}) \times d^2\}$	$= 0{\cdot}027$
$206 \times 10^{-10} \times d^2$	$= +\ \ 1{\cdot}1$		
$h_2 - h_1$	$= +556{\cdot}3$		

Elev. of mark at B	= 571·3
Instrument above mark	= 4·1
Elev. of instrument	= 575·4
$h_2 - h_1$	= +556·3
Elev. of signal at A	= 1131·7
Signal above mark at A	= − 10·5
Elev. of ground mark at A	= 1121·2

In all cases in trigonometrical levelling it is advisable to observe vertical angles in both directions, simultaneously if possible, as this tends to eliminate the effects of curvature and vertical refraction.

15. Trigonometrical Resection: The Three-point Problem.

During the breaking-down process in trigonometrical survey it is often necessary to fix a point on the ground from observations to three fixed points, some or all of which are inaccessible and cannot be occupied as instrument stations. This happens, for instance, when the fixed points are vanes on spires or lightning conductors on the top of factory chimneys which have been fixed by intersection from other points. The same problem arises in hydrographical work when the positions of soundings have to be fixed from sextant observations made in a small boat to fixed points on shore, but here the solution is generally carried out by graphical means.

In fig. 6.12, A, B and C are three fixed points and P the point to be fixed. Then, provided P does not lie on or near the circle drawn

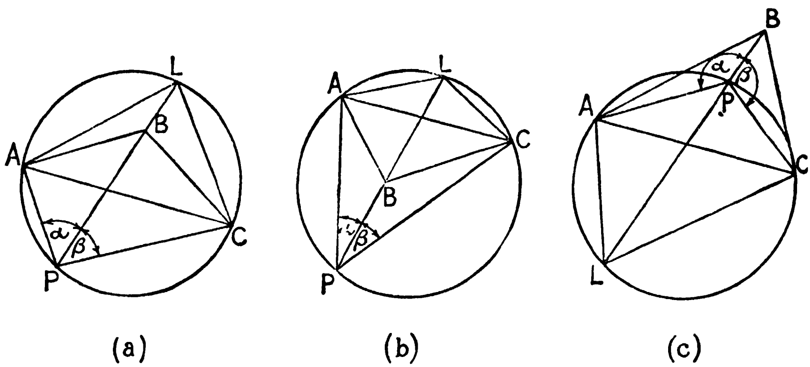


Fig. 6.12

through A, B and C, the co-ordinates of P can be found if the angles APB $= \alpha$ and BPC $= \beta$ are observed. In figs. 6.12*a* and *b*, P is outside the triangle ABC, and in *c* it is inside it.

There are a large number of different solutions possible, and the following, which is easy to remember, is known as the *Collins solution.*

Collins Solution.

Imagine a circle drawn through A, P and C and let PB produced meet this circle in L. Then, in figs. 6.12*a* and *b*, angle LAC $=$ LPC $= \beta$ and LCA $=$ LPA $= \alpha$. Since A, B and C are fixed points, we can calculate the lengths and bearings of the sides AB, BC and CA. In the triangle ALC we therefore know the length of the side AC and the angles adjacent to it. Hence, we can solve the triangle ALC for the sides AL and CL, and, knowing the bearing of AC and the angles LAC and LCA, we can find the co-ordinates of L from A and check from C. The co-ordinates of L and B are therefore known, and therefore the bearing LB can be computed. Hence, the angles ALP and CLP can be calculated from the bearings of the containing rays. Consequently, in the triangles ALP and CLP we know the angles at P and L and the sides LA and LC, so that we can solve the triangles for AP and CP. Moreover, since CAP = CLP and ACP = ALP, we can calculate the bearings of AP and CP and thus find the co-ordinates of P from those of A and check from those of B.

A similar construction and argument hold for fig. 6.12*c*, where P is inside the triangle ABC, but in this case the angles LPA and LPC are $180° - \alpha$ and $180° - \beta$ respectively.

It is important to note that, when the observations at a resected station are booked, the figures should be accompanied by a clear sketch showing roughly the position of the resected point with reference to the three fixed points. Also, it is advisable to observe to a fourth fixed point, as the observed angle will enable a bearing to be computed which can be compared with the bearing calculated from the co-ordinates of the fourth point and the computed co-ordinates of the point being fixed. In this way, a check is obtained on both observations and computations.

It should also be particularly noted that a solution of this problem becomes impossible if the point to be fixed lies on or near the circle which passes through the three fixed points.

Example.—Assume that the co-ordinates of A, B and C in fig. 6.12*a* are:

Point	x	y
A	34,358·1	27,948·3
B	39,117·2	39,106·1
C	27,178·9	55,718·7

and the observed angles are $\alpha = 61° \ 15' \ 27''$, $\beta = 53° \ 46' \ 54''$. Find the co-ordinates of P.

The first step is to compute the bearings and distances between the fixed points. The computation gives:

Line	Bearing	Log distance
AB	66° 54′ 03·3″	4·083 871
AC	104 29 41·1	4·457 630
BC	125 42 05·2	4·310 845

Then:

Triangle ACL

$A = \beta =$	53° 46′ 54″	log CL	= 4·407 244
$C = \alpha =$	61 15 27	log sin A	= 9·906 751
	115 02 21	log cosec L	= 0·042 863
	180	log AC	= 4·457 630
L =	64 57 39	log sin C	= 9·942 896
		log AL	= 4·443 389

L from A

Bg. AC	=	104	29	41·1
β	=	53	46	54
Bg. AL	=	50	42	47·1

log Δx	=	4·244 933
log cos AL	=	9·801 544
log AL	=	4·443 389
log sin AL	=	9·888 732
log Δy	=	4·332 121

L from C

Bg. CA	=	284	29	41·1
α	=	61	15	27
Bg. CL	=	345	45	08·1

log Δx	=	4·393 675
log cos CL	=	9·986 431
log CL	=	4·407 244
log sin CL	=	9·391 139
log Δy	=	3·798 383

	x	y
A =	34,358·1	27,948·3
	+17,576·5	+21,484·3
L =	51,934·6	49,432·6

	x	y
C =	27,178·9	55,718·7
	+24,755·7	− 6,286·1
L =	51,934·6	49,432·6

Bearing LB

B =	39,117·2	39,106·1
L =	51,934·6	49,432·6
	−12,817·4	−10,326·5

log Δy	=	4·013 953
log Δx	=	4·107 800
log tan LB	=	9·906 153

Bearing LB = 218 51 25·5

LB − 180° = 38 51 25·5

Triangle ALP

Bg. LA	=	230 42 47·1
Bg. LB	=	218 51 25·5
Angle ALP	=	11 51 21·6
L	=	11 51 21·6
P	=	61 15 27
		73 06 48·6
		180
A	=	106 53 11·4

log LP	=	4·481 352
log sin A	=	9·980 859
log cosec P	=	0·057 104
log AL	=	4·443 389
log sin L	=	9·312 712
log AP	=	3·813 205

Triangle CLP

Bg. LB	=	218 51 25·5
Bg. LC	=	165 45 08·1
Angle CLP	=	53 06 17·4
L	=	53 06 17·4
P	=	53 46 54
		106 53 11·4
		180
C	=	73 06 48·6

log LP	=	4·481 352
log sin C	=	9·980 859
log cosec P	=	0·093 249
log CL	=	4·407 244
log sin L	=	9·902 946
log CP	=	4·403 439

P from A		*P from C*	
Bg. AL	= 50 42 47·1	Bg. CL	= 345 45 08·1
Angle LAP	= 106 53 11·4	Angle LCP	= 73 06 48·6
Bg. AP	= 157 35 58·5	Bg. CP	= 272 38 19·5
log Δx	= 3·779 133	log Δx	= 3·066 559
log cos AP	= 9·965 928	log cos CP	= 8·663 120
log AP	= 3·813 205	log CP	= 4·403 439
log sin AP	= 9·581 013	log sin CP	= 9·999 539
log Δy	= 3·394 218	log Δy	= 4·402 978

	x	y		x	y
A =	34,358·1	27,948·3	C =	27,178·9	55,718·7
	− 6,013·6	+ 2,478·7		+ 1,165·6	−25,291·7
P =	28,344·5	30,427·0	P =	28,344·5	30,427·0

Hence,

Co-ordinates of P = 28,344·5; 30,427·0.

In working out this example we have introduced a number of checks which it is always well to use. These are:

1. Co-ordinates of L are computed from both A and C.
2. Partial check on solution of triangle ALP by solving triangle CLP and finding a second value for the common side LP.
3. Co-ordinates of P computed from both A and C.

Note also that, as can be seen from the geometry of the figure, angle A in triangle ALP is equal to the sum of the angles L and P in triangle CLP, that angle C in triangle CLP is equal to the sum of the angles L and P in triangle ALP, and that angle A in triangle ALP is the supplement of angle C in triangle CLP. Hence, in the solution of the triangles, sin A in triangle ALP is equal to sin C in triangle CLP.

A final check on the computation would be to work out the bearings of the lines PA, PB and PC from the co-ordinates and see if the differences of bearings give the observed angles α and β. A better check, however, as it is a check on both computation and observations, is to compare the computed bearing to a fourth fixed point with the observed bearing as deduced from the observation of an angle to that point from one of the others.

It will be seen that this problem is a particularly good exercise in the various computations dealt with in this and the previous two chapters.

16. Two-point Problem.

It is also possible, by choosing a suitable auxiliary point, to obtain a fixing from two fixed points only.

In fig. 6.13, A and B are the fixed points and P the point to be fixed. Set out the auxiliary point Q so that the figure gives four reasonably well-shaped triangles APQ, APB, ABQ and BPQ, with all four points intervisible from one another. At P observe the angles APB $= \alpha$ and BPQ $= \beta$, and at Q observe the angles PQA $= \gamma$ and AQB $= \delta$.

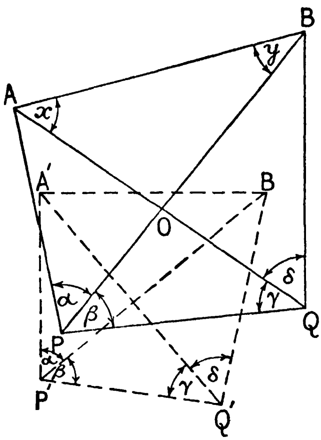


Fig. 6.13

Assume (1) approximate co-ordinates of point P, (2) a bearing for PQ, and (3) a convenient length, say 1000 or 10,000, for the length of PQ. Then the triangles PAQ and PBQ can be solved in terms of the assumed value of the length PQ and preliminary co-ordinates found for Q, A and B. From these preliminary co-ordinates a preliminary bearing and length can be computed for the line AB. But the figure so computed, P′A′B′Q′ in fig. 6.13, will be similar in all respects to the figure PABQ except that its position, scale and orientation will be different as shown. Hence, the difference between the computed bearing of A′B′ and the true bearing of AB will give a constant correction to be applied to the computed bearings of all lines to give true bearings, and the ratio of the true length of AB to the computed length of A′B′ will give a constant scale factor which can be applied to the computed lengths to give the true lengths. Accordingly, the true lengths and bearings of the lines AP and BP can be found, and the true co-ordinates of P computed from those of A and checked from those of B.

Although the method just described is easy to construct from simple principles and without remembering special formulæ, it is probably not quite so easy to compute as the following interesting analytical solution.

Analytical Solution.—Denoting the angles QAB and PBA by x and y (fig. 6.13), we see that

$$x + y = \beta + \gamma.$$

Moreover, if O is the point of intersection of the diagonals PB and QA, we have the identity

$$\frac{OA}{OP} \cdot \frac{OP}{OQ} \cdot \frac{OQ}{OB} \cdot \frac{OB}{OA} = 1.$$

Applying the sine rule to the triangles AOP, POQ, QOB and BOA in turn, and remembering that angle PAQ $= 180^\circ - (\alpha + \beta + \gamma)$ and angle PBQ $= 180^\circ - (\beta + \gamma + \delta)$, we get

$$\frac{\sin \alpha}{\sin (\alpha + \beta + \gamma)} \cdot \frac{\sin \gamma}{\sin \beta} \cdot \frac{\sin (\beta + \gamma + \delta)}{\sin \delta} \cdot \frac{\sin x}{\sin y} = 1.$$

$$\therefore \quad \frac{\sin y}{\sin x} = \frac{\sin \alpha \sin \gamma \sin (\beta + \gamma + \delta)}{\sin \beta \sin \delta \sin (\alpha + \beta + \gamma)} = k, \text{ say.}$$

But it is easy to show that

$$\tan \tfrac{1}{2}(x - y) \cot \tfrac{1}{2}(x + y) = \frac{\sin x - \sin y}{\sin x + \sin y} = \frac{1 - \dfrac{\sin y}{\sin x}}{1 + \dfrac{\sin y}{\sin x}} = \frac{(1 - k)}{(1 + k)}.$$

$$\therefore \quad \tan \tfrac{1}{2}(x - y) = \frac{(1 - k)}{(1 + k)} \tan \tfrac{1}{2}(x + y).$$

This gives the value of $(x - y)$ and hence, as we know the value of $(x + y)$, the angles x and y can be obtained. It is then easy to solve the triangle ABP for BP and AP and to deduce the bearings of these lines. Having done this, the co-ordinates of P can be calculated from those of A and B.

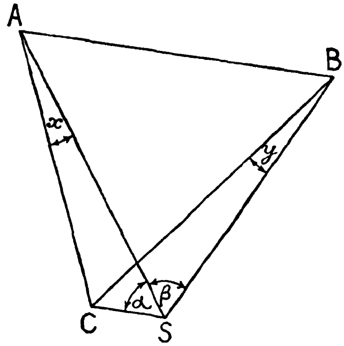


Fig. 6.14

17. Satellite or Eccentric Stations.

In fig. 6.14, the angles BAC and ABC have been observed from the fixed stations A and B, but, for certain reasons, it is impossible to set the instrument over station C to observe the angle ACB. The triangle ABC can be solved from the angles BAC and ABC, but the fixing will be strengthened if we can get a value for ACB which depends on observation. To get such a value, set up the instrument at the *satellite station* S, as close as possible to C, measure the distance SC, and observe the angles ASC $= \alpha$ and ASB $= \beta$.

Make a preliminary solution of the triangle ABC from the two observed angles and the known length AB, thus obtaining approximate values for the lengths AC and BC. From the figure, using these approximate values, we have

$$\sin x = \frac{\text{CS}}{\text{AC}} \sin \alpha,$$

or, if x is small and is expressed in seconds of arc,

$$x'' = \frac{\text{CS}}{\text{AC}} \frac{\sin \alpha}{\sin 1''}.$$

Similarly,

$$y'' = \frac{\text{CS}}{\text{CB}} \frac{\sin (\alpha + \beta)}{\sin 1''}.$$

But
$$\angle\text{BCS} = 180° - (\alpha + \beta) - y,$$
$$\angle\text{ACS} = 180° - x - \alpha.$$
$$\therefore \quad \angle\text{ACS} - \angle\text{BCS} = \angle\text{ACB} = C = \beta + y - x,$$
or
$$C - \beta = y - x.$$

Hence C can be found.

It should be noted that β should be observed with the same precision as angles BAC and ABC, but α need only be observed fairly approximately.

GRAPHIC TRIANGULATION WITH THE PLANE-TABLE

In small-scale topographical work, it is often necessary, after the minor instrumental triangulation has been completed, to fix additional points required for the detail survey by means of graphic triangulation executed by plane-table, or even to extend small chains of graphic triangulation into areas where the instrumental triangulation is not sufficiently dense. In other cases, where small areas only are involved and great accuracy is not needed, a graphic triangulation by plane-table may be used as a framework for a survey which is to be carried out entirely by graphic methods.

To commence a plane-table triangulation, it is necessary to start with a base line which is either a line between two points fixed by instrumental triangulation, or resection or traverse, or, if such a line is not available, one must be measured by chain or steel band. The direction of this line and the position of one end of it must also be known or determined.

Assuming that a suitable line is available, the two end points, A and B, are plotted on the plane-table sheet. The plotted positions are shown at **a** and **b** in fig. 6.15. The instrument is then set up and levelled over one point, say point A. The point to be fixed as the third apex of the first triangle and the other end of the base having been

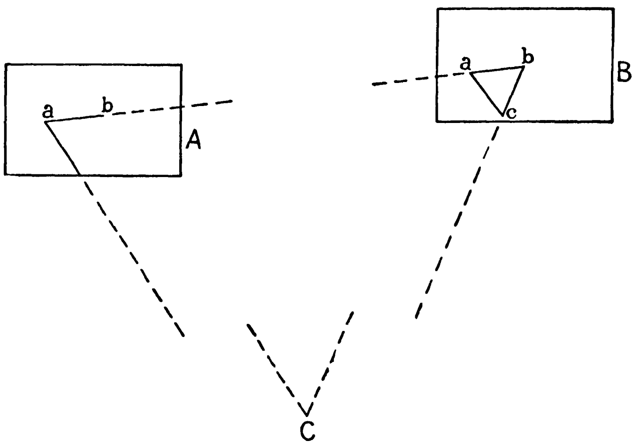


Fig. 6.15

marked with ranging poles or other signals, the edge of the alidade is laid carefully along the line representing the base line. The table is then unclamped and turned until the other end of the base, as viewed in the direction **ab** on the map, appears to be in line with the cross hairs of the alidade, after which the table is clamped. This means that the table is now oriented with the line **ab** on the map lying in the direction AB on the ground. A sharp-pointed pencil being held vertically with its point on **a**, the plotted position of the point A, so that the pencil point may act as a pivot, the alidade is laid against this point and rotated round it until the point C, the point to be fixed, is seen to be in line with the cross hairs. A line through **a** is then drawn against the edge of the alidade in the direction of C.

The instrument is next moved to B and the table oriented with the line **ba** on the map pointing in the direction of the line BA on the ground. With the point **b** on the map as pivot, the line of sight of the alidade is directed to the point C, and a line drawn through **b** in the direction of C. The intersection of this line with the line previously drawn through **a** will then give the position of **c** on the map. If necessary, the table can be moved to C, and either of the lines CA or CB used as a base for another triangle. In this case, of course, sights to additional points which have to be taken from either A or B would be taken before the instrument was moved from there. In this way, a whole chain or network of triangles can be built up graphically on the map.

As a point is fixed by the intersection of rays drawn from two previously fixed points, a single point fixed in the manner described above is said to be *fixed by intersection.* Plane-table resection, from three and two fixed points, is described in Chap. VII of *Principles and Use of Surveying Instruments.*

QUESTIONS ON CHAPTER VI

1. A line was measured with a 100-ft. band of which the true length was 99·875 ft., and the apparent length of the line was found to be 1964·66 ft. What is the true length of the line?

2. The following angles of slope were observed on a certain line, distances being measured along the slope:

From	To	Angle of slope	Distance
1	2	5° 25′	295·28
2	3	3° 30′	158·61
3	4	8° 45′	301·06
4	5	6° 40′	278·95

What is the correct length of the line?

3. A line was measured with a 300-ft. steel band whose true length at 62° F. was found to be 300·1482 ft. The mean temperature of measurement was 86° F., and the measured length of the line was 3606·984 ft. What is the corrected length of the line? Assume that the coefficient of thermal expansion of the band is 0·000 0065 per 1° F.

4. Describe how you would measure accurately the length of a base line for a minor triangulation survey, giving details of the corrections to be applied in order to obtain the true length referred to mean sea-level. (Inst. C.E., April, 1946.)

5. Calculate the sag correction for a 100-ft. band weighing 12 oz. per 100 ft., used under a pull of 15 lb.

6. A base line of approximately 500 ft. was measured in five bays with an Invar tape suspended in catenary under a tension of 20 lb. The results are tabulated below. The tape was standardized in catenary at a temperature of 50° F. under a tension of 15 lb. and the standard length was found to be 99·998 ft.

   Compute the true length of the base reduced to mean sea-level.

Bay No.	I	II	III	IV	V
Recorded length, ft.	100·235	100·035	100·394	99·890	100·752
Temperature, °F.	75·5	72·0	73·5	73·0	73·0
Difference in height between ends of bay, ft.	4·92	5·66	0·98	2·04	1·17

   Coefficient of expansion = 0·000 0003 per ° F.; sectional area of tape = 0·0025 sq. in.; Young's modulus = $2{\cdot}9 \times 10^7$ lb. per sq. in.; weight per foot run = 0·006 lb.; radius of earth = $20{\cdot}9 \times 10^6$ ft.; mean height of base = 1500 ft. (Inst. C.E., October, 1956.)

7. A nominal distance of 100 ft. was set out with a 100-ft. steel tape from a mark on the top of one peg to a mark on the top of another, the tape being in catenary under a pull of 20 lb. and at a mean temperature of 70° F. The top of one peg was 0·56 ft. below the top of the other. The tape had been standardized in catenary under a pull of 25 lb. and at a temperature of 62° F.

   Calculate the exact horizontal distance between the marks on the two pegs and reduce it to mean sea-level. The top of the higher peg was 800 ft. above mean sea-level.

   Radius of the earth = $20{\cdot}9 \times 10^6$ ft.
   Density of tape = 0·28 lb. per cub. in.
   Section of tape = $\frac{1}{8}$ in. × $\frac{1}{20}$ in.
   Coefficient of expansion = 0·000 00625 per 1° F.
   Young's modulus = 30×10^6 lb. per sq. in.

   (Inst. C.E., April, 1948.)

8. The following were the observed angles of a triangle ABC, the length of the side AB being 7619·82 ft.

   $A = 42° \ 14' \ 25''$; $B = 53° \ 54' \ 40''$; $C = 83° \ 51' \ 10''$.

   Adjust the angles, and solve the triangle for the sides AC and BC.

9. It was required to lay out the line of a tunnel between two points A and D which were not intervisible; the points were joined by two triangles ABC and BCD, of which the observed angles were:

Triangle ABC,

$A = 48° 15' 20''$; $B = 67° 39' 55''$; $C = 64° 04' 30''$.

Triangle BCD,

$B = 71° 14' 30''$; $C = 44° 38' 10''$; $D = 64° 07' 50''$.

The points A and D lie on opposite sides of BC, and the measured length of the side AB was 3246·33 metres. Find the distance AD, and the angle BAD which must be set out on a theodolite in order to set the instrument on the correct alignment for setting out the line AD.

10. It is required to find the bearings of two lines TA and TB from an inaccessible station T. A satellite station S was set up 12·7 ft. from T in an approximately S.E. direction and from it the following theodolite angles were measured:

Telescope pointing on A = 360° 00′ 00″
Telescope pointing on B = 70 40 00
Telescope pointing on T = 305 00 00

If point A lies N. 10° 04′ 32″ E. from station S, what are the true bearings of A and B from T?

AT is 9 miles; BT is 10 miles; log sin 1″ = $\bar{6}$·68557.
(Inst. C.E., October, 1953.)

11. Angles were read from a boat X at sea to three station points A, B and C on shore to the north of X. The angle AXB was 40° 36′, and the angle BXC was 34° 18′. AB was 1154·1 ft. in length, BC was 1000 ft. The bearing of AB from north was N. 30° 00′ 00″ E., and that of BC was S. 75° 06′ E. Show (without using graphical methods) that the position of the boat could not be fixed from the angles read from it, and calculate the maximum distance it could have been from station B, when the angles were read. (Inst. C.E., October, 1947.)

12. Three fixed points, A, B and C, were observed from a point X *inside* the triangle ABC, the measured angles being

AXB = 106° 53′ 20″ and BXC = 112° 29′ 40″.

The co-ordinates of A, B and C are

Point	x (northing)	y (easting)
A	37,032·6	15,050·1
B	41,121·8	22,984·6
C	29,974·3	29,538·4

Find the co-ordinates of X, and the bearing from X to A.

CHAPTER VII

TRAVERSING

1. General.

Traversing is an extension of the method of fixing a point by distance and bearing, but, instead of a single point and a single line, a traverse consists of a whole series of connected intersecting lines in which a forward station is fixed from an instrument station and then, in the next stage, becomes an instrument station to fix a new forward point. Hence, a traverse follows a zigzag course such as that shown in fig. 1.3 (p. 5).

Instrumental traversing, i.e. traversing in which angles or bearings are measured and are not supplied by graphical or chain survey methods, consists of many different kinds of work in which the fundamental principles remain the same, but in which details of measurement differ in order to obtain different degrees of accuracy. On the one hand, there is a first-order precise traverse, intended to replace first- or second-order triangulation, in which angles are measured with the utmost accuracy with a full-scale geodetic type of theodolite and distances with long Invar bands to much the same degree of accuracy. At the other extreme there are " rope and sound " traverses, executed for the survey of relatively unimportant detail in small-scale topographical work, in which distances are measured with a rope, and bearings are observed on a hand compass to a noise made at an unseen point, some distance away. Traversing, in some form or another, is one of the most commonly used survey methods: it is now used more commonly than triangulation, and is particularly suitable for flat or wooded country where triangulation is difficult or impossible.

In ordinary engineering and land surveying work, theodolite traverses are used for a number of purposes, mainly to establish some kind of control for detail work, principally in cases where triangulation is impossible or too expensive. Thus, the whole of the main framework for a town survey, or the survey of any large area, may consist of theodolite traverses of several degrees of accuracy. In the first place, there may be a theodolite " surround traverse " surrounding the whole area, with a number of internal criss-cross or radiating traverses of the same order to strengthen it. These traverses are surveyed with a fairly large theodolite and with careful chaining by steel band. Areas between are broken down by other traverses of a lesser degree of accuracy, also carried out by theodolite and steel band, or even by an ordinary land chain, but not so carefully measured as the principal framework traverses.

Long single theodolite traverses, as opposed to closed loops, occur in engineering work in location surveys for such things as railways, roads, pipe lines, etc. These traverses may or may not start and end at fixed points. Their function is to provide a strong " backbone " for the final plans and layout of the work. They are planned to enable a proper and final survey to be made of a line which has been laid down on paper from a preliminary approximate survey, or from topographical features shown on a map. In general they follow very closely the line which the projected work will follow, and it is from permanent or semi-permanent points established by means of them that the engineer will work when laying out the work in detail and putting in pegs for the guidance of the contractor.

Compass traverses are not used in engineering surveys as much as are theodolite traverses. For one thing, they are not nearly so accurate as the latter. Their main uses are two-fold. One is for preliminary and reconnaissance surveys where it is desired to get a general picture of the country involved without being concerned with too much detail. The other is to help to fill in less-important detail on surveys where framework of a higher order exists. Compasses, as a general rule, are not used for laying out engineering works to the same extent as theodolites are used.

In this chapter we shall consider theodolite traversing first because the principles and general procedure are best exemplified by it, and we shall then consider compass traversing, the procedure involved in it being mainly a modification of theodolite traversing to allow of a certain amount of relaxation in standards of accuracy and of simplification in method.

THEODOLITE TRAVERSING

Theodolite traversing has one advantage over triangulation in that no elaborate preliminary reconnaissance to ascertain if a scheme is possible is ordinarily necessary. Nevertheless, it is generally advisable, where possible, for a responsible surveyor to go over the route to be followed before work commences, with the object of finding out where special difficulties are likely to be encountered, and how the number of legs may be reduced to a minimum. Another advantage is that, in traverses, points are fixed in accessible positions where they can easily be used in later work, whereas triangulation points are generally on high hills which are often far off the beaten track and are very inaccessible. On the other hand, triangulation will enable points to be fixed over a given area fairly quickly, but a traverse is confined to a single line.

The great disadvantage of traversing as opposed to triangulation is the far greater possibility of making gross errors either in field measurements or in the computations. Hence the utmost care must be taken at all times in traverse work and in traverse computation to avoid the occurrence of such errors.

The various stages in traversing now to be considered may be grouped as follows:

1. Reconnaissance and laying-out.
2. Station marks and signals.
3. Angular observations.
4. Linear measurements.
5. Computations.

2. Reconnaissance and Laying-out.

The reconnaissance of the route to be followed by a traverse may be made some time ahead of the observing and measuring, or it may be done immediately ahead of the main measuring party, or it may not be done at all. Much will depend on the length of the traverse, the nature of the country involved and the strength of the party. In very many cases the route of a traverse will follow a main road or railway, thus avoiding unnecessary clearing or working over cultivated land.

The main things in selecting the route of a traverse are:

1. To make individual legs as long as possible.
2. To make them as equal in length as possible.
3. To avoid very short legs.
4. To avoid grazing rays.
5. To select lines over which chaining will be easy.
6. To select lines which will avoid heavy clearing or damage to crops or property.

In the more accurate types of work the chief difficulty in the field work is to maintain the accuracy of the angular work, and this can only be done by bringing bearings forward over long legs. Sometimes long sights can be obtained over several legs, although the chaining has to be taken over the more tortuous course. Thus, in fig. 7.1, the chaining follows the line ABCDE, but it is found that a sight may be obtained from A to E. In this case, the line ABCDE may be treated

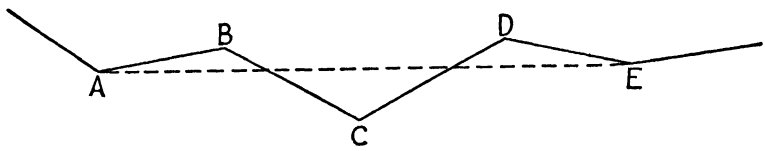


Fig. 7.1

as a subsidiary traverse from which the length of the leg AE can be computed. The main bearings are then brought forward along the line AE, which is treated as a single leg with a computed length for the distance. Other variations of this problem are possible.

In many cases, the initial and end points of a traverse are fixed points at which a line of known bearing already exists on the ground, or else they have to be tied to such points. If the end points are already fixed, nothing more has to be done than to sight along the lines of known bearing to get fixed bearings for the first and last legs. If an end point is not already fixed, both a fixing and a bearing can be obtained, either by intersection from two fixed points, or by a two-point resection, if two fixed points giving a well-shaped triangle are visible from it, or else by a three-point resection if three suitably placed fixed points are visible. If only one fixed point can be seen from it, it will be necessary to use some form of measured base to obtain a fixing. Thus, in fig. 7.2, A and B are the ends of a single leg, the fixed point X being visible from both points, and from X another fixed point Y can be seen as well as the points A and B. Then, by measuring the length of the leg AB and the angles XAB and XBA, the triangle XAB can be solved for XA or XB, and, by setting up the theodolite at X and observing the angle AXY or BXY, the bearing of XA or XB, and hence of AX or BX, can be obtained from the bearing of XY, thus

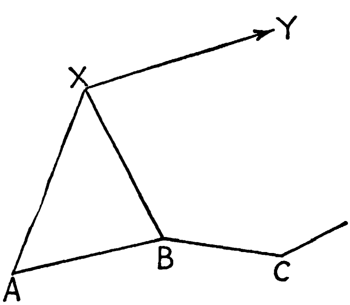


Fig. 7.2

giving all the data necessary to fix the points A and B. If only a bearing is required, all that is necessary is to observe the angles AXY and XAB, or BXY and XBA.

In fig. 7.3, the fixed point X is visible from A and B, but A and B are not intervisible. Here a subsidiary traverse A**ab**B, known as a *traverse base*, is run between A and B, and this enables the length of the side AB and the angles **a**AB and **b**BA to be calculated. The angles XA**a** and XB**b** are observed, and hence the angles XAB and XBA are known. Consequently, the triangle XAB can be solved as before. The traverse base can be computed by using assumed values for the co-ordinates of A, and an assumed false value, say zero, for the bearing of A**a**. This will give the data to compute the distance and the false bearing of AB.

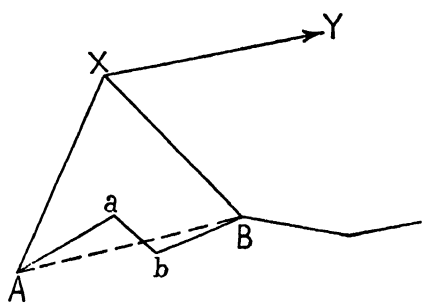


Fig. 7.3

Fig. 7.3 also represents the case where A and B are intervisible, but the direct distance AB is impossible for ordinary chaining and has to be found by computation from a traverse base run between the two points.

If fixed points can be seen from different intermediate points on the traverse, angular observations to them should be taken whenever possible, or at least at reasonable intervals. These observations form a useful check, because, when the traverse has been computed, a bearing from the instrument station to the fixed point can be computed from co-ordinates, and this computed value compared with the value calculated from the bearings of the traverse lines and the angular observations to the point.

If no fixed point exists in the neighbourhood of a traverse, it is possible to obtain initial and closing bearings from astronomical observations for azimuth. Otherwise, a rough initial bearing may be obtained by magnetic compass, but a compass bearing at the end may not be sufficiently accurate to use as a bearing on which to close bearings brought forward by theodolite. In many cases it is necessary or sufficient to use an assumed value as the initial bearing of a traverse, and assumed co-ordinates as the co-ordinates of the initial point.

3. Station Marks and Signals.

The nature of the marks to be put in at traverse stations will depend on whether or not permanent or semi-permanent marks will be required. If permanent ground marks are needed, they may be iron posts or concrete pillars with suitable centre marks. In most cases it will be advisable to put in good permanent marks at the beginning and end points; other permanent marks can also be put in at reasonable intervals, not necessarily at every single traverse station, the length of these intervals depending on the lengths of the traverse legs and the purpose for which the traverse is required. Intermediate marks can then consist of ordinary wooden pegs, preferably creosoted, with tacks or nails driven in on top to mark the exact centre.

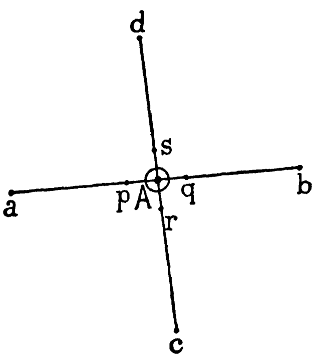


Fig. 7.4

When putting in permanent marks of traverses, it is well to put them in in sets of three, one at each of three consecutive traverse stations. This preserves two bearings and the included angle on the ground. Hence, if at any future time there is a doubt about whether or not a mark has been moved, the theodolite can be set over the middle mark and the angle between the other two measured and compared with the original angle. If the two do not agree, the presumption is that at least one of the three marks has been moved.

On constructional work especially, marks have often to be put in at points where they are likely to be moved, and where it may be necessary to re-establish them later. Let A in fig. 7.4 be a point which is likely to be moved during constructional work. Pegs **a** and **c** are put in at convenient points on lines A**a** and A**c**, approximately at right angles to one another. The theodolite is set over A and sighted on **a**. The telescope is then transited and a peg **b** put in on the line **a**A produced. Similarly, the peg **d** is put in on the line **c**A produced. If the point A is subsequently moved, and it is desired to replace it, the theodolite is set over **a**, sighted on **b**, and two pegs **p** and **q** are put in on the line **ab**, one on either side of where it is seen that the original mark must have been. Similarly, from **c** two pegs **r** and **s** are put in on the line **cd**. The intersection of strings stretched between **p** and q and between r and s will be the true position of the original mark. This operation is known as *referencing a point.*

Signals for sighting on will normally be flagged ranging poles held temporarily by a labourer while observations are in progress or else held in a ranging-pole support,* or supported by wire guys over the mark. In all cases it is important to ensure that the pole is truly vertical at the instant of observation and, to minimize errors due to lack of verticality, it should be sighted low down near the base. A better arrangement than ranging poles, however, is to use the equipment and targets of the three-tripod system of observing (p. 72) if these are available.

If lines are very long, a signal consisting of a post and vanes such as is used in triangulation (fig. 6.6, p. 73) may be necessary. If they are very short, and if the three-tripod equipment is not available, the string of a plumb bob suspended from a tripod, with the plumb bob hanging freely over the station mark and a piece of white paper on the string, makes a good signal.

In all theodolite traverse work it is essential to see that signals are vertical and properly centred over the station mark, as nothing will make for inaccurate work more than signals not being truly vertical and faulty centring of signals and instruments.

4. Angular Observations.

Angular observations on traverse work are generally made in much the same way as in triangulation, but, except on precise traverses, smaller instruments are used and fewer observations are taken at each station. Occasionally, the method of repetitions (p. 88) is used on more accurate work and a special method, known as the *direct bearing* method, on minor traverses; but by far the most common method is a modification of the method of directions (p. 88). The usual instrument used in engineering work is a small theodolite reading direct to anything from 2″ to 1′.

If the method of directions is employed, the instrument is set with one vernier to read something near 0° or 180°, and pointed and clamped with the telescope sighting the back station. After both verniers have been read, the upper clamp is loosened, and the instrument turned and clamped to sight the forward station, when the verniers are again read. In traverse observations it is not usual to close back on the R.O., and this therefore gives one measure on one face of the angle

* Chap. II, fig. 2.1 (p. 5) of *Principles and Use of Surveying Instruments.*

measured clockwise from the back station to the forward station. Face or circle is now changed and the instrument set to a reading near 90° or 180°, when the previous operation is repeated with the swing made in the direction opposite to that used in the first observation. This completes the observations at the station, the mean of the face-right and face-left results being taken as the measure of the angle. An example of the booking of such an observation is given on p. 121. Here it is most important to see that the readings to rear and front stations are correctly shown against their proper station. If, through an error, the reading to the back station is booked as being to the forward station and vice versa, the angle worked out from the readings will be 360° minus the true value.

The above supposes that the instrument available is graduated on the whole-circle system, and that angles also are measured whole-circle from the rear station. If the instrument is graduated on the half-circle system, i.e. through 90° on either side of zero to 180°, it is more convenient to measure angles of deflection, measured right or left, *from the forward direction* of the rear leg, and the procedure in observing is slightly different. The upper plate being set and clamped to read 0°, the lower clamp is loosened, and the instrument set and then clamped with the line of collimation intersecting the rear station; both verniers are read. The telescope is then transited to point in the forward direction of the rear line and, the lower circle being kept clamped, the upper circle is unclamped and the telescope pointed to the forward station. The upper circle is clamped and the line of collimation brought on to the forward station by means of the upper horizontal tangent screw, when both verniers are again read. This gives one measure of the angle of deflection. The operation is then repeated with opposite face and swing, and this gives a second value, the mean of the two values being taken.

In both of the above methods, the stations are observed in the same order on both faces—that is to say, rear station, forward station, change of face, rear station, forward station—but, when whole-circle angles are measured, it is also easy, and it is possibly a little quicker in practice, to observe the forward station after the change of face, and then to swing back to and observe the rear station, the usual care of course being taken to see that readings are not booked against an incorrect station.

The direct-bearing method of observing is normally used on work of a minor nature only; it consists of an arrangement by which at every set-up of the instrument the reading obtained on the upper plate, when the telescope is sighted to the forward station, is the actual bearing of that station. Several slight variations in procedure are possible, but there is little to choose between them, and the following is probably as good as any:

Let A, B, C, D, etc., be the different instrument stations, the bearing of the line AB being known. The instrument is set up at station B and clamped with the upper circle reading the bearing of AB on one vernier. The lower circle is unclamped, and the instrument turned to sight on A, the line of sight being brought on to A by clamping the lower circle and using the lower-circle tangent screw, the upper circle being kept clamped. The telescope is then transited so that it points in the forward direction AB. Hence, it points along the direction of AB produced with the vernier reading the correct bearing of AB. The upper clamp is then unloosened and the telescope turned to point to station C, the line of collimation being brought into coincidence with C by means of the upper clamp and tangent screw. A little consideration will now show that the telescope and upper circle have been turned through an angle equal to the difference between the bearings of AB and BC, so that the angle read on the same vernier as before will be the bearing of BC. The upper circle is kept clamped, while the theodolite is moved to C, when the previous operation is repeated.

This method is only suitable when a reading on one vernier and one face is sufficient. Hence, it should only be used for short traverses of a minor kind, as in breaking down traverses on town and estate surveys. The instrument used should be in good adjustment for horizontal collimation. In addition, although the same vernier will always register the forward bearings of lines, provided the upper circle is not shifted at all relative to the lower during the process of moving from one station to another, it is always well to verify that the horizontal circle is registering correct bearings after the instrument has been set up at the new station.

5. Linear Measurements.

In theodolite traverse work, the ordinary surveyor's chain is very little used, except perhaps on very short "breaking down" traverses, a long, thin steel band up to about 300 ft. in length being used instead for all main linear measurements. The most convenient system of graduation for all except precise framework traverses is a band graduated either at every foot, with the first ten feet graduated in tenths, or else at every ten feet, with the first ten divided in feet and the first foot in tenths.

In deciding on methods, the first point to be considered is whether to use the tape "on the flat" along the ground or "in catenary" as described on pp. 25 and 27 of *Principles and Use of Surveying Instruments* respectively. For most ordinary engineering work, surface taping will be more convenient and quicker, provided the ground is not too rough and there is not much clearing to be done. Otherwise, if greater accuracy is desired, or if the ground is very rough or there is a lot of bush clearing, some form of catenary taping may be the better method to use.

The next point for consideration is what subsidiary observations—slope, temperature, etc.—will be necessary, and whether or not the tape should be used with a spring balance by means of which a constant tension can be applied. The answer to this lies in the purpose of the traverse and consequently the accuracy required. First- and second-order precise traverses, for instance, require the use of Invar bands very accurately standardized, accurately applied tension, and good observations of temperature, slope and elevation above sea-level. For the lowest class of work, an unstandardized band used without a spring balance, with no observations for temperature or slope, will serve the purpose, slopes in this case being allowed for by step-chaining. The location survey for a road or railway is usually made by such means used in conjunction with an ordinary small engineer's vernier theodolite.

A short summary, giving particulars of the order of accuracy to be expected in ordinary theodolite traversing by the use of different instruments and different methods, will be found on pp. 130–131; this may assist in giving some idea of the most suitable methods to adopt for different classes of work, and to reach a required degree of accuracy.

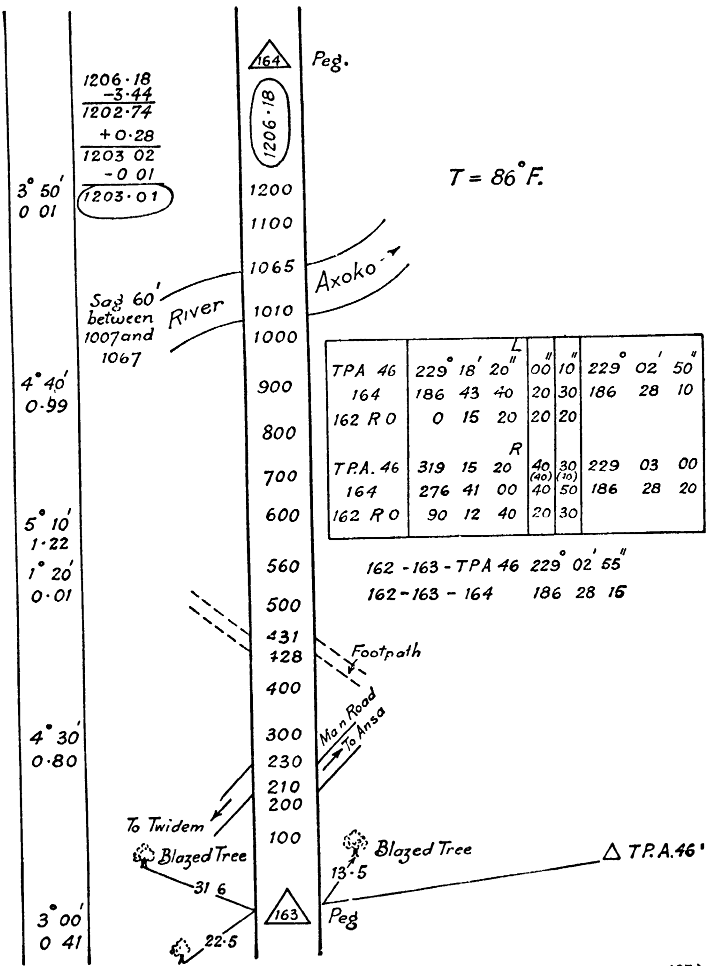
6. Booking the Field Measurements.

The field book used for booking a theodolite traverse is generally an ordinary oblong note-book with parallel lines about half an inch apart running up the centre. Booking is from the bottom of the page upward, and usually not more than two legs—more often one—are booked on the same page.

An example of booking a line measured with a 300-ft. band used on the flat under a tension of 10 lb. is given opposite, the angular measurements being made with a vernier theodolite reading to 20″ of arc. In this case an additional angular observation has been taken to a trigonometrical point T.P.A. 46 which lies to the right of the line. The observed angles are shown on the right-hand side of the page, the full figures immediately following the station numbers being the readings on "A" vernier, and the figures in the next column the seconds only of the reading on " B " vernier, the mean of the seconds being given in the next column. The last column gives the mean angle on a single face measured at station 163 from the rear station 162 as R.O. The top figures are the observations with face left and the lower ones with face right, the mean result of the two faces being below.

The figures in between the central vertical lines relate to the chainage, the total measured length, which is 1206·18, being shown ringed below

EXAMPLE OF PAGE OF TRAVERSE FIELD BOOK



the number of the forward station 164. A road, a footpath and a river cross the line, and the chainage of the cuts are noted. On the extreme left of the page are the angles of slope, as read on an Abney level, with the corresponding slope correction written below. Thus, the slope for the first 300 ft. is 3°, for which length the correction (obtained as explained on p. 81) is 0·41. There are decided changes of slope at chainages 560 and 600, the slope from chainage 300 to 560 being 4° 30′, and from 560 to 600 it is 1° 20′, which is succeeded by a slope of 5° 10′ between chainages 600 and 900. On the right of the page is the temperature, 86° F. at the time of measurement. The band concerned was of standard length under the applied pull at 49° F., so that the combined temperature and standardization correction is 1206 × 37 × 0·000 0062 = 0·28, and, as the band was too long, the measured length is too short and the correction is additive.

There is one sag of 60 ft. noted between chainage 1007 and 1067, where the line crosses a river, and, with tension of 10 lb. and a band weighing 16 oz. per 100 ft., the corresponding sag correction is 0·01.

The sum of the slope corrections is −3·44, and this, and the combined temperature and standardization correction of +0·28 and the sag correction of −0·01 are applied to the observed length as shown at the top left-hand corner of the page, the corrected length, shown ringed, being 1203·01.

This example illustrates one way of keeping the field book of an ordinary traverse executed by the methods described, but the entries may easily be varied to suit other methods. Often it will be found more convenient to keep the notes relating to the angular work in one book and those relating to linear measurements in another. This will be necessary if one party is responsible for the angular work and another for the linear work.

7. Traverse Computations.

The computations of a theodolite traverse consist of the following:

(i) Application of corrections to observed lengths.
(ii) Abstract of angles, and computation and adjustment of bearings.
(iii) Preparation of latitude and departure forms, and computation of latitudes and departures or co-ordinates.
(iv) Adjustment of co-ordinates.

(i) *Application of Corrections to Observed Lengths.*

The corrections to be applied to the measurements of traverse legs are similar to those described on pp. 79–86 in connection with the measurement of base lines by means of Invar bands, exactly the same formulæ being used; in the case of traverses some of the corrections may be omitted, the particular ones which are unnecessary depending on the accuracy of the field methods and the final accuracy desired. The corrections most commonly applied are·

1. Combined correction for standardization and temperature.
2. Correction for slope.

Sag correction will be applicable if the tape was standardized on the flat and is used in catenary, and, on precise work in country of high elevation, it may be necessary to correct for height above sea-level. If sea-level correction is necessary, the best method of applying it is either to combine it with the standardization and temperature correction over a section of 4 or 5 miles, taking the mean elevation of the section as the basis for computation, or else to apply it to the co-ordinates of the end point of a section and embody it in the proportional correction for closing error which is applied to the co-ordinates of intermediate points.

The correction to observed lengths are best made in the field book as shown in the example on p. 121.

(ii) *Abstract of Angles, and Computation and Adjustment of Bearings.*

The mean values of the angles are worked out and shown in the field books; they are then either entered on sheets of paper or on the latitude and departure forms described below. When the bearing of one line at the initial station is known, the bearings of all the lines are worked out from the included angles by the rules given in Chapter IV. This can be done direct on the latitude and departure forms as shown in the specimen form facing p. 125.

In most cases a traverse will either begin and end at the same point, or else it will begin at one fixed point and end at another fixed point, at each of which there exists a bearing already laid out on the ground. Consequently, when the end of the traverse is reached, it will be possible either to see if the observed included angles fulfil the rule for the sum of the angles in a closed figure given in Chap. IV, pp. 43–44, or else to see if a bearing brought forward through the intermediate legs of the traverse agrees with the fixed bearing at the end station. In practice it almost invariably happens that there is a small discrepancy between the sum of the observed angles and the theoretical sum in the one case, or between the calculated and fixed bearings in the other, and this discrepancy must be distributed among the angles or bearings. The method of doing this can best be seen from examples.

Example 1.—The observed internal angles in a closed figure of six sides are given in the second column of the following table:

Angle	Observed value	Adjustment	Adjusted value
A	92° 48′ 10″	+5″	92° 48′ 15″
B	136 18 00	+5	136 18 05
C	161 17 40	+5	161 17 45
D	79 33 10	+5	79 33 15
E	162 57 30	+5	162 57 35
F	87 05 00	+5	87 05 05
	719 59 30	+30	720 00 00
	720 00 00		
	+30		

Here there are 6 sides, so that, by the rule given on p. 43, the angles should add up to 6 × 180° − 360° = 720°. The actual sum is 30″ less than this, and hence 30/6 = 5″ must be added to each angle, giving the adjusted values shown in the last column of the table.

Example 2.—A traverse ABCDEF starts at a point A and ends at a point F. At A it is tied to a fixed line AP, of which the bearing is 332° 18′ 10″, and at F it is tied to a fixed line FQ, of which the bearing is 286° 21′ 00″. The observed angles PAB, ABC, BCD, CDE, DEF, EFQ, all measured clockwise from the rear station, are as set out in the third column of the following table. Find the adjusted bearings of the legs of the traverse.

From	To	Observed angle	Bearing	Adjustment	Adjusted bearing
A	P	—	332° 18′ 10″		332° 18′ 10″
A	B	131° 16′ 50″	103 35 00	−5″	103 34 55
B	C	152 51 30	76 26 30	−10	76 26 20
C	D	212 43 40	109 10 10	−15	109 09 55
D	E	176 22 10	105 32 20	−20	105 32 00
E	F	119 34 20	45 06 40	−25	45 06 15
F	Q	61 14 50	286 21 30	−30	286 21 00
		854 03 20	286 21 00		
		332 18 10	−30		
		1186 21 30			
		286 21 30			
		900 00 00			
5 × 180° =		900			
		0 00 00			

The unadjusted bearings are worked out by the rules given on pp. 39–43, and are set out in the fourth column, the value obtained for the bearing

FQ being 286° 21′ 30″. This is 30″ too much. Hence, by subtracting 30/6 = 5″ from each angle, or by subtracting 5″ from the bearing of AB, 10″ from the bearing of BC, 15″ from the bearing of CD, etc., the closing error is distributed equally among the observed angles. In the table, the adjustment has been applied directly to the bearings worked out from the unadjusted angles, the adjusted bearings being given in the last column of the table.

In the third column, the observed angles have been added together, and the bearing of AP added to the sum. From the resulting quantity, the unadjusted bearing of the last leg has been subtracted, leaving 900° 00′ 00″, which is exactly 5 × 180°. This checks the minutes and seconds, and the units in the degrees in the unadjusted bearing of FQ

(iii) *Preparation of Latitude and Departure Forms, and Computation of Latitudes and Departures.*

The latitude and departure form shown in the folder makes provision for the computation and adjustment of bearings and co-ordinates, and gives the final values of the latter.

The station numbers or letters are entered in order in the first two columns, and the figures in the subsequent columns refer to a good-quality traverse made with a small micrometer theodolite reading direct to 10″ and by estimation to single seconds, distances to two decimal places of a foot having been measured with a standardized steel band. Corrections for combined standardization and temperature and for slope have been applied to all distances, so that the lengths entered in column 6 are the corrected lengths.

The traverse starts at a fixed point, 1, whose co-ordinates are known and are to be held fixed, and it ends at another point, 10, whose co-ordinates are likewise known and are to be held fixed. The bearings from Trig. point 146 to point 1 and from 10 to T.P. 122 are also known and are to be held fixed.

For purposes of illustration, the angles entered in column 3 are deflection angles measured left (L) or right (R) from the forward direction of the rear leg. The unadjusted bearings worked out from the observed angles are first entered in column 4, but underneath each bearing is written the adjustment and the adjusted bearings. The fixed bearing of the line T.P. 146—station 1 is 331° 14′ 26″ and from station 10—T.P. 122 it is 134° 31′ 05″. The bearing of this last line computed through the traverse is 134° 30′ 24″, so that there is a closing error in bearing of 41″; as this error has to be distributed among 10 legs, the correction to the bearing of the rth leg is $\frac{1}{10} \times 41'' \times r = 4''\cdot1r$. These corrections are applied to the nearest second as shown on the form.

As a check on the working out of the bearings, the left and right deflection angles in column 3 have been added together, giving 246° 42′ 39″ L and 49° 58′ 37″ R. The difference is 196° 44′ 02″ L and, when this is subtracted from the bearing of the line T.P. 146—station 1, we get 134° 30′ 24″, which agrees with the bearing worked out in column 4. The reduced bearings in column 5 are obtained from the adjusted bearings in column 4.

The remainder of the form needs little explanation. Although nearly all traverse forms sold by instrument makers provide no special space or column for the actual logarithmic calculation of latitudes and departures, provision here is made for this in column 7, the latitudes and departures being entered in columns 8–11. These latitudes and departures, when added to the co-ordinates of the instrument station, give the co-ordinates of the station at the end of the forward leg. Hence the co-ordinate values appearing in columns 12 and 13 are the co-ordinates of the station appearing in column 2, and the bearings in column 4 are the bearings in the direction of the stations given in column 1 to those given in column 2. Thus, the unadjusted bearing of the line 4 to 5 is 276° 52′ 20″ and the unadjusted co-ordinates of station 5 are 34,873·73; 57,122·97.

As a check on the working out of the co-ordinates in columns 12 and 13, the latitudes and departures in columns 8, 9, 10 and 11 are summed, and the difference between the northings and southings when applied with its correct sign to the x co-ordinate of station 1 should give the unadjusted x co-ordinate of station 10. Similarly, the difference between the eastings and westings when applied to the y co-ordinate of station 1 should give the unadjusted y co-ordinate of that station. This test, shown on the form, should be repeated on every sheet.

In this form, in order to provide as complete an example as possible, six-figure logarithms have been used and bearings have been taken out to seconds. For many classes of theodolite traverses, however, where angles have only been observed to minutes and distances measured to a corresponding degree of accuracy, it is sufficient to work with five- or even four-figure logarithms and bearings to minutes or half minutes. Direct-bearing theodolite traverses and compass traverses may be computed with the special traverse tables described on pp. 138–139.

(iv) *Adjustment of Co-ordinates.*

When a traverse closes on the same point from which it started, the co-ordinates of the initial point calculated through the traverse should, if there were no errors, agree with the initial co-ordinates of that point; or, if the traverse starts at one fixed point and ends on another fixed point, the co-ordinates of the terminal point calculated through the traverse should agree with the fixed co-ordinates of that point. In practice, there are always errors of some kind and there is consequently a discrepancy (which is small unless there is a gross error somewhere) between the co-ordinates worked through the traverse and the accepted co-ordinates of the closing station. This discrepancy is known as the *closing error*, and the adjustment of the co-ordinates means distributing the error among the co-ordinates of the different stations. As there are two co-ordinates for each point, there is a closing error, which we shall call δx, in x, and one, which we shall call δy, in y. The accuracy of a traverse is judged by the *fractional closing error*, which is given by $1/\{L \div \sqrt{(\delta x)^2 + (\delta y)^2}\}$, where L is the total length of the traverse.

Traverses may be adjusted by applying corrections

1. to the lengths and bearings;
2. to the latitudes and departures; and
3. to the co-ordinates themselves.

There is no simple method which is really sound in theory, and some of those which are sometimes used are purely empirical. Perhaps the one most commonly adopted is Bowditch's rule, which has a theoretical basis although it is doubtful if some of the assumptions with regard to the nature of the errors involved are really sound. This rule, as originally stated by Bowditch, involves applying corrections to the individual latitudes and departures, but the following simple modification enables the corrections to be applied to the co-ordinates themselves. The modified rule is:

Correction to x co-ordinate (y co-ordinate) = closing error in x (y), multiplied by length of traverse up to point concerned, divided by the total length of traverse.

In the example given on the form, the fixed co-ordinates of the point 10 are $x = 34{,}574{\cdot}55$; $y = 51{,}527{\cdot}01$, so that the closing error is $+0{\cdot}22$ in x and $-0{\cdot}75$ in y. In column 6 of the form, the total lengths from station 1 are written in brackets below the lengths of the different legs, the total length of the traverse being 13,530 ft. approximately. Hence the corrections to point 6, say, are

$$\delta x_6 = -0{\cdot}22 \times \frac{7630}{13{,}530} = -0{\cdot}12,$$

and

$$\delta y_6 = +0{\cdot}75 \times \frac{7630}{13{,}530} = +0{\cdot}42.$$

These corrections are written below the co-ordinate values and the adjusted values are written underneath. The co-ordinates of other points are treated in the same way and the traverse form thus completed.

In this example, the fractional closing error is

$$1/\{13{,}530 \div \sqrt{(0{\cdot}22)^2 + (0{\cdot}75)^2}\} = 1/(13{,}530 \div 0{\cdot}78) = 1/17{,}300$$

approximately.

Another rule, involving corrections to the latitudes and departures, which is very often used in adjusting a traverse, although it has less theoretical justification than Bowditch's rule, is:

Correction to latitude (departure) = closing error in latitude (departure), multiplied by the latitude (departure) to be adjusted, divided by the arithmetical sum of all the latitudes (departures).

8. Errors and Standards of Accuracy in Theodolite Traversing.

The main difficulty with regard to traversing is to prevent the occurrence of gross or systematic errors in the field work, and gross error in the computations. For this reason, great care must be taken at all stages of the work both in the field and in the office. Gross errors in the field work are most likely to occur in the chainage by dropping a chain length or by misreading the graduations on a band, and in the angular work by booking observations taken to a rear station as if they were taken to a forward station, and vice versa, or by booking a deflection angle as being left (right) when it is actually right (left). Gross errors may also arise through sighting a wrong signal or through wrong readings of the degrees and minutes when reading the verniers, but this is less likely to happen if the angle is read in full with every change of face.

In computing, errors can occur at almost every stage of the work, and the only really safe way of preventing them is to get a separate computer to make an entirely independent computation, including the abstraction of the data from the field books.

Systematic error is likely to arise in the chaining by using bands or chains not properly standardized, by using a band with the wrong pull, or by using a thermometer which gives faulty readings.

Apart from the prevention of gross and systematic error, accurate traverse work depends on the elimination, as far as possible, or the reduction of the small accidental errors of observations, such as errors in pointing of the telescope or in reading the verniers or micrometers. The most common source of minor inaccuracy is probably bad centring and bad plumbing of signals, and particular attention should be paid to this point.

The closing error of a bearing computed through a theodolite traverse, as compared with the fixed bearing at the end, or the closure of the angles in a closed surround, is a test of the accuracy of the angular measurements. The closing error in bearing or angles normally depends on the square root of the number of angles observed and, when each angle is the mean of a single face-right and face-left observation, should not exceed $2{\cdot}5\alpha\sqrt{N}$ seconds, where α is the smallest number of seconds that can be read or estimated on the vernier or micrometer, and N is the number of angular stations. If angles are the means of n observations, each observation being the mean of a face-right and face-left measurement, the closing error should not exceed $2{\cdot}5\alpha\sqrt{(N/n)}$. Normally, the actual closing errors should be less than values given by these rules.

By differentiating the expressions $\Delta x = l \cos \alpha$ and $\Delta y = l \sin \alpha$, we get the errors $\delta(\Delta x)$ and $\delta(\Delta y)$ in Δx and Δy produced by errors δl in l and $\delta\alpha$ in α, viz.

$$\delta(\Delta x) = \delta l \cos \alpha - \delta\alpha \sin 1'' \,.\, l \sin \alpha,$$
$$\delta(\Delta y) = \delta l \sin \alpha + \delta\alpha \sin 1'' \,.\, l \cos \alpha,$$

or

$$\delta(\Delta x) = \delta l \cos \alpha - \Delta y \,.\, \delta\alpha \sin 1'',$$
$$\delta(\Delta y) = \delta l \sin \alpha + \Delta x \,.\, \delta\alpha \sin 1'',$$

where $\delta\alpha$ is a small angle expressed in seconds of arc.

The accuracy of theodolite traversing is ordinarily judged by the closing error in position expressed as a fraction of the length when the traverse closes on itself or on points fixed by a higher order of survey. The closing errors of closed surrounds, however, are inclined to be somewhat better than those of open traverses between fixed points, not only because of the shape of the figure but also because systematic errors, such as those due to inaccuracies of standardization, do not show up on the closing error of a closed surround, whereas they will do so in the closing error of an open traverse.

It is very difficult to lay down definite rules about the closing errors to be expected in different classes of traverse because so much depends on the particular instrument and methods used, the skill and experience of the surveyor, atmospheric conditions, the nature of the ground, etc., and, moreover, the closing error of a traverse is, strictly speaking, not directly proportional to the length of the traverse, although it is generally assumed to be so. The following notes, however, may be useful to the student in giving him a very rough general idea of the order of accuracy to be expected with different instruments and different methods.

(i) *First-order Precise Traverse.*

Horizontal angles observed with a geodetic-type theodolite reading direct to single seconds and by estimation to fifths or tenths. About 8 to 16 observations on different zeroes, each observation the mean of a face-right and face-left measurement. Linear measurements to 0·001 ft. with Invar bands used in catenary. Pull by weights over straining trestles. Temperatures observed at every tape length. Angles of slope, for slope correction and elevation above sea-level, by theodolite. Corrections applied for standardization, temperature, slope and height above sea-level. Astronomical observations for azimuth about every 10th station, and allowance made in the computations for the curvature of the earth. Accuracy of order 1/70,000 to 1/200,000.

(ii) *Third-order Traverse.*

Horizontal angles observed by micrometric theodolite reading direct to 10″, and by estimation to 1″ or 2″. Four observations on different zeroes, each mean of face right and face left. Linear measurement to 0·01 ft. with 300-ft. steel band used in catenary. Pull by spring balance. Temperature observed every tape length. Angles of slope by Abney level. Corrections applied for standardization, temperature and slope, and, at high elevations, for height of each section above sea-level. Astronomical observations for azimuth at end of 20 to 30 stations. Accuracy 1/10,000 to 1/30,000.

(iii) *Major Theodolite Traverse.*

Horizontal angles by vernier instrument reading to 30″. One observation mean of face right and face left with change of zero between faces. Linear measurements to 0·01 ft. with 300-ft. steel band used in catenary or on the flat. Pull by spring balance. Temperature observed every 10th set-up of band. Angles of slope by Abney level. Corrections applied for standardization, slope and temperature. Accuracy 1/3000 to 1/10,000.

(iv) *Minor Theodolite Traverse.*

Horizontal angles by vernier instrument reading to 1′. One face-right and one face-left observation with change of zero between face. Linear measurements to 0·01 ft. with 300-ft. steel band used on flat. No spring balance used. One temperature taken for the day. Angles of slope not observed, but slopes step-chained carefully. Corrections applied for standardization (if error is appreciable) and temperature. Accuracy 1/1000 to 1/5000.

(v) *Direct-bearing Traverse.*

Bearings observed directly on one face only with small vernier instrument reading to 1′. Linear measurements to 0·1 ft. with 300-ft. steel band or 100-ft. chain used on flat. No spring balance used and temperature not observed. Slopes by step-chaining. No corrections applied. Accuracy 1/500 to 1/1000.

As examples of the class of work to which these traverses might be applied, (ii) might be used for the main framework of a town survey; (iii) for breaking down framework of a town survey or for the main surround of a large estate; (iv), but with no temperature observed, for a railway location survey; (v) for short traverses for survey of detail and of such things as banks of streams and lakes.

When gauging the accuracy necessary in angular work, it is useful to remember that a minute of arc subtends a length of 0·03 ft., or approximately a third of an inch, at 100 ft., or a foot and a half at a mile. As a fraction, this is approximately 1/3400.

9. Miscellaneous Problems in Theodolite Work.

Before closing these sections on theodolite traversing we shall conclude by discussing a few problems which are of importance in general theodolite work.

(i) *Prolonging Straight Lines by Theodolite.*

In prolonging a straight line by theodolite it is generally advisable to do so on both faces. In fig. 7.5 the instrument is set up at B, and it is desired to prolong the line AB.

Sight on A and, after transiting the telescope, line in a mark at C near where it is desired to put in a peg on line, C being a point on the line of collimation. If the instrument is slightly out of adjustment in collimation, the line BC will not be on the line AB produced but will lie to one side of it as shown. Change face and sight back on A. Then transit telescope and sight in direction of C. Owing to the error in collimation, C will not lie on the line of collimation. Put in point D beside C but on line of collimation. Measure distance CD and drive a nail or a peg and nail at point E, where EC = ED. E will then be a point on the line AB produced.

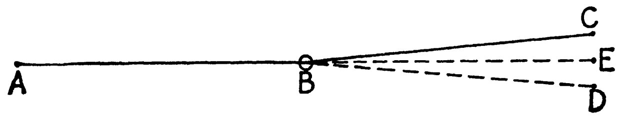


Fig. 7.5

In such work as laying out long straights on railways, lines should always be prolonged by using both faces of the instrument in the manner just described.

(ii) *Laying out Straight Lines between Two Points which are not Intervisible.*

In many cases the best method of doing this is to fix the positions of both points by theodolite traverse, and then calculate the bearing between them from the co-ordinates so obtained. This bearing can then be laid out from one of the points by turning off the requisite angle from a line of known bearing.

If the direction of the line is known approximately, an alternative method is to range out a random line BD as nearly as possible in the required direction as shown in fig. 2.5 (p. 12). Choose a point E on BD when the random line is seen to be close to A and measure the angle BEA and the distance EA. Then an intermediate point f can be ranged in by laying off the angle BFf = BEA and measuring Ff such that $Ff = EA \times \frac{BF}{BE}$.

(iii) *Obstacles Obstructing Chaining.*

The principle of triangulation is often very easy to apply in cases where a theodolite and chain or steel band are available, and it is necessary to find the distance between two points, direct chaining between those points being obstructed by obstacles. Figs. 7.6*a*, *b*, *c* and *d* illustrate four such cases. Here AB is the distance required. The lines shown by full lines are measured, the angles α, β, γ and δ observed and the distance AB computed.

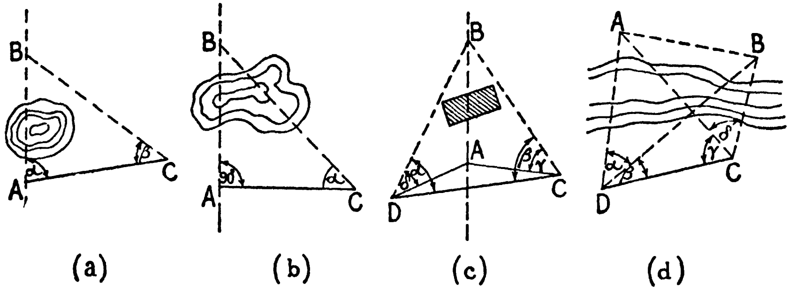


Fig. 7.6

(iv) *To lay out a Perpendicular on a Given Line from a Given Point.*

Let AB be a given line, P a given point, and let it be required to lay out from P a line perpendicular to AB. In fig. 7.7*a*, P is accessible. Observe α and calculate β from $\beta = 90° - \alpha$. The perpendicular PC can then be laid out by setting up the instrument at P, sighting on A, and setting out the angle β.

In fig. 7.7*b*, P is inaccessible. Observe α and β and measure AB. Then it is easy to show that

$$AC = \frac{AB \tan \beta}{\tan \alpha + \tan \beta} = \frac{AB \cos \alpha \sin \beta}{\sin (\alpha + \beta)}.$$

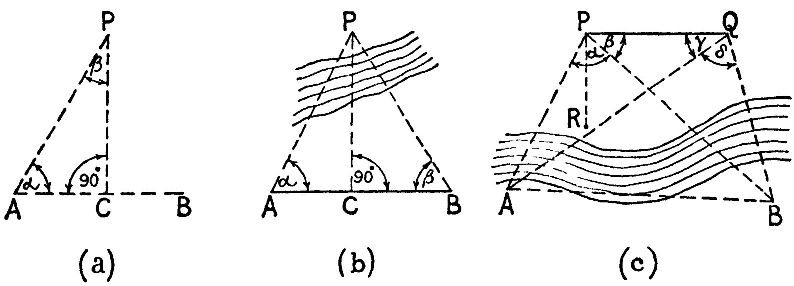


Fig. 7.7

In fig. 7.7*c*, AB is inaccessible, but P is accessible. Choose an auxiliary point Q, measure PQ, and observe angles α, β, γ and δ. In triangle PAQ compute PA, and in triangle PQB compute PB. Then, in triangle PAB, PA, PB and α are known, and angle PAB can be computed. Lay off point R such that $\angle APR = 90° - \angle PAB$. Then R is a point on the perpendicular from P on AB.

(v) *To lay out a Parallel to a Given Line through a Given Point.*

Let P be the given point and AB be the given line.

In fig. 7.8*a* both AB and P are accessible. Measure α and at P lay off $\angle APC = \beta = 180° - \alpha$. In fig. 7.8*b*, P is inaccessible, but AB is accessible. Measure AB and observe angles α and β. Then, if PF is the perpendicular from P on AB, $PF = AB \sin \alpha \sin \beta / \sin (\alpha - \beta)$.

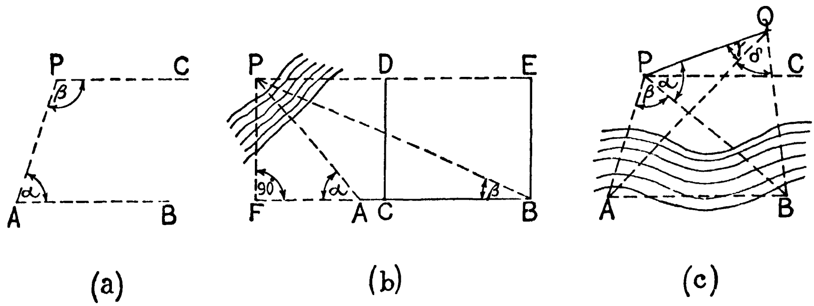


Fig. 7.8

At points C and B on AB lay out CD and BE perpendicular to AB and equal to PF. Then D and E are points on the line through P parallel to AB. In fig. 7.8*c*, P is accessible, but AB inaccessible. Choose an auxiliary point Q, measure PQ, and observe angles α, β, γ and δ. From these compute angle PAB by using the triangles PAQ, PBQ and PAB in turn and lay off PC such that $\angle APC = 180° - \angle PAB$. PC is then the parallel required.

COMPASS TRAVERSING

The fact that bearings can be read on a compass very much less accurately than angles can be observed with a theodolite severely restricts the use of the former instrument in engineering work. Thus, it is impossible to read even a large compass closer than to about a quarter of a degree, whereas a direction can be read on even a very small theodolite to at least a single minute of arc. Nevertheless, the compass can be used for many purposes such as preliminary road and railway surveys, surveys of rivers, streams and lakes, and for topographical work on small scales.

10. Magnetic Bearings and their Variations.

At most places on the earth's surface the direction of magnetic north does not coincide with the direction of the true or geographical north. The difference varies from place to place and at any one place it also varies slightly with time.

In fig. 7.9, PN is the true meridian, or direction of true north, and PM the direction of magnetic north, here shown as west of true north. The angle MPN $= d$ between PM and PN is called the *magnetic declination* or *magnetic variation*. The *magnetic bearing* M of the point Q is the angle MPQ, and the azimuth or true bearing A of Q is NPQ, each measured clockwise, the one from PM, the direction of the *magnetic meridian*, and the other from PN, the direction of the true or geographical meridian at P. From the figure we see that

$$M = A + d \text{ and } \therefore\ A = M - d$$

for all points in which the magnetic variation is west of north; while

$$M = A - d \text{ and } \therefore\ A = M + d$$

for all points in which the magnetic meridian lies east of north.

At the present time, the magnetic declination is about 8° west of true north in the east of England and about $13\frac{1}{2}°$ west of true north in the west of Ireland, and these variations are decreasing by about $7\frac{1}{2}'$ per annum. In some other parts of the world, the declination is east instead of west. Charts showing the *isogonic lines*, or *isogons*, which are curves passing through all places where the magnetic declination has the same value at any given time, can be obtained from the agents for Admiralty charts. Lines of zero declination, which separate parts of the earth where the declination is west from parts where it is east of north, are called *agonic* lines.

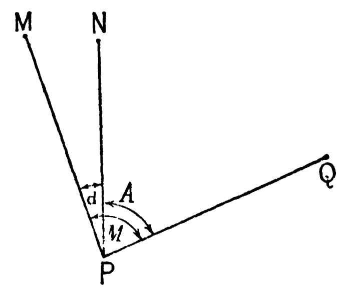


Fig. 7.9

Besides the yearly change in declination, there is also a small daily variation which is hardly large enough to be appreciable with an ordinary compass. At present, in England, this daily variation has an amplitude (total swing) of about 12′ in summer and 7′ in winter. In addition, irregular and sometimes appreciable variations in the value of the magnetic declination occur from time to time as the result of magnetic storms.

The correct amount of the magnetic declination is indicated, when the instrument is set on a true south to north line, only by a compass which is without error, and which consequently reads zero when the magnetic axis lies in the direction of the magnetic meridian. As, however, the magnetic and geometrical axis of their needles seldom coincide, most compasses have their own individual errors, and hence they do not always indicate the real magnetic north when the instrument reads zero. The resulting error (which is also the difference between the real magnetic bearing of any line and the bearing indicated by the compass) is known as the *compass error*, and it affects every single reading of the compass by the same constant amount. Its value can be found by observing the bearing of a line whose magnetic bearing is already known. Unfortunately, because of the difficulty of establishing the true magnetic meridian without special instruments, such a line will seldom be available, but, if a line of known azimuth or true bearing is available, a combined correction for magnetic declination and compass error can be ascertained by observing the compass bearing of the line, and comparing it with the true bearing. When this correction is applied with its proper sign to the compass readings, the result will be true bearings.

It will be realized from the above that, as magnetic bearings change with long intervals of time, it is always advisable, when a survey is based on magnetic bearings, either to show the value of the magnetic declination on the plan, or else to indicate the date of the survey if the magnetic declination is not known.

11. Field Methods used in Compass Traversing.

There are two main classes of compass traverse, one executed with a large surveyor's compass mounted on a stand, and used in conjunction with a surveyor's chain or steel band; the other with a small compass held in the hand, and used in conjunction with some rough method of measuring distance, such as by pacing, by pedometer or by rope.

In traversing with the surveyor's compass, the instrument is set up at the initial station, pointed to a ranging pole held on the forward station, and, when the needle has settled down after having been released, the reading is taken to the nearest half—or quarter—degree. While this is being done, no metal object should be allowed near the compass, the chain or band being held well clear of it. The distance is then measured, and the compass brought forward and set over the next station.

Sometimes, after removal to a fresh station, the instrument is set to read the bearing of the rear station as well as that of the forward station, so that the forward and rear bearings of each leg are observed.

In this case the two bearings should, of course, differ by 180°; if not, half the discrepancy is applied as a correction to the forward bearing. Having read the rear bearing, the surveyor observes the forward bearing of the forward leg. Often, however, only forward bearings are observed, but reading both forward and rear bearings give a check against a gross error in reading and adds a little, but not a great deal, to accuracy.

Distances on major compass traverses may be measured to a tenth of a foot with either chain or steel band used on the flat. Here, no spring balance is used, and no temperature or slopes are observed, allowance for slope being made automatically by step-chaining. Consequently, no corrections are applied to measured distances. The errors of closure of traverses of this kind between points fixed by surveys of higher order may be expected to lie between about 1/100 to 1/500.

When a hand compass is used, forward bearings only are observed unless the surveyor decides to read back bearings as a check against gross error. Bearings are usually taken to ranging poles held at the forward station, but, in the case of "rope and sound" traverses in forest or bush country, they are taken to a sound made by the man holding the forward end of the rope. No corrections are applied to measured lengths but, in order to allow for twists and bends in the path or line, the rope is generally a little longer than its nominal length. The accuracy of such a traverse cannot be expected to be much greater than about 1/50. If distances are measured to the nearest foot with a chain, accuracy may be increased to about 1/100 or a little more.

12. Plotting and Computation of Compass Traverses.

Compass traverses may either be plotted direct from the observed bearings and distances, or else co-ordinates may be computed in the ordinary way or by means of traverse tables, and the work plotted from the co-ordinates. Minor surveys made with hand compasses are hardly worth computing and are almost invariably plotted direct from the bearings and distances.

The best instrument to use for plotting from bearings and distances is a large brass protractor about 8 to 12 in. in diameter. This consists of a circular ring of brass, with the otherwise flat top bevelled slightly on top towards the outer edge. This bevelled portion carries the graduations, which usually are on the whole-circle system and go to half degrees. The two inside edges of the ring opposite the 90° and 270° graduations are joined by a brass arm forming part of the main casting, the edge of this arm on the side of the zero graduation of the arc coinciding with a diameter of the ring. In the centre of the arm is a small notch marking the centre of the circle.

To use the protractor, a line parallel to the direction of magnetic north (or true north if the readings have been corrected to give true bearings) is drawn on the paper through the point representing the rear station of the leg. The protractor is then set with its centre point exactly over this point and its zero mark on the line representing the direction of the meridian. It is thus oriented, and a tick can be drawn on the paper against the graduation representing the bearing of the line. A pencil line drawn through the point representing the station and through the tick gives the direction of the traverse leg, and a plotted distance along this line gives the position of the forward point of the leg. A line parallel to the direction of the meridian is then drawn through the latter point and the previous operation repeated for the next forward leg.

Plotting by co-ordinates is most conveniently carried out on squared paper, a suitable point being chosen as origin and the vertical and horizontal lines through this point chosen as the axes of x and y respectively. Plotting is then done by choosing or measuring out from the origin a distance along the axis of y equal in length to the y co-ordinate and then erecting a perpendicular at this point equal in length to the x co-ordinate. If the traverse gets too far away from the origin for convenient scaling, some other point can be chosen as a local origin, and differences of co-ordinates between it and the other points plotted from it.

Computation of Latitudes and Departures.

When compass traverses are to be computed, the accuracy of the work hardly justifies the labour of computing by the cosine and sine formulæ in the manner used in the computation of theodolite traverses. Instead of this, traverse tables can be used. These tables give, for every degree or for every minute of arc, the natural cosines and natural sines of the angle, each multiplied in turn by the numbers 1, 2, 3, . . . 9. Thus, the following is part of the entry of a traverse table for 64°.

64°	1		2		3		4	
	Lat.	Dep.	Lat.	Dep.	Lat.	Dep.	Lat.	Dep.
0′	0·4384	0·8988	0·8767	1·7976	1·3151	2·6964	1·7535	3·5952
1	0·4381	0·8989	0·8762	1·7978	1·3143	2·6968	1·7524	3·5957
2	0·4378	0·8990	0·8757	1·7981	1·3135	2·6971	1·7514	3·5962

If we wish to find the latitude and departure of a line whose bearing and length are 64° 02′ and 2143, we proceed as follows:

	Lat.	Dep.
2000	875·7	1798·1
100	43·78	89·90
40	17·51	35·96
3	1·31	2·70
2143	938·3	1926·7

In this case, for the 2000 in the number we look out the latitude and departure for 2 under 64° 02′ and multiply the tabulated values by 1000. For the 100, we multiply the latitude and departure for 1 by 100, those for 4 by 10, and so on. The sum of each set of quantities is then the latitude or departure for the whole line.

Latitude and departure tables may also be used for checking the computations of theodolite traverses against gross error, or for locating a gross error when it is known that one exists. They can also be used for computing minor theodolite traverses.

If the observed bearings have had a correction applied to them to give true bearings, and these corrected bearings are used in the computation of the latitudes and departures, the direction of the axis of x will be the true north line. If the observed bearings are used without any correction for the magnetic declination, the direction of the axis of x will, of course, be magnetic north.

Adjusting Compass Traverses.

Apart from the correction for compass error, there is no adjustment for bearings in magnetic traverse work, as there is in a theodolite traverse, but it is necessary to adjust it for the closing error in position of the last point.

Fig. 7.10*a* (p. 140) shows a traverse surround closing on itself at A, while *b* shows a traverse beginning at a fixed point A and closing on another fixed point F′. When the traverses are plotted, the position of the last point falls at F, whereas in *a* it should fall at A and in *b* it should fall at F′. The lengths FA and FF′ are then the closing errors of the traverses.

If co-ordinates have been computed, they can be adjusted by the rules already given for adjusting the co-ordinates of theodolite traverses, but, if the traverse has been plotted direct from bearings and distances, the easiest way is to use the following graphical method of adjustment.

Join AF or FF′, and through the points B, C, D and E draw lines

BB′, CC′, DD′ and EE′ parallel to AF or FF′, and make the lengths of these lines equal to $C \times l/L$, where C is the magnitude of the closing error at F, l the length of traverse from A up to the point being adjusted, and L is the total length of the traverse. Join AB′, B′C′, C′D′, D′E′ and E′A or E′F′. Then AB′C′D′E′A (fig. 7.10*a*) and AB′C′D′E′F′ (fig. 7.10*b*) are the adjusted traverses.

The lengths of the lines BB′, CC′, DD′, EE′ can easily be calculated by slide rule, but, for a long traverse, a graphical method is probably quicker. In fig. 7.10*c* set off the horizontal line AF proportional to the

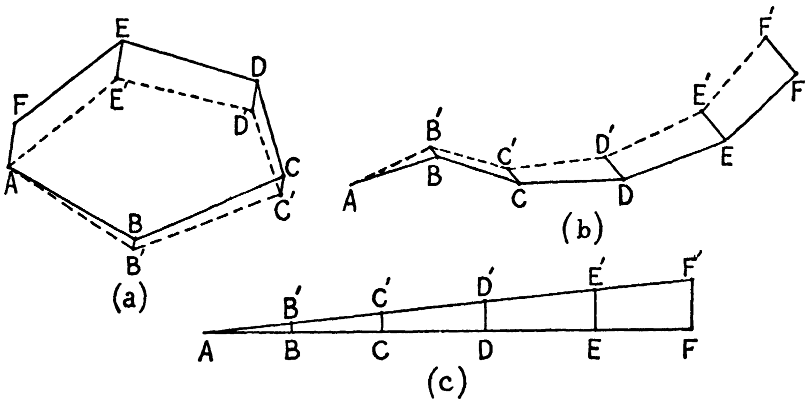


Fig. 7.10

total length of the traverse, and at F draw the line FF′ perpendicular to AF and proportional in length to the closing error at F. Join F′ to A and on AF lay off the lengths AB, BC, CD, DE and EF proportional to the lengths of the different legs. Draw lines BB′, CC′, DD′, EE′ perpendicular to AF at B, C, D and E to meet AF′ in B′, C′, D′ and E′. Then the lengths of these perpendiculars represent the amount of the adjustment necessary at each point.

GRAPHIC TRAVERSING WITH THE PLANE-TABLE

When great accuracy is not required, a graphic traverse by plane-table may sometimes be used instead of an instrumental traverse.

Let the traverse start at a point B whose position is fixed and from which a line BA of known direction already exists on the ground. Let the position of B and the direction of A be plotted as **ba** on the plan (fig. 7.11).

The plane-table is set up at B and oriented and clamped in the usual manner, so that the line **ba** is set in the direction BA. The direction of C, the next station, is then drawn in by setting one end of the alidade against the point of a pencil held at **b** and pivoting it about this point until the line of sight intersects C. The distance BC is chained and this distance laid off on the line **bc** that has just been drawn, so giving the plotted position **c** of C. The next leg can then be drawn by moving the instrument to C, orienting it with the line **cb** directed in the direction of CB and proceeding as before.

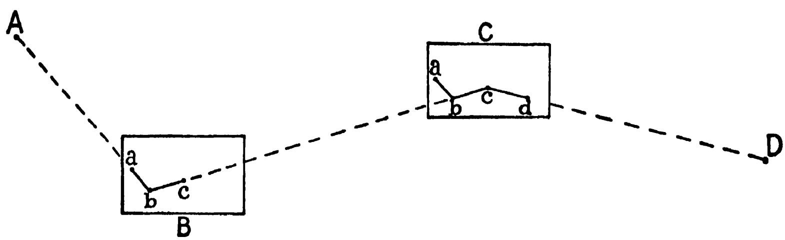


Fig. 7.11

If the plan on the plane-table is on a very small scale, the plotted legs of the traverse will be very short, and hence there would be a very short length of line to lay the alidade against when orienting the table. Consequently, in order to avoid unnecessary errors, it is advisable in all cases to mark the directions of the various legs on the edge of the plan, so as to have reasonably long lengths against which the alidade can be laid when the table is being oriented at a new station.

QUESTIONS ON CHAPTER VII

1. The following were the observed interior angles of a closed figure ABCDEA:

$$A = 122° 14' 50''; \quad B = 136° 13' 10''; \quad C = 121° 54' 30'';$$

$$D = 129° 45' 40''; \quad E = 29° 47' 10''.$$

Adjust the angles of the figure.

2. In a compass traverse the following bearings were taken:

Line	Bearing	Line	Bearing
AB	S. 35° 30′ E.	DE	N. 40° 10′ W.
BA	N. 35° 32′ W.	ED	S. 38° 45′ E.
BC	N. 70° 15′ E.	EF	S. 48° 46′ W.
CB	S. 73° 19′ W.	FE	N. 50° 12′ E.
CD	N. 17° 02′ E.	FA	N. 81° 13′ W.
DC	S. 13° 57′ W.	AF	S. 81° 15′ E.

Give the corrected bearing of each line and state where local attraction is present (if any). On plotting the traverse it is found that there is a closing error: explain how this error can be distributed by a graphical application of Bowditch's rule. (Inst. C.E., October, 1945.)

3. Using the data of a closed traverse given below, calculate the lengths of the lines BC and CD.

Line	Length	Whole circle beginning from N	Reduced bearing	Latitude	Departure
AB	344 ft.	14° 31′	N 14° 31′ E	+333·0	+86·2
BC	—	319° 42′	N 40° 18′ W	—	—
CD	—	347° 15′	N 12° 45′ W	—	—
DE	300 ft.	5° 16′	N 5° 16′ E	+298·8	+27·6
EA	1958 ft.	168° 12′	S 11° 48′ E	−1916·4	+400·4

(Inst. C.E., October, 1955.)

4. The calculated co-ordinates for a closed traverse were as follows:

Line	Departure	Latitude
AB	−170·5	+164·4
BC	−321·2	−359·6
CD	+340·8	−250·2
DA	+155·7	+440·0

Distribute the closing error among the latitudes and departures, and calculate the area enclosed by the traverse, using the Double Meridian Distance or other method. (Inst. C.E., April, 1946.)

5. A tunnel ran from a point A to a point D. It was required to find the length and bearing of the line AD, so a traverse ABCD was run as follows: AB = 438 ft., BC = 341 ft., CD = 491·5 ft. Angle ABC = 118° 15′, angle BCD = 108° 40′, and the line AB ran due north. Calculate the length and bearing from north of the line AD. (Inst. C.E., October, 1947.)

6. A traverse ran from station A to a fixed station F, the following being the measured distances and the observed angles measured whole-circle clockwise from the last station:

Angle	Observed value	Line	Distance
PAB	187° 15′ 00″	AB	1,728·6
ABC	234° 55′ 50″	BC	3,369·2
BCD	168° 32′ 45″	CD	976·4
CDE	197° 55′ 10″	DE	1,214·2
DEF	236° 17′ 50″	EF	2,011·9
EFQ	188° 28′ 50″		

The fixed bearings of the lines AP and FQ were 152° 14′ 20″ and 105° 40′ 15″, and the fixed co-ordinates of A and F were:

Point	x (northing)	y (easting)
A	26,948·6	97,478·2
F	32,913·3	101,948·6

Compute the traverse after adjusting the bearings, and then adjust the co-ordinates of the intermediate points to the fixed co-ordinates of A and F.

CHAPTER VIII

LEVELLING AND CONTOURING

In levelling we are concerned with the measurement of differences of height between points, or with the determination of the elevation of certain points above some given plane or surface known as the *datum plane* or *surface*. This datum may be a purely arbitrary one, but for many purposes it is convenient to take the mean level of the sea, known as *mean sea-level* (M.S.L.), as the fundamental and natural surface above which elevations should be measured.

Some form of levelling is one of the most common operations in engineering surveying, as levels are required for all kinds of purposes, such as deciding depths of excavations for foundations; giving pegs to enable a contractor to know when he has reached the required depth; setting out the limits, depths and heights of cuttings and embankments on railway and road construction; setting out gradients for pipe lines; mapping contours as a guide in planning and designing works and in estimating costs. In fact, there is no branch of civil engineering in which levels in some form or another are not required.

On p. 7 we have named the different methods available for determining elevations and differences of height, but by far the most important, so far as the civil engineer is concerned, is the method known as *spirit levelling,* and accordingly we shall consider it first.

SPIRIT LEVELLING

In Chapter V of *Principles and Use of Surveying Instruments* we have described the ordinary surveyors' level and have indicated how it is used. The principle of levelling is that by the use of the level we establish a line, the line of collimation of the instrument, which lies in a horizontal plane passing through the horizontal hair of the instrument. The operation of levelling consists essentially in determining the vertical distance from this line to points whose elevations or differences of height relative to one another are required.

In fig. 8.1 we suppose that the elevation of a point A is given and we wish to determine the elevation of a point D some distance away. The level is set up at L_1 and sighted on a levelling staff held vertically on A. After the instrument has been properly levelled, the line of collimation is a horizontal line **ab** which intersects the staff held on A at **a**. Consequently, A**a** is the reading on the staff. After this reading has been taken, the telescope is revolved about its vertical axis to point at a staff held vertically at B, and, the instrument still being level, the reading B**b** is taken on this staff. Through B draw the horizontal line BA′ to meet **a**A at A′. Then AA′ is the difference in elevation between A and B.

But, $\quad AA' = A\mathbf{a} - A'\mathbf{a} = A\mathbf{a} - B\mathbf{b}$

$= \text{(staff reading at A)} - \text{(staff reading at B)}.$

Taking AB as the forward direction of the line, the sight $L_1\mathbf{a}$ is a *backsight*, as it is a sight in the rear direction of the line, and $L_1\mathbf{b}$, being in the forward direction of the line, is called a *foresight*. Hence,

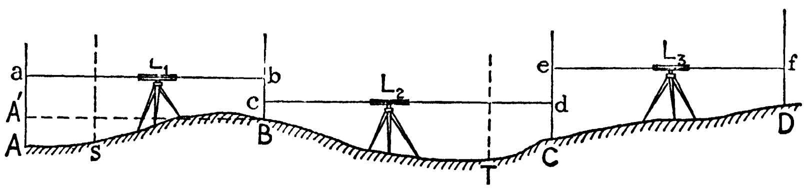


Fig. 8.1

we see that the difference in elevation between two points is equal to the backsight reading minus foresight reading, and it is also obvious that the elevation of the forward point relative to that of the rear point is a rise if the backsight reading is greater numerically than the foresight reading; similarly, the elevation of the forward point relative to that of the rear point is a fall if the backsight reading is less numerically than the foresight reading.

The observations from L_1 having been completed, the staff at B is still held there and the instrument is moved to L_2, where, after re-levelling, a new backsight reading B**c** is taken to the staff at B, and a new foresight reading C**d** is taken to a staff held at C, some distance ahead of the instrument. Similarly, the instrument is set up and levelled at L_3 and a backsight reading taken to a staff held at C and a foresight reading to one held at D. It will then be obvious that

Difference of elevation A − D

$= A\mathbf{a} - B\mathbf{b} + B\mathbf{c} - C\mathbf{d} + C\mathbf{e} - D\mathbf{f}$

$= (A\mathbf{a} + B\mathbf{c} + C\mathbf{e}) - (B\mathbf{b} + C\mathbf{d} + D\mathbf{f})$

$= \text{(sum of backsight readings)} - \text{(sum of foresight readings)}.$

This principle may be extended to any number of points so that we can finally write:

Difference of height between points A and P
= (sum of backsight readings) − (sum of foresight readings),

and it also follows that (P being the forward point) P is higher (lower) than A if the sum of the backsight readings is numerically greater (less) than the sum of the foresight readings. Finally, we have:

Elevation of P above datum
= elevation of A + (sum of backsight readings) − (sum of foresight readings).

Points such as B, C, and D, where the staff is held and sighted for two successive set-ups of the level, are called *turning points*, and the accuracy of levelling depends, very largely, on these points not moving at all between the readings. Consequently, good solid points must be used as turning points. They can be stout wooden pegs, or iron pins driven firmly into the ground, or any well-defined point on a solid object, such as a part of a rock, top of a step, top of a man-hole, etc. The point where the staff is held should be well defined so that there is no doubt about the bottom of the staff resting on exactly the same point for the two readings. Instrument makers can provide special small triangular plates with a spike on top which can be used as turning points on hard firm ground.

When the point A is a fixed permanent or semi-permanent point whose elevation above the datum plane is known, or is assumed for purposes of calculating elevations, it is called a *bench mark*.

If the elevations of additional points which are not used as turning points are needed, they can be found by setting up a staff and taking a reading there. Thus, in fig. 8.1, S and T are additional points of this kind, and it can be seen that their elevation can be found by subtracting the reading taken at them from the backsight reading, and adding the result to the elevation of the point used for the backsight. Additional points of this kind which are not bench marks or turning points are termed *intermediate points*. It will also be seen that the difference of elevation between A and B is

(staff reading at A − staff reading at S)
+ (staff reading at S − staff reading at B)
= (rise A to S) + (rise S to B).

Similarly, the fall from B to C =

(staff reading at B − staff reading at T)
+ (staff reading at T − staff reading at C)
= (fall B to T) + (rise T to C),

falls being reckoned as negative quantities as compared with rises, which are taken to be positive quantities.

1. Booking and Reducing Levels.

There are two main methods of booking levels and working out elevations. These are (1) the *rise and fall* method, and (2) the *height of collimation* method. An example of the *rise and fall method* is given below.

The first entry is a backsight 4·23 taken at a bench mark (B.M.) whose elevation, or *reduced level*, is 363·98.

Backsight	Inter.	Fore-sight	Rise	Fall	Reduced level	Distance	Remarks
4·23					*363·98*	*1042+41*	B.M. on culvert M.47.
	1·6		*2·63*		*366·61*	*1044*	
	2·8			*1·20*	*365·41*	*1045*	
9·28		*1·47*	*1·33*		*366·74*		T.P.1.
	6·6		*2·68*		*369·42*	*1046*	
	3·1		*3·50*		*372·92*	*1047*	
	2·48		*0·62*		*373·54*	*1047+21*	Nail on stump. temp. B.M.
5·48		*3·26*		*0·78*	*372·76*		T.P.2.
	8·7			*3·22*	*369·54*	*1048*	
	2·1		*6·60*		*376·14*	*1049*	
	10·34			*8·24*	*367·90*	*1050*	Peg.
		7·78	*2·56*		*370·46*	*1050+36*	B.M. on platform Arboy stn.
18·99		*12·51*	*19·92*	*13·44*			
12·51			*13·44*		*363·98*		
+6·48			*+6·48*		*+6·48*		

This backsight is entered in the first line of the first column, and in the second column and second and third lines are the readings at two intermediate points taken on pegs at chainages 1044 and 1045. The first foresight, to T.P. (Turning Point) 1, follows in the fourth line in the third column, and this completes the observations at the first set-up of the level.

The instrument is then moved, and, after having been levelled up, is sighted back on the staff held at T.P.1, the reading being 9·28. This is entered in column 1 on the same line as the previous foresight. Then follow three intermediate sights and the foresight of 3·26 on T.P.2. At the next set-up of the instrument, the backsight is entered on the same line as the foresight from the last set-up, and the page closes with a foresight on a bench mark.

In this line of levels the backsights and foresights were read to two decimal places so as to maintain accuracy in the fixing of bench marks, but some of the intermediate sights, having been taken to ground level only to provide data for estimating earthwork quantities, were read and entered only to the nearest single decimal place. Those at chainages 1047 + 21 and 1050 were, however, read to the second decimal place as these points were intended to serve as temporary bench marks later on.

Note that all observations taken to a particular point are entered on the same line. Thus, the backsight and foresight on T.P.1 appear on the same line, though in different columns.

In working out the elevations or reduced levels, the first step is to work out the rises and falls. These are reckoned rises or falls with reference to the previous point, whether backsight or intermediate point. Finally, the reduced levels are obtained by adding or subtracting each rise or fall to or from the reduced level of the last point. Thus, the reading 1·6 at the first intermediate point is less than the backsight of 4·23, and hence the rise to this point is (4·23 — 1·6) = 2·63, so that the reduced level at the intermediate point is 366·61. Similarly, the reading 2·8 at the second intermediate point is greater than the reading 1·6 at the first intermediate point, and there is a fall of 1·20 from the first point to the second, the reduced level of the latter thus being 366·61 — 1·20 = 365·41. In the same way, there is a rise of 1·33 to T.P.1, giving a reduced level for that point of 366·74, which is the same quantity as that found directly by adding to the elevation of the bench mark the difference between the backsight of 4·23 and the foresight of 1·47 to the first turning point.

There are two checks on the arithmetical work, as the difference between the sum of the backsights and the sum of the foresights should be equal to the difference between the rises and falls, and both differences should be equal to the difference between the final and initial reduced levels. These tests should be worked out, as shown in the example, at the bottom of every page as it is completed.

The *height of collimation method* is even easier to follow. Referring to fig. 8.1 (p. 145), we see that, if the reading A**a** of the backsight to A is added to the reduced level of A, the result is the reduced level of **ab**, the line of collimation. The reduced level of any other point is then obtained by subtracting the staff reading at that point from the reduced level of the height of collimation.

In the specimen page given here it will be seen that the entries in the

Backsight	Intermediate	Foresight	Height of instrument	Distance	Reduced level	Remarks
6·98			*1050·22*		*1043·24*	B.M. south abutment of bridge at M.*24·3*.
	8·4				*1041·8*	N. bank R. Asore.
	12·1				*1038·1*	Centre of stream.
	9·3				*1040·9*	S. bank R. Asore.
5·16		*4·34*	*1051·04*		*1045·88*	T.P.
3·64		*2·23*	*1052·45*		*1048·81*	T.P.
	3·9				*1048·6*	
		9·91			*1042·54*	B.M. stump at *1397* + *24*
15·78		*16·48*			*1043·24*	
		15·78				
		−0·70			*−0·70*	

first three columns are made exactly as before, but in the fourth column the first backsight has been added to the elevation of the B.M. to give the height of instrument, or height of line of collimation. The intermediate readings and the first foresight are subtracted from this height to give the reduced levels at the points to which the readings were taken. In this way, the reduced level of the first T.P. is obtained, and, when the second backsight is added to this, we get the height or reduced level of the instrument at the second set-up, and so on.

There is one check on the reduced level of the point to which the last foresight is taken, for the difference of the sums of the backsights and foresights should be equal to the difference of the reduced levels of the final and initial points. This test, like the other, should be applied on every page of the field book.

In both cases, the initial entry on a page should be a backsight and the final entry a foresight. If the page ends at an intermediate point, the reading at that point should be entered as the last foresight on that page and as the first backsight on the next page.

It will be seen that the checks on the rise and fall method of reducing levels act as a check on the reduction of levels of the intermediate points, whereas in the case of the height of collimation method this is not so, and there is no check on the reduction of the intermediate levels. This is the one disadvantage of the height of collimation method; otherwise, it is simpler and rather quicker in use, and for this reason is the more generally favoured method of the two. It is particularly suitable for use in lines run primarily for establishing bench marks, where intermediate sights are not needed or taken, and all the entries are backsights and foresights.

2. Precautions necessary in Levelling.

Good levelling can only be obtained by careful work. The instrument should be kept in good adjustment, and particular care taken to see that the bubble is in the centre of its run when sights are taken to a bench mark or turning point. Turning points should be well-defined objects on which there is only one point where the bottom of the staff can rest, and they should be solid enough not to move or give under the weight of the staff while sights are being taken or between successive sights. The staff should be held vertical; if it is not fitted with a level bubble or plumb bob to indicate verticality, it should be waved gently to and fro about its base in the direction of the observer and the smallest reading taken. Failure to hold the staff on the same point for both foresight and backsight after the instrument has been moved will, of course, cause a gross error to be made.

If backsights and foresights are kept as nearly as possible equal in length, the effects of collimation error, curvature of the earth, and refraction will tend to cancel out. This can be seen from figs. 8.2*a* and *b*. In *a*, AB is a horizontal line through the cross hairs of the instrument; now suppose that, when the bubble is central and the instrument sights the left-hand staff, the line of sight, instead of intersecting the staff at A, intersects it at A′ below A. When the instrument is turned to sight the staff on the right and the bubble is central, the line of sight will intersect the staff at B′. If $LA = LB$, then $A'A = B'B$, and it follows that the true difference in height between C and D $= CA - DB = CA' - DB'$, or the difference between the actual readings on the staff.

In fig. 8.2*b* the staffs at C and D lie along the verticals at these points, and, owing to curvature, these verticals lie on the radii of the circle representing the curved surface of the earth as represented by the mean level of the sea. A horizontal line at L, tangential to the vertical there, will intersect the staffs at C and D at A and B, while a circle through L concentric with the circle representing mean sea-level will intersect the staffs at A′ and B′. As elevations are distances above sea-level, or a spherical surface concentric with it, measured along the vertical at each point, (CA′ − DB′) represents the true difference of level between C and D. If LA = LB, then A′A = B′B and hence (CA − DB) = (CA′ − DB′). But (CA − DB) = difference of readings on staff = apparent difference of elevation. Consequently A′A and B′B cancel out when the difference between the staff readings is taken, so that this difference represents the true difference of elevation between

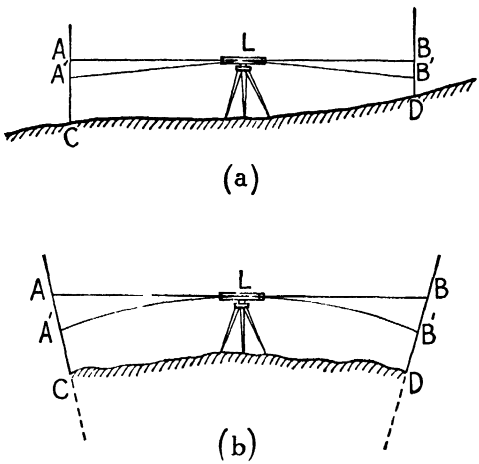


Fig. 8.2

the two points. Similarly, a ray bent by refraction will be equally bent between A and B; hence the amount of bending observed on the staffs will be equal and will be cancelled when the difference of readings is taken.

With the length of sight used in ordinary work, the effects of curvature and refraction would not be appreciable on single sights, but in precise work they would tend to accumulate to an extent that would be appreciable if back and forward sights were not kept reasonably equal in length. The length of sight used in levelling depends on the resolving power of the telescope, and should not be greater than that at which the telescope can resolve the smallest divisions on the staff. This will usually be somewhere between 200 and 300 ft. Very short sights, involving a considerable alteration in focusing, should be avoided as far as possible, though on steep ground this may be impossible. A good average length of sight is about 150 ft.

Lines of levels of any importance, and particularly those run for the purpose of establishing bench marks, are generally levelled twice —once in each direction—and it is usual to lay down rules for the allowable discrepancy between the two levellings. The rule generally adopted is of the form $d = k\sqrt{M}$, where d is the allowable discrepancy in *feet*, M is the length of a single levelling in *miles*, and k is a constant. For lines observed with an ordinary engineer's lead, k is taken as 0·05 for moderately flat country and 0·10 for hilly country. In the case of first-order geodetic levels k is taken as 0·010. If d exceeds the limits laid down, the line must be re-levelled.

3. Bench Marks.

Bench marks are permanent or semi-permanent marks of fixed or known elevation which can be used for determining the elevations of points in the immediate neighbourhood. They should be made on permanent objects which are not likely to settle or to be disturbed or

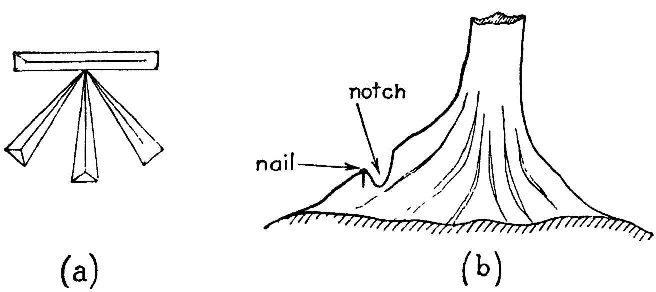


FIG. 8.3

moved; and they should be very well-defined points which can easily be found and recognized, and on which a staff can conveniently be held.

A mark chiselled on solid rock or a metal bolt set into solid rock makes the most stable form of bench mark but, when rock is not available, a metal pin set on top of a solid concrete pillar with good foundations or on a solid building can be used instead. In Great Britain, the Ordnance Survey put most of their bench marks on buildings. The older type consists of a broad arrow cut into a wall as shown in fig. 8.3*a*. In this form, the centre of the groove in the horizontal bar on top of the arrow is the mark to be taken. The new type of bench mark consists of a special bronze tablet let into a wall. On construction work, where no rock or building is available, a semi-permanent bench mark may be made by cutting a notch in the root of a tree and driving a nail on the part of the root left pointed, as shown in fig. 8.3*b*

4. Datum for Levelling.

Where no bench mark of known elevation is available, a datum must be assumed This means establishing an initial bench mark and calling the elevation of it some convenient whole number, such as 500 or 1000, above datum. This datum is then described as being so many feet below the bench mark. The number chosen should be such that no point in the area in which the survey is to be made is likely to be *below* the assumed datum.

Most national surveys adopt mean sea-level as the datum for levelling. This means the average level of the sea as determined by observations of sea-level at one-hour intervals spread over a long period of time. The advantage of this datum is that it is a natural one which can be re-established very approximately at any time if every single bench mark on shore should disappear or be destroyed by such calamities as earthquakes, etc. In addition, drainage, hydraulic and similar engineering schemes and works are often related to sea-level, and hence a datum in terms of it is needed. Moreover, a datum based on sea-level may be important from a scientific point of view in studying slow changes of the land level relative to the sea. Mean sea-level is a better datum to use than the levels of high or low waters, which are often difficult to determine. Unfortunately, mean sea-level is not absolutely constant over limited periods of time, as monthly, or even yearly mean values show appreciable differences among themselves. Consequently, for a first-order determination, observations should cover a fairly considerable period—at least one complete lunar year of 354⅓ days for a reasonably good determination, but the longer the better, provided the period taken is a multiple of the lunar month of 29·53 days.

Mean sea-level is determined by means of a tide gauge. In its most simple form, this consists of a metal float working inside a long cast-iron pipe, the lower part of which is open at the bottom and lies below the level of the lowest of low waters. This pipe generally has small holes bored in its side, and its function is to damp down the oscillations of the water outside the pipe so that the float works in water which, apart from the rise and the fall of the tide, is motionless. The float carries a vertical rod, or else it is attached by a wire which rides over a frictionless pulley and carries a counterweight at its other end. Readings of the rise and fall of the water are made by means of a pointer on the rod or on the counterweight as it works against a graduated scale. The zero on the scale is connected by levelling to a nearby bench mark and to the level of the water in the pipe. Consequently, the levels of the water in the pipe can be converted into levels below the level of the bench mark, and the latter expressed in terms of feet above mean sea-level.

A gauge of this type requires the attendance of an observer at every hour of the day, which is expensive and a great inconvenience. Hence, an automatic self-registering tide gauge is generally employed when observations have to be taken over an extended period. In this type of gauge, the rise and fall of the water are transmitted from the float to a recording pen working over a special chart carried on a revolving drum operated by clockwork. In this way, a graphical record of the movement of the water relative to a fixed bench mark can be kept.

5. Levelling over wide Gaps: Reciprocal Levelling.

Occasionally a line of levels has to be carried over a wide gap, such as a wide river or a lake, over which it would be impossible in the ordinary way to maintain equality of backsights and foresights to eliminate the effects of curvature, refraction, and errors of collimation. If the water is stagnant, or nearly stagnant, as in some lakes or very sluggish rivers, pegs can be put in flush with water surface at each side of the obstacle and the line carried to the peg on one

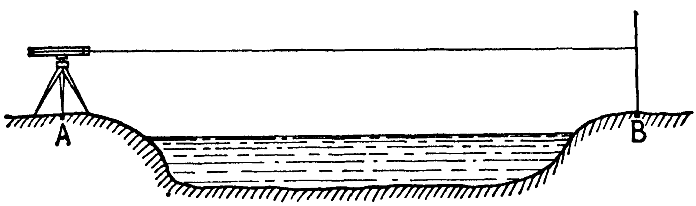


Fig. 8.4

side, and then continued from the one on the other side, it being assumed that the water level is the same at each peg. In the case of swiftly moving rivers it is unlikely that the water will be at the same level on both sides, and in that case the best method of procedure is *reciprocal levelling*.

In fig. 8.4 a peg is put in at A on one bank of a wide river and another at B on the opposite bank. The instrument is set up over the peg at A, the height of the cross hairs above the peg is measured, and a sight taken to a staff held at B. This will give one measure of the difference of elevation between A and B. The instrument is then moved and set up over the peg at B, and a similar set of observations taken with the staff held on A. This will give another measure of the difference of elevation between the two pegs, and the mean of the two results will give a value from which the effects of curvature, refraction and collimation error have been eliminated.

The principle of this method will be understood from a description of the "two-peg" method of adjusting a dumpy level described on p. 125 of *Principles and Use of Surveying Instruments.*

If the gap is too wide for the graduation on the staff to be read accurately, a number of sights may be taken to a target staff, or to a target held against the staff and moved up and down it until it is intersected by the line of sight. The same number of sights is taken at the other side of the gap, and the mean of the two sets of results accepted as the difference of height.

6. Precise Levelling.

Precise levelling does not differ greatly from ordinary levelling except that a larger instrument, fitted with a parallel-plate micrometer enabling readings to be taken to 0·0001 ft., and a standardized Invar staff in which the effects of temperature changes are reduced to a minimum are used. Great care is taken here to maintain the equality of the lengths of backsights and foresights. The instrument is generally fitted with a horizontal hair and two stadia hairs; readings on the latter serve not only as a check against gross error in the reading of the middle hair, but also as a check on the equality of the length of backsight and foresight at each set-up. Observations to points lower than about 18 in. from the bottom of the staff are not allowed, and the permissible limits of lengths of sights are strictly defined. The staffs are carefully standardized from time to time against a standardized Invar tape used under a definite tension and kept specially for the purpose, corrections for errors of standardization of the staffs being made in the computations. Air temperatures are taken at intervals during the course of the field work, and the necessary corrections applied. In addition, corrections have to be introduced for the convergence towards the pole of the equipotential or level surfaces due to the variation of gravity with latitude and height above sea-level. These corrections, known as the *orthometric* and *dynamic height* corrections, are not appreciable in ordinary levelling, but have to be taken into account in precise work. They will be found described in books on advanced surveying or in works on geodesy.

As an example of the accuracy of modern precise levelling, it might be mentioned that the greatest closing error in a closed circuit in the geodetic levelling of England (1912–21) was 0·4771 ft. in a circuit of 290 miles, and the smallest was 0·0073 in a circuit of 60 miles.

LEVELLING WITH TACHEOMETER

One of the principal uses of tacheometry in surveying is in contouring, as here it is necessary to fix the positions as well as to determine the elevations of a number of points round and near the instrument. The ordinary level is normally fitted with stadia hairs which enable distances from the instrument to be determined to points not too far away, and not too much above or below it, but this in itself is not sufficient to fix the position of a point on a plan. Consequently, apart from checking the equality of backsights and foresights, the stadia hairs in a level only have limited applications, and for contouring it is more convenient to use a theodolite fitted with stadia hairs or a tacheometer.

The theory of tacheometry is fully explained in Chapter VI of *Principles and Use of Surveying Instruments*, so that here it is only necessary to recall the principal formulæ. For a staff held vertically, these are:

$$d = k_1 s \cos^2 \theta + k_2 \cos \theta, \quad \ldots \ldots \quad (1)$$

$$E = e - R + k_1 s \cos \theta \sin \theta + k_2 \sin \theta, \quad \ldots \quad (2)$$

where d = the horizontal distance from instrument to staff,
s = staff intercept,
k_1 = the stadia constant,
k_2 = a constant equal to the focal length of the objective plus the distance of the objective from the vertical axis of the tacheometer,
θ = inclination to the horizontal of line of collimation through central hair,
E = elevation above datum of point where staff is held,
e = elevation of horizontal axis of instrument above datum,
R = reading of central hair on staff.

Hence, knowing k_1, k_2 and e, and by observing s, θ and R, we can find d and E, and, if the bearing to the staff is also observed, we can fix the position of the point, as well as determine its elevation above datum.

In formulæ (1) and (2) the constant k_2 will seldom exceed much more than eighteen inches, and for many purposes, such as ordinary contouring, the terms containing it can be neglected, in which case the formulæ may be written:

$$d = k_1 s \cos^2 \theta, \qquad (3)$$
$$E = e - R + \tfrac{1}{2} k_1 s \sin 2\theta. \qquad (4)$$

The determination of the bearing of any point may be obtained by measuring the horizontal angle between a line of fixed bearing and the line joining the instrument station to the point; but, for many purposes, including contouring, it will be sufficient to observe compass bearings, if the instrument is fitted with a compass, or else to use the method of observing bearings directly which is described on pp. 118–119.

For a staff held perpendicular to the line of collimation, the formulæ are:

$$d = k_1 s \cos \theta + k_2 \cos \theta + R \sin \theta, \qquad (5)$$
$$E = k_1 s \sin \theta + k_2 \sin \theta - R \cos \theta + e. \qquad (6)$$

which, when k_2 is very small, become

$$d = k_1 s \cos \theta + R \sin \theta, \qquad (7)$$
$$E = k_1 s \sin \theta - R \cos \theta + e. \qquad (8)$$

In an instrument fitted with an anallactic lens, k_2 is zero and hence the formulæ applicable are (3), (4), (7) and (8).

LEVELLING WITH PLANE-TABLE AND INDIAN CLINOMETER

Levelling with the plane-table and Indian clinometer involves the measurement of vertical angles with the Indian clinometer, using the plane-table as a stand. If the position and elevation of the plane-table have not already been fixed from other stations, the position must be fixed by resection, or intersection, or by plane-table traverse, and the elevation found by levelling the Indian clinometer and measuring the angle of elevation to a fixed point of known elevation. The distance from the plane-table to the fixed point is scaled off the plan, and the difference in elevation between this point and the peep hole of the clinometer is calculated from the expression $h = d \tan \theta$, where d is the horizontal distance to the signal and θ the angle of elevation or depression. If the angle is an angle of elevation, the signal is higher than the plane-table and h must be subtracted from the elevation of the signal to give the elevation of the clinometer: if θ is an angle of depression, h is additive to the elevation of the signal. In both cases, the elevation of ground level or station mark is obtained by subtracting the height of the peep hole above the ground point from the elevation of the clinometer.

The simple rule $h = d \tan \theta$ only holds for short sights. For long sights, the effects of curvature and refraction must be taken into account by using the formula

$$h = d \tan \theta + 0{\cdot}574 \left(\frac{d}{5280}\right)^2,$$

where h and d are in feet.

Having determined the elevation of the instrument, the elevations of other points whose positions have been, or can be, fixed may be obtained by the measurement of vertical angles with the Indian clinometer.

LEVELLING WITH ANEROID BAROMETER

Although the aneroid barometer carries a scale of absolute heights referred to mean sea-level, and in theory absolute heights may thus be determined by direct readings on the scale, in practice it would be unsound, except for very rough geographical purposes or when no alternative exists, to rely on direct scale readings. This is because the atmosphere is never static, and it very rarely happens that the physical conditions at mean sea-level which are assumed for the purposes of working out the scale of heights actually exist at the time of observation. Hence, levelling by aneroid is best done by observing the recorded elevations at two points not too far apart, and taking the difference between them as the difference in elevation. If the elevation of one point is known, and this difference is added to or subtracted from it, the elevation of the second point is obtained. By proceeding from a point of known elevation and observing differences between readings there and readings at other points, or between readings from point to point, a line of aneroid heights may be taken over a considerable distance.

If, as almost always happens, there is an appreciable interval of time between the readings at two points, the atmospheric conditions at the first station observed will probably have changed appreciably since the observation was taken there, and another observation would give a different reading. Similar changes will also have occurred at the second station. This will vitiate the results, since it has been assumed that conditions at the first station remain constant during the interval between the observations there and those at the second station. Hence, the best results with the aneroid are obtained when two observers, each with his own aneroid, are employed. The observer at the first station takes a series of observations at different times, and notes the time of each, while the second observer, who also takes a reading with his instrument at the first station, takes other observations at the points required and notes the time of each. In this way a correction for time of observation can be worked out from the readings taken by the first observer. Thus, suppose readings by No. 1 observer at station A were 364 at 7 a.m. and 376 at 11 a.m., and by No. 2 observer they were 371 at 7 a.m. at station A and 496 at 11 a.m. at station B. Then No. 2 observer's instrument would read $371 + (376 - 364) = 383$ at 11 a.m. at station A. Consequently, difference in elevation between stations A and B $= 496 - 383 = 113$ ft.

Usually the station from which differences are reckoned is in camp, and results will be better if both observers are provided, not with a single aneroid, but with batteries of two or more.

If two observers are not available, the surveyor takes a reading, or set of readings, at the first station, and then proceeds as quickly as possible to the second station, where he takes other readings. He then returns to the first station and repeats his original observations, the mean of the two sets being taken as the accepted reading at that station. This, of course, is not always possible on exploratory and reconnaissance surveys, particularly where there is only one party constantly on the move.

In certain parts of the tropics the variations of the barometer during the day are very regular and remain the same, or practically the same, day after day over fairly long periods. Here a series of observations can be taken throughout the course of the day at some point in the centre of the area under survey, and the results can be plotted against time. This gives a curve which can be used for correcting the field observations for time of observation, and does away with the need for maintaining observations daily at one station. In this case, it is necessary to construct fresh curves at fairly regular periods of time, say once weekly. This method, of course, is only advisable in places where the daily barometric wave is known to be very constant for long periods.

CONTOURING

A *contour* may be defined as a continuous line or curve joining points at the same elevation above datum; otherwise, it is the curve which the surface of the water would trace out along its points of contact with the land if the whole of the latter were covered with still water up to the elevation of the contour. Owing to the sloping surface of the land, contours at different elevations will trace out different curves, and the process known as *contouring* means a survey carried out to enable the contours to be plotted on a map or plan. The curves to be plotted will represent contours corresponding to equal intervals of elevation, the difference in elevation between successive plotted contours being known as the *vertical interval.*

Plans showing contours are required for very many different classes of work in civil engineering, as they are necessary in almost all cases where extensive earthwork excavations are involved, not only for choosing the site and for planning the work in the most economical way, but also as a basis for estimating and calculating earthwork quantities. Contours may be surveyed either from the air or on the ground, but it is with ground survey methods that the following pages are concerned.

The amount of work involved, and hence the time spent on survey, depends mainly on the vertical interval chosen. This in turn depends on the scale of the plan, the average slope of the ground, and the purpose for which the contours are needed. In small-scale maps, the vertical interval may be anything from 25 to 100 ft., and in very large-scale plans it may be no more than 1 to 5 ft. The smaller the vertical interval, the greater will be the accuracy required in the survey and the amount of work involved.

There are two main methods by which contours may be surveyed. One is to trace out and mark the line of each contour on the ground and then to make a detail survey of the curves so formed, so that these can be plotted on the plan. The second method is to observe a number of *spot heights,* i.e. levels at different points whose horizontal positions together with their elevations above datum are observed, and then to *interpolate* the contours on the plan from the plotted positions and elevations of these points. The first method is only used for work for a plan on a large scale of a limited area. It is much the slower, but at the same time it is the more accurate method of the two, since the "run" of the contours is seen on the ground at the time of survey. The second method is not so accurate because, unless the work is for a medium- or small-scale map and is done by plane-table, the actual tracing of the contours is carried out in the office on the assumption that the slope of the ground between two successive spot heights is uniform.

7. Laying Out Contours on the Ground.

In this method the engineer's level is used to enable a number of points at exactly the same level to be laid out on the ground. The points are marked by pegs or pins, and their positions surveyed subsequently by any of the methods used for the survey of ordinary detail.

Working from the nearest bench mark, the surveyor runs a line of levels until he reaches a point where the height of collimation of the instrument is a few feet above the elevation of the contour required, the difference between these two elevations giving the staff reading. Having taken a trial observation with the staff held a little distance away, he directs the staff man to go backwards or forwards until a point is reached where the reading on the staff is the correct amount. The staff man here drives a peg into the ground and marks it with a number corresponding to the elevation of the contour. He then moves in a direction as closely as he can judge at right angles to the line of maximum slope of the ground until he arrives at a point at a convenient distance where the staff again gives the correct staff reading, when he drives and numbers another peg. The operation is repeated a number of times, at different points along the contour, and in this way a line of pegs, following the line of the contour, is laid out which can be surveyed in the ordinary way, usually by some form of traverse, other contours being set out and surveyed in a similar manner.

The distance between pegs marking a contour will depend on the accuracy required, which, in turn, depends on the vertical interval, the slope of the ground, and the scale of the plan. If the vertical interval is small, say a couple of feet, the presumption is that accurate contouring is necessary for plotting on a very large scale, and here the interval between successive pegs should not exceed about 25 ft., and should be closer in places where there is a very decided change in the direction of the contours. If the vertical interval is 5 ft., the intervals between pegs may be anything up to 100 ft., with closer spacing at all places where the directions of the contours are changing rapidly.

8. Contouring by Spot Heights.

In contouring by spot heights, three methods may be used:

1. By spot heights along lines laid out at right angles to a surveyed line.
2. By dividing the area under survey into a series of roughly square or rectangular blocks by means of chained lines, and taking spot heights along these lines.
3. By spot heights radiating along lines in different directions from the instrument, and measuring the bearing and distance from the fixed instrument station to the point where the spot height is taken.

The first method is commonly used in such work as preliminary surveys in connection with the construction of roads or railways, where a contoured plan of a long narrow stretch of country is needed for locating the best possible alignment. The surveyor or engineer goes ahead, choosing by eye what appears to be a good general line to be followed. He makes a major compass traverse of his route, using a surveyor's compass and chain or steel band, and putting in pegs at every chain. At the same time he carries a line of spirit levels along the main chainage lines, taking spot heights at the end of every chain length and at every decided change of slope, and establishing bench marks at convenient intervals. This traverse and the line of levels form the framework of his survey. Lines about 100 to 300 ft. apart, and about 200 to 300 ft. long, are then laid out by optical square on either side of, and at right angles to, the main traverse lines. Spot heights are taken along these lines, which are known as *cross-sections,* generally by means of a hand level and levelling staff, or else by means of a theodolite or Abney level or clinometer. The spot heights are plotted on the plan as shown in fig. 8.5, where AB and BC are parts of two traverse lines and **aa**, **bb**, **cc** and **dd** are cross-sections at right angles to them, with spot heights plotted at intervals. The contours are drawn in by interpolation between the spot heights. Thus, in the cross-section **bb**, there is a spot height of 393·4, where the cross-section is intersected by the line AB and below it is a spot height of 399·2. Obviously, then, the contour 395 crosses the cross-section somewhere between these two points. The difference of level between the two spot heights is 5·8 ft. The distance from the first spot height to the second is 68 ft. and the difference in elevation between 393·4 and 395 is 1·6 ft. Hence, the contour will cross the section at a point $68 \times 1{\cdot}6/5{\cdot}8 = 18{\cdot}8$ ft. from the 393·4 spot height. This point is marked off on the plan and the cuts of the same contour plotted on the other sections, the points so obtained being drawn in as a free-hand curve to give the line of the contour. Other contours are then drawn in in a similar manner. On the plan, contours are usually drawn in brown ink.

In fig. 8.5 the cross-sections are at regular intervals but often marked irregularities in the ground, indicating sudden bends or changes in the contours, occur between cross-sections. When this happens, additional cross-sections should be run between the main ones to pick up the irregularities. If necessary, a cross-section may be inclined at an angle to the main line, the direction of the section being fixed by chain survey methods or by instrument.

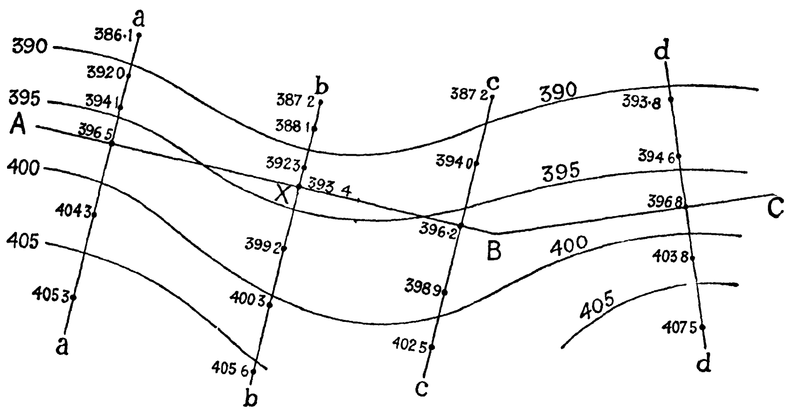


Fig. 8.5

In fig. 8.6 the survey covers an area, not a narrow strip, and the block shown is bounded by traverses AB, BC, CD and DA which have also been spirit-levelled. The straight lines **aa**, **bb**, **cc**, **dd**, **ee**, **ff**, **gg** and **hh** have been laid out as closely as possible parallel to one another and are tied at each end to traverses so that their positions on the plan can be plotted. On each of these lines a series of spot levels has been observed at every chain length, giving a fairly close mesh of spot heights from which contours can be interpolated in the usual way.

In the figure, the line XY has been established between the traverses DA and CB, and a line of spirit levels, with levelled pegs at every chain length, has been run along it. This serves as an intermediate check on the chainage and levelling along the lines **aa**, **bb**, **cc**, etc., and helps to stiffen the network.

The third method of surveying contours by spot heights is most conveniently carried out by tacheometer. Thus, in fig. 8.7 the tacheometer is set up at the point A, whose position is fixed, and sights at different points are taken along a series of radiating lines, of which the bearing can be obtained from readings of the compass on the instrument, or from readings on the horizontal circle and sights from

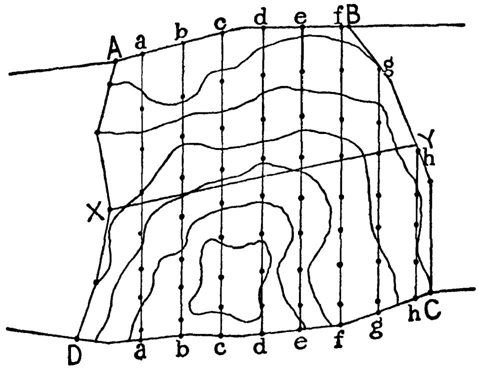


Fig. 8.6

other fixed stations. The staff readings enable the horizontal distances and the elevations of the various points to be computed. These points are later plotted on the plan, their elevations written against them, and the cuts of the contours interpolated as before. The instrument is next set up at some adjoining point which, if necessary, has been

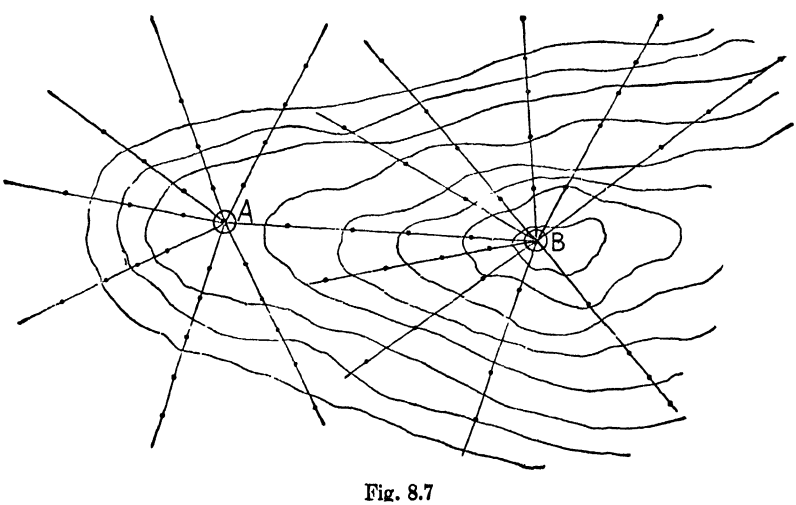


Fig. 8.7

fixed by bearing and distance from the last point, and a similar series of radiating lines of spot heights observed. In this way, the ground is covered with a series of spot heights which, in turn, are plotted on the plan, and from these plotted heights the contours are interpolated.

9. Interpolation of Contours between Spot Heights.

The method of interpolating contours described above involves a little arithmetic which, however, is not great, especially if a slide rule is used. Interpolation may, however, be done graphically. In fig. 8.8 the horizontal line YY is taken as the level 380 ft. above datum, and the vertical line XX as the vertical through the point where the

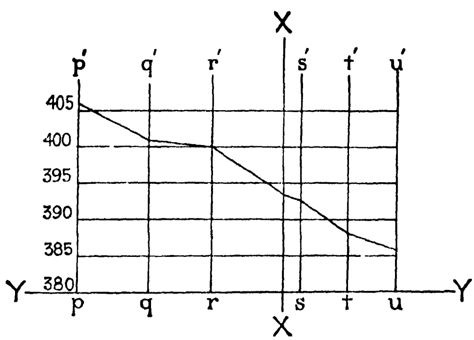


Fig. 8.8

traverse line AB in fig. 8.5 intersects the cross-section **bb.** From the point X lay out on the same scale as the plan X**p**, X**q**, X**r**, X**s**, X**t**, X**u** equal in length to the distances of the different spot heights from the traverse line, and erect perpendiculars at these points. On these perpendiculars lay off, on a scale of, say, 10 ft. to the inch, ordinates proportional to the heights of the corresponding spot heights above the 380 datum line. Join the points so obtained by straight lines. The result will be a section showing graphically, though with vertical heights exaggerated relative to horizontal distances, the slopes of the ground along the cross-section. On XX lay off horizontal lines to represent horizontal planes at the 385, 390, 395, 400 and 405 ft. levels. The points where these lines intersect the line representing ground surface will give the points where the corresponding contours cut this surface, and, by scaling the distances of these points from XX and laying them out on the section **bb** in fig. 8.5, we obtain on the plan the cuts of the contours along that section.

10. Cross and Longitudinal Sections.

In fig. 8.8 we have shown a cross-section representing the slope of the ground in a direction at right angles to a main traverse line, and we have used it to obtain the distances of the points where the contours cut the surface from the point of intersection of the two lines. Cross-sections of a similar kind, however, are often required for the purpose of obtaining areas for the calculation of earthwork quantities. For such a purpose, the horizontal and vertical scales will normally be the same, so that a true area will be obtained from the scaled dimensions. The plotting is best done on squared paper.

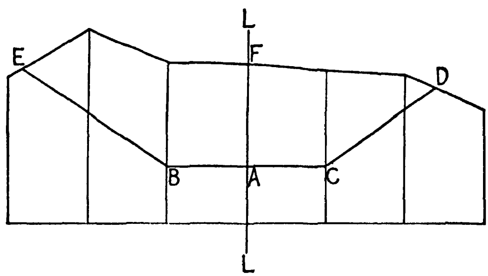


Fig. 8.9

Fig. 8.9 is a cross-section used for taking out quantities during the construction of a railway. The line LL is the centre of the line, which here is in a cutting, the depth of cut being given by the line AF, where A is a point on the centre line at the proposed height of bottom of *formation*, or bottom of cutting, above datum.

A horizontal line BAC is drawn through A such that AB = AC = half width of cutting, and through B and C are drawn lines BE and CD on a slope of $1\frac{1}{2}$ to 1 to represent the sides of the cutting, and meeting the line representing the surface of the ground at E and D. The section EBACDFE represents a vertical section of the earth to be removed.

Longitudinal sections are vertical sections along a line representing the main axis of a survey. In railway work, for instance, a preliminary reconnaissance is followed by the preliminary contoured survey mentioned on p. 162. A suitable line is chosen and drawn in on the plan to represent the centre line of the projected work. The construction engineer, taking the plan out on the ground, proceeds to lay out on the ground the line laid down on paper. For this work he uses a theodolite and steel band and, during the course of it, he corrects such errors as may be due to the preliminary survey having been made by a compass.

He carries forward a *through chainage,* i.e. a chainage which is continuous from some fixed starting point, and in which points are identified by a whole number of chain lengths plus a certain number of hundredths of a chain. Thus, there may be a culvert at station 1096 + 46·2, meaning 1096 full chain lengths plus 46·2 hundredths of a chain (feet or links), or 109,646·2 ft. or links from the starting point. After setting out the straights and curves, he runs a line of levels along the line he has laid out, fixing bench marks at convenient intervals as he goes, connecting in to the bench marks established during the preliminary survey, and taking spot heights at every chain length and at every

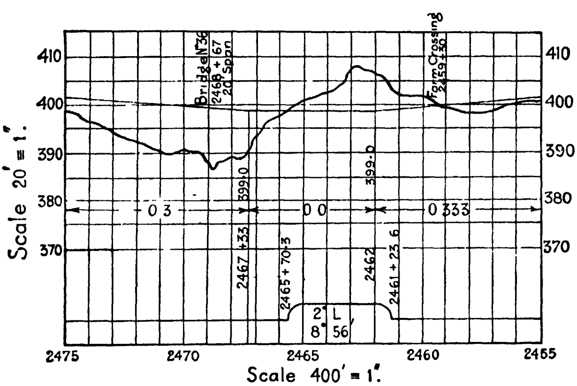


Fig. 8.10

decided change in slope. These spot heights are plotted as a *longitudinal section* or *profile* which not only shows the general slope, rises and falls in the ground, but also the gradients which the bottom of the formation will follow.

In plotting a longitudinal section or profile, the horizontal and vertical scales are not generally the same, the horizontal scale being usually 400 ft. to the inch and the vertical scale 20 ft. to the inch. Such a section is shown in fig. 8.10, in which these were the scales of the original drawing. There is a down gradient of 0·333 per cent from chainage 2455 to a reduced level of 399·0 at chainage 2462. Then follows a level length to chainage 2467 + 33, whence there is an up gradient of 0·3 per cent. These gradients are clearly marked in the diagram. There is a farm crossing at chainage 2459 + 30 and a bridge at chainage 2468 + 67. Details of the horizontal alignment of the centre line of formation are shown diagrammatically at the bottom, and these indicate that a 2° left-hand curve, with a total deflection angle of 8° 56′, commences at chainage 2461 + 23·6 and ends at chainage 2465 + 70·3. The values of ground elevation, formation level, and the cut or fill at every station are also often written on longitudinal sections, but are not shown on the section illustrated above, except in the case of stations where there is a change of grade, when the formation level is given.

11. Contouring on Small Scales.

Contouring on small-scale maps is generally carried out by plane-table in open country which is being systematically and accurately mapped, or by aneroid heights in forest country or where the map is on a very small scale and is little more than a rough geographical or reconnaissance sketch.

Contouring by plane-table is done by establishing the positions of a number of instrument stations by resection or plane-table traverse, and observing with the Indian clinometer the vertical angles to points of known height. The difference of height between table and distant point is obtained by multiplying the distance to the point, as scaled from the map, by the tangent of the angle of elevation or depression. A number of spot heights near the table are then established; they are plotted by measuring the distance to a staff, plotting this along the direction indicated by sighting through the alidade, and writing the observed height beside the plotted position. This work will be facilitated if a telescope alidade is used, as distance can then often be obtained by stadia reading. The operation is repeated at a number of other stations, so that the topographer has a whole series of spot heights from which he can draw in the contours as he moves from point to point.

It is impossible to lay down any hard-and-fast rule about the intervals between instrument stations and between spot heights, as this will depend very largely on the experience of the plane-tabler, as well as on the nature of the country and the density of control points. Plane-tabling, in general, is skilled work, in which efficiency can be gained only by extensive experience.

The aneroid barometer is not nearly such an accurate instrument as the plane-table and telescopic alidade, but it is quicker to use, and it serves for establishing spot heights for rough contouring on geographical and reconnaissance maps on very small scales, where the contour interval is large, and where time or lack of framework points do not permit of the use of the plane-table. It is also used occasionally for establishing spot heights for contours in densely wooded country where the ordinary resection methods by plane-table would not be possible. In this case, for work on scales less than about 1/25,000, the country is split up into blocks bounded by paths or cut lines, anything from 500 ft. to 1 mile apart. These paths and lines are surveyed by rope and sound, or other rough traverse, and lines of aneroid heights which are tied to traverses of higher order and to lines of spirit levels about 4 to 8 miles apart. All main streams and water courses, tops of ridges, etc., are also surveyed, as these are important guides to the run of the contours. In this way, the area is covered with a series of spot heights from which rough contours at, say, 50 ft. vertical intervals may be drawn.

QUESTIONS ON CHAPTER VIII

1. The following readings were taken with a level:

Point	Backsight	Foresight
B.M. " A "	9·16	
T.P.1	3·42	2·08
T.P.2	8·61	6·64
B.M. " B "		11·32

The elevation of B.M. "A" was 948·78. What is the elevation of B.M. " B "?

2. Prove that the effects of collimation error in the instrument can be eliminated by keeping the backsights and foresights of equal length.

3. The following levels were taken on the ground along a survey line:

Distance (ft.)	B.S.	Int.	F.S.	Rise	Fall	R.L.
0	8·21					105·63
100		5·96				
200	10·33		3·28			
300		8·19				
400		8·95				
500		6·27				
600			4·93			

Reduce the levels, applying the usual checks. How would you test whether the line of collimation were in adjustment? (Inst. C.E., October, 1945.)

4. The figures given below represent readings with an engineer's level from stations A, B, C, D and E. Write down the readings in the form in which you would book them, both by the "Height of Instrument" method and the "Rise and Fall" method. In each case assume that you must start a fresh page in your levelling book after taking the reading to point "*l*" from Station C.

Readings taken from	Back sight	Intermediate sights	Forward sight
Station A	(*a*) 6·72	(*b*) 10·84, (*c*) 9·51, (*d*) 7·17	(*e*) 8·35
B	(*e*) 13·12	(*f*) 4·69	(*g*) 2·32
C	(*g*) 5·87	(*k*) 11·23, (*l*) 10·98*, (*m*) 7·62	(*n*) 6·59
D	(*n*) 4·45	—	(*p*) 12·86
E	(*p*) 3·07	(*q*) 5·42, (*r*) 8·71	(*s*) 10·59

Reduced level of ground at (*a*) = 206·42 ft.

* Leave a blank line in your booking and assume that you have begun a new page. (Inst. C.E., April, 1954.)

5. A modern dumpy level was set up at a position equidistant from two pegs A and B. The bubble was adjusted to its central position for each reading, as it did not remain quite central when the telescope was moved in azimuth from A to B. The readings on A and B were 4·86 and 5·22 respectively. The instrument was then moved to D so that the distance DB was about 5 times the distance DA, and the readings with the bubble central were 5·12 and 5·43 respectively. Was the instrument in adjustment? If not, how would the necessary adjustments be made? Describe any other method of testing for this adjustment. (Inst. C.E., April, 1948.)

6. A tacheometric theodolite was set to point at a staff held vertically, the vertical circle of the instrument being set to read zero. The readings of the stadia hairs on the staff were 6·721 and 4·296. What was the distance of the instrument to the staff if the instrumental constants were $k_1 = 100{\cdot}3$ and $k_2 = 1{\cdot}1$?

7. A dumpy level fitted with stadia wires was sighted on a staff, and the readings on the staff were 4·282 and 3·368, the measured distance between instrument and staff being 92·1 ft. The staff was then moved to a point 205·4 ft. away from the instrument, and the readings on the stadia hairs were 7·433 and 5·385. Calculate the values of the constants k_1 and k_2.

8. Calculate the average gradient between two points P and Q from the following observations, which were taken from a station O:

Point	Horizontal circle	Vertical circle	Stadia readings
P	53° 00′	+4° 00′	6·70, 2·30
Q	143° 00′	+5° 30′	7·85, 1·15

The middle reading on the staff in each case was 4·50 ft., the staff being held vertical. Multiplying constant = 100; additive constant, zero. (Calculate distances to the nearest foot.) (Inst. C.E., April, 1946.)

9. Tacheometric readings were taken from a survey station S on to a staff held vertically at two pegs A and B, and the following readings were recorded:

Point	Horizontal circle	Vertical circle	Stadia readings		
			r1	r2	r3
A	62° 00′	+4° 10′ 30″	4·10	6·17	8·24
B	152° 00′	−5° 05′ 00″	2·89	6·17	9·45

The multiplying constant of the instrument was 100, and the addition constant zero. Calculate the horizontal distance from A to B, and the height of peg A above the axis level of the instrument. (Inst. C.E., April, 1947.)

10. It is required to contour an area of country in which there is a good distribution of well-determined spot heights, but they are not sufficiently close to enable contours to be drawn at the required interval of 10 ft. Enumerate the various methods by which the area could be contoured, assuming that the spot heights are easily recognizable both on the ground, and if necessary on air photographs. Describe very briefly the principle of the method used in each case. (Inst. C.E., April, 1947.)

11. An irrigation scheme is to be extended into a flat area about 150 miles by 30 miles, where there are no recorded levels and the gradient is approximately 5 ft. in 100 miles. Part of the area is swampy but it can be assumed that in the dry season it is possible to work on dry ground almost down the centre of the area.

Describe the instruments you would use and the field procedure you would adopt to run a line of what, owing to the extreme flatness of the country, would have to be precise levels, as close to the central axis of the area as possible. There are no available bench marks and all heights must be referred to an assumed water level at the supply end. Vertical air photographs of the area are available but no maps have been made from them. (Inst. C.E., April, 1955.)

CHAPTER IX

SETTING OUT WORK

Setting out work for a contractor or foreman is one of the most important branches of engineering surveying. It is a subject which is more closely related to simple field geometry than to ordinary surveying, and it should present no difficulties to a student who knows his elementary geometry and is familiar with the use of the steel band, theodolite and dumpy level. The main problems to be considered relate to the setting out of railway and road curves but, before proceeding to deal with these, we shall consider a few problems of a more elementary kind.

1. Setting Out Gradients by Boning Rods.

If a couple of pegs have been set out at the ends of a predetermined gradient or on a level surface, intermediate pegs can be established by means of three boning rods. These consist (fig. 9.1) of three T-shaped

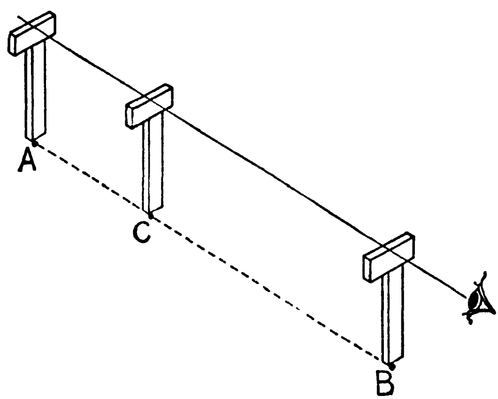


Fig. 9.1

wooden rods, each made up on a short board, about 12 in. long and 3 in. deep, set on, and perpendicular to, a wooden slat about 3 ft. high and 3 in. wide. All these rods must be of the same height from the top of the T-piece to the bottom of the supporting rod.

Let A and B be the pegs at the end of the gradient which it is required to set out. Set boning rods on A and B and sight along the top of them. An intermediate peg can be set to the gradient between A and B by setting a third boning rod on top of it and driving it into the ground until the tops of all three rods appear to be on line.

2. Setting Out Excavations for Foundations.

This work consists in putting in pegs to show the foreman the limits of excavation. The points to be fixed are the main corners, or changes of direction, although it is well to put in intermediate pegs on all very long straights. The simplest method is to use short offsets or offset triangles from well-fixed chain lines.

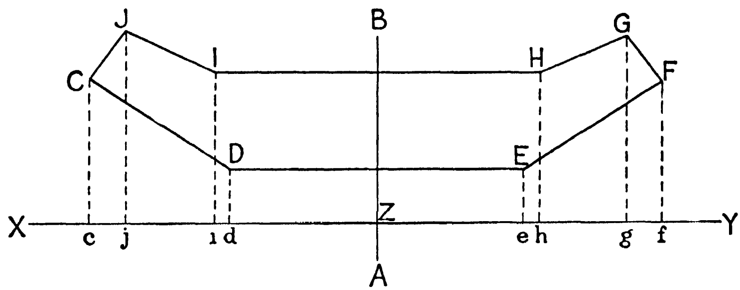


Fig. 9.2

Fig. 9.2 shows a simple case of laying out the excavation for the abutment for a bridge, where pegs have to be put in at the points C, D, E, F, G, H, I, J. The main face of the abutment is to be perpendicular to the line AB, and a line XZY is laid out perpendicular to this line as close as possible to the edge of excavation. From this line, which should be marked by a taut string stretched between tacks on pegs, perpendicular offsets of proper lengths are laid off at the corresponding distances from Z, the point where XZY is intersected by AB. In this case the line XZY would best be set out by theodolite, but the offset lines could be set out by the methods described on pp. 13–14 or by a special wooden "set square" with long sides of 4 or 5 ft.

3. Setting Out Slope Stakes.

One of the railway or road engineer's most common tasks during construction is to lay out slope stakes marking the limits of excavation for cuts or of fills for embankments. The sides of the cut or fill in this case have to lie on planes of given slope; the given data are the width of grade, or formation, and the reduced level of the bottom of the cut or of the top of the embankment.

In fig. 9.3 we have a section at right angles to the centre line of the projected railway. The ground surface is indicated by the line DEFGH and the earth has to be excavated to the line DBCAH, where ACB is the bottom of formation, and C is the centre of AB vertically under a centre peg at F. The sloping sides BD and AH are to lie on a slope of K units horizontal to 1 unit vertical.

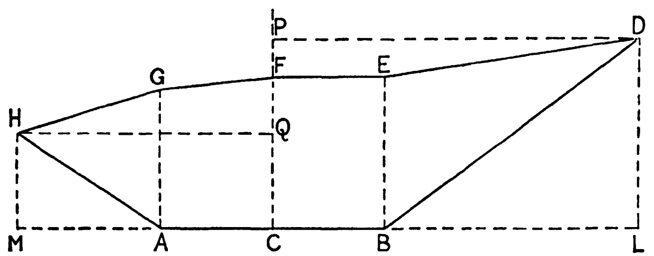


Fig. 9.3

Continue the line AB right and left until it is intersected at the points M and L by the verticals HM and DL from H and D, and through H and D draw the horizontal lines HQ and DP to meet CF at Q and P. Then HQ = MC and DP = LC are the horizontal distances for the slope stakes at H and D from the centre line, and we see at once that

$$PD = CB + K \times DL,$$
$$QH = CA + K \times HM.$$

The work is done with a dumpy level and a box tape, and the procedure will best be illustrated by an example.

Let AC = CB = 11 ft. and $K = 1\frac{1}{2}$, and let the reduced level or elevation of AB be 373·66. We therefore want to find points D and H such that PD = 11 + $\frac{3}{2}$DL and QH = 11 + $\frac{3}{2}$HM.

Starting at a point of known elevation, carry levels forward to a point near the projected cross-section. Let the height of collimation of the level be 380·14. Then the *grade reading*, i.e. depth of formation below height of collimation, is 380·14 − 373·66 = 6·48, and, if a staff were placed at any point E, say, and the reading on it were 3·16, the cut there would be (380·14 − 3·16) − 373·66 = 3·32, which, as can be seen from geometrical considerations, is also equal to 6·48 − 3·16. Hence, the cut at any point is equal to grade reading minus staff reading, and the horizontal distance from the centre line of a slope stake at a point such as D must be equal to 11 + $\frac{3}{2}$(grade reading − staff reading). Accordingly, the process consists in taking readings at various points along the line ED until one is found where this relation between horizontal distance and staff reading is satisfied. For example, after some trials it was found that the distance of D from the centre line was 16·4 ft. and the staff reading 2·9. Hence, as $16{\cdot}4 = 11 + \frac{3}{2}(6{\cdot}5 - 2{\cdot}9)$, D is the point where the slope of the cutting meets ground surface, and is the point where the right-hand slope stake should be put. Similarly, at H the staff reading was 4·7 at distance 13·7 ft. from the centre line, and this is the point where the left-hand slope stake should be driven. At these two points the surface of the ground is 3·6 ft. and 1·8 ft. above formation level, and these points would be indicated by a small peg level with the ground and with witness stakes alongside. The stakes would be marked on one side with the chainage of the section and with a minus sign (to indicate cut) and the depth from formation level to ground level, i.e. −3·6 and −1·8 respectively, and on the other side with the distances 16·4 R and 13·7 L.

It should be noted that, when formation is on a gradient, the grade reading will change at every section, or at different chainages. So long as formation is level, there will be no change of grade reading from section to section for a single set-up of the level.

If the ground level were very uneven, levels would be taken at the points E and G at distance 11 ft. from the centre, and in any case one would also be taken on a peg flush with the surface at F. If the staff readings at the points were 3·2, 3·9 and 3·7 respectively, the cuts would be 3·3, 2·6 and 2·8. These would be marked with the chainage and with a minus sign on one side of the witness stakes, and with the figures 11 R, 11 L and C.L. on the other. The whole section would then be shown on the right-hand page of the level book as follows:

$$\frac{-1{\cdot}8}{13{\cdot}7} \quad \frac{-2{\cdot}6}{11} \quad \frac{-2{\cdot}8}{0} \quad \frac{-3{\cdot}3}{11} \quad \frac{-3{\cdot}6}{16{\cdot}4},$$

the booking on the left-hand page being in the usual form. The top figures indicate depth of formation below ground level, i.e. the necessary cut, and the bottom figures the distances left and right from the centre.

In a similar manner, it is easy to show that, for a fill,

$$\text{Fill} = \text{staff reading} - \text{grade reading},$$

and horizontal distance from centre line to slope stake = half width of formation + K (staff reading − grade reading). Hence, *ground level is below formation level (fill) when the staff reading is greater than the grade reading, and ground level is above formation level (cut) when the grade reading is greater than the staff reading.*

When cuts and fills are light, and the longitudinal slope of the ground is small and fairly regular, the interval between sections where slope stakes are inserted is generally 100 ft.; where, however, the slope is very uneven, or cuts and fills are heavy, the interval between sections is reduced to anything from 10 to 50 ft. Slopes stakes should also be put in at all places where the change from cut to fill takes place at both sides of formation and on the centre line.

4. Transferring Bearings and Levels from Ground Surface to Lines Underground.

The problem of transferring bearings or directions and levels from ground surface to lines underground arises in mining and tunnelling work. For long railway tunnels, where work started from both ends has to meet accurately in the middle, the instruments used for setting out are generally larger and more accurate than the ordinary engineer's instruments.

Transferring a bearing down a shaft requires very careful and accurate work because it necessitates prolonging a line underground from marks whose distance apart is not greater than the diameter of a shaft. The method consists in lining in on the surface two fine piano wires carrying heavy plumb bobs, and then setting a theodolite underground in line with these wires. The wires are either directly attached to, or pass over, a pulley carried by a frame which is supported on heavy timber baulks laid across the top of the shaft and bolted down to rigid supports. This frame is usually provided with a fine-motion lateral movement enabling the wire to be moved at right angles to the direction of the line to be transferred. The plumb bobs at the bottom of the shaft swing in vessels containing water or oil so that their oscillations may be damped down as much as possible. The tops of the wires are lined in by theodolite on the surface and another theodolite is lined in underground as far away as possible from the wires. After the lower theodolite has been lined in and sighted on the wires, intermediate points and forward points can be lined in as required*. It is, of course, necessary to ensure that the wires do not touch the sides of the shaft at any point of their length.

The wires, if need be, can be illuminated by lamps screened by tracing cloth. Some form of illumination of the cross hairs of the theodolite is also necessary when it is used underground.

Levels can be transferred by making marks at the top and bottom of the wires and levelling to and from these marks, the distance between

* For cases where space underground prevents the theodolite being lined in with the wires see Appendix, pages 268–269.

being carefully measured on the surface with steel tapes or bands. Alternatively, levels may be brought down by measurements vertically along the slides or guides in which the cage works. When the depth to be measured exceeds a tape length, temporary platforms must be fixed in the shaft to enable a chain man to hold the end of the tape as the cage descends to the end of the next tape length.

SETTING OUT CURVES BY THEODOLITE

Setting out curves, or curve ranging, is necessary during the excavation stage of railway construction to give the centre line of the earthworks to the contractor or foreman. Later, during the plate-laying stage, it is necessary to lay out the line on the completed formation for the guidance of the plate layer when laying down the rails. Curves are necessary when the direction of the line changes.

There are two main classes of curve with which the railway or road engineer has to deal. These are

(A) Circular Curves.
(B) Transition Curves.

Circular curves, as their name implies, consist of arcs of circles, and they may be divided into three classes:

(1) Simple curves; (2) Compound curves; and (3) Reverse curves.

A simple curve consists of a single circular curve tangential at its ends to two intersecting *straights* or *tangents*.

A compound curve consists of two circular arcs of different radii with curvatures of the same sign, the centres of the circles both lying on the same side of the curve. The curves join on directly to one another; they have a common tangent at their point of contact and at their other ends are tangential to the intersecting tangents.

A reverse curve consists of two circular curves, tangential at their point of contact, but curving in opposite directions. At their other ends the circular curves are tangential to the intersecting tangents.

Transition curves are special curves, not circles, introduced where a tangent joins a curve or where two curves of different curvature, or different direction of curvature, meet. The object of transition curves is to introduce cant or superelevation and change of curvature gradually, and so to avoid sudden changes in the value of the horizontal acceleration due to the centrifugal force caused by curvature.

5. Specification of Circular Curves.

In England curves are generally defined by the lengths of their radii, but on the American continent they are specified by the number of degrees in the angle subtended at the centre by a chord of a specified length, usually 100 ft. Thus a 3° curve is a circle, or part of one, in which a chord 100 ft. long subtends an angle of 3° at the centre.

6. Relation between Radius and Degree of a Circular Curve.

In fig. 9.4, ACB is a chord of length l which subtends an angle D at the centre O of a circle of radius R. From O draw OC perpendicular to ACB. Then

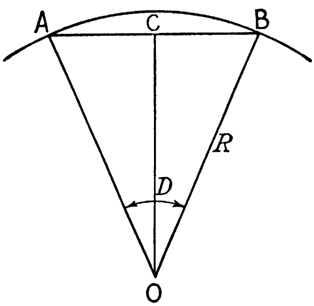


Fig. 9.4

$$AC = \tfrac{1}{2}l = R \sin \tfrac{1}{2}D.$$

$$\therefore \quad R = \frac{l}{2 \sin \frac{1}{2}D}$$

$$= \frac{50}{\sin \frac{1}{2}D}, \text{ when } l = 100.$$

This gives the exact relationship between R and D, but in practice an approximate, but far more convenient, rule is generally used in all cases where extreme accuracy is not needed. This rule depends on the fact that the sine of a small angle is not very different from the radian measure of the angle itself.

Let R_1 be the radius of a curve of degree D_1. Then

$$R_1 = \frac{l}{2 \sin \frac{1}{2}D_1},$$

and so

$$\frac{R}{R_1} = \frac{\sin \frac{1}{2}D_1}{\sin \frac{1}{2}D} = \frac{D_1}{D} \text{ approximately.}$$

If $D_1 = 1°$ and $l = 100$, $R_1 = 5729{\cdot}6 = 5730$ approximately, and hence

$$R = \frac{5730}{D}.$$

This rule may be used in all ordinary railway and road work so that, if $D = 3°$, we take R as $5730/3 = 1910$.

Again, when D is a small angle expressed in degrees,

$$R = \frac{l}{Dr},$$

where r is the number of radians in one degree. Also, if l_1 is another chord length on the same curve and D_1 the corresponding angle at the centre,

$$R = \frac{l_1}{D_1 r}.$$

$$\therefore \quad \frac{l}{Dr} = \frac{l_1}{D_1 r},$$

$$\therefore \quad D_1 = \frac{l_1}{l} D.$$

7. Tangent Lengths of a Circular Curve.

The first thing to be done in ranging out a circular curve is to find the intersection of the tangents, measure the angle of deflection, and calculate the distances of the beginning and end of the curve from the

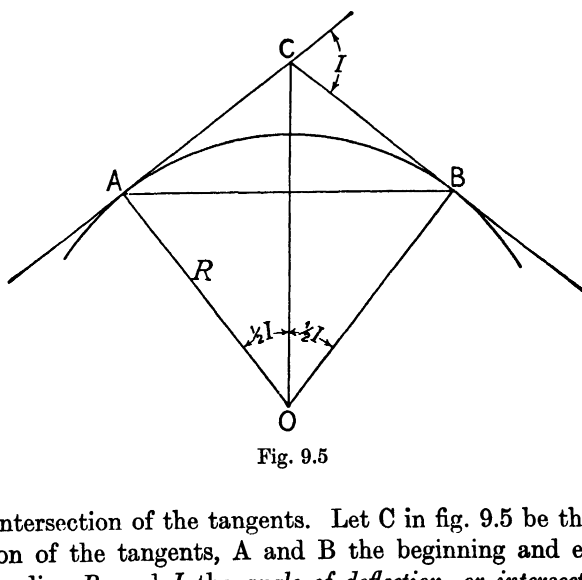


Fig. 9.5

point of intersection of the tangents. Let C in fig. 9.5 be the point of intersection of the tangents, A and B the beginning and end of the curve of radius R, and I the *angle of deflection, or intersection angle,* of the tangents AC and BC. We need the distances CA and CB, which are called the *tangent lengths,* A and B being called the *tangent points.*

From the figure $\angle AOB = I$, and $\angle AOC = \angle BOC = \frac{1}{2}I$. Hence

$$\begin{aligned} AC &= AO \tan AOC \\ &= R \tan \tfrac{1}{2}I \\ &= BC. \end{aligned}$$

Also, it will be seen that the chord length AB is given by

$$AB = 2R \sin \tfrac{1}{2}I,$$

and the arc length AB by

$$\text{arc } AB = RI \times \frac{\pi}{180^\circ} = 0{\cdot}01745RI,$$

or

$$\text{arc } AB = l \times \frac{I}{D} \text{ approximately.}$$

The intersection of the tangents AC and BC can be found by lining in with the theodolite two pegs with tacks on the line AC, well on either side of where CB appears to cross it. Two similar pegs and tacks are driven in on CB, when strings stretched between opposite tacks will give the point of intersection. Hence, having established C and measured the angle of deflection there, distances CA and CB chained from C such that $CA = CB = R \tan \frac{1}{2}I$ or $(5730/D) \tan \frac{1}{2}I$ will give the first and second tangent points, or end points, of the curve. These points should each be marked by a stout peg with a tack on the centre line, and they should also be referenced (see p. 116), as should be point C. For the case where C is inaccessible, see p. 183.

8. Setting Out a Circular Curve by Deflection Angles.

In setting out flat circular curves, i.e. circular curves of large radius and small degree, it is usual to put in pegs at the end of every chain length on a through chainage. The point A will seldom be at the end of a chain length and hence, to complete the first chain length, the first point on the curve will be at a distance less than l from A. Let **b** in fig. 9.6 be the first point on the curve, and let the chord A**b** $= k_1 l$, where k_1 is a fraction less than unity. This chord will subtend an angle D_1 at the centre of the circle, where $D_1 = (k_1 l/l)D = k_1 D$, D being the degree of the curve or the angle subtended at the centre by a chord of length l. Now, by a well-known theorem in geometry, the angle between the chord and the tangent at one end of it is one-half of the angle subtended by the chord at the centre of the circle.

Hence, as AC is tangential at A, the first deflection angle $\delta_1 = \text{CA}\mathbf{b} = \frac{1}{2}D_1 = \frac{1}{2}k_1D$. Accordingly, **b** can be found by setting up the theodolite at A, sighting along AC, turning off the angle $\delta_1 = \frac{1}{2}k_1D$ and measuring the distance $\text{A}\mathbf{b} = k_1l$ along the direction so laid out.

Again, **bc** is the next full chord length, and hence $\angle\mathbf{b}\text{O}\mathbf{c} = D$. Consequently $\angle\text{AO}\mathbf{c} = D_1 + D$, and hence the deflection angle at A is $\angle\text{CA}\mathbf{c} = \frac{1}{2}(D_1 + D) = \delta_1 + \delta$, where $\delta = \frac{1}{2}D$. Thus, **c** may be found by turning off the deflection angle $\delta_1 + \delta$ and chaining out $\mathbf{bc} = l$ from **b** so that **c** lies on the direction indicated by the theodolite.

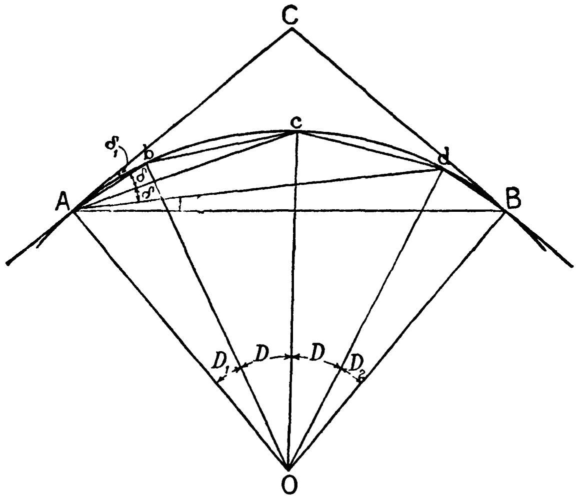


Fig. 9.6

In a similar manner, the deflection angle for the third point will be $\delta_1 + \delta + \delta = \delta_1 + 2\delta$, and so on.

Proceeding in this way, we come to a point near the end of the curve, say at the end of n chain lengths or points, where the next chord length is less than l. Let k_2l be the length of this short chord length, where k_2 is a fraction less than unity, and let **d**B in fig. 9.6 be this chord. Then $k_1l + (n - 1)l + k_2l$ should be very approximately equal to the length of the curve, which is $0{\cdot}01745RI$, or Il/D. Also, angle $\text{BO}\mathbf{d} = (\mathbf{d}\text{B}/l)D = k_2D$, and the deflection angle $\text{BA}\mathbf{d}$ is $\frac{1}{2}\angle\text{BO}\mathbf{d} = \frac{1}{2}k_2D = k_2\delta$. Accordingly, the last deflection angle is k_2 times the deflection angle for a standard chord length l, and the total deflection angle from the tangent AC is $\text{CAB} = \delta_1 + (n - 1)\delta + \delta_2 = \frac{1}{2}I$. Point B has already been fixed as a tangent length from C and hence the setting out of the curve can be checked by laying out angle CAB and measuring $dB = k_2 l$.

Example.—The chainage of the intersection point of two tangents is 3428 + 46·2 ft. and the angle between them is 8° 56′. It is desired to join them with a 2° left-hand curve. Find the chainages of the beginning and end points of the curve, and draw up a table of deflection angles for every chain length (100 ft.) along the curve.

By the precise rule $R = \frac{50}{\sin 1^\circ} = 2864{\cdot}93.$

By the approximate rule $R = \frac{5730}{2} = 2865.$

Taking the latter value, the tangent distance

$$AC = 2865 \tan 4^\circ\, 28' = 223{\cdot}8.$$

$$\text{Hence chainage of A} = (3428 + 46{\cdot}2) - (2 + 23{\cdot}8)$$
$$= 3426 + 22{\cdot}4.$$

$$\text{Length of curve} = \frac{8^\circ\, 56'}{2^\circ} \times 100 = 446{\cdot}7.$$

$$\text{Hence chainage of B} = (3426 + 22{\cdot}4) + (4 + 46{\cdot}7)$$
$$= 3430 + 69{\cdot}1.$$

Deflection angle per 100 ft. $= \frac{1}{2} \times 2^\circ = 1^\circ.$

Deflection angle for first chord of 77·6 ft. $= \frac{77{\cdot}6}{100} \times 1^\circ = 0^\circ\, 46\frac{1}{2}'$; and

deflection angle for last chord $= \frac{69{\cdot}1}{100} \times 1^\circ = 0^\circ\, 41\frac{1}{2}'.$

Hence total deflection angles for different through chainages are:

Chainage	*Deflection angle*
3427	0° 46½′
3428	1° 46½′
3429	2° 46½′
3430	3° 46½′
3430 + 69·1	4° 28′

In this example, we have taken a curve defined in terms of D, the degree of the curve. If the curve is defined in terms of R, the radius, D, can be calculated and the computation proceeds as before.

9. Obstacles to Curve Ranging when using Deflection Angles.

If the point C in fig. 9.5 (p. 179) is inaccessible, as in fig. 9.7, put pegs at intervisible points D and E on the tangents AC and BC, and at D and E observe the angles CDE and CED. Measure the distance DE. Then $I = \angle CDE + \angle CED$ and

$$\frac{CE}{\sin CDE} = \frac{CD}{\sin CED} = \frac{DE}{\sin I}.$$

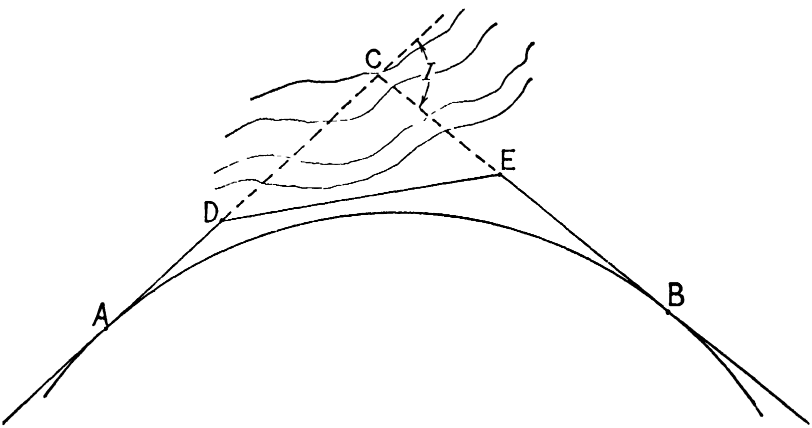


Fig. 9.7

Thus, the angle I and the tangent lengths CA and CB can be calculated. CD and CE can also be calculated, and from these distances the lengths DA = CA − CD and EB = CB − CE can be chained, and A and B located.

Now suppose that A and B are not intervisible, and that the point C in fig. 9.8 having been laid out, other points beyond C cannot be seen from A. In the figure, CD is the tangent at C and DAC = δ_c is the deflection angle of C at A. By simple geometry, angle DCA = DAC = δ_c. Hence, if the instrument is set up at C and sighted at A, it can be set to point along the tangent at C by turning off the angle δ_c to the right. On transiting the telescope, the latter will point in the direction DC produced, and hence other points can be laid out by turning off the same deflection angles as would be used if C were the commencing point of a new curve to which DC was a tangent.

In this last case it has been possible to set out the curve as far as a point C, from which B and all points lying between C and B can be seen. In fig. 9.9, however, having got as far as **d**, other points between **d** and **g** cannot be seen owing to a large building. One way of dealing with the problem would be to run the remainder of the curve in the reverse direction from B, but this may not be convenient.

An alternative method is to set off the deflection angles at A until one is obtained in which the line of sight clears the obstacle. Let

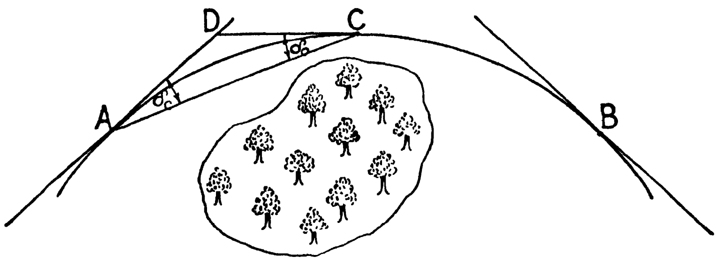


Fig. 9.8

CA**g** $= \alpha$ be this deflection angle. Then the angle subtended at the centre by the chord A**g** is 2α, and the length of A**g** can be accurately calculated from A**g** $= 2R \sin \alpha$. Hence, chain this distance from A, so obtaining **g**. Set up the theodolite at **g**, sight A, and turn off an angle α to the right of **g**A. This will bring the line of sight tangential to the curve at **g**, and hence points to the left and right of **g** may be set out in the usual manner.

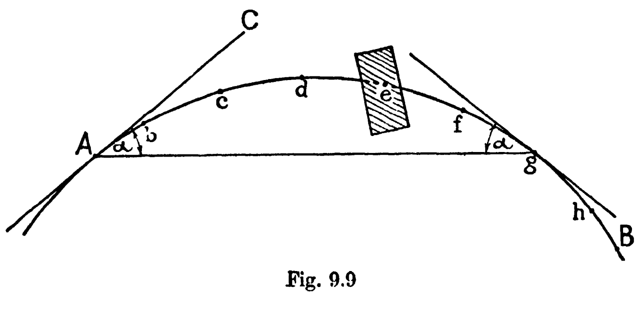


Fig. 9.9

Three cases of inaccessible end points may arise. In the first, the first tangent point A is not accessible. Set out points D and E, fig. 9.10, and determine the distance DE by one of the methods already described in Chapter VII. This will give the chainage of E and, as the chainage of C is known, and that of A can be calculated, the distance AE is known. Calculate the offset EF to the curve from

$$\text{EF} = \text{AO} - \text{GO} = R - \sqrt{(R^2 - \text{AE}^2)} = R - \sqrt{\{(R - \text{AE})(R + \text{AE})\}}$$

and lay out a peg at F. Now run in the curve from B and check on F, where the chainage of F can be calculated from the chainage of A and the angle θ subtended by AF at the centre, where

$$\sin\theta = \frac{GF}{OF} = \frac{AE}{OF} = \frac{AE}{R}.$$

Alternatively, assume δ as the value of the deflection angle of a point F on the curve (fig. 9.11) such that the tangent at F will intersect the tangent AC at A at a point G which is clear of the obstacle. Then $AG = R \tan \delta$. Hence, since the chainage of A can be calculated, we know the chainage of G. Put in a peg at G and set the instrument

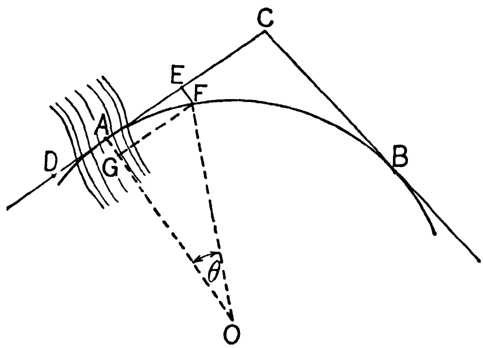


Fig. 9.10

there. From G lay out GF = AG making angle CGF = 2δ. This gives the point F on the curve. If the instrument is set up at F, and sighted on G, it will be aligned on the tangent at F and the curve can now be laid out in both directions from F.

If the second tangent point is inaccessible, the curve can be laid out as far as possible from the first tangent point. The problem now is to establish the chainage on the second tangent beyond the second tangent point.

In fig. 9.10 let B be the *first* tangent point, F the last chainage point to be established on the curve, and let D be a point on the tangent CA whose chainage is required. From F drop a perpendicular FE on the tangent AC, measure the distance CE, and determine the distance ED. Then, distance CD = CE + ED and distance CA can be calculated. Hence AD = CD − CA. The chainage of A can be computed from the chainage of B and the length of the curve, and so the through chainage of D can be found.

When both tangent points are inaccessible, establish the point F as in figs. 9.10 or 9.11, and run in the curve from that point to a chainage point as close to B, the end tangent point, as possible. Then proceed as before to determine the chainage of a point on the tangent on the far side of B.

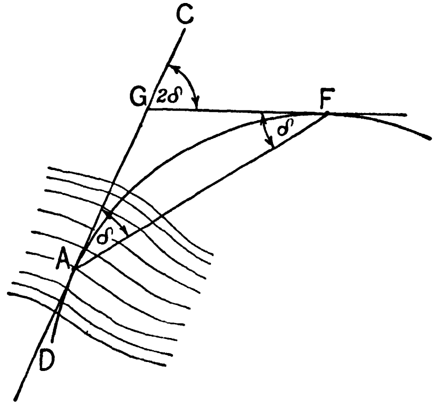


Fig. 9.11

10. Setting Out Circular Curves by Two Theodolites.

This method consists in setting up a theodolite at each tangent point and working out the deflection angles from the tangents. The theodolites are then set to read corresponding deflection angles and points set out to lie on both lines of sight (fig. 9.12).

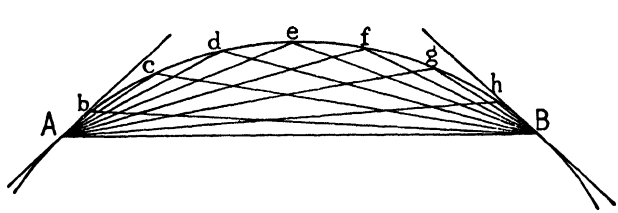


Fig. 9.12

SETTING OUT CURVES BY CHAIN AND STEEL TAPE

Curves on railways should normally be set out by theodolite and chain but short curves may be set out by the chain and steel tape alone; this method is also applicable where great accuracy is not required, or when short curves marking curves of buildings, curved wing walls, etc., have to be set out.

11. Location of Tangent Points.

If the positions of the tangent points are not known, find and mark the intersection C of the tangents. From C, fig. 9.13, lay off equal distances CD and CE along the two tangents. Measure DE, bisect it at F and measure the distance CF. Then, by similar triangles CDF and COA,

$$\frac{CA}{AO} = \frac{CF}{DF}.$$

$$\therefore \quad CA = R\frac{CF}{DF}.$$

Hence A and B can be found by computing CA and measuring this distance along the tangents from C in the directions CA and CB.

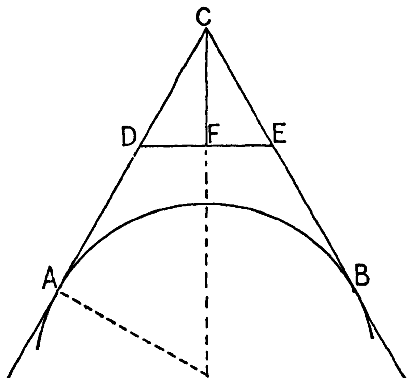


Fig. 9.13

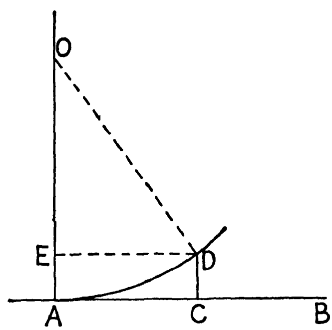


Fig. 9.14

12. Setting Out Circular Curves by Offsets from Tangents.

In fig. 9.14, AB is tangent at A to curve. Offset CD at point C is given by

$$CD = AO - EO = R - \sqrt{R^2 - ED^2} = R - \sqrt{(R - AC)(R + AC)}.$$

Again, from the properties of the circle,

$$AE(2R - AE) = ED^2.$$

Hence, for short offsets, $CD = AE = \frac{ED^2}{2R} = \frac{AC^2}{2R}$ approximately when AE is small in comparison with R.

13. Setting Out Circular Curves by Offsets from Chords.

In fig. 9.15 we require the offset EF at the point E on the chord AB. Then, C being the mid-point of AB,

$$CD = R - \sqrt{(R - CB)(R + CB)},$$
$$GD = R - \sqrt{(R - GF)(R + GF)}$$
$$= R - \sqrt{(R - CE)(R + CE)}.$$
$$\therefore \quad EF = CG = CD - GD$$
$$= \sqrt{(R - CE)(R + CE)} - \sqrt{(R - CB)(R + CB)}.$$

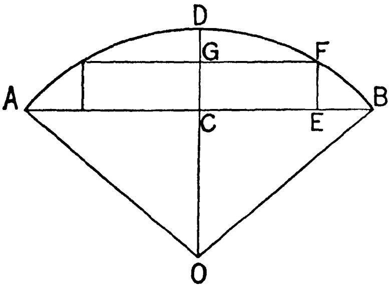


Fig. 9.15

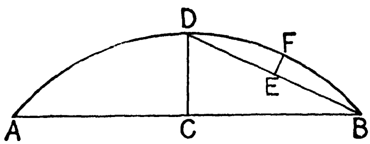


Fig. 9.16

When CD and GD are small in comparison with R,

$$CD = \frac{CB^2}{2R} \text{ and } GD = \frac{CE^2}{2R}.$$

$$\therefore \quad EF = \frac{CB^2 - CE^2}{2R} = \frac{(CB - CE)(CB + CE)}{2R}.$$

The curve can also be set out by laying out the effect CD at the centre point C of AB using

$$CD = \frac{CB^2}{2R}.$$

Join DB, fig. 9.16, and at E, the mid-point of DB, erect offset EF, where

$$EF = \frac{EB^2}{2R}.$$

Similarly, points may be established at the mid-points of DF and FB, and so on.

14. Setting Out Circular Curves by Deflection Distances.

In many cases when laying out curves by chain and tape a through chainage is not needed, and we can start straight away with a chord of a whole chain length.

Hold one end of the chain at the tangent point A, fig. 9.17, and swing the other end until the right-angled offset M**b** from the tangent is equal to $\frac{l^2}{2R}$, where l is the length of the chain. This locates the first point **b** on the curve.

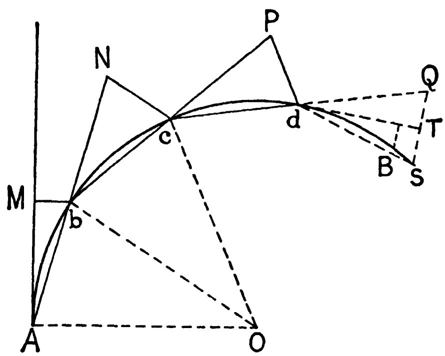


Fig. 9.17

Produce A**b** to N, making **b**N = A**b** = l, and pivot one end of the chain about **b** until the distance N**c** from N to the other end is equal to l^2/R. Then **c** is the second point on the curve.

In a similar manner produce **bc** to P so that **c**P = **bc** = l, and make P**d** = l^2/R and **cd** = l, thus locating **d**, a third point on the curve; and so on.

To check the work at the end point B, the distance from the last point to B will be less than l. Let **d** in fig. 9.17 be the last point before reaching B. Produce **cd** to Q and locate S such that **d**Q = l and QS = l^2/R. Bisect SQ at T. Then **d**T is the tangent to the curve at **d**. Let l_1 be the distance **d**B. Chain this distance from **d** along **d**T, and at the end of it lay out a right-angled offset = $l_1{}^2/(2R)$. The point so obtained should coincide with B.

To show that N**c** = l^2/R, we have, since OA = O**b** = O**c** and A**b** = **bc** = l, ∠O**b**A = ∠O**bc**. Hence, ∠N**bc** = 180° − (O**bc** + O**b**A) =

180° − 2O**bc** = **b**O**c**. Consequently, triangles **b**N**c** and **b**O**c** are similar and

$$\frac{\mathrm{Nc}}{\mathrm{bc}} = \frac{\mathrm{bc}}{\mathrm{Ob}}.$$

$$\therefore\ \mathrm{Nc} = \frac{\mathrm{bc}^2}{\mathrm{Ob}} = \frac{l^2}{R}.$$

Note also that this relation is exact, but that the value M**b** = $l^2/(2R)$ for the right-angled offset at M is only approximate.

If, as in the case of a railway or road survey, it is desired to maintain a through chainage with pegs at every chain length, and A is not at a whole chainage mark, let l_1 be the length between A and the next whole chainage mark. Hold one end of chain at A and find point **b**, so that A**b** = l_1 and the right-angled offset M**b** to **b** from the tangent at A is equal to $l_1^2/(2R)$ (see fig. 9.18). Similarly, find point **b**′ on the other side of A, such that A**b**′ = $l - l_1$ and the right-angled offset M′**b**′ to **b**′ is equal to $(l - l_1)^2/(2R)$. Then **b** and **b**′ are both points on the curve. Produce **bb**′ to N making **b**N = **bb**′ = l, and find **c** so that N**c** = l^2/R and **bc** = l. The point **c** is thus another point on the curve and the work from there proceeds as before.

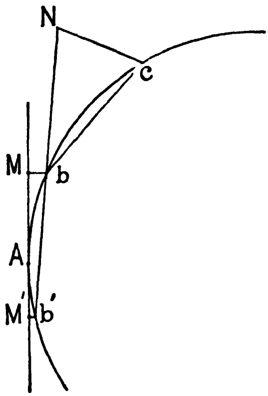


Fig. 9.18

COMPOUND CURVES

Let the tangents AD and CD in fig. 9.19, which intersect at D, be joined by the circular curves AB of radius R_1 and BC of radius R_2, and let EF be the common tangent at B meeting AD and CD at E and F respectively. Then O_1 and O_2, the centres of the curves, lie on the straight line BO_1O_2 perpendicular to EB.

The angle of intersection ω of the tangents having been observed, the known elements are R_1 and R_2 and ω. In addition, we must know either the chainage of A, or of B, or else one of the angles ϕ or θ which the tangent EF makes with AD and DC. Usually, the chainage of one tangent point, say A, will be scaled from the plan and, the chainage of D having been found in the field, the tangent length AD = T_1 can be obtained. The parts now required are θ, ϕ and CD $= T_2$, and, when these have been obtained, we have all the data required to lay out the two circular curves by the methods already described.

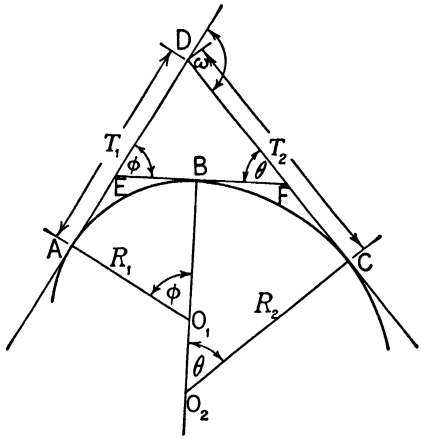


Fig. 9.19

From the figure,

$$\omega = \theta + \phi; \quad \frac{\text{ED}}{\text{EF}} = \frac{\sin \theta}{\sin \omega},$$

and

$$\text{AD} = \text{AE} + \text{ED}$$
$$= R_1 \tan \tfrac{1}{2}\phi + \text{EF} \frac{\sin \theta}{\sin \omega}.$$

But

$$\text{EF} = \text{EB} + \text{BF} = R_1 \tan \tfrac{1}{2}\phi + R_2 \tan \tfrac{1}{2}\theta.$$

Hence

$$T_1 = R_1 \tan \tfrac{1}{2}\phi + (R_1 \tan \tfrac{1}{2}\phi + R_2 \tan \tfrac{1}{2}\theta) \frac{\sin \theta}{\sin \omega}.$$

$$\begin{aligned}
\therefore\ T_1 \sin \omega &= R_1 \tan \tfrac{1}{2}\phi\,(\sin \theta + \sin \omega) + R_2 \tan \tfrac{1}{2}\theta \sin \theta \\
&= R_1 \tan \tfrac{1}{2}(\omega - \theta)(\sin \theta + \sin \omega) + 2R_2 \sin^2 \tfrac{1}{2}\theta \\
&= R_1 \frac{\sin \tfrac{1}{2}(\omega - \theta)}{\cos \tfrac{1}{2}(\omega - \theta)} \left\{2 \sin \tfrac{1}{2}(\omega + \theta) \cos \tfrac{1}{2}(\omega - \theta)\right\} + R_2(1 - \cos \theta) \\
&= 2R_1 \sin \tfrac{1}{2}(\omega - \theta) \sin \tfrac{1}{2}(\omega + \theta) + R_2(1 - \cos \theta) \\
&= R_1 (\cos \theta - \cos \omega) + R_2(1 - \cos \theta) \\
&= R_1\{(1 - \cos \omega) - (1 - \cos \theta)\} + R_2(1 - \cos \theta) \\
&= (R_2 - R_1)(1 - \cos \theta) + R_1(1 - \cos \omega) \\
&= (R_2 - R_1) \operatorname{versin} \theta + R_1 \operatorname{versin} \omega.
\end{aligned}$$

T_1, ω, R_2 and R_1 being known, θ can be found from this equation and ϕ from $\phi = \omega - \theta$. The other tangent distance T_2 can then be calculated from a similar expression to that involving T_1, namely

$$T_2 \sin \omega = (R_1 - R_2) \text{ versin } \phi + R_2 \text{ versin } \omega.$$

If the curve is started at A, the length of the curve AB can be calculated and the point B laid out on the ground. The theodolite is then moved to B, sighted on A, and angle $\frac{1}{2}\phi$ turned off to the right. This will cause the instrument to sight along the tangent BE or BF, and from there the curve BC may be run in to check on the point C which has been fixed from D. The point B can also be fixed by measuring off AE $= R_1 \tan \frac{1}{2}\phi$ and CF $= R_2 \tan \frac{1}{2}\theta$, and then chaining EB $=$ AE on the line EF, thus affording another check.

REVERSE CURVES

Reverse curves may be considered to be special cases of a compound curve in which the curvature of one curve is negative with respect to that of the other, and for which the formulæ of compound

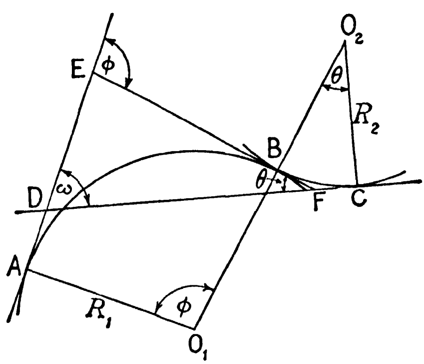


Fig. 9.20

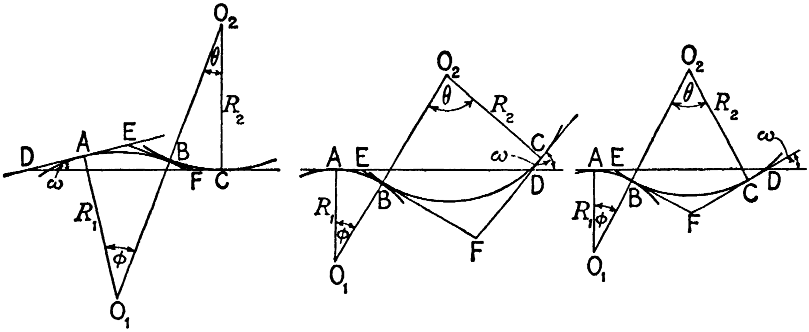
curves hold with a change of sign in some of the terms. Thus, for the case shown in fig. 9.20, where AD $= T_1$ and DC $= T_2$,

$$T_1 \sin \omega = R_1 \text{ versin } \omega - (R_1 + R_2) \text{ versin } \theta,$$

$$T_2 \sin \omega = (R_1 + R_2) \text{ versin } \phi - R_2 \text{ versin } \omega,$$

and we also see from the geometry of the figure that $\omega = \phi - \theta$. These formulæ can easily be derived in the same way as the formulæ for compound curves were derived.

Three other cases are illustrated in fig. 9.21, and the reader may be interested to verify for at least one or two of them that the same formulæ hold for these cases, except that the signs of the terms on the right in the expression for $T_1 \sin \omega$ are reversed in all three cases,



(a)	(b)	(c)
$T_1 \sin \omega = -a + b$	$T_1 \sin \omega = -a + b$	$T_1 \sin \omega = -a + b$
$T_2 \sin \omega = +c - d$	$T_2 \sin \omega = -c + d$	$T_2 \sin \omega = +c - d$

Fig. 9.21

and the signs of the terms on the right in the expression for $T_2 \sin \omega$ are reversed in the cases of fig. 9.21*b*. The signs of the terms are given under each diagram in which $a = R_1$ versin ω, $b = (R_1 + R_2)$ versin θ, $c = (R_1 + R_2)$ versin ϕ, $d = R_2$ versin ω.

CROSS-OVERS

A special case arises when the tangents are parallel and ω is zero. Both T_1 and T_2 are then infinite and the case becomes that of a cross-over from two parallel sets of rails (fig. 9.22). Here, if EF is the common tangent at B, ∠BAE = ∠EBA and ∠BCF = ∠FBC. But, since AL and CN are parallel, ∠FEL = ∠EFN. Now ∠FEL = 2 ∠BAE and ∠EFN = 2 ∠BCN, so that ∠BAE = ∠BCN, and consequently ABC is a straight line. Also, since AO_1 is parallel to CO_2, $\theta = \phi$.

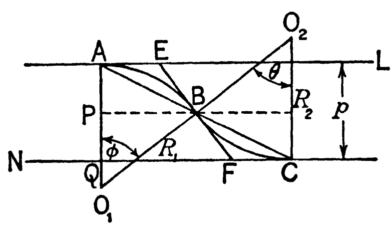


Fig. 9.22

Through B draw BP parallel to AL and CN. If p is the perpendicular distance AQ between the two tangents

$$p = AQ = AP + PQ$$
$$= R_1 \text{ versin } \phi + R_2 \text{ versin } \theta$$
$$= (R_1 + R_2) \text{ versin } \phi.$$
$$\therefore \text{ versin } \phi = p/(R_1 + R_2),$$

and, knowing p, R_1 and R_2, ϕ can be found. This gives the data to compute the positions of B and C, and to enable both curves to be set out. Moreover, as both curves are short, setting out can, if necessary, be done by short offsets from the tangents or from the chords.

TRANSITION CURVES

Transition curves are curves of varying curvature which are used in combination with cant or superelevation (i.e. making the outer rail higher than the inner), to minimize the effects of sudden changes in centrifugal force when the path in which a vehicle is travelling undergoes a sudden change in curvature. Transition curves are therefore introduced at the beginning and ends of circular curves. Their curvature is zero where they join on to a tangent; it is equal to the curvature of the curve where they join on to a circular curve, and it assumes intermediate values in between.

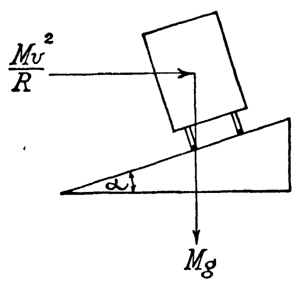


Fig. 9.23

15. Cant on Circular Curve.

In fig. 9.23 the centrifugal force acting on a vehicle of mass M travelling with a velocity v ft. per second on a circular curve of radius R is Mv^2/R. This force acts in a horizontal direction, and the force in the vertical direction is Mg, where g is the acceleration due to gravity. If there is to be no sideways thrust along the road surface, which is inclined to the horizontal at an angle α, the resolved forces acting in a direction parallel to the road surface must equalize one another. Hence,

$$\frac{Mv^2}{R} \cos \alpha = Mg \sin \alpha$$

or

$$\tan \alpha = \frac{v^2}{gR};$$

or, where α is small and expressed in radians,

$$\alpha = \frac{v^2}{gR}.$$

If d is the width of the road bed in inches and c is the amount of cant or superelevation in inches on the outer edge,

$$\begin{aligned} c &= d \sin \alpha \\ &= d \,.\, \alpha \text{ approximately} \\ &= \frac{dv^2}{gR}. \end{aligned}$$

Assuming that g is 32·2 ft. per second per second, R is in feet, V is the velocity of the vehicle in *miles per hour*, and that, with a standard gauge of 4′ 8½″ between rails, the width between centres of rails is 4′ 11″, this gives for a railway curve with c in *inches*:

$$c = \frac{3{\cdot}9V^2}{R}.$$

If no transition curve is provided, cant may be introduced in several different ways. One method is to introduce it gradually on the straight, to reach its full value at the point of tangency. Otherwise, it may be introduced gradually on the curve, or else a combination of the two methods may be used. All of these methods are objectionable because they ignore the rule that the amount of cant should be proportional to the curvature at every point. Consequently, the usual practice is to introduce transition curves in which the proper relation between cant and curvature is maintained throughout for a certain definite and assumed standard velocity. In practice, however, in railway work cant is seldom allowed to exceed 6 in.

16. Lengths of Transition Curves.

The length of a railway transition curve is now generally calculated in this country by a rule which was proposed by Mr. W. H. Shortt in 1908 and is based on the assumption that a passenger will suffer no inconvenience if the rate of change of acceleration does not exceed 1 ft. per second per second per second. Let a be this rate of change, l the length of the transition curve at any point P measured along the curve from the tangent point, L the total length of the transition curve up to the point of tangency with the circular curve, r the radius of curvature of the transition curve at P, R the radius of the circular curve, and t the time taken to travel from the origin to the point P. Then, in differential notation,

Rate of change of acceleration at P

$$= \frac{d}{dt}\left(\frac{v^2}{r}\right).$$

But $$v = \frac{dl}{dt},$$

consequently, putting the rate of change of acceleration equal to a and substituting for dt,

$$d\left(\frac{v^2}{r}\right) = a\frac{dl}{v}.$$

Integrating, $$\frac{v^2}{r} = \frac{al}{v} + \text{a constant}.$$

But $r = \infty$ when $l = 0$, and $r = R$ when $l = L$. Hence

$$\frac{aL}{v} = \frac{v^2}{R}.$$

$$\therefore \quad L = \frac{v^3}{aR}.$$

When L and R are in *feet*, V is the velocity in *miles per hour*, and $a = 1$ ft. per sec.^{3}, we have

$$L = \frac{3{\cdot}155V^3}{R}.$$

In the United States, the length of a transition curve is commonly made to depend on the time rate of increase of cant, the maximum allowable for the usual 4 ft. 8½ in. gauge being 7/6 in. per sec. This gives

$$c = \frac{7}{6}t,$$

or, substituting the value of c found above,

$$\frac{3{\cdot}9V^2}{R} = \frac{7}{6}\frac{L}{v}.$$

But $$v = \frac{88}{60}V.$$

$$\therefore \quad L = \frac{3{\cdot}9V^3}{R} \times \frac{6}{7} \times \frac{88}{60}$$

$$= \frac{5V^3}{R}$$

Thus, the American rule, which is that recommended by the American Railway Engineering Association, is exactly similar in form to the rule commonly used in this country but embodies a higher valued constant, giving a longer curve for the same speed of vehicle and the same radius of the circular curve.

17. Derivation of Equation of Transition Curve.

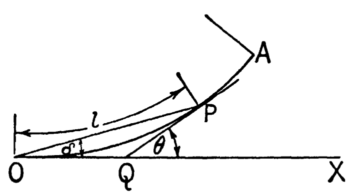


Fig. 9.24

In fig. 9.24 let P be any point on a transition curve, distant l measured along the arc from the tangent point at O, and let A be the point where the transition curve joins the circular curve of radius R. Let L be the length of the arc OA, r the radius of curvature of the transition at P, and θ the angle which the tangent at P makes with the horizontal axis OX. Then, by the ordinary formula for curvature,

$$\frac{1}{r} = \frac{d\theta}{dl}.$$

If the cant is taken as increasing directly as l, its angular value at point P is $\alpha = kl$, where k is a constant representing the increase per unit length. Hence

$$kl = \frac{v^2}{gr}.$$

But at A,

$$kL = \frac{v^2}{gR},$$

$$\therefore \quad lr = LR$$

and

$$\frac{d\theta}{dl} = \frac{l}{LR}.$$

$$\therefore \quad \theta = \frac{l^2}{2LR},$$

the constant of integration vanishing as $\theta = 0$ when $l = 0$. This is the general equation, in intrinsic form, of a transition curve.

By putting $dy = dl \sin\theta$, $dx = dl \cos\theta$, and expanding $\sin\theta$ and $\cos\theta$ in the forms

$$\sin\theta = \theta - \frac{\theta^3}{6} + \ldots;\ \cos\theta = 1 - \frac{\theta^2}{2} + \frac{\theta^4}{24} - \ldots$$

and then using the equations

$$\theta = \frac{l^2}{2LR} \text{ and } dl = \frac{LR}{l}\,d\theta$$

to eliminate l, we get by integration

$$y = \sqrt{2RL}\left(\frac{\theta^{3/2}}{3} - \frac{\theta^{7/2}}{7.6} + \ldots\right),$$

$$x = \sqrt{2RL}\left(\theta^{1/2} - \frac{\theta^{5/2}}{5.2} + \ldots\right).$$

If δ is the deflection angle from O to the point P, we have

$$\tan\delta = \frac{y}{x} = \frac{\theta}{3} + \frac{\theta^3}{105} + \ldots,$$

which, as θ may be taken as a small angle, is very approximately equal to the expansion

$$\tan\frac{\theta}{3} = \frac{\theta}{3} + \frac{\theta^3}{81} + \ldots,$$

so that we can take $\delta = \frac{1}{3}\theta$. Hence, for small angles, *the deflection angle from the tangent point to a point on the curve is one-third of the angle which the tangent to the curve at the point concerned makes with the main tangent.* This property is of considerable value in laying out the curve by deflection angles.

From the above equations we may derive

$$y = \frac{l^3}{3(2RL)} - \frac{l^7}{7.6(2RL)^3} + \ldots,$$

$$x = l - \frac{l^5}{5.2(2RL)^2} + \ldots,$$

and so, after reversing the last series to get l in terms of x, obtain

$$y = \frac{x^3}{3(2RL)} + \frac{16x^7}{7.6.5(2RL)^3} + \ldots.$$

The second term in both expressions for y is very small, so that we can take with sufficient approximation

$$y = \frac{l^3}{6RL} \text{ or } y = \frac{x^3}{6RL}.$$

The first of these equations is that of the cubic spiral and is a little more exact than the second equation, which is that of a cubic parabola. The latter curve, often called *Froude's transition curve*, is commonly used as a transition curve because of the ease with which it may be set out by offsets. For both curves, we may take $\delta = \frac{1}{3}\theta$ and angle QPO in fig. 9.24 (p. 197) as $\frac{2}{3}\theta$.

18. Geometry of the Transition Curve.

In fig. 9.25 let A be the beginning of the transition curve joining the tangent AT to the circular curve of radius R and centre O which commences at B, and let BF, the tangent common at B to both circular and transition curves, make angle ϕ with the tangent AT.

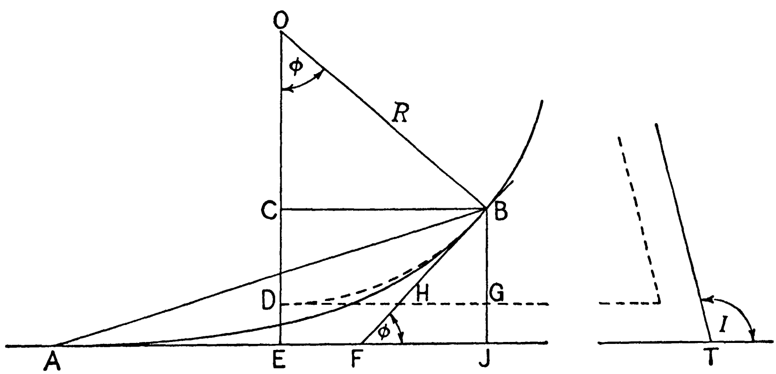


Fig. 9.25

From O draw the perpendicular OE on the tangent AT and produce the circular curve in the direction of OE to meet OE at D. Through D draw DHG parallel to AT, meeting BF in H, and through B draw BC parallel to EF.

Then $\angle BOC = \angle BHG = \angle BFT = \phi$,

and $CD = R(1 - \cos\phi) = R \text{ versin } \phi$.

Also, by substituting L for l in the intrinsic equation of the transition curve,

$$\phi = \frac{L}{2R}.$$

Again, the ordinate BJ from the tangent AT to B will be obtained by substituting L for l in the formula for y. Hence

$$BJ = \frac{L^3}{6RL} = \frac{L^2}{6R}.$$

But $CD = R(1 - \cos\phi) = R \times \frac{1}{2}\phi^2$ approximately

$$= \frac{L^2}{8R}.$$

$$\therefore \quad DE = CE - CD = BJ - CD = \frac{L^2}{6R} - \frac{L^2}{8R} = \frac{L^2}{24R}.$$

This quantity DE is called the *shift*. It is the amount by which the circular curve has to be moved above the main tangent to make room for the transition, and it is denoted by s.

Again $CB = EJ = R\sin\phi = R \cdot \frac{L}{2R} = \frac{1}{2}L$, so that, when the angle BAJ is small, $AE = AJ - EJ = L - \frac{1}{2}L = \frac{1}{2}L$ approximately, and the ordinate to the curve at E is

$$y = \frac{L^3}{48(RL)} = \frac{L^2}{48R} = \frac{1}{2}DE = \frac{1}{2}s.$$

Again, it will be seen from fig. 9.25 that, if a circular curve is to be joined to the tangents at each end by similar transition curves, and if the tangents intersect at an angle I, the angle subtended at its centre by the circular curve will be $(I - 2\phi)$ degrees, and the length of the circular arc will be $(I - 2\phi) \times \frac{\pi R}{180°}$. In addition, if T is the intersection of the tangents, we have

$$ET = OE \tan \tfrac{1}{2}I,$$

and, as AE is approximately equal to $\frac{1}{2}L$, AT, the tangent distance, is given by

$$AT = (R + s)\tan \tfrac{1}{2}I + \tfrac{1}{2}I$$

19. Setting Out the Transition Curve.

The first step is to determine the tangent distance AT from the intersection of the main tangents at T to A, the beginning of the transition curve. For this purpose, calculate L from

$$L = \frac{3{\cdot}155V^3}{R},$$

and then determine the shift s from

$$s = \text{DE} = \frac{L^2}{24R}.$$

If I is the angle of intersection of the tangents, the distance AT can be obtained from the expression

$$\text{AT} = (R + s)\tan \tfrac{1}{2}I + \tfrac{1}{2}L,$$

and so the chainage of A is determined.

If the transition is to be set out by deflection angles, these can be computed from

$$\delta \text{ (in minutes of arc)} = \frac{l^2}{6RL} \times \frac{180 \times 60}{\pi} = 573\frac{l^2}{RL},$$

and for the point B, where the transition curve joins the circular curve,

$$\delta_B = 573\frac{L}{R}.$$

In using these expressions, l is taken as some convenient chord length, say 25 ft. The first deflection angle will therefore be $\delta_1 = 573 \times (25)^2 \div (RL)$. With the instrument at A and sighted along the tangent AT in fig. 9.25, this angle is laid out and 25 ft. chained along the chord, thus fixing the first point on the curve. The second deflection angle will be $\delta_2 = 573 \times (50)^2 \div (RL) = 4\delta_1$. This angle is laid out from A and a distance of 25 ft. chained from the first point so that the end point of the second chord lies on the line of sight of the instrument. The process is carried on with the other chords until the final deflection angle is $\delta_B = 573L/R$ and the chord distance from the end of the last whole chord length is $L - 25n$, where n is the number of chords already laid out.

When B is reached, the instrument is set up there, sighted at A, and the angle $2\delta_B$ is turned off to the left. This brings the line of sight on to the direction of the tangent to the circular curve, when the latter may be laid out by the usual methods.

If the curve is to be laid out by offsets from the tangent, the offsets are calculated from

$$y = \frac{x^3}{6RL},$$

where y is the offset and x is the distance from A along the tangent. The offset of B is very approximately

$$y_B = \frac{L^2}{6R},$$

where L has been written for x in the last equation.

Example.—Two tangents which intersect at an angle of 37° 46′ are to be connected by a circular curve of 2000 ft. radius with a transition curve at either end. The chainage of the point of intersection is 3436 + 46. Find the chainages of the beginnings and ends of the three curves, and draw up a table of deflection angles and inches of cant for chords of 50 ft. for each transition curve. Assume that the velocity for which the curve is to be designed is 50 m.p.h.

$$\text{Length of each transition curve} = 3{\cdot}155 \times \frac{V^3}{R} = 3{\cdot}155 \times \frac{(50)^3}{2000}$$

$$= 197{\cdot}2 \text{ ft.}$$

$$\text{Shift} = s = \frac{L^2}{24R} = \frac{(197{\cdot}2)^2}{24 \times 2000} = 0{\cdot}8 \text{ ft.}$$

$$\text{Hence tangent distance} = (R + s) \tan \tfrac{1}{2}I + \tfrac{1}{2}L$$

$$= 2000{\cdot}8 \tan 18^\circ\, 53' + 98{\cdot}6$$

$$= 684{\cdot}4 + 98{\cdot}6 = 783{\cdot}0 \text{ ft.}$$

$$\therefore \text{Chainage of A} = (3436 + 46) - (7 + 83) = (3428 + 63).$$

$$\text{Chainage of B} = (3428 + 63) + (1 + 97{\cdot}2) = (3430 + 60{\cdot}2).$$

$$\delta_B = 573 \frac{L}{R} = 56{\cdot}5'.$$

$$\phi = 3\delta_B = 2^\circ\, 49{\cdot}5'.$$

Angle subtended by circular curve

$$= I - 2\phi = 37^\circ\, 46' - 2 \times (2^\circ\, 49{\cdot}5')$$

$$= 32^\circ\, 07'.$$

$$\therefore \text{Length of circular curve} = 2000 \times \frac{\pi \times 32^\circ\, 07'}{180^\circ}$$

$$= 1121{\cdot}1.$$

$\therefore$ Chainage of end of circular curve

$$= (3430 + 60{\cdot}2) + (11 + 21{\cdot}1)$$
$$= 3441 + 81{\cdot}3.$$

Chainage of end of second transition curve

$$= (3441 + 81{\cdot}3) + (1 + 97{\cdot}2)$$
$$= 3443 + 78{\cdot}5.$$

Deflection angles:

$$\delta_1 = \frac{573 \times (50)^2}{2000 \times 197{\cdot}2} = 03{\cdot}63'.$$

$$\delta_2 = 4 \times 03{\cdot}63 = 14{\cdot}5'.$$

$$\delta_3 = 9 \times 03{\cdot}63 = 32{\cdot}7'.$$

$$\delta_B = \left(\frac{197{\cdot}2}{50}\right)^2 \times 03{\cdot}63 = 56{\cdot}5' \text{ (check)}.$$

$$\text{Cant on circular curve} = \frac{3{\cdot}9V^2}{R} = \frac{3{\cdot}9 \times (50)^2}{2000} = 4{\cdot}9 \text{ in.}$$

$$\text{Cant at Point 1} = \frac{50}{197{\cdot}2} \times 4{\cdot}9 = 1{\cdot}24 \text{ in.}$$

$$\text{Cant at Point 2} = 2 \times 1{\cdot}24 = 2{\cdot}48 \text{ in.}$$

$$\text{Cant at Point 3} = 3 \times 1{\cdot}24 = 3{\cdot}72 \text{ in.}$$

$$\text{Cant at B} = 4{\cdot}9 \text{ in.}$$

20. Vertical Curves.

Corresponding to transition curves in a horizontal plane, special curves in a vertical plane are fitted on railways at all places where there is an appreciable change in gradient. These are required to ease the passage from one gradient to another.

The curve ordinarily chosen for a vertical curve is part of a parabola, the length of the curve depending on the algebraic difference between the gradients. If the gradients are expressed as percentages, one commonly used rule is that

$$\text{length of curve in feet} = 100 \times (a - b)/k,$$

where a and b are the gradients as percentages and k is a factor which is taken as 0·1 on summits and 0·5 in sags. Thus, at a summit, if the two gradients meeting are a rise of 1 per cent and a fall of 0·3 per cent,

$(a - b) = 1{\cdot}3$, since the second gradient is in a different direction to the first and is therefore negative with respect to it, and the length of the curve will be $(100 \times 1{\cdot}3)/0{\cdot}1 = 1300$ ft. A length of 650 ft. is then chained in each direction from the point where the gradients meet, and this will give the chainage of the beginning and end of the curve, thus fixing the points A and B in fig. 9.26. The elevation of the mid-point D of AB is now calculated, and from this, and from the known

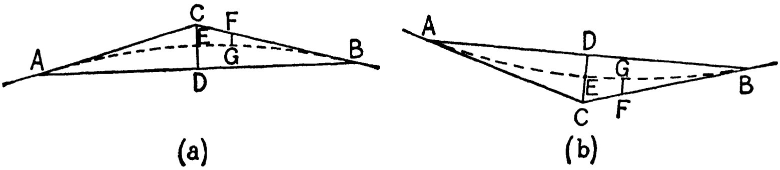


Fig. 9.26

elevation of C, the point of intersection of the gradients, the depth CD can be calculated. The middle of the curve will then be the point E such that CE $= \frac{1}{2}$CD. Now, in a parabola, the ordinates from a tangent are proportional to the squares of the distances from the tangent point, and hence at the point F, the ordinate FG is given by

$$\frac{\mathrm{FG}}{\mathrm{FB}^2} = \frac{\mathrm{CE}}{\mathrm{CB}^2}.$$

$$\therefore \quad \mathrm{FG} = \mathrm{CE}\left(\frac{\mathrm{FB}}{\mathrm{CB}}\right)^2.$$

The elevation of F having been calculated from the position of F on the gradient, the elevation of G can now also be calculated.

21. Transition Curves on Roads.

Transition curves are now commonly used on roads. Here, if there is no cant, the condition for stability is that, where the radius of the curve is R, the centrifugal force Mv^2/gR should not exceed μM, where μ is a coefficient, known as the *coefficient of adhesion*, which depends on the nature of the surface of the road. Assuming $\mu = 0{\cdot}25$ and that R is in feet and V is the velocity in miles per hour, this gives

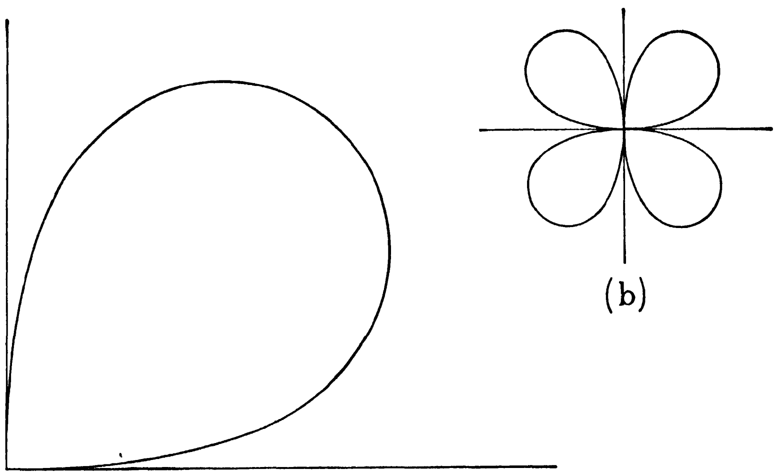
$$R = 0{\cdot}2672V^2.$$

If we also assume Mr. Shortt's criterion of 1 foot per second[3] as a suitable value for a comfortable rate of change of acceleration, we obtain as before for the length L in feet of the transition curve

$$L = \frac{3{\cdot}155V^3}{R},$$

and, substituting in this the above relation between R and V, we have

$$L = 12V \text{ approximately}$$
$$= 23\sqrt{R} \text{ approximately.}$$



(a) (b)

Fig. 9.27

The transition itself may be either a cubic spiral or a cubic parabola, the latter not being suitable if the deflection angle from the point of tangency exceeds 9°, because the curvature then reaches a maximum, after which it starts to decrease. The spiral

$$\theta = \frac{l^2}{2RL} \text{ or } l = \sqrt{2RL\theta}$$

is suitable up to a deflection angle of about 45°, but, for deflection angles greater than that, the lemniscate

$$\rho = C\sqrt{\sin 2\delta},$$

where ρ is the deflection ray or chord for a deflection angle δ and C is a constant, is often used. This curve, fig. 9.27a, is a closed curve in which the relation $\delta = \frac{1}{3}\theta$ is rigorously true, and it repeats itself in each of the four quadrants as in fig. 9.27*b*. Space prohibits a detailed description of it, but details will be found in Vol. I of Clark's *Plane and Geodetic Surveying for Engineers*, or, in more complete form, and with tables for its calculation, in Professor F. G. Royal-Dawson's book *Elements of Curve Design for Road, Railway and Racing Track on National Transition Principles.*

QUESTIONS ON CHAPTER IX

1. Calculate the radius of each of the following circular curves:

   $1\frac{1}{2}°$, 2°, 3°, 4°, 5°.

   Calculate the radii of the last three by the rigid formula, and compare the result with that obtained from the approximate formula.

2. What are the "degrees" of the circular curves of radii 2292 ft.; 1637·2 ft.; 1273·3 ft.?

3. The chainage of the point of intersection of two tangents is 264 + 69 ft., and the deflection angle measured to the right from the forward direction of the first tangent, is 60° 28′. The tangents are to be joined by a circular 3° curve. Calculate the chainages of the tangent points, and tabulate the deflection angles from the first tangent point to give a through chainage with pegs at the end of every 100 ft. of the chainage.

4. What superelevation would you recommend for a 3° curve for an average velocity of 40 miles per hour? Assume that the distance between rail centres is 4′ 11″.

5. A circular curve for a mineral railway is to be set out, with a radius of 1500 ft., connecting two straights which intersect at 150° (i.e. one straight deflects from the other at an angle of 30°). The chainage at the intersection point is 84 + 42. Calculate the chainage at the tangent points, and the tangential angles necessary for setting out the curve with a 100-ft. chain and theodolite. (Inst. C.E., April, 1946.)

6. In setting out a circular railway curve it is found that the curve must pass through a point 50 ft. from the intersection point, and equidistant from the tangents. The chainage of the intersection point is 280 + 80, and the intersection angle (i.e. deflection angle) 28°.

   Calculate the radius of the curve, the chainage at the beginning and end of the curve, and the degree of curvature. (Inst. C.E., October, 1947.)

7. A circular curve of 2250 ft. radius, with cubic parabola transition curves each 180 ft. long, is to connect two straight lengths of railway line which intersect at an angle of 135° (i.e. one straight deflects from the other by 45°). If the chainage of the intersection of the straights is 79 + 42 ft., calculate the chainage of the beginning and end of each part of the circular curve, and the tangential angles for setting out the circular curve with a theodolite and chain. (Inst. C.E., October, 1945.)

8. Calculate the lengths of a spiral transition curve on a railway to join on to a 3° circular curve suitable for a velocity of 60 m.p.h., and calculate the deflection angles necessary for setting out the curve at the ends of 50-ft. chords. Describe how you would set out the circular curve from the end of the transition curve.

9. An up-grade of 1 per cent joins a down-grade of 0·5 per cent at a point whose chainage is 374 + 50 ft., and whose reduced level is 468·26 ft. Calculate the chainages of the beginning and end of a suitable vertical curve to join these gradients, and tabulate the reduced levels of points on the curve at 100-ft. intervals.

10. It is required to range a simple curve which will be tangential to three straight lines YX, PQ and XZ, where PQ is a straight, joining the two intersecting lines YX and XZ. Angles YPQ = 134° 50′; YXZ = 72° 30′; PQZ = 116° 10′ and the distance XP = 5·75 chains.

    Compute the tangent distance from X along the straight YX and the radius of curvature. (Inst. C.E., October, 1953.)

11. An uphill gradient of 1 in a 100 meets a downhill gradient of 0·45 in a 100 at a point where the chainage is 61 + 00 and the reduced level is 126 ft. If the rate of change of gradient is to be 0·18% per 100 ft., perpare a table for setting out a vertical curve at intervals of 100 ft. (Inst. C.E., October, 1956.)

12. Two straights AI and BI meet at I on the far side of a river. On the near side of the river a point E was selected on the straight AI and a point F on the straight BI and the distance from E to F measured and found to be 3·40 chains. The angle AEF was found to be 165° 36′ and the angle BFE 168° 44′. If the radius of a circular curve joining the straights is 20 chains, calculate the distance along the straights from E and F to the tangent points. (Inst. C.E., October, 1952.)

CHAPTER X

GROUND AND AIR PHOTOGRAPHIC SURVEYING

GROUND PHOTOGRAPHIC SURVEYING

The circumstances in which ordinary ground photographic surveying is mostly useful are when small-scale surveys have to be made of very mountainous, or similar, country which in the ordinary way is inaccessible. In such conditions, either ground or air photographic surveys are sometimes the only practicable method. Ordinary ground photographic surveying—usually simply called *photographic surveying*—is not much used in Great Britain, but the principles upon which it is based serve as a useful introduction to some of those involved in air survey methods.

1. Formation of the Image in a Surveying Camera.

The ordinary surveying photo-theodolite, as described in Chap. VIII of *Principles and Use of Surveying Instruments*, consists of a special camera mounted on what is essentially a special form of theodolite. At the back of the camera, immediately in front of the photographic plate, are a vertical and a horizontal hair which intersect at right angles at the centre of the plate, and images of these hairs are formed on the plate when it is exposed and developed. The intersection of the images of the hairs is called the *principal point* and, when the camera is in proper adjustment, this point coincides with the point where the optical axis of the lens meets the plate. The image of the horizontal hair in the plate is called the *horizon line* and that of the vertical hair the *principal line.* A horizontal plane through the horizon line is called the *horizon plane* and a vertical plane through the principal line is called the *principal plane.*

In some cameras the hairs are replaced by *collimating marks*, which show up on the top and bottom and side edges of the photograph. Lines drawn through opposite collimating marks give the horizon and principal lines.

The lens of the camera behaves in a similar manner to the lens of a telescope in that the images of distant points are brought to a focus in a vertical plane at a constant distance, the *focal length* of the lens, behind the latter, and rays passing in any direction through the optical centre of the lens continue as straight lines in their passage through and out of it.

In fig. 10.1, O is the optical centre of the lens and **bpa**, which is at right angles to **p**OP, the optical axis of the lens, is the line of intersection of the photographic plate with a plane containing **p**OP and a distant point A, **p** being the principal point of the plate and O**p** the focal length of the lens. Then the ray AO from A will form an image at the point **a**, where the line AO continued meets the plate. Similarly,

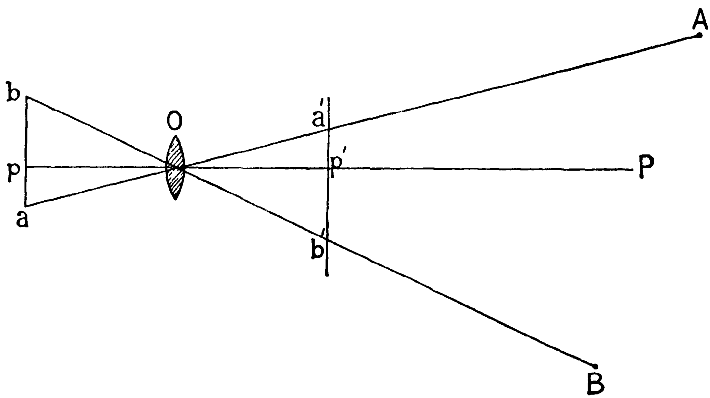


Fig. 10.1

any point B lying in the plane AOP will form an image at **b**, where BO**b** is a straight line, and other points in different planes passing through **p**OP will yield similar images. Hence, we see that the picture on the plate is a reversed and upside-down image of the scene being photographed, situated behind the lens, and, moreover, the angle which the ray from O to the image of a point makes with the optical axis is equal to the angle which the ray to the object makes with the same axis. In addition, the image lies in the plane containing the object and the optical axis.

In considering the theory of photographic surveying it is useful to replace the plane of the plate by a plane parallel to it and at a distance equal to the focal length of the lens in *front* of the latter. This plane is called the *picture trace*. Let **a′** in fig. 10.1 be the point where the ray AO**a** intersects this plane, and **p′** and **b′** the points where OP and the ray BO**b** intersect it. It will then be seen that on it we have a complete reproduction of the image on the plate, but direct and right side up. This reproduction is also what an observer at O would see through a transparent sheet of glass held in the position of the plane, or what would be seen from O on a print held right side up at **a′p′b′**.

2. Fixing Points with the Photo-theodolite by Intersection.

Suppose that O and O′ in fig. 10.2 are the plotted positions of two fixed camera stations at which photographs were taken, O**p** and O′**p′** the plotted positions of the optical axis, and **apb** and **a′p′b′** the plotted positions of the picture traces of the horizon lines. Then **a**

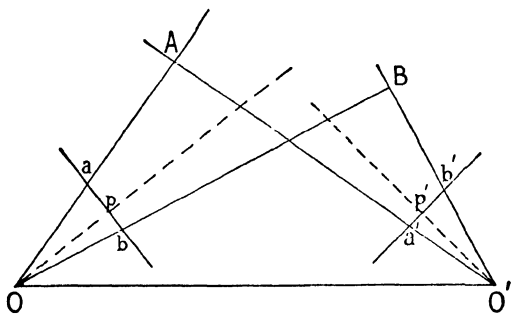


Fig. 10.2

will represent the position on the picture trace of the horizontal projection on the horizon line of the image of the point A in the photograph. Similarly, **b** will represent the position on the picture trace of the horizontal projection on the horizon line of the image of the point B in the photograph. In the photograph taken from O′, the horizontal projections on the horizon line of the images of points A and B will be represented by **a′** and **b′** respectively. The two photographs are shown in fig. 10.3.

It will be seen from this and from fig. 10.2 that, if we can plot on the plan O**p** and O′**p′** and the picture traces passing through these points, we can plot the position of the point A by scaling off the distances **pa** and **p′a′** on the traces and drawing rays through **a** and **a′** from O and O′ respectively to intersect at A. As the angles **a**OO′, **b**OO′, **a′**O′O, **b′**O′O, etc., in fig. 10.2, are independent of the scale of the map, the distances O**p** and O′**p′** may both be drawn full-size on the plan, and in that case the plotted lengths **pa**, **pb**, **p′a′**, **p′b′**, etc., are the corresponding lengths measured direct from the photographs, in the same units as f, the focal length of the lens. In this way, we can plot the positions of a number of points which appear and can be clearly identified on both photographs.

Again, the elevations of different points can be determined by scaling the ordinates of the images above or below the horizon line,

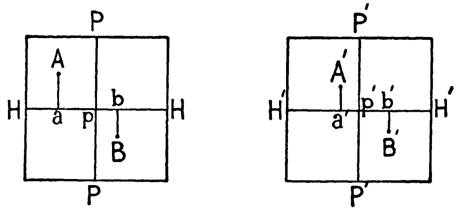


Fig. 10.3

and then scaling the distance of the point on the plan, for, if α is the angle of elevation of A and d is the length of Oa in fig. 10.2, we have, from fig. 10.3,

$$\tan \alpha = \frac{\text{Aa}}{d},$$

and height of point A above the level of the horizon line is

$$h = \frac{\text{Aa}}{d} \times \text{distance OA}.$$

3. Orienting the Picture Traces.

The orientation of Op and hence of the picture trace will be known if the angle O′Op has been measured on the horizontal circle of the instrument or, the bearing of OO′ being known and plotted, if the bearing of Op has been found by measuring the angle through which the instrument has been turned from its position where the telescope was directed to a point of known bearing.

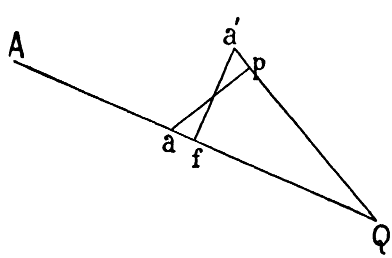


Fig. 10.4

Alternatively, the orientation of the picture trace can be found if one or more points, whose positions are known and are plotted on the plan, appear on the photograph. If only one point appears, join Q, the plotted position of O on the plan, to A, the plotted position of the fixed point, and on QA mark off Qf, fig. 10.4, equal in length to the focal length of the lens. From **f** draw the perpendicular **fa′** equal in length to the abscissa of the known point A on the photograph (**pa** in fig. 10.3). Join Q**a′**. On this line mark off Q**p** = Q**f** = f, and through **p** draw **pa** = **fa′** perpendicular to Q**a′**. Then it is obvious that **pa** is part of the picture trace, and Q**pa′** is the trace of the principal plane.

If more than one point of known position appears on the photograph, orientation can be obtained very simply by marking off on the straight edge of a sheet of paper the abscissæ **pa**, **pb**, **pc**, etc., as measured direct from the photograph, and then laying down this sheet so that the points **a**, **b**, **c**, etc., coincide with the lines QA, QB, QC joining the plotted position of O to the plotted positions of the points A, B, C (fig. 10.5). The edge **abc** of the paper will then represent the position of the picture trace, while the line Q**p** will give the direction of the trace of the principal plane. This line should be perpendicular to **abc** and Q**p** should be equal to f, the focal length of the lens. If there are only two fixed points on the photograph, the sheet edge must be kept touching a circle drawn with Q as centre and radius equal to f and moved about until the points **a** and **b** on the sheet coincide with the lines QA and QB on the plan.

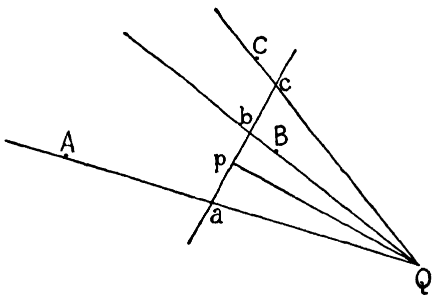


Fig. 10.5

AIR SURVEY

In recent years the method of constructing maps and plans from photographs taken from an aeroplane in flight has made great progress and has become of considerable importance in both military and civil work. The principal advantages of the method are (1) the speed with which (given favourable weather) the field work may be completed; (2) the photographs form a permanent record of the ground; and (3) the photographs are often of considerable use for purposes other than pure survey. The principal disadvantages are (1) the relatively long time required for plotting the results; (2) the expense of the initial equipment which make it essential for the work to be handled by an organization specializing in this class of work; and (3) the dependence of the work on the weather, since in many countries the periods during which satisfactory air photography is possible are of very limited duration.

In Great Britain, and in most parts of the Empire, the method usually used is vertical photography, that is, the photographs are taken with the plate in a more or less horizontal position and with the lens pointing vertically below. The plane flies in a series of parallel lines so that the photographs overlap each other by about 30 to 60 per cent on each side, and exposures are arranged so that there is an overlap of about 60 per cent in the direction of flight. This means that, provided there are no gaps caused by errors of navigation, each piece of ground is photographed at least four times.

There are a number of reasons for providing ample overlaps. The main ones are:

1. To ensure the fitting together of different photographs.
2 To overcome the distortions which occur principally at the outer edges of photographs.
3. To facilitate interpretation so that objects may be viewed from different angles.
4. To provide pairs of photographs for stereoscopic examination.
5. To provide an alternative picture if one is defective by reason of excessive tilt, cloud shadows, etc.
6. To ensure that there are no gaps in the photography.

One of the most difficult operations in air photography is the navigation of the aircraft so that it travels on a reasonably even keel on the predetermined paths without leaving gaps in the photography or insufficient overlaps. During the last few years the application of radar has provided a means of control which makes the navigation very much easier and has lessened the possibilities of awkward gaps in the side overlaps.

Work is much simplified and is more accurate if tilt, or the deviation of the axis of the lens from the vertical, or of the plate from the horizontal, is small. In addition, as the scale of photography depends on the height of the camera at the instant of exposure, variations in the flying height must be reduced to a minimum.

Air photography in itself is not sufficient for mapping purposes, and it is necessary to provide a certain number of *ground control points* which will show on the photographs and whose positions are fixed; if contours are required, the elevations of some at least of these points must also be fixed or known. These points must, of course, be fixed by ground survey methods.

4. Relation between Scale of Photograph and Height of Camera.

Let O in fig. 10.6 be the position of the lens of the camera at height h above the level surface AB, and let **apb** be the position of the plate, supposed horizontal, where **Op** is perpendicular to **apb** and equal to f, the focal length of the lens. A ray from A to O will intersect the plate at **a** on AO produced, and a ray from B to O will intersect the

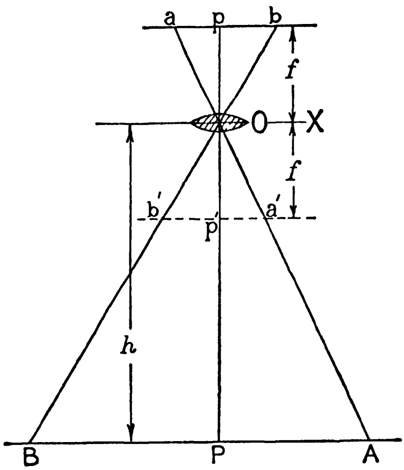


Fig. 10.6

plate at **b** on BO produced. These points will therefore be the images of A and B, and their positions on the picture trace will be at **a′** and **b′**, where **b′p′a′** is parallel to **apb** at distance O**p′** $=f$ below the lens. From the figure,

$$\frac{\mathbf{ab}}{\mathrm{AB}} = \frac{\mathrm{O}\mathbf{p}}{\mathrm{OP}} = \frac{f}{h},$$

where P is the foot of the perpendicular from O on the plane AB.

But **ab** is the distance between the images of A and B on the plate, and the true distance between these points is AB. Hence

$$\text{scale of photograph} = \frac{\mathbf{ab}}{\mathrm{AB}} = \frac{f}{h},$$

and

$$\mathbf{ab} = \mathrm{AB}\,\frac{f}{h},$$

f and h, of course, being expressed in the same units and **ab** in the same units as AB.

It follows from this expression that the scale of a photograph will vary at different points if the ground level varies in height between the limits of the photograph. The point P vertically below O is called the *ground plumb point* or *ground nadir point*, and its image **p** on the plate is called the *plate plumb point* or *plate nadir point*. When the optical axis of the lens is vertical as in the figure, the plate plumb point and the principal point coincide.

5. Displacement of Detail due to Elevation above Datum.

Let PAC in fig. 10.7 be the horizontal datum plane to which the map is drawn, and let B be a point on the ground whose elevation above datum is *e*. From B draw BA perpendicular to PAC. Then A is the point where B would be shown on the map, and BA $= e$. Through B and A draw lines to O, the position of the lens, intersecting the picture trace at **b′** and **a′**. Then **a′** is the position of the point A on the photographic print and **b′** the position where the point B actually appears in the photograph. Hence the distortion is **a′b′**, and, from similar triangles, it follows that

$$\frac{a'b'}{AC} = \frac{f}{h}.$$

But

$$\frac{AC}{e} = \frac{PC}{h} = \frac{p'b'}{f},$$

$$\therefore\ a'b' = \frac{e}{h} \cdot p'b',$$

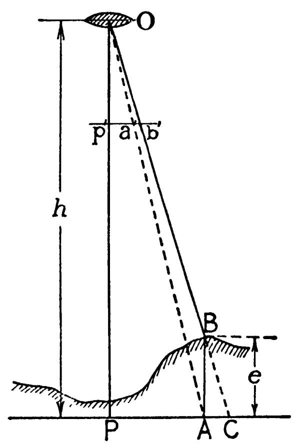


Fig. 10.7

so that the displacement is directly proportional to the distance of the image of the point from the plate plumb point of the photograph, and to the elevation of the point above datum, and is inversely proportional to the height of the lens. For points whose elevation is above datum, the displacement is radial outwards from the plate point of the photograph, and for points whose elevation is below datum it is radial towards the plate plumb point.

6. Displacement due to Tilt.

In fig. 10.8*a* the optical axis O**p** of the lens is tilted through an angle $t =$ **n′**O**p** from the vertical O**n′**. As a result, the axis CD of the picture trace will be tilted from the horizontal AB through an angle BiD $= t$ about an *axis of tilt* through **i** perpendicular to the plane containing On′ and Op. Op, the perpendicular from O on CD, meets CD at **p**, the principal point on the plate, and the vertical On meets the horizontal plane through AB in **n**′, where On′ = Op $= f$, and CD in the plate plumb point **n**. If the plate were horizontal, the image of a point lying on the ray OD would meet it at B, but, as it is, the image of this point will be at D. The displacement of the image will then be iD — iB. Similarly, the displacement at C will be iC — iA, and in fig. 10.8*b* a rectangle A′AA″B″BB′ in the horizontal plane will become a trapezium C′CC″D″DD′ on the plate, where iC and iD are equal to iC and iD in fig. 10.8*a* respectively.

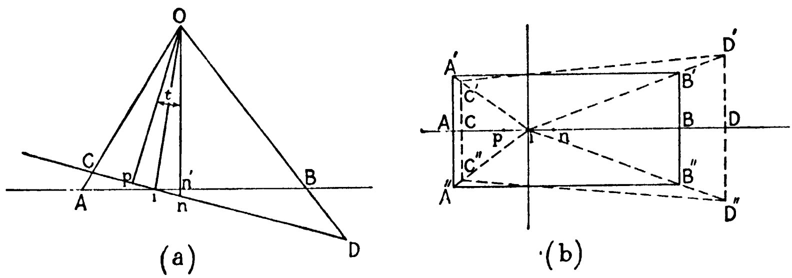


Fig. 10.8

The point **i** in fig. 10.8*b* is called the *isocentre* (or *centre of distortion*). It is the point where a plane through the plumb line from O, the optical centre of the lens, and the optical axis meets the axis of tilt, and it can be shown that the displacement at any point on the photograph is in a direction radial from or towards the isocentre, with zero displacement along the axis of tilt. Thus, in fig. 10.8*b*, the points A′ and B′ are displaced to C′ and D′ respectively along the lines A′**i** and iB′.

Displacements due to tilt therefore differ from displacements due to height of ground above datum, as in this last case the displacements are radial from the plate plumb point. When tilts are very small, however, say two or three degrees, the principal point, isocentre and plate plumb point practically coincide, and it may be assumed that the displacements due to elevation and tilt are both radial from the principal point. In such circumstances, angles measured from the principal point on the photograph will be equal to the corresponding angles on the ground. This is the basis of a simple method of plotting from air photographs, known as *Arundel* or *radial-line method*, which is now extensively used, and which avoids the use of elaborate and expensive plotting machines.

7. Plotting from Air Photographs by the Radial-line Method.

In plotting by the radial-line method, at least three points whose positions have been fixed on the ground must appear on the first two photographs of a strip. The first step is to determine the scale of photography. This is done by measuring the distance between two points on the photograph and comparing it with the corresponding distance on the ground. For this purpose, a line should be taken in which the end points lie nearly equidistant on either side of the principal point of the photograph. A skeleton map is then drawn on the scale of the photograph and all the fixed ground points plotted on it.

In order to orient and fix the position of the first photograph on the map, radial lines are drawn from the principal point to images of three fixed points and a tracing of these lines made on tracing cloth or, better still, on a sheet of transparent film base. This tracing is transferred to the skeleton map and manipulated until the three rays pass through the corresponding fixed points on the map. The point where the rays meet is then pricked through on to the map and this gives the map position of the principal point of the photograph as well as the orientation of the latter, the process being a form of resection.

Before transferring the tracing to the map, and while the tracing paper is still on the photograph, rays are drawn from the principal point to any other fixed points and also to any clearly defined points of detail which show on the next photograph. These rays are transferred to the map by pricking through after the trace has been oriented on the latter and are used to fix other points by intersection.

The second photograph is then treated in the same manner as the first and, after orientation, rays to fixed points and sharply defined points of detail appearing on both photographs are transferred to the map. Intersections of rays on the map from the two principal points to the same point of detail will give a fixation by intersection for that point, and rays to fixed points will check the plotting or strengthen fixes. In this manner, points fixed by intersection may be used as *picture control points* to fix points on adjoining photographs in which there are insufficient control points fixed by ground methods. After sufficient control points have been plotted on the map, other detail can be transferred to it from the photographs, using the control points to adjust for varying scale, etc. The best results will be obtained when the picture control points are fixed by intersections from the principal points of three successive overlapping photographs.

This method, which is a form of graphical triangulation, may be used to bridge quite extensive gaps, where there are few, if any, fixed control points, by providing sufficient picture control points to serve as additional fixed points from which to fix new control points, or on which to hang detail. When the gap begins at one set of fixed points and then ends at another set some distance away, because of small errors in plotting, etc., the positions of the end series of points will not coincide exactly with their accepted positions on the map, so that some adjustment of the intermediate points becomes necessary. This may be done somewhat on the lines of the method of adjusting compass traverses described on pages 139–140 but the more satisfactory and easier method is to use the *slotted templet* method described below.

If difficulty is experienced in plotting, and checks are not satisfactory, the trouble may be due to varying heights of photography or of the ground or to too much tilt. In this case, *rectified prints* may be obtained by using special rectifying apparatus in which corrections for tilt may be made by tilting the negative or the board on which the printing paper is fastened, and corrections for scale may then be made by small variations in the distance between negative and board.

8. The Slotted Templet Method.

This method of establishing and adjusting auxiliary control points is a mechanical development of the radial line plot. In this case, each photograph is placed, picture side up, on a sheet of cardboard and the principal point and ground control points, and points to be used as picture control points, are pricked through to the cardboard. Radial lines are then drawn from the point representing the principal point to all these ground and picture control points, and the cardboard is next put on a special punch which punches a circular hole at the principal point and cuts slots along the lines radiating from this point. These cardboard sheets, or *templets*, are used in conjunction with special studs (fig. 10.9), which consist of a small cylinder fitted on to a metal disc, the whole being bored through its centre with a vertical hole to take a needle or pin. The outside diameter of the cylinder is only very slightly less than the width of a radial slot so that the stud can move freely, but without play, in the direction of the slot.

The fixed ground control points are plotted on a large sheet of drawing paper representing the map, and needles stuck at these points, a stud being slipped down over each needle. Studs are then fitted in the hole representing the principal point of the first templet and in the slots radiating to the picture control points, and the templet is laid down so that the slots to the ground control points fit over the corresponding studs on the map. Provided there are at least three ground control points on the first photograph, this will fix the position and orientation of the first templet. The second templet is similarly fixed and orientated and studs in the first templet relating to picture control points are moved radially to fall within the slots relating to the same points on the second templet. Since each stud will now be at the point

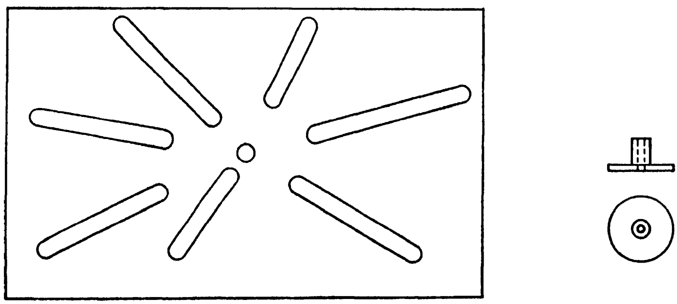


Fig. 10.9

of intersection of the two rays which fix the point concerned, this fixes the positions of the picture control points common to the two templets. Work is thus carried on using such ground control points, and any picture control points fixed from the previous templets, as are needed to fix and orientate each new templet in turn. In this way, a system of triangulation may be carried for quite long distances over photographs covering ground in which ground control points are sparse or non-existent. Finally, if the work ends in country which is reasonably supplied with ground control points, the triangulation can be fitted on to them. The positions of these points as brought forward by the air photographs will not agree exactly with their plotted positions, there being the usual small discrepancies due to small and uncontrollable errors in the work. However, by moving the assembly as a whole slightly, the last templet can be adjusted to fit on to the studs over the end control points, the intermediate studs being adjusted proportionately. The positions of all the intermediate picture control points can then be transferred to the map by pricking through the holes in the studs.

If the assembly cannot be fitted between the terminal points without some bending or distortion, the indications are that there is an error somewhere, or tilt for which insufficient adjustment has been made.

An alternative to the cardboard templets just described is a set of *spider templets*. These consist of narrow metal strips with slots cut in them through which studs of the kind already described can be passed to move freely along the slot. A stud with a special threaded boss is used to mark the principal point of each photograph, and one end of each strip has two or three holes bored in it to fit exactly over the boss. The different strips are now laid down to radiate from the principal point along the directions of the rays to the ground and picture control points, and the whole is then firmly secured in position by a lock nut screwed down tightly over the threaded boss to hold all the strips in their proper relative positions. In this way are formed templets equivalent to slotted templets and used just as they are used. The advantage of the method is that no special punch for punching holes or slots is needed, and the labour of setting and cutting the templets in the punch is saved.

9. Principles of Binocular Vision and Stereoscopic Fusion.

In binocular vision the same object is seen as a whole with both eyes at the same time and, for this to be possible, the lines of sight from the object to each eye must be inclined slightly to one another in order that the images formed on the individual retinas should appear to coincide in space. For very distant objects, the lines of sight are parallel, but for nearer objects they make an appreciable angle with one another, the muscles which operate the eyes rotating them very slightly if necessary, without conscious effort on our part, to form the single image. It is this relative convergence of the lines of sight on a single object which enables us to judge distances. In the case of solid objects, appearance in depth is given by the slightly different views obtained by the two eyes, different parts of the object requiring different degrees of accommodation by the eyes to form single combined images.

Fig. 10.10

In fig. 10.10 are shown four dots **a**, $\mathbf{a}_1$, **b** and $\mathbf{b}_1$. Look at some fairly distant object and, keeping the eyes at the same focus, interpose the page, with the paper between the eyes and the object, about 15 in. from the former. At first two pairs of dots will be seen by each eye, but after a time the left-eye images of **a** and **b** can be made to coincide with the right-eye images of a_1 and b_1 respectively. When this happens, the combined image of **a** and a_1 will appear to stand out in relief above the combined image of **b** and b_1. This is an example of *stereoscopic fusion.* When fusion takes place, other images of the dots will be seen to the left and right of the fused images. These *ghost* or *satellite* images should be ignored, and attention concentrated on the fused images in the centre.

If difficulty is experienced at first in obtaining fusion, it may help if a piece of card is placed half-way between the pairs of dots with its plane perpendicular to the plane of the paper, so as to prevent the left and right eyes from seeing the right- and left-hand dots respectively.

The theory of this phenomenon is illustrated in fig. 10.11. Here the combined images of **a** and a_1, as seen by the eyes at E and E_1, appear to be at **m** and the combined images of **b** and b_1 at **n**. The distance **mn** is called the *stereoscopic depth,* and is a measure of the amount by which the positions of the fused images appear to stand out in depth relative to one another. The angles ϕ_1 and ϕ_2 at **m** and **n** are called the *parallax* or *parallactic* angles, and the stereoscopic depth depends on the difference between these angles, i.e. on the sum of the angles α and β at E and E_1, and also on the distance EE_1—the *eye base*—between the eyes, the smaller $(\alpha + \beta)$ and the shorter the eye base the greater being the stereoscopic depth. At the same time, as the eye cannot resolve angles less than about 20″ of arc, the impression of stereoscopic depth is lost for values of $\alpha + \beta$, ϕ_1 or ϕ_2 less than about 20″, the exact amount differing slightly with different observers.

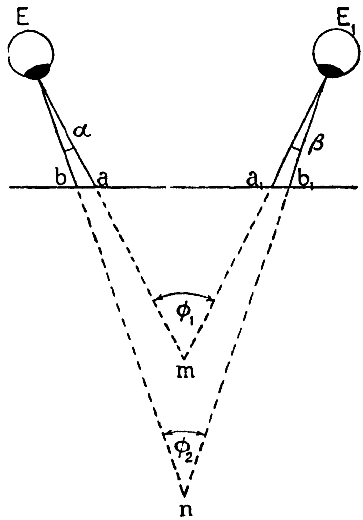


Fig. 10.11

Now suppose that **a** and **b** are two points on a photograph lying to the left of the observer, and a_1 and b_1 are the same points as they appear on an overlapping photograph lying to the right of the observer. It will then be seen that, when the images of the points are fused, the one point will appear to stand out in relief with respect to the other.

Stereoscopic fusion is best obtained by the use of a *stereoscope*, of which there are two main kinds. The principal function of the stereoscope is to accommodate a wide separation of the points in the left- and right-hand photographs to the fixed length of the eye base. In the lens or prism stereoscope (fig. 10.12*a*) this is done by two lenses or narrow small-angled prisms. In the second type of stereoscope (fig. 10.12*b*), mirrors are used to bring the images together. From

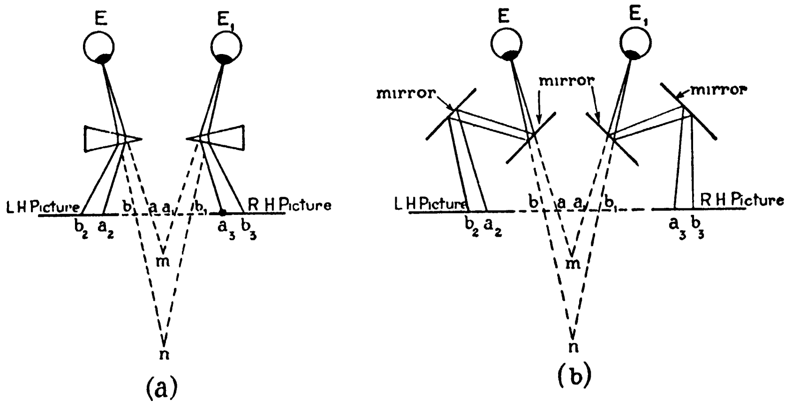


Fig. 10.12

these diagrams, in which the paths of the rays are clearly indicated, it will be seen that in both cases the effective separation between the same points on the photographs has been widened, for the same eye base, from $\mathbf{aa_1}$ to $\mathbf{a_2a_3}$, and from $\mathbf{bb_1}$ to $\mathbf{b_2b_3}$.

The lens or prism stereoscope has the disadvantage that the two photographs must be placed rather close together, but the mirror stereoscope allows them to be more widely separated and a somewhat wider photograph to be used. A slightly enlarged image of each picture can be obtained directly by means of the refracting stereoscope when suitable lenses are used instead of small-angled prisms. The image from the mirror stereoscope is of natural size, but magnifiers placed between the eye and the inner mirrors will yield magnified images.

10. Application of Stereoscopic Principles to Overlapping Photographs for determining Differences of Elevation.

In fig. 10.13, L and R are the picture traces of two overlapping vertical photographs taken with the lens in the positions O_1 and O_2, both positions being at the same height H above datum. The images of the point A, of elevation h above datum, are $\mathbf{a}_1$ in photograph L and $\mathbf{a}_2$ in photograph R. $\mathbf{p}_1$ and $\mathbf{p}_2$ are the principal points of the photographs, which in this case coincide with the plate plumb points, and P_1 and P_2 are the ground plumb points. The direction of flight

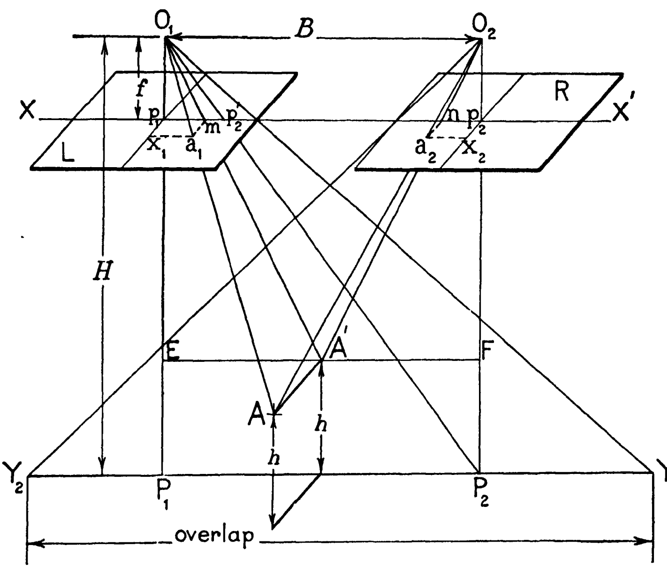


Fig. 10.13

is horizontal and in the direction $\mathbf{p}_1\mathbf{p}_2$ or P_1P_2, and $\mathbf{p}'_2$ is the image of P_2 in photograph L. The distance $O_1O_2 = B$ is the *air base*, which here is horizontal. The amount of overlap is given by the lines O_1Y_1 and O_2Y_2 which pass through the adjoining edges of the photographs.

Through O_1 and O_2 draw the vertical plane $O_1O_2P_2P_1$, and from A draw AA′ perpendicular to this plane. The datum plane is the horizontal plane through P_1P_2. Also, if $\mathbf{m}$ and $\mathbf{n}$ are the intercepts on the plate principal lines of lines joining O_1 and O_2 to A′, the x co-ordinates of $\mathbf{a}_1$ and $\mathbf{a}_2$ are $x_1 = \mathbf{p}_1\mathbf{m}$ and $x_2 = \mathbf{p}_2\mathbf{n}$. Then from the figure,

$$\frac{x_1}{f} = \frac{\text{EA}'}{(H-h)}; \quad \frac{x_2}{f} = \frac{\text{A}'\text{F}}{(H-h)}.$$

$$\therefore\ x_1 + x_2 = \frac{f(\text{EA}' + \text{A}'\text{F})}{(H-h)} = \frac{f\text{B}}{(H-h)}.$$

But

$$\frac{p_1p_2'}{f} = \frac{B}{H}.$$

$$\therefore\ P = x_1 + x_2 = \frac{p_1p_2'H}{(H - h)} = \frac{bH}{(H - h)},$$

where $p_1p_2' = b$.

The quantity $x_1 + x_2$, which in generalized form is written $P = x_1 - x_2$ because of the negative sign of x_2, is called the *stereoscopic parallax*, or simply the *parallax*, of the point A, and it is given by the algebraic difference of the x co-ordinates of the images of A in the two photographs. It follows that, if P and b could be measured and H were known, we could determine h. In practice, the full stereoscopic parallax is not very easy to measure, and it is much simpler to measure differences of parallax between points. By differentiating the previous expression with respect to h, we get

$$dP = \frac{bH}{(H - h)^2}\,dh$$

or

$$dh = \frac{dP(H - h)^2}{bH}.$$

It should be noted that this formula only holds for small values of dh and dP. For larger values of dh and dP it is easy to show that

$$\Delta h = \frac{\Delta P(H - h)^2}{(H - h)\Delta P + bH}$$

and, when h is zero and Δh is measured from the datum plane,

$$\Delta h = \frac{\Delta P\,.\,H}{(\Delta P + b)}.$$

The differences of stereoscopic parallax can be measured on the photographs by special instruments provided with micrometers. The quantity b can be measured in the photographs, and H can be obtained from the altimeter readings recorded on the photographs. Hence, by measuring differences of parallax, we can obtain differences of elevation, and, working from points of known elevation, we can determine the elevations of other points.

Different types of instruments for measuring differences of parallax can be obtained, and these vary from very simple instruments to others of some complexity. In general, they consist of two parts which may be separate or may be combined to form a single instrument. The one part consists of the actual measuring apparatus and the other of a stereoscope. Even when the two parts are not combined in the one instrument, the measuring device is generally used in conjunction with some form of simple stereoscope.

The measuring instrument, or measuring part of the instrument, consists of a micrometer screw and two adjustable marks mounted on transparent material, such as glass or lucite, which can be fused under the stereoscope to form a *floating mark*. After the two photographs have been properly set and oriented, a relief model is seen in the stereoscope. The marks are first fused and, after having been set in contact with the image of one of the points in the relief model from which the difference in elevation is to be measured, the fused image is brought into contact with the fused image of a second point. The difference in parallax can then be read on the micrometer screw to which the adjustable marks are attached. This screw has a range of anything from about 7 mm. to 25 mm. or more, and readings may be taken to 0·01 mm. When the measuring device is separate from the stereoscope, it is called a *parallax* or *micrometer bar*, but, when the measuring device is combined with a stereoscope and special stages for holding the photographs to form a combined instrument, the latter is usually called a *stereo-comparator*.

11. Stereoscopic Plotting Equipment.

A number of different types of instruments, some of them very complicated and expensive, have been devised for direct plotting from stereoscopic pairs of photographs. In these, the photographs can be set and adjusted in such a way as to remove the effects of height and tilt, and a stereoscopic image is obtained in which the ground appears to stand out in its correct relief. A floating mark can be adjusted to be brought into contact with any given point on the three-dimensional image, and this mark is made to operate a pencil, so that, as the mark moves, the pencil also moves and can be made to trace the movement on a plan. Thus, by setting the floating mark to coincide with a point whose elevation is known, contours can be traced out on the plan. There are a number of instruments on the market of this type, among which may be mentioned the Swiss Wild Autograph A5 and the more recently developed British instrument designed by Professor E. H. Thompson and manufactured by Messrs. Hilger & Watts, Ltd. These instruments can be used for mapping on almost any scale, large or small, and are capable of very accurate work, even on scales such as 1/200, but they are somewhat complicated and costly.

A much simpler apparatus, particularly useful for mapping on moderate and small scales, has been developed under the name of *Multiplex*, and has been much used in this country and in the United States during and since the Second World War. This consists of a number of small projectors mounted on a horizontal beam. Each projector has a number of adjustable movements so that each photograph, in the form of a considerably reduced transparent positive of the original called a *diapositive*, can be adjusted in such a manner that it occupies the same position relative to adjoining diapositives and to the drawing paper as the original photograph occupied relative to the other photographs and to the ground at the moment of photography. Transparent green and red filters are put in with alternate diapositives so that overlapping green and red images are formed below the projectors a short distance above the drawing paper. These images are viewed through spectacles in which one glass is coloured green and the other red. A small adjustable stand with a flat circular table-top of about 2 in. diameter can be moved over the drawing paper, and, when the image received on the table is viewed through the coloured spectacles, the ground is seen standing out in stereoscopic relief. This stand has a very small illuminated hole in the centre of the disc to serve as a floating mark, and a pencil or pricker immediately below the hole can be pressed to form a mark on the paper. By lowering or raising the table slightly, the operator can make the illuminated hole coincide with the image of any point on the ground image seen in space, and, by pressing the pricker, he can register the horizontal position of the point on the plan. After the illuminated hole has first been set to coincide with a point of known elevation, the elevation of any other point can be found by reading on a vertical scale attached to the stand the amount by which the hole has been raised or lowered when it is brought into contact with the image of the second point. In this way, the detail can be plotted direct and a good contoured map drawn with little difficulty.

The diapositives used in the ordinary Multiplex apparatus are reductions from the original negatives to about $1\frac{3}{4} \times 2\frac{1}{4}$ in. and accuracy is lost through this reduction. The Kelsh Plotter operates on much the same principles as the Multiplex but uses diapositives of the same size as the negatives of the taking camera, and this, combined with other features such as concentrating the illumination on the part of the photograph being examined instead of over the whole of it, results in improved accuracy. Like the ordinary Multiplex, the Kelsh Plotter operates with transmitted instead of reflected light, which ensures a brighter image.

12. Minimum Number of Photographs Required to Cover a Given Area, and Time Interval between Successive Exposures.

Let the width of a photograph measured in the direction of flight be w, the depth t, and let there be specified a 60% overlap in the direction of flight and a 30% overlap laterally. Let $a_1b_1c_1d_1a_1$ (fig. 10.14) be the first photograph in a strip, O_1 its principal point and e_1 and f_1 the points where the line of flight through O_1 intersects its rear and

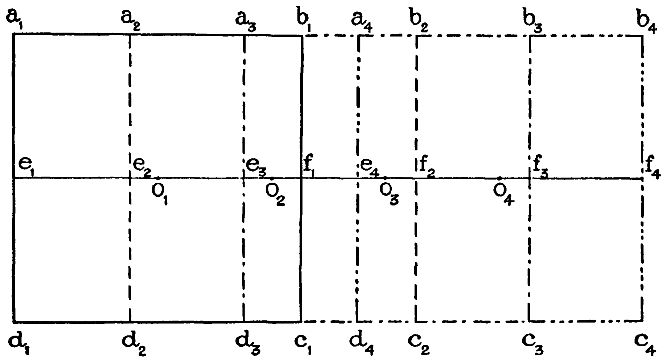


Fig. 10.14

front edges, and let $a_2b_2c_2d_2a_2$ be the second photograph in the strip, O_2 its principal point and e_2 and f_2 the points where the line of flight through O_1 and O_2 intersects its rear and front edges. Then the overlap between the two photographs is the rectangle $a_2b_1c_1d_2a_2$ and, since this is to be 60% in the direction of the line of flight, $a_2b_1 = e_2f_1 = 0{\cdot}6w$. But the distance $O_1f_1 = 0{\cdot}5w$. Therefore the distance e_2O_1 is $0{\cdot}1w$. Similarly, the distance e_2O_2 is $0{\cdot}5w$. Hence, the distance $O_1O_2 = 0{\cdot}4w$.

Again, let $a_3b_3c_3d_3a_3$ be the third photograph, O_3 its principal point and e_3 and f_3 the points of intersection of the line of flight $O_1O_2O_3$ with its rear and front edges. Then the overlap on the second photograph is $a_3b_2c_2d_3a_3$ and e_3f_2 is $0{\cdot}6w$. Distance e_3O_3 is $0{\cdot}5w$ and distance O_3f_2 is $0{\cdot}6w - 0{\cdot}5w = 0{\cdot}1w$. Similarly, distance e_3O_2 is $0{\cdot}1w$ and thus the distance O_2O_3 is $0{\cdot}4w$. Thus it will be seen that

$$\text{Distance between principal points} = 0{\cdot}4w.$$

More generally, if the overlap is $r \times w$, the distance between the centres or principal points of successive photographs is $w(1 - r)$.

As an example, suppose that flying is at 15,000 ft., the focal length of the lens is 10 in., that each plate measures 9 × 9 in. and that the length of the rectangle on the ground to be covered is 60 miles. How many photographs are required to the strip, the stipulated overlap being 60%?

Then ground covered by each photograph in the direction of flight

$$= \frac{9 \times 15{,}000}{10} = 13{,}500 \text{ ft.}$$

$$\text{Distance between successive photographs} = 0{\cdot}4 \times 13{,}500 = 5400 \text{ ft.}$$

$$\text{Number of photographs} = 60 \times 5280/5400 = 59,$$

and, allowing for 2 extra photographs to provide for sufficient overlaps at the ends, say 61 photographs per strip.

Again, to get the number of strips, a similar argument to the last will show that the distance between the centres of parallel strips to secure a 30% overlap will be $0{\cdot}7t$. Hence, with flying at 15,000 ft. with a focal length of 10 in. and with 9 × 9 in. plates, the distance between strips will be 0·7 × 13,500 ft. Then, if the depth of the area to be covered is 45 miles, the number of strips will be 45 × 5280/9450 = 25·2, say 27 after allowing for sufficient coverage at the edges of the area covered. Thus, the minimum number of photographs needed to cover the area of 60 × 45 miles will be 61 × 27 = 1647.

To work out the time interval between successive exposures, suppose that the aircraft is travelling at 200 miles per hour. Then it will travel 5400 ft., the distance between successive photographs, in (5400 × 60 × 60)/(200 × 5280) = 18·41 sec., the required time interval, or, if the interval between exposures were to be 20 sec., the speed of the aircraft would have to be (18·41 × 200)/20 = 184·1 miles per hour.

QUESTIONS ON CHAPTER X

1. The angle at a camera station O, between two points A and B, was measured with a theodolite, and its value was found to be 40°. A photograph was then taken, and the abscissæ a and b of the images of A and B measured on the horizon line from the principal point P were found to be Pa = 1·607 in. and Pb = 2·798 in. respectively, a and b being on opposite sides of P. What was the focal length of the lens?

2. State briefly what are considered to be the advantages to the civil engineer of mapping from air photographs, and indicate its limitations. (Inst. C.E., April, 1948.)

3. In a photo-theodolite survey, a factory chimney is shown on photographs taken from two stations C and D, C being 1200 ft. due west of D. In each case the instrument was sighted on a third station E, north of the line CD; the angle ECD was 54°, and EDC was 45° 30′. The chimney appeared 1·45 in. to the left of the vertical line on the print obtained from C, and 1·66 in. to the right of the vertical line on the print from D. Find the horizontal distance of the chimney from E, and its direction from that station. The focal length of the camera was 6 inches. (Inst. C.E., April, 1946.)

4. Two consecutive air photographs were taken with a 14-in. focal length camera at a height of 20,000 ft. The overlap was exactly one-third, and the prints were 9 in. by 9 in. The height was constant for both exposures; the aircraft flew on an even keel with no drift; and the ground, which was approximately 4000 ft. above sea-level, was almost flat.

   Determine the scale of the photograph and the length of the air base. How would you determine these factors from a map of the area if the flying height were not known?

   Describe briefly the advantages and disadvantages of short and long focal length cameras in air survey. (Inst. C.E., April, 1947.)

5. (*a*) What do you understand by "parallax measurements" in connection with air survey? Assume a pair of overlapping photographs to be taken with the camera axis vertical, and at a constant height; derive an expression for the parallax of a point R ft. above the datum level, taking the focal length as f in., the "air base" as B ft., and the flying height of the aircraft above datum as H ft.

   (*b*) A pair of overlapping vertical photographs show a large pylon carrying high-tension wires across the Thames. From the following measurements and data determine the height of the pylon above datum level. The base of the pylon was at datum level.

	Photo 1	Photo 2
x co-ordinate from principal point to top of pylon.	+89·5 mm.	+37·2 mm.
x co-ordinate from principal point to bottom of pylon.	+84·1 mm.	+35·3 mm.

Focal length = 507 mm.; air base = 780 ft.; flying height above datum = 8100 ft.

Note that in both prints the photo image of the pylon was forward of the principal point, and all parallax measurements are therefore positive. (Inst. C.E., October, 1947.)

6. Draw a diagram to show that the height of an object appearing in a pair of overlapping vertical photographs can be determined from the x co-ordinates of the images of the top and bottom of the object. It can be assumed that the x co-ordinates are measured on the photograph from the principal point along the direction of flight.

A cliff rising from sea-level appears to the right of the principal point of each two overlapping vertical photographs which have been taken with a 507 mm. camera at a height of approximately 8000 ft. The distance flown between the exposures was 780 ft. and the following are the x co-ordinates measured on the photographs:

	L.H. Photograph	R.H. Photograph
P.P. to top of cliff	94·6 mm.	41·0 mm.
P.P. to bottom of cliff	88·38 mm.	40·4 mm.

What is the height of the cliff above sea-level? (Inst. C.E., April, 1956.)

7. The detail of a large number of modern topographical maps is copied or traced from air photographs on each of which there is a number of minor control points. The latter are plotted by radial line methods making use of the " radial assumption ". What is the " radial assumption " and why is it necessary? (Inst. C.E., October, 1953.)

8. As a representative of an air survey firm how would you explain to a civil engineering client the essential differences, economic and technical, between making a map from air photographs to show contours at 2-ft. intervals and making one to show contours at 100-ft. intervals? (Inst. C.E., October, 1956.)

9. In a pair of overlapping vertical air photographs the mean distance between the two principal points (both of which lie at datum level) is 1·95 in. At the time of photography the aircraft was 8000 ft. above datum level, and the camera had a focal length of 20 in.

In the common overlap, and lying between the two principal points, there was a pylon 500 ft. high, the base of which was at datum level. Determine the difference in the absolute parallax measurements (i.e. the x co-ordinates in the direction of flight) for the top and bottom of the pylon.

Why under ideal conditions should the y co-ordinates of any point, measured on two photographs overlapping in the direction of flight, be exactly the same? What is the most likely cause of a discrepancy, generally known as a " want of correspondence ", between the two y measurements? (Inst. C.E., October, 1952.)

CHAPTER XI

TOPOGRAPHICAL AND HYDROGRAPHICAL SURVEYING

TOPOGRAPHICAL SURVEYING

In previous chapters we have considered methods of detail surveying and contouring, mainly applicable to surveys of small areas which are to be plotted on large-scale plans; we now proceed to describe very briefly some of the principles and methods involved in making topographical surveys of fairly extensive areas which are to be plotted on small scales.

In all cases, the first thing to be done is to establish a framework of fixed points and known heights if no such framework exists already. In open country this is best done by triangulation, which should be broken down into triangles having sides one to five miles long, the density of control points varying with the scale on which the work is to be plotted. The heights of these points above datum must be fixed either by trigonometrical means or by spirit levelling. If the area involved is not too extensive, the triangulation need not be of geodetic accuracy, and in many cases it will be sufficient to measure the base line with ordinary steel tapes used stretched along the ground and to measure the angles with a small theodolite reading to 20″ or 30″. In other words, a triangulation of geodetic third-order accuracy is all that is needed.

For very large areas, owing to the way in which error accumulates, the main triangulation should be based on work of geodetic first-order standard broken down with second- and third-order work. Elevations determined by vertical heights should be tied in at intervals and adjusted to lines of spirit levels.

The detail survey is best executed by plane-table and Indian clinometer. Points are fixed by resection or intersection with heights determined by clinometer observations, and detail and contours put in by sketching in between the plane-table fixings. These plane-table fixings may be so arranged as to divide the area between trigonometrical points up into rectangular blocks of convenient size, with a plane-table fixing at the corner of each block.

In dense forest country, ordinary plane-tabling, and triangulation for that matter, will be impossible. Here the framework will consist of theodolite traverses and lines of spirit levels run along such roads or paths as may exist; if no roads or paths are available, it is necessary to cut lines. Other traverses and lines of levels split the areas between the main framework traverses and levels up into a series of blocks, and these blocks are cut up into smaller blocks by major compass traverses (traverses measured with a large stand compass and a chain or long steel band) or by plane-table traverses, in which intermediate elevations are determined by Abney or Indian clinometer. Finally, the blocks bounded by the compass or plane-table traverses are broken down by minor compass or plane-table detail traverses, run along cut lines or partals if necessary, with elevations derived from clinometer heights or from aneroid barometer runs.

The distance between successive minor detail traverses and between major compass or plane-table traverses will depend on the scale of the map and the accuracy desired. For work on the one-inch scale (1/63,360), for instance, the distance between major compass or plane-table traverses may be anything from four to eight miles and distances between detail traverses may be anything from 500 ft. to a mile. Supplementary major compass traverses should be run along main rivers and along main roads on which there is not a theodolite traverse, and detail traverses along small streams and waterways. Detail traverses are also advisable along main ridges, with others cutting across the ridge. The topography on either side of the traverse lines should be roughly sketched in in the field books or on the plane-table sheet as a guide to the final drawing, and the chainage of all important "cuts" of detail, such as cuts of streams and paths, should be noted and indicated.

All traverse work is generally plotted on *auxiliary sheets* on a scale larger than the scale of the map, and then reduced and plotted on the latter. For work on the one-inch scale, the scale for the auxiliary sheets may be about 1/25,000. It is a convenience if these sheets consist of sheets of squared paper.

The first stage in drawing is to plot a *grid* or *graticule*. A grid is used if the co-ordinates of the framework points are given in terms of rectangular co-ordinates, and it consists of a mesh of squares formed by lines parallel to the co-ordinate axes. Points are plotted with reference to the lower left-hand corner of the smallest squares within which they fall, this corner thus becoming a local origin for the plotting of the point concerned. In this way, small errors of plotting are reduced to a minimum.

When the co-ordinates of framework points are given in terms of latitude and longitude, a graticule is used instead of a grid. A graticule consists of a mesh formed by lines representing parallels of latitude and meridians of longitude. In general, the parallels of latitude are represented by curves, and the meridians may consist of straight lines or of curves, according to the *map projection* used. The parallels and meridians are best plotted by rectangular co-ordinates or in certain cases by ordinates from tangents to the parallels at the points where they cut the central meridian of the sheet, the values of these co-ordinates or ordinates being obtained from special tables supplied for the particular projection used.

In order to eliminate as far as possible errors caused by contraction and expansion of the paper, important maps are often plotted on metal sheets with specially prepared surfaces. In the Ordnance Survey, for example, the drawings for all maps and plans, including the sheets of the new 1/1250 plans and the new 1/25,000 maps, are drawn on metal, no paper being used at any stage before printing.

HYDROGRAPHICAL SURVEYING

The branch of surveying known as hydrographical surveying is a particularly wide one, since it includes national surveys for the charting of the depths of the waters in the immediate vicinity of land, as well as very minor surveys for the determination of the discharge of rivers and streams. The main engineering applications are in connection with harbour works, water supply, irrigation, water power, flood control, etc.

1. Sounding.

So far as ordinary hydrographic surveying is concerned, sounding means determining the depth of the water at various points in the waters adjoining the land, or in lakes, rivers and streams inland. These depths are used for plotting charts in the one case, or for planning under-water works, or determining volumes of discharge, etc., in the other.

Soundings for marine charting are more difficult than soundings in lakes or rivers because allowance has to be made for the height of the tide at the time of observation. In the Royal Navy most sounding is now done by sonic methods, which consist in measuring the time taken for a sound impulse to be transmitted to the bottom of the sea and thence to return, after reflection, to a special receiver mounted alongside the transmitter. This method can be adapted to give a continuous graphic record of the profile of the sea bed, and it is also used to locate wrecks, etc. If no echo-sounding apparatus is available, either a sounding rod or a lead line must be used.

A sounding rod is a long graduated wooden pole of from 2 to 3 in. diameter and 15 to 25 ft. long, which is plunged into the water until bottom is reached, the depth of the bed below water surface being read on the graduations on the pole. The lead line consists of a graduated line or chain, to which a heavy lead weight is attached. This is lowered into the water and run out until touch and a slackening of the line indicate that the bottom has been reached, when the depth of the weight below water surface is read on the line. Usually, in marine work, the bottom of a sounding rod or of a weight is hollowed out slightly to take some tallow, so that specimens of the material of the sea bed may be obtained.

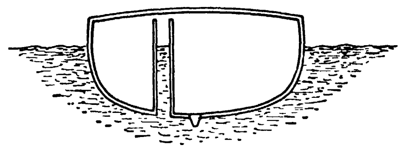


Fig. 11.1

Soundings are best taken from a special sounding boat in the bottom of which is a well communicating with the water outside (fig. 11.1). The work is also very considerably simplified if a special sounding machine, of which there are several different varieties, is used.

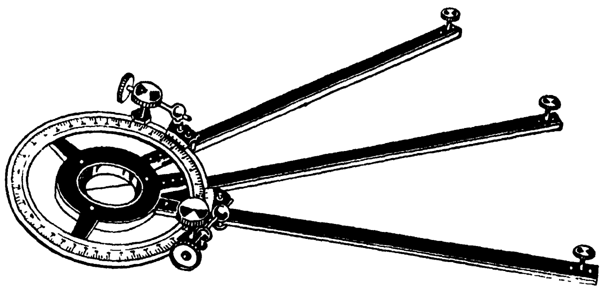
When soundings are taken at sea, the time of observation must be carefully recorded and a correction applied later for the state of the tide. This correction may be obtained from the results of readings on a tide gauge or tide pole on shore, on which the height of water above or below some definite datum is observed at regular intervals of time while soundings are in progress, or it can be obtained by scaling from the chart of an automatic recording tide gauge. The datum to which marine soundings are ordinarily referred is L.W.O.S.T. (Low Water Ordinary Spring Tides).

In river and lake surveys water level should be checked at intervals by readings on a river gauge consisting of a graduated board set vertically in the water and connected by levelling to a fixed bench mark on land.

2. Fixing the Positions of Soundings.

Offshore soundings are fixed with reference to trigonometrical or other fixed points on shore. The most common method is to use resection from three fixed points, the angles being observed from the sounding boat at the time a sounding is taken by means of a sounding sextant. Alternatively, the position of a sounding can be fixed by intersection from simultaneous observations with two theodolites set over two trigonometrical stations on shore.

If soundings are fixed by resection, their positions can be plotted on the chart by means of the *station pointer* shown in fig. 11.2. This consists of three long arms, one fixed and two hinged to the centre of a circular protractor, to which they can be clamped at any reading. The arms are set to read the observed angles and the instrument is moved about until the edge of each arm lies alongside the plotted position of



(By courtesy of Messrs. Hilger and Watts, Ltd.)

Fig. 11.2

the point to which the corresponding observation was taken. The position of the centre of the protractor can then be pricked through a hole in the axis on to the chart, and this gives the position of the point from which the soundings were taken.

Various other methods of fixing off-shore soundings are possible, some of them depending on time intervals between soundings fixed by other means. Thus, in fig. 11.3, a series of poles, 1, 2, 3, 4, 5, 6, 7, 8, 9 and 10 are ranged out and fixed on shore so that pairs, such as 10 and 1, 9 and 2, etc., define straight lines more or less perpendicular to the coast line. The sounding boat in its travels from point to point is kept on these lines in turn by sighting on each pair of points. A theodolite at T is used to measure the bearing to the boat as each sounding is taken. Then, the points 1, 2, 3, etc., being plotted on the plan, the intersection of the line of the observed bearing with the line on which the sounding is taken fixes the position of the sounding boat.

An alternative method is to take the sounding when the boat is at a point such as *s*, where the poles 10 and 1 and the poles 8 and 4 respectively appear to be in line. Intermediate soundings between points such as *s* and *t* may be located by noting the time of travel between soundings and the total time spent in actual travelling between *s* and *t*.

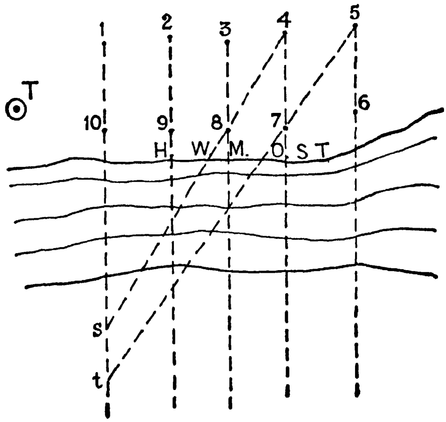


Fig. 11.3

When sounding across rivers, the simplest way, if the stream is not too wide, is to stretch a rope across it between fixed points with marks at regular known intervals on the rope. The boat is ferried across the river, one man holding on to the rope, and soundings taken as each of the marks is reached. If the stream is too wide for a marked rope to be stretched across it, soundings may be fixed by lines of suitable placed poles such as those described above, or by two theodolites set up at suitable fixed points on the bank.

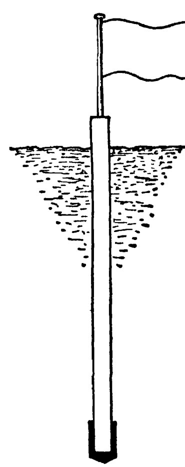


Fig. 11.4

3. Measurement of Tidal Currents.

In measuring tidal currents it is necessary to determine the speed, direction and location of the current. This can be done by using special floats which present as little above-water surface as possible, but which can easily be seen from a distance. Fig. 11.4 shows a float made of a wooden rod about 2½ to 4½ ft. long and 3 in. square, weighted at the bottom with a lump of lead and carrying a small flag on top. The float is released on its course and its position determined at observed intervals of time. Fixings may be made by intersection by two theodolites from the shore or by resection from a boat accompanying the float.

4. Measurement of Discharge of Rivers and Streams.

The measurement of the discharge of rivers and streams means the measurement of the volume of water passing a given cross-section per second; there are two cases to be considered. The first is the measurement of the discharge of a wide river across which a temporary dam cannot be built, and the other is the case of a narrow stream which can be dammed and water allowed to flow over a special weir.

The principle of measurement in the case of a wide river is very simple, but the actual work is, perhaps, not quite so easy. The first thing is to obtain a cross-section by levelling and sounding, and the next step is to measure the velocity of the water across the section at fairly close intervals. The easiest and best method of doing this is to use a *current meter* and observe the velocity of the water at different depths below the surface. A current meter consists of a small screw like a ship's propeller mounted to move up and down a graduated pole, or suspended by a graduated cable at different depths. The screw is made to assume a direction at right angles to the direction of flow by means of a four-bladed vane attached behind it, and the revolutions over a certain observed time are recorded by gearing on a dial on the instrument, or else they are caused to make a noise in a telephone buzzer. The main cross-section is divided into a number of equally spaced sub-sections, and a series of readings taken at different depths at the middle of each sub-section. The area of each sub-section having been calculated from the plotted cross-section, the total discharge is the sum of the quantities obtained by multiplying the average velocity at each sub-section by the area of that sub-section.

If no current meter is available, floats must be used and the times taken for them to travel between lines defined by pairs of poles on either side of the section observed. These lines may be 50 to 300 ft. apart and cross-sections must be taken at them and at other points 50 to 100 ft. apart if the distance between them exceeds these limits. A series of observations at approximately equal distances apart along the cross-sections is taken, and the mean velocity for each run deduced. The sum of the quantities obtained by multiplying each average velocity by the mean area of the corresponding sub-section gives the total discharge.

In carrying out these observations, a length of the river should be chosen in which the banks are fairly straight and parallel to one another, and in which the flow of the river is parallel to the banks and is not too fast. In all cases the length used in calculating the velocity is the distance between the end sections, not the distance actually travelled by the float.

The velocity of the water in a straight stretch of river varies with the depth, being zero on the bed and varying according to a more or less parabolic law with a maximum at a point about 0·2 of the depth below the surface. The mean velocity has a value somewhere between 0·7 and 0·95 of the surface velocity, the exact value increasing as the velocity and depth increase. This value can be found by means of a current meter, or by observations with surface floats and with others which are only slightly shorter than the depth of the water, and which can therefore be assumed to be carried along with a velocity equal to the average velocity of the water.

5. Stream Discharge over a Weir.

In this method, water is allowed to flow over an aperture or notch in a special weir, and the head of water a few feet upstream from the weir is measured.

The opening or notch through which the water flows can be of various shapes—triangular, rectangular or trapezoidal. This opening

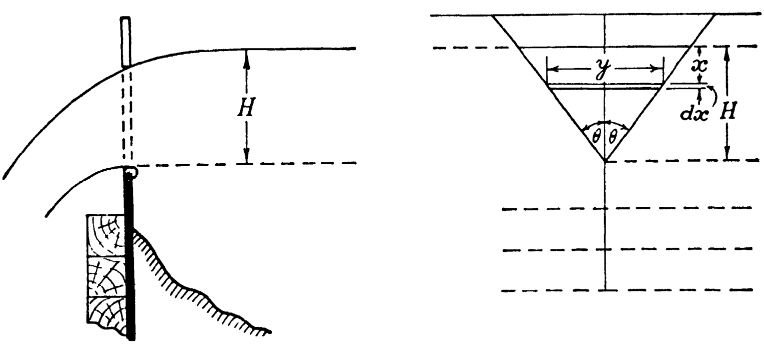


Fig. 11.5

is cut in a metal plate with sharp edges which is fastened to a dam of planks or other material built across the stream at right angles to the direction of flow.

The principle of the method will be understood from fig. 11.5 which shows a weir with a triangular notch, the angle at the apex of the notch being 2θ. Consider a horizontal strip of width y and depth dx

at depth x below the surface of the stream. The head of water is then x, so that, from the principles of hydraulics, the velocity of the water flowing through the strip is given by

$$v^2 = 2gx,$$

where g is the acceleration due to gravity, and the volume of water flowing is

$$dQ = vy\,dx = y\sqrt{2gx}\,dx.$$

But $y = 2(H - x)\tan\theta$, where H is the depth of the bottom of the notch below the surface of the stream. Hence, the total volume for the whole notch is

$$\begin{aligned} Q &= \int_0^H 2\sqrt{(2gx)}\,.\,(H - x)\tan\theta\,dx \\ &= 2\sqrt{(2g)}\,.\,\tan\theta\int_0^H x^{1/2}(H - x)\,dx \\ &= 2\sqrt{(2g)}\,.\,\tan\theta\left[\tfrac{2}{3}Hx^{3/2} - \tfrac{2}{5}x^{5/2}\right]_0^H \\ &= \tfrac{8}{15}\sqrt{(2g)}\,.\,\tan\theta\,.\,H^{5/2} = 4{\cdot}3\tan\theta\,.\,H^{5/2}, \end{aligned}$$

when for g we substitute 32 ft. per second per second.

In actual practice, the constant multiplier is found not to be 4·3 but to vary from about 2·48 to 2·56. Assuming a value of 2·5, we can write

$$Q = 2{\cdot}5\tan\theta\,.\,H^{5/2},$$

and for a notch in which $2\theta = 90°$

$$Q = 2{\cdot}5H^{2{\cdot}5} \text{ cub. ft. per second,}$$

a formula which is easily remembered.

In the case of rectangular notches, there are two cases to be considered. One is when the notch is the same width as the stream, in which case there are no *end constrictions*, and the other is when the notch is not so wide as the stream and there are end constrictions.

The theoretical formula for a notch or weir of length L with no end constrictions is easily found, for, if x is the depth below the surface of a strip of depth dx, the discharge through the strip is $L\sqrt{(2gx)}\,.\,dx$ and the discharge for the whole weir is

$$\begin{aligned} Q &= \int_0^H L\sqrt{(2gx)}\,dx \\ &= \tfrac{2}{3}L\sqrt{(2g)}\,.\,H^{3/2} = 5{\cdot}35LH^{3/2} \text{ cub. ft. per second.} \end{aligned}$$

In practice, the discharge has not the theoretical value, and it has been found that, instead of the factor 5·35, we must use a *coefficient of discharge* c which varies slightly with H and L. Values of this coefficient for different values of H and L are generally given in text-books on hydraulics, but a good average value is 3·33. Accordingly, we may write

$$Q = 3{\cdot}33LH^{3/2} \text{ cub. ft. per second.}$$

This formula assumes discharge by a stream which is static a short distance from the weir. Usually there is also a small *velocity head* due to the movement of the stream as a whole. This head is given by $h = v^2/(2g)$, where v is the velocity of the stream in feet per second. This velocity is found by obtaining an approximate value of Q from the above formula and dividing it by the area of a cross-section of the whole stream a short distance above the weir. The formula then becomes

$$Q = 3{\cdot}33L\{(H + h)^{3/2} - h^{3/2}\}.$$

When a notch is not as wide as the stream, the lateral motion of the water along the side of the weir causes the flow to be constricted at the edges as shown in fig. 11.6. Each end constriction has the effect

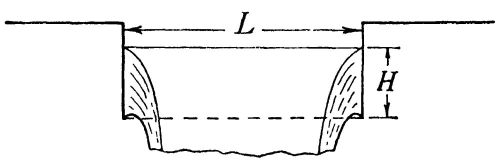


Fig. 11.6

of shortening the length of the notch by about $0{\cdot}1H$, so that with two end constrictions the discharge is given by the above formulæ with $(L - 0{\cdot}2H)$ substituted for L; then

$$Q = 3{\cdot}33(L - 0{\cdot}2H)H^{3/2},$$

or

$$Q = 3{\cdot}33(L - 0{\cdot}2H)\{(H + h)^{3/2} - h^{3/2}\}.$$

The measurement of the head is best done by means of a special *hook gauge*, which consists of a pointed hook attached to a graduated rod that works up and down against an index and vernier carried on a post or arm. This arm is fixed about twelve inches upstream above the weir, in such a way that it does not interfere with the flow of the water. The hook is immersed in the water and raised until the sharp point is just level with the water surface. The gauge having been set so that readings on the scale correspond to heights of the points of the hook above the bottom of the notch, readings of the head of water can be obtained to the nearest thousandth part of a foot by reading the scale and vernier.

QUESTIONS ON CHAPTER XI

1. Describe a method which you would use for the survey of the banks of a very wide river, and how you would take cross-sections at different points along it.
2. In making a chart of a small harbour the positions of soundings were fixed by taking horizontal angles with a sextant to known points on the shore. What method would you use for plotting the positions? Explain how your method differs from plotting by the usual resection method adopted in a topographical survey using a plane-table and alidade. (Inst. C.E., October, 1946.)
3. Describe three different methods of fixing the positions of soundings off a coast-line.
4. Calculate the discharge in cubic feet per second over a triangular notch in which the sides are inclined at an angle of 45° to the vertical, the measured depth of the water above the bottom of the notch being 33 in.
5. Calculate the discharge in cubic feet per second over a rectangular notch 6 ft. wide, with end constrictions, and $H = 2$ ft. Compare the values obtained when "velocity head" is neglected, and when it is taken into account assuming that the area of a cross-section a short distance upstream from the notch is 50·2 sq. ft.
6. To determine the discharge of a sluggish stream at a particular section, a dam was built and the water made to flow over a 90° triangular weir with sharp edges. The level of the water above the bottom of the weir was found to be 2·75 ft. Calculate the discharge of the stream at the section in *cusecs*.

   Describe any method you know for determining the level of the surface above the bottom of the weir. (Inst. C.E., April, 1953.)
7. Describe briefly three methods of sounding depths on a broad river estuary and three methods of fixing the positions of such soundings. State which of the latter methods you would prefer to use, giving your reasons. The shores of the estuary are gently rising and are well mapped. (Inst. C.E., April, 1957.)

CHAPTER XII

AREAS AND EARTHWORK QUANTITIES

CALCULATION OF AREAS

The computation of areas can be done (1) by calculation from measurements made on a plan; (2) by direct calculation from co-ordinates; and (3) by the use of an instrument called a *planimeter.*

1. Calculation of Areas from Measurements on a Plan.

An area can easily be calculated from measurements taken on a plan if the boundaries are straight lines which permit of the whole figure being divided into a series of simple geometrical figures consisting of triangles, squares or rectangles. Often, however, the boundaries are irregular, and in that case the procedure is to divide the greater part of the interior into a series of simple geometrical figures whose areas can be calculated by ordinary simple formulæ, leaving

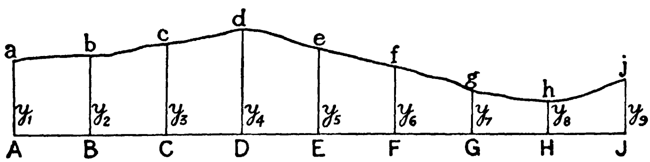


Fig. 12.1

other figures, which have an irregular line as one boundary, and whose areas must be calculated by special means and formulæ. As it is these irregular figures which present the greater difficulty, we shall consider them first.

In fig. 12.1 we have an irregularly shaped boundary **a, b, c, d . . . j** close to a line AJ. Divide AJ into a number of *equal* parts AB, BC, CD, . . . and at the points A, B, C, D . . . erect ordinates A**a**, B**b**, C**c**, D**d**, . . . to meet the boundary in **a, b, c, d,** . . . . If the points of division are close enough together, the part of the boundary between ordinates will be very approximately a straight line, and the whole figure will be divided into a number of trapeziums, AB**ba**, BC**cb**, CD**dc**, etc.

Let the ordinates **Aa**, **Bb**, **Cc**, **Dd**, . . . be of length y_1, y_2 y_3, y_4, . . . and let d be distances AB, BC, CD . . . .

Then area of first trapezium $= \frac{1}{2}d(y_1 + y_2)$,

„ 2nd „ $= \frac{1}{2}d(y_2 + y_3)$,

„ 3rd „ $= \frac{1}{2}d(y_3 + y_4)$,

. . . .

. . . .

„ $(n-2)$th „ $= \frac{1}{2}d(y_{n-2} + y_{n-1})$,

„ $(n-1)$th „ $= \frac{1}{2}d(y_{n-1} + y_n)$.

Hence total area

$$A = \frac{1}{2}d[y_1 + 2y_2 + 2y_3 + 2y_4 + \ldots + 2y_{n-2} + 2y_{n-1} + y_n].$$

This formula is known as the *trapezoidal formula for areas.* The approximation used in it is, of course, that the lines **ab**, **bc**, **cd**, . . . are straight lines, whereas in actual fact they may be curved. To allow for this, another alternative rule—known as *Simpson's rule*—is often used instead. This formula is based on the assumption that the lines **ac**, **ce**, **eg**, etc. . . . are parabolic arcs of the form $y = a + bx + cx^2$. With such a curve it can be shown that the area of a figure such as AB**Ccba**A, is

$$\tfrac{1}{2}(y_1 + y_3) \times 2d + \tfrac{2}{3}\{y_2 - \tfrac{1}{2}(y_1 + y_3)\} \times 2d = \tfrac{1}{3}d\{y_1 + 4y_2 + y_3\},$$

the second term in the first expression being the area included between the parabolic arc and the chord, and the first term the area of the trapezium bounded by y_1 and y_3. Accordingly, if the area is divided into an even number of parts so that there is an odd number of ordinates, we have for the area of the whole figure,

$$A = \tfrac{1}{3} \cdot d[y_1 + 4y_2 + 2y_3 + 4y_4 + 2y_5 + \ldots + 2y_{n-2} + 4y_{n-1} + y_n].$$

In words, Simpson's rule is therefore:

Divide the area into an even *number of* equally spaced *vertical strips. Add together the first and last ordinates, four times the sum of the even ordinates, and twice the sum of the remaining odd ordinates, and multiply the result by one third of the distance between successive ordinates.*

It is obvious that both the trapezoidal and Simpson's formulæ can be applied to a figure such as that shown in fig. 12.2, where the ordinates are drawn from one irregular boundary to the other, and the area of the whole of the figure is included in the calculation. Owing to some account being taken of the curvature of the boundaries, Simpson's rule is rather more accurate than the trapezoidal rule. The latter is included in Simpson's rule as a special case when the boundary between alternate ordinates is a straight line.

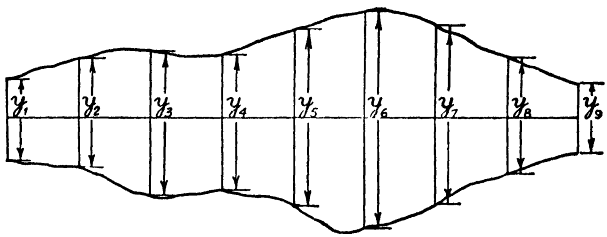


Fig. 12.2

In other cases it is convenient to divide the central part of the figure into a series of triangles, as in fig. 12.3, and calculate the areas of the triangles, the areas of the small portions between the boundary and the sides of the triangles being taken out by the trapezoidal or by Simpson's rule. If ABCDEFGA is a closed traverse, the area included in it may also be calculated by co-ordinates by the rules given in Chapter V, p. 59.

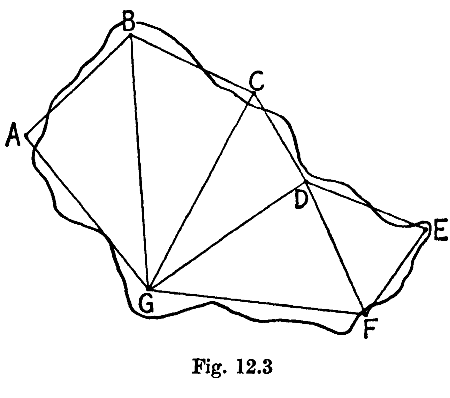


Fig. 12.3

If the area is divided into triangles, the sides of which are measured, the area of any triangle is given by

$$A = \sqrt{s(s-a)(s-b)(s-c)},$$

where a, b and c are the lengths of the sides and $s = \frac{1}{2}(a + b + c)$. Alternatively, the length of one side a and of the perpendicular p to it from the apex opposite to it may be scaled from the plan, when, of course, $A = \frac{1}{2}ap$.

A sheet of tracing paper or celluloid with a square grid, such as that shown in fig. 12.4, enables an area to be found rapidly and reasonably accurately. Each square contains a known area, and the method

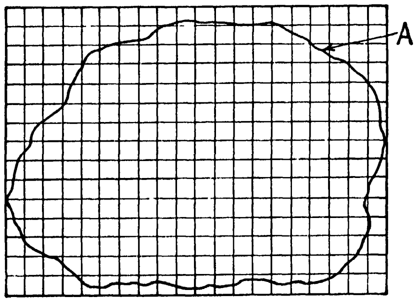


Fig. 12.4

consists in counting the number of squares included in the figure but not crossed by any part of the boundary. The areas of the figure included in squares such as A which are crossed by the boundary can be estimated by eye as a decimal part of the area of the whole square, and the results added together and to the total area of the complete squares.

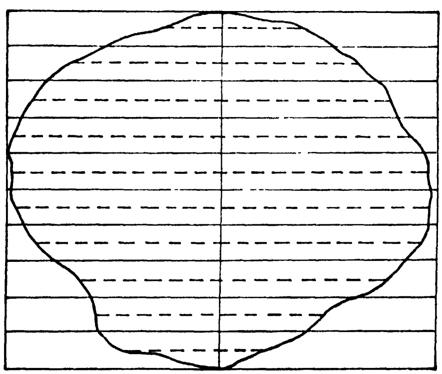


Fig. 12.5

Another method is to use a piece of tracing paper (fig. 12.5) on which has been drawn a series of strips bounded by parallel lines a fixed distance apart. This is placed over the plan, and the distance between the boundaries at the middle of each strip is measured. The sum of these distances multiplied by the width of each strip gives the total area of the figure. This method can be simplified somewhat by the use of a special *computing scale*. This consists of a scale on which is mounted a rider with a square aperture provided with a vertical cross hair. The cross hair is set so that the small portion of the area which is included by it appears to be equal to the area of the portion excluded, as in fig. 12.6. The reading on the scale is taken, and the rider moved over to the other side of the figure, when the hair is once more set to equalize the areas cut off by it. The scale is again read and the difference of readings then gives the mean length of the strip. As the instrument is made to work over strips of standard width, the scale can be graduated to give readings direct in terms of area.

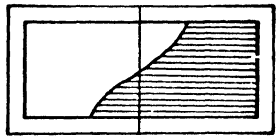


Fig. 12.6

2. Computation of Areas from Co-ordinates.

This is the most accurate method of determining areas, and is applicable when the figure, or the greater part of it, consists of a closed traverse surround and the co-ordinates of the various stations are known. The method is described and the formulæ given on pp. 58–61.

3. Determination of Area by Amsler's Planimeter.

In order to understand the working of the planimeter, consider the motion of a *tracing bar* AB (fig. 12.7) freely jointed at A to a *pole arm* AO which rotates about a fixed centre O. Let the end A move through a small angle to A′, and the end B to a point B′ close to B. As AA′ is a small arc, small enough to be considered a straight line, the motion of B can be considered to be a lateral displacement BB″ parallel and equal to AA′, and a rotation through the angle B″A′B′ $= d\alpha$ about A′. Hence, if l is the length of AB and $dh =$ AA′ sin BAA′ is the perpendicular from A′B″ to AB, the

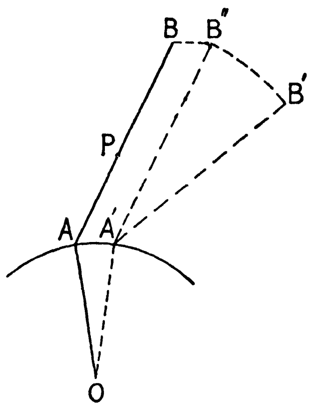


Fig. 12.7

area swept out by the arm as it moves from AB to A′B′ will be

$$dA = l\,dh + \tfrac{1}{2}l^2\,d\alpha.$$

Now suppose there is a small wheel or roller at P on AB which will only register displacements at right angles to AB, and let AP $= k$AB. The displacement of the point P perpendicular to AB, and hence the displacement registered on the wheel, will be

$$dp = dh + kl\,d\alpha, \text{ or } dh = dp - kl\,d\alpha.$$

Hence, substituting this value of dh in the original equation,

$$dA = l\,dp + (\tfrac{1}{2}l^2 - kl^2)\,d\alpha,$$

and, for finite area, displacement and rotation,

$$A = lp + (\tfrac{1}{2}l^2 - kl^2)\alpha,$$

where p is the total lateral displacement registered on the wheel and α is the total angle through which AB has rotated about A.

Now take the cases shown in figs. 12.8 and 12.9, where the end B travels clockwise round a closed curve C_1, while A describes part of a circular arc C_2 in fig. 12.8 and a complete circle C_2 exterior to the

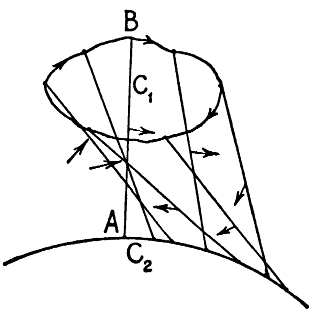


Fig. 12.8

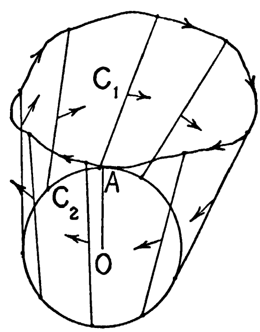


Fig. 12.9

curve C_1 in fig. 12.9. In both cases the direction of its motion is shown by an arrow for different positions of the arm AB, the movement registered on the drum, of course, being the component perpendicular to AB, and it will be seen that, as AB returns to its original position, the total rotation of B about A is zero, so that $\alpha = 0$. In fig. 12.8, during the left to right travel of B, AB will sweep out a positive area bounded on the top and on the bottom by the upper part of C_1 and an arc of C_2 respectively, while during the right to left travel of B, AB will sweep out a negative area bounded on the top by the lower part of C_1 and on the bottom by the same arc of C_2, the left and right boundaries of both areas being the extreme left and right positions of AB. Consequently, the area recorded will be the algebraic sum of these two areas, which is the area contained by C_1. In this case $\alpha = 0$ and

$$A = lp. \qquad \qquad (1)$$

In the case of fig. 12.9, the positive area swept out by AB is the area of the figure bounded on the top by the upper part of C_1 and on the bottom by the upper part of the circle C_2, while the negative area is bounded on the top by the lower part of C_1 and on the bottom by

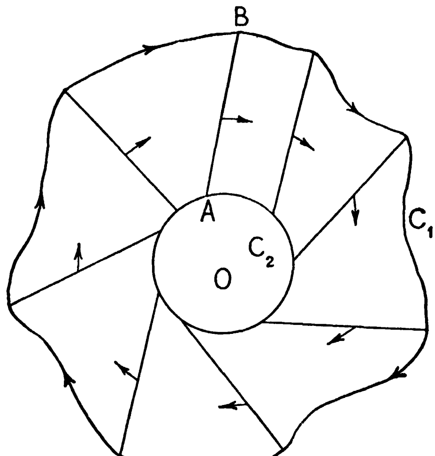


Fig. 12.10

the lower part of the circle C_2. The area recorded by the planimeter is the algebraic sum of these two areas, which is $(A_1 - A_2)$, where A_1 is the area of the figure C_1 and A_2 is the area of the circle C_2. Hence, as $\alpha = 0$ in this case also,

$$A_1 - A_2 = lp,$$

or

$$A_1 = A_2 + lp. \qquad \qquad (2)$$

Next, let the figure traced by B totally enclose the circle described by A as in fig. 12.10. In this case, AB will complete a whole revolution before it returns to its original position and hence $\alpha = 2\pi$, so that the original equation gives

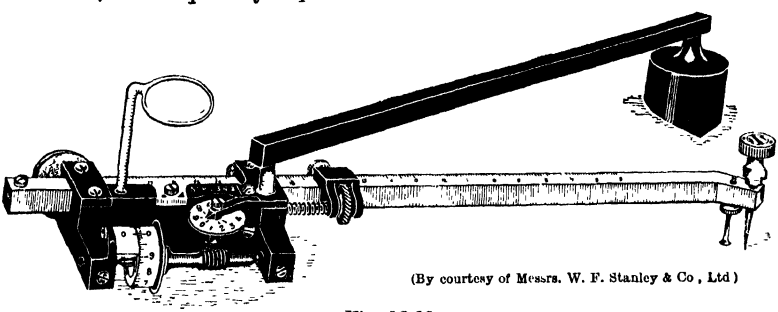
$$A = lp + (\tfrac{1}{2}l^2 - kl^2)2\pi.$$

But the area traced out by AB will be the area lying between the curves C_1 and C_2, so that, if A_1 and A_2 are the areas contained by these curves,

$$A_1 - A_2 = lp + (\tfrac{1}{2}l^2 - kl^2)2\pi.$$

$$\therefore\ A_1 = A_2 + lp + (\tfrac{1}{2}l^2 - kl^2)2\pi. \quad . \quad . \quad . \quad (3)$$

In these equations A_2, l and k are constants for the instrument and the only observed quantity is p, which is given by the revolutions of the drum; consequently A_1 can be found.



(By courtesy of Messrs. W. F. Stanley & Co., Ltd.)

Fig. 12.11

Fig. 12.11 shows an instrument designed on these principles. In this planimeter the tracing bar is adjustable in a carriage which carries the graduated rotating drum and to which the end of the pole arm is pivoted. Complete revolutions of the drum are counted on a small disc worked by a worm screw, but parts of a revolution are read on the drum and a vernier, a magnifying glass being provided to enable the vernier index to be seen clearly. A scale and setting arrangement enable the tracing bar to be set with different effective lengths to suit different scales, and a table supplied with the instrument gives the positions at which the carriage must be set on the tracing bar to measure areas on any required scale, and also the area corresponding to each complete revolution of the drum.

The drum itself can only register movements at right angles to the tracing bar. If there is any longitudinal movement in the direction of the tracing bar the drum simply slides along the paper without registering. The point of the tracing bar which is used to trace along the boundary is an adjustable pin which is kept just clear of the paper by an adjustable support.

The end of the pole arm is pivoted on a needle-pointed weight which carries the centre about which the arm rotates.

To use the instrument, set the tracing bar in the carriage at the reading necessary to give the result in the desired units on the scale on which the plan is drawn, and then place the weight outside the area to be measured. Mark some point on the boundary from which to start and, having set the tracing bar point at this point, read the counter, drum and vernier. Now move the tracing point clockwise along the boundary until it returns to the starting point and read the counter, drum and vernier again. The difference in the two sets of readings will give a number which, when multiplied by the constant given in the tables, will give the area required.

If the area is large, it may be necessary to set the pole O inside it, in which case the end A of the tracing bar will trace out the circle C_2, fig. 12.10, as the end of the tracing bar B moves completely round the boundary. In this case, the area of the circle C_2, which is given in the tables or is marked on the instrument, together with the quantity $(\frac{1}{2}l^2 - kl^2)2\pi$, which is also given in the tables, must be added to the area derived from the instrument readings as in equation (3) above.

When a new instrument is being used for the first time, it is well to test the constants and settings. This can be done by drawing squares or rectangles of given sides and then comparing the calculated area from the area obtained by planimeter.

4. Calculation of Areas of Cross-sections of Cuttings and Embankments.

The calculation of areas of cross-sections of cuttings and embankments from recorded levels follows from very elementary problems in mensuration. In fig. 12.12 we have an ideal section, level throughout along the surface, the side slopes being k units horizontally to 1 unit vertically. This is the case which arises when rough quantities are to be computed from a longitudinal section, and it can easily be seen that the area is made up of the areas of the triangles ABE and DCF and the rectangle BCFE. This gives

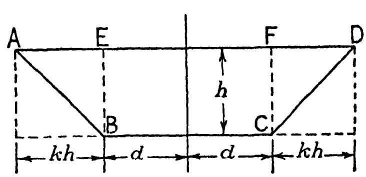


Fig. 12.12

$$A = \tfrac{1}{2}kh^2 + \tfrac{1}{2}kh^2 + 2dh$$
$$= h(2d + kh).$$

If much work of this kind is involved the area can be computed from special earthwork tables. If none is available, special tables can easily be computed for given values of k and d and for different values of h.

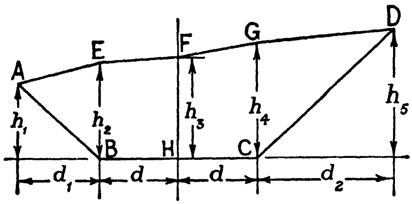


Fig. 12.13

In fig. 12.13, where the heights h_1, h_2, h_3, h_4, and h_5, and the distances d_1 and d_2 are known from the levelling operations,

$$\begin{aligned} A &= \tfrac{1}{2}d_1h_2 + \tfrac{1}{2}d(h_2 + h_3) + \tfrac{1}{2}d(h_3 + h_4) + \tfrac{1}{2}d_2h_4 \\ &= \tfrac{1}{2}h_2(d_1 + d) + h_3d + \tfrac{1}{2}h_4(d + d_2). \end{aligned}$$

Other cases can be worked out by simple mensuration.

CALCULATION OF EARTHWORK QUANTITIES

5. Volumes of Railway Cuttings and Embankments.

Let AGgaA in fig. 12.14 represent part of a longitudinal section for a road or railway, and let cross-sections be taken at equal distances d apart at A, B, C, D, E, F, and G. Let the areas of these sections

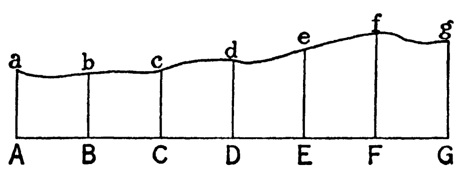


Fig. 12.14

be A_1, A_2, A_3, A_4, ... Then if we take the mean of the end areas of the solid comprised between A and B, which is equal to $\frac{1}{2}(A_1 + A_2)$, and multiply it by d, the distance between end sections, it is reasonable to suppose that this will give us an approximate estimate of the cubic contents of the solid. Hence

$$V_1 = \tfrac{1}{2}(A_1 + A_2)d.$$

Similarly,

$$V_2 = \tfrac{1}{2}(A_2 + A_3)d,$$
$$V_3 = \tfrac{1}{2}(A_3 + A_4)d,$$
$$V_4 = \tfrac{1}{2}(A_4 + A_5)d,$$
$$V_5 = \tfrac{1}{2}(A_5 + A_6)d,$$
$$V_6 = \tfrac{1}{2}(A_6 + A_7)d,$$

and, by addition,

$$V = \tfrac{1}{2}d(A_1 + 2A_2 + 2A_3 + 2A_4 + 2A_5 + 2A_6 + A_7).$$

This is the trapezoidal rule which is often used for obtaining approximate volumes, but it is only valid when the area of the section at the middle is the mean of the end areas, and this is only true when the solid is composed of wedges and prisms. Most of the solids involved in earthwork cuttings or embankments are prismoids, which are solids having two parallel end faces, each of which is a polygon of any number of sides, the side faces of the solid being planes. Accordingly, we now proceed to deduce a more rigorous formula known as the *prismoidal formula.*

6. Prismoidal Formula.

In fig. 12.15, ABCD and EFGH are the parallel end faces of a prismoid, the area of these faces being A_1 and A_3 respectively. Let

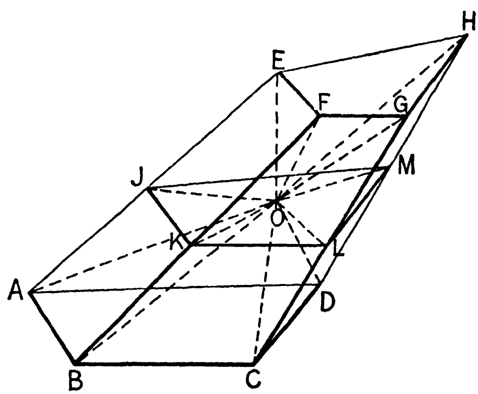


Fig. 12.15

JKLM be a section midway between the end sections, and parallel to both, and let the distance between this section and either of the end sections be d. In JKLM take any pole O and join O to A, B, C and

D, and to E, F, G and H. Then the total volume may be considered to be made up of two pyramids OABCD and OEFGH, of height d and with bases of areas A_1 and A_3 respectively, and four other pyramids OABFE, OBFGC, OCDHG and OADHE, whose bases are the sides of the solid. The volumes of the pyramids OABCD and OEFGH are equal to $\frac{1}{3}A_1d$ and $\frac{1}{3}A_3d$ respectively.

Now take the pyramid OABFE. The volume of this pyramid is $\frac{1}{3}$(area ABFE) × (height of pyramid), which we may take equal to $\frac{1}{3}$ × JK × $2d$ × p, where p is the perpendicular from O on JK. But p × JK = 2 × area of triangle JOK. Consequently, the volume of the pyramid = $\frac{4}{3}$ × area of triangle JOK × d. Similarly, the volumes of the pyramids OBFGC, OCDHG and OADHE are $\frac{4}{3} \times d$ multiplied by the areas of the triangles KOL, LOM and MOJ. But the sum of the areas of the triangles JOK, KOL, LOM and MOJ is equal to the area of the cross-section JKLM = A_2. Hence the volume of the whole prismoid is equal to

$$V = \tfrac{1}{3}A_1d + \tfrac{1}{3}A_3d + \tfrac{4}{3}A_2d$$
$$= \tfrac{1}{3}d(A_1 + A_3 + 4A_2).$$

Now, apply this result to the sections at A, B, C, D, E, F, G in fig. 12.14 and the result is

$$V = \tfrac{1}{3}d(A_1 + A_3 + 4A_2) + \tfrac{1}{3}d(A_3 + A_5 + 4A_4) + \tfrac{1}{3}d(A_5 + A_7 + 4A_6)$$
$$= \tfrac{1}{3}d(A_1 + 4A_2 + 2A_3 + 4A_4 + 2A_5 + 4A_6 + A_7)$$
$$= \tfrac{1}{3}d[(A_1 + A_7) + 4(A_2 + A_4 + A_6) + 2(A_3 + A_5)],$$

a result which may be compared with Simpson's Rule for Areas.

In general, a prismoid may be divided up into a series of prisms, wedges and pyramids so that the rule holds for any shape of cross-section, and the different sections need not have the same number of sides. Hence, for any prismoid:

Divide the cutting or embankment into an even *number of* evenly *spaced parts by an* odd *number of cross-sections. Then the volume required is equal to one-third of the distance between sections, multiplied by the sum formed by the sum of the areas of the first and last sections, plus four times the sum of the areas of the even sections, plus twice the sum of the areas of the remaining odd sections.*

It will be noted that it is assumed in the derivation of this formula that the ground surface between the odd sections is a plane. This is not always the case, and in such an event there will be a small error.

Consequently, in selecting the positions for cross-sections it is well to aim at siting them so that there is no great change in slope at even sections. If necessary, the spacing apart of the cross-sections can be altered and the whole volume divided into several parts, the prismoidal rule being applied to each part. When an excavation or embankment is on a curve, the prismoidal rule is not strictly applicable because radial sections are no longer parallel. A correction for curvature can be made if desired, although, as the curves are generally flat, the application of the prismoidal formulæ in the ordinary way does not lead to serious error, and it is usual to neglect the curvature correction. This correction will be found described in some of the more advanced textbooks.

7. Computation of Volumes of Large Excavations from a Rectangular Mesh of Spot Heights.

In dealing with large excavations covering a large area, a convenient plan is to divide the area up into a mesh of squares or rectangles, as shown in fig. 12.16, with spot heights at the intersections of the

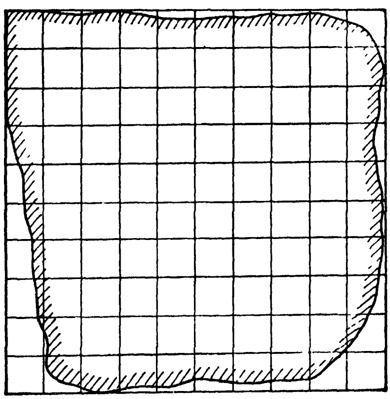


Fig. 12.16

lines forming the mesh. For computing purposes, each square or rectangle is best divided into two triangles by a diagonal, and the volume computed by multiplying the area of each triangle by the mean of the depths of excavation at the apices. This gives a better result than that obtained by multiplying the area of each square by the means of the depths of excavation at the four corners, since it deals more effectively with any change of slope of the ground surface along and at right angles to the diagonal.

A simplification in computing results if we note that, taking any triangles which do not abut on the boundary, and thus have the same area, some depths may occur once at an apex, some twice, some three times, and so on up to eight times. Let

$$\Sigma h_1 = \text{sum of depths used once,}$$

$$\Sigma h_2 = \text{sum of depths used twice,}$$

$$\Sigma h_3 = \text{sum of depths used thrice,}$$

. . . .

$$\Sigma h_8 = \text{sum of depths used eight times,}$$

$$A = \text{area of a single triangle.}$$

Then, the volume of that part of solid which does not contain any prisms abutting on the boundary is

$$V = \tfrac{1}{3}A[\Sigma h_1 + 2\Sigma h_2 + 3\Sigma h_3 + \ldots + 7\Sigma h_7 + 8\Sigma h_8].$$

The volume of the whole solid is then this quantity plus the sum of the volumes of all prisms abutting on the boundary, the volume at each prism being the area of the base multiplied by the mean height.

8. Computation of Volumes from Contours.

A simple case of this is to find out the storage capacity of a reservoir from a contoured plan, as in this case the surface of the water lies in a horizontal plane parallel to the level surfaces enclosed by the contour lines. Here, the area enclosed by each contour line is computed, and the volume enclosed between each pair of contours is taken as the mean of the areas of the surfaces enclosed by the two contours multiplied by the vertical distance between them. The same method can be used in computing the volumes of excavations when the bottom of the excavation is to be taken as a flat surface. Alternatively, a mesh of rectangles or squares can be drawn on the plan and the depth of excavation at each corner point interpolated from the contours. This last method may also be used if the new surface of the excavation or fill is to be taken out or made up to a specified gradient, but in this there is an alternative procedure.

In fig. 12.17*a* are shown a number of contours of a hill which has to be excavated to the gradient ABCDE in the longitudinal section along the line XY which is shown in fig. 12.17*b*. This means that all the earth lying to the left of ABCDE is to be removed. On the longi tudinal section draw the horizontal lines A′A, B′B, C′C, D′D and E′E, to represent the planes of the 320, 315, 310, 305 and 300-ft. contours, and to meet the line representing the gradient at the points A, B, C, D and E. Through A and B draw lines perpendicular to XY to meet the 320- and 315-ft. contours (fig. 12.17*a*) in the lines **aa′** and **bb′** respectively. Then the areas lying to the left of **aa′** and **bb′**, and bounded by these lines and by the 320- and 315-ft. contours respectively, represent

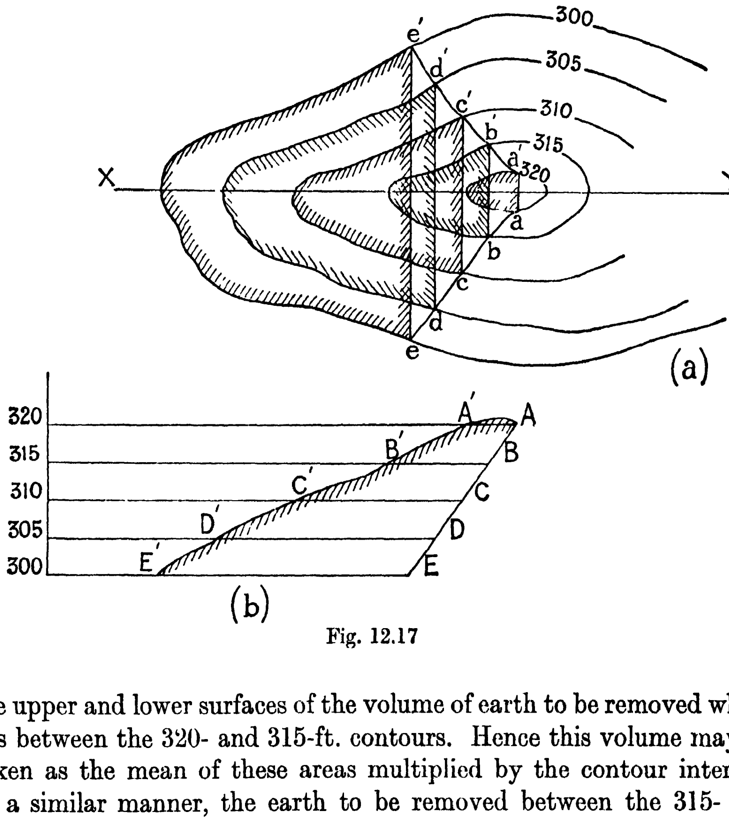


Fig. 12.17

the upper and lower surfaces of the volume of earth to be removed which lies between the 320- and 315-ft. contours. Hence this volume may be taken as the mean of these areas multiplied by the contour interval. In a similar manner, the earth to be removed between the 315- and 310-ft. contours is the contour interval multiplied by the mean of the areas bounded by the lines **bb′** and **cc′** and the portions of the 315- and 310-ft. contours which lie to the left of them. Proceeding in this way, by taking out the areas indicated in the plan by shading, the whole volume may be found.

It may be noted that the curve formed by joining **e**, **d**, **c**, **b**, **a**, **a′**, **b′**, **c′**, **d′** and **e′** represents the face of the cutting in plan.

QUESTIONS ON CHAPTER XII

1. Calculate the area of a cross-section of a cutting in flat country in which the depth is 6 ft., the total width of formation of the bottom 25 ft., and the side slopes one and a half horizontal to one vertical.

2. Calculate the area of a cross-section of a railway embankment resting on ground sloping sideways at a slope of 1 : 10. Depth of embankment at centre is 8 ft., total width on top 25 ft., and side slopes one horizontal to one vertical.

3. The following offsets were taken to a boundary from points along a chainage line, all measurements being in feet:

Chainage	Offset	Chainage	Offset
0	12	250	38
50	15	300	31
100	22	350	22
150	29	400	17
200	36		

Calculate the area of the figure by Simpson's rule.

4. A chainage line was run through the centre of a closed figure, and the total width of the figure was measured at different points along the line, and in a direction at right angles to it, the following being the results in feet:

Chainage	Width of figure	Chainage	Width of figure
0	0	400	205
100	120	500	140
200	180	600	80
300	216		

Calculate the area by Simpson's rule, and compare the result with the area calculated by the trapezoidal rule.

5. What is a prismoid, and why is the prismoidal formula particularly suitable for determining volumes of such shapes as the end-portions of cuttings and embankments?

Calculate the cubic contents of the length of embankment of which the cross-sectional areas at 50-ft. intervals are as follows:

Distance:	0	50	100	150	200	250	300
Area (sq. ft.):	110	425	640	726	1590	1790	2600

Make a similar calculation using the trapezoidal formula, and explain why the results differ. (Inst. C.E., April, 1948.)

6. Calculate the volume of the embankment described in the last example, but assume that the area at zero distance is zero instead of 110 sq. ft. Give the results calculated by both the prismoidal and trapezoidal formulæ.

7. Calculate the area of a triangle of which the measured sides are 1306 ft., 2248 ft., 1559 ft.

8. The following symbols relate to the dimensions of a railway cutting:

b = width of cutting at formation level;
h = depth of cutting on centre line;
s_1 and s_2 = side widths or half breadths;
h_1 and h_2 = the vertical heights above formation level of the upper and lower limits of the transverse slope;
n horizontal to 1 vertical = the inclination of the side slopes;
m horizontal to 1 vertical = the transverse inclination of the ground.

The transverse slope of the ground does not cut the formation level.

Derive an expression for the area of the cross-section and indicate how the dimensions s_1 and s_2 are obtained. (Inst. C.E., April, 1953.)

9. The formation level of the centre point, C, of a particular cross-section of a proposed cutting is 220·5 ft. The proposed formation width is 20 ft. and the side slopes are to be $1\frac{1}{2}$ to 1. A level is set up nearby and its height of collimation is 225·85 ft.

Describe how you would locate the side slopes of the section.

If the staff reading at the ground-level centre point D is 3·10 and the readings at the slope stakes are 3·85 and 2·25 respectively, what is the sectional area of the cutting? (Inst. C.E., October, 1954.)

APPENDIX TO PART I

SUPPLEMENTARY NOTES

1. Proof of Formulæ for the Sag Correction of a Metal Band with End Supports at the Same Level.

Consider an infinitesimally small length δs of the tape, or band, as in fig. A.1, and let the tangents at the ends of this length make angles θ and $\theta + \delta\theta$ with the horizontal tangent to the band at the centre of the span. Let the corresponding pulls on the ends be P and $P + \delta P$. Assuming that the band is perfectly flexible,* these pulls, together with the weight of the element, keep the latter in equilibrium, so that by resolving horizontally and vertically we have the following equations:

$$P \cos \theta - (P + \delta P) \cos (\theta + \delta\theta) = 0,$$

$$P \sin \theta + w\,\delta s - (P + \delta P) \sin (\theta + \delta\theta) = 0.$$

As $\delta\theta$ is a very small angle, we may write $\cos \delta\theta = 1$ and $\sin \delta\theta = \delta\theta$.

$$\therefore\ P \cos \theta - (P + \delta P)(\cos \theta - \delta\theta \sin \theta) = 0,$$

and $$P \sin \theta + w\,\delta s - (P + \delta P)(\sin \theta + \delta\theta \cos \theta) = 0.$$

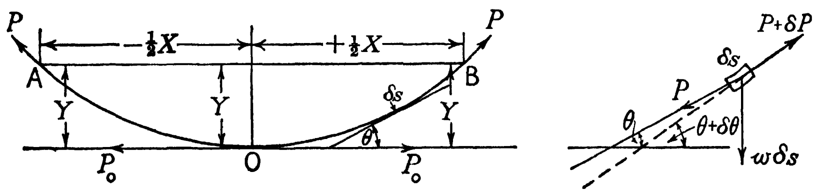


Fig. A.1

Also, as δP and $\delta\theta$ are both very small, we can put $\delta P \,.\, \delta\theta = 0$, and so

$$P\,\delta\theta \sin \theta - \delta P \cos \theta = 0,$$

$$P\,\delta\theta \cos \theta + \delta P \sin \theta = w\,\delta s,$$

or $$d(P \cos \theta) = 0,$$

and $$d(P \sin \theta) = w\,ds.$$

* This means that there are no internal bending moments or shearing stresses.

Integrating,
$$P \cos \theta = A,$$
$$P \sin \theta = ws + B,$$

where A and B are constants of integration.

But if we take the origin of co-ordinates at the centre of the span at its lowest point, and if P_0 is the pull there, which will be in a horizontal direction,
$$P_0 = A; \quad B = 0.$$

Consequently
$$P \cos \theta = P_0,$$
$$P \sin \theta = ws,$$

and
$$\tan \theta = \frac{ws}{P_0}.$$

But
$$dx = ds \cos \theta = \frac{ds}{\sec \theta} = \frac{ds}{\sqrt{[1 + \tan^2 \theta]}} = \frac{ds}{\sqrt{[1 + (ws/P_0)^2]}}$$
$$= \left[1 - \tfrac{1}{2}\left(\frac{ws}{P_0}\right)^2 + \tfrac{3}{8}\left(\frac{ws}{P_0}\right)^4 - \ldots\right] ds.$$

If X is the total chord length and l the total length of the band, $x = \frac{1}{2}X$ when $s = \frac{1}{2}l$, and $x = -\frac{1}{2}X$ when $s = -\frac{1}{2}l$. Hence, taking these as the limits of integration,
$$X = l - \frac{1}{24}\frac{w^2 l^3}{P_0^2} + \frac{3}{640}\frac{w^4 l^5}{P_0^4} - \ldots.$$

This gives X in terms of l, the length of the band, and of P_0, the unknown horizontal pull at the middle of the sag, while we want X in terms of l and of P, the observed pull at the end of the band.

At the end of the band, where $P = P$ and $s = \frac{1}{2}l$,
$$\tan \theta = \frac{wl}{2P_0}$$

and
$$P = P_0 \sec \theta.$$
$$\therefore \frac{1}{P_0^2} = \frac{1}{P^2} \sec^2 \theta = \frac{1}{P^2}(1 + \tan^2 \theta) = \frac{1}{P^2}\left(1 + \frac{w^2 l^2}{4P_0^2}\right)$$
$$= \frac{1}{P^2}\left(1 + \frac{w^2 l^2}{4P^2}\right).$$
$$\therefore X = l - \frac{w^2 l^3}{24P^2}\left(1 + \frac{w^2 l^2}{4P^2}\right) + \frac{3}{640}\frac{w^4 l^5}{P^4} - \ldots$$
$$= l - \frac{1}{24}\frac{w^2 l^3}{P^2} - \frac{11}{1920}\frac{w^4 l^5}{P^4} - \ldots.$$

In ordinary cases the third term is negligible and we have

$$X = l - \frac{1}{24}\frac{w^2l^3}{P^2}.$$

In order to get the sag, we have

$$dy = ds \sin\theta = ds\frac{\tan\theta}{\sec\theta} = ds \tan\theta[1 + \tan^2\theta]^{-\frac{1}{2}}$$

$$= ds \tan\theta[1 - \tfrac{1}{2}\tan^2\theta + \ldots]$$

$$= ds[\tan\theta - \tfrac{1}{2}\tan^3\theta + \ldots]$$

$$= \frac{ws}{P_0}ds - \tfrac{1}{2}\frac{w^3s^3}{P_0^3}ds + \ldots.$$

Now $y = 0$ when $s = 0$, and $y = Y$ when $s = \frac{1}{2}l$, Y being the sag at the centre of the span. Hence

$$Y = \int_0^{\frac{1}{2}l}\left[\frac{ws}{P_0} - \tfrac{1}{2}\frac{w^3s^3}{P_0^3} + \ldots\right]ds$$

$$= \tfrac{1}{8}\frac{wl^2}{P_0} - \tfrac{1}{128}\frac{w^3l^4}{P_0^3} + \ldots.$$

But

$$\frac{1}{P_0^2} = \frac{1}{P^2}\left(1 + \tfrac{1}{4}\frac{w^2l^2}{P^2} + \ldots\right),$$

$$\therefore \frac{1}{P_0} = \frac{1}{P}\left(1 + \tfrac{1}{8}\frac{w^2l^2}{P^2} + \ldots\right).$$

$$\therefore Y = \tfrac{1}{8}\frac{wl^2}{P} + \tfrac{1}{128}\frac{w^3l^4}{P^3} + \ldots,$$

in which the second term on the right is negligible in practice.

If this series is reversed so as to obtain $\frac{wl^2}{P}$ in terms of Y, we have

$$\frac{wl^2}{P} = 8Y - \frac{32Y^3}{l^2} + \ldots,$$

and substitution of this in the expression for sag correction already found gives

$$X = l - \frac{8Y^2}{3l} - \frac{32}{15}\frac{Y^4}{l^3} - \ldots.$$

When the last term on the right, which is very small numerically, is neglected, we get the formula given on p. 87, which can be used when sag correction is determined in the field by measuring the actual sag at the centre of the curve.

2. Approximate Adjustment of a Braced Quadrilateral.

Let ABCD (fig. A.2) be a braced quadrilateral in which all the angles marked 1, 2, 3, 4, 5, 6, 7 and 8 have been observed. Owing to small accidental errors of observation the sum of the angles in any triangle will not be exactly equal to the theoretical sum of 180°. Moreover, as pointed out on page 94, even after the angles have been adjusted to close the triangles, different values for the length of a side or diagonal will usually be obtained when, starting with the length of one side as base, the length of any other side is computed by a different route. Thus, the length of CD may be computed from the length of AB by using in turn the triangles ABC and BCD or by using the triangles ABD and ADC. In general, there will be a small difference between the two computed lengths and it is desired to "iron out" this difference as well as to make all three angles in each triangle add up to 180°. An approximate adjustment of the figure which does not require the use of the method of least squares may be obtained as follows:

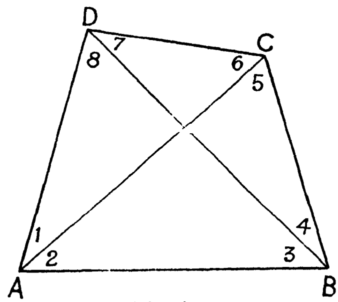


Fig. A.2

1. Add up the eight observed angles, the sum of which should be equal to 360°. Let there be a small difference, say e_1, and let the sum be greater than 360°, so that e_1 is positive. Then subtract one-eighth of e_1 from each angle so that the new values of the angles add up to 360° exactly. If the original sum is less than 360°, so that e_1 is negative, one-eighth of e_1 must be added to each angle.

2. Similarly, add angles 4 and 5 together and subtract the sum from the sum of 1 and 8. Let the difference be e_2. If e_2 is positive, subtract one-fourth of it from 1 and 8 and add one-fourth of it to 4 and 5 so that $1 + 8 - 4 - 5 = 0$. If e_2 is negative, add one-fourth of it to 1 and 8 and subtract one-fourth of it from 4 and 5.

3. Add angles 6 and 7 together and subtract the sum from the sum of angles 2 and 3. Let the difference be e_3. If e_3 is positive, subtract one-fourth of it from 2 and 3 and add one-fourth of it to 6 and 7 so that $2 + 3 - 6 - 7 = 0$. If e_3 is negative, add one-fourth of it to 2 and 3 and subtract one-fourth of it from 6 and 7.

4. Now look out the log sines of all the adjusted angles, at the same time booking the difference in the log sine for an increase of 1″ in the value of

the angle. (These differences, or multiples of them, are given in most Tables of log sines.) Calculate

$$(\log \sin 2 + \log \sin 4 + \log \sin 6 + \log \sin 8)$$
$$-(\log \sin 1 + \log \sin 3 + \log \sin 5 + \log \sin 7).$$

This sum should, in theory, be equal to zero. If not, let it be e_4. Also let $r_1, r_2, r_3, r_4, r_5, r_6, r_7$ and r_8 be the corrections to be added algebraically to the angles and let $d_1, d_2, d_3, d_4, d_5, d_6, d_7$ and d_8 be the differences in the last place of the log sines for an increase of 1″ in the angle. Then apply the following corrections

$$r_1 = \frac{e_4 d_1}{[d^2]},\ r_2 = -\frac{e_4 d_2}{[d^2]},\ r_3 = \frac{e_4 d_3}{[d^2]},\ r_4 = -\frac{e_4 d_4}{[d^2]},\ \ldots,\ r_7 = \frac{e_4 d_7}{[d^2]},\ r_8 = -\frac{e_4 d_8}{[d^2]}$$

to the various angles adjusted as above, $[d^2]$ being

$$d_1^2 + d_2^2 + d_3^2 + d_4^2 + d_5^2 + d_6^2 + d_7^2 + d_8^2.$$

This method is superior to, and in the end quicker than, the method of meaning results described in page 95.

A worked example is given on the facing pages 266 and 267. In this example, the observed angles are set out in column 2, the preliminary adjusted angles after the application of corrections for e_1 e_2, and e_3 in column 6 and the final corrected values in column 11. From this it will be seen that the discrepancy in the sum of the eight angles is reduced from —4″ to zero, the excess of the sum of 1 and 8 over the sum of 4 and 5 from +6″ to +0·8″, that of the sum of 2 and 3 over the sum of 6 and 7 from —12″ to —0·2″, while the excess in the difference of the log sines is reduced from —21 in the 6th place to zero. The adjustment does not always show such good results as this and in some cases, if the residual discrepancies are larger than desirable, it may be advisable to proceed with a second adjustment, using the finally adjusted angles from the first adjusment as observed angles.

The finally adjusted log sines, given in column 12, are obtained by multiplying together the relevant r and d and adding the result to the unadjusted log sine; they are checked by looking out the log sines of the finally adjusted angles.

The *side equation*, that is the equation for the difference of sums of log sines, can be deduced as follows:

The length of the side DC in terms of that of AB can be obtained from the triangles ADB and ADC, as well as from the triangles ABC and DBC. Thus, from the triangles ADB and ADC,

$$\frac{\text{AD}}{\text{AB}} = \frac{\sin 3}{\sin 8}.$$

$$\frac{\text{DC}}{\text{AD}} = \frac{\sin 1}{\sin 6}.$$

$$\therefore \text{ DC} = \text{AB}\frac{\sin 1}{\sin 6} \cdot \frac{\sin 3}{\sin 8}.$$

Similarly, from the triangles ABC and ADC,

$$\text{DC} = \text{AB}\frac{\sin 2}{\sin 5} \cdot \frac{\sin 4}{\sin 7}.$$

Hence, equating the two values of DC and taking logarithms,

$$(\log\sin 2 + \log\sin 4 + \log\sin 6 + \log\sin 8) -$$
$$(\log\sin 1 + \log\sin 3 + \log\sin 5 + \log\sin 7) = 0.$$

This equation would be rigorously satisfied if there were no errors in the observed angles. Owing to small errors of observation, however, the equation is not satisfied entirely when the values of the observed angles are substituted in it, and the difference in the sums of the log sines is e_4, so that 1, 2, 3, . . . 8, being observed values,

$$(\log\sin 2 + \log\sin 4 + \log\sin 6 + \log\sin 8) -$$
$$(\log\sin 1 + \log\sin 3 + \log\sin 5 + \log\sin 7) = e_4$$

In order to satisfy the equation we apply corrections $r_1, r_2, r_3, \ldots, r_8$, so that the corrected values are $1 + r_1$, $2 + r_2$, $3 + r_3, \ldots, 8 + r_8$. But as $r_1, r_2, r_3, \ldots, r_8$ are small quantities,

$$\log\sin(1 + r_1) = \log\sin 1 + r_1 d_1,$$
$$\log\sin(2 + r_2) = \log\sin 2 + r_2 d_2,$$
$$\log\sin(3 + r_3) = \log\sin 3 + r_3 d_3, \text{ etc.},$$

d_1, d_2, d_3, etc., being the differences in the log sines for 1″ increase in the value of the angles. Hence, we must have,

$$(\log\sin 2 + \log\sin 4 + \log\sin 6 + \log\sin 8) -$$
$$(\log\sin 1 + \log\sin 3 + \log\sin 5 + \log\sin 7) +$$
$$(r_2d_2 + r_4d_4 + r_6d_6 + r_8d_8) - (r_1d_1 + r_3d_3 + r_5d_5 + r_7d_7) = 0.$$

Hence

$$e_4 + (r_2d_2 + r_4d_4 + r_6d_6 + r_8d_8) - (r_1d_1 + r_3d_3 + r_5d_5 + r_7d_7) = 0.$$

EXAMPLE OF THE APPROXIMATE ADJUSTMENT

1	2	3	4	5	6
Angle No.	Observed values	$\frac{1}{8}e_1$	$\frac{1}{4}e_2$ and $\frac{1}{4}e_3$	Total corrn.	Preliminary adjusted angles
1	*81° 24′ 27″*	*+0·5″*	*−1·5″*	*−1·0″*	*81° 24′ 26·0″*
2	*35 25 17*	*+0·5*	*+3·0*	*+3·5*	*35 25 20·5*
3	*25 19 40*	*+0·5*	*+3·0*	*+3·5*	*25 19 43·5*
4	*61 42 14*	*+0·5*	*+1·5*	*+2·0*	*61 42 16·0*
5	*57 32 38*	*+0·5*	*+1·5*	*+2·0*	*57 32 40·0*
6	*32 34 20*	*+0·5*	*−3 0*	*−2·5*	*32 34 17·5*
7	*28 10 49*	*+0·5*	*—3·0*	*−2·5*	*28 10 46 5*
8	*37 50 31*	*+0·5*	*−1·5*	*−1·0*	*37 50 30·0*
	359 59 56	*+4·0*	*0·0*	*+4·0*	*360 00 00*

$e_1 = -4''$; $e_2 = +6''$; $e_3 = -12''$; $e_4 = -21$.

Formulæ.—

$(1 + 2 + 3 + 4 + 5 + 6 + 7 + 8) - 360° = e_1$. Apply $-\frac{1}{8}e_1$ to each angle.

$1 + 8 - 4 - 5 = e_2$. Apply $-\frac{1}{4}e_2$ to 1 and 8 and $+\frac{1}{4}e_2$ to 4 and 5.

$2 + 3 - 6 - 7 = e_3$. Apply $-\frac{1}{4}e_3$ to 2 and 3 and $+\frac{1}{4}e_2$ to 6 and 7.

$$(\log\sin 2 + \log\sin 4 + \log\sin 6 + \log\sin 8) - (\log\sin 1 + \log\sin 3 + \log\sin 5 + \log\sin 7) = e_4.$$

OF A BRACED QUADRILATERAL

7	8	9	10	11	12
log sines	d	d^2	r	Finally adjusted angles	Adjusted log sines
9·995 097	+0·32	0·102	−0·13	81° 24′ 25·9	9·995 097
9·763 128	+2·97	8·821	+0·96	35 25 21·5	9·763 131
9·631 253	+4·45	19·803	−1·43	25 19 42·1	9·631 246
9·944 736	+1·13	1·277	+0·36	61 42 16·4	9·944 736
9·926 243	+1·33	1 769	−0·43	57 32 39·6	9·926 243
9·731 066	+3·28	10 758	+1·05	32 34 18·6	9·731 070
9·674 160	+3·93	15·445	−1·26	28 10 45·2	9·674 155
9·787 802	+2·72	7·398	+0·87	37 50 30·9	9·787 804
e_4 = −21		65·373	−0·01	360 00 00·0	0

$e_1 = 0{\cdot}0''$; $e_2 = +0{\cdot}8''$; $e_3 = -0{\cdot}2''$; $e_4 = 0$.

Let d = difference in the log sine for an increase of 1″ in the angle; $[d^2]$ = sum of squares of the d's; $r_1, r_2, r_3 \ldots r_8$ the corrections to be applied to the angles after corrections for e_1, e_2 and e_3 have been applied to them. Then

$$r_1 = +\frac{e_4d_1}{[d^2]};\quad r_2 = -\frac{e_4d_2}{[d^2]};\quad r_3 = +\frac{e_4d_3}{[d^2]};\quad r_4 = -\frac{e_4d_4}{[d^2]};$$

$$r_5 = +\frac{e_4d_5}{[d^2]};\quad r_6 = -\frac{e_4d_6}{[d^2]};\quad r_7 = +\frac{e_4d_7}{[d^2]};\quad r_8 = -\frac{e_4d_8}{[d^2]}.$$

If e_1, e_2, e_3, e_4 are negative, the signs of the corrections to the angles are reversed in all the above formulæ.

Remember that d is positive for all angles less than 90° but negative for all angles between 90° and 180°.

This equation will be satisfied if r_1, r_2, r_3, etc., are numerically proportional to d_1, d_2, d_3, etc., or

$$\frac{r_1}{d_1} = -\frac{r_2}{d_2} = \frac{r_3}{d_3} = -\frac{r_4}{d_4} = \frac{r_5}{d_5} = -\frac{r_6}{d_6} = \frac{r_7}{d_7} = -\frac{r_8}{d_8} = k.$$

Then, substituting for $r_1, r_2, r_3, \ldots, r_8$ in the last equation,

$$e_4 - k(d_1^2 + d_2^2 + d_3^2 + d_4^2 + d_5^2 + d_6^2 + d_7^2 + d_8^2) = 0.$$

$$\therefore\ k = \frac{e_4}{(d_1^2 + d_2^2 + d_3^2 + \ldots + d_8^2)} = \frac{e_4}{[d^2]}.$$

$$\therefore\ r_1 = \frac{d_1e_4}{[d^2]},\quad r_2 = -\frac{d_2e_4}{[d^2]},\quad r_3 = \frac{d_3e_4}{[d^2]},\quad r_4 = -\frac{d_4e_4}{[d^2]}\ldots,$$

where $[d^2] = d_1^2 + d_2^2 + d_3^2 + \ldots + d_8^2$.

3. Further Note on Transferring Bearings from Ground Surface to Lines Underground.

On page 176 we have described a method of transferring bearings and levels from ground surface to lines underground, and the assumption there was that a theodolite could be set exactly in line with two wires suspended down a shaft. Space, however, may not permit setting a theodolite on the exact line of the wires, and several methods have been devised to overcome this difficulty, and so to carry on an underground traverse.

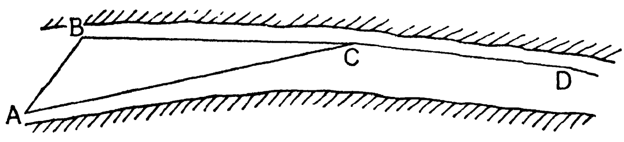


Fig. A.3

In fig. A.3, AB is the base line defined by the wires but there is not space to set up a theodolite on the line of AB continued. The first point of the underground traverse is therefore established at a point C as far away as possible along the line of the traverse and from which both A and B are visible. The lengths AB, BC, and AC are measured very carefully, and the angles ACB and ACD are observed, D being the next traverse station. The angles A, B, and C of the triangle can then be computed from the measured sides by the formulæ

$$r = \sqrt{\left[\frac{(s-a)(s-b)(s-c)}{s}\right]},$$

$$\tan \tfrac{1}{2}A = \frac{r}{s-a}; \quad \tan \tfrac{1}{2}B = \frac{r}{s-b}; \quad \tan \tfrac{1}{2}C = \frac{r}{s-c},$$

where $s = \frac{1}{2}(a+b+c)$, and hence the bearings of BC and AC can be computed from the bearing of AB. The angle C is computed as a check on the work, as this computed value should agree closely with the observed value and also with the value computed from

$$C = 180° - (A + B).$$

The triangle ABC is sometimes known as the *Weisbach Triangle.* Maximum accuracy is obtained when the angles at A and C are small.

Another method which is useful when the line AB lies more or less at right angles to the line of the traverse is shown in fig. A.4, where ABCD is a well-shaped quadrilateral of which the line AB between the wires is one side and the angles marked 1, 2, 3, 4, and 5 are observed and the sides and diagonals measured. This provides sufficient data, as well as several checks, for the solution of the quadrilateral and for determining the bearing of the first leg, DE, of the traverse.

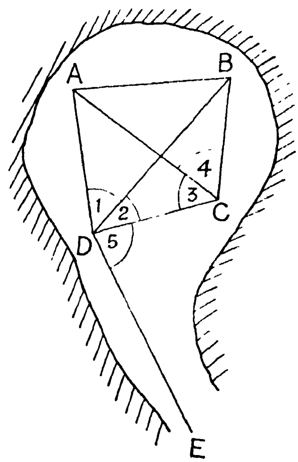


Fig. A.4

4. Adjustment of the Bearings of a Traverse between Azimuth Stations.

It sometimes happens that a traverse begins at one station where there is an observed azimuth and ends at a second station where there is also an observed azimuth, and it is necessary to adjust the bearings of the intermediate legs.

In fig. A.5 AB is a traverse which begins at station A where there is an observed azimuth, the direction of the meridian at A, that is the direction of true north at A, being AN, and the traverse is carried forward to the point B, where there is also an observed azimuth. It is assumed that the bearings of the intermediate legs will be used to compute rectangular co-ordinates in which AN will be the direction of the axis of X.

At B the direction of the meridian is BN, but, supposing that the bearings of the traverse are based on the observed azimuth at A so that the angle NAm $= \alpha$ is the bearing of the first leg of the traverse, then the bearing, β, of the last leg, sB, as calculated through the intermediate angles, supposed without error, will be the angle N′BC, where BC is a continuation of sB and BN′ is parallel to AN. The meridians at A and B, however, are not parallel but converge towards the earth's pole, and hence BN, and not BN′, will be the direction of the meridian

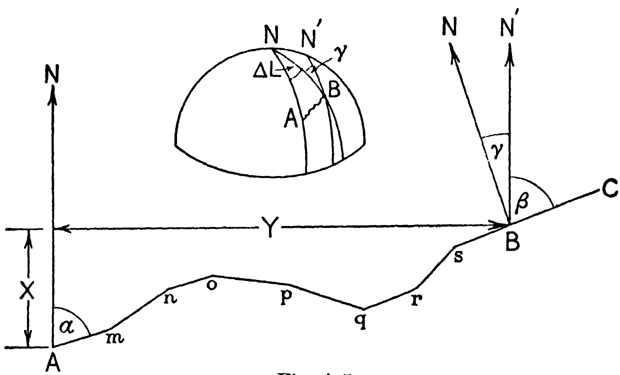


Fig. A.5

at B, the angle γ between BN and BN′ being called the *convergence at B*. (See the small inset in fig. A.5 representing the earth's northern hemisphere in which N is the north pole, NA and NB the meridians through A and B, and BN′ is a small circle through B, the plane containing it being parallel to the plane containing the meridian AN.) Thus, the bearing of sB brought forward from A will be the angle N′BC, while the azimuth of sB will be the angle NBC. Hence if we are to get a true comparison between the azimuth of sB and the bearing of sB brought forward through the intermediate angles from A, or a bearing to which intermediate bearings are to be adjusted, we must deduct the angle γ from the azimuth of BC, the value of γ being given in seconds of arc by

$$\gamma'' = \frac{Y \tan \phi_B}{R \sin 1''},$$

in which Y is the Y rectangular co-ordinate of B from A, ϕ_B is the latitude of B, and R is the radius of the earth expressed in the same linear units as Y. Strictly speaking, the R to be used here depends on the latitude of B, but, for minor work, $\log \dfrac{1}{R \sin 1''}$ may be taken as $\bar{3}{\cdot}99373$ when Y is in feet, this being the value for latitude 45°.

If we call AN the standard meridian to which all traverse bearings are referred, and azimuths are reckoned clockwise from 0° to 360° from north, we have the following rules for applying convergence:

For points in the northern hemisphere

Bearing sB = Azimuth of sB − γ when B is east of A.
Bearing sB = Azimuth of sB + γ when B is west of A.

For points in the southern hemisphere

Bearing sB = Azimuth of sB + γ when B is east of A.
Bearing sB = Azimuth of sB − γ when B is west of A.

The latitude of B, if not known from special astronomical observations, may generally be obtained with sufficient accuracy for minor work by scaling from a map, and the value of Y from a preliminary computation of the traverse using the unadjusted bearings.

Suppose, for instance, that the observed bearing of sB brought forward through a traverse of 10 legs from A is 68° 13′ 21″, the observed azimuth at B is 68° 19′ 59″, and that $Y = +\,48\,954$ ft. and $\phi_B = 37°\ 19'\ 40''$ north, the point B being east of A since Y is positive. Then

$$\begin{aligned} \log Y &= 4{\cdot}689\ 79 \\ \log \tan \phi_B &= 9{\cdot}882\ 28 \\ \log 1/R \sin 1'' &= \underline{\bar{3}{\cdot}993\ 73} \\ \log \gamma'' &= 2{\cdot}565\ 80 \\ \gamma &= 367{\cdot}96'' \\ &= 6'\ 08{\cdot}0'' \end{aligned}$$

$$\begin{aligned} \text{Azimuth of sB} &= 68°\ 19'\ 59'' \\ \gamma &= \underline{\quad -\ 6\ \ 08''} \\ \text{Correct bearing of sB} &= 68\ \ 13\ \ 51 \end{aligned}$$

But the traverse bearing is 68° 13′ 21″, which is 30″ too low. Hence the discrepancy must be distributed between the 10 bearings, or 3″ must be added to the bearing of the first leg, 6″ to the bearing of the second leg, 9″ to the bearing of the third leg, and so on, and finally 30″ to the bearing of the last leg. The bearings having been adjusted in this way, the next step is to re-compute the co-ordinates, using the adjusted bearings.

It should be noted that the formula for γ is approximate only and should only be used for minor work of limited extent, say 60 miles, in

Y. For more extensive surveys, or surveys of greater accuracy, the reader should consult books on geodesy or advanced surveying. Thus, the convergence mentioned on page 36 relates to the convergence applicable to the National Survey of Great Britain and is the angular difference between the direction of the meridian at any point and the direction of the central meridian for the whole country. In this case, therefore, the Y may be very large and the simple formula for convergence given above would not be nearly accurate enough.

It should also be noted that, if the meridian through a point A is the direction of the axis of X co-ordinates to which bearings are referred, and if B and C are two other points between which a traverse has been run and at which azimuths have been observed, then the initial bearing at B from which the bearings for the traverse start will be the azimuth there $\pm$ the convergence between B and A, and the final bearing at C to which they will be adjusted is the azimuth there $\pm$ the convergence between C and A.

5. Correction to Traverse Latitudes and Departures for Small Corrections to Lengths and Bearings.

In the last section we noticed that it was necessary to have a value for Y before the convergence at a traverse azimuth station could be calculated and the bearings finally adjusted. Final co-ordinates, however, could not be obtained without finally adjusted bearings. Since the value of Y required to compute γ need only be approximate, a preliminary computation might be done with four-figure logs and approximate lengths and bearings, the final computation being with full-figure logs and the fully measured lengths and adjusted bearings. An alternative is to compute the traverse at the very beginning with full-figure logs, the complete and accurate measures of the lengths, and the observed, but so far unadjusted, values of the bearings, and then, when the corrections to the bearings are known, to apply corrections, computed as follows, to the preliminary values of the latitudes and departures. The corrections in question are numerically small and can quite well be computed with slide rule or four-figure logs.

On page 129 we showed that the errors $\delta(\Delta x)$ and $\delta(\Delta y)$ in latitude, Δx, and departure, Δy, produced by errors δl in l and $\delta\alpha$ in α are, when $\delta\alpha$ is expressed in seconds of arc,

$$\delta(\Delta x) = \delta l \cos\alpha - \Delta y \,.\, \delta\alpha \sin 1''$$
$$\delta(\Delta y) = \delta l \sin\alpha + \Delta x \,.\, \delta\alpha \sin 1''$$

Instead of treating $\delta(\Delta x)$, $\delta(\Delta y)$, δl and $\delta\alpha$ as errors, we can treat them as corrections, so that, if $\delta\alpha$ is the correction to be applied to a bearing, the consequential corrections $\delta(\Delta x)$, $\delta(\Delta y)$, to be applied to the latitudes and departures are:

$$\begin{aligned}
\delta(\Delta x_1) &= -\Delta y_1\,\delta\alpha_1 \sin 1''; & \delta(\Delta y_1) &= +\Delta x_1\,\delta\alpha_1 \sin 1'',\\
\delta(\Delta x_2) &= -\Delta y_2\,\delta\alpha_2 \sin 1''; & \delta(\Delta y_2) &= +\Delta x_2\,\delta\alpha_2 \sin 1'',\\
\delta(\Delta x_3) &= -\Delta y_3\,\delta\alpha_3 \sin 1''; & \delta(\Delta y_3) &= +\Delta x_3\,\delta\alpha_3 \sin 1'',\\
&\cdots & &\cdots\\
\delta(\Delta x_n) &= -\Delta y_n\delta\alpha_n \sin 1''; & \delta(\Delta y_n) &= +\Delta x_n\delta\alpha_n \sin 1''.
\end{aligned}$$

It is much easier to apply these corrections to the latitudes and departures than it is to re-compute the whole traverse with the original lengths and the corrected bearings, because the corrections are only small quantities which can be computed with a slide rule or four-figure logs, whereas to re-compute the whole traverse from the beginning means the use of the full-figure logarithms and the full values of the lengths and bearings. In these formulae $\sin 1''$ may be taken as 0·000 00485 and $\log \sin 1''$ as $\bar{6}{\cdot}6856$. When latitudes and departures are computed by machine using the natural cosines and sines, it may be simpler to jot down, when interpolating, the differences, d_c and d_s, in the values of the cosine and sine for $1''$ difference in α for each value of α and then to replace the terms $-\Delta y\,\delta\alpha \sin 1''$ and $+\Delta x\,\delta\alpha \sin 1''$ by $-l.d_c\,\delta\alpha$ and $+l.d_s\,\delta\alpha$ respectively.

Similarly, the corrections to the latitudes and departures for given corrections to lengths are given by

$$\delta(\Delta x) = \delta l \cos\alpha; \quad \delta(\Delta y) = \delta l \sin\alpha.$$

But corrections to lengths are usually constant fractions of the lengths, so that $\delta l_1 = kl_1$, $\delta l_2 = kl_2$, $\delta l_3 = kl_3$, etc., where k is a constant. Hence the corrections to the latitudes and departures on account of corrections kl_1, kl_2, kl_3, etc., are

$$\begin{aligned}
\delta(\Delta x_1) &= kl_1 \cos\alpha_1 = k\,\Delta x_1; & \delta(\Delta y_1) &= kl_1 \sin\alpha_1 = k\,\Delta y_1,\\
\delta(\Delta x_2) &= kl_2 \cos\alpha_2 = k\,\Delta x_2; & \delta(\Delta y_2) &= kl_2 \sin\alpha_2 = k\,\Delta y_2,\\
\delta(\Delta x_3) &= kl_3 \cos\alpha_3 = k\,\Delta x_3; & \delta(\Delta y_3) &= kl_3 \sin\alpha_3 = k\,\Delta y_3,\\
&\cdots & &\cdots\\
\delta(\Delta x_n) &= kl_n \cos\alpha_n = k\,\Delta x_n; & \delta(\Delta y_n) &= kl_n \sin\alpha_n = k\,\Delta y_n.
\end{aligned}$$

These results can be seen at once from first principles, as the corrections are obviously simply a matter of a change of scale, and it can also be seen that the corrections to the co-ordinates, or total latitude and departure, of any point are kx and ky.

PART II

Elementary Field Astronomy

CHAPTER XIII

INTRODUCTORY

1. Functions of Field Astronomy.

The object of astronomical observations in survey work is to fix the position and orientation of a survey on the earth's surface when no fixed points are available on the ground of which the latitudes, longitudes, and the azimuths, or true bearings, of certain of the lines between different points are known. It is not necessary, of course, for the exact position of every survey on the earth's surface to be known, or for the bearings used to be based on a true bearing or azimuth at one point, but in some cases this is necessary and consequently astronomical observations to the sun or stars have to be taken to obtain time, latitude, longitude, and azimuth. Often, however, it is only necessary to obtain azimuth. Such a case will arise when the latitude and longitude can be scaled from a map with sufficient precision Azimuth, however, cannot be obtained from a map and, if no lines of known azimuth exist on the ground, special astronomical observations must be taken to determine it. In some methods of doing this, it is necessary to have an approximate value of the latitude of the point of observation in order to compute the azimuth, and in some it is necessary to know the time of observation. Here, the approximate latitude, if not known from the map, and time, if not obtainable from radio time signals, must be determined astronomically.

2. Solar and Stellar Observations.

The celestial bodies to which observations are taken for survey work are the sun and stars; the moon and planets are virtually useless for this purpose. The sun is a little easier to observe than the stars because there is no difficulty in finding it or in the identification of the right body, but observations to it are never so accurate as careful observations to a star, and they are rather more troublesome to compute. In addition, since the sun's position in the heavens changes very rapidly with time when compared with that of a star, it is generally necessary to observe the times of solar observations, and, as the sun's position is tabulated in Almanacs and Ephemerides in terms of Greenwich time, it is necessary to be able to convert the recorded times of observation to Greenwich times. This can be done by comparing the clock with radio time signals, or else, when no suitable wireless set or chronometer keeping Greenwich time is available, by observing local time astronomically and then, knowing the longitude, converting local into Greenwich time.

Solar observations have the great advantage that they can be taken by day instead of at night, and hence they are very commonly used by engineers and surveyors when great accuracy is not required. In geodetic work, however, stellar observations are always used instead of solar observations because of their greater accuracy and the possibility of taking observations to a number of different stars, thus varying the conditions of observation.

3. Factors affecting Choice of Method.

There are a number of methods available for determining time, latitude, longitude, and azimuth. Some are more accurate, and some easier and more convenient than others. Hence the methods adopted vary according to the accuracy required and the instruments available, more care in observation naturally being needed for the more accurate work.

4. Instruments.

The instruments most commonly used in field and geodetic astronomy are the sextant, prismatic astrolabe, theodolite, chronometer, stop watch, and chronograph (recording), and, for working out corrections to observed altitudes, an aneroid barometer graduated in inches of pressure, and a thermometer. A suitable receiving set is a useful addition for obtaining Greenwich time, and one capable of receiving the scientific time signals is now a necessity for the accurate determination of longitude.

The prismatic astrolabe is designed specially for the determination of latitude and longitude and cannot be used for the determination of azimuth. The recording chronograph also is a special instrument which is used in geodetic work for the precise measurement of short intervals of time. Neither of these instruments is used to any extent in ordinary minor survey work, and hence descriptions of them are not given in these pages.

The sextant, complete with artificial horizon, is used for determining latitude and time in navigation and in reconnaissance, geographical, and exploratory surveys where great accuracy is not needed. It cannot be used for azimuth determinations and for these a theodolite must be employed. For a description of the sextant see *Principles and Use of Surveying Instruments*, Chapter IV.

In all cases where a sextant or theodolite is used for solar observations it must be provided with a dark-glass shade or cover for the eyepiece, as very serious damage can be done to the eye if the sun is viewed through a telescope, even for a second or two, without some sort of protection for the eye.

The theodolite is a good all-round instrument suited for most classes of astronomical work as well as for the measurement of horizontal and vertical angles. When it is to be used for astronomical work, it should be provided with a dark shade for solar observations and with a diagonal eyepiece. A striding level for measuring slight inclinations of the trunnion or horizontal axis is also a useful accessory. For night observations, it is necessary to have some means of illuminating the cross hairs and micrometers or verniers. Many of the modern instruments have built-in electric illumination, operated by an accumulator or dry battery, for illuminating the cross hairs and micrometers; in others, it is possible to obtain a special cap to fit over the objective end of the telescope with a slit and a small reflecting prism at one side. By this means light from an electric torch held at the side can be reflected down the barrel to illuminate the cross hairs, the torch also being employed to light up the micrometers or verniers. The diagonal eyepiece is needed for observing bodies of high altitude, as, when an altitude of about 45° or over has to be observed, the horizontal circle gets in the way of the ordinary eyepiece, which cannot then be used.

When the observations concerned involve accurate measurements of time, as when azimuths are being determined from observed hour angles, a good chronometer, either pocket or boxed, is needed, and, when the surveyor is alone and has not got a reliable booker who can note the time as he observes the heavenly body, a good stop watch, readable to a tenth of a second, is useful for recording the short interval between the time when the body appears to be on the cross hair and the time when the chronometer can be read a few seconds later. If much astronomical work has to be done, it is a good plan to have two chronometers, one rated to keep ordinary mean time and the other rated to keep mean sidereal time in which 24h. mean sidereal time is equal to 23h. 56m. 04·09s. ordinary mean time. Not only are two such chronometers useful if both stellar and solar observations are being taken, but also they give a spare in the event of one breaking down.

A good wireless set for the reception of the wireless time signals is now practically a necessity for field astronomy work. In addition to the usual provision for the reception of the ordinary Greenwich time signals on the long and medium wave-lengths, it is advisable to have a set capable also of short-wave reception, so that, if necessary, advantage can be taken of the 24-hours-a-day service from the American National Bureau of Standards Station WWV, which radiates on frequencies of 2·5 to 25 megacycles.

CHAPTER XIV

SOME BASIC DEFINITIONS AND FORMULAE OF SOLID GEOMETRY AND SPHERICAL TRIGONOMETRY

As a necessary preliminary to the study of field astronomy it is essential that the reader should clearly understand certain geometrical properties of the sphere and have some knowledge of elementary spherical trigonometry. Accordingly, this chapter will be devoted to a résumé of those parts of these subjects which we shall require later on.

1. The Sphere: Definitions and Properties.

A *sphere* is a surface such that every point of it is at a constant distance from a fixed point inside it which is known as its *centre*, the constant distance being known as its *radius*. Thus, in fig. 14.1, P, P′,

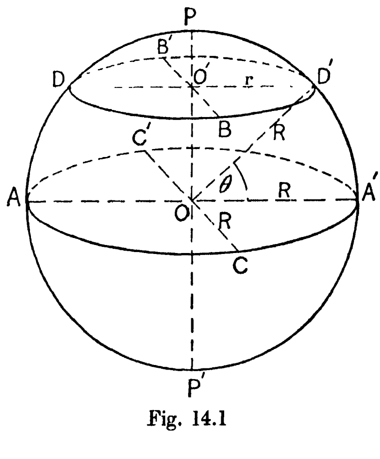


Fig. 14.1

A, A′, B and C are fixed points on the sphere whose centre is O, and the distances of these points from the centre, OP, OP′, OA, OA′, OB and OC, are each equal to the radius R. Any line passing through the centre of the sphere will intersect the surface in two points, and such a line is known as a *diameter*. Thus, the lines POP′ and AOA′, which intersect the surface of the sphere at the points P, P′, and A, A′ are diameters.

Lines such as AP or BO′B′ which join two points on the surface but do not pass through the centre are not diameters.

Any plane passing through the centre of the sphere will intersect the surface in a circle, which is called a *great circle*. If the plane does not pass through the centre of the sphere it will intersect the surface of the sphere in a circle known as a *small circle*. Thus, in fig. 14.1, the plane containing the points ACA′C′ passes through the centre of the sphere and traces out the circle ACA′C′ on the surface. Hence, this circle is a great circle. The plane containing the points DBD′B′ does not contain the centre of the sphere but intersects the surface in the circle DBD′B′, and this circle is therefore a small circle.

If a diameter POP′ is drawn through the centre of the sphere perpendicular to the great circle ACA′C′, i.e. perpendicular to the plane containing the circle, it will intersect the surface of the sphere in two points, P and P′. These points are known as the *poles* of the great circle. Similarly, if POP′ is perpendicular to the small circle DBD′B′, i.e. perpendicular to the plane containing the circle, and passes through the centre O′ of this circle, the points P and P′ where the line intersects the surface of the sphere are the *poles* of the small circle.

The radius of a great circle is equal to the radius, R, of the sphere. The radius, r, of the small circle is the length O′B = O′B′ = O′D = O′D′. If the great circle PD′A′P′ADP is drawn through D′ perpendicular to the planes containing the small circle DBD′B′ and the great circle ACA′C′, both planes being parallel to one another, and the line OD′ makes angle θ with OA′, it can be seen that

$$r = O'D' = OD' \,.\, \cos\theta = R \,.\, \cos\theta. \quad . \quad . \quad . \quad (1)$$

2. The Spherical Triangle.

In fig. 14.2*a*, ABXA′ and ACX′A′ are parts of two great circles of which the containing planes intersect along the diameter AOA′. Let AT and AT′ be the tangents to the circles at the point A. These tangents will, of course, be in the planes containing the great circles. From O draw OX perpendicular to OA in the plane ABXA′ and OX′ perpendicular to OA in the plane ACX′A′. Then the points X and X′ lie on a great circle whose pole is A. The angle between the great circles is the angle between the planes containing them, and this is measured by the angle TAT′ = XOX′, or, if the sphere is of unit radius, the circular measure of the angle will be given by the length of the arc XX′.

Through the point B on the great circle ABXA′ draw a great circle BCB′ intersecting the great circle ACX′A′ in the point C. Then the arcs

AB, BC, and AC form a figure on the sphere known as a *spherical triangle* which is shown in fig. 14.2*b*. This figure is the equivalent on the sphere of an ordinary plane triangle on a plane, and, like the plane triangle, it consists of six *parts*, three of which are known as the angles and three as the sides. The angles are (1) the angle at A between the great circles ABXA′ and ACX′A′, (2) the angle at B between the great circles ABXA′ and BCB′, and (3) the angle at C between the great circles ACX′A′ and BCB′. The sides are the angles represented by the arcs AB, BC, CA, as we are not concerned with the lengths of these arcs but only with the angles they subtend at the centre of the sphere.

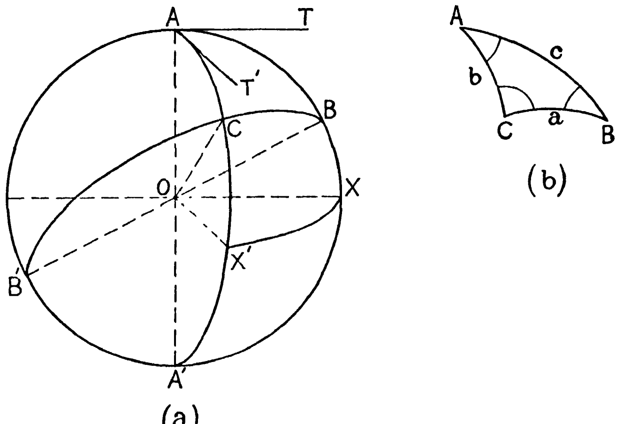


Fig. 14.2

Hence, the side AB is the angle AOB which lies in the plane of the great circle ABXA′ containing AB; the side BC is the angle BOC which lies in the plane containing the great circle BCB′; and the side AC is the angle AOC, which lies in the plane containing the great circle ACX′A′.

As in the case of the plane triangle, we denote the angles of a spherical triangle by capital letters, such as A, B and C, and the sides by the small letters, a, b and c, corresponding to the capital letters denoting the angles at the apices of the triangle which lie opposite to them (see fig. 14.2*b*). Note again, however, that in the plane triangle the sides are lengths but that in the case of the spherical triangle they are angles.

3. Solution of Spherical Triangles.

In a plane triangle the value of any one of the six parts can be calculated if we know the values of three other parts of which at least one must be a side, and the same rule holds with regard to spherical triangles though here the three unknown parts may consist of three angles or three sides as well as different combinations of angles and sides. Thus, in fig. 14.2*b*, we can calculate the side a if we are given (1) the angles A and B and the side b, or (2) the sides b and c and the angle A, or (3) the three angles A, B and C. Hence, corresponding to the formulæ of plane trigonometry for the solution of plane triangles, in spherical trigonometry we have analogous expressions which resemble the others in form but in which we have trigonometrical functions of the angles representing the sides instead of linear values of the sides themselves. Many of these formulæ are set out below and the student should compare them with the corresponding formulæ of plane trigonometry.

$$\frac{\sin A}{\sin a} = \frac{\sin B}{\sin b} = \frac{\sin C}{\sin c}, \qquad (2)$$

$$\cos a = \cos b \cos c + \sin b \sin c \cos A, \qquad (3)$$

$$\cos A = -\cos B \cos C + \sin B \sin C \cos a, \qquad (4)$$

$$\tan \tfrac{1}{2}(a + b) = \frac{\cos \frac{1}{2}(A - B)}{\cos \frac{1}{2}(A + B)} \tan \tfrac{1}{2}c, \qquad (5)$$

$$\tan \tfrac{1}{2}(a - b) = \frac{\sin \frac{1}{2}(A - B)}{\sin \frac{1}{2}(A + B)} \tan \tfrac{1}{2}c, \qquad (6)$$

whence we get a and b from $a = \frac{1}{2}(a + b) + \frac{1}{2}(a - b)$ and $b = \frac{1}{2}(a + b) - \frac{1}{2}(a - b)$.

$$\tan \tfrac{1}{2}(A + B) = \frac{\cos \frac{1}{2}(a - b)}{\cos \frac{1}{2}(a + b)} \cot \tfrac{1}{2}C, \qquad (7)$$

$$\tan \tfrac{1}{2}(A - B) = \frac{\sin \frac{1}{2}(a - b)}{\sin \frac{1}{2}(a + b)} \cot \tfrac{1}{2}C, \qquad (8)$$

whence we get A and B from $A = \frac{1}{2}(A + B) + \frac{1}{2}(A - B)$ and $B = \frac{1}{2}(A + B) - \frac{1}{2}(A - B)$.

$$\tan \tfrac{1}{2}A = \sqrt{\left[\frac{\sin (s - b) \sin (s - c)}{\sin s \sin (s - a)}\right]}, \qquad (9)$$

$$\sin \tfrac{1}{2}A = \sqrt{\left[\frac{\sin (s - b) \sin (s - c)}{\sin b \sin c}\right]}, \qquad (10)$$

where $s = \frac{1}{2}(a + b + c)$.

When all three angles A, B, and C are needed, calculate

$$\tan r = \sqrt{\left[\frac{\sin (s-a) \sin (s-b) \sin (s-c)}{\sin s}\right]}, \quad . \quad (11)$$

and then

$$\tan \tfrac{1}{2}A = \frac{\tan r}{\sin (s-a)}; \quad \tan \tfrac{1}{2}B = \frac{\tan r}{\sin (s-b)}; \quad \tan \tfrac{1}{2}C = \frac{\tan r}{(s-c)}. \quad (12)$$

Also

$$\cot a \sin c = \cos c \cos B + \sin B \cot A, \quad . \quad . \quad . \quad (13)$$

$$\sin a \cos B = \sin c \cos b - \cos c \sin b \cos A, \quad . \quad . \quad (14)$$

$$\sin A \cos b = \sin C \cos B + \cos C \sin B \cos a. \quad . \quad (15)$$

Other formulæ can, of course, be obtained from suitable interchanges of a, b, and c and A, B, and C. Thus, interchanging b for a and B for A in formula (3), we get the analogous expression

$$\cos b = \cos a \cos c + \sin a \sin c \cos B. \quad . \quad . \quad (16)$$

It should also be noted that, when an angle is obtained from the sine, as when A is computed from (2) or (15), there is an ambiguous case because $\sin A = \sin (180° - A)$. The value to be taken can often be determined at once by inspection of a diagram showing the conditions of the problem or from remembering that the larger angle of a triangle is subtended by the larger side, but in cases of doubt it is better to use a formula involving the tangent or cosine of the unknown instead of the sine.

4. The Spherical Right-angled Triangle.

When one angle of a spherical triangle is a right angle the formulæ are considerably simplified since in them we can then substitute unity for the sine of the right angle and zero for the cosine. Thus, if C is the right angle, we have:

$$\sin a = \sin c \sin A = \tan b \cot B, \quad . \quad . \quad . \quad (17)$$

$$\sin b = \sin c \sin B = \tan a \cot A, \quad . \quad . \quad . \quad (18)$$

$$\cos A = \cos a \sin B = \cot c \tan b, \quad . \quad . \quad . \quad (19)$$

$$\cos B = \cos b \sin A = \cot c \tan a, \quad . \quad . \quad . \quad (20)$$

$$\cos c = \cos a \cos b = \cot A \cot B. \quad . \quad . \quad . \quad (21)$$

CHAPTER XV

APPARENT MOTIONS OF CELESTIAL BODIES
THE CELESTIAL SPHERE

1. Apparent Daily Motions of Stars and Sun.

An observer watching the sky by night will see that the stars appear to move relative to the horizon; some will appear to sweep out paths from east to west and others from east to west for a time, and then, as they sink lower in the heavens, from west to east. Some will be visible all the time but others will appear to rise above the horizon in the east and then, after seeming to ascend higher and higher in the heavens until they reach a maximum elevation, they will appear to get lower and lower until they finally set, or sink below the horizon, in the west. Further, it will be noted that stars do not appear to alter their positions relative to one another. Moreover, closer examination will show that all stars appear to revolve during the course of the 24 hours about a fixed point in the heavens which, in the northern hemisphere, is north of the observer's zenith and is very close to one particular star—the pole star. Unless the observer is on, or very close to the earth's equator, some stars near to the elevated pole* will remain above the horizon all the time and will neither rise or set. These stars—called *circum-polar stars*—will seem for a time to travel from east to west and then from west to east.

Similarly, in the northern hemisphere south of latitude $67\frac{1}{2}°$ N. an observer will see the sun rise each day in the east and continue to rise until the middle of the day, when it will start to get lower, finally to set in the west. For observers north of latitude $67\frac{1}{2}°$N., during certain seasons of the year the sun will neither rise nor set but will be visible for the whole of the 24 hours: at other seasons, it will not appear above the horizon at all and it will be dark for the whole of the 24 hours, and for seasons in between it will rise and set each day, but its times of rising and setting and its elevation above the horizon at midday will vary greatly from month to month. In addition, if the sun's position is com-

* The *elevated pole* is the pole which is above the observer's horizon. To an observer in the northern hemisphere it is the north pole and to one in the southern hemisphere it is the south pole.

pared day after day and night after night with the positions of the stars, it will be seen that its position changes considerably relative to theirs in a comparatively short time.

The apparent daily motion from east to west of all heavenly bodies, and then of some from west to east, is due to the earth's daily rotation about its polar axis. The earth actually rotates from west to east, but, as the observer is unaware of his own motion, a heavenly body which rises and sets appears to him to move from east to west. This is the same effect as the one we experience sometimes when a train in which we are travelling is standing still at a station and a moving one passing us appears to us to be stationary, while we appear to be travelling in the direction opposite to that of the moving one.

2. The Celestial Sphere.

Since an observer on the earth is not conscious of the relative distances of the stars, although some appear to be brighter than others, what he sees is what he would see if the stars were situated on the inside surface of a very large sphere and he were at the centre of the sphere,

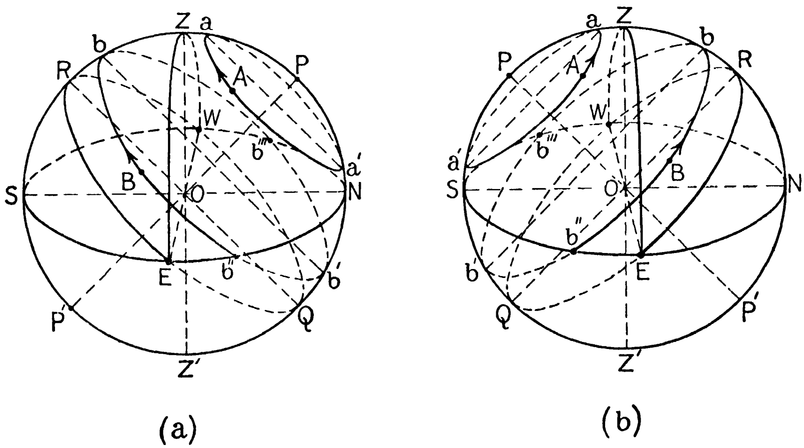


Fig. 15.1

a conception which is invaluable in the study of spherical astronomy. Thus, in fig. 15.1*a* and *b*, the stars A and B can be considered to be on the surface of the sphere whose centre O coincides with the observer's position. This sphere is called the *celestial sphere.* If the observer's eye were level with a perfectly horizontal plane, this plane, called the *horizon plane*, would intersect the sphere in the great circle NESWN, called the *horizon circle*. The point Z immediately above him where a straight line drawn vertically through O perpendicular to the plane NESWN meets the surface of the sphere, is the observer's *zenith*, and the point Z′ on the surface of the sphere immediately below him is the *nadir point*, or simply the *nadir*. Owing to his rotation from west to east, the stars A and B will appear to him to describe the small circles a′Aaa′ and b′Bbb′, in the direction shown in the diagram, about the *celestial pole* P where the line OP is a prolongation of the earth's axis of rotation to meet the celestial sphere at P. Owing to the great distances of the stars compared with the diameter of the earth, what the observer sees from his position on the surface of the earth is, to all intents and purposes, exactly what he would see if he could be imagined to be at the centre of the earth. Hence, the centre of the celestial sphere is considered also to be the centre of the earth.

The plane ZPNZ′SZ which contains the zenith Z and the pole P and is perpendicular to the horizon plane NESWN is the observer's *meridian plane*, and the great circle ZPNZ′SZ which the plane traces out on the surface of the sphere is his *meridian*. Hence, the meridian is the great circle traced out on the surface of the celestial sphere by the plane perpendicular to the horizon plane which passes through both zenith and pole. Also, as the latitude of the point on the earth where the observer is standing is the angle which the earth's axis of rotation makes with the observer's horizon plane, the angle NOP in the meridian plane in fig. 15.1*a*, which represents the case where the observer is in the earth's northern hemisphere, or the angle SOP in the meridian plane in fig. 15.1*b*, which represents the case where the observer is in the earth's southern hemisphere, is the observer's *latitude* ϕ. The complement of the latitude, $90° - \phi$, is the arc PZ and is known as the *co-latitude*, *c*.

The points N and S where the meridian intersects the horizon are the *north* and *south* points respectively. The plane ZEOWZ drawn through Z perpendicular to the horizon plane NESWN and the meridian plane ZPNZ′SZ is the *prime vertical* plane. This plane intersects the sphere in a great circle EZW, the *prime vertical*, and this circle meets the horizon circle NESWN in the *east point* E and the *west point* W. When the observer in the northern hemisphere faces the elevated pole—the northern pole in his case—the east point lies to his right and the west point to his left, as in fig. 15.1*a*. On the other hand, when an observer in the southern hemisphere faces the elevated pole lying above the horizon plane—the southern pole—the east point will lie to his left and the west point to his right as in fig. 15.1*b*.

A plane through O perpendicular to OP will trace out a great circle ERWQE which passes through the west point W and the east point E. This circle is called the *celestial equator.*

It will be seen from fig. 15.1 that the star A completes a whole revolution above the observer's horizon, and hence it will never set, or disappear below the horizon. The condition for this is that the angle QOa′ should be greater than the angle QON in fig. 15.1*a* or QOS in fig. 15.1*b*. But the angle QOa′, which is the angle that the star's position makes with the plane of the celestial equator, is a fixed angle for the star which is known as the star's *declination* and is denoted by δ. Similarly, the angles QON in fig. 15.1*a* and QOS in fig. 15.1*b* are each equal to $90° - \phi$. Hence the condition that the star should neither rise nor set in the ordinary way is that

$$\delta > 90° - \phi \text{ or } c. \qquad \qquad (1)$$

Such a star is called a *circum-polar star,* though in Field Astronomy the term is usually only applied to stars which neither rise nor set and which at the same time are not more than a few degrees—about 10°—from the elevated pole.

Again, the star B will rise at b″ and set at b‴ and will be below the horizon during its passage from b‴ to b′ and from b′ to b″. In this case, the angle QOb′ is the star's declination δ, and, as QON, fig. 15.1*a* and QOS, fig. 15.1*b* are each equal to $90° - \phi$, we have for a star that sets and rises

$$\delta < 90° - \phi \text{ or } c. \qquad \qquad (2)$$

When a celestial body during its daily passage around the pole crosses the meridian it is said to *transit.* Thus, since ZPNZ′SZ in fig. 15.1 is the observer's meridian, the star A transits at a and a′ and the star B at b and b′. When the body crosses the meridian on the same side of the pole as the zenith the transit is called *upper transit*; when it crosses the meridian on the opposite side of the pole to the zenith the transit is called *lower transit.* Thus, the upper transits of the stars A and B in fig. 15.1 are at a and b respectively, and the lower transits at a′ and b′, the latter transit being below the horizon and hence not visible to the observer.

3. Apparent Motion of the Sun during the Year.

One important difference between the apparent motions of the sun and stars is that, whereas the positions of the stars appear for all practical purposes to remain fixed relative to one another, so that they seem to complete their daily revolution about the celestial pole all together as a whole, the sun's position relative to the stars appears to undergo a continuous change during the course of the year. In summer, for instance, if we are in the northern hemisphere we notice that the sun is much higher in the heavens at midday than it is in winter, and it rises and sets much farther north than it does in winter. The reason for this is that the earth travels around the sun in an elliptical path in a plane called *the plane of the ecliptic*, or simply *the ecliptic*, the sun being at one focus of the ellipse, and the polar axis of the earth remains all the time at a practically fixed angle of about $66\frac{1}{2}°$ with the plane of the ecliptic. As a consequence, the plane of the celestial equator is inclined at an almost constant angle of about $23\frac{1}{2}°$ to the plane of the ecliptic.

Fig. 15.2*a* shows the motion of the earth around the sun and figs. 15.2*b* and *c* show sections of the path along the lines SW and VA. In each case, the letters NP denote the north pole of the earth and the letters EQ the earth's equator. The sun will appear to be at its maximum elevation above the equator, about $23\frac{1}{2}°$, on June 21 of each year,

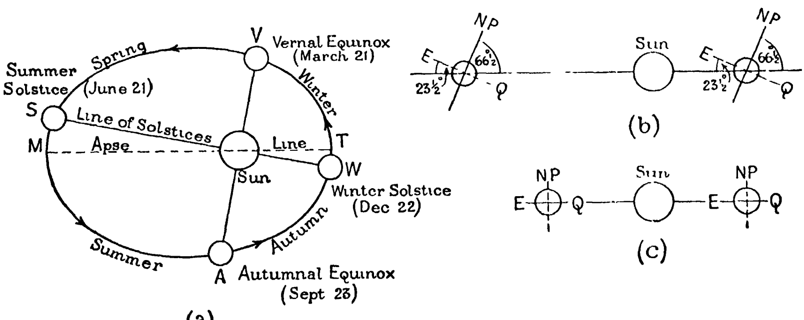


Fig. 15.2

when the earth is at the point S, and this point is called the *summer solstice*: on December 22, when the earth is at the point W, the sun will appear to be below the equator by its maximum amount, $23\frac{1}{2}°$, and the point W is called the *winter solstice*: on March 21 and September 23 it will be on the equator, as well as on the ecliptic, at the points V and A, and these points are called the *vernal equinox* and the *autumnal equinox* respectively, the line VA, which is the line of intersection of the planes of the horizon and the ecliptic, being called the *line of equinoxes*. The earth will be farthest from the sun and moving most slowly in its orbit when it is at *aphelion* at the point M, and it will be nearest to the sun and moving most rapidly in its orbit when it is at *perihelion* at the point T, the line MT being called the *apse line.*

The apparent annual motion of the sun as seen by an observer in the northern hemisphere is shown in fig. 15.3, where P is the celestial (north) pole and EAQVE the celestial equator. The great circle SAWVS is the great circle traced out on the celestial sphere by the plane of the ecliptic. This plane and the plane containing the equator will intersect in the line VA, and these two planes are inclined to one another at an angle of approximately 23½°. At S, at the summer solstice, June 21, the sun is at its highest point in the heavens and we have the longest days, or amount of daylight in the year; at W, the winter solstice, December 22, it is at its lowest and we have the shortest days of the year.

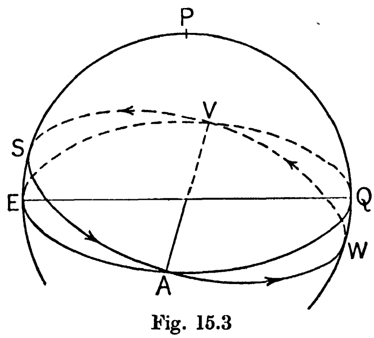


Fig. 15.3

When it is at V, the vernal equinox (March 21) and at A, the autumnal equinox (September 23) the sun is on the equator and day and night are then of equal length. At V the sun crosses the equator, and its declination, or elevation above or below the equator, changes from negative to positive. In the southern hemisphere, of course, the seasons are reversed and summer occurs while the sun's declination is negative, or south of the equator, and winter when it is positive, or north of it,

The time in which the earth completes a whole revolution from one vernal equinox to the next is about 365·2422 mean solar days, the apparent yearly motion of the sun relative to the stars being in a direction (west to east) opposite to its apparent daily motion. As a result, the interval between two successive upper transits of the sun (a solar day) is a little greater than the interval between two successive upper transits of a star (a sidereal day), and, in fact, the sun appears to slip back relative to the stars by about 3 minutes 57 seconds of time (3m 57s) per day on an average. We shall return to this point later when we consider the question of time.

The point V on the line of intersection of the equatorial and ecliptic planes, the line of equinoxes, where the sun passes from south to north of the equator, is a very important one in astronomy and is called the *First Point of Aries*, and is denoted by the symbol ♈. At the time it was so called it was in the constellation of Aries but it has moved since then and is now in the constellation Pisces. The point A at the other end of the line of equinoxes where the sun passes from north to south of the equator is known as the *First Point of Libra* and is denoted by the symbol ♎. When the motion of ♈ is studied, it is found that it, and the line of equinoxes ♈♎, move relative to the stars by about 50″ per annum in a direction opposite to that of the yearly motion of the sun, this movement being called the *precession of the equinoxes.* The ordinary surveyor, however, does not need to take account of it in his calculations as the tables he uses take care of it.

CHAPTER XVI

CELESTIAL CO-ORDINATES
THE ASTRONOMICAL TRIANGLE

1. Celestial Co-ordinates.

In astronomical computational work it is necessary to be able to define accurately the position of a celestial body at any time. There are several methods available but the following are the ones which concern the surveyor;

1. The Azimuth and Altitude System.
2. The Hour Angle and Declination System.
3. The Right Ascension and Declination System.

The first two systems are peculiar to the observer since at any instant of time the azimuth, altitude, and hour angle of a star are different for observers differently situated on the earth's surface. Moreover, these three co-ordinates change very rapidly with time.

The third system is independent of the position of the observer and the co-ordinates, except in the case of the sun and other members of the solar system, change very slowly with time. This enables tables to be prepared that give fixed values for the co-ordinates which depend neither on place nor (except within very small limits) on the time of observation.

(i) *The Azimuth and Altitude System.*

This system fixes or defines the position of a star or sun relative to the observer's meridian and horizon. In fig. 16.1*a*, Z is the observer's zenith, SZPN his meridian and NESWN his horizon, P as before being the earth's pole. Let S_1 be the celestial body whose position is to be defined. Through S_1 and the zenith Z draw a great circle to meet the plane of the horizon at L_1. As Z is the pole of the horizon circle, this great circle—the *vertical circle*—will be perpendicular to the horizon circle. Then the *azimuth A* of the star is the angle between the plane of the meridian ZPN and the plane of the great circle drawn through Z and S_1, *measured from* 0° *to* 360° *clockwise from the north.* Hence, it is the angle NOL_1, or NZS_1, in fig. 16.1*a*, or NZL_1 in fig. 16.1*b* which shows a plan of the sphere projected on the horizon plane when looking down from Z.

The *altitude* h of the star is the angle between it and the horizon plane as measured in the plane of the vertical circle. It is positive, and measured from 0° at the horizon plane to 90° at the zenith, when it

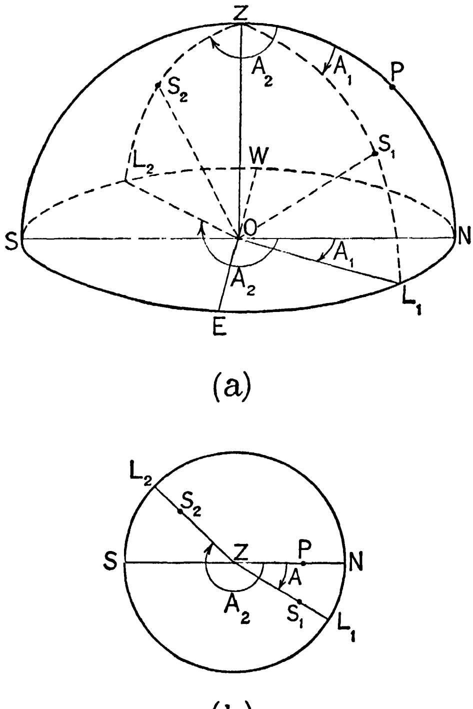


Fig. 16.1

is above the horizon plane between that plane and the zenith, and it is negative and similarly measured when it is below the horizon between that plane and the nadir. Hence it is the angle L_1OS_1 in fig. 16.1*a*. The angle ZOS_1 is the *zenith distance* of S_1 and is denoted by z. Hence, it can be seen that

$$z = 90° - h. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

If S_1 in fig. 16.1*a* had been below the horizon circle NESWN, that is between the horizon circle and the nadir, h would be negative and z

would be $90° - (-h) = 90° + h$, or numerically greater than 90°. Fig. 16.1*a* also shows a star S_2 situated between south and west, i.e. it is in the third quadrant. The azimuth, being always measured clockwise from north, is the angle NOL_2 or NZS_2 in fig. 16.1*a*, or the angle NZL_2 in fig. 16.1*b*, as shown by the arrows. As drawn, this angle is about 220°. The altitude is the angle L_2OS_2 in fig. 16.1*a*.

Owing to the star's daily circular motion about the pole both azimuth and altitude are constantly changing. In both hemispheres the azimuth is 0° when the star crosses the meridian north of the zenith and 180° when it crosses the meridian south of the zenith. It lies between 0° and 180° when the star is east of the meridian and between 180° and 360° when the star is west of the meridian. h is positive in practically all cases which affect the surveyor, whether he is in the northern or southern hemisphere. One case where h is negative is when one is computing the time of sunrise or sunset. Here, owing to the effect of atmospheric refraction, the sun appears to be on the horizon when it is actually about 30′ below it. Hence, h in this case is taken as $-30'$ and z is $90° - (-30')$, or 90° 30′.

(ii) *The Hour Angle and Declination System.*

In this system elevations are measured from the celestial equator, and not from the horizon plane, and the apex of the angle forming the other co-ordinate is the celestial pole, and not the zenith. In fig. 16.2*a*, PS_1M is the great circle, the *hour circle*, which passes through the celestial pole P and the star S_1 and is perpendicular to the celestial equator EMRWQE at M. Then the elevation MS_1 of S_1 above the celestial equator is the angle MOS_1 and this angle is known as the star's *declination*, δ, and the angle ZPS_1 drawn westwards from the position of upper transit of the star, from 0° to 360° as shown by the arrow, is the star's *hour angle*, H. The vertical circle ZS_1L_1 meets the horizon circle in the point L_1, the angle PZS_1 being the star's azimuth as already defined.

Fig. 16.2*b* is a projection on the horizon plane looking down from a distant point outside Z. Since upper transit occurs above P on the same side of the meridian as Z, the hour angle is the angle ZPS_1, as indicated by the arrow. Figs. 16.2*a* and 16.2*b* are drawn for an observer in the northern hemisphere and figs. 16.2*c* and 16.2*d* are the equivalent diagrams for an observer in the southern hemisphere, and these show how the co-ordinates H and δ are reckoned in each case.

Declination is measured from 0° to 90° above or below, or north or south of, the celestial equator, and is positive if the star is on the same side of the equator as the north pole, and negative if it is on the same side of it as the south pole. In figs. 16.2*a* and 16.2*b* the declination is positive because S_1 is on the same side of the equator as the north pole, and in figs. 16.2*c* and 16.2*d* it is taken as negative because it is on the same side of the equator as the south pole.

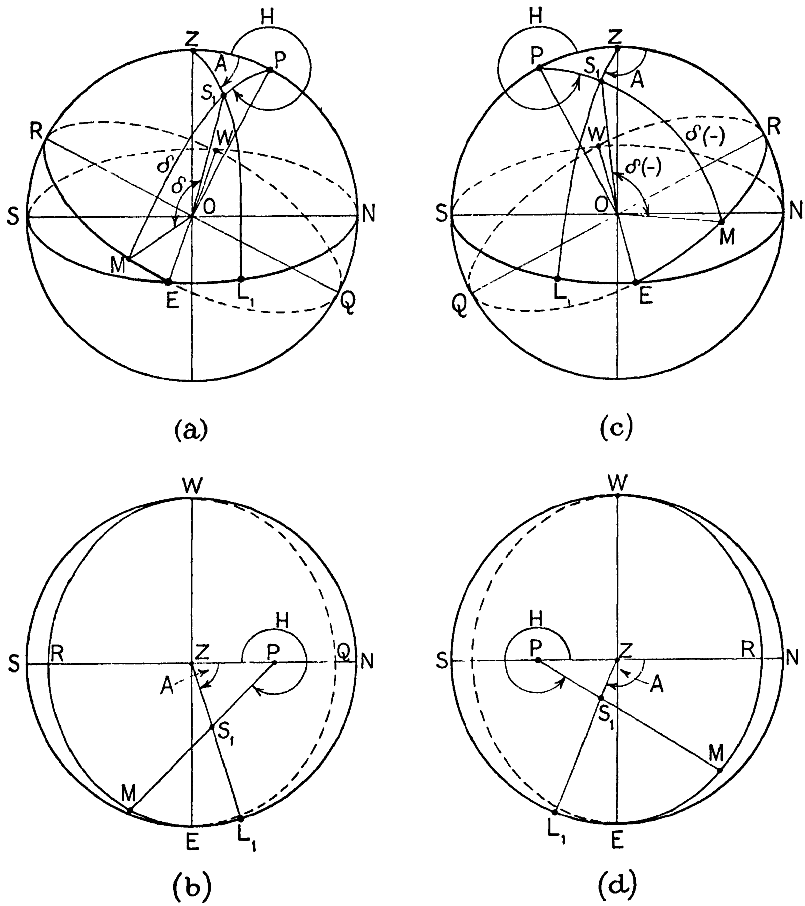


Fig. 16.2

The angle POS_1, or the angle subtended at O between the elevated pole and the star, is called the *polar distance* and is denoted by p. When declinations are positive for stars north of the celestial equator and negative for stars south of it we have

$$p = 90° - \delta \qquad (2)$$

at places in the northern hemisphere, and, now measuring p from the southern celestial pole,

$$p = 90° + \delta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

at places in the southern hemisphere, the sign of δ being of course reversed in both cases when δ is south or negative.

Note that, whereas to an observer facing the equator in the northern hemisphere the stars appear to move *clockwise* from east to west, in the southern hemisphere they appear to an observer facing the equator to move *anti-clockwise* from east to west. Azimuth measured clockwise from north, and hour angle measured westwards from 0° to 360° from upper transit in the direction of the apparent motion of the stars, are then the angles marked A and H in all four diagrams.

(iii) *The Right Ascension and Declination System.*

In this system right ascensions are angles measured in the plane of the celestial equator and declinations are, as before, angles of elevation north or south of it.

In figs. 16.3*a* and 16.3*b* REQWR is the celestial equator and P the pole. Through P and S_1, the apparent position of the star, draw the hour circle PS_1M perpendicular to the celestial equator at M. Then the angle MOS_1 is the declination δ reckoned positive from 0° to 90° if the star is north of the equator and negative from 0° to 90° if it is south of it. The *Right Ascension*, or *R.A.*, is the angle ♈OM, where ♈ is the First Point of Aries, reckoned along the equator in the direction *opposite* to the direction of the apparent daily motion of the stars. To an observer at the north celestial pole looking down at the celestial equator as in figs. 16.3*a* and 16.3*b* this direction would appear to be anti-clockwise: to an observer at the southern celestial pole looking down on the celestial equator, as in figs. 16.3*c* and 16.3*d*, the direction would appear to be clockwise.

If $H_♈$ and H_s are the hour angles of ♈ and the star at any moment, it will be seen from the figures that

$$H_♈ = H_s + \text{R.A.}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

or, in words,

Hour angle of ♈ = hour angle of star + the star's right ascension.

This is a very important relation which should be memorized. If the expression on the right exceeds 360°, 360° or 24 hours, if hour angles and right ascension are being reckoned in time, should be subtracted from it.

As we shall see later, hour angle and right ascension are closely connected with time and can be expressed either in hours, minutes and seconds of time or in ordinary angular measure. For some purposes it is more convenient to express hour angles in units of time, and for others

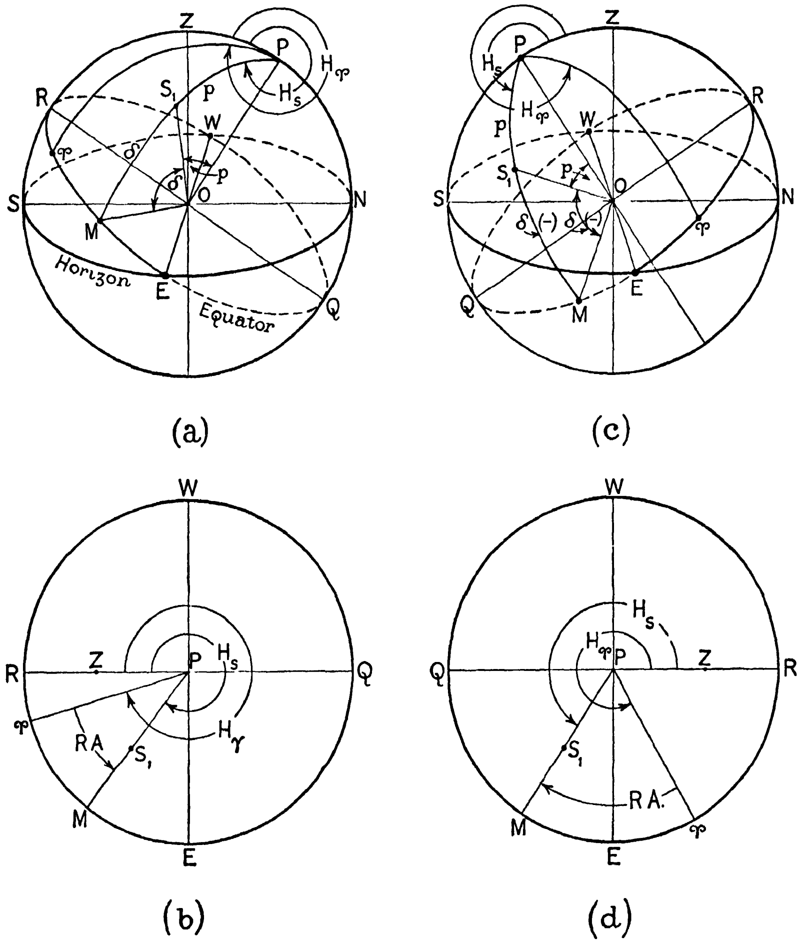


Fig. 16.3

it is more convenient to express them in angular measure. Right ascensions are generally expressed in units of time and are normally given in time units in Tables and Almanacs such as *The Star Almanac for Land Surveyors*.

2. Summary of Directions and Signs.

It is very important to remember the directions of apparent motion of celestial bodies and the direction in which celestial co-ordinates are reckoned. Accordingly, we summarize them as follows:

1. East to west means the direction of the apparent daily motion of an equatorial star, or star near the equator, from rising to setting, or the direction of the apparent motion of a star at upper transit.
2. Altitude, h, is positive when the body is above the horizon and negative when it is below it. It is nearly always positive and is assumed to be positive in what follows. Zenith distance is $90° -$ altitude.
3. Azimuth, A, is reckoned clockwise from 0° to 360° from due north in both hemispheres.
4. Hour angle, H, is reckoned from 0° to 360°, or from 0 h. to 24 h., westwards in the direction of apparent motion of the body as it crosses the meridian from the position of upper transit.
5. Declination, δ, is positive from 0° to 90° when the body is north of the celestial equator and negative from 0° to 90° when it is south of it. Polar distance, for positive declination, is $90° - \delta$ when referred to the northern celestial pole and $90° + \delta$ when referred to the southern celestial pole, the sign of δ being reversed in each case when it is negative.
6. Right ascension, R.A., is reckoned along the equator from 0° to 360°, or more usually from 0 h. to 24 h., anti-clockwise from γ as seen by an observer viewing the celestial equator from the northern celestial pole, or clockwise from γ as seen by an observer viewing the celestial equator from the southern celestial pole.
7. The apparent daily motion of the sun and stars from east to west is clockwise to an observer viewing the celestial equator from the northern celestial pole and anti-clockwise from east to west to an observer viewing the celestial equator from the celestial southern pole.
8. The apparent annual motion of the sun relative to γ and the stars is anti-clockwise to an observer viewing the celestial equator from the northern celestial pole and clockwise to an observer viewing the celestial equator from the southern celestial pole.
9. The precession of the equinoxes is in the direction opposite to the apparent annual motion of the sun, i.e. clockwise to an observer viewing the celestial equator from the northern celestial pole and anti-clockwise to an observer viewing the celestial equator from the southern celestial pole.

Applying these rules to what we know about the apparent motion of the sun, we note the following points:

1. The sun's declination changes from — to + at the vernal equinox (March 21) and from + to — at the autumnal equinox (September 23.)
2. The sun's declination reaches its maximum positive value of approximately $23\frac{1}{2}°$ at the summer solstice (June 21) and its maximum negative value of approximately $-23\frac{1}{2}°$ at the winter solstice (December 22).
3. The sun's right ascension increases by approximately 4 m. a day and 2 h. per month from 0 h. at the vernal equinox to 24 h., when it returns to the same point. The right ascensions and declinations of the stars vary slightly during the year owing to the precession of the equinoxes and other causes, but over very short periods of time, i.e. for a few days, they may be considered for our purpose to be constant.

3. The Astronomical Triangle.

The intersection, two by two, of the great circles meridian, hour circle, and vertical circle, define a spherical triangle on the celestial sphere with apices at the celestial pole, star and zenith. This triangle is known as the *astronomical triangle* and is of great importance in astronomical theory. Thus, in fig. 16.4*a*, the meridian is the great circle ZPN, the vertical circle is the great circle ZS_1L_1 and the hour circle is the great circle PS_1M_1. The intersections of these great circles give the astronomical triangle ZPS_1 (fig. 16.4), the sides of which are the co-latitude, c, the zenith distance, z, and the polar distance, p. The interior angles are here called the *zenith* or *azimuth angle*, α, at Z, the *polar* or *time angle*, β, at P, and the *parallactic angle*, γ, at S, respectively. The solution of this triangle by the ordinary rules for the solution of spherical triangles given in Chapter XIV leads to the working formulæ of field astronomy.

Fig. 16.4*a* and *b* show the astronomical triangle for the case where the observer is in the northern hemisphere and the star is east of the meridian ZP, and fig. 16.4*c* shows it when the star is west of the meridian. Figs. 16.4*d*, *e* and *f* show the triangle when the observer is in the southern hemisphere, figs. *d* and *e* showing the star east of the meridian and fig. 16.4*f* when it is west of the meridian.

The parts c, z, p, α and β of the triangle are functions of latitude, ϕ; altitude, h; declination, δ; azimuth, A; and hour angle, H, respectively. Let us adopt the following conventions:

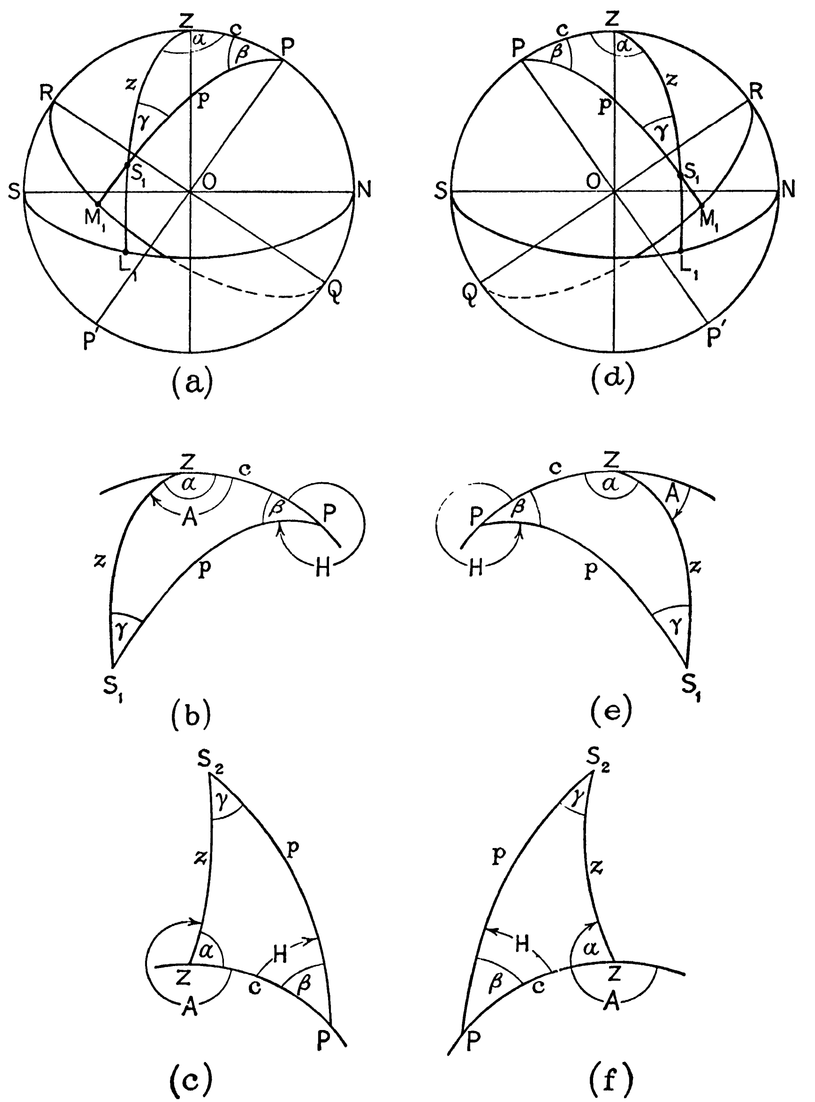


Fig. 16.4

ϕ = latitude, positive when north, negative when south, reckoned from 0° to ±90°.

δ = declination, positive when north of the celestial equator, negative when south of it. 0° to ±90°.

h = altitude of star, positive from horizon to zenith, negative from horizon to nadir. 0° to ±90°. (Nearly always positive.)

H = hour angle, positive westwards from 0° to 360°, or 0 h. to 24 h., from that part of the observer's meridian which contains his zenith.

A = azimuth, positive eastwards, or clockwise, 0° to 360°, from north in both hemispheres.

z = zenith distance = $90° - h$.

p = polar distance = $\begin{matrix} 90° - \delta \\ 90° + \delta \end{matrix}$ for observer in the $\begin{matrix} \text{northern} \\ \text{southern} \end{matrix}$ hemisphere.

c = co-latitude = $\begin{matrix} 90° - \phi \\ 90° + \phi \end{matrix}$ for observer in the $\begin{matrix} \text{northern} \\ \text{southern} \end{matrix}$ hemisphere.

In the expressions for p and c the signs of δ and ϕ must be reversed, as usual, when they are negative.

With these conventions, we have the following relations between α and A and β and H, and between p, c and z and δ, ϕ and h:

OBSERVER IN THE NORTHERN HEMISPHERE

Star east of meridian	*Star west of meridian*
$\alpha = A$	$\alpha = 360° - A$
$\beta = 360° - H$	$\beta = H$
$p = 90° - \delta$	$p = 90° - \delta$
$c = 90° - \phi$	$c = 90° - \phi$
$z = 90° - h$	$z = 90° - h$

OBSERVER IN THE SOUTHERN HEMISPHERE

Star east of meridian	*Star west of meridian*
$\alpha = 180° - A$	$\alpha = A - 180°$
$\beta = 360° - H$	$\beta = H$
$p = 90° + \delta$	$p = 90° + \delta$
$c = 90° - \phi$	$c = 90° - \phi$
$z = 90° - h$	$z = 90° - h$

In the expressions for p, it is to be understood that the sign of δ must be reversed when δ is negative.

In solving astronomical problems involving the astronomical triangle, the best plan is to draw the triangle and to convert the given data into the sides and internal angles of the triangle; then, having solved the latter, convert the calculated parts of the triangle into their astronomical equivalents. Thus, for an observer in the northern hemisphere and with the star in the east, to find the azimuth given the corrected observed altitude, h, the declination, δ, and the observed hour angle, H,

$$z = 90° - h; \quad p = 90° - \delta; \quad \beta = 360° - H.$$

From fig. 16.4*b* and formula (2), page 284,

$$\frac{\sin \alpha}{\sin p} = \frac{\sin \beta}{\sin z},$$

$$\sin \alpha = \frac{\sin p \sin \beta}{\sin z} = \frac{\cos \delta \sin \beta}{\cos h}, \quad . \quad . \quad . \quad . \quad (5)$$

and

$$A = \alpha.$$

For a star west of the meridian in the southern hemisphere, but otherwise with the same data as before, (fig. 16.4*f*), we have

$$z = 90° - h, \quad p = 90° + \delta, \quad \beta = H,$$

$$\sin \alpha = \frac{\sin p \sin \beta}{\sin z} = \frac{\cos \delta \sin \beta}{\cos h},$$

and $A = 180° + \alpha$.

It should be noted that there is an ambiguous case here because the sine has the same value and sign for $180° - \alpha$ as for α. In most cases the doubt about which value to accept can be resolved from the surveyor's notes by knowing not only on which side of the meridian the star lay at the time of observation but also the actual quadrant in which it was situated, or else by remembering that the largest side subtends the largest angle. However, it is not always possible to resolve the doubt in this way, and, if sufficient other data exist, it is then advisable to use a formula involving a tangent or a cosine. Thus, if the star is close to the prime vertical so that the zenith angle α is close to 90°, the sine rule will not enable one to decide whether to accept, say, α or $180° - \alpha$, and, in any case, if logarithms are being used, the log sine becomes somewhat insensitive for angular values near 90°. Hence, if the three sides of the triangle are known, we can find α by using the formula

$$\tan \tfrac{1}{2}A = \sqrt{\left[\frac{\sin (s-b) \sin (s-c)}{\sin s \sin (s-a)}\right]}$$

in spherical trigonometry, so that, substituting for A, a, b, c, and s their equivalents α, p, z, c, and $\frac{1}{2}(p + z + c)$ in the astronomical triangle, we have

$$\tan \tfrac{1}{2}\alpha = \sqrt{\left[\frac{\sin (s - z) \sin (s - c)}{\sin s \sin (s - p)}\right]}. \quad . \quad . \quad . \quad (6)$$

If both azimuth and hour angle are required we use formulæ (11) and (12) on page 285 for the spherical triangle and substitute in them the astronomical equivalents for the sides and angles, so getting

$$\tan r = \sqrt{\left[\frac{\sin (s - p) \sin (s - z) \sin (s - c)}{\sin s}\right]}, \quad . \quad (7)$$

$$\tan \tfrac{1}{2}\alpha = \frac{\tan r}{\sin (s - p)}; \quad \tan \tfrac{1}{2}\beta = \frac{\tan r}{\sin (s - z)}. \quad . \quad . \quad (8)$$

Then, having found α and β, we use the rules on page 302 to obtain A and H. Thus, if the star is west of the meridian and the place of observation is in the southern hemisphere, we have

$$A = 180° + \alpha; \quad H = \beta.$$

The positive sign for $\tan \frac{1}{2}\alpha$ must always be taken so that $\frac{1}{2}\alpha$ is equal to or less than 90°. If it were greater than 90°, α would be greater than 180° and hence the star would be on the other side of the meridian, and, as a general rule, we know which side it is on. A similar argument holds when β is obtained from the analogous formulæ involving the three sides of the triangle.

Other examples of the solution of the astronomical triangle are:

$$\sin p \cos \beta = \sin c \cos z - \cos c \sin z \cos \alpha, \quad . \quad . \quad (9)$$

$$\cos z = \cos p \cos c + \sin p \sin c \cos \beta, \quad . \quad . \quad (10)$$

which come from the spherical trigonometry formulæ (14) and (16) on page 285 and

$$\tan \alpha = \frac{\sin \beta}{\cot p \sin c - \cos c \cos \beta}, \quad . \quad . \quad . \quad (11)$$

which is derived from the spherical trigonometry formula (13) on page 285.

$$\cot a \sin c = \cos c \cos B + \sin B \cot A,$$

in which A is replaced by α, B by β, and a by p, so that the astronomical triangle ZPS corresponds to the spherical triangle ABC.

Some numerical examples of problems involving the solution of the astronomical triangle will be found on pages 347 to 355.

4. The Right-Angled Astronomical Triangle.

An important case arises when the star is a circum-polar one and the parallactic angle is 90°. In these circumstances, the star is said to be at *elongation*. It is then in the most suitable position for azimuth

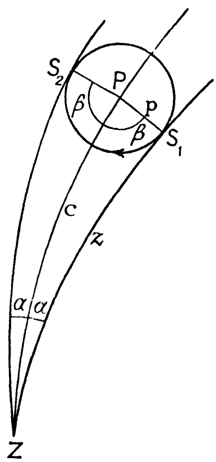


Fig. 16.5

observations as the line of sight is tangential to the path of the star and hence a small error in the observation of the hour angle or the altitude will have little or no effect on the azimuth. (See fig. 16.5 and the example on pages 354 and 355.)

Adapting spherical trigonometry formulæ (17)—(21), page 285, to the astronomical triangle, we have for a circum-polar star at elongation,

$$\sin p = \sin c \sin \alpha = \tan z \cot \beta. \quad (12)$$

$$\sin z = \sin c \sin \beta = \tan p \cot \alpha. \quad (13)$$

$$\cos \alpha = \cos p \sin \beta = \cot c \tan z. \quad (14)$$

$$\cos \beta = \cos z \sin \alpha = \cot c \tan p. \quad (15)$$

$$\cos c = \cos p \cos z = \cot \alpha \cot \beta. \quad (16)$$

CHAPTER XVII

TIME

1. Measures of Time.

The apparent diurnal motion, east to west, of the stars and sun about the pole affords a measure of time. For universal use, it is necessary to adopt a certain meridian of longitude as a standard meridian, so that the time of passage of a selected celestial body across this meridian can be accepted as the zero datum from which time is reckoned. This meridian is internationally accepted as the meridian passing through the main transit instrument at Greenwich Observatory, and times based on transits across this meridian are spoken of as *Greenwich Time* (G.T.) or, when Greenwich Mean Time (G.M.T.) is involved, *Universal Time* (U.T.).

Often in astronomical work it is convenient to reckon time from the time of passage of the body across the observer's own meridian. This gives *Local Time* (L.T.). In addition, since it would be very inconvenient, when there is a large difference between Greenwich and local times due to a large difference in longitude, to reckon time for ordinary civil purposes at all places in the world from time of transit at Greenwich, the earth has been divided into a series of time zones in which standard civil time is taken as so many hours before or after Greenwich time. This time is called *Zone Time*. Thus, standard civil time for the Eastern States of the United States of America is taken as 5 hours behind Greenwich time, so that, when it is noon at Greenwich, it is only 7 a.m. in the east of the United States.

Daylight and darkness on the earth are governed by the sun, and hence it appears to be natural to use the sun as man's time-keeper. Unfortunately, the sun's motion during the course of a year is not constant, so that, if the length of a day were to be defined as the time interval between successive upper or lower transits of the sun across any given meridian, this length, as measured on a mechanical device such as a clock, would not be the same at all seasons of the year. Hence, as a matter of convenience, it has been necessary to adopt the idea of *Mean Time*; this is the time kept by an imaginary *mean sun* which is assumed to move at a uniform rate along the celestial equator and for which the interval between successive upper or lower transits across the meridian at any place is constant throughout the year and is called the *mean solar day.* Mean time reckoned from 0 to 24 hours from the lower transit of the mean sun across the Greenwich meridian is known as *Greenwich Mean Time* (G.M.T.), or now, more commonly, *Universal Time* (U.T.).

The apparent motion of the stars throughout the year is very much more regular than that of the sun, and, in fact, such irregularities as there are are not noticeable except with very refined and accurate observations. Hence, for astronomical work when observations to stars are involved we use *Sidereal Time**. The interval between successive upper or lower transits of a star across the meridian is called the *sidereal day.*

2. Relation between Time and Angular Measure.

The mean solar day and the sidereal day are each divided into 24 hours, each hour into 60 minutes, and each minute into 60 seconds. During this time the body appears to complete a single whole revolution of 360° about the earth's axis of rotation. Hence, 24 hours of time are equivalent to 360° of angular measure, so that 1 hour of time is equivalent to 360°/24 = 15° of arc and 1° of arc = 1/15 hour. Again, since there are 60 minutes of time in an hour and 60 minutes of arc in a degree, 1 minute of time is equivalent to 1/15 of 1 minute of arc, or 4 seconds of arc, and 1 second of time is equivalent to 1/15 of a second of arc. Hence we have the rules

(1) *To convert degrees, minutes, and seconds of arc into hours, minutes, and seconds of time, divide by* 15.
(2) *To convert hours, minutes, and seconds of time into degrees, minutes, and seconds of arc multiply by* 15.
} (1)

Thus (33° 15′ 15″)/15 = 2 h. 13 m. 01 s. and (15 h. 04 m. 13 s.) × 15 = 226° 03′ 15″.

In practical computing, time is saved by using special tables for converting arc into time and time into arc. Such tables are given in various mathematical and astronomical tables, such as *Chambers's Seven-Figure Mathematical Tables, Chambers's Six-Figure Mathematical Tables, The Star Almanac for Land Surveyors,* etc.

* Astronomers also now use *Ephemeris Time.* This is a refinement designed to deal with very small variations in the earth's rate of rotation about its axis, but the effects are too small to be of direct concern to the ordinary surveyor.

3. Longitude and Time.

The longitude of any point on the earth's surface is the angle between the meridian passing through that point and the standard meridian. In fig. 17.1, which represents the earth, PP′ is the earth's axis of rotation, P and P′ the poles, and the great circle PSP′ the meridian cut out on the earth's surface by a plane passing through P, P′ and the point S, the latter being any point on the earth's surface. PGP′ is the meridian chosen as the standard meridian from which longitudes are reckoned, this usually being taken as the meridian which passes through

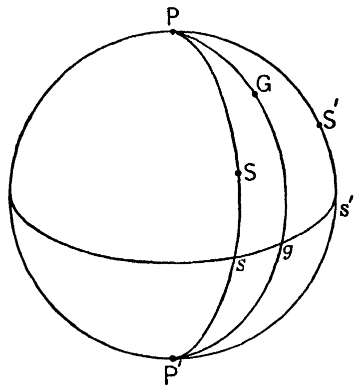


Fig. 17.1

the main transit instrument at Greenwich Observatory. Consequently, if G represents the position of the transit instrument at Greenwich, the longitude of S is the angle GPS, or the arc sg divided by the radius of the earth, s and g being the points where the meridians through S and G cut the earth's equator.

Longitudes are reckoned from 0° to 180° west or east of the Greenwich meridian. In the diagram, the longitude of S is west and that of the point S′ is east. West longitudes are taken as positive and east longitudes as negative.

Since the motion of the earth relative to the stars is west to east and the apparent motion of the stars relative to the earth east to west, and since the planes of the terrestrial and celestial meridians coincide, a celestial body such as the sun or a star would appear to be on the meridian through G sooner than it would appear to be on the meridian through S, and it would appear to cross the meridian through G later than it would appear to cross the meridian through S′. Hence, local time at S is later than time at G, and time at G is later than local time at S′. Consequently, when longitude in angular measure is converted into time, we see that, for points in west longitude,

$$\text{Greenwich time} = \text{Local time} + \text{longitude in time}, \quad . \quad (2)$$

and, for points in east longitude, after taking into account the negative sign for the longitude,

$$\text{Greenwich time} = \text{Local time} - \text{longitude in time}. \quad . \quad (3)$$

These relations hold for both solar and sidereal time.

4. Relation between Apparent Solar Time and Mean Solar Time.

We have already noted that the sun is an irregular time-keeper, with the result that the interval between successive upper or lower transits of the sun is not the same throughout the year, and, to overcome this difficulty so that mechanical devices such as clocks and hour glasses can be used to measure time, it has been necessary to adopt the expedient of an imaginary mean sun which is assumed to move along the celestial equator, and for which the interval between successive upper or lower transits is always the same and equal to the mean interval between successive upper or lower transits of the real or apparent sun throughout the year. The interval between two successive upper or lower transits of the real or apparent sun is called a *solar day*, and the interval between imaginary similar transits of the mean sun is called a *mean solar day*. Time kept by the apparent sun is called *apparent time*, and time kept by the mean sun is called *mean time*.

In astronomical work, the solar day, both apparent and mean, is measured from 0 h. to 24 h. from the moment of lower transit to the moment of the next lower transit, i.e. the day commences and ends at midnight. In civil reckoning in this country, however, the day is divided into two parts, one, *ante meridiem*, or a.m., from 0 h. to 12 h., and the other, *post meridiem*, or p.m., commencing with 0 h. instead of 12 h. and continuing to midnight as 12 h. Thus, ante-meridiem time is the same as astronomical time but post-meridiem time is astronomical time less 12 h.

The difference between apparent and mean time is known as the *equation of time* and this quantity is tabulated for different dates and times in many astronomical tables containing an ephemeris of the sun in the form

$$\text{A.T.} - \text{M.T.} = e, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

where A.T. stands for apparent time, M.T. for mean time, and e for the equation of time. In recent years, however, it has been found more convenient for surveyors and navigators to tabulate a variable E which is defined as the difference between the Greenwich hour angle of the apparent or true sun and the corresponding Universal, or Greenwich, mean time. Thus, we have the relation

$$\text{G.H.A. sun} = \text{U.T.} + E, \qquad (5)$$

which connects apparent with universal mean time. Also, from this, and from (4) above, and remembering that hour angle is measured westwards from upper transit, it follows that

$$E = e + 12\text{ h.} \qquad (6)$$

If U.T. + E exceeds 24 h., then 24 h. must be subtracted from it. Note also that

$$\text{U.T. of upper transit of the true sun across the Greenwich Meridian} = 24\text{ h.} - E. \qquad (7)$$

The Star Almanac for Land Surveyors tabulates E to a tenth of a second of time at 0 h., 6 h., 12 h. and 18 h. for every day of the year, and this makes it easy to convert apparent time and hour angle into mean time and vice versa.

5. Relation between Mean and Sidereal Time Intervals.

Owing to its completing one entire revolution in the ecliptic round the sun during the course of a *tropical year*, i.e. the interval between two successive vernal equinoxes, the earth appears to complete one more revolution in this time about its axis relative to the stars than it does relative to the sun. There are 365¼ mean solar days in the tropical year, and so the earth appears to complete 365¼ revolutions in this time relative to the sun and 366¼ revolutions relative to the stars. Hence, we have:

$$\begin{aligned} 365\tfrac{1}{4}\text{ mean solar days} &= 366\tfrac{1}{4}\text{ sidereal days.} \\ 1\text{ mean solar day} &= (366\tfrac{1}{4})/(365\tfrac{1}{4})\text{ sidereal days} \\ &= 24\text{ h. } 03\text{ m. } 56{\cdot}6\text{ s. sidereal time intervals} \\ &= 1\text{ sidereal day} + 4\text{ m.} - 3{\cdot}4\text{ s. approximately.} \qquad (8) \end{aligned}$$

$$\begin{aligned} 1\text{ sidereal day} &= (365\tfrac{1}{4})/(366\tfrac{1}{4})\text{ mean solar days} \\ &= 23\text{ h. } 56\text{ m. } 04{\cdot}1\text{ s. mean time} \\ &= 1\text{ mean day} - 4\text{ m.} + 4\text{ s. approximately.} \qquad (9) \end{aligned}$$

From these relations we have:

1 mean solar hour	= 1 h. + 9$\frac{5}{6}$ s. sidereal time approximately,
6 m. mean solar time	= 6 m. + 1 s. sidereal time approximately.
1 sidereal hour	= 1 h. − 9$\frac{5}{6}$ s. mean solar time approximately,
6 m. sidereal time	= 6 m. − 1 s. mean solar time approximately.

In practice, the conversion of intervals of mean solar time to intervals of sidereal time and vice versa is best carried out by means of special tables such as those given on page 433 of *Chambers's Seven-Figure Mathematical Tables*, etc. Thus, to find the sidereal interval corresponding to an interval of 13 h. 14 m. 36 s. mean time:

Mean time interval	=	13 h.	14 m.	36 s.
Correction for 13 h. (*Chambers*, *p.* 433)	=	+	2	08·13
Correction for 14 m.	=	+		2·30
Correction for 36 s.	=	+		0·10
Algebraic Sum = Sidereal interval	=	13	16	46·53

Again, to find the mean time interval corresponding to a sidereal interval of 22 h. 48 m. 42 s.:

Sidereal interval	=	22 h.	48 m.	42 s.
Correction for 22 h. (*Chambers*, *p.* 433)	=	−	3	36·25
Correction for 48 m.	=	−		7·86
Correction for 42 s.	=	−		0·11
Algebraic sum = Mean time interval	=	22	44	57·78

The "Interpolation Table for *R*" on pages 68–69 in *The Star Almanac for Surveyors* may also be used for converting mean to sidereal time intervals, and *vice versa*, as it gives, for a range of 6 h., the differences between mean time and sidereal time intervals at critical values of mean time corresponding to increases of 0·1 s. in the differences. Thus, in the first example given above, to convert 13 h. 14 m. 36 s. mean time to sidereal time, the difference to be added to the mean time is twice the difference for 6 hours plus the tabulated difference for 1 h. 14 m. 36 s., viz. 2 × 59·1 s. + 12·3 s. = 2 m. 10·5 s. and hence the sidereal time interval required is 13 h. 14 m. 36 s. + 2 m. 10·5 s. = 13 h. 16 m. 46·5 s. An example of the reverse process is given in the Introduction to the Tables, page xi. An error of 0·2 s. is possible with this method.

6. Sidereal Time and its Reckoning.

We have already seen that sidereal time is based on the intervals between successive upper or lower transits of a star across the meridian. In practice, we use as the time measurer a fictitious star whose position in the heavens at any instant is the position of γ, the First Point of Aries. The sidereal day is reckoned from 0 h. to 24 h. from the moment when the First Point of Aries crosses the meridian at *upper transit.* Thus, the method of reckoning sidereal time differs from that used in reckoning solar time in that the sidereal day commences at the upper transit of γ, whereas the solar day commences at the moment of lower transit of the true or apparent sun in the case of apparent time or of the mean sun in the case of mean solar time.

The sidereal day, like the solar day, is divided into 24 hours, each hour into 60 minutes, and each minute into 60 seconds. Also, like solar time, sidereal time is local sidereal time if it is measured from the moment of transit of γ over the local meridian, and Greenwich sidereal time if measured from the moment of transit over the Greenwich meridian; the rules for reducing local sidereal time (L.S.T.) to Greenwich sidereal time (G.S.T.) and vice versa are the same as those given on page 309.

Owing to slight changes in the apparent motion of γ throughout the year, the length of the sidereal day varies slightly from day to day, but, in contradistinction to the varying length of the solar day, the difference is so small as to be inappreciable in ordinary survey work as it is taken into account in the positions given in *The Star Almanac for Land Surveyors,* though in geodetic work star positions need to have small corrections applied which depend on the date and time of observation. Sidereal time based on the average length of the sidereal day over a considerable period of time is called *Mean Sidereal Time,* but, as it differs by so little from *Apparent Sidereal Time,* it is usual to speak of the time kept by a clock regulated to keep mean sidereal time simply as *Sidereal Time,* it being understood that it is Mean Sidereal Time which is meant.

7. Relations between Sidereal Time, Right Ascension and Hour Angle.

Fig. 17.2*a* shows a plan of the celestial equator as seen by an observer looking at it from above the north celestial pole and fig. 17.2*b* a similar plan as seen by an observer looking at it from the south celestial pole. In both diagrams P is the projection of the pole on the equator, NPZS or NZPS is the projection of the meridian, and Z the projection of the zenith. The direction of apparent motion of a heavenly body as seen by each observer is shown by an arrow and Υ is the position of the First Point of Aries.

Since hour angle is measured westwards in both cases from upper transit, the hour angle, θ, of Υ is the angle SPΥ in fig. 17.2*a* and NPΥ in fig. 17.2*b*. But since sidereal time is also measured westwards from upper transit of Υ, it corresponds to the angle SPΥ in fig. 17.2*a* and NPΥ in fig. 17.2*b*, and so, in either hemisphere,

$$\text{Sidereal time at any instant} = \text{Hour angle of } \Upsilon. \quad . \quad (10)$$

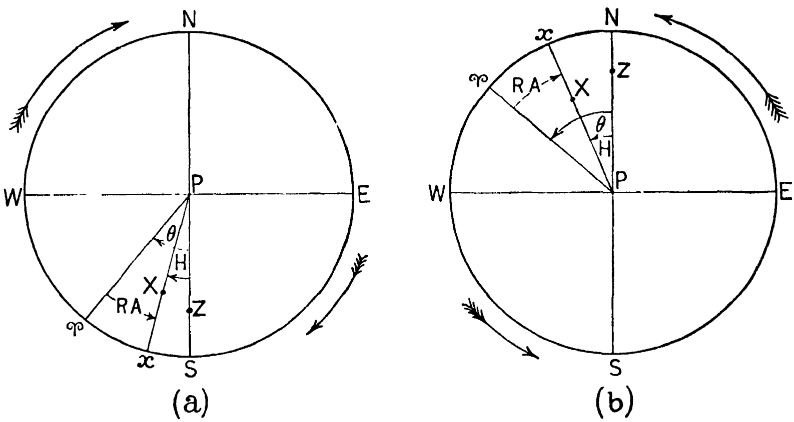


Fig. 17.2

Again, let X be the position in plan of a star and x the point where the great circle through P and X meets the celestial equator. Then the hour angle, H, of X is the angle SPx in fig. 17.2*a* and NPx in fig. 17.2*b*. But the right ascension of the star is measured in the direction opposite to the direction of its apparent daily motion and is the angle ΥPx. Hence, since SPx in fig. 17.2*a* = SPΥ − ΥPx and NPx = NPΥ − ΥPx in fig. 17.2*b*,

$$\text{Hour angle of star} = \text{hour angle of } \Upsilon - \text{star's R.A.} \quad (11)$$

$$= \text{sidereal time} - \text{star's R.A.} \quad . \quad (12)$$

The star is at upper transit when its hour angle is zero and hence

$$\text{Sidereal time of upper transit of star} = \text{star's R.A.} \quad (13)$$

The Star Almanac for Land Surveyors gives, for the first day of each month throughout the year, the right ascensions to a decimal of a second of time and declinations to a second of arc of 650 stars, arranged in order of ascending right ascension and also, but at 10-day intervals of time, the right ascension and declinations of three northern and two southern circum-polar stars.

8. Conversion of Sidereal into Mean Time and Vice Versa.

In the case of the sun the *Star Almanac* tabulates at 6-hour intervals for each day of the year a quantity R defined as the difference between the hour angle of Aries and the corresponding Greenwich Mean Time, or U.T., according to the equation

$$\text{G.H.A. Aries} = \text{G.S.T.} \quad . \quad . \quad . \quad . \quad . \quad (14)$$

$$= \text{U.T.} + R, \quad . \quad . \quad . \quad . \quad (15)$$

and, for time and hour angle referred to a local meridian,

$$\text{L.H.A. Aries} = \text{L.S.T.} \quad . \quad . \quad . \quad . \quad . \quad (16)$$

$$= \text{L.M.T.} + R. \quad . \quad . \quad . \quad (17)$$

R increases by the difference between a sidereal and mean time interval, or at the rate of 3 m. 56·6 s. per diem, and a table to facilitate the determination of intermediate values of R is given as "Interpolation Table for R" near the end of the *Star Almanac*.

Since

$$\text{L.M.T.} = \text{L.H.A. mean sun} \mp 12 \text{ h.},$$

then

$$\text{L.H.A. Aries} = \text{L.H.A. mean sun} \mp 12 \text{ h.} + R.$$

But

$$\text{L.H.A. Aries} - \text{L.H.A. mean sun} = \text{R.A. mean sun}.$$

$$\therefore \text{R.A. Mean sun} = R \mp 12 \text{ h.}$$

$$\therefore R = \text{R.A. Mean sun} \pm 12 \text{ h.} \quad . \quad . \quad . \quad (18)$$

Similarly, the reader may care to verify the following relations for himself:

$$\text{R.A. of apparent sun} = R - E. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

$$\text{G.H.A. of Aries} = \text{G.H.A. apparent sun} + R - E. \quad (20)$$

$$\text{L.H.A. of Aries} = \text{L.H.A. apparent sun} + R - E. \quad (21)$$

$$\text{G.H.A. of star} = \text{U.T.} + R - \text{R.A. star}. \quad . \quad . \quad . \quad . \quad (22)$$

$$\text{L.H.A. of star} = \text{L.M.T.} + R - \text{R.A. star}. \quad . \quad . \quad . \quad (23)$$

9. The Star Almanac for Land Surveyors.

This almanac is prepared annually by H.M. Nautical Almanac Office and is published by H.M. Stationery Office at 4/6 a copy about six months before the beginning of the year to which it refers. It is intended for use in ordinary survey work of a minor order, and not for geodetic work, for which another publication called *Apparent Places of Fundamental Stars* must be used. It opens with an Introduction which includes a description of the various tables contained in it, together with numerical examples of their use. Then follows an ephemeris of the sun in 24 pages, with data for half of the month on the left of two facing pages and data for the other half of the month on the right-hand page. This ephemeris gives values of *R*, declination, and *E* at U.T. 0 h., 6 h., 12 h., and 18 h. of each day. Below this main table are given the sun's semi-diameter for the month and times of sunrise and sunset at various latitudes from 0° to 60° N and 60° S at 5-day intervals, and, in a line below this, the days and hours of the moon's phases. The sun's semi-diameter is used for working out a correction to reduce observations taken to one edge, or limb, to what they would be if they were taken to its centre, and the data about times of sunrise and sunset and the moon's phases are useful for preparing programmes for stellar observations, so that the stars selected will be likely to be visible at the time when observations are to be taken.

It is to be noted that the argument in the ephemeris of the sun is G.M.T., or U.T., and the tabulated values of *R* and *E* are for the mean, not the true or apparent, sun. Hence if, for instance, we wish to convert L.S.T. into L.M.T., we must, before we can utilize the *Almanac*, first convert L.S.T. to G.M.T. and then work back from G.M.T. to local time.

The magnitudes,* right ascensions, and declinations of 650 stars are given for the beginning of each month in the 26 pages which follow the ephemeris of the sun, right ascensions being given to 0·1 s. and declinations to 1″. A table for three northern and two southern circumpolar stars gives the R.A. to 1 s. and the declination to 1″ for each of the five stars at 10-day intervals throughout the year.

Other tables include an Index to Places of Stars, a Pole Star Table giving factors for computing latitude and azimuth from observations to the Pole Star (*Polaris*, or α *Ursae Minoris*), Refraction Tables for

* The *magnitude*, or relative brightness of a star is indicated by a number, the smaller the number the brighter the star, so that a star of magnitude 2 is brighter than one of magnitude 4. *The Star Almanac* includes all stars not fainter than magnitude 4·0 and such other stars not fainter than magnitude 4·5 as are tabulated in *Apparent Places of Fundamental Stars*, except close circum-polar stars.

working out corrections to observed altitudes for atmospheric refraction, Interpolation Tables for the Sun and Star Tables, Interpolation Table for R, and a table for converting time to arc and arc to time.

In the examples given in Section 11 below and at the end of Chapter XVIII, the data from the *Star Almanac* for 1959 have been taken by permission of the Controller of H.M. Stationery Office.

It should also be noted that the accepted way of writing date and time is to put year, month, day of month, hours, minutes, and seconds in this order Thus, 3 h. 51 m. 22 s. on the 5th June, 1959, would be written 1959, June 5 d. 3 h. 51 m. 22 s.

10. Summary of Principal Formulæ for Time Conversion.

Problems in time and hour angle may be very confusing to a beginner and it is very important to be able to solve them quickly and confidently. Accordingly, the following is a list of formulæ which it would be well to commit to memory and which includes all those usually needed for the solution of practical problems:

G.T. = L.T. + West longitude. . . . . (24)

G.T. = L.T. − East longitude. . . . . (25)

Apparent − Mean solar time = e. . . . . . . (26)

G.H.A. true sun = U.T. + E. . . . . (27)

L.H.A. true sun = L.M.T. + E. . . . . (28)

L.S.T. of upper transit of a star = star's R.A. . (29)

G.H.A. Aries = G.S.T. . . . . (30)

G.S.T. = U.T. + R. . . . . (31)

L.H.A. Aries = L.S.T. . . . . (32)

L.S.T. = L.M.T. + R. . . . . (33)

G.H.A. Star = G.H.A. Aries − R.A. Star. . . (34)

G.H.A. Star = U.T. + R − R.A. Star. . . . (35)

L.H.A. Star = L.H.A. Aries − R.A. Star. . . (36)

L.H.A. Star = L.M.T. + R − R.A. Star. . . . (37)

In these formulæ 24 h. should be subtracted or added when necessary.

11. Examples.

(i) An observation of the sun on 1959, January 3, at a place in longitude 64° 45′ West gave the L.H.A. as 5 h. 34 m. 00 s. Find the corresponding L.M.T. (*Obtain G.H.A. and use formula* 27 *above.*)

From the *S.A.*, Table, p. 67, longitude 64° 45′ W	=	4 h.	19 m.	00 s.
L.H.A. Sun	=	5	34	00
G.H.A. Sun	=	9	53	00
−*E* for 1959, January 3 d. 18 h. (Table, p. 2)	=	−11	55	34·6
Approximate U.T. (Using formula 27)	=	21	57	25·4
Correction for *E* for 3 h. 57 m. 25 s. (Table, p. 70)	=		+	4·5
U.T. of observation	=	21	57	29·9
Longitude	=	− 4	19	00·0
L.M.T. of observation	=	17 h.	38 m.	29·9 s.

(In the above, the first step is to get an approximate value for the U.T. so as to get a value for *E*. A preliminary look at the Table shows that *E* is about 11 h. 56 m. and hence the approximate U.T. is somewhere about 21 h. 57 m., so that we then look out the tabulated value corresponding to the U.T. next below 21 h., viz. 18 h. We thus get a closer approximation to the true value of the U.T. A correction to allow for the correction to *E* for the difference between 18 h. and the approximate U.T. of 21 h. 57 m. 25·4 s. is then introduced and the corrected value for the U.T. so obtained.

Note also that, as *E* is here greater numerically than the G.H.A. sun, 24 h. are added to the latter before the subtraction takes place.)

(ii) An upper transit of the sun was observed at a place A in longitude 122° 14′ 36″ East on 1959, July 24, and the time registered on a clock set to keep L.M.T. was 12 h. 06 m. 53·8 s. What was the error of the clock? (*Obtain G.H.A. and use formula* 27.)

L.H.A. at upper transit	24 h.	00 m.	00 s.
From *S.A.*, p. 67, 122° 14′ 36″ E. longitude	− 8	08	58·4
G.H.A. at upper transit at A	15	51	01·6
−*E* at U.T. July 24 d. 0 h. (*S.A.*, p. 15)	−11	53	37·8
∴ Approximate U.T. of upper transit at A	3	57	23·8
Correction to *E* for 3 h. 57 m. 24 s. (*S.A.*, p. 70.)			+ 0·3
∴ U.T. of upper transit at A	3	57	24·1
East longitude	+ 8	08	58·4
L.M.T. of upper transit at A	12	06	22·5
Time by clock	12	06	53·8
Clock fast			31·3 s.

(iii) Find the L.M.T. of 2 h. 14 m. 33·4 s. L.S.T. on 1959, Sept. 15, at a place in longitude 4 h. 23 m. East. (*Use formula* 33, p. 316.)

		h.	m.	s.
Given L.S.T.		2 h.	14 m.	33·4 s.
R at 1959 Sept. 15 d. 0 h.		23	32	50·4
∴ Approx. L.M.T. (Using formula 33.)	1959, Sept. 15 d.	2	41	43·0
East longitude		−4	23	00·0
Approximate U.T.	1959, Sept. 14	22	18	43·0
Correction to R for 1 h. 41 m. 17 s.	(*S.A.*, p. 68.)			+16·6
∴ U.T.	1959, Sept. 14	22	18	59·6
East longitude		+4	23	00·0
∴ L.M.T.	1959, Sept. 15	2	41	59·6

(Note here that, owing to the negative longitude being greater than the approximate L.M.T., U.T. is on Sept. 14, not Sept. 15, and, having taken R out for Sept. 15 d. 0 h. as a preliminary value, the correction to the R is the correction for the difference between Sept. 14 d. 22 h. 18 m. 43·0 s. and Sept. 15 d. 00 h., or for 1 h. 41 m. 17 s.)

(iv) Find the L.S.T. corresponding to 1959, Jan. 5 d. 06 h. 12 m. 32·3 s. L.M.T. at a place in longitude 4 h. 23 m. East. (*Use formula* 31, p. 316.)

		h.	m.	s.
Given L.M.T.	Jan. 5 d.	06 h.	12 m.	32·3 s.
East longitude		−4	23	00·0
∴ U.T.		1	49	32·3
R at U.T. 1959, Jan. 5 d. 00 h.		6	55	22·2
Correction to R for 1 h. 49 m. 32 s.				+ 18·0
∴ G.S.T. (from formula 31.)		8	45	12·5
East longitude		4	23	00·0
∴ L.S.T.		13	08	12·5

For further examples involving time conversions see the examples at the end of Chapter XVIII.

CHAPTER XVIII

ASTRONOMICAL OBSERVATIONS

1. Limitations of Astronomically Observed Positions as Survey Control Points.

Although astronomically observed latitudes and longitudes have been, and still are, extensively used as a control for small-scale surveys and mapping, they have their limitations as a direct control or check on position in large-scale work. This is because of irregularities, or *anomalies*, in the earth's gravitational field which affect the apparent direction of the observer's zenith relative to the true celestial pole. These irregularities, commonly called the *deviation of the vertical,* can only be ascertained by extensive geodetic operations which are quite beyond the province of the ordinary engineer or surveyor. The result is that, if a survey is tied in to astronomically determined points, the differences between the astronomical fixings and fixings determined by direct survey may be much greater than the known accuracy of the survey methods adopted would lead us to expect. Hence, astronomically observed latitudes and longitudes have only limited value as checks for position on large- or medium-scale survey work of limited extent, though they are useful as a control for mapping on very small scales—say 1/250,000 or smaller—for control of rough preliminary reconnaissance and exploratory surveys and for locating the positions on the earth's surface of large-scale surveys and " putting them on the map ". Fortunately, these deviations of the vertical have only a slight effect on azimuths, so that in the case of ordinary engineering and large-scale surveys this effect can be neglected, although allowance for it is usually made in geodetic work of large extent. A knowledge of the astronomical latitude is, however, required in some azimuth computations and a fairly accurate knowledge of the time of observation, local or Greenwich, is needed in others. However, azimuth is the astronomical observation which is of most importance and occurs most frequently in engineering and similar surveys and hence in this chapter the emphasis will be on it with short descriptions of some of the simpler methods of determining time, latitude, and longitude, mainly as factors to be used in, or to assist in, azimuth work.

2. Equipment for Azimuth Observations.

In Chapter XIII we have described the instrumental equipment required for astronomical observations in general. For azimuth work we need a good station mark, or *Referring Object* (R.O.), at a distant point to which horizontal angle measurements between star and point may be referred. If the observations are being taken to the sun, there is no special difficulty about this as the signal will be one which can be clearly defined and seen by daylight, the only thing being that it should be as far away as possible, preferably inside the area to be covered by the survey. Thus, the point of a distant spire, a lightning conductor on a factory chimney, a carefully plumbed and distant ranging rod, a nail head in a tree, etc., will all do, there being no absolute necessity to fix the position of the R.O. accurately. At night time, signals or marks such as those just mentioned would be useless, and a luminous or illuminated signal becomes necessary. This can take several forms. The most convenient is one of the special targets, mounted on a tripod and illuminated by a battery-fed electric-light bulb, which are obtainable from makers as part of a " three-tripod " observing outfit, or as part of a mining-survey equipment for underground traversing. These targets generally show a cross of thin black vertical and horizontal lines against a bright background, but some show bright lines against a dark background. If such a target is not available, it will be necessary to improvise one. This can be a box containing an electric or oil lamp with a front of frosted glass or of tracing cloth on which a cross formed by thin black vertical and horizontal lines has been drawn. For a distant R.O., a box with a small circular aperature in an otherwise opaque front, which is illuminated from behind so that the aperature shows up at a distance as a fine point of light, will form a suitable R.O.

For night observing, if the instrument is not provided with internal electrical illumination of cross hairs and micrometers, or with a sun cap fitted with a small reflecting prism at the side to reflect light held at the side down the telescope to the cross hairs, other means must be improvised for illuminating these hairs. A narrow piece of white paper, fastened by a rubber band to the ordinary sun cap and bent over at an angle to reflect light shone on it from the side down the telescope, will serve to illuminate the cross hairs. The micrometers and verniers are usually provided with reflectors which can be illuminated by a hand-held electric torch.

Whenever times and altitudes are observed, it is well, as far as possible, not to attempt to use the tangent screw to bring the cross hair into contact with the image of the star, but rather to set the telescope to point a little ahead of the star and then to note the reading as the star appears to cross the hair. If, for instance, time observations are being taken by the method of ex-meridian altitudes, the telescope can be clamped when it is pointing a little ahead of the star and the time noted when the star appears to cross the horizontal hair, the angle of elevation, if not read immediately before the passage of the star, being then read immediately afterwards. If azimuth is being determined by ex-meridian altitudes, the telescope can be set as before at a fixed altitude, and, by using the horizontal-circle tangent screw, the image of the star should be kept on the vertical hair until the moment when the image appears to cross the horizontal hair. The object in all cases should be to avoid any unnecessary movement or touching of parts of the instrument by using the actual apparent motion of the star instead.

Again, when a series of altitude observations are being taken and the results will be worked out from the mean of the observed altitudes, these observations should be taken as quickly as possible so as to avoid the necessity for " curvature corrections " (pages 330 and 333) needed to allow for the fact that the rate of change of altitude of a star with regard to time is not quite constant.

3. Solar Observations.

The point in the sun to which the data given in Solar Ephemerides refer is the centre, but in practice it is not possible to point the telescope and cross hairs to the sun's exact centre. Instead, the observations are taken to a limb (edge) or limbs. It would be possible, for instance, when observing solar altitudes, to observe the altitude of the lower edge of the disc and then correct the observation for the semi-diameter of the sun, the angular value of which (about 16′) is given in *The Star Almanac for Land Surveyors* for each half-month in the year. Instead, it is usual to take two observations in quick succession, one to the lower limb and one to the upper as in fig. 18.1*a*, at the same time changing face between observations, and to take the mean of the two results as the altitude to be used in the computations. Similarly, if azimuth is being observed by time observations, observations would be taken in turn to the west and east limbs as in fig. 18.1*b*, while, if azimuth is being observed by altitudes, the sun would be observed in opposite quadrants as A and B or C and D as in fig. 18.1*c*.

If only one limb is observed, the correction to the azimuth angle can be derived as follows. In fig. 18.2 the correction to the azimuth angle is $\Delta\alpha$, ab is the sun's semi-diameter, S.D., and z the observed zenith distance. The great circle Za is the great circle through the zenith and

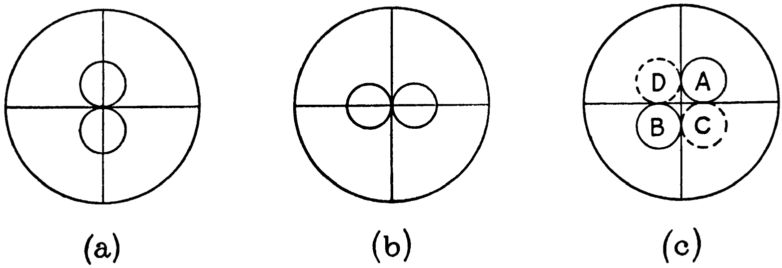


Fig. 18.1

the sun's centre a, and Zb the great circle tangential to the sun's limb at b, so that the angle Zba in the spherical triangle Zba is a right angle. Solving the triangle, we have

$$\sin \Delta\alpha = \sin \text{S.D.} \operatorname{cosec} \text{Za},$$

or, since $\Delta\alpha$ and S.D. are small angles and Za is approximately equal to Zb $= z$,

$$\Delta\alpha = \text{S.D.} \operatorname{cosec} z, \quad \ldots \ldots \quad (1)$$

$\Delta\alpha$ being in the same units, minutes or seconds, as S.D.

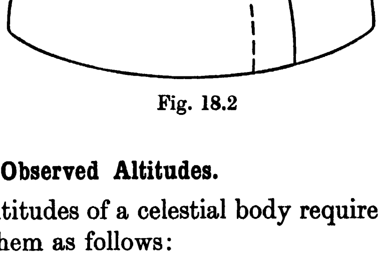


Fig. 18.2

4. Corrections to Observed Altitudes.

All observed altitudes of a celestial body require various corrections to be applied to them as follows:

(i) Correction for refraction.

(ii) Correction for dislevelment of the vertical axis in the direction of the line of sight,

and, in the case of the sun,

(iii) Correction for parallax.

(i) *Correction for Atmospheric Refraction.*

Fig. 18.3 shows a ray of light from a distant celestial body falling on the earth's atmosphere at A, where, owing to refraction, it is bent slightly towards the normal AN. As the density of the atmosphere increases with decrease in height, the bending becomes greater as the ray penetrates the atmosphere, and in the limit the ray travels along the curved path AB, with the result that the body is seen in the direction

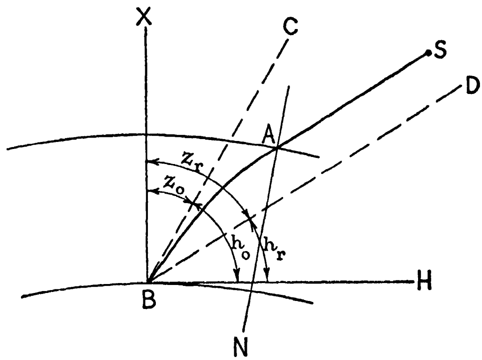


Fig. 18.3

BC, where BC is tangential at ground level at B to the curve BA. Draw BD parallel to AS. Then, as the body is at a very great distance away, a ray from it, in the absence of the earth's atmosphere, would reach B along the path DB. Hence, the true zenith distance of the star referred to the normal BX at B is the angle XBD, but, owing to refraction, the apparent direction is BC and the apparent zenith distance is the angle XBC. Hence the angle CBD is the correction for refraction which has to be applied to the observed zenith distance to give the true zenith distance. When applied to the observed zenith distance it is always positive, but when applied to the observed altitude it is always negative.

The correction for refraction varies with the altitude of the body and the atmospheric temperature and pressure at the time of observation. Its values for any given altitude, temperature, and pressure can be found from the refraction tables given on pages 60–62 of *The Star Almanac for Land Surveyors* or from the refraction tables given in *Chambers's Seven-Figure Mathematical Tables.* As surface observations of pressure and temperature can only give a comparatively rough indication of the conditions along the path of the ray, there is always a little doubt about the true value of the refraction correction, especially at low altitudes when the paths of rays of light traverse the greatest thickness of the atmosphere. Hence, *observations at very low altitudes —say less than* 15°*—are to be avoided,* and, whenever possible, observational methods should be arranged to minimize the effects of errors of refraction.

As a rough rule, for stars of moderate elevation and neglecting the small corrections for temperature and pressure, the refraction correction can be taken as

$$R = 58'' \cot h_0 \text{ seconds of arc,} \quad . \quad . \quad . \quad . \quad (2)$$

where h_0 is the observed altitude, the true altitude then being

$$h_r = h_0 - 58'' \cot h_0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

(ii) *Correction for Dislevelment of the Vertical Axis in the Direction of the Line of Sight.*

The observed elevation of a celestial body will be in error if the vertical axis of the theodolite is not truly vertical at the moment of observation, and for this reason the altitude level on the vernier arm should be read immediately before and after an observation is made. The correction to be added algebraically to the observed altitude is then given by

$$c = \frac{O - E}{2} d, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

where O = reading of the end of the bubble at the object glass end of the altitude level, E = reading of the end of the bubble at the eyepiece end of the altitude level, and d = the value of one division of the altitude level in angular measure. This supposes the bubble tube to be graduated in both directions from zero in the middle of the tube. If the latter is graduated continuously from the eyepiece end and G is the reading of the central graduation,

$$c = \frac{(O + E) - 2G}{2} d. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

If there are n observations, including both F.R. and F.L., the correction to the mean altitude is

$$c = \frac{\Sigma O - \Sigma E}{2n} d \quad \text{or} \quad c = -\frac{\Sigma(O + E) - 2nG}{2n} d, \qquad . \quad (6)$$

where ΣO and ΣE denote summations of the n readings of both O and E.

(iii) *Correction for Parallax.*

This correction, which is only applicable to solar observations, allows for the fact that, as the length of the earth's radius is not altogether negligible in comparison with the sun's distance from the earth, the altitude of the sun at a place on the earth's surface is not quite the same as it would be if it could be viewed from the centre of the earth.

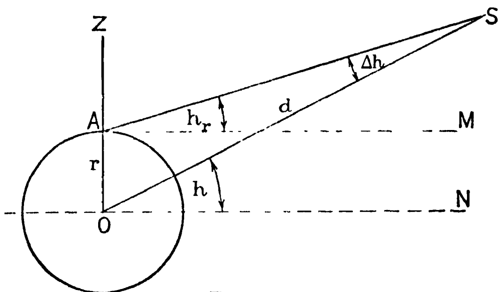


Fig. 18.4

In fig. 18.4, O is the centre of the earth, A the place of observation on the earth's surface, Z the zenith at A, AM the horizon plane at A, and ON is parallel to AM. Then the observed altitude of the sun S from A, corrected for refraction and dislevelment, is the angle SAM = h_r, but, if S were observed from O, the observed altitude would be the angle SON = h. Then the angle ASO = Δh is the correction which must be applied to h_r to give h.

Let OA = r and OS = d. Then in the triangle OAS,

$$\frac{\sin \Delta h}{r} = \frac{\sin (90° + h_r)}{d},$$

or

$$\sin \Delta h = \frac{r}{d} \cos h_r.$$

The quantity r/d, which we shall denote by P, is called the *sun's horizontal parallax* and is the value which Δh would have if the sun were on the observer's horizon and there was no refraction. Also, the angle Δh is small, so that we can write

$$\Delta h = P \cos h_r. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

P varies from about $8''\cdot95$ at perihelion early in January to $8''\cdot66$ at aphelion early in July, and its mean value can be taken to be $8''\cdot80$. Hence, the correction, which is always additive to h_r is very small and for much minor work can be neglected.

5. Correction to an Observed Direction for Dislevelment of the Horizontal Axis of the Theodolite.

When an instrument is used which is provided with a striding level for measuring the dislevelment of the horizontal axis, there is a small correction to be applied to observed directions when the axis is not truly horizontal, this correction being given by

$$c'' = e'' \tan h, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

where e'' is the dislevelment of the horizontal axis as obtained from readings on the striding level and h is the observed altitude of the point concerned.

For angles of elevation, the correction to the observed direction on the horizontal circle is positive when the left pivot is higher than the right and negative when it is lower, the signs being reversed for angles of depression.

6. Astronomical Observations in General.

In general, astronomical observations may be divided into the following classes according to the methods used:

1. Meridian Observations.
2. Circum-Meridian Observations.
3. Ex-Meridian Observations.
4. Equal-altitude Observations.
5. Circum-polar Star Observations.
6. Position Line Methods.

Meridian observations are observations taken when the celestial body is on the observer's meridian. Circum-meridian observations are observations taken when the body is not actually on the meridian but is only a short distance away from it. Ex-meridian observations are observations taken when the body is some distance away from the meridian. Equal-altitude observations are observations taken to the same body at equal altitudes on each side of the meridian or to two or more bodies at the same altitude. Circum-polar observations are observations to a circum-polar star, generally at or near *culmination* (or meridian transit) or at elongation. Position line methods give latitude and longitude and have the advantage that the reductions are mainly graphical and enable the most likely values of the unknowns to be determined graphically when a number of observations are involved.

The following are the main methods used for determining azimuth, time, latitude, and longitude, set out, as far as possible, in order of relative simplicity; this, in general, is the reverse of the order of relative accuracy.

Observations for Azimuth

1. By equal altitudes of star or sun.
2. By ex-meridian altitudes of star or sun.
3. By hour angles of sun or star.
4. By altitudes of *Polaris*, the pole star.
5. By observations to a close circum-polar star.
6. By observations of circum-polar stars at elongation.
7. By observations of a circum-polar star at culmination.

Observations for Time

1. By reception of the wireless time signals.
2. By equal altitudes of a star or of the sun.
3. By ex-meridian altitudes of stars or sun.
4. By meridian transits of stars or sun.

Observations for Latitude

1. By meridian altitudes of sun or star.
2. By circum-meridian altitudes of sun or star.
3. By ex-meridian altitudes of sun or star.
4. By altitudes of *Polaris*.
5. By meridian altitudes of a circum-polar star at upper and lower transits.
6. By Talcott's method.

Observations for Longitude

The determination of longitude involves a determination of the difference between local and Greenwich time. The local time can be determined by any of the methods mentioned above while Greenwich time is now easily obtained from the wireless time signals.

Combined Observations

In certain cases observations can be arranged to yield two unknowns from the one set of observations. In general, each set involves recording both elevation and time, or else observations to two or more suitably situated stars. Thus, timed observations of the altitude of *Polaris* will give both azimuth and latitude, and latitude and time may be obtained from timed observations of two stars of considerably different declinations as they cross an approximate local meridian. Other methods are equal altitudes of sun or star for rough determinations and ex-meridian altitudes for more accurate ones.

AZIMUTH OBSERVATIONS

In principle an azimuth observation is a little more complicated than any of the other ordinary astronomical observations because it consists of a purely astronomical observation, which, when computed, gives the azimuth of the celestial body at the moment of observation, combined with the measurement of the horizontal angle between the body and the mark or signal on the ground used as a R.O. The general procedure therefore consists in a pointing to the R.O., with a reading on the horizontal circle, followed by a pointing to the star for the purely astronomical part of the observation and another reading on the horizontal circle.

(i) *Azimuth by Equal Altitudes of Star or Sun.*

If the horizontal angle between a star and the R.O. is observed when the star is at equal altitudes east and west of the meridian, the mean of the horizontal angles gives the horizontal angle between the meridian and the R.O., and hence the azimuth of the latter. In practice, an even number of observations should be taken to the star when it is east of the meridian, with change of face between each observation of a pair, and then a similar series taken at the same altitudes when the star is west of the meridian, using in each case the same face as previously used for the same altitude.

When taking the observation, the lower circle of the theodolite is clamped and the telescope pointed so that the vertical hair intersects the image of the R.O., when the upper circle is clamped and the horizontal circle read. The upper clamp is then loosened and the telescope set to point a little ahead of the star, when both horizontal and vertical circles are clamped. The fine-motion tangent screw of the upper circle is then turned so that the vertical hair intersects the image of the star, and, by turning the tangent screw, it is kept on the image of the star until the latter appears to cross the horizontal hair, when both horizontal and vertical circles are read. The difference between the first and second readings of the horizontal circle gives the angle between the star and the R.O. when the star was at the elevation read on the vertical circle. Face is then changed and the operation repeated for a slightly higher elevation. This completes one set of observations. If greater accuracy is needed, more sets can be observed. These sets are then repeated when the star is west of the meridian, the vertical circle being set to the same elevations, on the corresponding faces, as were used when the star was west of the meridian. The mean of all the horizontal angles gives the zenith angle between the meridian and the R.O.

If the sun is used, as actual elevations are not needed, all observations can be made with the same limb, upper or lower, on the horizontal hair, say as in D and A or C and B in fig. 18.1*c*. If, however, observations are made with the sun in diagonally opposite quadrants of the reticule, as in D and C, and, in the afternoon, B and A in fig. 18.1*c*, the morning observation could be computed as an ex-meridian observation should the afternoon observations become impossible. As the declination of of the sun changes fairly rapidly with time, and if there is a large interval between morning and afternoon observations, it may be necessary to apply a correction for the change of declination during that interval to the mean of the morning and afternoon horizontal angles. The correction is given by

$$c = \tfrac{1}{2}(\delta_W - \delta_E) \sec\phi \operatorname{cosec} t, \quad . \quad . \quad . \quad . \quad (9)$$

where ϕ = observer's latitude,

δ_E = sun's declination at the mean time of the morning observations,

δ_W = sun's declination at the mean time of the afternoon observations,

t = half of the time interval between morning and afternoon observations.

When $(\delta_W - \delta_E)$ is positive, the mean of the observed horizontal angles lies west of the meridian when the place of observation is north of the equator and east when the place of observation is south of the equator, the directions being reversed when $(\delta_W - \delta_E)$ is negative. The declination δ is changing most rapidly at the equinoxes, when the rate of change is about 1′ per hour, and it is changing most slowly at the solstices. Hence, at an equinox, if $\phi = 45°$ and $t = 3$ h. $= 45°$, $c = \frac{1}{2} \times 6 \times \sqrt{2} \times \sqrt{2} = 6'$ and is certainly not negligible.

This method is a very simple one from the point of view of there being practically no computational work involved, and also it can be used when the latitude of the place of observation is known only very roughly or (and this may be a great advantage) if no tables are available. It is, however, an inferior method as regards accuracy because, if observations are taken shortly before and shortly after transit, errors in altitude produce unduly large errors in azimuth, and, if observations are taken some hours before and some hours after transit, the interval of time between them is rather large for convenience, and, moreover, there may be very appreciable changes in refraction during that time.

(ii) *Azimuth by Ex-meridian Observations of Star or Sun.*

The observations consist of observing a series of altitudes of a star or of the sun, preferably when it is on or near the prime vertical, and at the same time noting the horizontal angle between the star and the R.O., just as was done for one of the observations described in the last section. Observations should be made very rapidly in pairs, with change of face between the observations of each pair, several pairs being taken if possible. If the observations are close together, the azimuth can be computed for the mean altitude and the mean horizontal angle by the formula: *

$$\tan \tfrac{1}{2}\alpha = \sqrt{\left[\frac{\sin(s-z)\sin(s-c)}{\sin s \sin(s-p)}\right]}, \quad . \quad . \quad . \quad (10)$$

where $s = \frac{1}{2}(p + z + c)$.

If times of observation are also noted, both azimuth and clock error can be found by using formulæ (7) and (8) on page 304, viz.

$$\tan r = \sqrt{\left[\frac{\sin(s-z)\sin(s-c)\sin(s-p)}{\sin s}\right]}, \quad . \quad (11)$$

and

$$\tan \tfrac{1}{2}\alpha = \frac{\tan r}{\sin(s-p)}; \quad \tan \tfrac{1}{2}\beta = \frac{\tan r}{\sin(s-z)}, \quad . \quad (12)$$

and then finding A and H from the rules given on page 302.

* In refined work a correction for *curvature of path* is applied when observations are averaged in this way for computational purposes. This allows for the fact that the altitude of a celestial body is not exactly a linear function of azimuth or time. For most ordinary work it can be neglected.

The method calls for a fairly accurate knowledge of the latitude of the place, and, if this is not already known, it can be found by one of the methods described later. In order to minimize the effects of errors due to errors in the assumed latitude and of errors of the same sign in the observed altitudes, it is advisable to balance a series of observations of east stars with a series of observations of similarly placed west stars at approximately the same altitudes as before, or, in the case of the sun, a series of morning observations should be balanced by a similar number of afternoon observations with the sun at approximately the same altitudes as in the morning. Solar observations should be taken with the sun in diagonally opposite quadrants of the reticule, as A and B or C and D in fig. 18.1*c*. When the sun is used, it is necessary to know the approximate Greenwich time of each observation in order to be able to interpolate the declination of the sun reasonably accurately. Hence, the U.T. of each observation, as well as the E and O readings of the vernier arm level bubble, should be recorded. Also, owing to the fact that the difference between the azimuths of the centre and one limb of the sun is dependent on the sun's altitude, there is a small correction applicable to the azimuth angle calculated from the mean of the altitudes of a face-right and face-left observation when there is an appreciable difference, Δh, in altitude between the two observations. This correction is given by

$$c = \tfrac{1}{2}\text{S.D.}\sin 1' \tan h \sec h\, \Delta h, \qquad . \quad . \quad . \quad . \quad (13)$$

where S.D. is the sun's semi-diameter in minutes and c is in minutes or seconds according as Δh (*here of the sun's centre*) is in minutes or seconds. Taking $\tfrac{1}{2}$S.D. $\sin 1' = 0{\cdot}142$, the correction becomes

$$c = 0{\cdot}142 \tan h \sec h\, \Delta h, \qquad . \quad . \quad . \quad . \quad . \quad (14)$$

The sign of the correction depends on the order in which the side limbs are observed, and it tends to cancel out if each morning or afternoon set of observations is meaned with another similar set in which the order in which these limbs are observed is reversed, and also, when morning and afternoon *azimuths of the R.O.* are to be meaned, if the limbs are observed in the same order in both sets.

The method, when carefully carried out, gives almost as good a result as can be obtained from any that is normally used for work other than geodetic work. The errors in α due to errors Δh in h or $\Delta\phi$ in ϕ are

$$\Delta\alpha = \cot \gamma \sec h\, \Delta h, \qquad . \quad . \quad . \quad . \quad . \quad (15)$$

where γ is the parallactic angle given by

$$\sin \gamma = \frac{\cos \phi \sin \alpha}{\cos \delta}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

and

$$\Delta\alpha = \cot \beta \sec \phi \, \Delta\phi. \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

(iii) *Azimuth by Hour Angles of Star or Sun.*

The procedure in the case of a star observation by this method is exactly the same as in the last method except that, instead of measuring altitudes, the chronometer times of the observations are noted and the azimuth deduced from the hour angle by formula (11), page 304, viz.

$$\tan \alpha = \frac{\sin \beta}{\cot p \sin c - \cos c \cos \beta} \quad . \quad . \quad . \quad (18)$$

or, for logarithmic computation,

$$\tan \alpha = \tan \beta \sin \theta \operatorname{cosec} (c - \theta), \quad . \quad . \quad . \quad (19)$$

where

$$\tan \theta = \tan p \cos \beta.$$

This method is not so convenient or simple as the previous one because of the necessity for having an accurate knowledge of the chronometer error, but it is slightly more accurate because it is not affected by refraction, and, if the observations are near the prime vertical, as they should be, small errors of time have no unduly serious effects on the result. For accurate work, observations east of the meridian should be balanced by others to a similarly placed star west of the meridian, or, in the case of the sun, morning observations should, if possible, be balanced by observations in the afternoon when it is at approximately the same elevation as in the morning.

The errors in α due to errors $\Delta\beta$ in β and $\Delta\phi$ in ϕ are

$$\Delta\alpha = - \sin \alpha \cos \gamma \operatorname{cosec} \beta \, \Delta\beta, \quad . \quad . \quad . \quad (20)$$

where γ is the parallactic angle calculated from formula (16) above, and

$$\Delta\alpha = \cot z \sin \alpha \, \Delta\phi, \quad . \quad . \quad . \quad . \quad . \quad . \quad (21)$$

where z, when not observed, can be calculated from

$$\sin z = \sin \beta \sin p \operatorname{cosec} \alpha. \quad . \quad . \quad . \quad . \quad . \quad (22)$$

(iv) *Azimuth from Observations of Polaris, the Pole Star.*

This method, which can be used to obtain latitude as well as azimuth, is suitable for observations between latitudes 15° N. and 66° N. as the computations are greatly simplified by the use of the "Pole Star Table" given in *The Star Almanac for Land Surveyors.* The altitude and approximate sidereal time of observation are noted at any hour angle, and the latitude of the place and the azimuth of the star can then be obtained from

$$\text{Latitude} = \text{corrected observed altitude} + a_0 + a_1 + a_2, \quad (23)$$

$$\text{Azimuth angle} = (b_0 + b_1 + b_2) \sec \phi, \quad . \quad . \quad . \quad . \quad . \quad (24)$$

where the quantities a_0, a_1, a_2, b_0, b_1, b_2 are obtained from the table given in the *Star Almanac*, which also gives a table of natural secants for the factor $\sec \phi$. As before, of course, the observation includes the measurement of the horizontal angle between the R.O. and the star at the moment of each observation of altitude and sidereal time. If a mean-time chronometer is used, it will be necessary to convert the recorded mean times into sidereal times.

(v) *Azimuth by Observations to a Close Circum-polar Star.*

This method, which can be used for first-order work when suitable instruments are available, is similar to the method of ex-meridian observations of hour angles to an ordinary star already described, the only thing being the use of a circum-polar star near the pole instead of any ordinary star. The best results are obtained when the star is at or near elongation, i.e. when $\sin \alpha$ is equal to, or approximately equal to, $\sin p \operatorname{cosec} c$, for the star is then moving in the direction of the line of sight.

If a series of n observations are taken fairly close together before and after elongation, the mean of the hour angles can be used to compute α by the formula (18) for $\tan \alpha$ given on page 332, but, to the α so computed must be applied a curvature correction given in seconds of arc by

$$c'' = \frac{2 \tan \alpha \cos^2 p \sum \sin^2 \frac{1}{2}\Delta t}{n \sin 1''}, \quad . \quad . \quad . \quad . \quad (25)$$

where Δt is the sidereal time interval, expressed in angular measure, between the time of a single observation and the mean of the times, and $\Sigma \sin^2 \frac{1}{2}\Delta t$ is the sum of the squares of the sines of the half Δt's.

The correction is always subtractive from α whether the star is east or west of the meridian. Values of the correcting factor $2 \sin^2 \frac{1}{2}\Delta t \operatorname{cosec} 1''$ are given in the "Table for Circum-Meridian Observations" on page 66 of the *Star Almanac.*

Suitable stars for this method are *Polaris* (α Ursae Minoris), 51 H Cephei, and δ Ursae Minoris in the northern hemisphere and ζ Octantis and σ Octantis in the southern, the right ascensions and declinations of which are given for 10-day intervals in the special table for circum-polar stars given in *The Star Almanac for Land Surveyors.*

(vi) *Azimuth by Observations to a Circum-polar Star at Elongation.*

When the star is at elongation the azimuth can be computed from

$$\sin \alpha = \frac{\cos \delta}{\cos \phi}. \quad \ldots \ldots \quad (26)$$

If the star was not actually at elongation at the moment of observation but at a small interval, not exceeding 30 minutes of time, on either side of it, the α calculated by the formula just given should have applied to it a correction, c'', which is given in seconds of arc by

$$c'' = 1{\cdot}96 \tan \alpha \cos^2 p(t_E - t)^2, \quad \ldots \quad (27)$$

where $(t_E - t)$ is the interval, in minutes of sidereal time, between the time of observation and the time of elongation. This expression, it will be noted, is of the same form as the expression given for c'' in the last section but with n put equal to 1 and $\frac{1}{2}(t_E - t)$ 15 sin 1′ written for $\sin \frac{1}{2}\Delta t$.

The observation involves the preliminary calculation of the azimuth, elevation, and time at the moment of elongation from

$$\sin \alpha = \frac{\cos \delta}{\cos \phi}, \quad \ldots \ldots \quad (28)$$

$$\sin h = \frac{\sin \phi}{\sin \delta}, \quad \ldots \ldots \quad (29)$$

$$\sin \beta = \frac{\cos h}{\cos \phi}, \quad \ldots \ldots \quad (30)$$

$$\text{L.S.T.} = H + \text{Star's R.A.} \quad \ldots \quad (31)$$

A few minutes before the computed time the instrument is set to read on the R.O. and the horizontal circle read. The telescope is then set to point to the star, and, when the latter is on the vertical hair, clock time and horizontal circle are read, the lower circle being kept clamped, of course, all the time. The telescope is then reversed very quickly to give a change of face, and another set of readings taken. Each of these readings has the correction given above applied to the calculated value of α and the mean of the resulting azimuths of the R.O. taken as the value required.

A rough method is to direct the telescope to the star shortly before the calculated time of elongation and then to keep it on the vertical cross hair until its direction of motion appears to be steady, readings on the horizontal circle being, of course, taken both with the telescope pointed to the R.O. and then again when the star appears to be stationary on the vertical hair. This method has the disadvantage that a single observation on one face only is possible.

TIME OBSERVATIONS

Owing to the ease with which Universal or Greenwich time can now be obtained from the ordinary wireless time signals the necessity for astronomical observations to obtain time no longer arises as often as it did before the advent of wireless. In fact, the occasions on which precise time determinations are necessary are now usually when longitude is required, and the precise determination of longitude is a problem which concerns the geodesist rather than the ordinary engineer or surveyor. Hence in this section we shall only consider methods which are sufficient to give time to about a half of a second, which is amply accurate enough to yield a satisfactory azimuth when the azimuth observations are dependent on an observation of time, or are needed merely to give times for interpolation purposes when looking out data for the sun or for *Polaris* from *The Star Almanac.*

(i) *Time by Reception of Wireless Time Signals.*

Two types of wireless time signals are used in surveying. The first corresponds to the ordinary time signal radiated by the British Broadcasting Corporation at certain hours of the day, and the second are the special rhythmic time signals radiated by different stations at intervals throughout the day. The rhythmic time signals are used for accurate longitude work and enable Greenwich time to be obtained to a few hundredths of a second. The signals radiated in their ordinary transmissions by the B.B.C. consist of six dots or " pips " marking successive mean time seconds, the first dot being emitted at five seconds before the hour and the last at the hour itself. With practice and care time may be estimated from these signals correct to about a third of a second. Also, if a short-wave receiver is available, time can be obtained to a little higher degree of accuracy from the time signals radiated, 24 hours a day, by station W.W.V., Washington, U.S.A. on the frequencies 2·5 to 25 megacycles per second.

If local time is required, U.T. determined from the wireless time signals must be corrected for longitude when the latter is known with sufficient precision; otherwise, local time must be found by astronomical observations.

The rhythmic signals are devised for work of a higher order of accuracy than that considered in these pages. For the minor work with which we are immediately concerned, other stations besides the B.B.C. radiate suitable time signals at fixed times. Occasional changes in the signals are made from time to time, but full particulars and explanations of all the signals available will be found in Vol. II of *The Admiralty List of Radio Stations*, which is published annually by H. M. Stationery Office.

(ii) *Time by Meridian Observations.*

A refined form of this type of observation is used for primary time observations in fixed observatories where facilities exist for setting the line of collimation of the instrument very accurately in the meridian, and it is also much used in geodetic work. It is a convenient and simple method to use in minor work when, as a result of previous survey operations, lines of known bearings exist on the ground from which the instrument, a theodolite, can be set on the meridian with fair precision. The observation consists of observing the time as the celestial body, sun or star, crosses the vertical hair of the telescope, this hair having already been set to give a line of sight which lies in the plane of the meridian.

When the sun is used, the times of passage of the east and west limbs are noted, and the mean of the times taken as the time of passage of the sun's centre. The hour angle of the sun will then be 24 hours, and L.M.T. at the time of passage of the sun's centre will be given by

$$\text{L.M.T.} = 24\text{h.} - E, \quad . \quad . \quad . \quad . \quad . \quad . \quad (32)$$

where E is the quantity defined in page 310 which is to be found tabulated in *The Star Almanac.*

In order to calculate the value of E, the L.M.T. must be corrected for longitude to give U.T. corresponding to 12 h. L.M.T., and the value of E interpolated for this value. Thus, suppose the observed mean time of transit of the sun on 1959 April 24 was 12 h. 01 s. 14·8 s.

at a place in longitude 8 h. 36 m. 24·8 s. West, what was the error of the watch?

Then 12 h. L.M.T. = 20 h. 36 m. 24·8 s. U.T.

From the *S.A.*, *E* for 1959, April 24 d. 18 h.	=	12 h.	01 m.	48·8 s.
Correction for 2 h. 36 m.	=			+ 1·2
E at 12 h. L.M.T.	=	12	01	50·0
L.M.T. = 24 h. −*E*	=	11	58	10·0
Time by watch	=	12	01	14·8
Watch fast			3	04·8

If a star is used, the hour angle at upper transit is zero and the sidereal time of upper transit is the star's right ascension. If the watch used is one keeping sidereal time, its error is thus the difference between the observed time of transit of the star and the star's right ascension. If a watch keeping mean time is used,

$$\text{L.M.T.} = \text{L.S.T.} - R, \quad \ldots\ldots \quad (33)$$

in which, for purposes of interpolation from *The Star Almanac*, L.M.T. must be converted to U.T. and the value of *R* used in the equation interpolated for this time.

In making the observation when a booker is available, the latter can note the exact time on the watch when the observer calls out " now " as the star crosses the meridian. If no booker is available, the observer can use a stop watch to note the difference in time between that when he sees the star on the cross hair and the time when he reads the watch. If no stop watch is available, he can count half-seconds by calling out thus: " 0 and 1 and 2 and 3 and . . ."—to carry time from instrument to watch.

(iii) *Time by Equal Altitudes of Star or Sun.*

If the times at which a star is at equal altitudes east and west of the meridian are noted, the mean of these times will be the watch time of transit of the star and this, compared with the time computed from the star's right ascension, will give the error of the watch. The best time to take the observations is when the star's altitude is changing most rapidly with time, that is, when the star is at, or near, the prime vertical; but the objection to the method is the long interval between observations, during which there may be a very appreciable change in atmospheric refraction as well as an appreciable one in the clock error.

If the sun is used, the change in declination between observations may be appreciable and will necessitate a correction, c, which must be applied to the mean of the watch times at equal altitudes to give the watch time of apparent noon. This correction is given by

$$c = \frac{\delta_E - \delta_W}{2 \times 15} (\tan \delta \cot t - \tan \phi \operatorname{cosec} t), \quad . \quad . \quad (34)$$

where δ_E and δ_W are the sun's declination at the U.T. of the morning and afternoon observations, $\delta = \frac{1}{2}(\delta_E + \delta_W)$, t is one half of the interval between observations and ϕ is the observer's latitude, which need only be known very approximately.

The observations may consist of a number of readings to one limb of the sun taken close together in the morning followed by a number taken at the same altitudes of the same limb in the afternoon.

(iv) *Time by Ex-meridian Altitudes of Star or Sun.*

This method consists in recording the altitude of the star or sun when it is some way east or west of the meridian and the clock time of observation. In this case, the latitude must be known fairly accurately as the solution directly involves the three " sides " of the astronomical triangle and is given by the formula

$$\tan \tfrac{1}{2}\beta = \sqrt{\left[\frac{\sin (s-p) \sin (s-c)}{\sin s \sin (s-z)}\right]}, \quad . \quad . \quad (35)$$

where

$$s = \tfrac{1}{2}(z + p + c).$$

The error $\Delta\beta$ in β due to an error Δh in h is got by differentiating β with respect to h in

$$\sin h = \sin \phi \sin \delta + \cos \phi \cos \delta \cos \beta.$$

This gives

$$\Delta\beta = - \frac{\cos h}{\cos \phi \cos \delta \sin \beta} \Delta h = - \sec \phi \operatorname{cosec} \alpha \Delta h, \quad (36)$$

or, with $\Delta\beta$ expressed in seconds of time and Δh in seconds of arc,

$$\Delta\beta \text{ s.} = -\tfrac{1}{15} \sec \phi \operatorname{cosec} \alpha \, \Delta h'', \quad . \quad . \quad . \quad . \quad (37)$$

From this it is obvious that the error in β for a given error in h is least when α is near 90°, i.e. the best position for observing it when determining time by this method is when the celestial body is on or near the prime vertical.

LATITUDE OBSERVATIONS

(i) *Latitude by Meridian Altitudes of Sun or Star*

A good value for the latitude is easily obtained by measuring the altitude of a body of known declination, sun or star, as it crosses the meridian, and, even when the direction of the meridian is not accurately known, it will usually be possible to find the approximate direction of it from *Polaris* or by prismatic compass or other means. A body will be at its highest altitude when it crosses the meridian at upper transit, and it will be at its lowest altitude when it crosses the meridian at

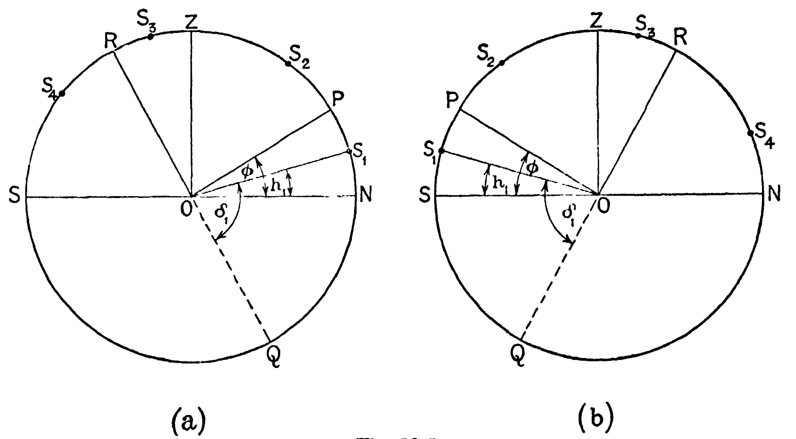


Fig. 18.5

lower transit, though if it is not a circum-polar star the lower transit will not be visible. Hence, if a series of altitudes of a body are observed for some little time before and after transit, the highest or lowest altitude can be interpolated and used to work out the latitude.

The relation between the latitude of the place of observation and the declination and altitude of the body at meridian passage can be studied from figs. 18.5*a* and *b* which represent a section of the meridian, fig. 18.5*a* being for the northern hemisphere and fig. 18.5*b* for the southern. Z, as usual, indicates the observer's zenith and P the celestial pole. SON is the section of the horizon plane and ROQ the section of the celestial equator. S_1 represents a (circum-polar) star at lower transit, S_2 a star crossing the meridian between the zenith and the elevated pole, S_3 one crossing the meridian between the zenith and the equator, and S_4 one which crosses the meridian below the equator. Then, ignoring negative signs of δ and ϕ and treating all values as positive, it is easy to show that, in both hemispheres,

$$\left.\begin{aligned}\phi &= 90^\circ + h_1 - \delta_1 \\ &= h_2 + \delta_2 - 90^\circ \\ &= 90^\circ + \delta_3 - h_3 \\ &= 90^\circ - \delta_4 - h_4,\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (38)$$

where δ_1, δ_2, δ_3, δ_4 and h_1, h_2, h_3 and h_4 are the declinations and elevations of S_1, S_2, S_3, and S_4 respectively.

If there are no lines of known bearing radiating from the point of observation from which the direction of the meridian can be set out, the instrument can be accurately orientated if the clock error in terms of local time is known by remembering that the sidereal time of transit of a star is the star's right ascension. Hence, by sighting on the star at this time, the observer can set the instrument very approximately on the meridian. If the clock error is not accurately known, the direction of the meridian should be determined approximately as described above and the highest or lowest altitude interpolated from observations before and after transit. This interpolation is easy if sufficient observations are taken because the altitude of a body changes only slowly as it crosses the meridian. Observations should start while the altitude is increasing or decreasing and be continued until it starts to decrease or increase as the case may be.

In the case of the sun, observations can be taken to one limb throughout and the altitude corrected for the sun's radius. Alternatively, a number of alternate F.R. and F.L. observations on upper and lower limbs should be taken in rapid succession and the altitude of the centre deduced from the means of pairs.

In the case of star observations, it is well to take a series with different stars transiting north of the zenith and to balance these with stars transiting south of the zenith at approximately the same altitudes.

This is probably the simplest method for the ordinary surveyor of determining latitude.

(ii) *Latitude by Meridian Altitudes of a Circum-polar Star at Upper and Lower Transits.*

If the altitude of a circum-polar star is observed at upper and lower transit, the latitude will be the mean of the two altitudes. This method is simple in theory but the main disadvantage is the long interval in time between the two observations. One feature of the method is that a knowledge of the star's declination is not needed.

(iii) *Latitude by Circum-meridian Altitudes of Stars or Sun.*

This method, which is an improved variation of Method (i), is simple and gives accurate results when the direction of the meridian and the clock error are known with moderate accuracy. It consists in measuring the altitudes of a series of stars at carefully timed intervals before and after transit. If the hour angle in each case is known from the observed times of observation, the observed altitude can be reduced to the altitude which the star would have when it crossed the meridian by applying the formula

$$z = z_0 - Am + Bn, \quad . \quad . \quad . \quad . \quad . \quad . \quad (39)$$

where

z_0 = observed zenith distance (corrected for refraction) and positive or negative according as to whether the star is south or north of the zenith,

z = required zenith distance when star is on the meridian,

z_1 = an approximate zenith distance as deduced from the maximum observed altitude.

ϕ_1 = an assumed approximate latitude as deduced from z_1, positive when north and negative when south,

β = polar angle of star at the moment of observation,

δ = declination of star,

$A = \cos\phi \cos\delta \operatorname{cosec} z = \cos\phi_1 \cos\delta \operatorname{cosec} z_1$ approximately,

$$m = \frac{2\sin^2 \frac{1}{2}\beta}{\sin 1''},$$

$$B = A^2 \cot z,$$

$$n = \frac{2\sin^4 \frac{1}{2}\beta}{\sin 1''}.$$

The number of observations, each of which must be reduced separately, should be the same for each side of the meridian, and, in order to reduce the effects of errors due to refraction, each set of observations to a northern star should be balanced by a similar set to a southern star at very approximately the same altitudes as the first set. The factor m can be obtained directly from the table on page 66 of the *S.A.*

In the case of solar observations, these should be taken to upper and lower limbs alternately, but, of course, observations cannot be paired by others taken on the other side of the zenith.

With this method, the observed body will be moving almost perpendicularly to the meridian, so that a small error in time will produce a very small error in the observed altitude.

(iv) *Latitude by Timed Altitudes of Polaris.*

In the northern hemisphere, when the clock error and rate are accurately known, latitudes between about 20° and 66° N. can be found fairly easily by timing the observations of the altitude of *Polaris*, the Pole Star (α Ursae Minoris). The latitude is then the observed altitude of the star with all corrections applied, plus the sum of three quantities, a_0, a_1, and a_2, which are specially tabulated in *The Star Almanac for Land Surveyors* in terms of L.S.T., latitude (only required approximately) and month.

(v) *Latitude by Talcott's Method.*

This method, sometimes called the Horrebow-Talcott method, is mainly used in first-order geodetic work but it necessitates the use of a special instrument called the *Zenith Telescope*, or else a theodolite fitted with a specially sensitive level bubble for the vernier arm and an eyepiece micrometer for measuring very accurately small differences in zenith distance. Hence it is little used in work other than geodetic work. It involves the measurement of the difference of the meridian zenith distances between two stars which cross the meridian at nearly equal altitudes with but a short interval of time between them, the one north and the other south of the zenith. The great advantage of the method is that, as small differences in zenith distances are used, and not zenith distances themselves, the effects of unknown and unpredictable errors of refraction are reduced to a minimum.

OBSERVATIONS FOR LONGITUDE

The determination of longitude involves astronomical observations for local time combined with observations for Greenwich time; for longitude, as we have seen, is no more than a difference between local and Greenwich time. Up to recent years the chief difficulty about a longitude observation has been the determination of Greenwich time, but the advent of the wireless time signals has done away with this difficulty to a very great extent.

For ordinary purposes, local time may be determined by any of the methods previously described, and Greenwich time by the reception of the ordinary wireless time signals as described on pages 335–336. In geodetic work, or when a high degree of accuracy is demanded, the rhythmic signals are used in conjunction with a chronometer beating half-seconds and a chronograph on which short periods of time are recorded graphically. The signals consist of a series of Morse dots extending over a period of five minutes, with 61 dots per 60 seconds of mean time, or 306 dots altogether in 300 seconds of mean time. By noting the coincidences of the beats of the wireless signals and the beats of the chronometer we get what is, in effect, a time vernier which enables time to be recorded to a hundredth of a second.

POSITION LINE METHODS

The position line method of determining position was originally devised for finding position at sea, but developments of it are now proving more and more popular among surveyors for finding position on land, so that this book would hardly be complete without some reference to it.

In fig. 18.6, LTMPS represents part of the celestial sphere, P being the celestial pole and LTM the celestial equator. Inside the sphere is a smaller concentric sphere representing the earth with pole p and

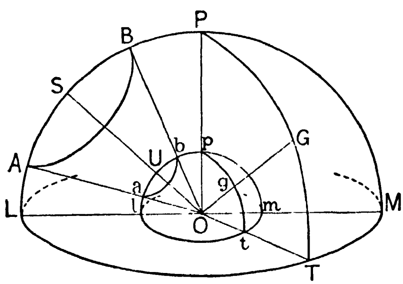


Fig. 18.6

terrestrial equator ltm. On the earth, pgt is the Greenwich meridian and on the celestial sphere this meridian is represented by PGT.

Let the altitude of a star be observed and the time of observation noted so that the Greenwich hour angle of the star can be computed. The declination of the star being known, its position, S, on the celestial sphere is known. Join OS and let OS cut the surface of the earth in

U. Then U is the position on the earth's surface at which the star would be vertically overhead at the time of observation, this point being called the *sub-stellar point*, or the star's *geographical position*. PSL is the celestial meridian through S, the meridian meeting the celestial equator in L.

Now the angle LOS, being the angular elevation of the star above the equator, is equal to the star's declination, and, if OL cuts the terrestial equator in l, the angle lOU = angle LOS = the terrestrial latitude of U. Thus, on the earth, the latitude of U is the star's declination. Also, the angle SPG is known from the Greenwich Hour Angle, and the angle Upg, which is the longitude of U with reference to Greenwich, is equal to it. Consequently, the latitude and longitude of U are known, and thus the position of U on the earth's surface is known.

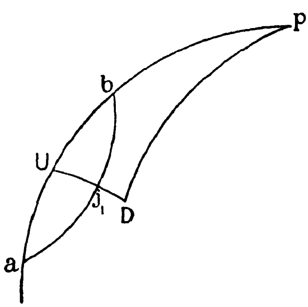


FIG. 18.7

This fixes the position of U, but what we want is the position of the observer. However, we note that all the places which would have z as the observed zenith distance of S must lie on a small circle ab on the earth's surface which has as its centre the point U and an angular radius Ua equal to z. This small circle is called the *position circle* for the star S. Similarly, if we observe the altitude and the time of observation of another star, we can get another position circle and the intersections of the two circles will give two points, one of which is the station of observation. It will always be easy to decide which of the two points must be taken since they will be widely separated, and the observer will have at least some rough idea of his position so that he will know without very much difficulty which point to accept.

Instead of observations to two stars, observations can be taken to the sun in two positions an hour or two apart, the point U at which the sun is overhead at the moment of observation being then called the *sub-solar point*.

In navigation the ship's position can be obtained very roughly by plotting the two position circles on a globe or a very small-scale chart, but this is not accurate enough for ordinary survey work. In this case, approximate values for the latitude and longitude of the point are assumed. Let ϕ_1 and L_1 be these assumed values and let D in fig. 18.7 be this point. Then in the spherical triangle UDp, Up $= 90° - \delta$, pD $= 90° - \phi_1$, and angle UpD $= (H - L_1)$. Hence, solving for UD,

$$\cos \text{UD} = \sin \delta \sin \phi_1 + \cos \delta \cos \phi_1 \cos (H - L_1). \qquad (40)$$

From this UD is found and Dj_1, the distance from D to the position circle, is UD $- z$, this distance being called *the intercept*. Again, the angle UDp, which is the azimuth of U from P, can be calculated from

$$\sin \text{UDp} = \frac{\cos \delta \sin (H - L_1)}{\sin \text{UD}}. \qquad (41)$$

The distance Dj_1 is normally very small and the problem can be completed graphically. Taking the point O as origin of rectangular co-ordinates (fig. 18.8) and giving it the assumed latitude ϕ_1 and longitude L_1, and letting distances scaled along OX and OY be represented by, say, 1 inch to 10 seconds of arc in latitude and 10 sec ϕ_1 seconds of arc in longitude,* lay off the intercept Oj_1 equal to UD $- z_1$ and making

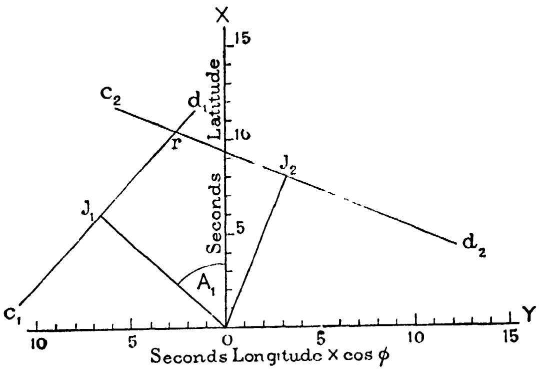


Fig. 18.8

angle A_1 = UDp with OX. Through j_1 draw the line $c_1j_1d_1$ perpendicular to Oj_1. Then $c_1j_1d_1$ is a tangent to the position curve of S_1 and is called the *position line* of S_1. For small distances on either side of j_1 the line $c_1j_1d_1$ will lie very close to the position curve, so that we can assume that the true position of the point of observation lies on this line. Now carry out the same procedure for another star S_2, or for another position of the sun, assuming the same value as before for the

* The reason for the factor sec ϕ_1 is that the radius of the small circle representing a parallel of latitude is proportional to the cosine of the latitude so that an angular displacement of ΔL corresponds to a linear displacement of $\Delta L \cos \phi_1$ on the parallel of latitude. Thus $\Delta L \cos \phi_1 = \Delta Y$ and $\Delta L = \Delta Y \sec \phi_1$.

point O, and so we get the second position line $c_2j_2d_2$ which intersects the first position line at r. Then r is the position of the real point of observation with reference to O, and the co-ordinates of r, as scaled from the diagram, will give the true position of the point of observation. Thus, in the diagram, if the assumed position of O is $\phi_1 = 51° 13' 28''$, $L_1 = 15° 16' 06''$ E., and the co-ordinates of r scaled from the diagram are $x = + 10{\cdot}4''$ and $y = - 2{\cdot}6'' \sec \phi_1 = - 4{\cdot}2''$, the true position of the point of observation is $\phi = 51° 13' 38{\cdot}4''$, $L = 15°16' 01{\cdot}8''$ E.

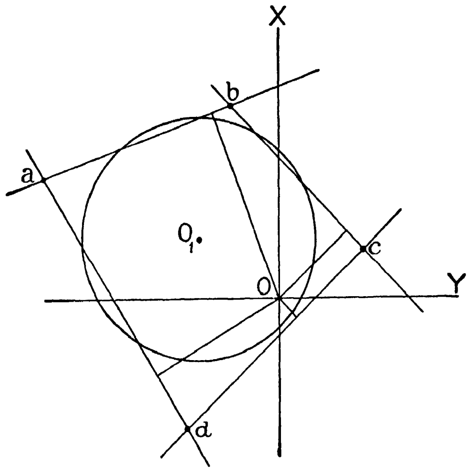


Fig. 18.9

If several stars, say 4, are observed, this will give several position lines such as ab, bc, cd, da in fig 18.9, enclosing a figure abcda, inside which can be drawn the circle with centre O_1 which most nearly touches all four lines. The rectangular co-ordinates of O_1 with reference to O yield the quantities which, added algebraically to the assumed co-ordinates of O, give the most likely fix for the point of observation.

EXAMPLES

(Note. In the following examples the astronomical data have been taken from *The Star Almanac for Land Surveyors* for 1959, which is referred to as the *S.A.*)

Example (i).—The star γ Pegasi was observed west of the meridian on 1959, October 5 and its altitude was found to be 42° 39′ 02″, atmospheric pressure and temperature being 29·9 in. and 69° F. respectively. The altitude bubble readings (page 324) were $O = 13{\cdot}7$ and $E = 10{\cdot}5$, and d, the angular value of one bubble division, was 10″. The latitude of the place of observation was 40° 07′ 15″ North and the longitude 0 h. 13 m. 41·2 s. West. Find the azimuth of the star, and, if the observed L M.T. of this observation was 2 h. 25 m. 09·1 s., what was the error of the clock?

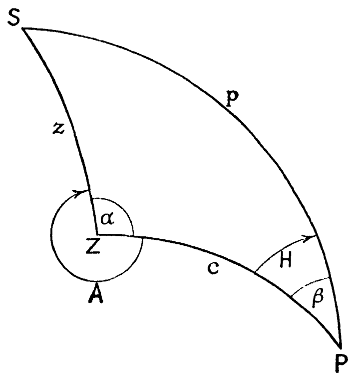


Fig. 18.10

From the *S.A.* pages 26 and 27, and the interpolation table on page 70, we see that the R.A. and declination of the star on 1959 October 5, were 0 h. 11 m. 11·1 s. and 14° 57′ 44″ N. respectively.

The data given are the three sides of the astronomical triangle, so we use formulæ (7) and (8) on page 304 to determine the angles α and β, viz.

$$\tan r = \sqrt{\left[\frac{\sin (s - p) \sin (s - z) \sin (s - c)}{\sin s}\right]},$$

$$\tan \tfrac{1}{2}\alpha = \frac{\tan r}{\sin (s - p)}; \quad \tan \tfrac{1}{2}\beta = \frac{\tan r}{\sin (s - z)}.$$

Then, from fig. 18.10 and the usual conventions regarding A and H,

$$A = 360° - \alpha, \quad H = \beta.$$

Thus,

From *S.A.*, p. 60, mean refraction for $h = 42° 39' = 63''$.
From p. 61, factor f for 29·9 in. and 69° F. = 0·96.

$$
\begin{array}{ll}
\text{Refraction correction} = -63 \times 0{\cdot}96 = -60{\cdot}5'' & = -00°\ 01'\ 00{\cdot}5'' \\
\text{Correction for dislevelment of vertical axis} & \\
\quad = (13{\cdot}7 - 10{\cdot}5)10/2 & = +00\ \ 00\ \ 16 \\
\text{Observed altitude} & = \quad 42\ \ 39\ \ 02 \\
\hline
\text{Corrected altitude} & = \quad 42\ \ 38\ \ 17
\end{array}
$$

$$
\begin{array}{rcl}
p = 90° - 14° 57' 44'' & = & 75° 02' 16'' \\
z = 90 - 42\ 38\ 17 & = & 47\ 21\ 43 \\
c = 90 - 40\ 07\ 15 & = & 49\ 52\ 45 \\
 & 2 | & 172\ 16\ 44 \\
s & = & 86\ 08\ 22 \\
(s - p) & = & 11\ 06\ 06 \qquad \log\sin = 9{\cdot}284\,544 \\
(s - z) & = & 38\ 46\ 39 \qquad \log\sin = 9{\cdot}796\,782 \\
(s - c) & = & 36\ 15\ 37 \qquad \log\sin = 9{\cdot}771\,921
\end{array}
$$

$$
\begin{array}{rl}
 & 8{\cdot}853\,247 \\
\log\sin s = & 9{\cdot}999\,013 \\
2 | & 8{\cdot}854\,234 \\
\log\tan r = & 9{\cdot}427\,117
\end{array}
$$

$$
\begin{array}{ll|ll}
\log\tan r & = 9{\cdot}427\,117 & \log\tan r & = 9{\cdot}427\,117 \\
\log\sin(s - p) & = 9{\cdot}284\,544 & \log\sin(s - z) & = 9{\cdot}796\,782 \\
\log\tan\tfrac{1}{2}\alpha & = 0{\cdot}142\,573 & \log\tan\tfrac{1}{2}\beta & = 9{\cdot}630\,335
\end{array}
$$

$$
\begin{array}{ll}
\tfrac{1}{2}\alpha = 54° 14' 24{\cdot}7''. & \tfrac{1}{2}\beta = 23° 07' 05{\cdot}0''. \\
\alpha = 108° 28' 49''. & \beta = 46° 14' 10''. \\
A = 360° - \alpha = 251° 31' 11''. & H = \beta = 3 \text{ h. } 04 \text{ m. } 56{\cdot}7 \text{ s.}
\end{array}
$$

From *S.A.*, p. 20, R on 1959, October 5 d. 0 h. = 0 h. 51 m. 41·5 s.
Hence, from equation,

$$\text{L.H.A. Star} = H = \text{L.M.T.} + R - \text{R.A. Star}$$

$$
\begin{array}{ll}
\text{Approx. L.M.T.} & = 3 \text{ h. } 04 \text{ m. } 56{\cdot}7 \text{ s.} - 0 \text{ h. } 51 \text{ m. } 41{\cdot}5 \text{ s.} \\
 & \qquad + 0 \text{ h. } 11 \text{ m. } 11{\cdot}1 \text{ s.} \\
 & = 2\quad 24\quad 26{\cdot}3 \\
\text{West longitude} & = 0\quad 13\quad 41{\cdot}2 \\
\hline
\text{Approx. U.T.} & = 2\quad 38\quad 07{\cdot}5
\end{array}
$$

Correction to R for 2 h. 38 m. 07·5 s. (p. 68 of *S.A.*) = + 26·0 s.*

* As the true value of U.T. was not available to begin with, the first value of R to be used was that for 1959 October 5 d. 0 h. The correction of +26·0 s. is the difference between this value and that for U.T. 2 h. 38 m. 07 s. Finally, there is a very small correction to R for this 26·0 s., so that the total final correction to R amounts to +25·9 s.

Corrected U.T. = 2 h. 38 m. 07·5 s. − 26·0 s.
= 2 37 41·5.
Corrected R for 2 h. 37 m. 41·5 s. = + 25·9 s.

Approximate L.M.T.	=	2 h. 24 m. 26·3 s.
Corrected L.M.T.	=	2 24 00·4
Time by clock	=	2 25 09·1
Hence Clock Fast	=	1 m. 08·7 s.

The above computation is by logarithms. For computation by machine we can use as our basic formulæ in spherical trigonometry:

$$\cos a = \cos b \cos c + \sin b \sin c \cos A.$$
$$\cos b = \cos a \cos c + \sin a \sin c \cos B.$$

From which, by the usual substitutions in the astronomical triangle, we get

$$\cos \alpha = \cos p \operatorname{cosec} z \operatorname{cosec} c - \cot z \cot c.$$
$$\cos \beta = \cos z \operatorname{cosec} p \operatorname{cosec} c - \cot p \cot c.$$

Arrange the work as follows:

$\cos p =$	0·258 182	$\cot z =$	0·920 773
$\operatorname{cosec} z =$	1·359 347	$\cot c =$	0·842 700
$\operatorname{cosec} c =$	1·307 724	$\cot z \cot c =$	0·775 935
$\cos p \operatorname{cosec} z \operatorname{cosec} c =$	0·458 957		
$-\cot z \cot c =$	−0·775 935		
$\cos \alpha =$	−0·316 978		

$$\alpha = 108° \, 28' \, 49''$$

$\cos z =$	0·677 365	$\cot p =$	0·267 243
$\operatorname{cosec} p =$	1·035 094	$\cot c =$	0·842 700
$\operatorname{cosec} c =$	1·307 724	$\cot p \cot c =$	0·225 206
$\cos z \operatorname{cosec} p \operatorname{cosec} c =$	0·916 893		
$-\cot p \cot c =$	−0·225 206		
$\cos \beta =$	0·691 687		

$$\beta = 46° \, 14' \, 10·3''$$

Example (ii).—On 1959 October 26, in latitude 36° 34·6′ N., longitude 3 h. 20 m. 50 s. E., it was only possible to observe the sun's lower and eastern limbs before cloud prevented further observations. The observed altitude of the lower limb at approximate L.M.T. 9 h. 20 m. 14 s., as registered by the surveyor's clock, was 30° 16·2′, and the horizontal angle between the eastern limb and the referring object, or R.O., measured clockwise from sun to R.O., was 48° 15·2′. The barometer read 30·7 in. and the thermometer 62° F. What was the azimuth of the R.O. and the approximate error of the clock?

In fig. 18.11, C is the centre of the sun, L the point where the vertical circle through Z touches the eastern limb. The angle CZL = $\Delta\alpha$ is thus the difference between the azimuths of the sun's centre and the sun's eastern limb at the time of observation.

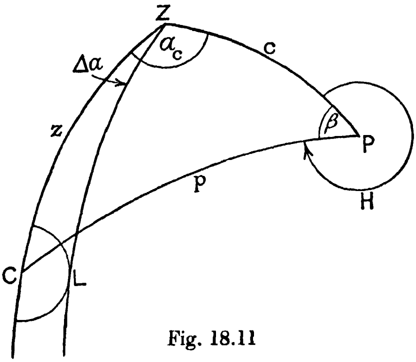


Fig. 18.11

Observed altitude of lower limb	= 30° 16·2′
From the *S.A.*, p. 60, mean refraction for 30° 16·2′ = 99″.	
Correction factor for 30·7 in. and 62° F. = 1·00.	
∴ Refraction = − 99″ × 1·00 = −99″ = −1·65′	= − 1·65
Correction for parallax = +8·8″ cos 30° 16′ = +7·6″	= + 0·13
From *S.A.*, p. 21, sun's S.D.	= +16·1
∴ Sum = corrected altitude of sun's centre	30° 30·8′
Clock L.M.T. of observation	= 9 h. 20 m. 14 s.
Longitude East	= 3 20 50
∴ Approx. U.T. of observation	= 5 59 24
From *S.A.*, p. 21, declination of sun at U.T. 1959 October 26 d. 5 h. 59 m. 24 s.	= −12° 11·6′

Then, using the tangent formulæ to determine α_c and β,

$p = 90° - (-12° 11{\cdot}6') = 102° 11{\cdot}6'$
$z = 90 - 30\ 30{\cdot}8 = 59\ 29{\cdot}2$
$c = 90 - 36\ 34{\cdot}6 = 53\ 25{\cdot}4$

2 | 215 06·2

s = 107 33·1

$(s - p)$ = 5 21·5	log sin =	8·97 027
$(s - z)$ = 48 03·9	log sin =	9·87 152
$(s - c)$ = 54 07·7	log sin =	9·90 866
		8·75 045
	log sin s =	9·97 930
	2	8·77 115
	log tan r =	9·38 558

log tan r	$= 9{\cdot}38\ 558$	log tan r	$= 9{\cdot}38\ 558$
log sin $(s - p)$	$= 8{\cdot}97\ 027$	log sin $(s - z)$	$= 9{\cdot}87\ 152$
log tan $\frac{1}{2}\alpha_c$	$= 0{\cdot}41\ 531$	log tan $\frac{1}{2}\beta$	$= 9{\cdot}51\ 406$

$$\tfrac{1}{2}\alpha_c = 68^\circ\ 58{\cdot}63' \qquad \tfrac{1}{2}\beta = 18^\circ\ 05{\cdot}33'$$
$$\alpha_c = 137^\circ\ 57{\cdot}3' \qquad \beta = 36^\circ\ 10{\cdot}7'$$
$$= 2\text{ h. }24\text{ m. }43\text{ s.}$$

$$\Delta\alpha = \text{sun's S.D.} \times \text{cosec } z \text{ (page 322)} = 16{\cdot}1' \times 1{\cdot}360 = 18{\cdot}7'$$

Azimuth of sun's eastern limb	= 137° 57·3′ − 18·7′	
	= 137	38·6
Angle eastern limb to R.O.	= 48	15·2
Azimuth of R.O.	= 185	53·8

L.H.A. Sun = 24 h. − β	= 21 h.	35 m.	17 s.
E for 1959 October 26 d. 5 h. 59 m. 24 s.	= 12	15	54
∴ L.M.T.	= 9	19	23
Observed time	= 9	20	14
∴ Clock fast	= 0	00	51

Example (iii).—At a station in latitude 30° S., longitude 18° E., on 1959 July 4, it is desired to observe a star whose R.A. is 18 h. 28 m. and declination 20° S. at about 9 p.m. L.M.T. Work out an approximate altitude and azimuth for setting the instrument for this time.

L.M.T.	= 21 h.	00 m.
Longitude = 18° E.	= − 1	12
Corresponding U.T.	= 19	48
R for 1959 July 4 d. 19 h. 48 m. (*S.A.*, p. 14)	= 18	48

L.H.A. Star = H = 21 h. 00 m. + 18 h. 48 m. − 18 h. 28 m.
= 21 h. 20 m.
= 320°.

Hence, the star lies east of the meridian and the angle β in the diagram is

$$\beta = 360^\circ - 320^\circ = 40^\circ.$$

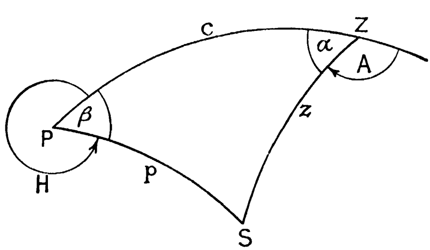


Fig. 18.12

We are therefore given

$$PZ = c = 90° - 30° = 60°, \quad PS = p = 90° - 20° = 70°$$

and $\beta = 40°$ and want to find the side $ZS = z$ and the angle α.

We have

$$\cos z = \cos p \cos c + \sin p \sin c \cos \beta.$$

This formula as it stands is all right for machine computation, but, for computation by logarithms, it is more convenient to get the right-hand side into a form involving the cosine of the difference of two angles by introducing the auxiliary angle θ such that

$$\cos p \;\; = k \cos \theta,$$

$$\sin p \cos \beta = k \sin \theta.$$

Hence, by division,

$$\tan \theta = \tan p \cos \beta. \quad . \; . \; . \; . \; . \; . \; . \quad (42)$$

Then, having found θ from (42),

$$\cos z = k\{\cos \theta \cos c + \sin \theta \sin c\}$$

$$= \frac{\cos p \cos (\theta - c)}{\cos \theta}. \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (43)$$

Hence,

$$\begin{aligned} \log \tan p &= 10{\cdot}43\,893 \\ \log \cos \beta &= \underline{\;9{\cdot}88\,425} \\ \log \tan \theta &= 10{\cdot}32\,318 \\ \theta &= 64°\ 35{\cdot}1', \\ (\theta - c) &= 4°\ 35{\cdot}1' \end{aligned}$$

$$\begin{aligned} \log \cos p &= 9{\cdot}53\,405 \\ \log \cos (\theta - c) &= \underline{9{\cdot}99\,861} \\ & \quad 9{\cdot}53\,266 \\ \log \cos \theta &= \underline{9{\cdot}63\,263} \\ \log \cos z &= 9{\cdot}90\,003 \\ z &= 37°\ 24{\cdot}2', \\ h &= 52°\ 35{\cdot}8'. \end{aligned}$$

To find α, use the tangent formula (6) on page 304 in order to avoid the ambiguity which arises when the simpler sine formula is used.

$$\tan \tfrac{1}{2}\alpha = \sqrt{\left[\frac{\sin (s - c) \sin (s - z)}{\sin s \sin (s - p)}\right]}.$$

$$\begin{aligned} z &= 37°\ 24{\cdot}2' \\ c &= 60\ \ 00 \\ p &= \underline{70\ \ 00} \\ 2 &| \underline{167\ \ 24{\cdot}2} \\ s &= 83\ \ 42{\cdot}1 \end{aligned}$$

$s - c = 23° \ 42{\cdot}1'$		$\log \sin =$	$9{\cdot}60\,420$
$s - z = 46 \quad 17{\cdot}9$		$\log \sin =$	$9{\cdot}85\,911$
			$9{\cdot}46\,331$
$s - p = 13 \quad 42{\cdot}1$		$\log \sin =$	$9{\cdot}37\,450$
			$0{\cdot}08\,881$
		$\log \sin s =$	$9{\cdot}99\,737$
		2	$0{\cdot}09\,144$
		$\log \tan \frac{1}{2}\alpha =$	$0{\cdot}04\,572$
		$\frac{1}{2}\alpha =$	$48° \ 00{\cdot}6'$
		$\alpha =$	$96° \ 01{\cdot}2'$

and $A = 180° - \alpha = 83° \ 58{\cdot}8'$.

Example (iv).—On 1959 June 6, the mean of six observed altitudes of *Polaris* corresponding to a mean L.M.T. of 20 h. 40 m. 20 s. was 38° 19·8′, the barometer reading 30·2 in. and the thermometer 58° F The mean horizontal angle between the star and the R.O. was 292° 17·6′ measured clockwise from the star, and the means of the bubble readings were $O = 11{\cdot}8$ and $E = 13{\cdot}6$, with $d = 9{\cdot}5''$. What was the latitude of the place of observation and what was the azimuth of the R.O.? The longitude of the point of observation was approximately 1 h. 30 m. 14 s. East.

		h.	m.	s.
L.M.T.	=	20 h.	40 m.	20 s.
Longitude East	=	1	30	14
U.T.	=	19	10	06
R at 1959 June 6 d. 18 h.	=	16	57	35·8
Increment for 1 h. 10 m. 06 s.	= +			11·5
G.S.T.	=	12	07	53.3
Longitude East	= +	1	30	14
L.S.T.	=	13	38	07·3

Mean refraction for altitude 38° 19·8′ = 73″.
Correction factor f for 30·2 in. and 58° F. = 0·99.
Refraction correction = − 73″ × 0·99 = −72″ = − 0° 01·2′.
Bubble correction (11·8 − 13·6) × 9·5/2 = −8·6″ = −0·1.

From *S.A.* Pole Star Table, p. 58,

For L.S.T. 13 h. 38 m. 07 s.	$a_0 = +55{\cdot}3$	$b_0 = -4{\cdot}6$
For latitude 38°	$a_1 = 0{\cdot}0$	$b_1 = 0{\cdot}0$
For June	$a_2 = +\ 0{\cdot}4$	$b_2 = +0{\cdot}4$
Sum	$+\ 55{\cdot}7$	$-4{\cdot}2$
Observed altitude	$= 38\ 19{\cdot}8$	
Refraction	$= -\ 1{\cdot}2$	
Bubble Correction	$= -\ 0{\cdot}1$	sec ϕ (*S.A.*, p. 63)
∴ Latitude	$= 39° \ 14{\cdot}2'$	$= 1{\cdot}291$

Azimuth of *Polaris* $= (b_0 + b_1 + b_2) \sec \phi = -\ \ 0° 05{\cdot}4'$
Angle star to R.O. $= 292\ \ 17{\cdot}6$

∴ Azimuth of R.O. $= 292° 12{\cdot}2'$

Example (v).—What will be the altitude, azimuth, and L.M.T. of eastern and western elongation of the circum-polar star ζ Octantis on 1959 July 14, at a place in latitude 28° 14′ 13″ S. and longitude 60° 30′ 15″ W.

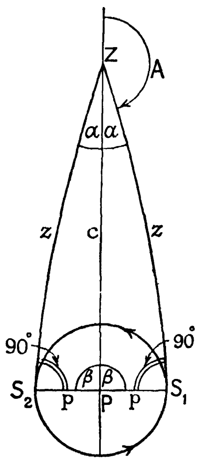


FIG. 18.13

From the *S.A.*, p. 52, the R.A. and declination of ζ Octantis on 1959 July 14, are:

R.A. = 9 h. 02 m. 47 s.; Declination = 85° 30′ 31″ S.

Since we know from the figure that α must be less than 90° in either position of the star we can use a sine formula to determine it, but we shall use a cosine or tangent formula to determine β, though it is pretty obvious from the figure that in both positions it, too, must be less than 90°. Then, by applying the rules 12, 16 and 15 on page 305 for the solution of the right-angled astronomical triangle:

$$\sin \alpha = \sin p \operatorname{cosec} c,$$
$$\cos z = \cos c \sec p,$$
$$\cos \beta = \cot c \tan p,$$

formulæ which, on their right-hand sides, contain only p and c,

$$p = 90° - 85° 30' 31'' = 4° 29' 29''; \quad c = 90° - 28° 14' 13'' = 61° 45' 47''.$$

$$\log \sin p = 8{\cdot}893\,812 \qquad \log \cos c = 9{\cdot}674\,970$$
$$\log \sin c = 9{\cdot}944\,975 \qquad \log \cos p = \bar{9}{\cdot}998\,664$$
$$\log \sin \alpha = 8{\cdot}948\,837 \qquad \log \cos z = 9{\cdot}676\,306$$

$$\alpha = 5° 05' 58'' \qquad z = 61° 40' 06''$$

$$\log \cot c = 9{\cdot}729\,995$$
$$\log \tan p = 8{\cdot}895\,148$$
$$\log \cos \beta = 8{\cdot}625\,143$$

$$\beta = 87° 34' 56''$$
$$= 5 \text{ h. } 50 \text{ m. } 20 \text{ s.}$$

At eastern elongation

$$A = 180° - \alpha = 174° 54' 02''; \quad h = 90° - z = 28° 19' 54'';$$
$$H = 24 \text{ h.} - \beta = 18 \text{ h. } 09 \text{ m. } 40 \text{ s.}$$

From formula

$$\text{L.H.A. Star} = \text{L.M.T.} + R - \text{R.A. Star.}$$

L.M.T. + R = 18 h. 09 m. 40 s. + 9 h. 02 m. 47 s.	= 27 h.	12 m.	27 s.
R for G.M.T. 1959 July 14 d. 00 h.	= 19	24	27·5
∴ Approx. L.M.T.	= 7	48	00
Longitude West	= 4	02	01
∴ Approx. U.T.	= 11	50	01

Correction to R for 11 h. 50 m. 01 s. = 59·1 + 57·5 = 0 h. 01 m. 56·6 s.

Approx. L.M.T.	= 7 h.	48 m.	00 s.
Correction for correction to R	= −	1	57
∴ L.M.T. of Eastern elongation	= 7	46	03

For western elongation:

$$A = 180° + \alpha = 185° \; 05' \; 58''; \quad H = \beta = 5 \text{ h. } 50 \text{ m. } 20 \text{ s.}$$

Difference in hour angle from eastern elongation to western elongation = 5 h. 50 m. 20 s. − 18 h. 09 m. 40 s. + 24 h. = 11 h. 40 m. 40 s. (or 2β).

Correction for mean time interval to 11 h. S.T.	= −		1 m.	48·13 s.
40 m. S.T.	= −			6·55
40 s. S.T.	= −			0·11
Sum = Difference in mean time intervals between Eastern and Western elongations	= 11	38		45
L.M.T. of Eastern elongation	= 7	46		03
∴ L.M.T. of Western elongation	= 19	24		48

Alternatively (as for Eastern elongation)

L.M.T. + R = 5 h. 50 m. 20 s. + 9 h. 02 m. 47 s.	= 14 h.	53 m.	07 s.
R for G.M.T. 1959 July 14 d. 18 h.	= 19	27	25
Approx. L.M.T.	= 19	25	42
Longitude West	= 4	02	01
Approx. G.M.T. of observation	= 23	27	43
Correction to R for 5 h. 27 m. 43 s.	=	—	54
Approx. L.M.T.	= 19	25	42
∴ Corrected L.M.T. of Western elongation	= 19	24	48

QUESTIONS ON CHAPTERS XIII–XVIII

1. What is the altitude of the sun's centre at upper transit in latitude 56° N. if the declination at the instant is 13° N.?

   At the same instant, in what latitude in the southern hemisphere would the sun have the same meridian altitude?

   What are the various corrections that have to be applied to the observed altitude of the sun's lower limb in order to get the corrected altitude of the sun's centre? (Inst. C.E., April, 1953.)

2. Describe briefly how you would determine the azimuth of a long survey line, using (*a*) an approximate method, and (*b*) a more accurate method. A theodolite reading to 10 sec. is available.

   Explain what data you would need and how you would obtain them; and state the degree of accuracy you would expect to obtain by each method. (Inst. C.E., October, 1953.)

3. In taking a round of theodolite angles from station A, an altitude of the sun's centre was recorded and the approximate time noted at the instant. The horizontal reading when pointing on station B was 220° 30′ 00″ and that on the sun's centre was 265° 00′ 00″. From the data given below compute the azimuth from N. of the line AB.

   What degree of accuracy would you expect to achieve with such an observation and how could the value of the result be increased?

   Corrected mean altitude of sun's centre (East sun) 36° 30′ 00″.

   Latitude of Station A 51° 30′ 00′ N.

   Declination of sun at time of observation 5° 00′ 00″ N.

   Aide Memoire $\sin \frac{A}{2} = \sqrt{\left[\frac{\sin (s-b) \sin (s-c)}{\sin b \sin c}\right]}$,

   where $2s = a + b + c$. (Inst. C.E., April, 1954.)

4. Draw a diagram showing the visible portion of the celestial sphere for an observer in latitude 50° N. Indicate in the diagram the celestial triangle PZS with a star " S " of declination 10° N. and an hour angle of 45°. What is the azimuth angle in the celestial triangle and how would you determine its value? (Inst C.E., October, 1954.)

5. Two methods of determining azimuth are as follows: (i) Knowing the latitude of the station, to observe the altitude of the sun (in the E. or W. sky), noting the approximate time of intersection, and to read the horizontal angle between the sun and a reference object. (ii) Knowing the latitude and longitude to record only the horizontal angle between the sun and reference object, and to note the exact time when the sun was intersected. With the aid of diagrams, show how with the altitude

in one case, and the time of intersection in the other, the azimuth of the reference object can be obtained. Formulæ for the solution of the triangles need not be given. (Inst. C.E., April, 1955.)

6. What are the particular merits of determining azimuth by observations on a circum-polar star at elongation?

It is given that at the instant of elongation

$$\cos P = \cot \delta \tan \phi$$

where P is the time angle ZPS in the celestial triangle; δ the declination of the star at elongation; ϕ the observer's latitude.

Compute the G.M.T. when Polaris will be at elongation in the following conditions:

Latitude 51° 29′ 57″ N.: Longitude 41·8 sec. of time West.
Declination of Polaris 89° 03′ 38″ N.
Right ascension of Polaris 01 h. 51 m. 48 s.
R at the instant of observation 10 h. 08 m. 04 s.
(G.H.A. Aries = U.T. + R). (Inst. C.E., October, 1955.)

7. A common method of determining azimuth is to note the time of observation and make use of the formula

$$\cot Z = \frac{\sin (c - x) \cot P}{\sin x} \text{ where } \tan x = \tan p \cos P.$$

This entails the preliminary computation of the angle P which is the hour angle ZPS of the celestial triangle.

Compute the values for the angle P (in time) in the following cases making use of the expressions: Greenwich Hour Angle Aries = U.T. + R and Greenwich Hour Angle Sun = U.T. + E, where U.T. is the same as G.M.T.

	East Star			West Star		
	h	*m*	*s*	*h*	*m*	*s*
Right ascension	10	04	50·7	00	09	46·5
G.M.T. of observation	19	05	18·0	19	15	39·1
R at instant of observation	10	40	27·8	10	40	29·5
Longitude of station		21	22·7 W.		21	22·7 W.

	East Sun			West Sun		
	h	*m*	*s*	*h*	*m*	*s*
G.M.T. of observation	10	24	40·2	14	38	12·8
E at instant of observation	12	14	16·4	12	14	17·2
Longitude of station		21	22·7 W.		21	22·7 W.

Explain briefly the derivation of the terms R and E. (Inst. C.E., April, 1956.)

8. Draw a diagram of the celestial sphere indicating clearly the celestial triangle PZS. State briefly any three methods you know by which the azimuth angle PZS can be determined and give the essential differences in the field procedure between the different methods. (Inst. C.E., October, 1956.)

9. Using the formula (or any other you may know) and information given below, compute the azimuth of the reference object from the observer's position.

   Mean observed altitude (h) of the sun's lower limb in the East sky $= 36° 51' 43''$.
   Latitude (ϕ) $= 51° 24' 00''$ N.
   Declination of sun (δ) $= 5° 30' 00''$ N. and $p = 90° - \delta$.
   Correction for refraction $= 1' 14''$.
   Correction for semi-diameter $= 15' 24''$.
   Correction for parallax $= 07''$.
   Horizontal angle measured clockwise from reference object to sun $= 86° 40' 30''$.

   Formula:

$$\tan \frac{A}{2} = \sqrt{[\sec s \sin (s - h) \sin (s - \phi) \sec (s - p)]},$$

where

$$s = \frac{h + \phi + p}{2}.$$

(Inst. C.E., April, 1957.)

10. Show that the formula for $\tan \frac{1}{2}A$ in the last example is another form of formula (6), Chapter XVI, page 304, viz.

$$\tan \tfrac{1}{2}\alpha = \sqrt{\left[\frac{\sin (s - z) \sin (s - c)}{\sin s \sin (s - p)}\right]},$$

where

$$s = \tfrac{1}{2}(p + z + c).$$

11. On 1959, July 9, at a place in latitude 15° 30′ S. and longitude 62° 10′ W., it is desired to observe a star, τ Virginis, whose R.A. is 13 h. 59 m. 36 s. and declination 1° 44′ 28″ N., at (approximately) 21 h. L.M.T. Work out the approximate azimuth and altitude to which the theodolite should be set in order to " pick up " the star. Allow 50″ for refraction.

12. At a place in the Southern hemisphere the circum-polar star, ζ Octantis, was observed at upper and lower transits and the altitudes found to be 49° 46′ 39″ and 40° 47′ 57″ respectively. Assuming refraction to be 49″ at upper transit and 67″ at lower transit, what were the latitude of the place and the declination of the star?

13. Assuming that, when the sun first appears to be on the horizon, it is actually, because of refraction, some 33′ below it, what was the approximate L.M.T. of sunrise at a place in latitude 42° 15′ 30″ N. on 1959, May 24? Take sun's declination at time of sunrise = 20° 58′ 40″ N. and E = 12 h. 03 m. 10 s.

14. Discuss the suitability of astronomically determined latitudes and longitudes as control points for mapping on large and small scales respectively.

ANSWERS TO QUESTIONS

CHAPTERS II AND III (p. 30)

2. 40 ft.

6. Lay out lengths of 100 ft. from given points and offsets at end = 100 tan α = (1) 100 ft.; (2) 173·21 ft.; 228·53 ft.

CHAPTER IV (p. 44)

1. 36° 12′ R; 76° 14′ R; 122° 50′ L; 63° 45′ 09″ L; 136° 12′ 18″ R; 99° 11′ 41″ L.

2. Station B: 63° 53′; 115° 59′ 26″; 73° 22′ 06″; 26° 13′ 12″.
   „ C: 167° 54′; 241° 28′ 50″; 97° 06′ 55″; 107° 04′ 06″.
   „ D: 234° 47′; 282° 18′ 12″; 112° 53′ 36″; 186° 05′ 26″.
   „ E: 331° 40′; 342° 23′ 13″; 210° 57′ 07″; 266° 00′ 37″.

3. (1) 274° 54′; 111° 22′; 227° 35′; 175° 38′.
   (2) 85° 06′; 248° 38′; 132° 25′; 184° 22′.

4. 196° 14′ 42″; 54° 17′ 28″; 16° 36′ 13″; 148° 16′ 54″; 273° 56′ 27″; 111° 06′ 21″; 178° 12′ 16″.

5. N 16° 14′ 42″ E; S 54° 17′ 28″ W; S 16° 36′ 13″ W; N 31° 43′ 06″ W; S 86° 03′ 33″ E; N 68° 53′ 39″ W; N 1° 47′ 44″ W.

6. 289° 30′; 212° 49′; 20° 25′; 355° 06′.

7. 18° 31′; 323° 46′; 237° 48′; 131° 13′; 249° 48′; 164° 48′.

8. $214\frac{1}{2}$°; $27\frac{1}{4}$°; $287\frac{3}{4}$°; $93\frac{1}{2}$°; $157\frac{1}{2}$°; $311\frac{3}{4}$°; $218\frac{1}{2}$°; $78\frac{3}{4}$°; 244° 14′; 278° 04′.

9. 97° 46′ 25″.

10. Sum of angles = 719° 58′ 30″, so they fail to close by 1′ 30″.

CHAPTER V (p. 61)

1. $x_B = +11{,}321{\cdot}5$; $y_B = -4{,}883{\cdot}5$.

2. $x_B = +7{,}620{\cdot}5$; $y_B = -10{,}758{\cdot}2$.

3. $\alpha = 275° \; 03' \; 18''$; $l = 6624{\cdot}6$.

4. Bearing AB = 155° 18′ 15″; log AB = 3·875 046.
   „ AC = 164° 44′ 35″; log AC = 3·658 838.
   „ BC = 321° 19′ 24″; log BC = 3·490 586.

5. Bearing BC = 32° 10′ 01″; log BC = 3·644 732; BC = 4413·0.

6. 47·17 acres.

CHAPTER VI (p. 108)

1. 1962·20 ft.
2. 1026·89 ft.
3. 3609·329 ft.
5. 0·0104 ft.
6. True length = 500·9821 ft.
7. 99·9803 ft.
8. log AC = 3·791 911; log BC = 3·711 964.
9. log AD = 3·701 134; angle BAD = 15° 57′ 59″.
10. TA = N 10° 05′ 17″ E; TB = N 82° 45′ 11″ E.
11. Angle ABC + angle AXC = 180°. Maximum distance of BX = 1774·5 ft.
12. x = 36,864·8; y = 21,571·6. Bearing XA = 271° 28′ 25″.

CHAPTER VII (p. 142)

1. Add 56″ to each angle.
2. Local attraction at C and E.

Corrected bearings:

AB	S 35° 31′ E	DE	N 40° 10′ W
BA	N 35° 31′ W	EO	S 40° 10′ E
BC	N 70° 15′ E	EF	S 50° 11·5′ W
CB	S 70° 15′ W	FE	N 50° 11·5′ E
CD	N 13° 57·5′ E	FA	N 81° 14′ W
DC	S 13° 57·5′ W	AF	S 81° 14′ E

3. BC = 472 ft.; CD = 948 ft.
4. 151,550 sq. ft.
5. 710·1 ft.; N 68° 12′ E.
6.

Station	x	y
B	28,567·9	96,872·5
C	31,347·6	98,776·9
D	32,246·7	99,157·9
E	33,164·8	99,952·6

CHAPTER VIII (p. 169)

1. 949·93.
4. Height of Instrument Method: For checking at bottom of first page, reading 10·98 must be considered as F.S. At top of new page 10·98 is booked as a B.S. to facilitate final checking by difference of sums of F.S. and B.S.

   Rise and Fall Method: Check in usual way at bottom of each page.
6. 244·3.
7. k_1 = 99·9: k_2 = 0·8.
8. 1 in 23·9.
9. Horizontal distance = 770·1 ft.; height of A above axis level = 23·89 ft.

CHAPTER IX (p. 206)

1. 3820; 2865; 1910; 1432·5; 1146. 1910·1; 1432·7; 1146·3.
2. $2\frac{1}{2}°$; $3\frac{1}{2}°$; $4\frac{1}{2}°$.
3. Chainage of beginning of curve = 253 + 55·9; chainage of end of curve = 273 + 71·6.

Deflection angles:

Chainage	Angle	Chainage	Angle
254	0° 39·7′	. . . . .	. . . . .
255	2° 09·7′	270	24° 39·7′
256	3° 39·7′	271	26° 09·7′
257	5° 09·7′	272	27° 39·7′
258	6° 39·7′	273	29° 09·7′
. . . . .	. . . . .	273 + 71·6	30° 14·1′

4. 3·3 in.
5. 80 + 40; 88 + 25; 1° 54′ 37″.
6. 1633 ft.; 276 + 73; 284 + 73; 3° 30′.
7. 69 + 20; 71 + 00; 86 + 87; 88 + 67; 1° 16′ 24″.
8. Length = 356·8 ft.

Distance	δ	Distance	δ
50	00° 02·1′	250	00° 52·6′
100	00° 08·4′	300	01° 15·7′
150	00° 18·9′	350	01° 43·0′
200	00° 36·6′	356·8	01° 47·0′

9. Chainage of beginning of curve = 367 + 00; of end 382 + 00.

Reduced levels:

Chainage	R L.	Chainage	R L.
367	460·76	375	465·55
368	461·71	376	465·70
369	462·56	377	465·76
370	463·31	378	465·71
371	463·96	379	465·56
372	464·51	380	465·31
373	464·95	381	464·96
374	465·30	382	464·51
374 + 50	465·44		

10. Tangent distance = 8·268 chains; radius = 6·037 chains.
11.

Chainage	Ordinate	Gradient level	Curve level
57	0·0′	122′	122′
58	0·09	123	122·91
59	0·36	124	123·64
60	0·82	125	124·18
61	1·45	126	124·55
62	0·82	125·55	124·73
63	0·36	125·1	124·74
64	0·09	124·65	124·56
65	0·0	124·20	124·20

12. EX = 3·022 chains: FY = 2·604 chains.

CHAPTER X (p. 229)

1. 6 in.
3. 538·2 ft.; 9′ E of N.
4. Scale = 1/13,714; length of air base = 6857 ft.
5. (a) $fB/(H - h)$; (b) height of pylon = 542 ft.
6. 875 ft.
9. P for top = 2·08 in.
   P for bottom = 1·95 in.
   dP = 0·13 in.

CHAPTER XI (p. 242)

4. 31·35 cub. ft. per sec.
5. 52·74 cub. ft. per sec.; 53·39 cub. ft. per sec.
6. 31·3 cusecs, using constant = 0·6.

CHAPTER XII (p. 258)

1. 204 sq. ft.
2. 268·3 sq. ft.
3. 10,383 sq. ft.
4. By Simpson's rule A = 91,800 sq. ft. By trapezoidal rule A = 90,100 sq. ft.
5. 11,687 cub. yd.; 12,085 cub. yd.
6. 11,620 cub. yd.; 11,983 cub. yd.
7. 22·77 acres.
8. Area $= \frac{1}{2}\left\{(s_1 + s_2)\left(h + \frac{b}{2n}\right) - \frac{b^2}{2n}\right\}$.

$$s_1 = \frac{b}{2} + \left(h + \frac{b}{2m}\right)\frac{mn}{m - n}.$$

$$s_2 = \frac{b}{2} + \left(h - \frac{b}{2m}\right)\frac{mn}{m + n}.$$

9. 53·26 sq. ft. (side slopes $1\frac{1}{2}$ horizontal to 1 vertical).

CHAPTERS XIII–XVIII (p. 356)

1. 47°; 30° S.
3. Azimuth of the line AB: N 94° 37′ 12″ E.
6. 21 h. 39 m. 42 s.
7. 4 h. 40 m. 27·6 s; 5 h. 24 m. 59·4 s.; 01 h. 42. m. 26·1 s.; 02 h. 31 m. 07·3 s.
9. N 52° 19′ 48″ E.
11. $h = 53° 42·3′$; $A = 244° 29·3′$ reckoned clockwise from North.
12. $\phi = 45° 16′ 20″$ S; $\delta = 85° 30′ 30″$ S.
13. L.M.T. Sunrise = 4 h. 32 m.; azimuth 60° 30·2′ reckoned clockwise from North.

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