ECLECTIC EDUCATIONAL SERIES.

NEW ELEMENTARY ALGEBRA.

PRIMARY ELEMENTS

OF

ALGEBRA,

FOR

COMMON SCHOOLS AND ACADEMIES.

By JOSEPH RAY, M. D.,
LATE PROFESSOR OF MATHEMATICS IN WOODWARD COLLEGE.

REVISED ELECTROTYPE EDITION.

VAN ANTWERP, BRAGG & CO.,
CINCINNATI: 137 WALNUT ST.
NEW YORK: 28 BOND ST.

Ray's Mathematical Series.

Embracing a Thorough, Progressive, and Complete Course

in Arithmetic, Algebra, and the Higher Mathematics.

PREFACE.

The object of the study of Mathematics is two fold—the acquisition of useful knowledge, and the cultivation and discipline of the mental powers. A parent often inquires, "Why should my son study Mathematics? I do not expect him to be a surveyor, an engineer, or an astronomer." Yet, the parent is very desirous that his son should be able to reason correctly, and to exercise, in all his relations in life, the energies of a cultivated and disciplined mind. This is, indeed, of more value than the mere attainment of any branch of knowledge.

The science of Algebra, properly taught, stands among the first of those studies essential to both the great objects of education. In a course of instruction properly arranged, it naturally follows Arithmetic, and should be taught immediately after it.

In the following work, the object has been to furnish an elementary treatise, commencing with the first principles, and leading the pupil, by gradual and easy steps, to a knowledge of the elements of the science. The design has been, to present these in a brief, clear, and scientific manner, so that the pupil should not be taught merely to perform a certain routine of exercises mechanically, but to understand the *why* and the *wherefore* of every step. For this purpose, every rule is demonstrated, and every principle analyzed, in order that the mind of the pupil may be disciplined and strengthened so as to prepare him, either for pursuing the study of Mathematics intelligently, or more successfully attending to any pursuit in life.

Some teachers may object, that this work is too simple, and too easily understood. A leading object has been, to make the pupil feel, that he is not operating on unmeaning symbols, by means of arbitrary rules; that Algebra is both a rational and a practical subject, and that he can rely upon his reasoning, and the results of his operations, with the same confidence as in arithmetic. For this purpose, he is furnished, at almost every step, with the means of testing the accuracy of the principles on which the rules are founded, and of the results which they produce.

Throughout the work, the aim has been to combine the clear explanatory methods of the French mathematicians with the practical exercises of the English and German, so that the pupil should acquire both a practical and theoretical knowledge of the subject.

While every page is the result of the author's own reflection, and the experience of many years in the school-room, it is also proper to state, that a large number of the best treatises on the same subject, both English and French, have been carefully consulted, so that the present work might embrace the modern and most approved methods of treating the various subjects presented.

With these remarks, the work is submitted to the judgment of fellow laborers in the field of education.

WOODWARD COLLEGE, *August*, 1848.

In this NEW ELECTROTYPE EDITION, the whole volume has been subjected to a careful and thorough revision. The oral problems, at the beginning, have been omitted; the number of examples reduced, where they were thought to be needlessly multiplied; the rules and demonstrations abridged; other methods of proof, in a few instances, substituted; and questions for GENERAL REVIEW introduced at intervals, and at the conclusion. It is confidently believed that these modifications, while they do not impair the integrity or change the essential features of the book, will materially enhance its value, and secure the approbation of all intelligent teachers.

March, 1866.

TO TEACHERS.—The following subjects may be omitted by the younger pupils, and passed over by those more advanced, until the book is reviewed: Observations on Addition and Subtraction, Articles 60—64; the greater part of Chapter II.; supplement to Simple Equations, Articles 164—177; properties of the Roots of an Equation of the Second Degree, Articles 215—217.

The pupil should be exercised in the solution of examples, until the principles are thoroughly understood; and, in the review, he should be required to demonstrate the rules on the blackboard.

CONTENTS.

I.—FUNDAMENTAL RULES.

II.—THEOREMS, FACTORING, ETC.

III.—ALGEBRAIC FRACTIONS.

IV.—SIMPLE EQUATIONS.

ELEMENTS OF ALGEBRA.

I. DEFINITIONS.

NOTE TO TEACHERS.—Articles 1 to 15 may be omitted until the pupil reviews the book.

Article 1. In **Algebra,** quantities are represented by letters of the alphabet.

2. **Quantity** is any thing that is capable of increase or decrease; as, numbers, lines, space, time, etc.

3. Quantity is called *magnitude*, when considered in an undivided form; as, a quantity of water.

4. Quantity is called *multitude*, when made up of individual and distinct parts; as, three cents, a quantity composed of three single cents.

5. One of the single parts of which a quantity of multitude is composed, is called the *unit of measure;* thus, 1 cent is the *unit of measure* of the quantity 3 cents.

The *value* or *measure* of any quantity is the number of times it contains its unit of measure.

6. In quantities of magnitude, where there is no natural unit, it is necessary to fix upon an artificial unit as a standard of measure; then, to find the value of the quantity, we ascertain how many times it contains its *unit of measure.* Thus,

To measure the length of a line, take a certain assumed

REVIEW.—1. How are quantities represented in Algebra? 2. What is quantity? 3. When called magnitude?
4. When multitude? 5. What is the unit of measure? 6. How find the value of a quantity when there is no natural unit?

distance called a foot, and, applying it a certain number of times, say 5, it is found that the line is 5 feet long; in this case, 1 foot is the *unit of measure.*

7. The **Numerical Value** of a quantity is the number that shows how many times it contains its unit of measure.

Thus, the numerical value of a line 5 feet long, is 5. The same quantity may have different numerical values, according to the unit of measure assumed.

8. A **Unit** is a single thing of an order or kind.

9. Number is an expression denoting a unit, or a collection of units. Numbers are either abstract or concrete.

10. An **Abstract Number** denotes how many times a unit *is to be taken.*

A **Concrete Number** denotes the units that *are taken.*

Thus, 4 is an abstract number, denoting merely the number of units taken; while 4 feet is a concrete number, denoting what unit is taken, as well as the number taken.

Or, *a concrete number is the product of the unit of measure by the corresponding abstract number.* Thus, $6 equal $1 multiplied by 6, or $1 taken 6 times.

11. In algebraic computations, *letters* are considered the representatives of *numbers.*

12. There are two kinds of questions in Algebra, *theorems* and *problems.*

13. In a **Theorem,** it is required to demonstrate some relation or property of numbers, or abstract quantities.

14. In a **Problem,** it is required to find the value of some *unknown* quantity, by means of certain given relations existing between it and others, which are *known.*

REVIEW.—7. Define numerical value. 8. What is a unit? 9. A number? 10. What does an abstract number denote? A concrete number? 11. What do the letters used in Algebra represent? 12. How many kinds of questions in Algebra? 13. What is a theorem? 14. A problem?

15. **Algebra** is a general method of solving problems and demonstrating theorems, by means of *figures*, *letters*, and *signs*. The letters and signs are called *symbols*.

EXPLANATION OF SIGNS AND TERMS.

16. **Known Quantities** are those whose values are given; **Unknown Quantities,** those whose values are to be determined.

17. Known quantities are generally represented by the first letters of the alphabet, as a, b, c, etc.; unknown quantities, by the last letters, as x, y, z.

18. The principal signs used in Algebra are

$$=,\quad +,\quad -,\quad \times,\quad \div,\quad (\ ),\quad >,\quad \surd.$$

Each sign is the representative of certain words. They are used to express the various operations in the clearest and briefest manner.

19. The **Sign of Equality,** $=$, is read *equal to*. It denotes that the quantities between which it is placed are equal. Thus, $a=3$, denotes that the quantity represented by a is equal to 3.

20. The **Sign of Addition,** $+$, is read *plus*. It denotes that the quantity to which it is prefixed is to be added.

Thus, $a+b$, denotes that b is to be added to a. If $a=2$ and $b=3$, then $a+b=2+3$, which $=5$.

21. The **Sign of Subtraction,** $-$, is read *minus*. It denotes that the quantity to which it is prefixed is to be subtracted.

Thus, $a-b$, denotes that b is to be subtracted from a. If $a=5$ and $b=3$, then $5-3=2$.

REVIEW.—15. What is Algebra? What are symbols? 16. What are known quantities? Unknown quantities? 17. By what are known quantities represented? Unknown quantities?

18. Write the principal signs used in Algebra. What does each represent? For what used?

19. How is the sign $=$ read? What does it denote? 20. How is the sign $+$ read? What denote? 21. How is the sign $-$ read? What denote?

22. The signs $+$ and $-$ are called *the* signs. The former is called the *positive*, the latter the *negative* sign: they are said to be *contrary* or *opposite*.

23. Every quantity is supposed to be preceded by one of these signs. Quantities having the positive sign are called *positive;* those having the negative sign, *negative.*

When a quantity has no sign prefixed, it is positive.

24. Quantities having the same sign are said to have *like* signs; those having different signs, *unlike* signs.

Thus, $+a$ and $+b$, or $-a$ and $-b$, have like signs; while $+c$ and $-d$ have unlike signs.

25. The **Sign of Multiplication,** $\times$, is read *into,* or *multiplied by.* It denotes that the quantities between which it is placed are to be multiplied together.

The product of two or more letters is sometimes expressed by a dot or point, but more frequently by writing them in close succession without any sign. Thus, ab expresses the same as $a\times b$ or $a.b$, and $abc=a\times b\times c$, or $a.b.c$.

26. Factors are quantities that are to be multiplied together.

The *continued product* of several factors means the product of the first and second multiplied by the third, this product by the fourth, and so on.

Thus, the continued product of a, b, and c, is $a\times b\times c$, or abc. If $a=2$, $b=3$, and $c=5$, then $abc=2\times 3\times 5=6\times 5=30$.

27. The **Sign of Division,** $\div$, is read *divided by.* It

REVIEW.—22. What are the signs plus and minus called, by way of distinction? Which is positive? Which negative?

23. What are quantities preceded by the sign plus said to be? By the sign minus? When no sign is prefixed? 24. When do quantities have like signs? When unlike signs?

25. How is the sign $\times$ read, and what does it denote? What other methods of representing multiplication? 26. What are factors? How many in a? In ab? In abc? In $5abc$?

27. How is the sign $\div$ read, and what does it denote? What other methods of representing division?

denotes that the quantity preceding it is to be divided by that following it. Division is oftener represented by placing the dividend as the numerator, and the divisor as the denominator of a fraction.

Thus, $a \div b$, or $\frac{a}{b}$, means, that a is to be divided by b. If $a=12$ and $b=3$, then $a \div b=12 \div 3=4$; or $\frac{a}{b}=\frac{12}{3}=4$.

Division is also represented thus, $a | b$, or $b \underline{| a}$, a denoting the dividend, and b the divisor.

28. The Sign of Inequality, $>$, denotes that one of the two quantities between which it is placed is greater than the other. The *opening* of the sign is toward the *greater* quantity.

Thus, $a>b$, denotes that a is greater than b. It is read, a greater than b. If $a=5$, and $b=3$, then $5>3$. Also, $c<d$, denotes that c is less than d. It is read, c less than d. If $c=4$ and $d=7$, then $4<7$.

29. The Sign of Infinity, ∞, denotes a quantity greater than any that can be assigned, or one indefinitely great.

30. The Numeral Coefficient of a quantity is a number prefixed to it, showing how many times the quantity is taken.

Thus, $a+a+a+a=4a$; and $ax+ax+ax=3ax$.

31. The Literal Coefficient of a quantity is a quantity by which it is multiplied. Thus, in the quantity ay, a may be considered the coëfficient of y, or y the coëfficient of a.

The literal coëfficient is generally regarded as a known quantity.

32. The coëfficient of a quantity may consist of a number and a literal part. Thus, in $5ax$, $5a$ may be re-

REVIEW.—28. What is the sign $>$ called, and what does it denote? Which quantity is placed at the opening?

29. What does the sign ∞ denote? 30. What is a numeral coëfficient? How often is ax taken in $3ax$? In $5ax$? In $7ax$?

31. What is a literal coëfficient? 32. When a quantity has no coëfficient, what is understood?

garded as the coëfficient of x. If $a=2$, then $5a=10$, and $5ax=10x$.

When no numeral coëfficient is prefixed to a quantity, its coëfficient is understood to be unity. Thus, $a=1a$, and $bx=1bx$.

33. The **Power** of a quantity is the product arising from multiplying the quantity by itself one or more times.

When the quantity is taken twice as a factor, the product is called its *square*, or *second* power; when three times, the *cube*, or *third* power; when four times, the *fourth* power, and so on.

Thus, $a\times a=aa$, is the *second* power of a; $a\times a\times a=aaa$, is the *third* power of a; $a\times a\times a\times a=aaaa$, is the *fourth* power of a.

An **Exponent** is a figure placed at the right, and a little above a quantity, to show how many times it is taken as a factor.

Thus, $aa=a^2$; $aaa=a^3$; $aaaa=a^4$; $aabbb=a^2b^3$.

When no exponent is expressed, it is understood to be unity. Thus, a is the same as a^1, each expressing the first power of a.

34. To raise a quantity to any given power is to find that power of the quantity.

35. The **Root** of a quantity is another quantity, some power of which equals the given quantity. The root is called the square root, cube root, fourth root, etc., according to the number of times it is taken as a factor to produce the given quantity.

Thus, a is the second or square root of a^2, since $a\times a=a^2$. So, x is the third or cube root of x^3, since $x\times x\times x=x^3$.

36. To *extract* any root of a quantity is to find that root.

REVIEW.—33. What is the power of a quantity? What is the second power of a? The third power of a?

33. What is an exponent? For what used? How many times is x taken as a factor in x^2? In x^3? In x^5? Where no exponent is written, what is understood? 35. What is the root of a quantity?

37. The **Radical Sign**, $\sqrt{}$, placed before a quantity, indicates that its root is to be extracted.

Thus, $\sqrt[2]{a}$, or $\sqrt{a}$, denotes the square root of a; $\sqrt[3]{a}$, denotes the cube root of a; $\sqrt[4]{a}$, denotes the fourth root of a.

38. The number placed over the radical sign is called the *index* of the root. Thus, 2 is the index of the square root, 3 of the cube root, 4 of the fourth root, and so on. When the radical has no index over it, 2 is understood.

39. Every quantity or combination of quantities expressed by means of symbols, is called an *algebraic expression*.

Thus, $3a$ is the algebraic expression for 3 times the quantity a; $3a-4b$, for 3 times a, diminished by 4 times b; $2a^2+3ab$, for twice the square of a, increased by 3 times the product of a and b.

40. A **Monomial, or Term,** is an algebraic expression, not united to any other by the sign $+$ or $-$.

A monomial is sometimes called a *simple quantity*. Thus, a, $3a$, $-a^2b$, $2any^2$, are monomials, or simple quantities.

41. A **Polynomial** is an algebraic expression, composed of two or more terms.

Thus, $c+2d-b$ is a polynomial.

42. A **Binomial** is a polynomial composed of two terms. Thus, $a+b$, $a-b$, and c^2-d, are binomials.

A **Residual Quantity** is a binomial, in which the second term is negative, as $a-b$.

43. A **Trinomial** is a polynomial consisting of *three terms*. Thus, $a+b+c$, and $a-b-c$, are trinomials.

44. The **Numerical Value** of an algebraic expression

REVIEW.—37. What is the sign $\sqrt{}$ called, and what does it denote? 38. What is the number placed over the radical sign called? 39. What is an algebraic quantity? 40. A monomial? A simple quantity? 41. A polynomial? 42. A binomial? A residual quantity? 43. A trinomial? 44. What is the numerical value of an algebraic expression?

is the number obtained, by giving particular values to the letters, and then performing the operations indicated.

In the algebraic expression $2a+3b$, if $a=4$, and $b=5$, then $2a=8$, and $3b=15$, and the numerical value is $8+15=23$.

45. The value of a polynomial is not affected by changing the order of the terms, provided each term retains its respective sign. Thus, $a^2+2a+b=b+a^2+2a$. This is self-evident.

46. Each of the *literal* factors of any simple quantity or term is called a *dimension* of that term. The *degree* of a term depends on the number of its literal factors.

Thus, ax consists of two literal factors, a and x, and is of the *second degree.* The quantity a^2b contains three literal factors, a, a, and b, and is of the *third degree.* $2a^3x^2$ contains 5 literal factors, a, a, a, x, and x, and is of the *fifth degree;* and so on.

47. A polynomial is said to be *homogeneous,* when each of its terms is of the same degree.

Thus, the polynomials $2a-3b+c$, of the first degree, $a^2+3bc+xy$, of the second degree, and x^3-8ay^2, of the third degree, are homogeneous: a^3+x^2 is not homogeneous.

48. A **Parenthesis,** ( ), is used to show that all the included terms are to be considered together as a single term.

Thus, $4(a-b)$ means that $a-b$ is to be multiplied by 4; $(a+x)(a-x)$ means that $a+x$ is to be multiplied by $a-x$; $10-(a+c)$ means that $a+c$ is to be subtracted from 10; $(a-b)^2$ means that $a-b$ is to be raised to the second power; and so on.

49. A **Vinculum,** ———, is sometimes used instead of

REVIEW.—46. What is the dimension of a term? On what does the degree of a term depend? What is the degree of the term xy? Of xyz? Of $2axy$? Of x^2? 47. When is a polynomial homogeneous? 48. For what is a parenthesis used? 49. What is a vinculum, and for what used?

a parenthesis. Thus, $\overline{a-b}\times x$ means the same as $(a-b)x$. Sometimes the vinculum is placed vertically: it is then called a *bar*.

Thus, $\begin{array}{r|l} a & y^2 \\ -x & \\ +4 & \end{array}$ has the same meaning as $(a-x+4)y^2$.

50. **Similar** or **Like** quantities are those composed of the same letters, affected with the same exponents.

Thus, $7ab$ and $-3ab$, also $4a^3b^2$ and $7a^3b^2$, are similar terms; but $2a^2b$ and $2ab^2$ are not similar; for, though composed of the same letters, these letters have different exponents.

51. The **Reciprocal** of a quantity is unity divided by that quantity. Thus, the reciprocal of 2 is $\frac{1}{2}$, of a is $\frac{1}{a}$.

The reciprocal of $\frac{2}{3}$ is 1 divided by $\frac{2}{3}$, or $\frac{3}{2}$. Hence, *the reciprocal of a fraction is the fraction inverted.*

52. The same letter *accented* is often used to denote quantities which occupy similar positions in different equations or investigations.

Thus, a, a', a'', a''', represent four different quantities; read a, a prime, a second, a third, and so on.

EXAMPLES.

The following examples are intended to exercise the learner in the use and meaning of the signs.

Copy each example on the slate or blackboard, and then express it in common language.

Let the numerical values be found, on the supposition that $a=4$, $b=3$, $c=5$, $d=10$, $x=2$, and $y=6$.

1. $c+d-b$. . . Ans.
2. $4a-x$. . . . Ans.
3. $-3ax$. . . Ans.
4. $6a^2x$ . . . Ans.
5. $\frac{ay}{b}+\frac{cd}{x}$. . . . . Ans.
6. $3a^2+2cx-b^3$ . . Ans.
7. $a(a+b)$. . . . . Ans.

REVIEW.—50. What are similar or like quantities? 51. The reciprocal of a quantity? 52. What the use of accented letters?

8. $a+b\times a-b$. . . . . Ans. 13.
9. $(a+b)(a-b)$. . . . . Ans. 7.
10. $x^2-3(a+x)(a-x)+2by$. . . . . Ans. 4.
11. $\frac{2ax^2}{(a-x)^2}-6x\sqrt{a}$. . . . . Ans. —16.
12. $3(a+c)(a-c)+3a^2-3c^2$. . . . . Ans. —54.
13. $\frac{a^2-x^2}{a+x}+a-x$. . . . . Ans. 4.

In the following, convert the words into algebraic symbols:

1. Three times *a*, plus *b*, minus four times *c*.
2. Five times *a*, divided by three times *b*.
3. *a* minus *b*, into three times *c*.
4. *a*, minus three times *b* into *c*.
5. *a* plus *b*, divided by three *c*.
6. *a*, plus *b* divided by three *c*.
7. *a* squared, minus three *a* into *b*, plus 5 times *c* into *d* squared.
8. *x* cubed minus *b* cubed, divided by *x* squared minus *b* squared.
9. Five *a* squared, into *a* plus *b*, into *c* minus *d*, minus three times *x* fourth power.
10. *a* squared plus *b* squared, divided by *a* plus *b*, squared.
11. The square root of *a*, minus the square root of *x*.
12. The square root of *a* minus *x*.

ANSWERS.

1.	7.
2.	8.
3.	9.
4.	10.
5.	11.
6.	12.

ADDITION.

53. Addition, in Algebra, is the process of finding the simplest expression for the sum of two or more algebraic quantities.

CASE I.

When the Quantities are similar, and have the same Sign.

1. James has 3 pockets, each containing apples: in the first he has 3 apples, in the second 4, and in third 5.

In order to find how many apples he has, suppose he proceeds to find their sum in the following manner:

3 *apples*,
4 *apples*,
5 *apples*,
12 *apples*.

But, instead of writing the word *apples*, suppose he should use the letter a, thus:

$$\begin{array}{r} 3a \\ 4a \\ 5a \\ \hline 12a \end{array}$$

It is evident that the sum of 3 times a, 4 times a, and 5 times a, is 12 times a, or $12a$, whatever a may represent.

2. In the same manner the sum of $-3a$, $-4a$, and $-5a$ would be $-12a$. Hence,

$$\begin{array}{r} -3a \\ -4a \\ -5a \\ \hline -12a \end{array}$$

TO ADD SIMILAR QUANTITIES WITH LIKE SIGNS,

Rule.—*Add together the coëfficients of the several quantities; to their sum prefix the common sign, and annex the common letter or letters.*

NOTE.—When a quantity has no coëfficient, 1 is understood; thus, $a=1a$.

REVIEW.—53. What is algebraic addition? When quantities are similar and have the same sign, how are they added together?

When several quantities are to be added together, is the result affected by the order in which they are taken?

EXAMPLES.

	(3)	(4)	(5)	(6)
	$3a$	$-6xy$	$2a^2$	$-3a^2b$
	$2a$	$-xy$	$3a^2$	$-4a^2b$
	a	$-4xy$	$5a^2$	$-5a^2b$
	$5a$	$-3xy$	$7a^2$	$-2a^2b$
Sum	$=11a$	$-14xy$	$17a^2$	$-14a^2b$

In the third example, suppose $a=2$, then $3a=3\times2=6$, $2a=2\times2=4$, $a=2$, $5a=5\times2=10$; their sum is $6+4+2+10=22$.

But the sum, 22, is more easily found from the algebraic sum, $11a$, for $11a=11\times2=22$.

In the fourth example, let $x=3$ and $y=2$; then,

$$-6xy=-6\times3\times2=-36$$
$$-xy=-3\times2=-6$$
$$-4xy=-4\times3\times2=-24$$
$$-3xy=-3\times3\times2=-18$$

the sum of their values $=-84$.

But this is more easily found thus: $-14xy=-14\times3\times2=-84$.

In the fifth example, let a represent three feet; then,

$2a^2=2aa=2\times3\times3=18$ square feet,
$3a^2=3aa=3\times3\times3=27$ " "
$5a^2=5aa=5\times3\times3=45$ " "
$7a^2=7aa=7\times3\times3=63$ " "

and their sum is 153 " "

Or the sum $=17a^2=17\times3\times3=153$ square feet.

NOTE.—Let the learner test the following examples numerically, by assigning values to the letters.

7. What is the sum of $3b$, $5b$, $7b$, and $9b$? Ans. $24b$.
8. Of $2ab$, $5ab$, $8ab$, and $11ab$? Ans. $26ab$.
9. Of abc, $3abc$, $7abc$, and $12abc$? Ans. $23abc$.
10. Of $-by$, $-2by$, $-5by$, and $-8by$? Ans. $-16by$.

(11)	(12)	(13)
$3ay+7$	$8x-4y$	$3a^2-2ax$
$ay+8$	$5x-3y$	$5a^2-3ax$
$2ay+4$	$7x-6y$	$7a^2-5ax$
$5ay+6$	$6x-2y$	$4a^2-4ax$

CASE II.

54. *When Quantities are alike, but have Unlike Signs.*

1. James receives from one man 6 cents, from another 9, and from a third 10. He spends 4 cents for candy and 3 for apples: how much will he have left?

If the quantities he *received* be considered *positive*, those he *spent* may be considered *negative;* and the question is, to find the sum of $+6c$, $+9c$, $+10c$, $-4c$ and $-3c$, which may be written thus:

$$\begin{array}{r} +6c \\ +9c \\ +10c \\ -4c \\ -3c \\ \hline +18c \end{array}$$

It is evident the true result will be found by collecting the positive quantities into one sum, and the negative quantities into another, and taking their difference. It is thus found that he received $25c$, and spent $7c$, which left $18c$.

2. Suppose James should receive 5 cents, and spend 7 cents, what sum would he have left?

If we denote the $5c$ as positive, the $7c$ will be negative, and it is required to find the sum of $+5c$ and $-7c$.

In its present form it is evident that the question is impossible. But if we suppose that James had a certain sum of money before he received the $5c$, we may inquire what effect the operation had upon his money.

The answer obviously is, that his money was *diminished* 2 cents; this would be indicated by the sum of $+5c$ and $-7c$, being $-2c$.

Hence, we say that the sum of a positive and negative quantity is equal to the *difference* between the two; the object being to find what the *united effect* of the two will be upon some third quantity.

This may be further illustrated by the following example:

3. A merchant has a certain capital; during the year it is *increased* by $3a$ and $8a$ \$'s, and *diminished* by $2a$ and $5a$ \$'s: how much will it be increased or diminished at the close of the year?

If we call the *gains* positive, the *losses* will be negative. The sum of $+3a$, $+8a$, $-2a$, and $-5a$, is $+11a-7a=+4a$.

Hence, we say that the merchant's capital will be *increased* by $4a$ \$'s, which is the same as to increase it by $3a$ and $8a$ \$'s, and then diminish it by $2a$ and $5a$ \$'s.

Had the loss been greater than the gain, the capital would have been *diminshed*, and the result would have been *negative*.

If the gain and loss were equal, the capital would neither be increased nor diminished, and the sum of the positive and negative quantities would be 0. Thus, $+3a-3a=0$.

Hence, to add a negative quantity is the same as to subtract a positive quantity. In such cases, the process is called *algebraic addition*, and the sum the *algebraic sum*, to distinguish them from arithmetical addition and arithmetical sum. Hence,

TO ADD LIKE QUANTITIES WHICH HAVE UNLIKE SIGNS,

Rule.—1. *Find the sum of the coëfficients of the positive quantities; also, the sum of the coëfficients of the negative quantities.*

2. *Subtract the less sum from the greater; to the difference prefix the sign of the greater, and annex the common literal part.*

4. What is the sum of $+3a$, $-5a$, $+9a$, $-6a$, and $+7a$?

Sum of positive coëfficients, $3+9+7=+19$.

Sum of negative coëfficients, $-5-6=-11$.

Difference, $=19-11=8$. Prefixing the sign of the greater, and annexing the literal part, we have for the required sum $+8a$.

In practice, it is most convenient to write the different terms under each other. Thus,

$$\begin{array}{r} 3a \\ -5a \\ 9a \\ -6a \\ 7a \\ \hline \text{Sum}=8a \end{array}$$

EXAMPLES.

5. What is the sum of $8a$ and $-5a$?
6. Of $5a$ and $-8a$?
7. Of $-7ax$, $3ax$, $6ax$, and $-ax$?
8. Of $5abx$, $-7abx$, $3abx$, $-abx$, and $4abx$?
9. Of $6a-4b$, $3a+2b$, $-7a-8b$, and $-a+9b$?

REVIEW.—54. What is case 2d in Addition? The rule?

10. Of $4a^2-2b$, $-6a^2+2b$, $2a^2-3b$, $-5a^2-8b$, and $-3a^2+9b$?

11. Of $xy-ac$, $3xy-9ac$, $-7xy+5ac$, $4xy+6ac$, and $-xy-2ac$?

NOTE.—The operation of collecting an algebraic expression into one sum is called the *Reduction of Polynomials.* The following are examples:

12. Reduce $3ab+5c-7ab+8c+8ab-14c-2ab+c$ to its simplest form.

13. Reduce $5a^2c-3b^2+4a^2c+5b^2-8a^2c+2b^2$ to its simplest form.

CASE III.

55. *When the Quantities are Unlike, or partly Like and partly Unlike.*

1. Thomas has a marbles in one hand, and b marbles in the other: what expression will represent the number in both?

If a is represented by 3, and b by 4, then the number in both would be represented by $3+4$, or 7.

Or, the number in both would be represented by $a+b$; but unless the numerical values of a and b are given, it is evidently impossible to represent their sum more concisely than by $a+b$.

So, the sum of $a+b$ and $c+d$, is represented by $a+b+c+d$.

If, in any expression, there are like quantities, it is obvious that they may be added by the preceding rules. Case 3d, therefore, embraces the two preceding cases. Hence,

TO ADD ALGEBRAIC QUANTITIES,

General Rule.—1. *Write the quantities to be added, placing those that are similar under each other.*

2. *Add similar terms, and annex the others, with their proper signs.*

REVIEW.—55. What is the general rule for the addition of algebraic quantities? In writing them, why are similar quantities placed under each other?

REMARK.—It is not absolutely necessary to place similar terms under each other; but as we can add only similar terms, *it is a matter of convenience* to do so.

EXAMPLES.

2. Add $6a-4c+3b$, and $-2a-3c-5b$.

3. $2ab+c$, $4ax-2c+14$, $12-2ax$, and $6ab+3c-x$.

4. $14a+x$, $13b-y$, $-11a+2y$, and $-2a-12b+z$.

NOTE.—Since the quantities in parenthesis are to be considered as one quantity, it is evident that 3 times, 5 times, and 7 times *any quantity whatever*=15 times that quantity.

Add together

5. $2c(a^2-b^2)$, $-3c(a^2-b^2)$, $6c(a^2-b^2)$, and $-4c(a^2-b^2)$.

6. $3az-4by-8$, $-2az+5by+6$, $5az+6by-7$, and $-8az-7by+5$.

7. $8a+b$, $2a-b+c$, $-3a+5b+2d$, $-6b-3c+3d$, and $-5a+7c-2d$.

8. $7x-6y+5z+3-g$, $-x-3y-8-g$, $-x+y-3z-1+7g$, $-2x+3y+3z-1-g$, and $x+8y-5z+9+g$.

9. $5a^3b^2-8a^2b^3+x^2y+xy^2$, $4a^2b^3-7a^3b^2-3xy^2+6x^2y$, $3a^3b^2+3a^2b^3-3x^2y+5xy^2$, and $2a^2b^3-a^3b^2-3x^2y-3xy^2$.

SUBTRACTION.

56. Subtraction, in Algebra, is the process of finding the difference between two algebraic quantities.

The quantity to be subtracted is called the *subtrahend;* that from which the subtraction is to be made, the *minuend;* the quantity left, the *difference* or *remainder.*

REMARK.—The word subtrahend means, *to be subtracted;* the word minuend, *to be diminished.*

1. Thomas has $5a$ cents; if he give $2a$ cents to his brother, how many will he have left?

Since 5 times any quantity, diminished by 2 times the same quantity, leaves 3 times the quantity, the answer is evidently $3a$; that is, $5a-2a=3a$. Hence,

To find the difference between two positive similar quantities,

Find the difference between their coëfficients, and to it annex the common letter or letters.

	(2)	(3)	(4)	(5)
From	$5x$	$7ab$	$8xy$	$11a^2x$
Take	$3x$	$3ab$	$5xy$	$5a^2x$
Remainder	$2x$	$4ab$	$3xy$	$6a^2x$

6. From $9a$, take $4a$. . . . . Ans.
7. From $11b$, take $11b$. . . . . Ans.
8. From $3a^2$, take $2a^2$. . . . . Ans.
9. From $7b^2xy$, take $4b^2xy$. . . . . . Ans.

57.—1. Thomas has a apples; if he give away b apples, what expression will represent the number he has left?

If a represents 6, and b 4, the number left will be represented by 6—4, or 2; and whatever numbers a and b represent, it is evident that their difference may be expressed in the same way; that is, by $a-b$. Hence,

To find the difference between two quantities not similar,

Place the sign minus before the quantity to be subtracted.

Observe that the sign of the quantity to be subtracted is changed from *plus* to *minus.*

REVIEW.—56. What is Subtraction, in Algebra? What is the quantity to be subtracted called? The quantity from which the subtraction is to be made? What does subtrahend mean? Minuend?

56. How find the difference between two positive similar quantities? 57. How between two quantities not similar?

2. From c, take d. . . . . Ans.
3. From $2m$, take $3n$. . . . . Ans.
4. From a^2x, take ax^2 . . . . Ans.
5. From x^2, take x. . . . . Ans.

58.—1. Let it be required to subtract $5+3$ from 9.

If we subtract 5 from 9, the remainder will be $9-5$; but we wish to subtract, not only 5, but also 3. Hence,

After we have subtracted 5, we must also subtract 3; this gives for the remainder, $9-5-3$, which is equal to 1.

2. Let it be required to subtract $5-3$ from 9.

If we subtract 5 from 9, the remainder is $9-5$; but the quantity to be subtracted is 3 less than 5. Hence,

We have subtracted 3 too much; we must, therefore, add 3 to $9-5$, which gives for the true remainder, $9-5+3$, or 7.

3. Let it now be required to subtract $b-c$ from a.

If we take b from a, the remainder is $a-b$; but, in doing this, we have subtracted c too much; hence, to obtain the true result, we must add c. This gives the true remainder, $a-b+c$.

If $a=9$, $b=5$, and $c=3$, the operation and illustration by figures would stand thus:

From	a	from	9	$=9$
Take	$b-c$	take	$5-3$	$=2$
Remainder,	$a-b+c$	Rem.	$9-5+3$	$=7$

For further illustration, take the following:

4. $a-(c-a)=a-c+a=2a-c$.
$a-(a-c)=a-a+c=c$.
$a+b-(a-b)=a+b-a+b=2b$.

Observe that in each of the preceding examples, the signs of the subtrahend are changed from *plus* to *minus*, and from *minus* to *plus*. Hence,

TO FIND THE DIFFERENCE BETWEEN TWO ALGEBRAIC QUANTITIES,

Rule.—1. *Write the quantity to be subtracted under that from which it is to be taken, placing similar terms under each other.*

2. *Conceive the signs of all the terms of the subtrahend to be changed, and then reduce by the rule for Addition.*

Note.—Beginners can write the example a second time, then actually change the signs, and add, as in the following example, until they become familiar with the rule.

From	$5a+3b-c$	The same with the	$5a+3b-c$
Take	$2a-2b-3c$	signs of the subtra-	$-2a+2b+3c$
Rem.	$3a+5b+2c$	hend changed.	$3a+5b+2c$

EXAMPLES.

	(6)	(7)	(8)
From	$3ax-2y$	$4cx^2-3by^2$	$8xyz+3az-8$
Take	$2ax+3y$	$2cx-3by^2$	$5xyz-3az+8$
Rem.	$ax-5y$	$4cx^2-2cx$	$3xyz+6az-16$

	(9)	(10)	(11)
From	$7x+4y$	$3a-2b$	$6ax-4y^2+3$
Take	$6x-y$	$5a-3b$	$3ax-6y^2+2$

12. From 14, take $ab-5$. . . . .
13. From $a+b$, take a. . . . .
14. From a, take $a+b$. . . . .
15. From x, take $x-5$. . . . .
16. From $x+y$, take $x-y$. . . . . . .
17. From $x-y$, take $x+y$. . . . . . .
18. From $x-y$, take $y-x$. . . . . .
19. From $x+y+z$, take $x-y-z$. . .
20. From $5x+3y-z$, take $4x+3y+z$. .
21. From a, take $-a$. . . . .
22. From $8a$, take $-3a$. . . . .
23. From $5b$, take $11b$. . . . . . .
24. From a, take $-b$. . . . .
25. From $-9a$, take $3a$. . . . .
26. From $-7a$, take $-7a$. . . . . . .
27. From $-6a$, take $-5a$. . . . . . .

28. From -13, take 3. . . . .
29. From -9, take -16. . . . . . .
30. From 12, take -8. . . . . . .
31. From -14, take -5. . . . .
32. From $13a-2b+9c-3d$, take $8a-6b+9c-10d+12$.
33. From $-7a+3m-8x$, take $-6a-5m-2x+3d$.

34. From $6a+5-3b$, take $-2a-9b-8$.

35. From $3ax-2y^2$, take $-5ax-8y^2$.
36. From $4x^2y^3-5cz+8m$, take $-cz+2x^2y^3-4cz$.

37. From $x^3-11xyz+3a$, take $-6xyz+7-2a-5xyz$.

38. From $5(x+y)$, take $2(x+y)$. . .
39. From $3a(x-z)$, take $a(x-z)$ . .
40. From $7a^2(c-z)-ab(c-d)$, take $5a^2(c-z)-5ab(c-d)$.

59. It is sometimes convenient to simply *indicate* the subtraction of a polynomial. This may be done by inclosing it in a parenthesis, and then placing the sign minus before it.

Thus, to subtract $a-b$ from $2a$, write it $2a-(a-b)$, which is equal to $2a-a+b$, and reduces to $a+b$.

By this transformation, the same polynomial may be written in several different forms, thus:

$$a-b+c-d=a-b-(d-c)=a-d-(b-c)=a-(b-c+d).$$

In the following examples, introduce all the quantities except the first into a parenthesis, and precede it by the sign minus, without altering the value of the expressions.

REVIEW.—58. In subtracting $b-c$ from a, after taking away b, have we subtracted too much, or too little? What must be added, to obtain the true result? Why? What is the general rule for finding the difference between two algebraic quantities?

59. How can the subtraction of an algebraic quantity be indicated?

1. $a-b+c$. . . . .
2. $b+c-d$. . . . .
3. $ax+bc-cd+h$. . . . .
4. $m-n-z-s$. . . . . . .

Let the pupil take the preceding polynomials, and write them in all possible modes, by including either two or more terms in a parenthesis.

OBSERVATIONS ON ADDITION AND SUBTRACTION.

60. It has been shown that algebraic addition is the process of collecting two or more quantities into one.

If these quantities are either all positive or all negative, the sum will be greater than either of the individual quantities.

If some of the quantities are positive and others negative, the aggregate may be less than either of them, or it may be nothing.

Thus, the sum of $+4a$ and $-3a$, is a; while that of $+a$ and $-a$, is zero, or 0.

As the introduction of the minus sign makes the operations of algebraic addition and subtraction differ materially from those of arithmetic, it will be proper to enter into a further explanation of them.

61. In arithmetical addition, when we say the sum of 5 and 3 is 8, we mean that their sum is 8 greater than 0.

In Algebra, when we say the sum of 5 and -3 is 2, we mean that the aggregate effect of adding 5 and subtracting 3, is the same as that of adding 2. When we say the sum of -5 and $+3$ is -2; we mean that the result of subtracting 5 and adding 3, is the same as that of subtracting 2.

Some say that numbers, with a negative sign, such as -3, represent quantities *less than nothing*. This phrase, however, is objectionable. If we understand by it that any negative quantity, added to a positive, will produce a result *less than if nothing had been added to it;* or, subtracted, will produce a result *greater than if nothing had been taken from it*, then the phrase has a correct meaning. Thus,

REVIEW.—60. When is the sum of two algebraic quantities less than either of them? When equal to zero?

If we take any number, as 10, and add to it the numbers 3, 2, 1, 0, —1, —2, and —3, it will be seen that adding a negative number produces a *less* result than adding zero.

10	10	10	10	10	10	10
3	2	1	0	—1	—2	—3
13	12	11	10	9	8	7

Hence, adding a negative number produces the same result as subtracting an equal positive number.

Again, if from any number, as 10, we subtract 3, 2, 1, 0, —1, —2, and —3, it will be seen that subtracting a negative number produces a *greater* result than subtracting zero:

10	10	10	10	10	10	10
3	2	1	0	—1	—2	—3
7	8	9	10	11	12	13

Hence, subtracting a negative number produces the same result as adding an equal positive number.

62. In consequence of the results they produce, it is customary to say, of negative algebraic quantities, that those which are *numerically* the greatest are really the least. Thus, —3 is less than —2, though numerically greater.

63. A correct idea of this subject may be gained by considering such questions as the following:

How will the money in a drawer be affected, if $20 are taken out, afterward $15 put in, after this $8 taken out, and then $10 put in?

Or, in other words, what is the sum of —20, +15, —8, and +10?

The answer, evidently, is —3; that is, the result of the whole operation diminishes the money in the drawer $3.

Had the answer been positive, the result of the operation would have been an increase of the amount of money in the drawer.

Again, suppose latitude north of the equator to be reckoned +, and that south —, in the following question:

A ship, in latitude 10 degrees north, sails 5 degrees south,

REVIEW.—61. What is meant by saying that the sum of +5 and —3, is +2? That the sum of —5 and +3, is —2?

61. Is it correct to say that any quantity is less than nothing? What is the effect of adding a positive quantity? A negative quantity? Of subtracting a positive quantity? A negative quantity?

62. In comparing two negative algebraic quantities, which is least? Which numerically greatest?

then 7 degrees north, then 9 degrees south, then 3 degrees north; what is her present latitude?

Here, +10, —5, +7, —9, and +3, are evidently +6; that is, the ship is in 6 degrees north latitude.

Had the sum of the negative numbers been the greater, the ship would have been in south latitude.

Other questions of a similar nature will readily suggest themselves.

64. Subtraction, in arithmetic, shows the method of finding the excess of one quantity over another of the same kind.

In this case, the subtrahend must be less than the minuend, and the signs are regarded as the same.

In algebraic subtraction, the two quantities may have either like or unlike signs, and the difference is often greater than either of the quantities. To understand this properly requires a knowledge of the nature of positive and negative quantities.

All quantities are to be regarded as positive, unless, for some special reason, they are otherwise designated. Negative quantities are, in their nature, the *opposite* of positive quantities.

Thus, if a merchant's gains are positive, his losses are negative; if latitude north of the equator is +, that south is —; if distance to the right of a certain line is +, that to the left is —; if elevation above a certain point is +, that below is —; if time after a certain hour is +, time before that hour is —; if motion in one direction is +, motion in an opposite direction is —; and so on.

With these illustrations of the use of the minus sign, it is easy to see how the difference of two quantities, having the same sign, is equal to their difference; and also how the difference of two quantities, having different signs, is equal to their sum.

1. One place is situated 10, and another 6 degrees north of the equator; what is their difference of latitude?

Here, the difference between +10 and +6, is +4; that is, the first place is 4 degrees farther north than the second.

2. Two places are situated, one 10, and the other 6 degrees south latitude; what is the difference of latitude?

REVIEW.—64. How does algebraic differ from arithmetical subtraction? How do negative quantities differ from positive? Illustrate the difference by examples.

Here, the difference between -6 and -10, is -4; that is, the first place is 4 degrees farther south than the second.

3. One place is situated 10 degrees north, and another 6 degrees south latitude; what is their difference of latitude?

Here, we are to find the difference between $+10$ and -6, or to take -6 from $+10$, which, by the rule for subtraction, leaves $+16$; that is, the first place is 16 degrees north of the other.

Thus, when properly understood, the results of algebraic subtraction are always capable of a satisfactory explanation.

MULTIPLICATION.

65. Multiplication, in Algebra, is the process of taking one algebraic expression as many times as there are units in another.

The quantity to be multiplied is called the *multiplicand;* the quantity by which we multiply, the *multiplier;* and the result, the *product.*

The multiplicand and multiplier are called *factors.*

66. Since a, taken once, is represented by a, taken twice, by $a+a$, or $2a$, taken three times, by $a+a+a$, or $3a$. Hence,

To multiply a literal quantity by a number,

Rule.— *Write the multiplier as the coëfficient of the literal quantity.*

1. If 1 lemon cost a cents, what will 5 lemons cost?

If *one* lemon cost a cents, *five* lemons will cost *five* times as much; that is, $5a$ cents.

2. If 1 orange cost c cents, what will 6 oranges cost?

3. Bought a pieces of cloth, each containing b yards, at c dollars per yard; what did the whole cost?

REVIEW.—65. What is Multiplication, in Algebra? What the multiplicand? Multiplier? Product? What are the multiplicand and multiplier generally called? 66. How multiply a literal quantity by a number?

In a pieces, the number of yards is represented by ab, or ba, and the cost of ab yards at c dollars per yard, is represented by c taken ab times; that is, by $ab\times c$, or abc.

67. It is shown in "Ray's Arithmetic, Third Book," Art. 30, that the product of two factors is the same, whichever be made the multiplier. Let us demonstrate this principle.

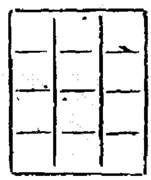
Suppose we have a sash containing a vertical, and b horizontal rows; there will be a panes in each horizontal row, and b panes in each vertical row; how many panes in the window?

The number of panes in the window is equal to the number in one row, taken as many times as there are rows. As there are a vertical rows, and b panes in each row, the number is represented by b taken a times; that is, by ab.

Again, since there are b horizontal rows, and a panes in each row, the whole number of panes is represented by a taken b times; that is, by ba.

As ab and ba each represents the same number, it follows that $ab=ba$. Hence,

The product of two factors is the same, whichever be made the multiplier.



By taking $a=3$ and $b=4$, the figure in the margin may be used to illustrate this principle.

So, the product of three or more quantities is the same, in whatever order taken.

Thus, $2\times3\times4=3\times2\times4=4\times2\times3$, since the product in each case is 24.

1. What will 2 boxes, each containing a lemons, cost at b cents per lemon?

One box will cost ab cents, and 2 boxes will cost twice as much as 1 box; that is, $2ab$ cents.

2. What is the product of $2b$, multiplied by $3a$?

The product will be represented by $2b\times3a$, or by $3a\times2b$, or by $2\times3\times ab$, since the product is the same, in whatever order the factors are placed. But $2\times3=6$; hence, $2\times3\times ab=6ab$.

REVIEW.—67. Prove that 3 times 4 is the same as 4 times 3. That a times b is the same as b times a. Is the product of any number of factors changed by altering their arrangement? In multiplying one monomial by another, how is the coefficient of the product obtained?

Hence, in the multiplication of one monomial by another,

The coëfficient of the product is obtained by multiplying together the coëfficients of the multiplicand and multiplier.

This is termed *the Rule of the Coëfficients.*

68. If we take any two factors, as 2×3, and multiply either by any number, as 5, the products will be 10×3, or 2×15, either of which is equal to 30, which is the true answer. Hence,

When either of the factors of a product is multiplied, the product itself is multiplied.

69.—1. What is the product of a by a?

As $b\times a=ab$, so $a\times a$ would be written aa; but this, Art. 33, for brevity, is written a^2.

2. What is the product of a^2 by a?

Since $a^2=aa$, the product of a^2 by a may be expressed thus, $aa\times a$, or aaa, which is written a^3. Hence,

The exponent of a letter in the product is equal to the sum of its exponents in the two factors.

This is termed *the Rule of the Exponents.*

3. What is the product of a^2 by a^2?
4. Of a^2b by ab? . . . . .
5. Of $2ab^2$ by $3ab$? . . . .

TO MULTIPLY ONE POSITIVE MONOMIAL BY ANOTHER,

Rule.—1. *Multiply the coëfficients of the two terms together.*

2. *To this product, annex all the letters in both quantities.*

3. *When the same letter occurs in both factors, add its exponents for the exponent of the product.*

NOTE.—Write the letters in the order of the alphabet; thus, $ab\times c=abc$.

6. Multiply ab by x. . . . .

REVIEW.—68. If you multiply one of the factors of a product, how does it affect the product? 69. How may the product of a by a be written? Of a^2 by a?

7. Multiply $2bc$ by mn. . . . . .
8. Multiply $4ab$ by $5xy$. . . . . .
9. Multiply $6by$ by $3ax$. . . . . .
10. Multiply $3a^2b$ by $4ab$. . . . .
11. Multiply $2xy^2$ by $3x^2y$. . . . .
12. Multiply $4ab^2x$ by $5ax^2y$. . . . .
13. Multiply $7xy^2z$ by $8x^3yz$. . . . .

NOTE.—Distinguish carefully between the coefficient and the exponent. To fix this in the mind, answer the following questions:

What is $2a—a^2$ equal to, when a is 1? . .
What is $a^2—2a$ equal to, when a is 5? . .
What is $a^3—3a$ equal to, when a is 4? . .

70.—1. If 5 oranges were purchased at 4 cents apiece, and 2 lemons at the same price; what did the whole cost?

The 5 oranges cost 20 cents, the 2 lemons cost 8 cents, and the whole cost was $20+8=28$ cents.

The work may be written thus:

$$\begin{array}{l} 5+2 \\ 4 \\ \hline 20+8=28 \text{ cents.} \end{array}$$

If you purchase a oranges at c cents apiece, and b lemons at c cents apiece, what is the cost of the whole?

The cost of a oranges at c cents each, is ac cents; the cost of b lemons at c cents each, is bc cents, and the whole cost is $ac+bc$ cents.

The work may be written thus:

$$\begin{array}{l} a+b \\ c \\ \hline ac+bc \end{array}$$

Hence, when the sign of each term is positive,

TO MULTIPLY A POLYNOMIAL BY A MONOMIAL,

Rule.—*Multiply each term of the multiplicand by the multiplier.*

NOTE.—It is most convenient to place the multiplier on the left.

EXAMPLES.

2. Multiply $a+d$ by b. . . . . .
3. Multiply $4x+5y$ by $3a$. . . . .
4. Multiply $m+2n$ by $3n$. . . .
5. Multiply x^2+y^2 by xy. . . . .
6. Multiply $2x+5y$ by abx. . . . .
7. Multiply $3x^2+2xz$ by $2xz$. . . .
8. Multiply $ab+ax+xy$ by $abxy$.

71.—1. Required the product of $x+y$ by $a+b$.

Here, the multiplicand is to be taken as many times as there are units in $a+b$, and the whole product will equal the sum of the two partial products. Thus,

$$\begin{array}{l} x+y \\ \underline{a+b} \\ ax+ay = \text{the multiplicand taken } a \text{ times.} \\ \underline{\quad bx+by} = \text{the multiplicand taken } b \text{ times.} \\ ax+ay+bx+by = \text{the multiplicand taken } (a+b) \text{ times.} \end{array}$$

If $x=5$, $y=6$, $a=2$, and $b=3$, the multiplication may be arranged thus:

$$\begin{array}{l} 5+6 \\ \underline{2+3} \\ 10+12 = \text{the multiplicand taken 2 times.} \\ \underline{\quad 15+18} = \text{the multiplicand taken 3 times.} \\ 10+27+18 = 55 = \text{the multiplicand taken 5 times.} \end{array}$$

Hence, when all the terms in each are positive,

TO MULTIPLY ONE POLYNOMIAL BY ANOTHER,

Rule.—*Multiply each term of the multiplicand by each term of the multiplier, and add the products together.*

$$\begin{array}{ll} (2) & (3) \\ a+b & a^2b+cd \\ \underline{a+b} & \underline{ab+cd^2} \\ a^2+ab & a^3b^2+abcd \\ \underline{\quad ab+b^2} & \underline{\quad +a^2bcd^2+c^2d^3} \\ a^2+2ab+b^2 & a^3b^2+a^2bcd^2+abcd+c^2d^3 \end{array}$$

4. Multiply $a+b$ by $c+d$.
5. $2x+3y$ by $3a+2b$.
6. $2a+3b$ by $3c+d$.
7. $m+n$ by $x+z$.
8. $4a+3b$ by $2a+b$.
9. $4x+5y$ by $2a+3x$.

10. $3x+2y$ by $2x+3y$.
11. a^2+b^2 by $a+b$.
12. $3a^2+2b^2$ by $2a^2+3b^2$.
13. a^2+ab+b^2 by $a+b$.
14. c^3+d^3 by $c+d$.
15. $x^2+2xy+y^2$ by $x+y$.

OF THE SIGNS.

72. In the preceding examples, it was assumed that the product of two positive quantities is positive. This, and the other possible cases, may be proved, as follows:

1st. Let it be required to find the product of $+b$ by a.

The quantity b taken once, is $+b$; taken twice, is $+2b$; taken 3 times, is $+3b$, and so on. Therefore, taken a times, it is $+ab$.

Hence, the product of two positive quantities is positive; or, more briefly, *plus* multiplied by *plus* gives *plus*.

2d. Let it be required to find the product of $-b$ by a.

The quantity $-b$ taken once, is $-b$; taken twice, is $-2b$; taken 3 times, is $-3b$; taken a times, is $-ab$. Hence,

A *negative* quantity multiplied by a *positive* quantity, gives a *negative* product; or, *minus* multiplied by *plus* gives *minus*.

3d. Let it be required to multiply b by $-a$.

REVIEW.—To what is the exponent of a letter in the product equal? Rule for multiplying one positive monomial by another.
70. What is the product of $a+b$, by c? When the signs are positive, how multiply a polynomial by a monomial? 71. How two polynomials?

When the multiplier is positive, we understand that the multiplicand is to be *added* to 0 as many times as there are units in the multiplier. Now, since the negative sign always expresses the opposite of the positive sign, when the multiplier is negative the multiplicand must be *subtracted* from 0 as many times as there are units in the multiplier.

The quantity b subtracted once, is $-b$; subtracted twice, it is $-2b$; subtracted a times, it is $-ab$. Hence,

A positive quantity multiplied by a negative quantity, gives a negative product; or, *plus* multiplied by *minus* gives *minus.*

4th. Let it be required to multiply $-b$ by $-a$.

According to the principle stated above, $-b$ is to be subtracted from 0 a times; subtracted once, it is $+b$; subtracted twice, it is $+2b$; subtracted a times, it is $+ab$. Hence,

The product of two negative quantities is positive; or, *minus* multiplied by *minus* gives *plus.*

NOTE.—The following proof of the last principle is generally regarded as more satisfactory than the preceding:

To find the product of $c-d$ by $a-b$.

Here, it is required to take $c-d$ a times, and then subtract from this product, $c-d$ taken b times.

The multiplication of $c-d$ by $a-b$ may be written thus:

$$\begin{array}{l} c-d \\ \underline{a-b} \\ ac-ad=c-d \text{ taken } a \text{ times.} \\ \underline{\quad +bc-bd=c-d} \text{ taken } b \text{ times. Subtract from the above.} \\ ac-ad-bc+bd, \text{ the true product.} \end{array}$$

Observing the answer, which we know to be the true product, we see that $+c\times+a$ must give $+ac$, $-d\times+a=-ad$, $+c\times-b=-bc$, and $-d\times-b=+bd$; which last result is the thing to be proved.

To illustrate by figures, find the product of 7—4 by 5—3.

$$\begin{array}{r} 7-4 \\ \underline{5-3} \\ 35-20 \\ \underline{+21-12} \\ 35-41+12 \end{array}$$

We first take 5 times 7—4; this gives a product too great, by 3 times 7—4, or 21—12, which being subtracted from the first product, gives for the true result, 35—41+12, which reduces to +6. This is evidently correct, for 7—4=3, and 5—3=2, and the product of 3 by 2 is 6. Hence,

THE GENERAL RULE,

FOR THE SIGNS.

1. *Plus multiplied by plus, or minus by minus, gives plus.*
2. *Plus multiplied by minus, or minus by plus, gives minus.*
3. Or, *the product of like signs gives plus, and of unlike signs gives minus.*

From all the preceding, we derive the following

GENERAL RULE,

FOR THE MULTIPLICATION OF ALGEBRAIC QUANTITIES.

1. *Beginning at the left hand, multiply each term of the multiplicand by each term of the multiplier, observing that like signs give plus and unlike signs give minus.*
2. *Add the several partial products together.*

NUMERICAL EXAMPLES,

TO VERIFY THE RULE OF THE SIGNS.

1. Multiply $8-3$ by 5. . . .
2. Multiply $9-5$ by $8-2$. .
3. Multiply $8-7$ by $5-3$. .

GENERAL EXAMPLES.

1. Multiply $3a^2xy$ by $7axy^3$. . .
2. Multiply $-5a^2b$ by $3ab^3$. . .
3. Multiply $-5x^2y$ by $-5xy^2$. .
4. Multiply $3a-2b$ by $4c$. . .
5. Multiply $3x+2y$ by $-2x$. .
6. Multiply $a+b$ by $x-y$. . .

REVIEW.—72. What is the product of $+b$ by $+a$? Why? The product of $-b$ by a? Why? The product of $+b$ by $-a$? Why? The product of -3 by -2?

72. What does a negative multiplier signify? What does minus multiplied by minus produce? General rule for the signs? For the multiplication of algebraic quantities?

7. Multiply $a-b$ by $a-b$.
8. Multiply a^2+ac+c^2 by $a-c$.
9. Multiply $m+n$ by $m-n$.
10. Multiply $a^2-2ab+b^2$ by $a+b$.
11. Multiply $3x^3y-2xy^3+y^4$ by $2xy+y^2$.

12. Multiply $a^2+2ab+b^2$ by $a^2-2ab+b^2$.

13. Multiply y^2-y+1 by $y+1$.
14. Multiply x^2+y^2 by x^2-y^2.
15. Multiply a^2-3a+8 by $a+3$.
16. Multiply $2x^2-3xy+y^2$ by x^2-5xy.

17. Multiply $3a+5b$ by $3a-5b$.
18. Multiply $2a^2-4ax+2x^2$ by $3a-3x$.

19. Multiply $5x^3+3y^3$ by $5x^3-3y^3$.
20. Multiply $2a^3+2a^2x+2ax^2+2x^3$ by $3a-3x$.

21. Multiply $3a^2+3ax+3x^2$ by $2a^2-2ax$.

22. Multiply $3a^2+5ax-2x^2$ by $2a-x$.

23. Multiply $x^6+x^4+x^2$ by x^2-1.
24. Multiply x^2+xy+y^2 by x^2-xy+y^2.
25. Multiply $a^3+a^2b+ab^2+b^3$ by $a-b$.

In the following, multiply together the quantities in the parentheses.

26. $(x-3)(x-3)(x-3)$.
27. $(x-4)(x-5)(x+4)(x+5)$.
28. $(a+c)(a-c)(a+c)(a-c)$.
29. $(a^2+b^2+c^2-ab-ac-bc)(a+b+c)$.

30. $(n^2+n+1)(n^2+n+1)(n-1)(n-1)$.

DIVISION.

73. Division, in Algebra, is the process of finding how many times one algebraic quantity is contained in another.

Or, having the product of two factors, and one of them given, Division teaches the method of finding the other.

The number by which we divide is called the *divisor;* the number to be divided, the *dividend;* the number of times the divisor is contained in the dividend, the *quotient.*

74. Since the divisor is the known factor and the quotient the one found, their product must always be equal to the dividend.

Division may be indicated by writing the divisor under the dividend in the form of a fraction, or as in arithmetic. Thus, ab divided by a, is written $\frac{ab}{a}$, or $a)ab$.

Note.—In solving the following, give the reason for the answer, as in the solution to the first question.

1. How many times is x contained in $4x$? Ans. $\frac{4x}{x}=4$.

$4x$ divided by x, equals 4, because the product of 4 by x is $4x$.

2. How many times is a contained in $6a$? .
3. Is a contained in ab? . . . . .
4. Is b contained in $3ab$? . . . . .
5. Is 2 contained in $4a$? . . . . .
6. Is $2a$ contained in $4ab$? . . . .
7. Is a contained in a^3? . . . . .
8. Is ab contained in $5a^2b$? . . . .
9. Is $4ab^2$ contained in $12a^3b^3c$? . .
10. Is $2a^2$ contained in $6a^5b$?

Solution, $\frac{6a^5b}{2a^2}=\frac{6}{2}a^{5-2}b=3a^3b$. Ans.

Review.—73. What is Algebraic Division? The divisor? The dividend? The quotient? 74. To what is the product of the quotient and divisor equal? Why? How is division indicated?

In obtaining the quotient, in the foregoing example, we readily see,

1st. That 2 must be multiplied by 3 to produce 6.

2d. That a^2 must be multiplied by a^3 to produce a^5; or, we must subtract 2 from 5 to find the exponent of a in the quotient.

3d. That since b is in the dividend, but not in the divisor, it must be in the quotient, so that the product of the divisor and quotient may equal the dividend.

75. It remains to ascertain the rule for the signs.

Since $+a\times+b=+ab$, $-a\times+b=-ab$, $+a\times-b=-ab$, and $-a\times-b=+ab$,

Therefore, $\frac{+ab}{+b}=+a$, $\frac{-ab}{+b}=-a$, $\frac{-ab}{-b}=+a$, and $\frac{+ab}{-b}=-a$.

Or, *like* signs give *plus*, and *unlike* signs give *minus*. Hence,

TO DIVIDE ONE MONOMIAL BY ANOTHER,

Rule.—1. *Divide the coëfficient of the dividend by that of the divisor; observing, that like signs give plus, and unlike signs minus.*

2. *For any letter common to the divisor and dividend, if it has the same exponent in both, suppress it; if not, subtract its exponent in the former from its exponent in the latter, for its exponent in the quotient.*

3. *Annex the letters found in the dividend, but not in the divisor.*

NOTE.—The pupil must recollect that a is the same as a^1.

EXAMPLES.

11. Divide $15a^3bc$ by $3a^2b$. . . . . .
12. Divide $27x^2y^2$ by $-3xy$. . . . . .
13. Divide $-18a^3x$ by $-6ax$. . . . . .
14. Divide $-12c^4x^3y^5$ by $-4c^4xy^2$. . . . .
15. Divide $6acx^2y^6v$ by $3ax^2y^4v$. . . . .

REVIEW.—75. When the signs of the dividend and divisor are alike, what will be the sign of the quotient? Why? When unlike? Why? Rule for dividing one monomial by another?

16. Divide $-10c^2x^5y^5v$ by $-2cy^4v$. . .
17. Divide $-28ac^2x^8y^4v^2$ by $14ax^5y^4$. .
18. Divide $30ac^4e^4x^4y^2$ by $-2aex^4$. . .

NOTE.—The following may be omitted until the book is reviewed:

19. Divide $(x+y)^2$ by $(x+y)$. . . .
20. Divide $(a+b)^4$ by $(a+b)$. . . .
21. Divide $6(m+n)^3$ by $2(m+n)$. . . .
22. Divide $6a^2b(x+y)^3$ by $2ab(x+y)^2$.
23. Divide $(x-y)^3(m-n)^2$ by $(x-y)^2(m-n)^2$.

76. It is evident that one monomial can not be divided by another, in the following cases:

1st. When the coëfficient of the dividend is not exactly divisible by the coëfficient of the divisor.

2d. When the same literal factor has a greater exponent in the divisor than in the dividend.

3d. When the divisor contains one or more literal factors not found in the dividend.

In each of these cases, the division is indicated by writing the divisor under the dividend, in the form of a fraction. This fraction may then be reduced, Art. 129.

77. It has been shown, Art. 68, that any product is multiplied by multiplying either of its factors; hence, conversely, *any dividend will be divided by dividing either of its factors.*

Thus, $\frac{4\times6}{2}=2\times6=12$; or, $\frac{4\times6}{2}=4\times3=12$.

78. In multiplying a polynomial by a monomial, we multiply each term of the multiplicand by the multiplier. Hence,

REVIEW.—76. In what cases is the exact division of one monomial by another impossible? 78. Rule for dividing a polynomial by a monomial?

TO DIVIDE A POLYNOMIAL BY A MONOMIAL,

Rule.—*Divide each term of the dividend by the divisor, according to the rule for the division of monomials.*

NOTE.—Place the divisor on the left, as in arithmetic.

1. Divide $6x+12y$ by 3. . . . . .
2. Divide $15x-20b$ by 5. . . . . .
3. Divide $21a+35b$ by -7. . . .
4. Divide $abc-acf$ by ac. . . . .
5. Divide $10ax-15ay$ by $-5a$. . .
6. Divide $a^2b^2-2ab^3x$ by ab. . . .
7. Divide $12a^2bc-9acx^2+6ab^2c$ by $-3ac$.

8. Divide $15a^5b^2c-21a^2b^3c^2$ by $3a^2bc$.

NOTE.—The following may be omitted until the book is reviewed:

9. Divide $6(a+c)+9(a+x)$ by 3.

10. Divide $a^2b(c+d)+ab^2(c^2-d)$ by ab.

11. Divide $ac(m+n)-bc(m+n)$ by $m+n$.
12. Divide $(m+n)(x+y)^2+(m+n)(x-y)^2$ by $m+n$.

79. To explain the method of dividing one polynomial by another, we will first find the product of two factors, and then reverse the operation.

Multiplication, or formation of a product.	Division, or decomposition of a product.
$2a^2-ab$	$2a^3-3a^2b+ab^2 \mid a-b$
$a-b$	$2a^3-2a^2b \qquad 2a^2-ab$
$2a^3-a^2b$	1st Rem. $-a^2b+ab^2$
$-2a^2b+ab^2$	$-a^2b+ab^2$
$2a^3-3a^2b+ab^2$	2d Rem. 0

In the foregoing illustration, let the pupil carefully observe, 1st. In the multiplicand, $2a^2-ab$, and the multiplier, $a-b$, a is called the *leading letter*, because its exponents decrease from left to right. It is evident that a will be the leading letter in the product also.

2d. In the division of $2a^3-3a^2b+ab^2$ by $a-b$, the dividend being the product, and the divisor one of the factors, both should be arranged with reference to the same leading letter, in order that the quotient, or remaining factor to be found, may have the same order of arrangement.

3d. If we divide $2a^3$ by a, the result, $2a^2$, will be the term of the quotient by which $a-b$ was first multiplied. If we now multiply $a-b$ by $2a^2$, and subtract the product from the dividend, there will remain $-a^2b+ab^2$, which is the product of $a-b$ by the other term of the quotient. Dividing $-a^2b$ by a, we find this unknown term. Multiplying $a-b$ by it, and subtracting the product, nothing remains.

4th. Had there been a second remainder, the third term of the quotient would have been found in the same manner, and so on for any number of terms.

5th. The divisor is placed on the right of the dividend for convenience in multiplying. Hence,

TO DIVIDE ONE POLYNOMIAL BY ANOTHER,

Rule.—1. *Arrange the dividend and divisor with reference to the leading letter, and place the divisor on the right of the dividend.*

2. *Divide the first term of the dividend by the first term of the divisor, for the first term of the quotient. Multiply the divisor by this term, and subtract the product from the dividend.*

3. *Divide the first term of the remainder by the first term of the divisor, for the second term of the quotient. Multiply the divisor by this term, and subtract the product from the last remainder.*

4. *Proceed in the same manner, and if you obtain* 0 *for a remainder, the division is said to be exact.*

REMARKS.—1. Bring down no more terms of the remainder, at each successive subtraction, than are necessary.

2. It is well to perform the same example in two ways: first, by making the powers of the letter *diminish* from left to right; and, secondly, *increase* from left to right.

3. When the first term of the arranged dividend, or of any remainder, is not exactly divisible by the first term of the arranged divisor, the exact division will be impossible.

1. Divide $6a^2-13ax+6x^2$ by $2a-3x$.

$$\begin{array}{l|l} 6a^2-13ax+6x^2 & 2a-3x \\ \underline{6a^2-9ax} & 3a-2x \text{ Quotient.} \\ \quad -4ax+6x^2 & \\ \quad \underline{-4ax+6x^2} & \end{array}$$

2. Divide x^2-y^2 by $x-y$.

$$\begin{array}{l|l} x^2-y^2 & x-y \\ \underline{x^2-xy} & x+y \text{ Quo.} \\ \quad xy-y^2 & \\ \quad \underline{xy-y^2} & \end{array}$$

3. Divide a^3+x^3 by $a+x$.

$$\begin{array}{l|l} a^3+x^3 & a+x \\ \underline{a^3+a^2x} & a^2-ax+x^2 \text{ Quo.} \\ \quad -a^2x+x^3 & \\ \quad \underline{-a^2x-ax^2} & \\ \qquad ax^2+x^3 & \\ \qquad \underline{ax^2+x^3} & \end{array}$$

4. Divide $5a^2x+5ax^2+a^3+x^3$ by $4ax+a^2+x^2$.

$$\begin{array}{l|l} a^3+5a^2x+5ax^2+x^3 & a^2+4ax+x^2 \\ \underline{a^3+4a^2x+ax^2} & a+x \text{ Quotient.} \\ \quad a^2x+4ax^2+x^3 & \\ \quad \underline{a^2x+4ax^2+x^3} & \end{array}$$

NOTE.—It is not absolutely necessary to arrange the dividend and divisor with reference to a certain letter; it should be done, however, as a matter of convenience.

In the above example, neither divisor nor dividend being arranged with reference to either a or x, we arrange them with reference to a, and then divide.

REVIEW.—79. What is meant by the leading letter? What is understood by arranging the dividend and divisor with reference to a certain letter? Explain the example given in illustration of division of polynomials.

79. Why is the divisor placed on the right? What is the rule for the division of one polynomial by another? When is the exact division impossible?

5. Divide $a^2+a^3-5a^4+3a^5$ by $a-a^2$.

Both quantities arranged according to the ascending powers of a.

$$\begin{array}{l|l} a^2+a^3-5a^4+3a^5 & a-a^2 \\ \hline a^2-a^3 & a+2a^2-3a^3 \\ \quad 2a^3-5a^4 & \text{Quotient.} \\ \quad 2a^3-2a^4 & \\ \qquad -3a^4+3a^5 & \\ \qquad -3a^4+3a^5 & \end{array}$$

Both quantities arranged according to the descending powers of a.

$$\begin{array}{l|l} 3a^5-5a^4+a^3+a^2 & -a^2+a \\ \hline 3a^5-3a^4 & -3a^3+2a^2+a \\ \quad -2a^4+a^3 & \text{Quotient.} \\ \quad -2a^4+2a^3 & \\ \qquad -a^3+a^2 & \\ \qquad -a^3+a^2 & \end{array}$$

It will be seen that the two quotients are the same, but differently arranged. If preferred, the divisor may be placed on the *left*, instead of on the *right*, of the dividend.

6. Divide $4a^2-8ax+4x^2$ by $2a-2x$.
7. Divide $2x^2+7xy+6y^2$ by $x+2y$.
8. Divide $2mx+3nx+10mn+15n^2$ by $x+5n$.

9. Divide $x^2+2xy+y^2$ by $x+y$.
10. Divide $8a^4-8x^4$ by $2a^2-2x^2$.
11. Divide $ac+bc-ad-bd$ by $a+b$.
12. Divide $x^3+y^3+5xy^2+5x^2y$ by $x^2+4xy+y^2$.

13. Divide $a^3-9a^2+27a-27$ by $a-3$.
14. Divide $4a^4-5a^2x^2+x^4$ by $2a^2-3ax+x^2$.

15. Divide x^4-y^4 by $x-y$.
16. Divide a^3-b^3 by a^2+ab+b^2.
17. Divide $x^3-y^3+3xy^2-3x^2y$ by $x-y$.

18. Divide $4x^4-64$ by $2x-4$.
19. Divide $a^5-5a^4x+10a^3x^2-10a^2x^3+5ax^4-x^5$ by $a^2-2ax+x^2$.
20. Divide $4a^6-25a^2x^4+20ax^5-4x^6$ by $2a^3-5ax^2+2x^3$.

21. Divide y^3+1 by $y+1$.

22. Divide $6a^4+4a^3x-9a^2x^2-3ax^3+2x^4$ by $2a^2+2ax-x^2$.

23. Divide $3a^4-8a^2b^2+3a^2c^2+5b^4-3b^2c^2$ by a^2-b^2.

24. Divide $x^6-3x^4y^2+3x^2y^4-y^6$ by $x^3-3x^2y+3xy^2-y^3$.

MISCELLANEOUS EXERCISES.

1. $3a+5x-9c+7d+5a-3x-3d-(4a+2x-8c+4d)$=what?
2. $a+b-(2a-3b)-(5a+7b)-(-13a+2b)$=what?
3. $(a+b)(a+b)+(a-b)(a-b)$=what? Ans. $2a^2+2b^2$.
4. $(a^2+a^4+a^6)(a^2-1)-(a^4+a)(a^4-a)$=what?
5. $(a^3+a^2b-ab^2-b^3)\div(a-b)-(a-b)(a-b)=$

II. ALGEBRAIC THEOREMS,

DERIVED FROM MULTIPLICATION AND DIVISION.

80. If we square $a+b$, that is, multiply $a+b$ by itself, the product will be $a^2+2ab+b^2$.

$$\begin{array}{r} \text{Thus: } a+b \\ a+b \\ \hline a^2+ab \\ +ab+b^2 \\ \hline a^2+2ab+b^2 \end{array}$$

But $a+b$ is the sum of the quantities, a and b. Hence,

Theorem I.—*The square of the sum of two quantities is equal to the square of the first, plus twice the product of the first by the second, plus the square of the second.*

NOTE.—Let the pupil apply the theorem by writing the following examples, enunciated thus: What is the square of $2+3$?

1. $(2+3)^2=4+12+9=25$.
2. $(2a+b)^2=4a^2+4ab+b^2$.
3. $(2x+3y)^2=4x^2+12xy+9y^2$.
4. $(ab+cd)^2=a^2b^2+2abcd+c^2d^2$.
5. $(x^2+xy)^2=x^4+2x^3y+x^2y^2$.
6. $(2a^2+3ax)^2=4a^4+12a^3x+9a^2x^2$.

81. If we square $a-b$, that is, multiply $a-b$ by itself, the product will be $a^2-2ab+b^2$.

$$\begin{array}{l} \text{Thus: } a-b \\ \phantom{\text{Thus: }} \underline{a-b} \\ \phantom{\text{Thus: }} a^2-ab \\ \phantom{\text{Thus: }} \underline{\quad -ab+b^2} \\ \phantom{\text{Thus: }} a^2-2ab+b^2 \end{array}$$

But $a-b$ is the difference of the quantities, a and b. Hence,

Theorem II.—*The square of the difference of two quantities is equal to the square of the first, minus twice the product of the first by the second, plus the square of the second.*

1. $(5-4)^2=25-40+16=1$.
2. $(2a-b)^2=4a^2-4ab+b^2$.
3. $(3x-2y)^2=9x^2-12xy+4y^2$.
4. $(x^2-y^2)^2=x^4-2x^2y^2+y^4$.
5. $(ax-x^2)^2=a^2x^2-2ax^3+x^4$.
6. $(5a^2-b^2)^2=25a^4-10a^2b^2+b^4$.

82. If we multiply $a+b$ by $a-b$, the product will be a^2-b^2.

$$\begin{array}{l} \text{Thus: } a+b \\ \phantom{\text{Thus: }} \underline{a-b} \\ \phantom{\text{Thus: }} a^2+ab \\ \phantom{\text{Thus: }} \underline{\quad -ab-b^2} \\ \phantom{\text{Thus: }} a^2-b^2 \end{array}$$

But $a+b$ represents the sum of two quantities, and $a-b$, their difference. Hence,

Theorem III.—*The product of the sum and difference of two quantities is equal to the difference of their squares.*

1. $(5+3)(5-3)=25-9=16=8\times2$.
2. $(2a+b)(2a-b)=4a^2-b^2$.
3. $(2x+3y)(2x-3y)=4x^2-9y^2$.
4. $(5a+4b)(5a-4b)=25a^2-16b^2$.
5. $(a^2+b^2)(a^2-b^2)=a^4-b^4$.
6. $(2am+3bn)(2am-3bn)=4a^2m^2-9b^2n^2$.

83. If we divide a^3 by a^5, observing the rule for the exponents, we have $\frac{a^3}{a^5}=a^{3-5}=a^{-2}$. But, Art. 127, $\frac{a^3}{a^5}=\frac{1}{a^2}$.

So, $\frac{a^m}{a^n}=a^{m-n}$; and $\frac{a^m}{a^n}=\frac{1}{a^{n-m}}$. Hence, $a^{m-n}=\frac{1}{a^{n-m}}$.

Also, $\frac{a}{b^m}=a\times\frac{1}{b^m}=ab^{-m}$; $\frac{a^m}{b^n}=a^mb^{-n}$; $\frac{a}{b}=ab^{-1}$; $\frac{1}{ab^2}=a^{-1}b^{-2}$; $\frac{ab^3}{c}=\frac{a}{b^{-3}c}$, etc. Hence,

Theorem IV.—**1.** *The reciprocal of a quantity is equal to the same quantity with the sign of its exponent changed.*

2. *Any quantity may be transferred from one term of a fraction to the other, if the sign of the exponent be changed.*

Thus: . . . . $\frac{a}{b}=ab^{-1}=\frac{b^{-1}}{a^{-1}}=\frac{1}{a^{-1}b}$;

$$\frac{a^2b^2}{c^2d^3}=a^2b^2c^{-2}d^{-3}=\frac{1}{a^{-2}b^{-2}c^2d^3}=\frac{c^{-2}d^{-3}}{a^{-2}b^{-2}}.$$

84. By the rule for the exponents, Art. 74, $\frac{a^2}{a^2}=a^{2-2}=a^0$; but since any quantity is contained in itself once, $\frac{a^2}{a^2}=1$.

Similarly, $\frac{a^m}{a^m}=a^{m-m}=a^0$; but $\frac{a^m}{a^m}=1$; therefore, $a^0=1$, since each is equal to $\frac{a^m}{a^m}$. Hence,

Theorem V.—*Any quantity whose exponent is 0 is equal to unity.*

85. If we divide a^2-b^2 by $a-b$, the quotient will be $a+b$. If we divide a^3-b^3 by $a-b$, the quotient will be a^2+ab+b^2.

In the same manner, we should find, by trial, that the quotients obtained by dividing the difference of the same powers of two quantities by the difference of those quantities, follow a simple law.

Thus: $(a^2-b^2)\div(a-b)=a+b.$
$(a^3-b^3)\div(a-b)=a^2+ab+b^2.$
$(a^4-b^4)\div(a-b)=a^3+a^2b+ab^2+b^3.$
$(a^5-b^5)\div(a-b)=a^4+a^3b+a^2b^2+ab^3+b^4.$
So, $(a^5-1)\div(a-1)=a^4+a^3+a^2+a+1.$
And $(1-b^5)\div(1-b)=1+b+b^2+b^3+b^4.$

The exponent of the first letter decreases by unity, while that of the second increases by unity. Hence, we have

Theorem VI.—*The difference of the same powers of two quantities is always divisible by the difference of the quantities.*

86. The two following theorems may also be readily shown to be true by trial:

Theorem VII.—*The difference of the even powers of two quantities of the same degree, is always divisible by the sum of the quantities.*

Thus: $(a^2-b^2)\div(a+b)=a-b.$
$(a^4-b^4)\div(a+b)=a^3-a^2b+ab^2-b^3.$
$(a^6-b^6)\div(a+b)=a^5-a^4b+a^3b^2-a^2b^3+ab^4-b^5.$
So, $(a^6-1)\div(a+1)=a^5-a^4+a^3-a^2+a-1.$
And $(1-b^6)\div(1+b)=1-b+b^2-b^3+b^4-b^5.$

REVIEW.—80. To what is the square of the sum of two quantities equal? 81. Of the difference of two quantities? 82. The product of the sum and difference of two quantities?

83. How may the reciprocal of any quantity be expressed? How may any quantity be transferred from one term of a fraction to the other? In what other form may a^m be written? a^{-m}?

84. What is the value of any quantity whose exponent is zero?

Theorem VIII.—*The sum of the odd powers of two quantities of the same degree, is always divisible by the sum of the quantities.*

Thus: $(a^3+b^3)\div(a+b)=a^2-ab+b^2$.

$(a^5+b^5)\div(a+b)=a^4-a^3b+a^2b^2-ab^3+b^4$.

$(a^7+b^7)\div(a+b)=a^6-a^5b+a^4b^2-a^3b^3+a^2b^4-ab^5+b^6$.

So, $(a^7+1)\div(a+1)=a^6-a^5+a^4-a^3+a^2-a+1$.

And $(1+a^7)\div(1+a)=1-a+a^2-a^3+a^4-a^5+a^6$.

NOTE.—For a complete demonstration of Theorems vi, vii, and viii, see RAY'S ALGEBRA, SECOND BOOK, Arts. 83, 84, 85, and 86.

FACTORING.

FACTORS, AND DIVISORS OF ALGEBRAIC QUANTITIES.

87. A **Divisor** or **Measure** of a quantity is any quantity that divides it without a remainder.

Thus, 2 is a divisor or measure of 6, and a^2 of a^2x.

88. A **Prime Number** is one which has no divisors except itself and unity.

A **Composite Number** is one which has one or more divisors besides itself and unity. Hence,

All numbers are either prime or composite; and every composite number is the product of two or more prime numbers.

The prime numbers are 1, 2, 3, 5, 7, 11, 13, 17, etc.

The composite numbers are 4, 6, 8, 9, 10, 12, 14, 15, 16, etc.

TO RESOLVE A COMPOSITE NUMBER INTO ITS PRIME FACTORS,

Rule.—1. *Divide by any prime number that will exactly divide it; divide the quotient again in the same manner, and so continue.*

REVIEW.—85. By what is the difference of the same powers of two quantities always divisible? 86. The difference of the even powers of the same degree? The sum of the odd powers?

2. *The last quotient and the several divisors will be the prime factors.*

REMARK.—It is most convenient to divide first by the smallest prime number that is a factor.—See RAY'S ARITHMETIC, THIRD BOOK, FACTORING.

1. What are the prime factors of 105? Ans.
2. What are the prime factors of 210? Ans.
3. Resolve 4290 into its prime factors. Ans.

89. A **Prime Quantity,** in Algebra, is one which is exactly divisible only by itself and by unity.

Thus, a, b, and $b+c$ are prime quantities.

90. Two quantities are *prime to each other* when no quantity except unity will exactly divide both.

Thus, ab and cd are prime to each other.

91. A **Composite Quantity** is one which is the product of two or more factors, neither of which is unity.

Thus, ax is a composite quantity, the factors being a and x.

92. To separate a monomial into its prime factors,

Rule.—*Resolve the coëfficient into its prime factors. To these annex the literal factors.*

Find the prime factors of the following monomials:

1. $15a^2bc$. . . . . Ans.
2. $21ab^2d$. . . . . Ans.
3. $35abc^2x$. . . . . Ans.

REVIEW.—87. What is the divisor of a quantity? 88. A prime number? A composite number? Name several of the prime numbers, beginning with unity. Composite numbers, beginning with 4. Rule for resolving any composite number into its prime factors?

89. What is a prime quantity? Example. 90. When are two quantities prime to each other? Example.

91. What is a composite quantity? Example. 92. Rule for separating a monomial into its prime factors?

93. To separate a polynomial into its factors when one of them is a monomial and the other a polynomial.

Rule.—*Divide the given quantity by the greatest monomial that will exactly divide each of its terms; the divisor will be one factor and the quotient the other.*

Separate the following expressions into factors:

1. $x+ax$. . . . .
2. $am+ac$. . . . .
3. bc^2+bcd. . . . . . .
4. $4x^2+6xy$. . . . . . .
5. $6ax^2y+9bxy^2-12cx^2y$. . .
6. $5ax^2-35ax^3y+5a^2x^3y$. . .
7. $a^3cm^2+a^2c^2m^2-a^2cm^3$. . .

94. To separate a quantity which is the product of two or more polynomials into its prime factors.

No general rule can be given for this case. When the given quantity does not consist of more than three terms, it may generally be accomplished by reversing some one of the preceding theorems.

1st. For a trinomial, whose extremes are squares and positive, and whose middle term is twice the product of the square roots of the extremes, reverse Arts. 80 and 81.

Thus: $a^2+2ab+b^2=(a+b)(a+b)$.
$a^2-2ab+b^2=(a-b)(a-b)$.

2d. For a binomial, which is the difference of two squares, verse Art. 82.

Thus: $a^2-b^2=(a+b)(a-b)$.

3d. For the difference of the same powers of two quantities, reverse Art. 85.

Thus: $a^3-b^3=(a-b)(a^2+ab+b^2)$.
$a^5-b^5=(a-b)(a^4+a^3b+a^2b^2+ab^3+b^4)$.

4th. The difference of the even powers of two quantities, higher than the second degree, may be separated thus:

$a^4-b^4=(a^2+b^2)(a^2-b^2)=(a^2+b^2)(a+b)(a-b)$.
$a^6-b^6=(a^3+b^3)(a^3-b^3)=(a+b)(a^2-ab+b^2)(a-b)(a^2+ab+b^2)$.

5th. For the sum of the odd powers of two quantities, see Art. 86.

Thus: $a^3+b^3=(a+b)(a^2-ab+b^2)$.

$a^7+b^7=(a+b)(a^6-a^5b+a^4b^2-a^3b^3+a^2b^4-ab^5+b^6)$.

Separate the following into their simplest factors:

1. $x^2+2xy+y^2$.
2. $9a^2+12ab+4b^2$.
3. $4+12x+9x^2$.
4. $m^2-2mn+n^2$.
5. $4x^2-20xz+25z^2$.
6. x^2-y^2.
7. $9m^2-16n^2$.
8. x^4-b^4.
9. y^3+1.
10. x^3-1.
11. $8a^3-27b^3$.
12. a^5+b^5.

ANSWERS.

1.
2.
3.
4.
5.
6.
7.
8.
9.
10.
11.
12.

95. To separate a quadratic trinomial into its factors.

A **Quadratic Trinomial** is of the form x^2+ax+b, in which the signs of the second and third terms may be either plus or minus.

Such quantities may be resolved into two binomial factors by inspection.

Thus: x^2-5x+6 will evidently be the product of $x-2$ by $x-3$.

REVIEW.—93. Rule for separating a polynomial into its prime factors, when one of them is a monomial and the other a polynomial. 94. When can a trinomial be separated into two binomial factors?

94. What are the factors of $m^2+2mn+n^2$? Of $c^2-2cd+d^2$? When can a binomial be separated into two binomial factors? What the factors of x^2-y^2? Of $9a^2-16b^2$? What is one of the factors of a^2-b^2? Of a^3-b^3? Of x^4-y^4? What are two of the factors of a^4-b^4? Of a^6-b^6?

Decompose each of the following trinomials into two binomial factors:

1. x^2+5x+6. . . . . Ans.
2. $a^2+7a+12$. . . . . Ans.
3. x^2-5x+6. . . . . Ans.
4. x^2+x-6. . . . . Ans.
5. x^2+x-2. . . . . Ans.
6. $x^2-13x+40$. . . . . Ans.
7. x^2-7x-8. . . . . Ans.
8. x^2-x-30. . . . . Ans.

We may often separate other trinomials into factors by first taking out the monomial factor common to each term.

Thus, $5ax^2-10ax-40a=5a(x^2-2x-8)=5a(x-4)(x+2)$.

9. $3x^2+12x-15$. . . . . Ans.
10. $2abx^2-14abx-60ab$. . Ans.
11. $2x^3-4x^2-30x$. . . . . Ans.

96. The principal use of factoring is to shorten algebraic operations by canceling common factors.

Whenever there is an opportunity of doing this, the operations to be performed should be merely indicated as in the following examples:

1. Multiply $a-b$ by $x^2+2xy+y^2$, and divide the product by $x+y$.

$$\frac{(a-b)(x^2+2xy+y^2)}{x+y}=\frac{(a-b)(x+y)(x+y)}{x+y}=(a-b)(x+y)$$
$$=ax+ay-bx-by.$$

2. Multiply $x-3$ by x^2-1, and divide the product by $x-1$, by factoring.

3. Divide z^3+1 by $z+1$, and multiply the quotient by z^2-1, by factoring.

4. Multiply x^2-5x+6 by $x^2-7x+12$, and divide the quotient by x^2-6x+9, by factoring.

REVIEW.—94. What is one of the factors of a^3+b^3? What is one of the factors of x^5+y^5? 95. What is a quadratic trinomial?

GREATEST COMMON DIVISOR.

97. A **Common Divisor,** or **Common Measure,** is any quantity that will exactly divide two or more quantities.

Thus, 2 is a common divisor of 8 and 12; and a is a common divisor of ab and a^2x.

REMARK.—Two quantities may sometimes have more than one common divisor. Thus, 8 and 12 have two common divisors, 2 and 4.

98. The **Greatest Common Divisor,** or **Greatest Common Measure,** of two or more quantities, is the greatest quantity that will exactly divide each of them.

Thus, the greatest common divisor of $4a^2xy$ and $6a^3x^2y^2$ is $2a^2xy$.

99. Quantities that have a common divisor are said to be *commensurable;* and those that have no common divisor *incommensurable.*

NOTE.—G.C.D. stands for greatest common divisor.

100. *To find the G.C.D. of two or more monomials.*

1. Let it be required to find the G.C.D. of $6ab$ and $15a^2c$.

By separating each quantity into its prime factors, we have $6ab=2\times3ab$, $15a^2c=3\times5aac$.

Here, 3 and a are the only factors common to both terms; hence, both can be divided either by 3 or a, or by their product $3a$, and by no other quantity; consequently, $3a$ is their G.C.D. Hence,

TO FIND THE GREATEST COMMON DIVISOR OF TWO OR MORE MONOMIALS,

Rule.—1. *Resolve the quantities into their prime factors.*

2. *Take the product of those factors that are common to each of the terms for the greatest common divisor.*

NOTE.—The G.C.D. of the literal parts will be the highest power of each letter which is common to all the quantities.

2. Find the G.C.D. of $4a^2x^3$, $6a^3x^2$, and $10a^4x$.

$4a^2x^3=2\times2a^2x^3$
$6a^3x^2=2\times3a^3x^2$
$10a^4x=2\times5a^4x$

Here, we see, that 2, a^2, and x are the only factors common to all the quantities. Hence, $2a^2x$ is the G.C.D.

Find the G.C.D. of the following quantities:

3. $4a^2x^2$, and $10ax^3$. . . . .
4. $4a^3b^2x^5y^3$, and $8a^5x^2y^2$. . . . .
5. $8ax^2y^4z^5$, $12x^5z^6$, and $24a^3x^3z^2$. . . .
6. $6a^2xy^2$, $12a^3y^4z^5$, $9a^5x^3y^4$, and $24a^3y^6z$. .

101. To find the G.C.D. of two polynomials.

Previous to the investigation of this subject, it will be necessary to state the following propositions:

Proposition I.—*A measure or divisor of a quantity is also a measure of any number of times that quantity.*

Thus, if 3 is a measure of 6, it is a measure of 2×6 or 12, of 3×6 or 18, etc.

Proposition II.—*A common measure of two quantities is also a measure of their sum.*

Thus, if 3 is a common measure of 27 and 18, it is also a measure of $27+18$ or 45, since $27+18$ divided by $3=9+6=15$, and $45\div3=15$.

Proposition III.—*A common measure of two quantities is also a measure of their difference.*

Thus, if 3 measures 27 and 18, it will also measure $27-18$ or 9, since $27-18$ divided by $3=9-6=3$, and $9\div3=3$. It is also evident that the greatest common measure of 27 and 18 is also the greatest common measure of 27, 18, and 9; that is, of the quantities themselves and their difference.

102. Let it be required to find the G.C.D. of 36 and 116.

If we divide 116 by 36, and there is no remainder, 36 is evidently the G.C.D., since it can have no divisor greater than itself. Dividing 116 by 36, we find the quotient is 3, and the remainder is 8, which is necessarily less than either of the quantities 116 and 36, and by PROP. 3, Art. 101, is exactly divisible by their G.C.D.; hence, their G.C.D. must divide 116, 36, and 8.

$$\begin{array}{l} 36)116(3 \\ \quad 108 \\ \quad\overline{\quad 8)36(4} \\ \qquad\quad 32 \\ \qquad\quad \overline{\ 4)8(2} \\ \qquad\qquad 8 \\ \qquad\qquad \overline{0} \end{array}$$

Now, if 8 will exactly divide 36, it will also exactly divide 116, since it must divide 3×36 or 108, and also $108+8$ or 116. PROP'S 1 and 2, Art. 101, and will be the G.C.D. sought. As it does not, it remains to find the G.C.D. of 36 and 8.

Dividing 36 by 8, the quotient is 4, with a remainder 4. Reasoning as before, the G.C.D. must divide 36, 8, and 4, and can not be greater than 4.

Now, if 4 will exactly divide 8, it will also exactly divide 36, since $36 = 4 \times 8 + 4$, and will be the G.C.D. sought. By trial, we find that 4 will divide 8, leaving no remainder.

Since, then, the G.C.D. of 116 and 36 must be the G.C.D. of 116, 36, 8, and 4, and since 4 has been shown to be a measure of 8, 36, and 116, and since 4 is the greatest measure of itself, it follows that 4 is the number sought.

103. Suppose, now, that it is required to find the G.C.D. of two polynomials, A and B, of which A is the greater.

Let the successive quotients and remainders be represented by Q, Q′, Q″, etc., and R, R′, R″, etc.

$$\begin{array}{l} B)A(Q \\ \quad BQ \\ \quad\overline{\ R)B(Q'} \\ \qquad RQ' \\ \qquad\overline{\ R')R(Q''} \\ \qquad\quad R'Q'' \\ \qquad\quad \overline{\quad 0} \end{array}$$

By the same process of reasoning as in the example above, it may be shown that R′ being the greatest measure of itself, and also a measure of R, is the greatest common measure of $R \times Q' + R'$ which is equal to B, and of $B \times Q + R$, which is equal to A, or that it is the G.C.D. of A and B.

REVIEW.—95. How can a quadratic trinomial be separated into binomial factors? 96. What is the principal use of factoring? 97. What is a common divisor of two or more quantities? Example.

98. What is the G.C.D. of two quantities? Example. 100. How find the G.C.D. of two or more monomials? 101. State the three propositions in Art. 101, and illustrate them.

103. To whatever extent the division is carried, the process of reasoning is the same, and the last divisor will be the G.C.D. When this last divisor is unity, or does not contain the letter of arrangement, there is no common divisor to the quantities.

104. If one of the quantities contain a factor not found in the other, it may be canceled without affecting the common divisor. (See Exam. 3.)

If both quantities contain a common factor, it may be set aside as a factor of the common divisor; and we may proceed to find the G.C.D. of the other factors of the given quantities. (See Exam. 2.)

Thus, in the fraction $\frac{2abx}{3abc}$, the G.C.D. of the two terms is evidently ab. If we cancel 2 or x in the numerator, or 3 or c in the denominator, ab is still the common divisor.

Again, in the fraction $\frac{27ab}{9ax}$, a is a part of the common divisor. Setting this aside, and finding the common divisor of $\frac{27b}{9x}$, which is 9, we have for the G.C.D. of the original fraction $a\times9=9a$.

105. We may multiply either quantity by a factor not found in the other, without affecting the G.C.D.

Thus, in the fraction $\frac{2abx}{3abc}$, whose G.C.D. is ab, if we multiply the dividend by 4, a factor not found in the divisor, we have $\frac{8abx}{3abc}$, of which the common divisor is still ab.

In like manner, if we multiply the divisor by any factor not found in the dividend, the common divisor will remain the same.

If, however, we multiply the numerator by 3, which is a factor of the denominator, the result is $\frac{6abx}{3abc}$, of which the G.C.D. is $3ab$, and not ab as before.

Hence, the G.C.D. may be changed by multiplying one of the quantities by a factor of the other.

106. In the general demonstration, Art's 101, 102, it has been shown that the G.C.D. of two quantities exactly divides each of the successive remainders. Hence, the preceding principles likewise apply to the successive remainders.

1. Find the G.C.D. of x^3-y^3 and $x^4-x^2y^2$.

Here, the second quantity contains x^2 as a factor, but it is not a factor of the first; we may, therefore, cancel it, and the second quantity becomes x^2-y^2. Divide the first by it.

After dividing, we find that y^2 is a factor of the remainder, but not of x^2-y^2, the dividend. Hence,

By canceling it, the divisor becomes $x-y$; then, dividing by this, we find there is no remainder; therefore, $x-y$ is the G.C.D.

$$\begin{array}{l|l} x^3-y^3 & x^2-y^2 \\ \underline{x^3-xy^2} & (x \\ \quad xy^2-y^3 & \\ \text{or, } (x-y)y^2 & \end{array}$$

$$\begin{array}{l|l} x^2-y^2 & x-y \\ \underline{x^2-xy} & (x+y \\ \quad xy-y^2 & \\ \quad \underline{xy-y^2} & \end{array}$$

2. Find the G.C.D. of $x^6+a^3x^3$ and $x^4-a^2x^2$.

The factor x^2 is common to both these quantities; it therefore forms part of the G.C.D., and may be taken out and reserved. Doing this, the quantities become x^4+a^3x and x^2-a^2.

The first quantity still contains a common factor, x, which the latter does not; canceling this, it becomes x^3+a^3. Then, proceeding as in the first example, we find the G.C.D. is $x^2(x+a)$.

$$\begin{array}{l|l} x^3+a^3 & x^2-a^2 \\ \underline{x^3-a^2x} & (x \\ \quad a^2x+a^3 & \\ \text{or, } (x+a)a^2 & \end{array}$$

$$\begin{array}{l|l} x^2 \quad a^2 & x+a \\ \underline{x^2+ax} & (x-a \\ \quad -ax-a^2 & \\ \quad \underline{-ax-a^2} & \end{array}$$

3. Find the G.C.D. of $5a^5+10a^4x+5a^3x^2$ and $a^3x+2a^2x^2+2ax^3+x^4$.

Here, $5a^3$ is a factor of the first quantity only, and x of the second only.

Suppressing these factors, and proceeding as in the previous examples, we find $a+x$ is the G.C.D.

$$\begin{array}{l|l} a^3+2a^2x+2ax^2+x^3 & a^2+2ax+x^2 \\ \underline{a^3+2a^2x+ax^2} & (a \\ \qquad ax^2+x^3 & \\ \text{or, } (a+x)x^2 & \end{array}$$

$$\begin{array}{l|l} a^2+2ax+x^2 & a+x \\ \underline{a^2+ax} & (a+x \\ \quad ax+x^2 & \\ \quad \underline{ax+x^2} & \end{array}$$

4. Find the G.C.D. of $2a^4-a^2x^2-6x^4$ and $4a^5+6a^3x^2-2a^2x^3-3x^5$.

In solving this example, there are two instances in which it is necessary to multiply the dividend, in order that the coëfficient of the first term may be exactly divisible by the divisor. See Art. 105.

The G.C.D. is found to be $2a^2+3x^2$.

$$\begin{array}{l|l} 4a^5+6a^3x^2-2a^2x^3-3x^5 & 2a^4-a^2x^2-6x^4 \\ \hline 4a^5-2a^3x^2-12ax^4 & (2a \\ \hline \quad 8a^3x^2-2a^2x^3+12ax^4-3x^5 & \\ \text{or, } (8a^3-2a^2x+12ax^2-3x^3)x^2 & \end{array}$$

$$\begin{array}{l} 2a^4-a^2x^2-6x^4 \\ \qquad 4 \end{array}$$

$$\begin{array}{l|l} 8a^4-4a^2x^2-24x^4 & 8a^3-2a^2x+12ax^2-3x^3 \\ \hline 8a^4-2a^3x+12a^2x^2-3ax^3 & (a \\ \hline \quad 2a^3x-16a^2x^2+3ax^3-24x^4 & \\ \qquad\qquad\qquad 4 & \\ \hline \quad 8a^3x-64a^2x^2+12ax^3-96x^4 & (x \\ \quad 8a^3x-\ \ 2a^2x^2+12ax^3-\ \ 3x^4 & \\ \hline \qquad -62a^2x^2 \qquad -93x^4 & \\ \text{or, } -31x^2(2a^2+3x^2) & \end{array}$$

$$\begin{array}{l|l} 8a^3-2a^2x+12ax^2-3x^3 & 2a^2+3x^2 \\ \hline 8a^3 \qquad\quad +12ax^2 & (4a-x \\ \hline \quad -2a^2x \qquad\quad -3x^3 & \\ \quad -2a^2x \qquad\quad -3x^3 & \text{Hence,} \end{array}$$

TO FIND THE GREATEST COMMON DIVISOR OF TWO POLYNOMIALS,

Rule.—1. *Divide the greater polynomial by the less, and if there is no remainder, the less quantity will be the divisor sought.*

Review.—102. Explain the rule for finding the G.C.D. of two numbers, as illustrated. 103. When do we conclude that there is no common divisor to two quantities?

104. How is the common divisor of two quantities affected by canceling a factor in one of them, not found in the other? When both quantities contain a common factor, how may it be treated?

105. How is the G.C.D. of two quantities affected by multiplying either of them by a factor not found in the other?

106. Rule for finding the G.C.D. of two polynomials? How find the G.C.D. of three or more quantities?

2. *If there is a remainder, divide the first divisor by it, and continue to divide the last divisor by the last remainder, until a divisor is obtained which leaves no remainder; this will be the greatest common divisor of the two given polynomials.*

NOTES.—1. When the highest power of the *leading* letter is the same in both, it is immaterial which of the quantities is made the dividend.

2. If both quantities contain a common factor, let it be set aside, as forming a factor of the common divisor, and proceed to find the G.C.D. of the remaining factors, as in Ex. 2.

3. If either quantity contain a factor not found in the other, it may be canceled before commencing the operation, as in Ex. 3. See Art. 104.

4. When necessary, the dividend may be multiplied by any quantity which will render the first term divisible by the divisor. See Art. 105.

5. If, in any case, the remainder does not contain the leading letter, that is, if it is independent of that letter, there is no common divisor.

6. To find the G.C.D. of three or more quantities, first find the G.C.D. of two of them; then, of that divisor and one of the other quantities, and so on. The last divisor thus found will be the G.C.D. sought.

7. Since the G.C.D. of two or more quantities contains all the factors common to these quantities, it may often be found most easily by separating the quantities into factors.

Find the G.C.D. of the following quantities:

5. $5a^2+5ax$ and a^2-x^2. . . . .
6. x^3-a^2x and x^3-a^3. . . . .
7. x^3-c^2x and $x^2+2cx+c^2$. . . . . . .
8. x^2+2x-3 and x^2+5x+6. . . . .
9. $6a^2+11ax+3x^2$ and $6a^2+7ax-3x^2$.
10. a^4-x^4 and $a^3+a^2x-ax^2-x^3$. . . . .
11. $a^2-5ax+4x^2$ and $a^3-a^2x+3ax^2-3x^3$.
12. $a^2x^4-a^2y^4$ and $x^5+x^3y^2$. . . . . .
13. a^5-x^5 and $a^{13}-x^{13}$. . . . . . .

LEAST COMMON MULTIPLE.

107. A **Multiple** of a quantity is that which contains it exactly.

Thus, 6 is a multiple of 2, or of 3; and 24 is a multiple of 2, 3, 4, etc.; also, $8a^2b^3$ is a multiple of $2a$, of $2a^2$, of $2a^2b$, etc.

108. A **Common Multiple** of two or more quantities is one that will exactly contain each quantity.

Thus, 12 is a common multiple of 2 and 3; and $6ax$ is a common multiple of 2, 3, a, and x.

109. The **Least Common Multiple** of two or more quantities is the least quantity that will contain them exactly.

Thus, 6 is the least common multiple of 2 and 3; and $10xy$ is the least common multiple of $2x$ and $5y$.

NOTE.—L.C.M. stands for *least common multiple.*

REMARK.—Two or more quantities can have but *one* L.C.M., while they may have an unlimited number of common multiples.

110. To find the L.C.M. of two or more quantities.

It is evident that one quantity will not contain another exactly, unless it contains all of its prime factors.

Thus, 30 does not exactly contain 14, because $30=2\times3\times5$, and $14=2\times7$; the prime factor 7 not being one of the prime factors of 30. It contains 6, because $6=2\times3$, the prime factors 2 and 3 being common to both numbers.

111. In order that any quantity may exactly contain two or more quantities, it must contain all the different prime factors of those quantities. And, to be the *least quantity* that shall exactly contain them, *it should contain these different prime factors only once, and no other factors beside.*

Thus, the L.C.M. of a^2bc and acx, is a^2bcx, since it contains all the factors in each of these quantities, and does not contain any other factor.

With this principle, let us find the L.C.M. of ax, bx, and abc.

a	ax	bx	abc
x	x	bx	bc
b	1	b	bc
	1	1	c

Arranging the quantities as in the margin, we see that a is a factor common to two of the terms; hence, it must be a factor of the L.C.M., and we place it on the left.

We then cancel this factor in each of the quantities in which it is found, by dividing by it, so that it may occur but once. As x is a common factor in the first and second terms, we divide by x for the same reason, and then by b.

We thus find, that a, x, b, and c are all the prime factors in the given quantities; therefore, their product, $abcx$, will be the L.C.M. of these quantities. Hence,

TO FIND THE LEAST COMMON MULTIPLE OF TWO OR MORE QUANTITIES,

Rule.—1. *Arrange the quantities in a horizontal line.*

2. *Divide them by any prime factor that will divide two or more of them without a remainder, and set the quotients, together with the undivided quantities, in a line beneath.*

3. *Continue dividing as before, until no prime factor, except unity, will divide two or more of the quantities without a remainder.*

4. *Multiply the divisors and the quantities in the last line*

REVIEW.—107. What is a multiple of a quantity? Example. 108. A common multiple of two or more quantities? Example.

109. What is the L.C.M. of two or more quantities? Example. How many common multiples may a quantity have?

110. When is one quantity not contained exactly in another? Example. 111. When contained exactly? Example. What is necessary, in order that one quantity may exactly contain two or more quantities?

111. What is necessary, in order that any quantity may be the least, that shall contain two or more quantities exactly? What factors does the L.C.M. of two or more quantities contain? Rule for finding the L.C.M.? For finding it by separating into factors?

together, and the product will be the least common multiple required.

Or, *Separate the quantities into their prime factors; then, to form a product,* 1st, *take each factor once;* 2d, *if any factor occurs more than once, take it the greatest number of times it occurs in either of the quantities.*

112. Since the G.C.D. of two quantities contains all the factors common to them, it follows, that *if we divide the product of two quantities by their G.C.D., the quotient will be their L.C.M.*

Find the L.C.M. in each of the following examples:

1. $4a^2$, $3a^3x$, and $6ax^2y^3$. . . .
2. $12a^2x^2$, $6a^3$, and $8x^4y^2$. . . .
3. 15, $6xz^2$, $9x^2z^4$, and $18cx^3$. .
4. $4a^2(a-x)$, and $6ax^4(a^2-x^2)$.
5. $10a^2x^2(x-y)$, $15x^5(x+y)$, and $12(x^2-y^2)$.

GENERAL REVIEW.

NOTE.—These GENERAL REVIEWS are not intended to be full and exhaustive, but simply suggestive to the teacher, who can extend the questions, making them as thorough and complete as is deemed desirable.

Define mathematics. Quantity. Algebra. Arithmetic. When is quantity called magnitude? When multitude? Define a problem. Theorem. Known quantities. Unknown. How are known quantities represented? Unknown?

Name the principal signs used in algebra. Define factor. Coëfficient. Power. Exponent. Root. A monomial. Binomial. Trinomial. Polynomial. Residual quantity. Numerical value. Dimension of a term. Degree. Reciprocal of a quantity.

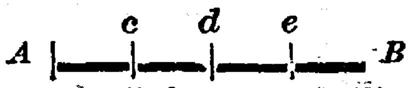
Define addition. Algebraic subtraction. What the difference between algebraic and arithmetical subtraction? Define multiplication. Division. What the rule of the Coëfficients? Rule of the exponents? Rule for the signs? Illustrate Theorem I. Theorem II.; III.; IV.; V.; VI.; VII.; VIII.

What the divisor of a quantity? A prime number? Composite? A prime quantity? Composite? Quadratic trinomial? A common divisor? Greatest common divisor? Illustrate Proposition I.; II.; III. Rule for the G.C.D. Define a multiple. Common multiple. Least common multiple. Rule for the L.C.M.

III. ALGEBRAIC FRACTIONS.

DEFINITIONS AND FUNDAMENTAL PROPOSITIONS.

113. A **Fraction** is an expression representing one or more of the equal parts into which a unit is supposed to be divided.



Thus, if the line *A B* be supposed to represent 1 foot, and it be divided into 4 equal parts, 1 of those parts, as *Ac*, is called one fourth $(\frac{1}{4})$; 2 of them, as *Ad*, are called two fourths $(\frac{2}{4})$; and 3 of them, as *Ae*, are called three fourths $(\frac{3}{4})$.

In the algebraic fraction $\frac{1}{c}$, if $c=4$ and 1 denotes 1 foot, then $\frac{1}{c}$ denotes one fourth of a foot. In the fraction $\frac{a}{c}$, if $a=3$ and $\frac{1}{c}=\frac{1}{4}$ of a foot, then $\frac{a}{c}$ represents three fourths $(\frac{3}{4})$ of a foot.

114. An **Entire Algebraic Quantity** is one not expressed under the form of a fraction.

Thus, $ax+b$ is an entire quantity.

115. A **Mixed Quantity** is one composed of an entire quantity and a fraction.

Thus, $a+\frac{b}{x}$ is a mixed quantity.

116. An **Improper Algebraic Fraction is one whose numerator can be divided by the denominator, either with or without a remainder.**

Thus, $\frac{ab}{a}$ and $\frac{ax^2+b}{x}$ are improper fractions.

117. A **Simple Fraction** is a single factional expression; as, $\frac{1}{3}$, $\frac{a}{b}$, or $\frac{c}{d}$. It may be either proper or improper.

REVIEW.—112. If the product of two quantities be divided by their G.C.D., what will the quotient be?
113. What is a fraction? 114. An entire algebraic quantity? Example. 115. A mixed quantity? Example. 116. An improper algebraic fraction? Example.

118. A **Compound Fraction** is a fraction of a fraction; as, $\frac{1}{2}$ of $\frac{2}{3}$, or $\frac{m}{n}$ of $\frac{a}{b}$.

119. A **Complex Fraction** is one that has a fraction either in its numerator or denominator, or in both.

Thus, $\frac{2\frac{1}{2}}{4}$, $\frac{3\frac{1}{2}}{2\frac{1}{3}}$, $\frac{a+\frac{b}{c}}{d}$, and $\frac{a+\frac{b}{c}}{e+\frac{m}{n}}$, are complex fractions.

120. Algebraic fractions are represented in the same manner as common fractions in Arithmetic.

The **Denominator** is the quantity below the line, and is so called because it *denominates* or shows the number of parts into which the unit is divided.

The **Numerator** is the quantity above the line, and is so called because it *numbers* or shows how many parts are taken.

Thus, in the fraction $\frac{3}{4}$, it is understood that the unit is divided into 4 equal parts, and that three of these parts are taken: $\frac{a}{c}$ denotes that a unit is divided into c equal parts, and that a of these parts are taken.

The numerator and denominator are called the *terms* of a fraction.

121. In the preceding definitions of numerator and denominator, reference is had to a *unit* only. This is the simplest method of considering a fraction; but there is another point of view in which it is proper to examine it.

If required to divide 3 apples equally, between 4 boys, it can be effected by dividing each of the 3 apples into 4 equal parts, and

REVIEW.—117. What is a simple fraction? Example. 118. A compound fraction? Example. 119. A complex fraction? Example.
120. In fractions, what is the quantity below the line called? Why? Above the line? Why? Example. What are the terms of a fraction?

giving to each boy 3 parts from 1 apple, or 1 part from each of the 3 apples; that is, $\frac{3}{4}$ of 1 unit is the same as $\frac{1}{4}$ of 3 units.

Thus, $\frac{2}{5}$ may be regarded as expressing *two* fifths of *one* thing, or *one* fifth of *two* things.

So, $\frac{m}{n}$ is either the fraction $\frac{1}{n}$ of one unit taken m times, or it is the nth of m units. Hence, the numerator may be regarded as showing the *number of units* to be divided; and the denominator, as showing the divisor, or *what part is taken* from each.

122. Proposition I.—*If we multiply the numerator of a fraction without changing the denominator, the value of the fraction is increased as many times as there are units in the multiplier.*

If we multiply the numerator of the fraction $\frac{2}{7}$ by 3, without changing the denominator, it becomes $\frac{6}{7}$.

Now, $\frac{2}{7}$ and $\frac{6}{7}$ have the same denominator, which expresses parts of the same size; but the second fraction, $\frac{6}{7}$, having three times as many of those parts as the first, is three times as large. The same may be shown of any fraction whatever.

123. Proposition II.—*If we divide the numerator of a fraction without changing the denominator, the value of the fraction is diminished as many times as there are units in the divisor.*

If we take the fraction $\frac{4}{5}$, and divide the numerator by 2, without changing the denominator, it becomes $\frac{2}{5}$.

Now, $\frac{4}{5}$ and $\frac{2}{5}$ have the same denominator, which expresses parts of the same size; but the second fraction, $\frac{2}{5}$, having only one half as many of those parts as the first, $\frac{4}{5}$, is only one half as large. The same may be shown of other fractions.

124. Proposition III.—*If we multiply the denominator of a fraction without changing the numerator, the value of the fraction is diminished as many times as there are units in the multiplier.*

Review.—121. In what two different points of view may every fraction be regarded? Examples. 122. How is the value of a fraction affected by multiplying the numerator only? Give the proof.
123. How is the value of a fraction affected by dividing the numerator only? Give the proof.

If we take the fraction $\frac{3}{4}$, and multiply the denominator by 2, without changing the numerator, it becomes $\frac{3}{8}$.

Now, the fractions $\frac{3}{4}$ and $\frac{3}{8}$ have the same numerator, which expresses the same number of parts; but, in the second, the parts being only one half the size of those in the first, the value of the second fraction is only *one half* that of the first. The same may be shown of any fraction whatever.

125. Proposition IV.—*If we divide the denominator of a fraction without changing the numerator, the value of the fraction is increased as many times as there are units in the divisor.*

If we take the fraction $\frac{2}{9}$, and divide the denominator by 3 without changing the numerator, it becomes $\frac{2}{3}$.

Now, the fractions $\frac{2}{9}$ and $\frac{2}{3}$ have the same numerator, which expresses the same number of parts; but, in the second, the parts being three times the size of those of the first, the value of the second fraction is three times that of the first. The same may be shown of other fractions.

126. Proposition V.—*Multiplying both terms of a fraction by the same number or quantity, changes the form of a fraction, but does not alter its value.*

If we multiply the numerator of a fraction by any number, its value, Prop. I., is *increased*, as many times as there are units in the multiplier; and, if we multiply the denominator, the value, Prop. III., is *decreased*, as many times as there are units in the multiplier. Hence,

The increase is equal to the decrease, and the value remains unchanged.

127. Proposition VI.—*Dividing both terms of a fraction by the same number or quantity, changes the form of the fraction but not its value.*

If we divide the numerator of a fraction by any number, its value, Prop. II., is *decreased*, as many times as there are units in the

REVIEW.—124. How by multiplying only the denominator? How proved? 125. By dividing the denominator only? How proved?

126. How is a fraction affected by multiplying both terms by the same quantity? Why?

divisor; and, if we divide the denominator, the value, Prop. IV., is *increased* as many times. Hence,

The decrease is equal to the increase, and the value remains unchanged.

CASE I.

TO REDUCE A FRACTION TO ITS LOWEST TERMS.

128. Since the value of a fraction is not changed by dividing both terms by the same quantity, Art. 127, we have the following

Rule.—*Divide both terms by their greatest common divisor.*

Or, *Resolve the numerator and denominator into their prime factors, and then cancel those factors common to both terms.*

REMARK.—The last rule is generally most convenient.

1. Reduce $\frac{4ab^2}{6bx^2}$ to its lowest terms.

$$\frac{4ab^2}{6bx^2}=\frac{2ab\times 2b}{3x^2\times 2b}=\frac{2ab}{3x^2} \text{ Ans.}$$

Reduce the following fractions to their lowest terms:-

2. $\frac{4a^3x^2}{6a^4}$. . . . Ans.

3. $\frac{6a^2x^2}{8ax^3}$. . . . Ans.

4. $\frac{9x^4y^3z^5}{12x^3y^4z^5}$. . . Ans.

5. $\frac{12x^2y^2z^4}{8x^2z^3}$. . . Ans.

6. $\frac{2a^2cx^2+2acx}{10ac^2x}$. Ans.

7. $\frac{5a^2b+5ab^2}{5abc+5abd}$. Ans.

8. $\frac{12x^2y-18xy^2}{18x^2y+12xy^2}$. . . . . Ans.

NOTE.—In the preceding examples, the greatest common divisor in each is a monomial; in those which follow, it is a polynomial.

REVIEW.—127. How by dividing both terms by the same quantity? Why? 128. How reduce a fraction to its lowest terms?

9. $\frac{3a^3-3ab^2}{5ab+5b^2}$. This is equal to

$$\frac{3a(a^2-b^2)}{5b(a+b)}=\frac{3a(a+b)(a-b)}{5b(a+b)}=\frac{3a(a-b)}{5b}.\text{ Ans.}$$

10. $\frac{3z^3-24z+9}{4z^3-32z+12}$. Ans.

11. $\frac{n^2-2n+1}{n^2-1}$. Ans.

12. $\frac{x^3-xy^2}{x^4-y^4}$. Ans.

13. $\frac{x^2-y^2}{x^2-2xy+y^2}$. Ans.

14. $\frac{x^3-ax^2}{x^2-2ax+a^2}$. Ans.

15. $\frac{x^2+2x-15}{x^2+8x+15}$. Ans.

129. Exercises in division, Art. 76, in which the quotient is a fraction, and capable of being reduced to lower terms.

1. Divide $5x^2y$ by $3xy^2$. . . . . Ans.

2. Divide amn^2 by a^2m^2n. . . . . Ans.

So, also, when one or both of the quantities are polynomials.

3. Divide $3m^2+3n^2$ by $15m^2+15n^2$. . . . . . Ans.

4. Divide $x^3y^2+x^2y^3$ by ax^2y+axy^2. . . . . Ans.

5. Divide x^2+2x-3 by x^2+5x+6. . . . . Ans.

CASE II.

TO REDUCE A FRACTION TO AN ENTIRE OR MIXED QUANTITY.

130. Since the numerator of the fraction may be regarded as a dividend, and the denominator as a divisor, this is merely a case of division. Hence,

Rule.—*Divide the numerator by the denominator for the entire part; and, if there be a remainder, place it over the denominator for the fractional part.*

NOTE.—The fractional part should be reduced to its lowest terms.

Reduce the following to entire or mixed quantities:

1. $\frac{3ax+b^2}{x}$. . . . .
2. $\frac{ab+b^2}{a}$. . . . .
3. $\frac{a^2+x^2}{a-x}$. . . . .
4. $\frac{2a^2x-x^3}{a}$. . . . .
5. $\frac{4ax-2x^2-a^2}{2a-x}$ . . . . . .
6. $\frac{a^3+x^3-x^4}{a+x}$. . . . . . .
7. $\frac{12x^3-3x^2}{4x^3-x^2-4x+1}$. . . .

CASE III.

TO REDUCE A MIXED QUANTITY TO THE FORM OF A FRACTION.

181.—1. In $2\frac{1}{3}$, there are how many thirds?

In 1 unit there are 3 thirds; hence, in 2 units there are 6 thirds; then, $2\frac{1}{3}$ or $2+\frac{1}{3}=\frac{6}{3}+\frac{1}{3}=\frac{7}{3}$.

So, $a+\frac{b}{c}=\frac{ac}{c}+\frac{b}{c}=\frac{ac+b}{c}$; and $a-\frac{b}{c}=\frac{ac}{c}-\frac{b}{c}=\frac{ac-b}{c}$. Hence,

TO REDUCE A MIXED QUANTITY TO THE FORM OF A FRACTION,

Rule.—*Multiply the entire part by the denominator of the fraction. Add the numerator of the fractional part to this* *product, or subtract it, as the sign may direct, and place the result over the denominator.*

REMARK.—Cases II. and III. are the reverse of, and mutually prove each other.

Before proceeding further, it is important to consider

THE SIGNS OF FRACTIONS.

132. The signs prefixed to the terms of a fraction affect only those terms; but the sign placed before the dividing line of a fraction, affects its whole value.

Thus, in $-\frac{a^2-b^2}{x+y}$, the sign of a^2, in the numerator, is plus; of b^2, minus; while the sign of each term of the denominator is plus. But the sign of the fraction, taken as a whole, is minus.

By the rule for the signs in division, Art. 75, we have $\frac{+ab}{+a}=+b$; or, changing the signs of both terms, $\frac{-ab}{-a}=+b$.

Changing the sign of the numerator, we have $\frac{-ab}{+a}=-b$.

Changing the sign of the denominator, we have $\frac{+ab}{-a}=-b$. Hence,

The signs of both terms of a fraction may be changed without altering its value or changing its sign; but, if the sign of either term of a fraction be changed, and not that of the other, the sign of the fraction will be changed.

Hence, *The signs of either term of a fraction may be changed, without altering its value, if the sign of the fraction be changed at the same time.*

REVIEW.—130. How reduce a fraction to an entire or mixed quantity? 131. A mixed quantity to the form of a fraction?
132. What do the signs prefixed to the terms of a fraction affect? The sign placed before the whole fraction? What is the effect of changing the signs of both terms of a fraction? Of one term, and not the other? The sign of the fraction, and of one of its terms?

Thus, . . . $\frac{ax-x^2}{c}=\frac{ax-x^2}{-c}=-\frac{x^2-ax}{c}$.

And, . . . $a-\frac{a-x}{b}=a+\frac{a-x}{-b}=a+\frac{x-a}{b}$.

1. Reduce $3a+\frac{ax-a}{x}$ to a fractional form.

$3a=\frac{3ax}{x}$ and $\frac{3ax}{x}+\frac{ax-a}{x}=\frac{3ax+ax-a}{x}=\frac{4ax-a}{x}$. Ans.

2. Reduce $4a-\frac{a-b}{3c}$ to a fractional form.

$4a=\frac{12ac}{3c}$, and $\frac{12ac}{3c}-\frac{a-b}{3c}=\frac{12ac-(a-b)}{3c}=\frac{12ac-a+b}{3c}$. Ans.

Reduce the following quantities to improper fractions:

3. $2+\frac{3}{5}$ and $2-\frac{3}{5}$. . . . . . .

4. $5c+\frac{a-b}{2x}$. . . . .

5. $5c-\frac{a-b}{2x}$. . . . .

6. $3x-\frac{4x^2-5}{5x}$. . . . .

7. $8y+\frac{3a-y^2}{5y}$. . . . .

8. $z-1+\frac{1-z}{1+z}$. . . . . . .

9. $\frac{4y}{2x+z}-5$. . . . .

10. $\frac{3+5c}{8}-6$. . . . .

11. $a-x+\frac{a^2+x^2-5}{a+x}$. . . . . . .

12. $a^3-a^2x+ax^2-x^3-\frac{a^4+x^4}{a+x}$. . .

CASE IV.

TO REDUCE FRACTIONS OF DIFFERENT DENOMINATORS TO EQUIVALENT FRACTIONS HAVING A COMMON DENOMINATOR.

133.—1. Reduce $\frac{a}{b}$ and $\frac{c}{d}$ to a common denominator.

Multiply both terms of the first fraction by d, the denominator of the second, and both terms of the second fraction by b, the denominator of the first. We shall then have $\frac{ad}{bd}$ and $\frac{bc}{bd}$.

In this solution, observe; *first*, the values of the fractions are not changed, since, in each, both terms are multiplied by the same quantity; and,

Second, the denominators must be the same, since they consist of the product of the same quantities.

2. Reduce $\frac{a}{m}$, $\frac{b}{n}$, and $\frac{c}{r}$, to a common denominator.

Here, we multiply both terms of each fraction by the denominators of the other two fractions. Thus,

$$\frac{a\times n\times r}{m\times n\times r}=\frac{anr}{mnr};\quad \frac{b\times m\times r}{n\times m\times r}=\frac{bmr}{mnr};\quad \frac{c\times m\times n}{r\times m\times n}=\frac{cmn}{mnr}.$$

It is evident that the value of each fraction is not changed, and that they have the same denominators. Hence,

TO REDUCE FRACTIONS TO A COMMON DENOMINATOR,

Rule.—*Multiply both terms of each fraction by the product of all the denominators except its own.*

Remark.—This is the same as to multiply each numerator by all the denominators except its own, for the new numerators; and all the denominators together, for the common denominator.

Review.—133. How do you reduce fractions of different denominators to equivalent fractions having the same denominator?
Why is the value of each fraction not changed by this process?
Why does this process give to each fraction the same denominator?

Reduce to fractions having a common denominator:

3. $\frac{a}{b}$, $\frac{c}{d}$, and $\frac{1}{2}$.

4. $\frac{x}{y}$, and $\frac{x+a}{c}$.

5. $\frac{2}{3}$, $\frac{3a}{4}$, and $\frac{x-y}{b}$.

6. $\frac{2x}{3y}$, $\frac{3x}{5z}$, and a.

7. $\frac{x+y}{x-y}$, and $\frac{x-y}{x+y}$.

8. a, $\frac{3b}{c}$, d, and 5.

134. When the denominators of the fractions to be reduced contain one or more common factors, the preceding rule does not give the *least* common denominator.

If we find the L.C.M. of all the denominators, and divide it by the denominators severally, it is easy to see that we shall obtain multipliers for each of the fractions, which will, without changing their value, make their denominators the same as the L.C.M.

1. Reduce $\frac{m}{b}$, $\frac{n}{bc}$, and $\frac{r}{cd}$, to equivalent fractions having the least common denominator.

The L.C.M. of the denominators is bcd; dividing this by b, bc, and cd, we obtain cd, d, and b. Multiplying both terms by these severally, we have $\frac{mcd}{bcd}$, $\frac{nd}{bcd}$, and $\frac{br}{bcd}$; or thus:

$$bcd \div b = cd, \text{ and } \frac{m}{b} \times \frac{cd}{cd} = \quad \ldots \quad \frac{mcd}{bcd}.$$

$$bcd \div bc = d, \text{ and } \frac{n \times d}{bc \times d} = \quad \ldots \quad \frac{nd}{bcd}.$$

$$bcd \div cd = b, \text{ and } \frac{r \times b}{cd \times b} = \quad \ldots \quad \frac{br}{bcd}$$

The process of multiplying the denominators may be omitted, as the product in each case is the same. Hence,

TO REDUCE FRACTIONS OF DIFFERENT DENOMINATORS TO EQUIVALENT FRACTIONS HAVING THE LEAST COMMON DENOMINATOR,

Rule.—1. *Find the least common multiple of all the denominators; this will be the common denominator.*

2. *Divide the least common multiple by the first of the given denominators, and multiply the quotient by the first of the given numerators; the product will be the first of the required numerators.*

3. *Proceed, in a similar manner, to find each of the other numerators.*

NOTE.—Each fraction should first be reduced to its lowest terms.

Reduce the following to equivalent fractions having the least common denominator:

2. $\frac{2a}{3bc}$, $\frac{3x}{cd}$, and $\frac{5y}{6bd}$.

3. $\frac{m}{ac}$, $\frac{n}{b^2c}$, $\frac{r}{c^2d}$.

4. $\frac{x+y}{x-y}$, $\frac{x-y}{x+y}$, $\frac{x^2+y^2}{x^2-y^2}$.

NOTE.—The two following Art's will be of frequent use, particularly in completing the square, in the solution of equations of the second degree.

135. To reduce an entire quantity to the form of a fraction having a given denominator.

1. Let it be required to reduce a to a fraction having b for its denominator.

Since $a=\frac{a}{1}$, if we multiply both terms by b, which will not change its value, Art. 126, we have $\frac{a}{1}=\frac{ab}{b}$. Hence,

TO REDUCE AN ENTIRE QUANTITY TO THE FORM OF A FRACTION HAVING A GIVEN DENOMINATOR,

Rule.—*Multiply the entire quantity by the given denominator, and write the product over it.*

2. Reduce x to a fraction whose denominator is 4.

3. Reduce m to a fraction whose denominator is $9a^2$.

4. Reduce $3c+5$ to a fraction whose denominator is $16c^2$.

5. Reduce $a-b$ to a fraction whose denominator is $a^2-2ab+b^2$.

136. To convert a fraction to an equivalent one, having a denominator equal to some multiple of the denominator of the given fraction.

1. Reduce $\frac{a}{b}$ to a fraction whose denominator is bc.

It is evident that this will be accomplished without changing the value of the fraction, by multiplying both terms by c. This multiplier, c, may be found by inspection, or by dividing bc by b. Hence,

TO CONVERT A FRACTION TO AN EQUIVALENT ONE HAVING A GIVEN DENOMINATOR,

Rule.—*Divide the given denominator by the denominator of the fraction, and multiply both terms by the quotient.*

REVIEW.—134. How reduce fractions of different denominators to equivalent fractions having the least common denominator?
134. If each fraction is not in its lowest terms before commencing the operation, what is to be done? 135. How reduce an entire quantity to the form of a fraction having a given denominator?

REMARK.—If the required denominator is not a multiple of the given one, the result will be a complex fraction. Thus, if it is required to convert $\frac{1}{2}$ into an equivalent fraction whose denominator is 5, the numerator of the new fraction would be $2\frac{1}{2}$.

2. Convert $\frac{3}{4}$ to an equivalent fraction, having the denominator 16.

3. Convert $\frac{b}{c}$ to an equivalent fraction, having the denominator a^2c^2.

4. Convert $\frac{m+n}{m-n}$ to an equivalent fraction, having the denominator $m^2-2mn+n^2$.

CASE V.

ADDITION AND SUBTRACTION OF FRACTIONS.

137.—1. Required to find the sum of $\frac{2}{5}$ and $\frac{4}{5}$.

Here, both parts being of the same kind, that is, fifths, we may add them together, and the sum is 6 fifths, ($\frac{6}{5}$).

2. Let it be required to find the sum of $\frac{a}{m}$ and $\frac{b}{m}$.

Here, the parts being the same, that is, $m$ths, we shall have $\frac{a}{m}+\frac{b}{m}=\frac{a+b}{m}$.

3. Let it be required to find the sum of $\frac{a}{m}$ and $\frac{c}{n}$.

Here, the denominators being different, we can not add the numerators, and call them by the same name. We may, however, reduce them to a common denominator, and then add.

Thus, $\frac{a}{m}=\frac{an}{mn}$; $\frac{c}{n}=\frac{cm}{mn}$; and $\frac{an}{mn}+\frac{cm}{mn}=\frac{an+cm}{mn}$. Hence,

TO ADD FRACTIONS,

Rule.—*Reduce the fractions, if necessary, to a common denominator; add the numerators together, and write their sum over the common denominator.*

138. It is obvious that the same principles would apply in finding the difference between two fractions. Hence,

TO SUBTRACT FRACTIONS,

Rule.—*Reduce the fractions, if necessary, to a common denominator; subtract the numerator of the subtrahend from the numerator of the minuend, and write the remainder over the common denominator.*

EXAMPLES IN ADDITION OF FRACTIONS.

4. Add $\frac{a}{2}$, $\frac{a}{3}$, and $\frac{a}{6}$ together. . .

5. Add $\frac{x}{3}$, $\frac{x}{5}$, and $\frac{x}{6}$ together. . .

6. Add $\frac{1}{a}$, $\frac{1}{b}$, and $\frac{1}{c}$ together. . .

7. Add $\frac{x}{2}$, $\frac{y}{3}$, and $\frac{z}{4}$ together. . .

8. Add $\frac{3x}{4}$, $\frac{4x}{5}$, and $\frac{5x}{6}$ together.

9. Add $\frac{x+y}{2}$ and $\frac{x-y}{2}$ together. .

10. Add $\frac{1}{a+b}$ and $\frac{1}{a-b}$ together. .

11. Add $\frac{5+x}{y}$, $\frac{3-ax}{ay}$, and $\frac{b}{3a}$ together.

12. Add $\frac{a-b}{ab}$, $\frac{b-c}{bc}$, and $\frac{c-a}{ac}$ together. .

Entire quantities and fractions may be added separately; or, the entire quantities may be put into the form of fractions by making their denominators unity. When mixed quantities occur, it is often better to reduce them to the form of improper fractions.

13. Add $2x$, $3x+\frac{3z}{5}$, and $x+\frac{2z}{9}$ together.

14. Add $5x+\frac{x-2}{3}$ and $4x-\frac{2x-3}{5x}$ together.

EXAMPLES IN SUBTRACTION OF FRACTIONS.

1. From $\frac{a}{2}$ take $\frac{a}{3}$. . . . .

2. From $\frac{3x}{4}$ take $\frac{2x}{3}$. . . .

3. From $\frac{a+b}{2}$ take $\frac{a-b}{2}$. . .

4. From $\frac{2ax}{3}$ take $\frac{5ax}{2}$. . . .

5. From $\frac{3}{4a}$ take $\frac{5}{2x}$. . . .

6. From $\frac{x+y}{x-y}$ take $\frac{x-y}{x+y}$. . .

7. From $\frac{2a+b}{5c}$ take $\frac{3a-b}{7c}$. .

8. From $5x+\frac{x}{b}$ take $2x-\frac{x-b}{c}$.

9. From $\frac{1}{a-b}$ take $\frac{1}{a+b}$. . .

10. From $a+b$ take $\frac{1}{a}+\frac{1}{b}$. . .

REVIEW.—186. How convert a fraction to an equivalent one having a given denominator? Explain the operation by an example.

187. When fractions have the same denominator, how add them together? When fractions have different denominators?

188. If two fractions have the same denominator, how find their difference? When they have different denominators?

11. From $\frac{x^3+y^3}{x-y}$ take $\frac{x^3-y^3}{x+y}$ . . .

12. From $x+\frac{1}{x-1}$ take $\frac{2}{x+1}$. . .

13. From $2a-3x+\frac{a-x}{a}$ take $a-5x+\frac{x-a}{x}$.

CASE VI.

MULTIPLICATION OF FRACTIONS.

139. To multiply a fraction by an entire quantity, or an entire quantity by a fraction.

It is evident, from Prop. I., Art. 122, that this may be done by multiplying the numerator.

Thus, $\frac{a}{b}\times2=\frac{2a}{b}$, and $\frac{a}{b}\times m=\frac{am}{b}$.

Again, as either quantity may be made the multiplier, Art. 67, to multiply 4 by $\frac{2}{3}$, is the same as to multiply $\frac{2}{3}$ by 4. Hence,

TO MULTIPLY A FRACTION BY AN ENTIRE QUANTITY, OR AN ENTIRE QUANTITY BY A FRACTION,

Rule.—*Multiply the numerator by the entire quantity, and write the product over the denominator.*

From Art. 125, it is evident that a fraction may also be multiplied by dividing its denominator by the entire quantity.

Thus, in multiplying $\frac{5}{8}$ by 2, we may divide the denominator by 2, and the result will be $\frac{5}{4}$, which is the same as to multiply the numerator by 2, and reduce the resulting fraction to its lowest terms. Hence,

REVIEW.—139. How multiply a fraction by an entire quantity, or an entire quantity by a fraction? When the denominator is divisible by the entire quantity, what is the shortest method?

In multiplying a fraction and an entire quantity together, we should always divide the denominator of the fraction by the entire quantity, when it can be done without a remainder.

REMARK.—The expression "$\frac{2}{3}$ of 6" is the same as $\frac{2}{3}\times 6$.

1. Multiply $\frac{2a}{bc}$ by ad. . .

2. Multiply $\frac{a+b}{c}$ by xy. .

3. Multiply $a-2b$ by $\frac{4c}{2a+c}$.

4. Multiply a^2-b^2 by $\frac{3c-a}{2a}$.

5. Multiply $\frac{2a+3xz}{a^2b}$ by ab. . . .

6. Multiply $\frac{5bc+3bx}{10x^3y^2-14x^4y^3}$ by $2x^2y^2$.

7. Multiply $\frac{ax+by}{4(a+b)(a-b)}$ by $2(a-b)$.

8. Multiply $\frac{5c+4d}{5(a-b)(c^2-d^2)}$ by $5(a-b)(c+d)$.

9. Multiply $\frac{a}{c}$ by c. . . . . . .

REMARK.—If a fraction is multiplied by a quantity equal to its denominator, the product will equal the numerator.

10. Multiply $\frac{a-b}{c+d}$ by $c+d$. . . . . .

11. Multiply $\frac{m^2-n^2}{2x+5y}$ by $2x+5y$. . . .

140. To multiply a fraction by a fraction.

1. Let it be required to multiply $\frac{4}{5}$ by $\frac{2}{3}$.

Since $\frac{2}{3}$ is the same as 2 multiplied by $\frac{1}{3}$, it is required to multiply $\frac{4}{5}$ by 2, and take $\frac{1}{3}$ of the product.

Now, $\frac{4}{5}\times2=\frac{8}{5}$, and $\frac{1}{3}$ of $\frac{8}{5}$ is $\frac{8}{15}$. Hence, the product of $\frac{4}{5}$ and $\frac{2}{3}$ is $\frac{8}{15}$.

So, to multiply $\frac{a}{c}$ by $\frac{m}{n}$, multiply $\frac{a}{c}$ by m, and take $\frac{1}{n}$ of the product. $\frac{a}{c}\times m=\frac{ma}{c}$, and $\frac{1}{n}$ of $\frac{ma}{c}=\frac{ma}{nc}$. Hence,

TO MULTIPLY A FRACTION BY A FRACTION,

Rule.—*Multiply the numerators together for a new numerator, and the denominators together for a new denominator.*

REMARKS.—1. The expression "one third of one fourth" has the same meaning as "$\frac{1}{4}$ multiplied by $\frac{1}{3}$;" or, $\frac{1}{3}$ of $\frac{1}{4}=\frac{1}{4}\times\frac{1}{3}$.

2. If either of the factors is a mixed quantity, reduce it to an improper fraction before commencing the operation.

3. When the numerators and denominators have common factors, indicate the multiplication, and cancel such factors.

Thus, $\frac{14a}{15b}\times\frac{5c}{21d}=\frac{2\times\not{7}\times\not{5}ac}{\not{5}\times3\times3\times\not{7}bd}=\frac{2ac}{9bd}$.

Also, $\frac{5a}{a^2-b^2}\times\frac{a+b}{2a}=\frac{5a(a+b)}{2a(a+b)(a-b)}=\frac{5}{2(a-b)}$.

1. Multiply $\frac{3a}{4}$ by $\frac{5x}{8}$. . . . .

2. Multiply $\frac{4a}{5x}$ by $\frac{3x}{7a}$. . . . .

3. Multiply $\frac{3(a+x)}{2}$ by $\frac{4x}{a+x}$. .

4. Multiply $\frac{2x+3}{5}$ by $\frac{10x}{7}$. . .

5. Multiply $\frac{x^2-y^2}{ab}$ by $\frac{a^2}{x+y}$. . .

6. Multiply $\frac{xyz}{x^4+y^3}$ by $\frac{x^4+y^3}{xyz}$. . .

REVIEW.—140. How do you multiply one fraction by another? Explain by analyzing an example. When one factor is a mixed quantity, what ought to be done? When the numerator and denominator have common factors? What the meaning of "one third of one fourth?"

7. Multiply $\frac{a-x}{x^2}$ by $\frac{a^2}{a^2-x^2}$. . . .

8. Multiply $\frac{x}{a+x}$, $\frac{a^2-x^2}{x^2}$, and $\frac{a}{a-x}$ together.

9. Multiply $\frac{a-b}{2}$, $\frac{2}{a^2-b^2}$, and $a+b$ together.

10. Multiply $\frac{x^2+y^2}{x^2-y^2}$ by $\frac{x-y}{x+y}$. . .

11. Multiply $c+\frac{cx}{c-x}$ by $\frac{c^2-x^2}{x+1}$. .

CASE VII.

DIVISION OF FRACTIONS.

141. To divide a fraction by an entire quantity.

It has been shown, in Arts. 123 and 124, that a fraction is divided by an entire quantity, by dividing its numerator, or multiplying its denominator.

Thus, $\frac{4}{5}$ divided by 2, or $\frac{1}{2}$ of $\frac{4}{5}$, is $\frac{2}{5}$; or, $\frac{4}{5}\div 2=\frac{4}{10}=\frac{2}{5}$.
So, $\frac{ma}{n}$ divided by m, or $\frac{1}{m}$ of $\frac{ma}{n}$ is $\frac{a}{n}$. Hence,

TO DIVIDE A FRACTION BY AN ENTIRE QUANTITY,

Rule.—*Divide the numerator by the divisor, if it can be done without a remainder; if not, multiply the denominator.*

To divide a number by 2 is to take $\frac{1}{2}$ of it, or to multiply it by $\frac{1}{2}$; to divide by m is to take $\frac{1}{m}$ of it, or to multiply it by $\frac{1}{m}$. Hence,

To divide a fraction by an entire quantity, we may write the divisor in the form of a fraction, as $m=\frac{m}{1}$, *then invert it, and proceed as in multiplication of fractions.*

REVIEW.—141. How divide a fraction by an entire quantity? Explain the reason of the rule by analyzing an example.

1. Divide $\frac{6a^2b}{7n}$ by $3ab$. . . . . . .

2. Divide $\frac{14ac^3m^2}{11xy}$ by $7acm^2$. . . .

3. Divide $\frac{a^2+ab}{3+2x}$ by a. . . . . . .

4. Divide $\frac{c^2+cd}{5}$ by $c+d$. . . . .

5. Divide $\frac{x^2+2xy+y^2}{c+d}$ by $x+y$. . . .

6. Divide $\frac{2a}{3c}$ by b. . . . .

7. Divide $\frac{3+5a}{a-b}$ by $a+b$. . . . . .

8. Divide $\frac{3a+5c}{2x+3y}$ by $2x-3y$. . . . .

9. Divide $\frac{b-c}{a^2+ab+b^2}$ by $a-b$. . . .

10. Divide $\frac{x-y}{x^2-xy+y^2}$ by $x+y$. . . .

11. Divide $\frac{a^2+abc}{b+c}$ by $a+bc$. . . . .

12. Divide $\frac{m^2-n^2}{b+c}$ by $am-an$. . . . .

13. Divide $\frac{a^3-b^3}{c}$ by a^2+ab+b^2. . . .

142. To divide an integral or fractional quantity by a fraction.

1. How many times is $\frac{2}{3}$ contained in 4, or what is the quotient of 4 divided by $\frac{2}{3}$?

$\frac{1}{3}$ is contained in 4 three times as often as 1 is contained in 4, because 1 is 3 times as great as $\frac{1}{3}$; therefore, $\frac{1}{3}$ is contained in 4, 12 times; $\frac{2}{3}$ is contained in 4 only one half as often as $\frac{1}{3}$, since it is twice as great; therefore, $\frac{2}{3}$ is contained in 4, 6 times.

2. How many times is $\frac{m}{n}$ contained in a?

Reasoning as above, $\frac{1}{n}$ is contained in a, na times, and $\frac{m}{n}$ is contained $\frac{na}{m}$ times.

3. How many times is $\frac{2}{3}$ contained in $\frac{3}{4}$?

$\frac{1}{3}$ is contained in $\frac{3}{4}$, three times as often as 1 is contained in $\frac{3}{4}$, that is, $\frac{9}{4}$ times; and $\frac{2}{3}$, half as often as $\frac{1}{3}$, that is, $\frac{9}{8}$ times.

4. How many times is $\frac{m}{n}$ contained in $\frac{a}{c}$?

Reasoning as before, $\frac{1}{n}$ is contained in $\frac{a}{c}$. $\frac{an}{c}$ times; and $\frac{m}{n}$ is contained $\frac{an}{cm}$ times.

An examination of each of these examples will show that the process consists in multiplying the dividend by the denominator of the divisor, and dividing it by the numerator. If, then, the divisor be inverted, the operation will be the same as that in multiplication of fractions. Hence,

TO DIVIDE AN INTEGRAL OR FRACTIONAL QUANTITY BY A FRACTION,

Rule.—*Invert the divisor, and proceed as in multiplication of fractions.*

NOTE.—After inverting the divisor, abbreviate by canceling.

1. Divide 4 by $\frac{a}{3}$. . . . .

2. Divide a by $\frac{1}{4}$. . . . .

3. Divide ab^2 by $\frac{2ab}{5c}$. . . . .

4. Divide $\frac{a}{3}$ by $\frac{c}{2}$. . . . .

REVIEW.—142. How divide an integral or fractional quantity by a fraction? Explain the reason of this rule, by analyzing the examples given. When and how can the work be abbreviated?

5. Divide $\frac{x^2y}{3a}$ by $\frac{xy^2}{2b}$. . . . .

6. Divide $\frac{16ax}{5}$ by $\frac{4x}{15}$ . . .

7. Divide $\frac{6z+4}{5}$ by $\frac{3z+2}{4y}$. . .

8. Divide $\frac{a^2-b^2}{5}$ by $\frac{a+b}{a}$. . .

9. Divide $\frac{z^2-4}{6}$ by $\frac{z-2}{2}$. . . .

10. Divide $\frac{x^2-2xy+y^2}{ab}$ by $\frac{x-y}{bc}$.

11. Divide $\frac{a}{a^2-1}$ by $\frac{a+1}{a-1}$. . .

12. Divide $\frac{2z+3}{x+y}$ by $\frac{10z+15}{x^2-y^2}$. .

13. Divide $\frac{3(a^2-x^2)}{x}$ by $\frac{2(a+x)}{a-x}$.

14. Divide $\frac{2x^2}{a^3+x^3}$ by $\frac{x}{a+x}$. . .

143. To reduce a complex to a simple fraction.

This may be regarded as a case of division, in which the dividend and the divisor are either fractions or mixed quantities.

Thus, $\frac{2\frac{1}{3}}{3\frac{1}{2}}$ is the same as $2\frac{1}{3}$ divided by $3\frac{1}{2}$.

Also, $\frac{a+\frac{b}{c}}{m+\frac{n}{r}}$ is the same as $a+\frac{b}{c}$ divided by $m+\frac{n}{r}$.

$$2\tfrac{1}{3}\div3\tfrac{1}{2}=\frac{7}{3}\div\frac{7}{2}=\frac{7}{3}\times\frac{2}{7}=\frac{2}{3}.$$

$$\left(a+\frac{b}{c}\right)\div\left(m+\frac{n}{r}\right)=\frac{ac+b}{c}\div\frac{mr+n}{r}=\frac{ac+b}{c}\times\frac{r}{mr+n}=\frac{acr+br}{cmr+cn}.$$

In like manner, reduce the following complex to simple fractions:

1. $\dfrac{\frac{a}{b}}{\frac{c}{d}}$. . .

2. $\dfrac{3\frac{1}{2}}{\frac{a}{3}}$. . . .

3. $\dfrac{a+\frac{1}{c}}{m}$. . .

4. $\dfrac{m}{a+\frac{1}{c}}$. . .

A complex fraction may also be reduced to a simple one, *by multiplying both terms by the least common multiple of the denominators of the fractional parts of each term.*

Thus, to reduce $\dfrac{4\frac{1}{3}}{5\frac{1}{2}}$, multiply both terms by 6; the result is $\frac{26}{33}$.

144. Resolution of fractions into series.

An **Infinite Series** is an unlimited succession of terms, which observe the same law.

The **Law of a Series** is a relation existing between its terms, such as that when some of them are known the others may be found.

Thus, in the infinite series, $1-ax+a^2x^2-a^3x^3+a^4x^4$, etc., any term may be found by multiplying the preceding term by $-ax$.

Any proper algebraic fraction whose denominator is a polynomial can, by division, be resolved into an infinite series; for the process of division never can terminate.

After a few of the terms of the quotient are found, the law of the series will, in general, be easily discovered.

REVIEW.—143. How reduce a complex fraction to a simple one by division? How, by multiplication?

144. What is an infinite series? What is the law of a series? Give an example. Why can any proper algebraic fraction whose denominator is a polynomial, be resolved into an infinite series by division?

1. Convert the fraction $\frac{1}{1-x}$ into an infinite series.

$$\begin{array}{l|l} 1 & 1-x \\ \underline{1-x} & 1+x+x^2+x^3+, \text{ etc.} \\ \quad +x & \\ \quad \underline{+x-x^2} & \\ \qquad +x^2 & \\ \qquad \underline{+x^2-x^3} & \\ \qquad\quad +x^3 & \end{array}$$

The law of this series evidently is, that each term is equal to the preceding term multiplied by $+x$.

From this, it appears that the fraction $\frac{1}{1-x}$ is equal to the infinite series, $1+x+x^2+x^3+x^4+$, etc.

Resolve the following into infinite series by division:

2. $\frac{1}{1+x}=1-x+x^2-x^3+x^4-$, etc., to infinity.

3. $\frac{ax}{a-x}=x+\frac{x^2}{a}+\frac{x^3}{a^2}+\frac{x^4}{a^3}+$, etc., to infinity.

4. $\frac{1-x}{1+x}=1-2x+2x^2-2x^3+$, etc., to infinity.

5. $\frac{x+2}{x+1}=1+\frac{1}{x}-\frac{1}{x^2}+\frac{1}{x^3}-$, etc., to infinity.

GENERAL REVIEW.

Define a fraction. An entire quantity. A mixed quantity. An improper fraction. Simple. Compound. Complex. Terms of a fraction. Denominator. Numerator. State Proposition I., and illustrate it. Proposition II.; III.; IV.; V.; VI.

How reduce a fraction to its lowest terms? To an entire or mixed quantity? How a mixed quantity to a fraction? Rule for the signs of fractions. How reduce to common denominators? To *least* common denominators?

How add fractions? Subtract? Multiply? Divide? How reduce a complex to a simple fraction? How resolve a fraction into an infinite series?

Define mathematics. Algebra. Theorem. Problem. Factor. Coëfficient. Exponent. Power. Root. Monomial. Binomial. Trinomial. Polynomial. Residual quantity. Reciprocal of a quantity. Prime quantity. Composite. Quadratic trinomial. The G.C.D. The L.C.M.

IV. SIMPLE EQUATIONS.

DEFINITIONS AND ELEMENTARY PRINCIPLES.

145. The most useful part of Algebra is that which relates to the solution of problems. This is performed by means of equations.

An **Equation** is an algebraic expression, stating the equality between two quantities; thus, $x-3=4$, is an equation, stating that if 3 be subtracted from x, the remainder will equal 4.

146. Every *equation* is composed of two parts, separated from each other by the sign of equality.

The **First Member** of an equation is the quantity on the left of the sign of equality.

The **Second Member** is the quantity on the right of the sign of equality.

Each member is composed of one or more terms.

147. There are generally two classes of quantities in an equation, the *known* and the *unknown*.

The known quantities are represented either by numbers or the first letters of the alphabet, as a, b, c, etc.; the unknown quantities, by the last letters of the alphabet, as, x, y, z, etc.

148. Equations are divided into degrees, called *first*, *second*, *third*, etc.

The **Degree** of an equation depends upon the highest power of the unknown quantity which it contains.

REVIEW.—145. What is an equation? Example. 146. Of how many parts composed? How are they separated? What is that on the left called? On the right? Of what is each member composed?

147. How many classes of quantities in an equation? How are the known quantities represented? How the unknown?

A **Simple Equation,** or an equation of the *first degree*, is one which contains no power of the unknown quantity higher than the first.

Thus, $2x+5=9$, and $ax+b=c$, are simple equations, or equations of the first degree.

A **Quadratic Equation,** or an equation of the *second degree*, is one in which the highest power of the unknown quantity is a square.

Thus, $4x^2-7=29$, and $ax^2+bx=c$, are quadratic equations, or equations of the second degree.

So, we have equations of the *third* degree, *fourth* degree, etc., distinguished by the highest power of the unknown quantity.

When any equation contains more than one unknown quantity, its degree is equal to the greatest sum of the exponents of the unknown quantities in any of its terms.

Thus, $xy+ax+by=c$, is an equation of the second degree.
$x^2y+x^2+cx=a$, is an equation of the third degree.

149. An **Identical Equation** is one in which the two members are identical; as, $5=5$, or $2x-1=2x-1$.

Equations are also distinguished as *numerical* and *literal.*

A **Numerical Equation** is one in which all the known quantities are expressed by numbers; as, $x^2+2x=3x+7$.

A **Literal Equation** is one in which the known quantities are represented by letters, or by letters and numbers; as, $ax-b=cx+d$, and $ax^2+bx=2x-5$.

REVIEW.—148. How are equations divided? On what does the degree depend? What is a simple equation, or an equation of the first degree? Example. What is a quadratic equation, or an equation of the second degree? Example.

148. When an equation contains more than one unknown quantity, to what is its degree equal? Example. 149. What is an identical equation? Examples. A numerical equation? Example. A literal equation? Example.

150. Every equation is to be regarded as the statement, in algebraic language, of a particular question.

Thus, $x-3=4$, may be regarded as the statement of the following question:

To find a number, from which, if 3 be subtracted, the remainder will be equal to 4.

Adding 3 to each member, we have $x-3+3=4+3$, or $x=7$.

An equation is said to be *verified*, when the value of the unknown quantity being substituted for it, the two members are rendered equal.

Thus, in the equation $x-3=4$, if 7, the value of x, be substituted instead of it, we have $7-3=4$, or $4=4$.

To *solve* an equation is *to find the value of the unknown quantity;* or, to find a number which, being substituted for the unknown quantity, will render the two members identical.

151. The **Root** of an equation is the value of its unknown quantity.

SIMPLE EQUATIONS CONTAINING ONE UNKNOWN QUANTITY.

152. All rules for finding the value of the unknown quantity in an equation are founded on this evident principle:

If we perform exactly the same operation on two equal quantities, the results will be equal.

This important principle may be otherwise expressed by the following

REVIEW.—150. How is every equation to be regarded? Example. When said to be verified? What is solving an equation? 151. What is the root of an equation?

152. Upon what principle are the operations used in solving an equation founded?

AXIOMS.

1. *If, to two equal quantities, the same quantity be added, the sums will be equal.*

2. *If, from two equal quantities, the same quantity be subtracted, the remainders will be equal.*

3. *If two equal quantities be multiplied by the same quantity, the products will be equal.*

4. *If two equal quantities be divided by the same quantity, the quotients will be equal.*

5. *If two equal quantities be raised to the same power, the results will be equal.*

6. *If the same root of two equal quantities be extracted, the results will be equal.*

REMARK.—An axiom is a self-evident truth. The preceding axioms are the foundation of a large portion of the reasoning in mathematics.

153. There are two operations of constant use in the solution of equations, *Transposition* and *Clearing an Equation of Fractions.* These we are now to consider.

TRANSPOSITION.

Suppose we have the equation $x-3=5$.

Since, by Art. 152, the equality will not be affected by adding the same quantity to both members, if we add 3 to each member, we have $x-3+3=5+3$.

But, $-3+3=0$; omitting these, we have $x=5+3$.

Now, the result is the same as if we had transposed the term -3, to the opposite member of the equation, and, at the same time, *changed* its sign.

Again, take the equation $x+b=c$.

REVIEW.—152. What are the axioms which this principle embraces? 153. What two operations are constantly used in the solution of equations?

If we subtract b from each side, Art. 152, Axiom 2, we have $x+b-b=c-b$, or $x=c-b$.

But, this result is also the same as if we had transposed the term $+b$, to the opposite side, changing its sign. Hence,

Any quantity may be transposed from one side of an equation to the other, if, at the same time, its sign be changed.

TO CLEAR AN EQUATION OF FRACTIONS.

154.—1. Let it be required to clear the following equation of fractions:

$$\frac{x}{2}+\frac{x}{3}=5.$$

If we multiply the first member by 2, the denominator of the first fraction will be removed; but if we multiply the first member by 2, we must multiply the other member by the same quantity, Art. 152, Axiom 3, in order to preserve the equality. Multiplying both sides by 2, we have

$$x+\frac{2x}{3}=10.$$

In like manner, multiplying both sides by 3, we have

$$3x+2x=30.$$

Instead of this, it is plain that we might have multiplied at once, by 2×3; that is, by the product of the denominators.

2. Again, clear the following equation of fractions:

$$\frac{x}{ab}+\frac{x}{bc}=d.$$

Reasoning as before, we first multiply both sides by ab, and then by bc, or at once by $ab\times bc$, and the equation will be cleared of fractions, and we shall have $bcx+abx=ab^2cd$.

REVIEW.—153. How may a quantity be transposed from one member of an equation to the other? Explain the principle by an example.

Instead of multiplying every term by $ab\times bc$, it is evident that if each term be multiplied by the L.C.M. of the denominators, which, in this case, is abc, the denominators will be removed; thus, $cx+ax=abcd$. Hence,

TO CLEAR AN EQUATION OF FRACTIONS,

Rule.—*Find the least common multiple of all the denominators, and multiply each term of the equation by it.*

Clear the following equations of fractions:

1. $\frac{x}{2}+\frac{x}{3}=5.$

2. $\frac{x}{3}-\frac{x}{4}=2.$

3. $\frac{x}{4}+\frac{x}{8}-\frac{x}{6}=\frac{5}{12}.$

4. $\frac{2x-3}{4}+\frac{x}{7}=\frac{x-3}{2}+\frac{5}{14}.$

5. $x-\frac{x-3}{2}=5-\frac{x+4}{3}.$

6. $\frac{x}{a}+\frac{x-5}{2}=b.$

7. $\frac{4}{x-3}+\frac{2a}{3}=\frac{3}{4}.$

8. $\frac{x+1}{x-3}+\frac{3-c}{a-b}=a.$

9. $\frac{x}{a+b}+\frac{x}{a-b}=\frac{c}{a^2-b^2}.$

10. $\frac{a}{bx}+\frac{c}{dx}+\frac{e}{fx}=h.$

REVIEW.—154. How clear an equation of fractions? Explain the principles by the examples given.

SOLUTION OF SIMPLE EQUATIONS, CONTAINING ONE UNKNOWN QUANTITY.

155. The unknown quantity in an equation may be combined with the known quantities, by addition, subtraction, multiplication, or division; or, in two or more of these methods.

1. Let it be required to find the value of x, in the equation $x+3=5$, where the unknown quantity is connected by addition.

By *subtracting* 3 from each side, we have $x=5-3=2$.

2. Let it be required to find the value of x, in the equation $x-3=5$, where the unknown quantity is connected by subtraction.

By *adding* 3 to each side, we have $x=5+3=8$.

3. Let it be required to find the value of x, in the equation $3x=15$, where the unknown quantity is connected by multiplication.

By *dividing* each side by 3, we have $x=\frac{15}{3}=5$.

4. Let it be required to find the value of x, in the equation $\frac{x}{3}=2$, where the unknown quantity is connected by division.

By *multiplying* each side by 3, we have $x=2\times3=6$.

From the solution of these examples, we see that

When the unknown quantity is connected by addition, it is to be separated by subtraction.

When connected by subtraction, it is separated by addition.

When connected by multiplication, it is separated by division.

When connected by division, it is separated by multiplication.

5. Find the value of x, in the equation $3x-3=x+5$.

By transposing the terms -3 and x, we have

$$3x-x=5+3$$

Reducing, $2x=8$

Dividing by 2, $x=\frac{8}{2}=4$.

Let 4 be substituted for x, in the original equation, and, if it is the true value, it will render the two members equal.

Original equation, . . . $3x-3=x+5$.

Substituting 4 in the place of x, it becomes

$$3\times4-3=4+5, \text{ or } 9=9.$$

This method of substituting the *value* of the unknown quantity instead of itself, is called *verification*.

6. Find the value of x in the equation $x-\frac{x-2}{3}=4+\frac{x+2}{5}$.

Multiplying both sides by 15, the L.C.M. of the denominators, we have $15x-(5x-10)=60+3x+6$;

or, $15x-5x+10=60+3x+6$.

By transposition, $15x-5x-3x=60+6-10$.

Reducing, $7x=56$.

Dividing, $x=8$.

7. Find the value of x in the equation $\frac{x}{b}-d=\frac{x}{a}+c$

Multiplying both sides by ab, $ax-abd=bx+abc$.

Transposing, $ax-bx=abc+abd$.

Separating into factors, $(a-b)x=ab(c+d)$.

Dividing by $(a-b)$, $x=\frac{ab(c+d)}{a-b}$.

From the preceding examples and illustrations, we derive the

REVIEW.—155. How may the unknown quantity in an equation be combined with known quantities? Examples.

155. When the unknown quantity is connected by addition, how is it to be separated? When, by subtraction? By multiplication? By division? What is verification?

RULE,

FOR THE SOLUTION OF SIMPLE EQUATIONS.

1. *If necessary, clear the equation of fractions.*

2. *Transpose all the terms containing the unknown quantity to one side, and the known quantities to the other.*

3. *Combine the terms in each member by the rule for addition.*

4. *Divide both members by the coëfficient of the unknown quantity.*

EXAMPLES FOR PRACTICE.

NOTE.—Verify the value of the unknown quantity in each example.

1. $3x-5=2x+7$. . . . .
2. $3x-8=16-5x$. . . . .
3. $3x-25=-x-9$. . . . .
4. $15-2x=6x-25$. . . . .
5. $5(x+1)+6(x+2)=9(x+3)$. . . .
6. $10(x+5)+8(x+4)=5(x+13)+121$.
7. $\frac{x}{2}-2=5-\frac{x}{5}$. . . . .
8. $\frac{x}{2}+\frac{x}{3}-\frac{x}{4}=14$. . . . .
9. $\frac{3x}{4}+\frac{2x}{3}-\frac{x}{6}=2\frac{1}{2}$. . . . .
10. $\frac{x-2}{4}-2=1-\frac{x+7}{3}$. . . . .
11. $\frac{3x+1}{2}-\frac{2x}{3}=10+\frac{x-1}{6}$. . . . . .
12. $\frac{3x-2}{4}-\frac{4-x}{2}=2x-\frac{7x-2}{3}$. . . . . .

REVIEW.—155. What is the rule for the solution of an equation of the first degree containing one unknown quantity?

13. $\frac{4}{5}x-\frac{5}{4}x+18=\frac{1}{9}(4x+1)$. . . .

14. $\frac{5x}{x+4}=1$. . . . .

15. $2x-\frac{x-2}{10}=x+\frac{x+18}{15}$. . . . .

16. $\frac{3}{4}-\frac{x-2}{3}=\frac{5}{4}-\frac{x+3}{4}$. . . . . .

17. $2x-\frac{2x+11}{5}-\frac{4x-6}{11}=\frac{7-8x}{7}$.

18. $\frac{x+7}{3}-5\frac{3}{4}=\frac{2x+5}{7}+\frac{10-5x}{8}$. . .

19. $\frac{x}{8}+\frac{2(x-1)}{5}=\frac{7x-4}{15}-\frac{x-1}{60}$. . .

20. $4x-b=2x-d$. . . . .

21. $ax+b=cx+d$. . . . . . .

22. $ax-bx=c+dx-e$. . . . .

23. $7+9a-5x=6x+5ax$. . . . .

24. $(a+b)(b-x)+(a-b)(a+x)=c^2$

25. $\frac{x}{a}+\frac{x}{b}=c$. . . . .

26. $\frac{a}{x}+\frac{b}{x}+\frac{c}{x}=1$. . . . . .

27. $\frac{x}{a}+\frac{x}{b}+\frac{x}{c}=d$. . . . . .

28. $\frac{ab}{x}+\frac{ac}{x}+\frac{bc}{x}=2$. . . . . .

29. $\frac{a}{x}+\frac{b}{c}-\frac{d}{e}=0$. . . . . .

30. $\frac{1+x}{1-x}=1+\frac{1}{x}$. . . . . .

31. $\frac{a^2}{x}=ab+b+\frac{1}{x}$. . . . .

32. $\frac{a-b}{x-c}=\frac{a+b}{x+2c}$. . . . .

QUESTIONS PRODUCING SIMPLE EQUATIONS, CONTAINING ONE UNKNOWN QUANTITY.

156. The solution of a problem, by Algebra, consists of two distinct parts:

1. *Expressing the conditions of the problem in algebraic language; that is, forming the equation.*

2. *Solving the equation, or finding the value of the unknown quantity.*

The first is the most difficult part of the operation.

Sometimes, the statement of the question furnishes the equation directly; and, sometimes, it is necessary, from the conditions given, to deduce others from which to form the equation. In the one case, the conditions are called *explicit* conditions; in the other, *implied* conditions.

It is impossible to give a precise rule for forming an equation. The first point is to understand fully the nature of the question or problem.

After this, the equation may generally be formed thus:

Rule.—*Denote the required quantity by one of the final letters of the alphabet; then, by means of signs, indicate the same operations that it would be necessary to make on the answer to verify it.*

REVIEW.—156. Of what two parts does the solution of a problem consist? What are explicit conditions? Implied conditions?
156. By what rule may the equation of a problem generally be formed?

EXAMPLES.

1. There are two numbers, the second of which is three times the first, and their sum is 48; what are the numbers?

Let $x=$ the first number.
Then, by the first condition, $3x=$ the second.
And, by the second condition, $x+3x=48$.
Reducing, $4x=48$.
Dividing by 4, $x=12$, the smaller number.
Then, $3x=36$, the larger number.

Proof, or verification. $12+36=48$.

2. A father said to his son, "The difference of our ages is 48 years, and I am 5 times as old as you." What were their ages?

Let $x=$ the son's age.
Then, $5x=$ the father's age.
And $5x-x=48$.
Reducing, $4x=48$.
Dividing, $x=12$, the son's age.
Then, $5x=60$, the father's age.

Verification. $60-12=48$, the difference of their ages.

3. What number is that, to which if its third part be added, the sum will be 16?

Let $x=$ the required number.
Then, the third part of it will be represented by $\frac{x}{3}$

And, by the conditions of the question, we have the equation

$$x+\frac{x}{3}=16.$$

Multiplying by 3, to clear it of fractions, $3x+x=48$.
Reducing, $4x=48$; and dividing, $x=12$.

Verification. $12+\frac{12}{3}=12+4=16$.

NOTE.—The pupil should *verify* the answer in every example.

4. What number is that, which being increased by its half, and then diminished by its two thirds, the remainder will be 105?

Let $x=$ the number.
Then, the one half equals $\frac{x}{2}$, and the two thirds, $\frac{2x}{3}$.

And, by the question, $x+\frac{x}{2}-\frac{2x}{3}=105$.
Multiplying by 6, $6x+3x-4x=630$.
Reducing, $5x=630$; and dividing, $x=126$.

It is sometimes better to simply indicate the multiplication, thus:

$$x+\frac{x}{2}-\frac{2x}{3}=105$$
$$6x+3x-4x=105\times6$$
$$5x=105\times6$$
$$x=21\times6=126.$$

5. It is required to divide a line 25 inches long, into two parts, so that the greater shall be 3 inches longer than the less.

Let $x=$ the length of the smaller part.
Then, $x+3=$ the greater part.
And by the question, $x+x+3=25$.
Reducing, $2x+3=25$.
Transposing 3, $2x=25-3=22$.
Dividing, $x=11$, the smaller part; and $x+3=14$, the greater.

6. It is required to divide \$68 between A, B, and C, so that B shall have \$5 more than A, and C \$7 more than B.

Let $x=$ A's share. Then, $x+5=$ B's share; and $x+12=$ C's.
Then, by the terms of the question, $x+(x+5)+(x+12)=68$.
Reducing, $3x+17=68$.
Transposing, $3x=68-17=51$.
Dividing, $x=17$, A's share.
$x+\ 5=22$, B's share.
$x+12=29$, C's share.

7. What number is that, which being added to its third part, the sum will be equal to its half added to 10?

Let x represent the number.

Then, the number, with its third part, is represented by $x+\frac{x}{3}$; and its half, added to 10, is expressed by $\frac{x}{2}+10$. By the conditions of the question, these are equal; that is, $x+\frac{x}{3}=\frac{x}{2}+10$.

Multiplying by 6, $6x+2x=3x+60$.
Reducing and transposing, $8x-3x=60$.
$5x=60$.
Dividing, $x=12$.

Verification. $12+\frac{12}{3}=\frac{12}{2}+10$; or, $16=16$.

Hereafter, we shall omit the terms, *transposing*, *dividing*, etc., as the steps of the solution will be evident by inspection.

8. A cistern was found to be one third full of water, and, after emptying into it 17 barrels more, it was found to be half full; what number of barrels will it contain when full?

Let $x=$ the number of barrels the cistern will contain.

Then, $\frac{x}{3}+17=\frac{x}{2}$.
$2x+102=3x$.
$102=x$; or, transposing, $-x=-102$.

And multiplying both sides by -1, we have $x=102$.

It is most convenient to make the unknown quantity stand on the left side of the sign of equality. If negative, the sign may be changed as above shown. Or, in general,

The signs of all the terms of both members of an equation may be changed at pleasure, since this would be the result of multiplying by -1.

9. A cistern is supplied with water by two pipes; the less alone can fill it in 40 minutes, and the greater in 30 min.; in what time will they fill it, both running at once?

Let $x=$ the number of min. in which both together can fill it.

Then, $\frac{1}{x}=$ the part which both can fill in 1 min.

Since the less can fill it in 40 min., it fills $\frac{1}{40}$ of it in 1 min. Since the greater can fill it in 30 min., it fills $\frac{1}{30}$ of it in 1 min. Hence, the part of the cistern which both can fill in 1 min., is represented by $\frac{1}{40}+\frac{1}{30}$, and also, by $\frac{1}{x}$.

Hence, $\frac{1}{40}+\frac{1}{30}=\frac{1}{x}$.

Multiply both sides by $120x$, and we have $3x+4x=120$.

$$7x=120.$$

$$x=\frac{120}{7}=17\tfrac{1}{7} \text{ min.}$$

10. A can perform a piece of work in 5 days, B in 6 days, and C in 8 days; in what time can the three perform it?

Let $x=$ the number of days in which all three can do it.

Then, $\frac{1}{x}=$ the part which all can do in 1 day.

If A can do it in 5 days, he does $\frac{1}{5}$ of it in 1 day.
If B " " 6 " " $\frac{1}{6}$ " "
If C " " 8 " " $\frac{1}{8}$ " "

Hence, the part of the work done by A, B, and C in 1 day, is represented by $\frac{1}{5}+\frac{1}{6}+\frac{1}{8}$, also by $\frac{1}{x}$. Hence, $\frac{1}{5}+\frac{1}{6}+\frac{1}{8}=\frac{1}{x}$.

Or, $24x+20x+15x=120.$

$$59x=120$$

$$x=\frac{120}{59}=2\tfrac{2}{59} \text{ days.}$$

11. How many pounds of sugar at 5 cents and at 9 cents per pound, must be mixed, to make a box of 100 pounds, at 6 cents per pound?

Let $x=$ the number of pounds at 5 cents.

Then, $100-x=$ the number of pounds at 9 cents.

Also, $5x=$ the value of the former in cents.

And $9(100-x)=$ the value of the latter in cents.

And $600=$ the value of the mixture in cents.

But the value of the two kinds must be equal to that of the mixture. Therefore, $5x+9(100-x)=600$

$$5x+900-9x=600$$

$$-4x=-300$$

$x=75$, the number of lbs. at 5 cts.

$100-x=25$, " " " 9 cts.

12. A laborer was engaged for 30 days. For each day he worked, he received 25 cents and his board; and, for each day he was idle, he paid 20 cents for his board. At the expiration of the time, he received \$3; how many days did he work, and how many was he idle?

Let $x=$ the number of days he worked.
Then, $30-x=$ the number of days he was idle.
Also, $25x=$ wages due for work.
And $20(30-x)=$ the amount to be deducted for boarding.
Therefore, $25x-20(30-x)=300$
$25x-600+20x=300$
$45x=900$
$x=20$, the number of days he worked.
$30-x=10$, the number of days he was idle.

Proof. $25\times20=500$ cents, = wages.
$20\times10=200$ cents, = boarding.
300 cents, = the remainder.

In the above example, we reduce the \$3 to cents; for it is evident we can add and subtract only quantities of the same denomination. And, since we can compare only quantities of the same name, therefore,

All the quantities in both members of an equation must be of the same denomination.

13. A hare is 50 leaps before a greyhound, and takes 4 leaps to the greyhound's 3; but 2 of the greyhound's leaps are equal to 3 of the hare's; how many leaps must the greyhound take to catch the hare?

Let x be the number of leaps taken by the hound. Then, since the hare takes 4 leaps while the hound takes 3, the number taken by the hare, after the hound starts, will be $\frac{4x}{3}$; and the whole number of leaps taken by the hare will be $\frac{4x}{3}+50$, which is equal, in extent, to the x leaps run by the hound. But 2 leaps of the hound are $=3$ of the hare's, or 1 leap $=\frac{3}{2}$ leaps of the hare; hence, x leaps of the hound $=\frac{3x}{2}$ leaps of the hare; and we have the equation

$$\frac{3x}{2}=\frac{4x}{3}+50$$

$$9x=8x+300$$

$x=300$, leaps taken by the greyhound.

14. The hour and minute hands of a watch are exactly together between 8 and 9 o'clock; required the time.

Let the number of min. more than 40 be denoted by x; that is, let $x=$ the min. from VIII to the point of coincidence, P; then, the hour hand moves from VIII to the point P, while the min. hand moves from XII to the same point; or, the former moves over x min., while the latter moves over $40+x$ min.; but the min. hand moves 12 times as fast as the hour hand.

Therefore, $12x=40+x$

$$11x=40$$

$x=\frac{40}{11}$ min. $=3$ min., $38\frac{2}{11}$ sec. Hence,

The required time is 43 min. $38\frac{2}{11}$ sec. after 8 o'clock.

15. A person spent one fourth of his money, and then received \$5. He next spent one half of what he then had, and found that he had only \$7 remaining; what sum had he at first?

Let $x=$ the number of dollars he had.

$\frac{x}{4}=$ what he spent the first time.

Subtracting and adding 5, $\frac{3x}{4}+5=$ what he had left.

One half of this, $\frac{3x}{8}+\frac{5}{2}=$ what he spent the second time.

Subtracting from above, $\frac{3x}{8}+\frac{5}{2}=$ what he had left, second time.

By the conditions $\frac{3x}{8}+\frac{5}{2}=7$

$$3x+20=56$$

$3x=36$, and $x=12$. Ans.

16. Divide 42 cents between A and B, giving to B twice as many as to A.

17. Divide 48 into three parts, so that the second may be twice, and the third three times the first.

18. Divide 60 into 3 parts, so that the second may be 3 times the first, and the third double the second.

19. A boy bought an equal number of apples, lemons, and oranges for 56 cents; for the apples he gave 1 cent, for the lemons 2 cents, and for the oranges 5 cents apiece; how many of each did he purchase?

20. Bought 5 apples and 3 lemons for 22 cents; gave as much for 1 lemon as for 2 apples; what did I give for each?

21. A's age is double that of B; the age of B is twice that of C; the sum of their ages is 98 years; what is the age of each?

22. A, B, C, and D have among them, 44 cents; A has a certain number, B three times as many as A, C as many as A and one third as many as B, and D as many as B and C; how many has each?

23. Divide 55 into two parts, in proportion to each other as 2 to 3.

Let $2x=$ one part; then, $3x=$ the other, since $2x$ is to $3x$ as 2 is to 3.

$$2x+3x=55$$
$$5x=55$$
$$x=11$$
$$\left.\begin{array}{r}2x=22\\3x=33\end{array}\right\}\text{Ans.}$$

Or, thus: Let $x=$ one part; then, $55-x=$ the other.

By the question, $x:55-x::2:3$. Then, since, in every proportion, the product of the means is equal to the product of the extremes, we have $3x=2(55-x)=110-2x$.

$$5x=110$$
$$x=22, \text{ and } 55-x=33, \text{ as before.}$$

Or, thus: Let x= one part; then, $\frac{3x}{2}$= the other.

And $x+\frac{3x}{2}=55$.

$2x+3x=110$, from which $x=22$, and $\frac{3x}{2}=33$.

The first method avoids fractions, and is of such frequent application, that we may give this general direction:

When two or more unknown quantities in any problem are to each other in a given ratio, assume each of them a multiple of an unknown quantity, so that they shall have to each other the given ratio.

24. The sum of two numbers is 60, and the less is to the greater as 5 to 7; what are the numbers?

25. Divide 60 into 3 parts, which shall be in proportion to each other as 2, 3, and 5.

26. Divide 60 into 3 such parts, that $\frac{1}{2}$ of the first, $\frac{1}{3}$ of the second, and $\frac{1}{5}$ of the third shall be equal.

Let $2x$, $3x$, and $5x$ represent the parts.

27. What number is that whose $\frac{1}{2}$, $\frac{1}{3}$, and $\frac{1}{4}$ part are together equal to 65?

28. What number is that, $\frac{1}{5}$ of which is greater than $\frac{1}{7}$ by 4?

29. The age of B is $2\frac{4}{5}$ times that of A, and the sum of their ages is 76 yr.; what the age of each?

30. Divide $440 between A, B, and C, so that the share of A may be $\frac{3}{5}$ that of B, and the share of B $\frac{3}{4}$ that of C.

31. Four towns are situated in the order of the letters A, B, C, D. From A to D is 120 mi.; from A to B is to the distance from B to C as 3 to 5; and one third of the distance from A to B, added to the distance from

B to C, is three times the distance from C to D; how far are the towns apart?

32. A merchant having engaged in trade with a certain capital, lost $\frac{1}{3}$ of it the 1st year; the 2d year he gained a sum equal to $\frac{2}{5}$ of what remained at the close of the 1st year; the 3d year he lost $\frac{1}{7}$ of what he had at the close of the 2d year, when he was worth $1236. What was his original capital?

33. The rent of a house this year is greater by 5% than it was last year; this year the rent is $168: what was it last year?

34. Divide the number 32 into 2 parts, so that the greater shall exceed the less by 6.

35. At an election, the number of votes given for two candidates was 256; the successful candidate had a majority of 50 votes; how many votes had each?

36. Divide $1520 among A, B, and C, so that B may receive $100 more than A, and C $270 more than B; what is the share of each?

37. A company of 90 persons consists of men, women, and children; the men are 4 more than the women, and the children 10 more than both men and women; how many of each?

38. After cutting off a certain quantity of cloth from a piece of 45 yards, there remained 9 yards less than had been cut off; how many yards had been cut off?

39. What number is that, which being multiplied by 7, gives a product as much greater than 20 as the number itself is less than 20?

40. A person dying left an estate of $6500, to be divided among his widow, 2 sons, and 3 daughters, so that each son shall receive twice as much as a daughter,

and the widow $500 less than all her children together; required the share of the widow, and of each son and daughter.

Ans. Widow $3000, each son $1000, each daughter $500.

41. Two men set out at the same time, one from London, the other from Edinburgh; one goes 20, the other 30 miles a day; in how many days will they meet, the distance being 400 miles?

42. A and B depart from the same place, to go in the same direction; B travels at the rate of 3, and A at the rate of 5 mi. an hr., but B has 10 hr. the start of A; in how many hr. will A overtake B?

43. Being asked the time of day, I replied, "If, to the time past noon, there be added its $\frac{1}{2}$, $\frac{1}{3}$, and $\frac{2}{5}$, the sum will $=$ $\frac{1}{6}$ of the time to midnight;" required the hour.

44. Divide 120 into two such parts that the less may be contained in the greater $1\frac{1}{2}$ times.

45. If I multiply a certain number by 7, add 3 to the product, divide this by 2, and subtract 4 from the quotient, the remainder is 15. What is the number?

46. What number is that, which, if you multiply it by 5, subtract 24 from the product, divide the remainder by 6, and add 13 to the quotient, will give the number itself?

47. A and B engaged in trade, the capital of B being $\frac{2}{3}$ that of A; B gained, and A lost, $100; after which, if $\frac{5}{7}$ of what A had left be subtracted from what B has, the remainder will be $134; what capital had each at first?

48. A man having spent $3 more than $\frac{2}{3}$ of his money, had $7 more than $\frac{1}{5}$ of it left; how much had he at first?

49. A and B have the same annual income; A saves $\frac{1}{5}$ of his, but B spends $25 per annum more than A, and

at the end of 5 years finds he has saved $200; what is the income of each?

50. In a quantity of gunpowder, $\frac{2}{3}$ of the whole, plus 10 lb., was niter; $\frac{2}{23}$ of the whole, plus 1 lb., was sulphur; and $\frac{1}{3}$ of the whole, minus 17 lb., was charcoal; how many lb. of gunpowder were there?

51. Bought a chaise, horse, and harness for $245; the horse cost 3 times as much as the harness, and the chaise $19 less than $2\frac{2}{3}$ times as much as both horse and harness; what the cost of each?

52. What two numbers are as 3 to 4, to each of which, if 4 be added, the sums will be as 5 to 6?

53. The ages of two brothers are now 25 and 30 years, or as 5 to 6; in how many years will they be as 8 to 9?

54. A cistern has 3 pipes; by the 1st it can be filled in $1\frac{1}{3}$ hr., by the 2d in $3\frac{1}{3}$ hr., and by the 3d in 5 hr.; in what time can it be filled by all running at once?

55. Find the time in which A, B, and C together can perform a piece of work, which requires 7, 6, and 9 days respectively, when done singly.

56. From a certain sum I took one third part, and put in its stead $50; from this sum I took the tenth part, and put in its stead $37; I then found I had $100; what was the original sum?

57. A spent $\frac{2}{5}$ of his salary for board, $\frac{1}{3}$ of the remainder for clothes, $\frac{1}{8}$ of the rest for books, and still saved $120 per annum; what was his salary?

58. A was engaged for a year at $80 and a suit of clothes; he served 7 mon., and received for his wages the clothes and $35; what was the value of the clothes?

59. A man and his wife can drink a keg of wine in 6 days, and the man alone in 10 days; how many days will it last the woman?

60. A steamboat that can run 15 mi. per hr. with the current, and 10 mi. per hr. against it, requires 25 hr. to go from Cincinnati to Louisville, and return; what is the distance between these cities?

61. In a mixture of wine and water, $\frac{1}{2}$ the whole, plus 25 gal., was wine, and $\frac{1}{3}$ of the whole, minus 5 gal., was water; required the quantity of each in the mixture.

62. Required to divide 72 into 4 such parts, that if the 1st be increased by 2, the 2d diminished by 2, the 3d multiplied by 2, and the 4th divided by 2, the sum, difference, product, and quotient shall be equal.

Let the four parts be represented by $x-2$, $x+2$, $\frac{1}{2}x$, and $2x$.

63. A merchant having cut 19 yd. from each of 3 equal pieces of silk, and 17 from another of the same length, found that the remnants taken together measured 142 yd.; what was the length of each piece?

64. For every 10 sheep I keep, I plow an acre of land, and allow 1 acre of pasture for every 4 sheep; how many sheep can I keep on 161 acres?

65. It is required to divide 34 into 2 such parts, that if 18 be subtracted from the greater, and the less be subtracted from 18, the first remainder shall be to the second as 2 to 3.

66. A person was desirous of giving 8 cents apiece to some beggars, but found that he had not money enough by 8 cents; he therefore gave each of them 2 cents, and then had 3 cents left; required the number of beggars.

67. A could reap a field in 20 days, but if B assisted him for 6 days, he could reap it in 16 days; in how many days could B reap it alone?

68. When the price of a bu. of barley wanted but 3 cents to be to the price of a bu. of oats as 8 to 5, nine bu. of oats were received as an equivalent for 4 bu. of barley and 90 cents in money; what was the price of a bu. of each?

69. Four places are situated in the order of the 4 letters, A, B, C, and D; the distance from A to D is 34 mi.; the distance from A to B is to the distance from C to D as 2 to 3; and $\frac{1}{4}$ the distance from A to B, added to $\frac{1}{2}$ the distance from C to D, is 3 times the distance from B to C. Required the distances.

70. The ingredients of a loaf of bread are rice, flour, and water, and the whole weighs 15lb; the weight of the rice, plus 5lb, is $\frac{2}{3}$ that of the flour; and the weight of the water is $\frac{1}{5}$ the weight of the flour and rice together; what is the weight of each?

GENERAL REVIEW.

What is an equation? Of what composed? What is the first member? The second? How separated? Of what is each composed? How many classes of quantities in an equation? By what represented?

How are equations divided? Upon what does the degree depend? Define a simple equation. A quadratic. Illustrate each. The degree of each. Define an identical equation. Numerical equation. Literal equation. Verification. Root of an equation.

State the six axioms. Define transposition. How clear an equation of fractions? How may an unknown quantity be combined with a known? How separated, when combined by addition? By subtraction? Multiplication? Division? Rule for solution of simple equations.

Define mathematics. Algebra. Theorem. Problem. Exponent. Coefficient. Factor. Power. Monomial. Binomial. Trinomial. Polynomial. Residual quantity. Reciprocal of a quantity. Prime quantity. Composite.

SIMPLE EQUATIONS,

CONTAINING TWO UNKNOWN QUANTITIES.

157. To find the value of any unknown quantity, we must obtain a single equation containing *it*, and known terms. Hence,

When we have two or more equations containing two or more unknown quantities, we must obtain from them a single equation containing only one unknown quantity.

The method of doing this is termed *elimination*, which may be defined thus:

Elimination is the process of deducing from two or more equations containing two or more unknown quantities, a less number of equations containing one less unknown quantity.

There are three methods of elimination.

1st. *Elimination by Substitution.*
2d. *Elimination by Comparison.*
3d. *Elimination by Addition and Subtraction.*

158. Elimination by Substitution consists in finding the value of one of the unknown quantities in one of the equations, in terms of the other unknown quantity and known terms, and substituting this, instead of the quantity, in the other equation.

To explain this, suppose we have the following equations, in which it is required to find the value of x and y.

NOTE.—The figures in the parentheses are intended to number the equations for reference.

$$x+2y=17. \quad (1.)$$
$$2x+3y=28. \quad (2.)$$

By transposing $2y$ in the equation (1), we have $x=17-2y$. Substituting *this* value of x, instead of x in equation (2), we have

$$2(17-2y)+3y=28;$$
or, $$34-4y+3y=28;$$
or, $$-y=28-34; \text{ or, } y=6;$$
and $$x=17-2y=17-12=5. \text{ Hence,}$$

TO ELIMINATE BY SUBSTITUTION,

Rule.—1. *Find an expression for the value of one of the unknown quantities in either equation.*

2. *Substitute this value in place of the same unknown quantity in the other equation; there will thus be formed a new equation containing only one unknown quantity.*

NOTE.—In finding an expression for the value of one of the unknown quantities, take that which is least involved.

Find the values of the unknown quantities in the following:

1. $x+5y=38.$
   $3x+4y=37.$
2. $2x+4y=22.$
   $5x+7y=46.$
3. $3x+5y=57.$
   $5x+3y=47.$
4. $4x-3y=26.$
   $3x-4y=16.$
5. $\frac{y}{5}-\frac{x}{4}=1.$
   $5x-3y=10.$
6. $\frac{2x}{7}-\frac{3y}{8}=0.$
   $\frac{2x}{3}+\frac{3y}{4}=26.$

159. Elimination by Comparison consists in finding the value of the same unknown quantity in two different equations, and then placing these values equal to each other.

REVIEW.—157. What is necessary in order to find the value of any unknown quantity? What, when we have two equations, containing two unknown quantities? What is elimination? How many methods? 158. Define elimination by substitution. Rule.

To illustrate this method, we will take the same equations which were used to explain elimination by substitution.

$$x+2y=17 \quad (1.)$$
$$2x+3y=28 \quad (2.)$$

From equation (1), $x=17-2y$. From (2), $x=\frac{28-3y}{2}$.

Therefore, $\frac{28-3y}{2}=17-2y$;

or, $28-3y=34-4y$; or, $y=6$.

Then, $x=17-2y=17-12=5$.

Or, the value of x may be found in like manner, by first finding the values of y, and placing them equal to each other. Hence,

TO ELIMINATE BY COMPARISON,

Rule.—1. *Find an expression for the value of the same unknown quantity in each of the given equations.*

2. *Place these values equal to each other; there will thus be formed a new equation containing only one unknown quantity.*

Find the values of the unknown quantities in the following:

1. $x+3y=16.$
   $x+5y=22.$

2. $5x-2y=4.$
   $2x-y=1.$

3. $\frac{x}{9}-\frac{y}{8}=1.$
   $\frac{x}{6}+\frac{y}{4}=12.$

4. $\frac{x}{4}-\frac{y}{4}=1.$
   $\frac{x}{3}+\frac{y}{2}=8.$

5. $\frac{x}{5}+\frac{y}{2}=14.$
   $\frac{x}{9}-\frac{y}{5}=3.$

6. $\frac{3x}{2}+2y-x+\frac{2y}{3}=42.$ . . . .
   $3x-\frac{4y}{3}=40+\frac{x}{5}.$ . . . .

160. Elimination by Addition and Subtraction consists in multiplying or dividing two equations, so as to render the coëfficient of one of the unknown quantities the same in both; and then, by adding or subtracting, to cause the term containing it to disappear.

To explain this method, we will take the same equations used to illustrate elimination by substitution and comparison.

$$x+2y=17 \quad (1.)$$
$$2x+3y=28 \quad (2.)$$

Multiplying equation (1) by 2, we have

$$2x+4y=34 \quad (3.)$$
$$2x+3y=28, \text{ equation (2) brought down.}$$

Subtracting, $y=6$

Then, substituting this value in (3), $2x+4\times6=34$; and $x=5$.

When the terms containing the unknown quantity to be eliminated have contrary signs, it is necessary to *add*. In illustration, take the following:

$$3x-5y=6 \quad (1.)$$
$$4x+3y=37 \quad (2.)$$

Multiplying (1) by 3, and (2) by 5, we have

$$9x-15y=18$$
$$20x+15y=185$$

Adding, $29x=203$

$x=7$

Then, from (1) $3\times7-5y=6$; or, $y=3$.

From this it will be seen, that after making the coëfficients of the quantity to be eliminated the same in both equations, if the signs are *alike*, we *subtract*; if *unlike*, we *add*. Hence,

REVIEW.—159. In what does elimination by comparison consist? Rule. 160. In what elimination by addition and subtraction? Repeat the rule.

TO ELIMINATE BY ADDITION AND SUBTRACTION,

Rule.—1. *Multiply or divide the equations, if necessary, so that one of the unknown quantities will have the same coëfficient in both.*

2. *Add or subtract the equations, according as the signs of the equal terms are alike or unlike, and the resulting equation will contain only one unknown quantity.*

Remark.—When the coëfficients of the unknown quantities to be eliminated are prime to each other, they may be equalized by multiplying each equation by the coëfficient of the unknown quantity in the other.

If the equations have fractional coëfficients, they ought to be cleared before applying the rule.

Find the value of the unknown quantities in the following:

1. $3x+2y=21$. Ans.
   $x-2y=-1$.
2. $3x-2y=7$. Ans.
   $5y-2x=10$.
3. $2x-y=3$. Ans.
   $3x+2y=22$.
4. $3x+2y=19$. Ans.
   $2x-3y=4$.
5. $\frac{x}{4}+\frac{y}{5}=8$. Ans.
   $\frac{x}{5}+\frac{y}{3}=9$.
6. $\frac{x}{2}-\frac{y}{3}=3$. Ans.
   $\frac{x}{6}+\frac{y}{9}=3$.

PROBLEMS PRODUCING EQUATIONS CONTAINING TWO UNKNOWN QUANTITIES.

161. The problems in Art. 156, were all capable of being solved by using one unknown quantity. Several of them, however, contained two, and some more than two unknown quantities; but the conditions were such that it was easy to express each one in terms of the other.

When this is not the case, it becomes necessary to use a separate symbol for each unknown quantity, and as many equations as there are symbols.

After the equations are obtained, they may be solved by either of the three methods of elimination.

Two examples are given below, which can be solved by using either one or two unknown quantities.

1. Given, the sum of two numbers equal to 25, and their difference equal to 9, to find the numbers.

Solution, by using one unknown quantity.

Let $x=$ the less number; then, $x+9=$ the greater.

And $x+x+9=25$.

$2x=16$.

$x=8$, the less number; and $x+9=17$, the greater.

Solution, by using two unknown quantities.

Let $x=$ the greater, and $y=$ the less.

Then, $x+y=25$ (1.)

And $\underline{x-y=9}$ (2.)

Adding (1) and (2), $2x=34$, and $x=17$, the greater number.

Subtracting (2) from (1), $2y=16$, and $y=8$, the less number.

2. The sum of two numbers is 44, and they are to each other as 5 to 6; required the numbers.

Solution, by using one unknown quantity.

Let $5x=$ the less number; then, $6x=$ the greater.

And $5x+6x=44$.

$11x=44$

$x=4$

$5x=20$, the less number.

$6x=24$, the greater number.

Solution, by using two unknown quantities.

Let $x=$ the less number, and $y=$ the greater.

Then, $x+y=44$ (1.)

And $x:y::5:6$

REVIEW.—161. In solving problems, when does it become necessary to use a separate symbol for each unknown quantity?

Or, $6x=5y$ (2.), by multiplying means and extremes.

$6x+6y=264$ (3.), by multiplying equation (1) by 6.

$6y=264-5y$ by subtracting equation (2) from (3).

$11y=264$; $y=24$, and $x=44-y=20$.

Several of the following problems may also be solved by using only one unknown quantity.

3. There is a certain number consisting of two places of figures; the sum of the figures equals 6; if from the double of the number, 6 be subtracted, the remainder is a number whose digits are those of the former in an inverted order; required the number.

In solving problems of this kind, observe that any number consisting of two places of figures is equal to 10 times the figure in the ten's place, plus the figure in the unit's place.

Thus, 23 is equal to $10\times2+3$. In a similar manner, 325 is equal to $100\times3+10\times2+5$.

Let $x=$ the digit in the place of tens, and $y=$ that in the place of units.

Then, $10x+y=$ the number.

And $10y+x=$ the number, with the digits inverted.

Then, $x+y=6$ (1.)

And $2(10x+y)-6=10y+x$ (2.)

Or, $20x+2y-6=10y+x$.

$19x=8y+6$

$8x=-8y+48$, multiplying (1) by 8, and transposing.

$27x=54$, by adding.

$x=2$, and $y=6-2=4$. Ans. 24.

4. What two numbers are those to which if 5 be added, the sums will be to each other as 5 to 6; but, if 5 be subtracted from each, the remainders will be to each other as 3 to 4?

By the conditions of the question, we have the following proportions:

$$x+5 : y+5 :: 5 : 6$$
$$x-5 : y-5 :: 3 : 4$$

Since, in every proportion, the product of the means is equal to the product of the extremes, we have the two equations,

$$6(x+5)=5(y+5)$$
$$4(x-5)=3(y-5)$$

From these equations, the values of x and y are readily found to be 20 and 25.

NOTE.—In solving the following, the values of the unknown quantities may be found by either method of elimination.

5. A grocer sold to one person 5 lb. of coffee and 3 lb. of sugar, for 79 cents; and to another, at the same prices, 3 lb. of coffee and 5 lb. of sugar, for 73 cents; what was the price of a lb. of each?

6. Sold to one person 9 horses and 7 cows, for $300; to another, at the same prices, 6 horses and 13 cows, for the same sum; what the price of each?

7. It is required to find two numbers, such that $\frac{1}{2}$ of the first and $\frac{1}{3}$ of the second shall be 22, and $\frac{1}{4}$ of the first and $\frac{1}{5}$ of the second shall be 12.

8. If the greater of two numbers be added to $\frac{1}{3}$ of the less, the sum will be 37; but if the less be diminished by $\frac{1}{4}$ of the greater, the difference will be 20; what are the numbers?

9. A farmer has 2 horses, and a saddle worth $25; if the saddle be put on the first horse, his value will be double that of the second; but, if put on the second horse, his value will be three times that of the first. Required the value of each horse.

10. A and B are in trade together; if $50 be added to A's property, and $20 taken from B's, they will have the same sum; and if A's property was 3 times, and B's 5 times as great as each is, they would together have $2350; how much has each?

11. A number consists of two digits, which, divided by their sum, gives 7; if the digits be written in inverse order, and the number so arising be divided by their sum plus 4, the quotient will be 3. What the number?

12. If we add 8 to the numerator of a certain fraction, its value becomes 2; if we subtract 5 from the denominator, its value becomes 3; required the fraction.

13. If to the ages of A and B 18 be added, the result will be double the age of A; but, if from their difference 6 be subtracted, the result will be the age of B; required their ages.

14. There are two numbers whose sum is 37, and if 3 times the less be subtracted from 4 times the greater, and the difference divided by 6, the quotient will be 6; what are the numbers?

15. Find a fraction, such that if 3 be subtracted from the numerator and denominator, the value will be $\frac{1}{4}$; and if 5 be added to the numerator and denominator, the value will be $\frac{1}{2}$.

16. A father gave his two sons, A and B, together $2400, to engage in trade; at the close of the year, A has lost $\frac{1}{4}$ of his capital, while B, having gained a sum equal to $\frac{1}{4}$ of his capital, finds that his money is just equal to that of his brother; what sum was given to each?

17. A said to B, "give me $100, and then I shall have as much as you." B said to A, "give me $100, and then I shall have twice as much as you." How much had each?

18. If the greater of two numbers be multiplied by 5, and the less by 7, the sum of their products is 198; but if the greater be divided by 5, and the less by 7, the sum of their quotients is 6; what are the numbers?

19. Seven years ago the age of A was just three times that of B; and seven years hence, A's age will be just double B's; what are their ages?

20. There is a certain number consisting of two places of figures, which being divided by the sum of its digits, the quotient is 4, and if 27 be added to it, the digits will be inverted; required the number.

21. A grocer has two kinds of sugar, of such quality that 1 lb. of each are together worth 20 cents; but if 3 lb. of the first, and 5 lb. of the second kind be mixed, a lb. of the mixture will be worth 11 cents; what is the value of a lb. of each sort?

22. A boy lays out 84 cents for lemons and oranges, giving 3 cents apiece for the lemons, and 5 cents apiece for the oranges; he afterward sold $\frac{1}{2}$ of the lemons and $\frac{1}{3}$ of the oranges for 40 cents, and cleared 8 cents on what he sold; what number of each did he purchase?

23. A owes $500 and B $600, but neither has sufficient money to pay his debts. A said to B, "lend me $\frac{1}{5}$ of your money, and I can pay my debts." B said to A, "lend me $\frac{1}{4}$ of your money, and I can pay mine." How much has each?

24. A son said to his father, "how old are we?" The father replied, "six years ago my age was $3\frac{1}{3}$ times yours, but 3 years hence my age will be only $2\frac{1}{6}$ times yours." Required their ages.

25. A farmer having mixed a certain number of bu. of oats and rye, found, that if he had mixed 6 bu. more of each, he would have mixed 7 bu. of oats for every 6 of rye; but if he had mixed 6 bu. less of each, he would have put in 6 bu. of oats for every 5 of rye. How many bu. of each did he mix?

26. A person having laid out a rectangular yard, observed that if each side had been 4 yd. longer, the length would have been to the breadth as 5 to 4; but, if each had been 4 yd. shorter, the length would have been to the breadth as 4 to 3; required the length of the sides.

27. A farmer rents a farm for $245 per annum; the tillable land being rented at $2 an acre, and the pasture at $1 and 40 cts. an acre; now the number of acres tillable is to the excess of the tillable above the pasture, as 14 to 9; how many were there of each?

28. After drawing 15 gal. from each of 2 casks of wine, the quantity remaining in the first is $\frac{2}{3}$ of that in the second; after drawing 25 gal. more from each, the quantity left in the first is only half that in the second; what number of gal. in each before the first drawing?

29. If 1 be added to the numerator of a certain fraction, and the numerator to the denominator, its value will be $\frac{1}{4}$; but if the denominator be increased by unity, and the numerator by the denominator, its value will be $\frac{8}{7}$; find it.

30. Find two numbers in the ratio of 5 to 7, to which two other required numbers, in the ratio of 3 to 5, being respectively added, the sums shall be in the ratio of 9 to 13, and the difference of their sums 16.

31. A farmer, with 28 bushels of barley, worth 28 cents per bushel, would mix rye at 36 cents, and wheat at 48 cents per bushel, so that the whole mixture may consist of 100 bushels, and be worth 40 cents a bushel; how much rye and wheat must be mixed with the barley?

32. A person has two horses, and two saddles, one of which cost \$50, and the other \$2. If he places the best saddle upon the first horse, and the other on the second, then the latter is worth \$8 less than the former; but if he puts the worst saddle upon the first, and the best upon the second horse, then the value of the latter is to that of the former as 15 to 4; required the value of each horse.

33. The weights of two loaded wagons were in the ratio of 4 to 5; parts of their loads, which were in the ratio of 6 to 7, being taken out, their weights were in the ratio of 2 to 3, and the sum of their weights was then 10 tons; what their weights at first?

34. A person had two casks and a certain quantity of wine in each; in order to have the same quantity in each cask, he poured as much out of the first cask into the second as it already contained; he next poured as much out of the second into the first as it then contained; and, lastly, he poured out as much from the first into the second as there was remaining in it; after this, he had 16 gal. in each cask; how many gal. in each at first?

GENERAL REVIEW.

Define elimination. How many methods of elimination? Of what does elimination by substitution consist? Rule. Elimination by comparison? Rule. Elimination by addition and subtraction? Rule. Define transposition. How are the signs affected by transposition? Explain by an example.

What is the dimension of a term? When is a polynomial homogeneous? For what is a parenthesis used? A vinculum? What does the same letter accented denote? What are similar or like quantities?

What the rule for addition of algebraic quantities? Subtraction? Multiplication? Division? Rule for the signs? For finding the L.C.M.? The G.C.D.? Define a fraction.

What the effect of multiplying the numerator of a fraction? The denominator? Both? Of dividing the numerator? The denominator? Both? Repeat the axioms.

SIMPLE EQUATIONS,

CONTAINING THREE OR MORE UNKNOWN QUANTITIES.

162. Equations involving three or more unknown quantities may be solved by either of the three methods of elimination already explained.

Suppose we have the following equations, in which it is required to find the values of x, y, and z.

$$x+2y+z=20 \quad (1.)$$
$$2x+y+3z=31 \quad (2.)$$
$$3x+4y+2z=44 \quad (3.)$$

SOLUTION BY SUBSTITUTION.

From equation (1), $x=20-2y-z$.

Substituting this in equation (2), we have

$$2(20-2y-z)+y+3z=31;$$
$$\text{or, } 40-4y-2z+y+3z=31$$
$$3y-z=9 \quad (4.)$$

Substituting the same value of x in equation (3), we have

$$3(20-2y-z)+4y+2z=44;$$
$$\text{or, } 60-6y-3z+4y+2z=44.$$
$$2y+z=16 \quad (5.)$$
$$3y-z=9 \quad (4.)$$

The values of y and z are found, by Rule, Art. 158, to be 5 and 6; substituting these values in equation (1), $x=4$.

SOLUTION BY COMPARISON.

From equation (1), $x=20-2y-z$.

" " (2), $x=\frac{31-y-3z}{2}$.

" " (3), $x=\frac{44-4y-2z}{3}$.

Comparing the first and second values of x, we have

$$20-2y-z=\frac{31-y-3z}{2};$$

or, $40-4y-2z=31-y-3z$;

or, $3y-z=9$ (4.)

Comparing the first and third values of x, we have

$$20-2y-z=\frac{44-4y-2z}{3};$$

or, $60-6y-3z=44-4y-2z$.

$2y+z=16$ (5.)

From equations (4) and (5), the values of y and z, and then x, may be found by the Rule, Art. 159.

SOLUTION BY ADDITION AND SUBTRACTION.

Multiplying equation (1) by 2, we have

$2x+4y+2z=40$

Equation (2) is $2x+y+3z=31$

By subtracting, $3y-z=9$ (4.)

Next, multiplying equation (1) by 3, we have

$3x+6y+3z=60$

Equation (3) is $3x+4y+2z=44$

By subtracting, $2y+z=16$ (5.)

$3y-z=9$ (4.)

By adding, $5y=25$

$y=5$

Then, $10+z=16$, and $z=6$.

And $x+10+6=20$, and $x=4$.

It will be found, in practice, that the method of elimination by addition and subtraction is generally to be preferred; we shall, therefore, illustrate it by another example.

$v+2x+3y+4z=30$ (1.)

$2v+3x+y+z=15$ (2.)

$3v+x+2y+3z=23$ (3.)

$4v+2x-y+14z=61$ (4.)

Let us first eliminate v; this may be done thus:

$2v+4x+6y+8z=60$, by multiplying equation (1) by 2.
$2v+3x+y+z=15$ (2.)
$x+5y+7z=45$ (5.), by subtracting.

$3v+6x+9y+12z=90$, by multiplying equation (1) by 3.
$3v+x+2y+3z=23$ (3.)
$5x+7y+9z=67$ (6.), by subtracting.

$4v+8x+12y+16z=120$, by multiplying equation (1) by 4.
$4v+2x-y+14z=61$ (4.)
$6x+13y+2z=59$ (7.), by subtracting.

Collecting into one place the new equations (5), (6), and (7), we find that the number of unknown quantities, as well as the number of equations, is *one* less.

$x+5y+7z=45$ (5.)
$5x+7y+9z=67$ (6.)
$6x+13y+2z=59$ (7.)

The next step is to eliminate x, in a similar manner.

$5x+25y+35z=225$, by multiplying equation (5) by 5.
$5x+7y+9z=67$, equation (6).
$18y+26z=158$ (8.), by subtracting.

$6x+30y+42z=270$, by multiplying equation (5) by 6.
$6x+13y+2z=59$, equation (7).
$17y+40z=211$ (9.), by subtracting.

Bringing together equations (8) and (9), the number of equations, as well as of unknown quantities, is *two* less.

$18y+26z=158$ (8.)
$17y+40z=211$ (9.)

$306y+720z=3798$, by multiplying equation (9) by 18.
$306y+442z=2686$, by multiplying equation (8) by 17.
$278z=1112$, by subtracting.
$z=4$

Substituting the value of z, in equation (9), we get

$17y+160=211$; and $17y=51$: and $y=3$.

Substituting the values of y and z, in equation (5), we get

$$x+15+28=45$$
$$x=2$$

Substituting the values of x, y, and z, in equation (1), we have

$$v+4+9+16=30$$
$$v=1$$

From the preceding example, we derive the following

GENERAL RULE,

FOR ELIMINATION BY ADDITION AND SUBTRACTION.

1. *Eliminate the same unknown quantity from each of the equations; the number of equations and of unknown quantities will be one less.*

2. *Proceed in the same way with another unknown quantity; the number of equations and of unknown quantities will be two less.*

3. *Continue this series of operations until a single equation is obtained, containing but one unknown quantity.*

4. *By going back and substituting, the values of the other unknown quantities may be readily found.*

Remark.—When one or more of the equations contains but one or two of the unknown quantities, the method of substitution will generally be found the shortest.

In literal equations and some others, the method of comparison may be most convenient. After solving several examples by each method, the pupil will be able to appreciate their relative excellence in different cases.

SOLVE BY EITHER METHOD OF ELIMINATION.

1. $\begin{cases} x+y=50. \\ x+z=28. \\ y+z=42. \end{cases}$ . . . . Ans. $\Bigg\{$

2. $\begin{cases} 3x+5y=76 \\ 4x+6z=108 \\ 5z+7y=106 \end{cases}$ . . . . Ans. $\Big\{$

3. $\begin{cases} x+y+z=26 \\ x+y-z=-6 \\ x-y+z=12 \end{cases}$ . . . . Ans. $\Big\{$

4. $\begin{cases} x+\frac{y}{2}=100 \\ y+\frac{z}{3}=100 \\ z+\frac{x}{4}=100 \end{cases}$ . . . . Ans. $\Big\{$

5. $\begin{cases} 2x-y+z=9 \\ x-2y+3z=14 \\ 3x+4y-2z=7 \end{cases}$ . . . . Ans. $\Big\{$

6. $\begin{cases} \frac{x}{3}-\frac{y}{2}+z=3 \\ \frac{x}{6}+\frac{y}{4}-\frac{z}{3}=1 \\ \frac{x}{2}-\frac{y}{4}+z=5 \end{cases}$ . . . . Ans. $\Big\{$

PROBLEMS PRODUCING EQUATIONS CONTAINING THREE OR MORE UNKNOWN QUANTITIES.

163. When a problem contains *three* or more unknown quantities, the equations may be formed according to the directions given in Art. 156 and 161.

Remark.—When one or more of the unknown quantities can be expressed in terms of another, it is best to reduce the number of equations and symbols by doing so.

Review.—162. What is the general rule for elimination by addition and subtraction? When is elimination by substitution to be preferred? When that by comparison?

1. A has 3 ingots, composed of different metals in different proportions; 1 lb. of the first contains 7 oz. of silver, 3 of copper, and 6 of tin; 1 lb. of the second contains 12 oz. of silver, 3 of copper, and 1 of tin; and 1 lb. of the third contains 4 oz. of silver, 7 of copper, and 5 of tin. How much of each must be taken to form an ingot of 1 lb. weight, containing 8 oz. of silver, $3\frac{3}{4}$ of copper, and $4\frac{1}{4}$ of tin?

Let x, y, z, represent the number of oz. taken of the 3 ingots respectively.

Then, since 16 oz. of the first contains 7 oz. of silver, 1 oz. will contain $\frac{7}{16}$ oz. of silver; and x oz. will contain $\frac{7x}{16}$ oz. of silver

In like manner, y oz. of the second will contain $\frac{12y}{16}$ oz. of silver; and z oz. of the third will contain $\frac{4z}{16}$ oz. of silver.

But, by the question, the number of oz. of silver in a pound of the new ingot, is to be 8; hence,

$$\frac{7x}{16}+\frac{12y}{16}+\frac{4z}{16}=8.$$

Or, by clearing it of fractions,

$$7x+12y+4z=128 \quad (1.)$$

Reasoning in a similar manner with reference to the copper and the tin, we have the two following equations:

$$3x+3y+7z=60 \quad (2.)$$
$$6x+y+5z=68 \quad (3.)$$

The terms containing y being the simplest, will be most easily eliminated.

Multiplying (2) by 4, and subtracting (1), we have

$$5x+24z=112 \quad (4.)$$

Multiplying (3) by 3, and subtracting (2), we have

$$15x+8z=144 \quad (5.)$$

REVIEW.—168. Upon what principle are equations formed, when a problem contains three or more unknown quantities? When may we reduce the number of symbols?

Multiplying (5) by 3, and subtracting (4), there results

$$40x=320$$
$$x=8$$

Substituting this value of x in equation (5), we have

$$120+8z=144$$
$$z=3$$

And substituting these values of x and z in equation (3),

$$48+y+15=68$$
$$y=5$$

Hence, the new ingot will contain 8 oz. of the first, 5 of the second, and 3 of the third.

2. The sums of three numbers, taken two and two, are 27, 32, and 35; required the numbers.

3. The sum of three numbers is 59; $\frac{1}{2}$ the difference of the first and second is 5, and $\frac{1}{2}$ the difference of the first and third is 9; required the numbers.

4. A person bought three silver watches; the price of the first, with $\frac{1}{2}$ the price of the other two, was \$25; the price of the second, with $\frac{1}{3}$ the price of the other two, was \$26; and the price of the third, with $\frac{1}{2}$ the price of the other two, was \$29; required the price of each.

5. Find three numbers, such that the first with $\frac{1}{3}$ of the other two, the second with $\frac{1}{4}$ of the other two, and the third with $\frac{1}{5}$ of the other two, shall each equal 25.

6. A boy bought at one time 2 apples and 5 pears, for 12 cts.; at another, 3 pears and 4 peaches, for 18 cts.; at another, 4 pears and 5 oranges, for 28 cts.; and at another, 5 peaches and 6 oranges, for 39 cts.; required the cost of each kind of fruit.

7. A and B together possess only $\frac{2}{3}$ as much money as C; B and C together have 6 times as much as A; and B has $680 less than A and C together; how much has each?

8. A, B, and C compare their money; A says to B, "give me $700, and I shall have twice as much as you will have left." B says to C, "give me $1400, and I shall have three times as much as you will have left." C says to A, "give me $420, and I shall have five times as much as you will have left." How much has each?

9. A certain number is expressed by three figures, whose sum is 11; the figure in the place of units is double that in the place of hundreds; and if 297 be added to the number, its figures will be inverted; required the number.

10. The sum of 3 numbers is 83; if from the first and second you subtract 7, the remainders are as 5 to 3; but if from the second and third you subtract 3, the remainders are to each other as 11 to 9; required the numbers.

11. Divide $180 among three persons, A, B, and C, so that twice A's share plus $80, three times B's share plus $40, and four times C's share plus $20, may be all equal to each other.

12. If A and B can perform a certain work in 12 days, A and C in 15 days, and B and C in 20 days, in what time could each do it alone?

13. A number expressed by three figures, when divided by the sum of the figures plus 9, gives a quotient of 19; the middle figure equals half the sum of the first and third; and if 198 be added to the number, we obtain a number with the same figures in an inverted order; what is the number?

14. A farmer mixes barley at 28 cents, with rye at 36, and wheat at 48 cents per bu., so that the whole is 100 bu., and worth 40 cents per bu. Had he put twice as much rye, and 10 bu. more wheat, the whole would have been worth exactly the same per bu.; how much of each was there?

15. A, B, and C killed 96 birds, which they wish to share equally; to do this, A, who has the most, gives to B and C as many as they already had; next, B gives to A and C as many as they had after the first division; lastly, C gives to A and B as many as they both had after the second division, and each then had the same number; how many had each at first?

GENERAL REVIEW.

What two parts in the solution of a problem? What are explicit conditions? Implied conditions? Rule for forming an equation. On what condition may you change the sign of one term in an equation?

Define elimination. How many methods of elimination? Define elimination by substitution—by comparison—by addition and subtraction. Rule for each method. How state a problem containing two unknown quantities? How one containing three or more unknown quantities? When is the first method of elimination preferred? When the second? The third? Rule for elimination in three or more unknown quantities.

Give two rules for rendering a complex fraction simple. State the eight theorems, Arts. 80 to 85. Rule for exponents in multiplication. In division. Difference between subtraction in algebra and in arithmetic. In clearing an equation of fractions, what is to be done when there is a minus sign before a fraction?

Define binomial. Term. Coefficient. Exponent. Factor. Prime number. Composite number. What is the reciprocal of a fraction? What are the factors of x^2-1, of x^3-1, of x^3+1, of x^2+3x+2? By how many different methods could you reduce $\frac{1}{2}$, $\frac{1}{3}$, $\frac{5}{6}$, and $\frac{7}{12}$ to a common denominator?

In what cases may cancellation be employed to advantage? What three methods of multiplying a fraction by a whole number? Of dividing a fraction by a whole number? What are infinite series? What the law of a series? How convert $\frac{5}{11}$ into an infinite series?

V. SUPPLEMENT TO SIMPLE EQUATIONS.

GENERALIZATION.

164. A **Literal Equation** is an equation in which the known quantities are represented, either entirely or partly, by letters.

Values expressed by letters are termed *general*, because, by giving particular values to the letters, the solution of one problem furnishes a *general* solution to all others of the same kind.

A **Formula** is the answer to a problem, when the known quantities are represented by letters.

A **Rule** is a formula expressed in ordinary language.

By the application of Algebra to the solution of general questions, a great number of useful and interesting truths and rules may be established.

We now proceed to illustrate this subject by a few examples.

165.—1. Let it be required to find a number, which being divided by 3, and by 5, the sum of the quotients will be 16.

Let $x=$ the number; then, $\frac{x}{3}+\frac{x}{5}=16.$

$$5x+3x=16\times15$$
$$8x=16\times15$$
$$x=2\times15=30$$

2. Again, let it be required to find another number, which being divided by 4, and by 7, the sum of the quotients will be 11.

By proceeding as in the above question, we find the number to be 28.

Instead of solving every example of the same kind separately, we may give a general solution, that will embrace all the particular questions; thus:

3. Let it be required to find a number, which being divided by two given numbers, a and b, the sum of the quotients may be equal to another given number, c.

Let x= the number; then, $\frac{x}{a}+\frac{x}{b}=c$.

$$bx+ax=abc$$
$$(a+b)x=abc$$
$$x=\frac{abc}{a+b}.$$

The answer is termed a formula; it shows that the required number is equal to the continued product of a, b, and c, divided by the sum of a and b. Or, it may be expressed thus:

Multiply together the three given numbers, and divide the product by the sum of the divisors; the result will be the required number.

The pupil may test the accuracy of this rule by solving the following examples, and verifying the results:

4. Find a number which being divided by 3, and by 7, the sum of the quotients may be 20.

5. Find a number which being divided by $\frac{1}{3}$ and $\frac{1}{4}$, the sum of the quotients may be 1.

166.—1. The sum of \$500 is to be divided between two persons, A and B, so that A may have \$50 less than B.

To make this question general, let it be stated as follows:

REVIEW.—164. What is a literal equation? When are values termed general? What is a formula? What is a formula called when expressed in ordinary language?

165. Example 3. What is the answer to this question, expressed in ordinary language?

2. To divide a given number, a, into two such parts, that their difference shall be b; or, the sum of two numbers is a, and their difference b; required the numbers.

Let $x=$ the greater number, and $y=$ the less.

Then, $x+y=a$

And $x-y=b$

By addition, $2x=a+b$

$$x=\frac{a+b}{2}=\frac{a}{2}+\frac{b}{2}$$

By subtraction, $2y=a-b$

$$y=\frac{a-b}{2}=\frac{a}{2}-\frac{b}{2}$$

This formula, expressed in ordinary language, gives the following

RULE,

FOR FINDING TWO QUANTITIES, WHEN THEIR SUM AND DIFFERENCE ARE GIVEN.

1. *To find the greater, add half the difference to half the sum.*

2. *To find the less, subtract half the difference from half the sum.*

Test the accuracy of the rule, by finding the two numbers in the following examples:

3. Sum 200, difference 50. . . . . . Ans. 125, 75.
4. Sum 100, difference 25. . . . . . Ans. $62\frac{1}{2}$, $37\frac{1}{2}$.
5. Sum 15, difference 10. . . . . . Ans. $12\frac{1}{2}$, $2\frac{1}{2}$.
6. Sum $5\frac{1}{2}$, difference $\frac{3}{4}$. . . . . . Ans. $3\frac{1}{8}$, $2\frac{3}{8}$.

167.—1. A can do a piece of work in 3 da., and B in 4 da.; in what time can both together do it?

Ans. $1\frac{5}{7}$ da.

To make this question general, let it be stated thus:

2. A can do a piece of work in m da., and B in n da.; in how many da. can they both together do it?

Let $x=$ the number of da. in which they can both do it.

Then, $\frac{1}{x}=$ the part of the work which both can do in one da.

Also, A can do $\frac{1}{m}$ part and B can do $\frac{1}{n}$ part of it in 1 da. Hence, the part of the work which both can do in 1 da. is represented by $\frac{1}{m}+\frac{1}{n}$, and also by $\frac{1}{x}$.

Therefore, ... $\frac{1}{m}+\frac{1}{n}=\frac{1}{x}$.

$$nx+mx=mn$$

$$x=\frac{mn}{m+n}.$$

This result, expressed in ordinary language, gives the following

Rule.—*Divide the product of the numbers expressing the time in which each can perform the work, by their sum; the quotient will be the time in which they can jointly perform it.*

The question can be made more general, thus:

A can produce a certain effect, e, in a time, t; B can produce the same effect, in a time, t'; in what time can they both do it?

The result and the rule would be the same as already given.

The following examples will illustrate the rule:

3. A cistern is filled by one pipe in 6, and by another in 9 hr.; in what time will it be filled by both together?

4. One man can drink a keg of cider in 5 da., and another in 7 da.; in what time can both together drink it?

REVIEW.—166. By what rule do we find two quantities, when their sum and difference are given?

167. When the times are given, in which each of two men can produce a certain effect, how is the time found in which they can jointly produce it?

168. Let it be required to find a rule for dividing the gain or loss in a partnership. First, take a particular question.

1. A, B, and C engage in trade, and put in stock in the following proportions: A put in \$3 as often as B put in \$4, and as often as C put in \$5. They gain \$60; required the share of each, it being divided in proportion to the stock put in.

Let $3x=$ A's share of the gain; then, $4x=$ B's, and $5x=$ C's. (See Example 24, page 126.)

Then, $3x+4x+5x=60$.

$$12x=60$$

$$x=5$$

$3x=15$, A's share; $4x=20$, B's; and $5x=25$, C's share.

2. To make this question general, suppose A puts in m \$'s as often as B puts in n \$'s, and as often as C puts in r \$'s, and that they gain c \$'s. Find the share of each.

Let the share of A be denoted by mx; then, $nx=$ B's, and $rx=$ C's share. Then, $mx+nx+rx=c$.

$$mx=\frac{mc}{m+n+r};\quad nx=\frac{nc}{m+n+r};\quad rx=\frac{rc}{m+n+r}.$$

If c had represented loss instead of gain, the same solution would have applied. Hence, to find each partner's share of the gain or loss, we have the following

Rule.—*Divide the whole gain or loss by the sum of the proportions of stock, and multiply the quotient by each partner's proportion, to obtain his respective share.*

When the times in which the respective stocks are employed are different, it becomes necessary to reduce them to the same time, to ascertain what proportion they bear to each other.

Review.—168. How is the gain or loss in fellowship found, when the times in which the stock is employed are the same? How, when different?

Thus, if A have \$3 in trade 4 mon., and B \$2 for 5 mon., we see that \$3 for 4 mon. are the same as \$12 for 1 mon.; and \$2 for 5 mon. are the same as \$10 for 1 mon. Therefore, in this case, the stocks are in the proportion of 12 to 10.

Hence, when time in fellowship is considered, we have the following

Rule.—*Multiply each man's stock by the time it was employed to find the proportions of stock; and then proceed according to the preceding rule.*

3. A, B, and C engaged in trade; A put in \$200, B \$300, and C \$700; they lost \$60; what was each man's share?

Since the sums engaged are to each other as 2, 3, and 7, we may either use these numbers, or those representing the stock.

4. In a trading expedition, A put in \$200 for 3 mon., B \$150 for 5 mon., and C \$100 for 8 mon.; they gained \$215; what was each man's share?

169.—1. Two men, A and B, can perform a certain piece of work in a da., A and C in b da., and B and C in c da.; in what time could each one alone perform it? In what time could they perform it, all working together?

Let x, y, and z represent the days in which A, B, and C can respectively do it.

Then, $\frac{1}{x}$, $\frac{1}{y}$, and $\frac{1}{z}$, represent the parts of the work which A, B, and C can each do in 1 da.

Since A and B can do it in a da., they do $\frac{1}{a}$ part of it in 1 da.

Hence, $\frac{1}{x}+\frac{1}{y}=\frac{1}{a}$ (1). In like manner, we have

$$\frac{1}{x}+\frac{1}{z}=\frac{1}{b} \quad (2)$$

$$\frac{1}{y}+\frac{1}{z}=\frac{1}{c} \quad (3)$$

$$\frac{2}{x}+\frac{2}{y}+\frac{2}{z}=\frac{1}{a}+\frac{1}{b}+\frac{1}{c}, \text{ adding (1), (2), and (3).}$$

$$\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=\frac{1}{2a}+\frac{1}{2b}+\frac{1}{2c} \quad \text{(4), dividing by 2.}$$

$$\frac{1}{x}=\frac{1}{2a}+\frac{1}{2b}-\frac{1}{2c}=\frac{bc+ac-ab}{2abc}, \text{ subtracting (3) from (4);}$$

or, $x(ac+bc-ab)=2abc$, by clearing of fractions.

$$x=\frac{2abc}{ac+bc-ab}.$$

In like manner, $y=\frac{2abc}{ab+bc-ac}$.

And $z=\frac{2abc}{ab+ac-bc}$.

Since $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}$, or $\frac{1}{2a}+\frac{1}{2b}+\frac{1}{2c}$, represents the part all can do in 1 da.; if we divide 1 by $\left(\frac{1}{2a}+\frac{1}{2b}+\frac{1}{2c}\right)$, the quotient, $\frac{2abc}{ab+ac+bc}$, will represent the number of da. in which all can do it.

170. In solving questions, it is sometimes necessary to use general values for particular quantities, to ascertain the relation which they bear to each other; as in the following:

If 4 A. pasture 40 sheep 4 wk., and 8 A. pasture 56 sheep 10 wk., how many sheep will 20 A. pasture 50 wk., the grass growing uniformly all the time?

The chief difficulty in solving this question, consists in ascertaining the relation that exists between the original quantity of grass on an A., and the growth on each A. in 1 wk.

Let $m=$ the quantity on an A. when the pasturage began, and $n=$ the growth on 1 A. in 1 wk.; m and n representing lb., or any other measure of the quantity of grass.

Then, $4n=$ the growth on 1 A. in 4 wk.

And $16n=$ the growth on 4 A. in 4 wk.

Also, $4m+16n=$ the whole amount of grass on 4 A. in 4 wk.

If 40 sheep eat $4m+16n$ in 4 wk., then 40 sheep will eat

$$\frac{4m+16n}{4}=m+4n \text{ in 1 wk.}$$

And 1 sheep eats $\frac{m+4n}{40}=\frac{m}{40}+\frac{n}{10}$ in 1 wk.

Again, $8m+80n=$ the whole amount of grass on 8 A. in 10 wk.

If 56 sheep eat $8m+80n$ in 10 wk.,

Then, 56 sheep eat $\frac{8m}{10}+8n$ in 1 wk.

And 1 sheep eats $\frac{8m}{560}+\frac{8n}{56}=\frac{m}{70}+\frac{n}{7}$ in 1 wk.

$$\text{Hence, } \frac{m}{40}+\frac{n}{10}=\frac{m}{70}+\frac{n}{7}.$$

$$\text{Or, } 7m+28n=4m+40n.$$

$$3m=12n$$

$$m=4n$$

Or, $n=\frac{1}{4}m$; hence, the growth on 1 A. in 1 wk., is equal to $\frac{1}{4}$ of the original quantity on 1 A.

Then, 1 sheep, in 1 wk., eats $\frac{m}{40}+\frac{n}{10}=\frac{m}{40}+\frac{m}{40}=\frac{m}{20}$.

And 1 sheep, in 50 wk., eats, $\frac{m}{20}\times 50=\frac{5m}{2}$

20 A. have an original quantity of grass, denoted by $20m$.

The growth of 1 A. in 1 wk. being $\frac{1}{4}m$, in 50 wk. it will be $\frac{50m}{4}$

And the growth of 20 A in 50 wk., will be $\frac{50m}{4}\times 20=250m$.

Then, $20m+250m=270m$, the whole amount of grass on 20 A. in 50 wk.

Then, $270m\div\frac{5m}{2}=\frac{540m}{5m}=108$, the number of sheep required.

GENERAL PROBLEMS.

1. Divide the number a into two parts, so that one of them shall be n times the other.

2. Divide the number a into two parts, so that m times one part shall be equal to n times the other.

3. Find a number which being divided by m, and by n, the sum of the quotients shall be equal to a.

4. What number must be added to a and b, so that the sums shall be to each other as m to n?

5. What number must be subtracted from a and b, so that the differences shall be to each other as m to n?

*6. After paying away $\frac{1}{m}$ and $\frac{1}{n}$ of my money, I had a dollars left: how many dollars had I at first?

7. A company paid for the use of a boat for an excursion, a cents each; if there had been b persons less, each would have had to pay c cents; how many persons were there?

8. A farmer mixes oats at a cents per bu., with rye at b cents per bu., so that a bu. of the mixture is worth c cents; how many bu. of each will n bu. of the mixture contain?

9. A person borrowed as much money as he had in his purse, and then spent a cents; again, he borrowed as much as he had in his purse, after which he spent a cents; he borrowed and spent, in the same manner, a third and fourth time, after which, he had nothing left; how much had he at first?

10. A person has 2 kinds of coin; it takes a pieces of the first, and b pieces of the second, to make \$1; how many pieces of each kind must be taken, so that c pieces, may be equivalent to \$1?

171. Sometimes in an equation of the first degree, the second, or some higher power of the unknown quantity occurs, but in such a manner that it may be made to disappear. The following examples belong to this class:

1. Given $2x^2+8x=11x^2-10x$, to find the value of x.

By dividing each side by x, we have

$$2x+8=11x-10, \text{ from which } x=2.$$

2. Given $(4+x)(3+x)-6(10-x)=x(7+x)$, to find x.

Performing the operations indicated, we have

$$12+7x+x^2-60+6x=7x+x^2.$$

Omitting the quantities on each side which are equal, we have

$$12-60+6x=0, \text{ from which } x=8.$$

3. $3x^2-8x=24x-5x^2$. . . . . .

4. $3ax^3-10ax^2=8ax^2+ax^3$. . . . .

5. $\frac{6x+13}{15}-\frac{3x+5}{5x-25}=\frac{2x}{5}$. . . . .

6. $(a+x)(b+x)-a(c-b)=x(b+x)$.

7. $x+a+b+c=\frac{x^2+a^2+b^2+c^2}{a+b-c+x}$. . .

8. If a certain book had 5 more pages, with 10 more lines on a page, the number of lines would be increased 450; if it had 10 pages less, with 5 lines less on a page, the whole number of lines would be diminished 450; required the number of pages, and of lines on a page.

NEGATIVE SOLUTIONS.

172. It sometimes happens, in the solution of a problem, that the value of the unknown quantity is found to be *minus*. Such a result is termed a *negative solution.* We shall now examine a question of this kind.

1. What number must be added to the number 5, that the sum shall be equal to 3?

Let x= the number.

Then, $5+x=3$; and $x=3-5=-2$.

Now, -2 added to 5, gives 3; thus, $5+(-2)=3$. The result, -2, is said to satisfy the question in an *algebraic sense;* but the problem is evidently impossible in an *arithmetical sense.*

Since adding -2 is the same as subtracting $+2$, Art. 61, the result is the answer to the following question: What number must be *subtracted* from 5, that the remainder may be equal to 3?

Let the question now be made general, thus:

What number must be added to the number a, that the sum shall be equal to b?

Let x= the number. Then, $a+x=b$; and $x=b-a$.

Now, since $a+(b-a)=b$, this value of x will always satisfy the question in an algebraic sense.

While b is greater than a, the value of x will be *positive*, and the question will always be consistent in an *arithmetical sense.* Thus, if $b=10$, and $a=8$, then $x=2$.

When b is less than a, the value of x will be *negative;* the question will then be true in its *algebraic*, but not in its *arithmetical sense*, and should be stated thus: What number must be *subtracted* from a, that the *remainder* may be equal to b? Hence,

1. *A negative solution indicates some inconsistency or absurdity in the question from which the equation was derived.*

2. *When a negative solution is obtained, the question to which it is the answer may be so modified as to be consistent.*

REVIEW.—172. What is a negative solution? When is a result said to satisfy a question in an algebraic sense? In an arithmetical sense? What does a negative solution indicate?

Let the pupil now read the OBSERVATIONS ON ADDITION AND SUBTRACTION, page 27, and then modify the following questions, so that they shall be consistent in an *arithmetical sense.*

2. What number must be *subtracted* from 20, that the *remainder* shall be 25? ($x=-5$.)

3. What number must be *added* to 11, that the *sum* being multiplied by 5, the product shall be 40? ($x=-3$.)

4. What number is that whose $\frac{2}{3}$ *exceeds* its $\frac{3}{4}$ by 3? ($x=-36$.)

5. A father, whose age is 45 yr., has a son aged 15; *in how many yr.* will the son be $\frac{1}{4}$ as old as his father? ($x=-5$.)

DISCUSSION OF PROBLEMS.

173. When a question has been solved in a general manner, that is, by representing the known quantities by letters, we may inquire what values the results will have when particular suppositions are made with regard to the known quantities.

The determination of these values, and the examination of the results, constitute what is termed the *discussion* of the problem.

Let us take, for example, the following question:

1. After subtracting b from a, what number, multiplied by the remainder, will give a product equal to c?

Let $x=$ the number.

Then, $(a-b)x=c$, and $x=\frac{c}{a-b}$.

REVIEW.—172. When a negative solution is obtained, how may the question to which it is the answer be modified?

173. What is understood by the discussion of a problem? The expression c divided by $a-b$, may have how many forms? Name them.

This result may have five different forms, depending on the values of a, b, and c.

We shall examine each of these in succession.

I. When b is less than a.

In this case, $a-b$ is a positive quantity, and the value of x is positive.

To illustrate this form, let $a=8$, $b=3$, and $c=20$; then, $x=4$.

II. When b is greater than a.

In this case, $a-b$ is a negative quantity, and the value of x will be negative.

To illustrate this case by numbers, let $a=2$, $b=5$, and $c=12$; then, $a-b=-3$, and $x=-4$.

III. When b is equal to a.

In this case, x becomes equal to $\frac{c}{0}$.

We must now inquire, what is the value of a fraction when the denominator is zero.

If we divide c successively by 1, $\frac{1}{10}$, $\frac{1}{100}$, $\frac{1}{1000}$, etc., the quotients will be c, $10c$, $100c$, $1000c$, etc.

As the denominator *decreases*, the value of the fraction *increases*. Hence, if the denominator be *less* than any assignable quantity, that is 0, the value of the fraction will be *greater* than any assignable quantity, that is, infinitely great. This is designated by the sign ∞, that is,

$$\frac{c}{0}=\infty.$$

IV. When c is o, and b is either greater or less than a.

If we put $a-b$ equal to d, then $x=\frac{0}{d}$.

We are now to determine the value of a fraction whose numerator is zero.

Review.—173. When is the value of x positive? When negative? When infinite? Show how the value of a fraction increases as its denominator decreases. Value of a fraction whose denominator is zero? Of x when c is 0, and b greater or less than a?

The value of a fraction *decreases* as the numerator *decreases*. Hence, if the numerator be less than any assignable quantity, that is 0, the value of the fraction is zero, or $\frac{0}{d}=0$.

V. When $b=a$, and $c=0$.

In this case, we have $x=\frac{c}{a-b}=\frac{0}{0}$.

If *any quantity* be put into the form of a fraction, and both terms be divided continually by the same quantity, the value of the fraction will remain unchanged, but the final result will be of the form $\frac{0}{0}$. This form is therefore expressive of any finite value whatever. Hence,

We say that $\frac{0}{0}$ is the symbol of indetermination; that is, the quantity which it represents has no particular value.

The form $\frac{0}{0}$ sometimes arises from a particular supposition, when the terms of a fraction contain a common factor. Thus, if $x=\frac{a^2-b^2}{a-b}$, and we make $b=a$, it reduces to $\frac{a^2-a^2}{a-a}=\frac{0}{0}$; but, if we cancel $a-b$, and then make $b=a$, we have $x=2a$. Hence,

Before deciding the value to be indeterminate, we must see that this form has not arisen from the existence of a factor whose value, by a particular supposition, is zero.

The discussion of the following problem, originally proposed by Clairaut, will serve to illustrate further the preceding principles.

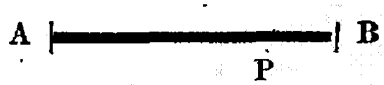
PROBLEM OF THE COURIERS.

Two couriers depart at the same time, from two places, A and B, distant a mi. from each other; the former travels m mi. an hour, and the latter n mi.; where will they meet?

There are two cases of this question.

I. When the couriers travel toward each other.

Let P be the point where they meet, and a=AB, the distance between the two places.



Let x=AP, the distance which the first travels.

Then, $a-x$=BP, the distance which the second travels.

The distance each travels, divided by the number of mi. traveled in 1 hr. will give the number of hr. he was traveling.

Therefore, $\frac{x}{m}$= the number of hr. the first travels.

And $\frac{a-x}{n}$= the number of hr. the second travels.

But they both travel the same number of hr., therefore,

$$\frac{x}{m}=\frac{a-x}{n}$$

$$nx=am-mx$$

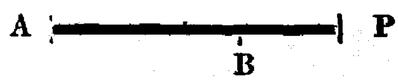
$$x=\frac{am}{m+n}$$

$$a-x=\frac{an}{m+n}$$

1st. Suppose $m=n$; then, $x=\frac{am}{2m}=\frac{a}{2}$, and $a-x=\frac{a}{2}$; that is, if they travel at the same rate, each travels half the distance.

2d. Suppose $n=0$; then, $x=\frac{am}{m}=a$; that is, if the second courier remains at rest, the first travels the whole distance from A to B. These results correspoud to the circumstances of the problem.

II. When the couriers travel in the same direction.



As before, let P be the point of meeting, each traveling in that direction, and let a=AB, the distance between the places.

x=AP, the distance the first travels.

$x-a$=BP, the distance the second travels.

Then, reasoning as in the first case, we have

$$\frac{x}{m}=\frac{x-a}{n}$$

$$nx=mx-am$$

$$x=\frac{am}{m-n}$$

$$x-a=\frac{an}{m-n}$$

1st. If we suppose m greater than n, the value of x will be positive; that is, the couriers will meet on the right of B. This evidently corresponds to the circumstances of the problem.

2d. If we suppose n greater than m, the value of x, and also that of $x-a$, will be negative. This negative value of x shows that the point of meeting is to the left of A.

Indeed, when m is less than n, it is evident that they can not meet, since the forward courier is traveling faster than the other. We may, however, suppose that they *had* met *previously*.

If we suppose the direction in which the couriers travel to be changed; that is, that the first travels from A, and the second from B toward P′; and put $x=AP'$, $a+x=BP'$, their values will be positive, and the question will be consistent; for we shall then have

$$\frac{x}{m}=\frac{a+x}{n}$$

$$x=\frac{am}{n-m}$$

$$a+x=\frac{an}{n-m}$$

3d. If we suppose $m=n$; then, $x=\frac{am}{0}$, and $x-a=\frac{an}{0}$.

These values of x, and $a-x$, being equal to infinity, Art. 173, it follows that if the couriers travel at the same rate, the one can *never* overtake the other. This is sometimes expressed by saying, they only meet at an *infinite* distance.

4th. If we suppose $a=0$; then, $x=\frac{0}{m-n}$, and $x-a=\frac{0}{m-n}$.

It has been shown already, that these values are equal to 0. Hence, if the couriers are *no* distance apart, they will have to travel *no* (0) distance to be together.

5th. If we suppose $m=n$, and $a=0$; then, $x=\frac{0}{0}$, and $x-a=\frac{0}{0}$.

It has already been shown that this form is expressive of *any finite value* whatever. Hence, if the couriers are *no* distance apart, and travel at the *same* rate, they will be *always* together.

REVIEW.—173. What is the value of x when $b=a$ and $c=0$? Of a fraction whose terms are both zero? How does this form sometimes arise?

173. Discuss the problem of the "Couriers," and show, that in every hypothesis the solution corresponds to the circumstances of the problem.

Lastly, if we suppose $n=0$; then, $x=\frac{am}{m}=a$; that is, the first courier travels from A to B, overtaking the second at B.

If we suppose $n=\frac{m}{2}$; then, $x=\frac{2am}{m}=2a$, and the first travels twice the distance from A to B, before overtaking the second.

CASES OF INDETERMINATION IN SIMPLE EQUATIONS, AND IMPOSSIBLE PROBLEMS.

174. An **Independent Equation** is one in which the relation of quantities which it contains, can not be obtained directly from others with which it is compared. Thus, the equations,

$$x+2y=11$$
$$2x+5y=26$$

are independent of each other, since the one can not be obtained from the other in a direct manner. But the equations,

$$x+2y=11$$
$$2x+4y=22$$

are not independent of each other, the second being derived directly from the first by multiplying both sides by 2.

175. An **Indeterminate Equation** is one that can be verified by different values of the same unknown quantity.

Thus, in the equation $x-y=5$, by transposing y, we have $x=5+y$.

If we make $y=1$, $x=6$. If we now make $y=2$, $x=7$, and so on; from which it is evident that an *unlimited* number of values may be given to x and y. that will verify the equation.

If we have two equations containing three unknown quantities, we may eliminate one of them; this will leave a single equation containing two unknown quantities, which, as in the preceding example, will be indeterminate. Thus, if we have

$x+3y+z=10$; and

$x+2y-z=6$; if we eliminate x, we have

$y+2z=4$; from which $y=4-2z$.

Putting $z=1$, $y=2$, and $x=10-3y-z=3$.
Putting $z=1\frac{1}{2}$, $y=1$, and $x=5\frac{1}{2}$; and so on.

Other examples might be given, but these are sufficient to show, that

When the number of unknown quantities exceeds the number of independent equations, the problem is indeterminate.

A question is sometimes indeterminate that involves only one unknown quantity; the equation deduced from the conditions, being of that class denominated identical; as the following:

What number is that, of which the $\frac{3}{4}$, diminished by the $\frac{2}{3}$, is equal to the $\frac{1}{20}$ increased by the $\frac{1}{30}$?

Let $x=$ the number.

Then, $\frac{3x}{4}-\frac{2x}{3}=\frac{x}{20}+\frac{x}{30}$.

Clearing of fractions, $45x-40x=3x+2x$; or, $5x=5x$, which will be verified by any value of x whatever.

176. The reverse of the preceding case requires to be considered; that is, when the number of equations is greater than the number of unknown quantities. Thus, we may have

$$x+y=10 \quad (1.)$$
$$x-y=4 \quad (2.)$$
$$2x-3y=5 \quad (3.)$$

Each of these equations being independent of the other two, one of them is unnecessary, since the values of x and y, which are 7 and 3, may be determined from any two of them.

When a problem contains more conditions than are necessary for determining the values of the unknown quantities, those that are unnecessary are termed *redundant conditions.*

The number of equations may exceed the number of unknown quantities, so that the values of the unknown quantities shall be incompatible with each other. Thus, if we have

$$x+y=9 \quad (1.)$$
$$x+2y=13 \quad (2.)$$
$$2x+3y=21 \quad (3.)$$

The values of x and y, found from equations (1) and (2), are $x=5$, $y=4$; from equations (1) and (3), are $x=6$, $y=3$; and from equations (2) and (3), are $x=3$, $y=5$. From this it is manifest, that only two of these equations can be true at the same time.

A question that contains only one unknown quantity is sometimes impossible; as the following:

What number is that, of which the $\frac{1}{2}$ and $\frac{1}{3}$ diminished by 4, is equal to the $\frac{5}{6}$ increased by 8?

Let $x=$ the number; then, $\frac{x}{2}+\frac{x}{3}-4=\frac{5x}{6}+8$.

Clearing of fractions, $3x+2x-24=5x+48$.

By subtracting equals from each side, $0=72$, which shows that the question is absurd.

177. Take the equation $ax-cx=b-d$, in which a represents the sum of the positive, and $-c$ the sum of the negative coëfficients of x; b the sum of the positive, and $-d$ the sum of the negative known quantities.

This will evidently express a simple equation involving one unknown quantity, in its most general form.

This gives $(a-c)x=b-d$.

Let $a-c=m$, and $b-d=n$.

We then have $mx=n$, or $x=\frac{n}{m}$.

Now, since n divided by m can give but one quotient, we infer that *an equation of the first degree has but one root;* that is, in a simple equation involving but one unknown quantity, there is but one value that will verify the equation.

Review.—174. When is an equation termed independent? Example. 175. When said to be indeterminate? Example. 176. What are redundant conditions?

VI. OF POWERS, ROOTS, AND RADICALS.

INVOLUTION, OR FORMATION OF POWERS.

178. The **Power** of a quantity is the product arising from multiplying the quantity by itself a certain number of times.

The **Root** of the power is the quantity to be multiplied.

Thus, a^2 is called the *second* power of a, because a is taken *twice* as a factor; and a is called the *second* root of a^2.

So, also, a^3 is the *third* power of a, because $a\times a\times a=a^3$, a being taken *three* times as a factor; and a is the *third* root of a^3.

The second power is generally called the *square*, and the second root, the *square* root. In like manner, the third power is called the *cube*, and the third root, the *cube* root.

The **Exponent** is the figure indicating the power to which the quantity is to be raised. It is written on the right, and a little above the quantity. See Arts. 33 and 35.

CASE I.

TO RAISE A MONOMIAL TO ANY GIVEN POWER.

179.—1. Let it be required to raise $2ab^2$ to the third power.

According to the definition, the third power of $2ab^2$ will be the product arising from taking it *three* times as a factor.

Thus, $(2ab^2)^3=2ab^2\times2ab^2\times2ab^2=2\times2\times2aaab^2b^2b^2$
$=2^3\times a^{1+1+1}\times b^{2+2+2}=2^3\times a^{1\times3}\times b^{2\times3}=8a^3b^6$.

The coëfficient of the power is found by raising the coëfficient, 2, of the root, to the given power; and the exponent of each letter, by multiplying the exponent of the letter in the root by 3, the index of the required power.

180. With regard to the signs of the different powers, there are two cases.

First, when the root is *positive;* and second, when *negative.*

1st. When the root is positive.

Since the product of any number of positive factors is always positive, it is evident that if the root is positive, all the powers will be positive.

Thus, $+a\times+a=+a^2$

$+a\times+a\times+a=+a^3$; and so on.

2d. When the root is negative.

Let us examine the different powers of a negative quantity, as $-a$.

$-a=$ first power, *negative.*

$-a\times-a=+a^2=$ second power, *positive.*

$-a\times-a\times-a=-a^3=$ third power, *negative.*

$-a\times-a\times-a\times-a=+a^4=$ fourth power, *positive.*

$-a\times-a\times-a\times-a\times-a=-a^5=$ fifth power, *negative.*

From this we see that the even powers of a negative quantity are *positive*, and the odd powers *negative.* Hence,

TO RAISE A MONOMIAL TO ANY GIVEN POWER,

Rule.—1. *Raise the numeral coëfficient to the required power, and multiply the exponent of each letter by the exponent of the power.*

2. *If the monomial is positive, all the powers will be positive; if negative, the even powers will be positive, and the odd powers negative.*

1. Find the square of $3ax^2y^3$. .
2. Square of $5b^2c^3$. . . . .
3. Cube of $2x^2y^3$. . . . . .
4. Square of $-ab^2c$. . . . .
5. Cube of $-abc^2$. . . . .
6. Fourth power of $3ab^3c^2$. .
7. Fourth power of $-3ab^3c^2$. .
8. Fifth power of ab^3cd^2. . .
9. Fifth power of $-ab^3cd^2$. .
10. Seventh power of $-m^2n^3$. .
11. Eighth power of $-mn^2$. .
12. Cube of $-3xy^2$. . . . .
13. Fourth power of $5a^2x^3$. . .
14. Fourth power of $7a^2x^3$. . .
15. Fifth power of $-3a^2xy^2z^3$. .

CASE II.

TO RAISE A POLYNOMIAL TO ANY POWER,

181. Rule.—*Find the product of the quantity, taken as a factor as many times as there are units in the exponent of the power.*

1. Find the square of $ax+cy$.

2. Square of $1-x$. . . .
3. Square of $x+1$. . . .
4. Square of $2x^2-3y^2$. . .
5. Cube of $a+x$. . . . .
6. Cube of $x-y$. . . . .
7. Cube of $2x-1$. . . .

REVIEW.—177. Show that in an equation of the first degree, the unknown quantity can have but one value. 178. What does the term power denote? The term root? What is the second power of a? Why? The third power of a? Why? What is the second power generally called? The second root? What is the exponent? Where should it be written?

8. Find the square of $a-b+c-d$.

9. Find the cube of $2x^2-3x+1$.

CASE III.

TO RAISE A FRACTION TO ANY POWER,

182. Rule.—*Raise both numerator and denominator to the required power.*

1. Find the square of $\frac{a+b}{c-d}$. . .

2. Find the square of $\frac{2x}{3y}$. . . .

3. Find the cube of $-\frac{ac}{x^2y}$. . . .

4. Find the square of $-\frac{2x^2}{3y}$. . .

5. Find the square of $\frac{x-2}{x+3}$. . .

6. Find the cube of $\frac{2a(x-y)}{3yz^2}$.

BINOMIAL THEOREM.

183. The Binomial Theorem, discovered by Sir Isaac Newton, explains the method of involving a binomial quantity without the tedious process of multiplication.

To illustrate it, we shall first, by means of multiplication, find the different powers of a binomial.

REVIEW.—179. In raising $2ab^2$ to the third power, how is the coëfficient of the power found? How the exponent of each letter? 180. When the root is positive, what is the sign of the different powers? When it is negative?

180. Rule for raising a monomial to any given power. 181. A polynomial. 182. A fraction. 183. What does the Binomial Theorem explain?

1. Let us first raise $a+b$ to the fifth power.

$$\begin{array}{l}
a+b \\
a+b \\ \hline
a^2+ab \\
\quad +ab+b^2 \\ \hline
a^2+2ab+b^2= \text{ . . . . . second power of } a+b, \text{ or } (a+b)^2. \\
a+b \\ \hline
a^3+2a^2b+ab^2 \\
\quad a^2b+2ab^2+b^3 \\ \hline
a^3+3a^2b+3ab^2+b^3= \text{ . . . third power of } a+b, \text{ or } (a+b)^3. \\
a+b \\ \hline
a^4+3a^3b+3a^2b^2+ab^3 \\
\quad +a^3b+3a^2b^2+3ab^3+b^4 \\ \hline
a^4+4a^3b+6a^2b^2+4ab^3+b^4= \text{ . . . . } (a+b)^4. \\
a+b \\ \hline
a^5+4a^4b+6a^3b^2+4a^2b^3+ab^4 \\
\quad +a^4b+4a^3b^2+6a^2b^3+4ab^4+b^5 \\ \hline
a^5+5a^4b+10a^3b^2+10a^2b^3+5ab^4+b^5= \text{ . . . . } (a+b)^5.
\end{array}$$

2. Let us next raise $a-b$ to the fifth power.

$$\begin{array}{l}
a-b \\
a-b \\ \hline
a^2-ab \\
\quad -ab+b^2 \\ \hline
a^2-2ab+b^2= \text{ . . . . } (a-b)^2. \\
a-b \\ \hline
a^3-2a^2b+ab^2 \\
\quad -a^2b+2ab^2-b^3 \\ \hline
a^3-3a^2b+3ab^2-b^3= \text{ . . . . } (a-b)^3. \\
a-b \\ \hline
a^4-3a^3b+3a^2b^2-ab^3 \\
\quad -a^3b+3a^2b^2-3ab^3+b^4 \\ \hline
a^4-4a^3b+6a^2b^2-4ab^3+b^4= \text{ . . . . } (a-b)^4. \\
a-b \\ \hline
a^5-4a^4b+6a^3b^2-4a^2b^3+ab^4 \\
\quad -a^4b+4a^3b^2-6a^2b^3+4ab^4-b^5 \\ \hline
a^5-5a^4b+10a^3b^2-10a^2b^3+5ab^4-b^5= \text{ . . . . } (a-b)^5.
\end{array}$$

The first letter, a, is called the *leading* quantity; and the second letter, b, the *following* quantity.

184. In examining these examples, we discover *four* laws, as follows:

1st. The *number* of terms of the power.
2d. The *signs* of the terms.
3d. The *exponents* of the letters.
4th. The *coëfficients* of the terms.

Let us examine these four laws separately.

1st. **Of the Number of Terms.** As the second power has *three* terms; the *third* power, *four* terms; the *fourth*, *five* terms; the *fifth*, *six* terms; we infer, that

The number of terms in any power of a binomial is one greater than the exponent of the power.

2d. **Of the Signs of the Terms.** *When both terms are positive, all the terms will be positive.*

When the first term is positive, and the second negative, all the ODD *terms will be* POSITIVE, *and the* EVEN *terms* NEGATIVE.

3d. **Of the Exponents of the Letters.** If we omit the coëfficients, the remaining parts of the fifth powers of $a+b$ and $a-b$, are

$(a+b)^5$ . . . . $a^5+a^4b+a^3b^2+a^2b^3+ab^4+b^5$.
$(a-b)^5$ . . . . $a^5-a^4b+a^3b^2-a^2b^3+ab^4-b^5$.

An examination of these and the other powers of $a+b$ and $a-b$, shows that

1. *The exponent of the leading letter in the first term is the same as that of the power of the binomial; in the other terms, it decreases by unity from left to right, and disappears in the last term.*

2. *The following letter begins with an exponent of one, in the second term; increases by unity from left to right; and, in the last term, is the same as the power of the binomial.*

3. *The sum of the exponents of the two letters in any term is always the same, and is equal to the power of the binomial.*

Write the different powers of the following binomials without the coëfficients:

$(x+y)^3$ . . . $x^3+x^2y+xy^2+y^3$
$(x-y)^4$ . . . $x^4-x^3y+x^2y^2-xy^3+y^4$.
$(x+y)^5$ . . . $x^5+x^4y+x^3y^2+x^2y^3+xy^4+y^5$.
$(x-y)^6$ . . . $x^6-x^5y+x^4y^2-x^3y^3+x^2y^4-xy^5+y^6$.
$(x-y)^7$ . . . $x^7-x^6y+x^5y^2-x^4y^3+x^3y^4-x^2y^5+xy^6-y^7$.
$(x+y)^8$ . . . $x^8+x^7y+x^6y^2+x^5y^3+x^4y^4+x^3y^5+x^2y^6+xy^7+y^8$.

4th. **Of the Coefficients of the Terms.** The coëfficient of the first and last terms is always 1; the coëfficient of the second term is the same as that of the power of the binomial.

The law of the other coëfficients is as follows:

If the coëfficient of any term be multiplied by the exponent of the leading letter, and the product be divided by the number of that term from the left, the quotient will be the coëfficient of the next term.

Omitting the coëfficients, the terms of $a+b$ raised to the sixth power, are

$$a^6+a^5b+a^4b^2+a^3b^3+a^2b^4+ab^5+b^6.$$

The coëfficients, according to the above principles, are

$$1,\quad 6,\quad \frac{6\times5}{2},\quad \frac{15\times4}{3},\quad \frac{20\times3}{4},\quad \frac{15\times2}{5},\quad \frac{6\times1}{6}.$$

Or, 1, 6, 15, 20, 15, 6, 1.

Hence, $(a+b)^6=a^6+6a^5b+15a^4b^2+20a^3b^3+15a^2b^4+6ab^5+b^6$.

REVIEW.—184. In examining the different powers of a binomial, how many laws are discovered? What is the number of terms in any power of a binomial? Examples.

184. When both terms of a binomial are positive, what is the law of the signs? When one term is positive, and the other negative? Give the law of the exponents in its three divisions.

From this, we see, that the coëfficients of the following terms are equal: the first and the last; the second from the first, and the second from the last; the third from the first, and the third from the last; and so on.

Hence, it is only necessary to find the coëfficients of half the terms, when their number is even, or one more than half, when odd; the others being equal to those already found.

1. Raise $x+y$ to the third power.

2. Raise $(x-y)$ to the fourth power.

3. Raise $m+n$ to the fifth power.

4. What is the sixth power of $x-z$?

5. What is the seventh power of $a+b$?

6. What is the eighth power of $m-n$?

7. Find the ninth power of $x-y$.

8. Find the tenth power of $a+b$.

185. The Binomial Theorem may be used to find the different powers of a binomial, when one or both terms consist of two or more factors.

Review.—184. To what is the coëfficient of the first and last terms equal?
184. Of the second term? How is the coëfficient of any other term found? What terms have their coëfficients equal?

1. Find the cube of $2x-ac^2$.

Let $2x=m$, and $ac^2=n$; then, $2x-ac^2=m-n$.

$$(m-n)^3=m^3-3m^2n+3mn^2-n^3$$

$$m=2x \qquad n=ac^2$$

$$m^2=4x^2 \qquad n^2=a^2c^4$$

$$m^3=8x^3 \qquad n^3=a^3c^6$$

Substituting these values of the different powers of m and n in the equation above, and we have

$$(2x-ac^2)^3=8x^3-3\times4x^2\times ac^2+3\times2x\times a^2c^4-a^3c^6$$

$$=8x^3-12ac^2x^2+6a^2c^4x-a^3c^6.$$

2. Find the cube of $2a-3b$.

3. Find the fourth power of $m+2n$.

4. Find the third power of $4ax^2+3cy$.

5. Find the fourth power of $2x-5z$.

186. The Binomial Theorem may likewise be used to raise a trinomial or quadrinomial to any power, thus:

1. Find the second power of $a+b+c$.

Let $b+c=x$; then, $a+b+c=a+x$.

$$(a+x)^2=a^2+2ax+x^2$$

$$2ax=2a(b+c)=2ab+2ac$$

$$x^2=(b+c)^2=b^2+2bc+c^2$$

Then, $(a+b+c)^2=a^2+2ab+2ac+b^2+2bc+c^2$.

2. Find the third power of $x+y+z$.

3. Find the second power of $a+b+c+d$.

EVOLUTION.

EXTRACTION OF THE SQUARE ROOT OF NUMBERS.

187. Evolution is the process of finding the *root* of a quantity.

The **Second, or Square Root** of a number, is that number which, being multiplied by *itself*, will produce the given number.

Thus, 2 is the square root of 4, because $2 \times 2 = 4$.

The **Extraction of the Square Root** is the process of finding the second root of a given number.

188. The first ten numbers and their squares are

1,	2,	3,	4,	5,	6,	7,	8,	9,	10,
1,	4,	9,	16,	25,	36,	49,	64,	81,	100.

The numbers in the first line are also the square roots of the numbers in the second.

Observing the ten numbers written above, we see that *when the number of places of figures in a number is not more than* TWO, *the number of places of figures in the square root will be* ONE.

Again, take the following numbers and their squares:

10,	20,	30,	40,	50,	60,	70,	80,	90,	100,
100,	400,	900,	1600,	2500,	3600,	4900,	6400,	8100,	10000.

From this we see, that *when the number of places of figures is more than* TWO, *and not more than* FOUR, *the number of places of figures in the square root will be* TWO.

In the same manner, it may be shown, that when the number of places of figures is more than *four*, and not more than *six*, the number of places in the square root will be *three*, and so on. Or, thus:

When the number of places of figures in the number is either *one* or *two*, there will be *one* figure in the root;

When the number of places is either *three* or *four*, there will be *two* figures in the root;

When the number of places is either *five* or *six*, there will be *three* figures in the root; and so on.

189. Every number may be regarded as being composed of tens and units. Thus, 23 consists of 2 tens and 3 units; 256 consists of 25 tens and 6 units. Therefore, if we represent the tens by t, and the units by u, any number will be represented by $t+u$, and its square, by the square of $t+u$, or $(t+u)^2$.

$$(t+u)^2=t^2+2tu+u^2=t^2+(2t+u)u.$$

Hence, *the square of any number is composed of the square of the tens, plus a quantity consisting of twice the tens plus the units, multiplied by the units.*

Thus, the square of 23, which is equal to 2 tens and 3 units, is

$$2 \text{ tens squared } =(20)^2=400$$
$$(\text{Twice 2 tens} + 3 \text{ units}) \times 3=(40+3)\times 3=\underline{129}$$
$$529$$

1. Let it be required to extract the square root of 529.

Since the number consists of three places of figures, its root will consist of two places, according to the principles in Art. 188; we therefore separate it into two periods, as in the margin.

	$\dot{5}2\dot{9}$	23
	400	
$20\times 2=$	40	129
	3	
	43	129

REVIEW.—187. What is the square root of a number? Example. 188. When a number consists of only one figure, what is the greatest number of figures in its square? Examples. When a number consists of two places of figures? Examples.

188. What relation exists between the number of places of figures in any number and the number of places in its square? 189. Of what may every number be regarded as being composed? Prove this, and then illustrate it.

Since the square of 2 tens is 400, and of 3 tens, 900, it is evident that the greatest square contained in 500, is the square of 2 tens (20); the square of 2 tens (20) is 400; subtracting this from 529, the remainder is 129.

Now, according to the preceding theorem, this number 129 consists of twice the tens plus the units, multiplied by the units; that is, by the formula, it is $(2t+u)u$.

The product of the tens by the units can not give a product less than tens; therefore, the unit's figure (9) forms no part of the double product of the tens by the units. Then, if we divide the remaining figures (12) by the double of the tens, the quotient will be (3) the unit's figure, or a figure greater than it.

We then double the tens, and add to it the unit figure (3), making $40+3=43$ $(2t+u)$; multiplying this by 3 (u), the product is 129, which is the double of the tens plus the units, multiplied by the units. As there is nothing left after subtracting this from the first remainder, we conclude that 23 is the exact square root of 529.

In squaring the tens, and also in doubling them, it is customary to omit the ciphers, though they are understood. Also, the unit's figure is added to the double of the tens, by merely writing it in the unit's place. The actual operation is usually performed as in the margin.

```
   529|23
   4
43|129
   129
```

2. Let it be required to extract the square root of 55225.

Since this number consists of five places of figures, its root will consist of three places, according to the principles in Art. 188; we therefore separate it into *three* periods.

In performing this operation, we find the square root of the number 552, on the same principle as in the preceding example. We next consider the 23 as so many tens, and proceed to find the unit's figure (5) in the same manner as in the preceding example. Hence,

```
     55225|235
     4
 43| 152
     129
465| 2325
     2325
```

TO EXTRACT THE SQUARE ROOT OF WHOLE NUMBERS,

Rule.—1. *Separate the given numbers into periods of two places each, beginning at the unit's place.*

2. *Find the greatest square in the left period, and place its root on the right, after the manner of a quotient in division. Subtract the square of the root from the left period, and to the remainder bring down the next period for a dividend.*

3. *Double the root already found, and place it on the left for a divisor. Find how many times the divisor is contained in the dividend, exclusive of the right hand figure, and place the figure in the root, and also on the right of the divisor.*

4. *Multiply the divisor, thus increased, by the last figure of the root; subtract the product from the dividend, and to the remainder bring down the next period for a new dividend.*

5. *Double the whole root already found for a new divisor, and continue the operation as before, until all the periods are brought down.*

NOTE.—If, in any case, the dividend will not contain the divisor, the right hand figure of the former being omitted, place a cipher in the root, also at the right of the divisor, and bring down the next period.

190. In division, when the remainder is greater than the divisor, the last quotient figure may be increased by at least 1; but in extracting the square root, the remainder may sometimes be greater than the last divisor, while the last figure of the root can not be increased.

To know when any figure may be increased, we must determine the relation between the squares of two consecutive numbers.

Let a and $a+1$ be two consecutive numbers.

Then, $(a+1)^2=a^2+2a+1$, is the square of the greater.

$(a)^2=a^2$ is the square of the less.

Their difference is $2a+1$. Therefore,

REVIEW.—189. Extract the square root of 529, and show the reason for each step, by referring to the formula.

The difference of the squares of two consecutive numbers is equal to twice the less number increased by unity. Hence,

When the remainder is less than twice the part of the root already found, plus unity, the last figure can not be increased.

Extract the square root of the following numbers:

1. 4225.
2. 289444.
3. 498436.
4. 950625.
5. 1525225.
6. 412252416.

OF THE SQUARE ROOT OF FRACTIONS.

191. Since $\frac{2}{3}\times\frac{2}{3}=\frac{4}{9}$, therefore, the square root of $\frac{4}{9}$ is $\frac{2}{3}$; that is, $\sqrt{\frac{4}{9}}=\frac{\sqrt{4}}{\sqrt{9}}=\frac{2}{3}$. Hence,

The square root of a fraction is found by extracting the square root of both terms.

Before extracting the square root of a fraction, it should be reduced to its lowest terms, unless both numerator and denominator are perfect squares. The reason for this will be seen by the following example:

Find the square root of $\frac{12}{27}$.

Here, $\frac{12}{27}=\frac{4\times3}{9\times3}$ Now, neither 12 nor 27 is a perfect square; but, by canceling the common factor 3, the fraction becomes $\frac{4}{9}$, the square root of which is $\frac{2}{3}$.

When both terms are perfect squares, and contain a common factor, the reduction may be made either before or after the square root is extracted.

REVIEW.—189. What is the rule for extracting the square root of numbers? 190. What is the difference between the squares of two consecutive numbers? When may any figure of the quotient be increased?

Thus, $\sqrt{\frac{16}{36}}=\frac{4}{6}=\frac{2}{3}$; or, $\frac{16}{36}=\frac{4}{9}$, and $\sqrt{\frac{4}{9}}=\frac{2}{3}$.

Find the square root of the following fractions:

1. $\frac{81}{625}$. . .
2. $\frac{64}{841}$. . .
3. $\frac{1071}{2975}$. . .
4. $\frac{572}{7007}$. . .
5. $\frac{1369}{10000}$. . .
6. $\frac{18225}{1000000}$. .

192. A **Perfect Square** is a number whose square root can be exactly ascertained; as, 4, 9, 16, etc.

An **Imperfect Square** is a number whose square root can not be exactly ascertained; as, 2, 3, 5, 6, etc.

Since the difference of two consecutive square numbers a^2 and a^2+2a+1, is $2a+1$; therefore, there are always $2a$ imperfect squares between them. Thus, between the square of 4(16), and the square of 5(25), there are $8(2a=2\times4)$ imperfect squares.

A **Surd** is a root which can not be exactly expressed. Thus, $\sqrt{2}$ is a surd; it is 1.414+.

The signs + and − are sometimes placed after an approximate root, to denote that it is less or greater than the true root.

It might be supposed, that when the square root of a whole number can not be expressed by a whole number, it exactly equals some fraction. We will therefore show, that

The square root of an imperfect square can not be a fraction.

Let c be an imperfect square, such as 2, and, *if possible*, let its square root be equal to a fraction, $\frac{a}{b}$, which is supposed to be in its lowest terms.

Then, $\sqrt{c}=\frac{a}{b}$; and $c=\frac{a^2}{b^2}$, by squaring both sides.

Now, by supposition, a and b have no common factor;

therefore, their squares, a^2 and b^2, can have no common factor, since to square a number, we merely repeat its factors. Consequently, $\frac{a^2}{b^2}$ must be in its lowest terms, and can not be equal to a whole number. Therefore, the equations $c=\frac{a^2}{b^2}$ and $\sqrt{c}=\frac{a}{b}$ are not true. Hence,

The square root of an imperfect square can not be a fraction.

APPROXIMATE SQUARE ROOTS.

193. To illustrate the method of finding the approximate square root of an imperfect square, let it be required to find the square root of 2 to within $\frac{1}{3}$.

Reducing 2 to a fraction whose denominator is 9 (the square of 3, the denominator of the fraction $\frac{1}{3}$), we have $2=\frac{18}{9}$.

The square root of 18 is greater than 4, and less than 5; and the square root of $\frac{18}{9}$ is greater than $\frac{4}{3}$, and less than $\frac{5}{3}$; therefore, $\frac{4}{3}$ is the square root of 2 to within less than $\frac{1}{3}$. Hence,

TO EXTRACT THE SQUARE ROOT OF A WHOLE NUMBER TO WITHIN A GIVEN FRACTION,

Rule.—1. *Multiply the given number by the square of the denominator of the fraction which determines the degree of approximation.*

2. *Extract the square root of this product to the nearest unit, and divide the result by the denominator of the fraction.*

Review.—191. How is the square root of a fraction found, when both terms are perfect squares? 192. When is a number a perfect square? Examples. When an imperfect square? How determine the number of imperfect squares between any two consecutive perfect squares?

192. What is a root called, which can not be exactly expressed? Prove that the square root of an imperfect square can not be a fraction. 193. How find the approximate square root of an imperfect square to within any given fraction? How, when the fraction is a decimal?

1. Find the square root of 5 to within $\frac{1}{5}$.
2. Of 7 to within $\frac{1}{13}$. . . . .
3. Of 27 to within $\frac{1}{30}$. . . . .
4. Of 14 to within $\frac{1}{10}$. . . . .
5. Of 15 to within $\frac{1}{100}$. . . . . . .

As the squares of 10, 100, etc., are 100, 10000, etc., the number of ciphers in the square of the denominator of a decimal fraction equals twice the number in the denominator itself. Therefore,

When the fraction which determines the degree of approximation is a decimal, add two ciphers for each decimal place required, extract the root, and point off from the right, one place of decimals for each two ciphers added.

6. Find the square root of 2 to six places of decimals.

7. Find the square root of 10.
8. Find the square root of 101.

194. To find the approximate square root of a fraction.

1. Let it be required to find the square root of $\frac{3}{7}$ to within $\frac{1}{7}$.

$$\frac{3}{7}=\frac{3}{7}\times\frac{7}{7}=\frac{21}{49}.$$

Now, since the square root of 21 is greater than 4, and less than 5, the square root of $\frac{21}{49}$ is greater than $\frac{4}{7}$, and less than $\frac{5}{7}$; therefore, $\frac{4}{7}$ is the square root of $\frac{3}{7}$ to within less than $\frac{1}{7}$. Hence,

If we multiply the numerator of a fraction by its denominator, then extract the square root of the product to the nearest unit, and divide the result by the denominator, the quotient will be the square root of the fraction to within one of its equal parts.

2. Find the square root of $\frac{4}{11}$ to within $\frac{1}{11}$.
3. Find the square root of $\frac{7}{15}$ to within $\frac{1}{15}$.

Since any decimal may be written in the form of a fraction having a denominator a perfect square, by adding ciphers to both terms (thus, $.4=\frac{40}{100}=\frac{4000}{10000}$, etc.), therefore, as shown in Art. 193, its square root may be found by the following

Rule.—*Annex ciphers, until the number of decimal places shall be double the number required in the root, extract the root, and point off from the right the required number of decimal places.*

Find the square root

4. Of .6 to six places of decimals. Ans.
5. Of .29 to six places of decimals. Ans.

The square root of a whole number and a decimal may be fcund in the same manner. Thus,

The square root of 2.5 is the same as the square root of $\frac{250}{100}$, which, carried out to 6 places of decimals, is 1.581138+.

6. Find the square root of 10.76 to six places of decimals.

7. Find the square root of 1.1025. . . .

When the denominator of a fraction is a perfect square, extract the square root of the numerator, and divide the result by the square root of the denominator; or, reduce the fraction to a decimal, and then extract its square root.

Review.—194. How find the approximate square root of a fraction to within one of its equal parts? How extract the square root of a decimal? Of a fraction, when both terms are not perfect squares?

When the denominator of the fraction is not a perfect square, the latter method should be used.

8. Find the square root of $\frac{3}{4}$ to five places of decimals.

$$\sqrt{3}=1.73205+,\ \sqrt{4}=2,\ \therefore\sqrt{\tfrac{3}{4}}=\tfrac{1.73205+}{2}=.86602+.$$

$$\text{Or, } \tfrac{3}{4}=.75. \text{ and } \sqrt{.75}=.86602+.$$

9. Find the square root of $3\frac{2}{9}$.
10. Find the square root of $\frac{7}{16}$.
11. Find the square root of $5\frac{8}{9}$.
12. Find the square root of $\frac{1}{7}$.

SQUARE ROOT OF MONOMIALS.

195. To square a monomial, Art. 179, we square its coëfficient, and multiply the exponent of each letter by 2. Thus,

$(3ab^2)^2=9a^2b^4$. Therefore, $\sqrt{9a^2b^4}=3ab^2$. Hence,

TO EXTRACT THE SQUARE ROOT OF A MONOMIAL,

Rule.—*Extract the square root of the coëfficient, and divide the exponent of each letter by* 2.

Since $+a\times+a=+a^2$, and $-a\times-a=+a^2$,

Therefore, $\sqrt{a^2}=+a$, or $-a$.

Hence, the square root of any positive quantity is either *plus* or *minus*. This is generally expressed by using the double sign. Thus, $\sqrt{4a^2}=\pm2a$, which is read, *plus or minus* $2a$.

If a monomial is *negative*, the extraction of the square root is impossible, since the square of any quantity, either positive or negative, is necessarily positive. Thus, $\sqrt{-9}$, $\sqrt{-4a^2}$, $\sqrt{-b}$, are algebraic symbols, which indicate impossible operations.

Such expressions are termed *imaginary quantities.* When they result from an equation or a problem, they indicate some absurdity or impossibility. See Art. 218.

Find the square root of the following monomials:

1. $4a^2x^2$.
2. $9x^2y^4$.
3. $36a^4b^6x^2$.
4. $49a^2b^4c^8$.
5. $625x^2z^4$.
6. $1156a^2x^4z^6$.

Since $\left(\frac{a}{b}\right)^2=\frac{a^2}{b^2}$, therefore, $\sqrt{\frac{a^2}{b^2}}=\pm\frac{a}{b}$; hence, *to find the square root of a fraction, extract the square root of both terms.*

7. Find the square root of $\frac{16x^2y^4}{25a^2z^2}$.

SQUARE ROOT OF POLYNOMIALS.

196. In order to deduce a rule for extracting the square root of a polynomial, let us first examine the relation that exists between the several terms of any quantity and its square.

$(a+b)^2=a^2+2ab+b^2=a^2+(2a+b)b.$

$(a+b+c)^2=a^2+2ab+b^2+2ac+2bc+c^2=a^2+(2a+b)b+(2a+2b+c)c.$

$(a+b+c+d)^2=a^2+2ab+b^2+2ac+2bc+c^2+2ad+2bd+2cd+d^2=a^2+(2a+b)b+(2a+2b+c)c+(2a+2b+2c+d)d.$

Hence, the square of any polynomial is formed according to the following law:

The square of any polynomial is equal to the square of the first term—plus twice the first term, plus the second, multi-

REVIEW.—195. How find the square of a monomial? How find its square root? What is the sign of the square root of any positive quantity?

195. Why is the extraction of the square root of a negative monomial impossible? Give examples of symbols that indicate impossible operations. What are they termed? What do they indicate?

plied by the second—plus twice the first and second terms, plus the third, multiplied by the third—plus twice the first, second, and third terms, plus the fourth, multiplied by the fourth; and so on.

Hence, by reversing the operation, we have the following

RULE,

FOR EXTRACTING THE SQUARE ROOT OF A POLYNOMIAL.

1. *Arrange the polynomial with reference to a certain letter.*

2. *Extract the square root of the first term, place the result on the right, and subtract its square from the given quantity.*

3. *Divide the first term of the remainder by double the root already found, and annex the result both to the root and the divisor. Multiply the divisor thus increased, by the second term of the root, and subtract the product from the remainder.*

4. *Double the terms of the root already found, for a partial divisor, and divide the first term of the remainder by the first term of the divisor, and annex the result both to the root and the partial divisor. Multiply the divisor thus increased, by the third term of the root, and subtract the product from the last remainder.*

5. *Proceed in a similar manner until the work is finished.*

REMARK.—If the first term of any remainder is not exactly divisible by double the first term of the root, the polynomial is not a perfect square.

1. Find the square root of $r^2+2rr'+r'^2+2rr''+2r'r''+r''^2$.

$$\begin{array}{l|l}
 & r^2+2rr'+r'^2+2rr''+2r'r''+r''^2 \;|\; \underline{r+r'+r'',\ \text{root.}} \\
 & r^2 \\ \hline
2r+r' & 2rr'+r'^2 \\
 & 2rr'+r'^2 \\ \hline
2r+2r'+r'' & 2rr''+2r'r''+r''^2 \\
 & 2rr''+2r'r''+r''^2 \\ \hline
\end{array}$$

The square root of the first term is r, which write as the first term of the root. Subtract the square of r from the given polynomial, and dividing the first term of the remainder $2rr'$, by $2r$, the double of the first term of the root, the quotient is r', the second term of the root.

Next, place r' in the root, and also in the divisor, and multiply the divisor, thus increased, by r', and subtract the product from the first remainder.

Double the terms $r+r'$, of the root already found, and proceed as before, until the work is finished.

2. Find the square root of $25x^2y^2-24xy^3+12x^3y+4x^4+16y^4$.

Arranging the polynomial with reference to x, we have

$$\begin{array}{r|l}
 & 4x^4-12x^3y+25x^2y^2-24xy^3+16y^4 \mid 2x^2-3xy+4y^2, \text{ root.} \\
 & 4x^4 \\
4x^2-3xy & -12x^3y+25x^2y^2 \\
 & -12x^3y+9x^2y^2 \\
4x^2-6xy+4y^2 & 16x^2y^2-24xy^3+16y^4 \\
 & 16x^2y^2-24xy^3+16y^4
\end{array}$$

Find the square root of the following polynomials:

3. x^2+4x+4. . . . .
4. $x^2y^2-8xy+16$. . . . . . .
5. $4a^2x^2+25y^2z^2-20axyz$. . . .
6. $x^4+4x^3+6x^2+4x+1$. . . .
7. $9y^4-12y^3+34y^2-20y+25$. . .
8. $1-4x+10x^2-20x^3+25x^4-24x^5+16x^6$.
9. $a^6-6a^5x+15a^4x^2-20a^3x^3+15a^2x^4-6ax^5+x^6$.
10. $x^2+ax+\frac{1}{4}a^2$. . . . .
11. $x^2-2x+1+2xy-2y+y^2$. . . .
12. $x(x+1)(x+2)(x+3)+1$. . . .

REVIEW.—196. What is the square of $a+b$? Of $a+b+c$? Of $a+b+c+d$? According to what law is the square of any polynomial formed? By reversing this law, what rule have we? When conclude that a polynomial is not a perfect square?

197. The following remarks will be found useful:

1st. *No binomial can be a perfect square;* for the square of a monomial is a monomial, and the square of a binomial is a trinomial.

Thus, a^2+b^2 is not a perfect square; but if we add to it $2ab$, it becomes the square of $a+b$. If we subtract from it $2ab$, it becomes the square of $a-b$.

2d. In order that a trinomial may be a perfect square, the extreme terms must be perfect squares, and the middle term twice the product of the square roots of the extreme terms. Hence, to find the square root of a trinomial when it is a perfect square,

Extract the square roots of the two extreme terms, and unite them by the sign plus or minus, according as the second term is plus or minus.

Thus, $4a^2-12ac+9c^2$ is a perfect square; since $\sqrt{4a^2}=2a$, $\sqrt{9c^2}=3c$, and $+2a\times-3c\times2=-12ac$.

But $9x^2+12xy+16y^2$, is not a perfect square; since $\sqrt{9x^2}=3x$, $\sqrt{16y^2}=4y$, and $3x\times4y\times2=24xy$, which is not equal to the middle term $12xy$.

GENERAL REVIEW.

What is meant by generalization? Explain by an example. Rule for finding two quantities when their sum and difference are given. Rule for fellowship without time. With time. What is a negative solution, and what does it imply?

Explain the Problem of the Couriers. How many cases may be supposed? Explain each case. When is an equation independent? When dependent? When redundant? When is a problem indeterminate? When impossible? Prove that a simple equation has but one root.

Rule for raising a monomial to any power. Rule for the signs in the involution of monomials. Why? Rule for a polynomial. A fraction. In Newton's theorem, show what is proved in regard to number of terms. Signs. Exponents. Coëfficients.

Why can no binomial be a perfect square? Example. What is necessary, in order that a trinomial may be a perfect square? How may its square root be found? Example.

RADICALS

OF THE SECOND DEGREE.

198. From Rule, Art. 195, it is evident that *when a monomial is a perfect square, its numeral coëfficient is a perfect square, and the exponent of each letter is exactly divisible by* 2.

Thus, $4a^2$ is a perfect square, while $5a^3$ is not.

When the exact division of the exponent can not be performed, it may be indicated, by writing the divisor under it, in the form of a fraction. Thus, $\sqrt{a^3}$ may be written $a^{\frac{3}{2}}$.

Since $a=a^1$, the square root of a may be expressed thus, $a^{\frac{1}{2}}$. Hence, the fractional exponent, $\frac{1}{2}$, is used to indicate the extraction of the square root.

Thus, $\sqrt{a^2+2ax+x^2}$ and $(a^2+2ax+x^2)^{\frac{1}{2}}$, also $\sqrt{4}$ and $4^{\frac{1}{2}}$, indicate the same operation; the radical sign $\sqrt{\ }$, and the fractional exponent $\frac{1}{2}$, being regarded as equivalent.

Radicals of the Second Degree are quantities affected by a square root sign whose root can not be exactly found; as, $\sqrt{a}$, $\sqrt{2}$, $a\sqrt{b}$, and $5\sqrt{3}$; or, as otherwise written, $a^{\frac{1}{2}}$, $2^{\frac{1}{2}}$, $ab^{\frac{1}{2}}$, and $5(3)^{\frac{1}{2}}$.

Radicals are also called *irrational quantities*, or *surds*.

The **Coefficient** of a radical is the quantity which stands before the radical sign. Thus, in the expressions $a\sqrt{b}$, and $3\sqrt{5}$, a and 3 are coëfficients.

Similar Radicals are those which have the same quantity under the radical sign.

Thus, $3\sqrt{2}$ and $7\sqrt{2}$ are similar radicals; so, also, are $b\sqrt{a}$ and $2c\sqrt{a}$.

Radicals that are not similar, may frequently become so by simplification. This gives rise to

REDUCTION OF RADICALS.

199. Reduction of radicals of the second degree consists in changing the form of the quantities without altering their value. It is founded on the following principle:

The square root of the product of two or more factors is equal to the product of the square roots of those factors.

That is, $\sqrt{ab}=\sqrt{a}\times\sqrt{b}$, and $\sqrt{36}=\sqrt{9}\times\sqrt{4}$; for $\sqrt{36}=\pm6$, and $\sqrt{9}\times\sqrt{4}=3\times2=6$.

Any radical of the second degree can, on this principle, be reduced to a simpler form, when it can be separated into factors, one of which is a perfect square.

Thus, $\sqrt{12}=\sqrt{4\times3}=\sqrt{4}\times\sqrt{3}=2\sqrt{3}$

$\sqrt{a^3b}=\sqrt{a^2\times ab}=\sqrt{a^2}\times\sqrt{ab}=a\sqrt{ab}$

$\sqrt{27a^3c^4}=\sqrt{9a^2c^4\times3a}=\sqrt{9a^2c^4}\times\sqrt{3a}=3ac^2\sqrt{3a}$. Hence,

TO REDUCE A RADICAL OF THE SECOND DEGREE TO ITS SIMPLEST FORM,

Rule.—1. *Separate the quantity into two parts, one of which shall contain all the factors that are perfect squares.*

2. *Extract the square root of the part that is a perfect square, and prefix it as a coëfficient to the other part placed under the radical sign.*

To determine whether any quantity contains a numeral factor that is a perfect square, ascertain if it is divisible by either of the perfect squares, 4, 9, 16, 25, 36, 49, 64, 81, etc.

Review.—198. When is a monomial a perfect square? How may the square root of a quantity be expressed without the radical sign?
198. What are radicals of the second degree? What is the coëfficient of a radical? What are similar radicals?

Reduce each of the following radicals to its simplest form:

1. $\sqrt{8a^2}$.
2. $\sqrt{12a^3}$.
3. $\sqrt{20a^3b^3c^3}$.
4. $4\sqrt{27a^3c^3}$.
5. $7\sqrt{28a^5c^2}$.
6. $\sqrt{32a^6b^2c^4}$.
7. $\sqrt{44a^5b^3c}$.
8. $\sqrt{48a^8b^6c^4}$.
9. $\sqrt{75a^3b^3c^3}$.
10. $\sqrt{243a^3b^2c}$.

In a similar manner, polynomials may sometimes be simplified.

Thus, $\sqrt{(2a^3-4a^2b+2ab^2)}=\sqrt{(a^2-2ab+b^2)2a}=(a-b)\sqrt{2a}$.

To reduce a fractional radical, multiply both terms by any quantity that will render the denominator a perfect square, and then separate the fraction into two factors, as before explained.

11. Reduce $\sqrt{\frac{2}{3}}$ to its simplest form.

$\sqrt{\frac{2}{3}}=\sqrt{\frac{2}{3}\times\frac{3}{3}}=\sqrt{\frac{6}{9}}=\sqrt{\frac{1}{9}\times 6}=\sqrt{\frac{1}{9}}\times\sqrt{6}=\frac{1}{3}\sqrt{6}$, Ans.

Reduce the following to their simplest forms:

12. $\sqrt{\frac{3}{5}}$.
13. $\sqrt{\frac{7}{8}}$.
14. $\sqrt{\frac{12}{25}}$.
15. $9\sqrt{\frac{16}{27}}$.
16. $5\sqrt{\frac{9}{10}}$.
17. $10\sqrt{\frac{8}{50}}$.

Since $a=\sqrt{a^2}$, $3=\sqrt{9}$, and $2\sqrt{3}=\sqrt{4}\times\sqrt{3}=\sqrt{4\times 3}=\sqrt{12}$, it is obvious that any quantity may be reduced to the form of a radical of the second degree, *by squaring it, and placing it under the radical sign.*

18. Reduce 5 to the form of a radical of the second degree.

19. Reduce $2a$ to the form of a radical of the second degree.

20. Express $3\sqrt{5}$, entirely under the radical.

21. Pass the coëfficient of $3c\sqrt{2c}$, under the radical.

ADDITION OF RADICALS.

200.—1. What is the sum of $3\sqrt{2}$ and $5\sqrt{2}$?

It is evident that 3 times and 5 times any certain quantity must make 8 times that quantity; therefore,

$$3\sqrt{2}+5\sqrt{2}=8\sqrt{2}.$$

2. What is the sum of $2\sqrt{3}$ and $5\sqrt{7}$?

Since dissimilar quantities can not be added, we can only find the sum of these expressions by placing the sign of addition between them; thus: $2\sqrt{3}+5\sqrt{7}$.

Sometimes radicals become similar after being reduced, and may then be added; thus: $\sqrt{2}+\sqrt{8}=\sqrt{2}+2\sqrt{2}=3\sqrt{2}$. Hence,

TO ADD RADICALS OF THE SECOND DEGREE,

Rule.—1. *Reduce the radicals to their simplest form.*

2. *If the radicals are similar, add the coëfficients, and annex the common radicals.*

3. *If they are not similar, connect them by their proper signs.*

Find the sum of the radicals in the following examples:

3. $\sqrt{8}$ and $\sqrt{18}$.

$$\sqrt{8}=\sqrt{4\times2}=2\sqrt{2}$$
$$\sqrt{18}=\sqrt{9\times2}=3\sqrt{2}$$

Adding, we have $5\sqrt{2}$, Ans.

4. $\sqrt{12}$ and $\sqrt{27}$. . . . .

5. $\sqrt{20}$ and $\sqrt{80}$. . . . .

6. $\sqrt{40}$, $\sqrt{90}$, and $\sqrt{250}$. . . . . .

7. $\sqrt{28a^2b^2}$ and $\sqrt{112a^2b^2}$. . . . . .

8. $\sqrt{\frac{1}{3}}$ and $\sqrt{\frac{3}{25}}$.

$$\sqrt{\tfrac{1}{3}}=\sqrt{\tfrac{3}{9}}=\sqrt{\tfrac{1}{9}\times3}=\tfrac{1}{3}\sqrt{3}$$
$$\sqrt{\tfrac{3}{25}}=\sqrt{\tfrac{1}{25}\times3}=\tfrac{1}{5}\sqrt{3}$$

Adding, we have $\frac{8}{15}\sqrt{3}$, Ans.

9. $2\sqrt{\frac{3}{4}}$ and $3\sqrt{12}$. . . . .

10. $\frac{1}{2}\sqrt{\frac{1}{2}}$ and $\frac{3}{4}\sqrt{2}$. . . . . .

11. $\sqrt{48a^2c^2x}$ and $\sqrt{12b^2x}$. . .

12. Find the sum of $\sqrt{(2a^3-4a^2c+2ac^2)}$ and $\sqrt{(2a^3+4a^2c+2ac^2)}$.

13. Find the sum of $\sqrt{a+x}+\sqrt{ax^2+x^3}+\sqrt{(a+x)^3}$.

SUBTRACTION OF RADICALS.

201.—1. Take $3\sqrt{2}$ from $5\sqrt{2}$.

It is evident that 5 times any quantity minus 3 times the quantity, will be equal to 2 times the quantity; therefore,

$$5\sqrt{2}-3\sqrt{2}=2\sqrt{2}.$$

In the same manner, $\sqrt{8}-\sqrt{2}=2\sqrt{2}-\sqrt{2}=\sqrt{2}$.

REVIEW.—199. In what does reduction of radicals of the second degree consist? On what principle is it founded? Illustrate this principle.

199. Rule for the reduction of a radical? How determine if any numerical quantity contains a factor that is a perfect square? How reduce a fractional radical? 200. Rule for addition of radicals.

If the radicals are dissimilar, their difference can only be indicated. Thus, to take $3\sqrt{a}$ from $5\sqrt{b}$, write $5\sqrt{b}-3\sqrt{a}$. Hence,

TO SUBTRACT RADICALS OF THE SECOND DEGREE,

Rule.—1. *Reduce the radicals to their simplest form.*

2. *If the radicals are similar, find the difference of their coëfficients, and annex the common radical.*

3. *If not similar, indicate their difference by the proper sign.*

2. $\sqrt{32}-\sqrt{8}$.

$$\sqrt{32}=\sqrt{16\times2}=4\sqrt{2}$$
$$\sqrt{8}=\sqrt{4\times2}=2\sqrt{2}$$

Subtracting, we have $2\sqrt{2}$, Ans.

3. $\sqrt{45a^2}-\sqrt{5a^2}$. . . . . . .
4. $\sqrt{54b}-\sqrt{6b}$. . . . . . .
5. $\sqrt{27b^3c^3}-\sqrt{12b^3c^3}$. . . . .
6. $\sqrt{49ab^3c^2}-\sqrt{25ab^3c^2}$. . . .
7. $5a\sqrt{27}-3a\sqrt{48}$. . . . . .
8. $2\sqrt{\frac{3}{4}}-3\sqrt{\frac{1}{3}}$. . . . .
9. $\sqrt{\frac{5}{6}}-\sqrt{\frac{10}{27}}$. . . . .
10. $3\sqrt{\frac{1}{2}}-\sqrt{2}$. . . . .
11. $\sqrt{4a^2x}-a\sqrt{x^3}$. . . . . .
12. $\sqrt{3m^2x+6mnx+3n^2x}-\sqrt{3m^2x-6mnx+3n^2x}$.

MULTIPLICATION OF RADICALS.

202. Since $\sqrt{ab}=\sqrt{a}\times\sqrt{b}$, therefore, $\sqrt{a}\times\sqrt{b}=\sqrt{ab}$. See Art. 199.

Also, $a\sqrt{b}\times c\sqrt{d}=a\times c\times\sqrt{b}\times\sqrt{d}=ac\sqrt{bd}$. Hence,

TO MULTIPLY RADICALS OF THE SECOND DEGREE,

Rule.—*Multiply the coëfficients together for a new coëfficient, and the quantities under the radical sign for a new radical.*

1. Find the product of $\sqrt{6}$ and $\sqrt{8}$.

$\sqrt{6}\times\sqrt{8}=\sqrt{48}=\sqrt{16\times3}=4\sqrt{3}$, Ans.

2. Find the product of $2\sqrt{14}$ and $3\sqrt{2}$.

$2\sqrt{14}\times3\sqrt{2}=6\sqrt{28}=6\sqrt{4\times7}=6\times2\sqrt{7}=12\sqrt{7}$, Ans.

3. Find the product of $\sqrt{8}$ and $\sqrt{2}$.
4. Find the product of $2\sqrt{a}$ and $3\sqrt{a}$.
5. Find the product of $\sqrt{27}$ and $\sqrt{3}$.
6. Find the product of $3\sqrt{2}$ and $2\sqrt{3}$.
7. Find the product of $2\sqrt{15}$ and $3\sqrt{35}$.

8. Find the product of $\sqrt{a^3b^5c}$ and $\sqrt{abc}$.
9. Find the product of $\sqrt{\frac{2}{5}}$ and $\sqrt{\frac{8}{9}}$.
10. Find the product of $2\sqrt{\frac{a}{5}}$ and $3\sqrt{\frac{a}{10}}$.

When two polynomials contain radicals, they may be multiplied as in multiplication of polynomials, Art. 72.

11. Find the product of $2+\sqrt{2}$ and $2-\sqrt{2}$.
12. Find the product of $\sqrt{x+2}$ by $\sqrt{x-2}$.

13. Find the product of $\sqrt{a+x}$ by $\sqrt{a+x}$.
14. Find the product of $\sqrt{x+2}$ by $\sqrt{x+3}$.

REVIEW.—201. Rule for the subtraction of radicals. 202. For the multiplication of radicals. Prove it.

Perform the operations indicated in the following:

15. $(c\sqrt{a}+d\sqrt{b})\times(c\sqrt{a}-d\sqrt{b})$. . .
16. $(7+2\sqrt{6})\times(9-5\sqrt{6})$. . . . .
17. $(\sqrt{a+x}+\sqrt{a-x})(\sqrt{a+x}-\sqrt{a-x})$.
18. $(x^2-x\sqrt{2}+1)(x^2+x\sqrt{2}+1)$. . . .

DIVISION OF RADICALS.

203. Since division is the reverse of multiplication, and since $\sqrt{a}\times\sqrt{b}=\sqrt{ab}$, therefore, $\sqrt{ab}\div\sqrt{a}=\sqrt{\frac{ab}{a}}=\sqrt{b}$.

Also, since, $2\sqrt{3}\times3\sqrt{15}=6\sqrt{45}$; therefore, $6\sqrt{45}\div2\sqrt{3}=3\sqrt{15}$. Hence,

TO DIVIDE RADICALS OF THE SECOND DEGREE,

Rule.—*Divide the coëfficient of the dividend by the coëfficient of the divisor for a new coëfficient, and the radical of the dividend by the radical of the divisor for a new radical.*

1. Divide $8\sqrt{72}$ by $2\sqrt{6}$.

$$\frac{8\sqrt{72}}{2\sqrt{6}}=\tfrac{8}{2}\sqrt{\tfrac{72}{6}}=4\sqrt{12}=4\sqrt{4\times3}=8\sqrt{3}, \text{ Ans.}$$

2. Divide $\sqrt{54}$ by $\sqrt{6}$. . . . . . .
3. Divide $6\sqrt{54}$ by $3\sqrt{27}$. . . . .
4. Divide $\sqrt{160}$ by $\sqrt{8}$. . . . . .
5. Divide $15\sqrt{378}$ by $5\sqrt{6}$. . . . .
6. Divide $ab\sqrt{a^3b^3}$ by $b\sqrt{ab}$. . . . .
7. Divide $\sqrt{\frac{a}{b}}$ by $\sqrt{\frac{d}{c}}$. . .
8. Divide $\sqrt{\frac{1}{2}}$ by $\sqrt{\frac{1}{3}}$. . . . .

REVIEW.—203. Rule for the division of radicals. Prove it.

9. Divide $\frac{2}{3}\sqrt{18}$ by $\frac{1}{2}\sqrt{2}$. . . . . . .

10. Divide $\frac{3}{5}\sqrt{\frac{1}{3}}$ by $\frac{1}{2}\sqrt{\frac{3}{5}}$. . . . . . .

11. Divide $\frac{1}{2}\sqrt{\frac{1}{2}}$ by $\sqrt{2}+3\sqrt{\frac{1}{2}}$. . . . . .

204. To reduce a fraction whose denominator contains a radical to an equivalent fraction having a rational denominator.

Since $\sqrt{a}\times\sqrt{a}=\sqrt{a^2}=a$, we conclude that *when the square root of a quantity is multiplied by itself, or squared, the radical sign is thrown off.*

Thus, $\sqrt{2}\times\sqrt{2}=2$, and $\sqrt{a+b}\times\sqrt{a+b}=a+b$.

When the fraction is of the form $\frac{a}{\sqrt{b}}$, if we multiply both terms by $\sqrt{b}$, the denominator will become rational.

$$\text{Thus,}\quad \frac{a}{\sqrt{b}}=\frac{a\times\sqrt{b}}{\sqrt{b}\times\sqrt{b}}=\frac{a\sqrt{b}}{b}.$$

Since the sum of two quantities, multiplied by their difference, is equal to the difference of their squares; if the denominator of the fraction is of the form $b+\sqrt{c}$, and we multiply both terms by $b-\sqrt{c}$, it will be made rational, since it will be b^2-c.

$$\text{Thus,}\quad \frac{a}{b+\sqrt{c}}\times\frac{b-\sqrt{c}}{b-\sqrt{c}}=\frac{ab-a\sqrt{c}}{b^2-c}.$$

For the same reason, if the denominator is $b-\sqrt{c}$, the multiplier will be $b+\sqrt{c}$. If it is $\sqrt{b}+\sqrt{c}$, or $\sqrt{b}-\sqrt{c}$, the multiplier will be $\sqrt{b}-\sqrt{c}$, or $\sqrt{b}+\sqrt{c}$.

These different forms may be embraced in the following

General Rule.—*If the denominator is a monomial, multiply both terms by it; but if it is a binomial, multiply both terms by the denominator, with the sign of the second term changed.*

Reduce the following fractions to equivalent fractions, having rational denominators:

1. $\frac{1}{\sqrt{2}}$. . . . .

2. $\frac{\sqrt{2}}{\sqrt{3}}$. . . . .

3. $\frac{3}{6-\sqrt{3}}$. . . . .

4. $\frac{5}{\sqrt{7}+\sqrt{6}}$. . . . .

REMARK.—The object of the above is to diminish the amount of calculation in obtaining the numerical value of a fractional radical.

Thus, suppose it is required to obtain the numerical value of the fraction $\frac{\sqrt{2}}{\sqrt{3}}$ in example 2 above, true to six places of decimals.

Here, we may first extract the square root of 2 and of 3 to seven places of decimals, and then divide the first result by the second. This operation is very tedious. If we render the denominator rational, the calculation merely consists in finding the square root of 6, and then dividing by 3.

5. Find the numerical value of $\frac{3}{\sqrt{5}}$.

6. Of $\frac{\sqrt{2}}{\sqrt{5}-\sqrt{3}}$. . . . . .

SIMPLE EQUATIONS CONTAINING RADICALS OF THE SECOND DEGREE.

205. In the solution of questions involving radicals, much will depend on the judgment and *practice* of the pupil, as almost every question can be solved in several ways.

The following directions will frequently be found useful:

1st. When the equation contains one radical expression, transpose it to one side of the equation, and the rational terms to the other, then involve both sides. Thus,

If we have the equation $\sqrt{(x-1)}-1=2$, to find x.

Transposing, $\sqrt{(x-1)}=3$
Squaring, $x-1=9$
$x=10$.

2d. When more than one expression is under the radical sign, the operation must be repeated.

Thus, $a+x=\sqrt{(a^2+x\sqrt{c^2+x^2})}$, to find x.

Squaring, $a^2+2ax+x^2=a^2+x\sqrt{c^2+x^2}$.

Reducing and dividing by x, $2a+x=\sqrt{c^2+x^2}$.

Squaring, $4a^2+4ax+x^2=c^2+x^2$; whence, $x=\frac{c^2-4a^2}{4a}$.

3d. When there are two radical expressions, place one of them alone on one side, before squaring.

Thus, $\sqrt{(x-5)}-3=4-\sqrt{(x-12)}$, to find x.

Transposing, $\sqrt{(x-5)}=7-\sqrt{(x-12)}$.

Squaring, $x-5=49-14\sqrt{(x-12)}+x-12$.

Reducing and transposing, $14\sqrt{(x-12)}=42$.

Dividing, $\sqrt{(x-12)}=3$.

Squaring, $x-12=9$, from which $x=21$.

1. $\sqrt{x+3}+3=7$. . . . .
2. $x+\sqrt{x^2+11}=11$. . . . . . .
3. $\sqrt{(6+\sqrt{x-1})}=3$. . . . . .
4. $\sqrt{x(a+x)}=a-x$. . . . . . .
5. $\sqrt{x}-2=\sqrt{x-8}$. . . . . . .
6. $x+\sqrt{x^2-7}=7$. . . . . . .
7. $\sqrt{x+7}=6-\sqrt{x-5}$. . . . . .
8. $\sqrt{x-a}=\sqrt{x}-\frac{1}{2}\sqrt{a}$. . . . . .

9. $\sqrt{x+225}-\sqrt{x-424}-11=0$. .

10. $x+\sqrt{2ax+x^2}=a$. . . . .

11. $\sqrt{x+a}-\sqrt{x-a}=\sqrt{a}$. . . . .

12. $\sqrt{x+12}=2+\sqrt{x}$. . . . . . .

13. $\sqrt{8+x}=2\sqrt{1+x}-\sqrt{x}$. . . . .

14. $\sqrt{5x}+\dfrac{12}{\sqrt{5x+6}}=\sqrt{5x+6}$. . . .

15. $\sqrt{x}-4=\dfrac{237-10x}{4+\sqrt{x}}$. . . . . . .

16. $\sqrt{x^2+\sqrt{4x^2+x+\sqrt{9x^2+12x}}}=1+x$.

17. $\sqrt{a+\sqrt{ax}}=\sqrt{a}-\sqrt{a-\sqrt{ax}}$. . .

18. $b(\sqrt{x}+\sqrt{b})=a(\sqrt{x}-\sqrt{b})$.

19. $\sqrt{x}+\sqrt{ax}=a-1$. . . . .

GENERAL REVIEW.

Define power. Root. Exponent. Index. Coëfficient. Factor. Term. Square. Square root. Cube. Cube root. Surd. Radical of the second degree. Rule for extracting the square root of whole numbers. Of common fractions. Of decimals.

What is a perfect square? An imperfect square? Prove that the square root of an imperfect square can not be a fraction. Rule for extracting the square root of an algebraic monomial. Of a polynomial. Prove that no binomial can be a perfect square.

To what is the square of a radical of the second degree equal? How reduce an integral radical to its simplest form? A fractional radical. Rule for addition of radicals. Subtraction. Multiplication. Division. How make a radical denominator rational?

Define elimination. How many methods of elimination? Define each. Rule for each. How state a problem containing two unknown quantities? Containing three or more? When is the first method of elimination preferred? The second? The third? What two parts in the solution of a problem?

When the denominator of a fraction contains a radical of the second degree, how may it be rendered rational? On what principle is this rule founded?

VII. QUADRATIC EQUATIONS.

DEFINITIONS AND ELEMENTARY PRINCIPLES.

206. A **Quadratic Equation** is an equation of the *second degree* in which the highest power of the unknown quantity is a square; as, $x^2=9$, and $5x^2+3x=26$.

An equation containing two or more unknown quantities is of the second degree, when the sum of the exponents of the unknown quantities in any term is 2; as, $xy=6$, $x^2+xy=8$, and $xy+x+y=11$.

207. Quadratic equations are of two kinds, *pure* and *affected.*

A **Pure Quadratic Equation** is one that contains the second power only of the unknown quantity, and known terms; as, $x^2=9$, and $8x^2-5x^2=12$.

A pure quadratic equation is also called *an incomplete equation of the second degree.*

An **Affected Quadratic Equation** is one that contains both the first and second powers of the unknown quantity, and known terms; as, $3x^2+4x=20$, and $ax^2-bx^2+dx-ex=f-g$.

An affected quadratic equation is also called *a complete equation of the second degree.*

208. Every quadratic equation may be reduced to one of the forms $ax^2=b$, or $ax^2+bx=c$.

REVIEW.—206. What is a quadratic equation? Examples. If an equation contains two unknown quantities, when is it of the second degree? Examples.

207. How many kinds of quadratic equations? What is a pure quadratic? Examples. By what other name called? What is an affected quadratic? Examples. By what other name called?

In a pure quadratic equation, all the terms containing x^2 may be collected together, and its form becomes $ax^2=b$, or $ax^2-b=0$.

An affected quadratic equation may be similarly reduced; for all the terms containing x^2 may be reduced to one term, as ax^2; and those containing x, to one, as bx; and the known terms to one, as c; then, the equation is

$$ax^2+bx=c.$$

PURE QUADRATICS.

209.—1. Let it be required to find the value of x in the equation

$$x^2-16=0.$$

Transposing, $x^2=16$.

Extracting the square root of both members,

$x=\pm4$; that is, $x=+4$, or -4.

Verification, $(+4)^2-16=16-16=0$.

Or, $(-4)^2-16=16-16=0$.

2. Let it be required to find the value of x in the equation

$$5x^2+4=49.$$

Transposing, $5x^2=45$.
Dividing, $x^2=9$.

Extracting the square root of both sides,

$$x=\pm3.$$

3. Let it be required to find the value of x in the equation

$$\frac{2x^2}{3}+\frac{3x^2}{4}=5\tfrac{2}{3}.$$

Clearing of fractions, $8x^2+9x^2=68$.
Reducing, $17x^2=68$.
Dividing, $x^2=4$.
Extracting the square root, $x=\pm2$.

4. Given $ax^2+b=cx^2+d$, to find the value of x.

$$ax^2-cx^2=d-b.$$

Or, $$(a-c)x^2=d-b.$$

$$x^2=\frac{d-b}{a-c}.$$

$$x=\pm\sqrt{\frac{d-b}{a-c}}.$$ Hence,

TO SOLVE A PURE QUADRATIC EQUATION,

Rule.—*Reduce the equation to the form $ax^2=b$. Divide both sides by the coëfficient of x^2, and extract the square root of both members.*

210. If we take the equation $ax^2=b$, we have

$$x^2=\frac{b}{a}.$$

$$x=\pm\sqrt{\frac{b}{a}};\ \text{that is, } x=+\sqrt{\frac{b}{a}},\ \text{and } x=-\sqrt{\frac{b}{a}}.$$

If we assume $\frac{b}{a}=m^2$; then, $x^2=m^2$.

By transposing, $x^2-m^2=0$.

By separating into factors, $(x+m)(x-m)=0$.

Now, when the product of two factors is 0, one of the factors must be 0. Hence, this equation can be satisfied in two ways, and in two *only*: that is, by making either of the factors equal to 0.

By making the second factor equal to 0, we have

$$x-m=0,\ \text{or } x=+m.$$

By making the first factor equal to 0, we have

$$x+m=0,\ \text{or } x=-m.$$

Since the equation $(x+m)(x-m)=0$, can be satisfied *only* in these two ways, it follows that the values of x obtained from these conditions are the only values of the unknown quantity. Hence,

REVIEW.—208. To what two forms may every quadratic equation be reduced? Why? 209. Rule for the solution of a pure quadratic equation.

1. *Every pure quadratic equation has two roots, and only two.*

2. *These roots are equal, but have contrary signs.*

Find the roots of the equation, or the values of x, in each of the following examples:

1. $x^2-8=28$. . . . .
2. $3x^2-15=83+x^2$. . . . . .
3. $a^2x^2-b^2=0$ . . . .
4. $\frac{1}{3}x^2-1=\frac{4x^2}{27}+\frac{2}{3}$. . . . . .
5. $\frac{5x^2}{3}+12=\frac{8x^2}{7}+37\frac{2}{3}$. . . . .
6. $ax^2-b=(a-b)x^2+c$. . . . .
7. $\frac{x-a}{a}-\frac{a-2x}{x-a}=\frac{x^2+bx}{x^2-a^2}$. . . .

QUESTIONS PRODUCING PURE QUADRATIC EQUATIONS.

211.—1. Find a number whose $\frac{2}{3}$ multiplied by its $\frac{2}{5}$, equals 60.

Let $x=$ the number; then, $\frac{2x}{3}\times\frac{2x}{5}=\frac{4x^2}{15}=60$.

$$4x^2=900.$$
$$x^2=225.$$
$$x=15.$$

2. What number is that, of which the product of its third and fourth parts is equal to 108?

3. What number is that whose square diminished by 16, is equal to half its square increased by 16?

REVIEW.—210. Show that every such equation has two roots, and only two. That they are equal, but have contrary signs.

4. What number is that, which being divided by 9, gives the same quotient as 16 divided by the number?

5. What two numbers are to each other as 3 to 5, and the difference of whose squares is 64?

Let $3x=$ the less number; then, $5x=$ the greater.

And $(5x)^2-(3x)^2=64$.

Or, $25x^2-9x^2=16x^2=64$.

$x=2$; hence,

$3x=6$, and $5x=10$.

6. What two numbers are to each other as 3 to 4, the difference of whose squares is 63?

7. The breadth of a lot is to its length as 5 to 9, and it contains 1620 sq. ft.; required the breadth and length.

8. Find two numbers whose sum is to the greater as 10 to 7, and whose sum, multiplied by the less, is 270.

Let $10x=$ their sum; then, $7x=$ the greater, and $3x=$ the less.

9. What two numbers are those, whose difference is to the greater as 2 to 9, and the difference of whose squares is 128?

10. A person bought a piece of muslin for $3 and 24 cts., and the number of cts. which he paid for a yd., was to the number of yd. as 4 to 9; how many yd. did he buy, and what was the price per yd.?

11. Find two numbers in the ratio of $\frac{1}{2}$ to $\frac{2}{3}$, the sum of whose squares is 225.

Reducing $\frac{1}{2}$ and $\frac{2}{3}$ to a common denominator, we find they are as 3 to 4. Then, let $3x$ and $4x$ represent the numbers.

12. Find three numbers in the proportion of $\frac{1}{2}$, $\frac{2}{3}$, and $\frac{3}{4}$, the sum of whose squares is 724.

AFFECTED QUADRATIC EQUATIONS.

1. Required to find the values of x in the equation

$$x^2-4x+4=1.$$

It is evident, from Art. 197, that the first member of this equation is a perfect square. By extracting the square root of both members, we have $x-2=\pm1$.

Whence, $x=2\pm1=2+1=3$, or $2-1=1$.

Verification, $(3)^2-4\times3+4=1$; that is, $9-12+4=1$.

Also, $(1)^2-4\times1+4=1$; that is, $1-4+4=1$.

Hence, x has *two* values, $+3$ and $+1$, either of which verifies the equation.

2. Required to find the value of x, in the equation

$$x^2+6x=16.$$

If the left member of this equation were a perfect square, we might find the value of x, by extracting the square root, as in the preceding example.

By a careful examination of the principle stated in Art. 197, we discover that the first member will become a perfect square if 9 be added to it.

Adding 9 to each member,

$$x^2+6x+9=25.$$

Extracting the square root, $x+3=\pm5$.

Whence, $x=-3\pm5=+2$, or -8.

Either of which values of x will verify the equation.

212. Every affected quadratic equation is of the form

$$ax^2+bx=c.$$

If we divide both sides by a, and make the term in x^2 positive, there can be but four possible forms, according as the signs of the other two terms are positive or negative, viz.:

$$x^2+2px=q \quad (1)$$
$$x^2-2px=q \quad (2)$$
$$x^2+2px=-q \quad (3)$$
$$x^2-2px=-q \quad (4)$$

In which $2p$ and q may be either integral or fractional.

We will now explain the principle by which the first member of this equation may always be made a perfect square.

The square of a binomial is equal to the square of the first term, plus twice the product of the first term by the second, plus the square of the second.

If, now, we consider x^2+2px as the first two terms of the square of a binomial, x^2 is the square of the first term (x), and $2px$, the double product of the first term by the second; therefore,

If we divide $2px$ by $2x$, the quotient, p (*half the coëfficient of* x), will be the second term of the binomial, and its square, p^2, added to the first member, will render it a perfect square. To preserve the equality, we must add the same quantity to both sides. This gives

$$x^2+2px+p^2=q+p^2$$

Extracting the square root, $x+p=\pm\sqrt{q+p^2}$

Transposing, $x=-p\pm\sqrt{q+p^2}$.

It is obvious, that the square may be completed in each of the other forms, on the same principle.

Collecting the values of x in each, we have the following table:

(1.)	$x^2+2px=q$.	$x=-p\pm\sqrt{q+p^2}$.
(2.)	$x^2-2px=q$.	$x=+p\pm\sqrt{q+p^2}$.
(3.)	$x^2+2px=-q$.	$x=-p\pm\sqrt{-q+p^2}$.
(4.)	$x^2-2px=-q$.	$x=+p\pm\sqrt{-q+p^2}$. Hence,

TO SOLVE AN AFFECTED QUADRATIC EQUATION,

Rule.—1. *Reduce the equation, by clearing of fractions and transposition if necessary, to the form* $ax^2+bx=c$.

2. *Make the first term positive, if it is not so already.*

3. *Divide each side of the equation by the coëfficient of* x^2.

4. *Add to each member the square of half the coëfficient of* x.

5. *Extract the square root of both sides.*

6. *Transpose the known term to the second member.*

1. Find the roots of the equation $x^2+8x=33$.

Completing the square by taking half the coëfficient of x, squaring it, and adding to each member, we have

$$x^2+8x+16=33+16=49.$$

Extracting the root, $x+4=\pm7$.
Transposing, $x=-4\pm7$.
Whence, $x=-4+7=+3$.
And $x=-4-7=-11$.

Verification. $(3)^2+8(3)=33$; that is, $9+24=33$.
Or, $(-11)^2+8(-11)=33$; that is, $121-88=33$.

2. Solve the equation $x^2-6x=16$.

Completing the square, $x^2-6x+9=16+9=25$.
Extracting the root, $x-3=\pm5$.
Transposing, $x=+3\pm5$.
Whence, $x=+3+5=+8$.
And $x=+3-5=-2$.

Both of which will be found to verify the equation.

3. Solve the equation $x^2+6x=-5$.

Completing the square, $x^2+6x+9=9-5=4$.
Extracting the root, $x+3=\pm2$.
Transposing, $x=-3\pm2$.
Whence, $x=-3+2=-1$.
And $x=-3-2=-5$.

4. Find the values of x, in the equation $x^2-10x=-24$.

Completing the square, $x^2-10x+25=25-24=1$.
Extracting the root, $x-5=\pm1$.
Transposing, $x=5\pm1$.
Whence, $x=5+1=6$.
And $x=5-1=4$.

The preceding examples illustrate the four different forms, when the equation is already reduced. Generally, however, equations are more complicated, and require to be reduced before completing the square.

5. Find the values of x, in the equation $3x-5=\frac{7x+36}{x}$.

Clearing of fractions, $3x^2-5x=7x+36$.
Transposing, $3x^2-12x=36$.
Dividing, $x^2-4x=12$.
Completing the square, $x^2-4x+4=16$.
Extracting the root, $x-2=\pm 4$.
Transposing, $x=+2\pm 4$.
Whence, $x=6$, or -2.

6. Find the values of x, in the equation $\frac{12x^2}{5}+x=52+\frac{13x}{5}$.

Clearing of fractions, $12x^2+5x=260+13x$.
Transposing and reducing, $12x^2-8x=260$.
Dividing, $x^2-\frac{2}{3}x=\frac{65}{3}$.

Here, the coëfficient of x is $-\frac{2}{3}$, the half of which is $-\frac{1}{3}$; the square of this is $\frac{1}{9}$, which being added to both sides, we have

$$x^2-\frac{2}{3}x+\frac{1}{9}=\frac{65}{3}+\frac{1}{9}=\frac{196}{9}.$$

Extracting the root, $x-\frac{1}{3}=\pm\frac{14}{3}$.
$x=+\frac{1}{3}\pm\frac{14}{3}$.
Whence, $x=+5$, or $-\frac{13}{3}$.

NOTE.—The following examples illustrate the four forms, to one of which every complete equation of the second degree may be reduced.

7. $x^2+8x=20$. . . . . .
8. $x^2+16x=80$. . . . . .
9. $x^2+3x=28$. . . . . .
10. $x^2-10x=24$. . . . . .
11. $x^2-5x=6$. . . . . . .
12. $x^2+6x=-8$. . . . . .
13. $x^2+7x=-12$. . . . .

REVIEW.—212. To what general form may every affected quadratic be reduced? What are the four forms that this gives, depending on the signs of $2p$ and q?

212. Explain the principle, by means of which the first member of the equation $x^2+2px=q$ may be made a perfect square. Rule for the solution of affected quadratic equations.

14. $x^2-8x=-15$. . . . . . .
15. $x^2-15x=-54$. . . . . . .
16. $3x^2-2x+123=256$. . . .
17. $2x^2-5x=12$. . . . .
18. $\frac{2x^2}{3}-\frac{5x}{2}=\frac{2}{3}$. . . . .
19. $x^2-x-40=170$. . . .
20. $\frac{x}{4}-\frac{44}{x-2}=4$. . . . .
21. $\frac{5}{8}x^2-\frac{1}{2}x+\frac{3}{4}=8-\frac{2}{3}x-x^2+\frac{273}{12}$.
22. $x^2+x=30$. . . . .
23. $\frac{x}{2}+\frac{2}{x}=\frac{x}{4}+\frac{3}{2}$. . . . . . .
24. $2x^2+92=31x$. . . . . .
25. $-x^2+x=\frac{6}{25}$. . . . .
26. $17x^2-19x=30$. . . . . .
27. $4x-3x^2=6x-8$. . . . . .
28. $x^2-4x=-1$.

29. $\frac{4x}{7}-\frac{2x^2}{3}=\frac{10x}{3}-\frac{20}{7}$. . .
30. $\frac{x}{x+8}=\frac{x+3}{2x+1}$. . . . . .
31. $x+\frac{24}{x-1}=3x-4$. . . .
32. $\frac{x+3}{x}+\frac{7x}{x+3}=\frac{23}{4}$. . . .
33. $\frac{2x+3}{10-x}=\frac{2x}{25-3x}-6\frac{1}{2}$. . .
34. $2ax-x^2=-2ab-b^2$. . . .
35. $x^2+3bx-4b^2=0$. . . .
36. $x^2-ax-bx=-ab$. . . .
37. $2bx^2+(a-2b)x=a$. . . .
38. $x^2-(a-1)x-a=0$. . . .
39. $x^2-(a+b-c)x=(a+b)c$.

213. THE HINDOO METHOD OF SOLVING QUADRATICS.—When an equation is brought to the form $ax^2+bx=c$, it may be reduced to a simple equation, without dividing by the coëfficient of x^2; thus avoiding fractions.

If we multiply both sides of the equation $ax^2+bx=c$, by a, the coëfficient of x^2, it becomes $a^2x^2+abx=ac$.

Now, we may regard a^2x^2+abx, or a^2x^2+bax, as the first and second terms of the square of a binomial, a^2x^2 being the highest power, ax the lower power with a coëfficient b.

Completing the square by adding the square of $\frac{b}{2}$ to each side, the equation becomes $a^2x^2+abx+\frac{b^2}{4}=ac+\frac{b^2}{4}$.

Now, the left side is a perfect square; but it will still be a perfect square, if we multiply both sides by 4, since the product of a square number by a square number is always a square number.

Multiplying by 4, the equation is cleared of fractions, and we have $4a^2x^2+4abx+b^2=4ac+b^2$.

Extracting the square root,

$$2ax+b=\pm\sqrt{4ac+b^2}.$$

Whence,

$$x=\frac{-b\pm\sqrt{4ac+b^2}}{2a}.$$

The equation $4a^2x^2+4abx+b^2=4ac+b^2$, may be derived directly from the equation $ax^2+bx=c$, by multiplying both sides by $4a$, the coëfficient of x^2, and then adding to each member, the square of b, the coëfficient of the first power of x. Hence,

TO SOLVE AN AFFECTED QUADRATIC EQUATION,

Rule.—1. *Reduce the equation to the form* $ax^2+bx=c$, *and multiply both sides by four times the coëfficient of* x^2.

2. *Add the square of the coëfficient of* x *to each side, and extract the square root.*

REVIEW.—213. Explain the Hindoo method of completing the square.

1. Given $3x^2-5x=28$, to find the values of x.

Multiplying both sides by 12, which is 4 times the coëfficient of x^2,

$$36x^2-60x=336.$$

Adding to each member 25, the square of 5, the coëfficient of x,

$$36x^2-60x+25=361.$$

Extracting the root, $6x-5=\pm19$.

$$6x=5\pm19=24, \text{ or } -14.$$

$$x=+4, \text{ or } -\tfrac{7}{3}.$$

By the same rule, find the values of the unknown quantity in each of the following examples:

2. $2x^2+5x=33$. . . . .
3. $5x^2+2x=88$. . . . . . .
4. $3x^2-x=70$. . . . . . .
5. $x^2-x=42$. . . . . . .
6. $\frac{1}{3}x^2+\frac{3x}{8}-5=9\frac{1}{4}$. . . . . . .

PROBLEMS PRODUCING AFFECTED QUADRATIC EQUATIONS.

214.—1. What number is that, whose square diminished by the number itself, is equal to 20?

Let $x=$ the number.

Then, $x^2-x=20$.

Completing the square, $x^2-x+\frac{1}{4}=20+\frac{1}{4}=\frac{81}{4}$.

Extracting the root, $x-\frac{1}{2}=\pm\frac{9}{2}$.

Whence, $x=+5$, or -4.

The negative value -4, will answer the conditions of the question in an arithmetical sense, if the question be changed, thus: what number is that, whose square *increased* by the number itself, is equal to 20?

2. A person buys several oranges for 60 cts.; had he bought 3 more for the same sum, each orange would have cost him 1 ct. less; how many did he buy?

Let $x=$ the number bought.

Then, $\frac{60}{x}=$ the price of each.

And $\frac{60}{x+3}=$ the price of one, had he bought 3 more for 60 cts.

Therefore, $\frac{60}{x}-\frac{60}{x+3}=1.$

Clearing of fractions, and reducing,

$$x^2+3x=180.$$

Completing the square, $x^2+3x+\frac{9}{4}=\frac{9}{4}+180=\frac{729}{4}.$

Extracting the root, $x+\frac{3}{2}=\pm\frac{27}{2}.$

Whence, $x=+12$, or -15.

Either of these values satisfies the *equation* from which it was derived; but only one satisfies the conditions of the question.

Since $\frac{60}{-15}=-4$ and $\frac{60}{-15+3}=-5$; and since buying and selling are opposite, the result, -15, is the answer to the following question:

A person *sells* several oranges for 60 cts. Had he sold 3 *less* for the same sum, he would have *received* 1 ct. *more* for each; how many oranges did he sell?

REMARK.—From the two preceding examples, we see that the positive root satisfies both the conditions of the question, and the equation derived from it; while the other root satisfies the equation only.

The negative value is the answer to a question, differing from the one proposed, in this; that certain quantities which were *additive*, have been *subtractive*, and *vice versa*.

Sometimes, however, as in the following example, both values of the unknown quantity satisfy the conditions of the question.

3. Find a number, whose square increased by 15, shall be 8 times the number.

Let $x=$ the number; then, $x^2+15=8x$.

Or, $x^2-8x=-15$.

Whence, $x=5$, or 3.

Either of which fulfills the conditions of the question.

When a problem contains two unknown quantities, and can be solved by the use of one symbol, the two values generally give the values of both unknown quantities, as in the following:

4. Divide 24 into two such parts, that their product shall be 95.

Let $x=$ one of the parts; then, $24-x=$ the other.

And $x(24-x)=95$.

Or, $x^2-24x=-95$.

Whence, $x=19$ and 5.

And $24-x=5$, or 19.

5. Find 3 numbers, such that the product of the first and third is equal to the square of the second; the sum of the first and second is 10, and the third exceeds the second by 24.

Let $x=$ the first; then, $10-x=$ the second.

And $10-x+24=34-x=$ the third.

Also, $(10-x)^2=x(34-x)$.

Or, $100-20x+x^2=34x-x^2$.

From which, $x=25$, or 2.

When $x=25$, $10-x=-15$, $34-x=9$, and the numbers are 25. -15, and 9.

When $x=2$, $10-x=8$, $34-x=32$, and the numbers are 2, 8, and 32.

Both sets of values satisfy the question in an algebraic sense; only the last in an arithmetical sense.

The first will satisfy the conditions of the following:

Find three numbers, such that the product of the first and third is equal to the square of the second; the *difference* of the first and second is 10; and the *sum* of the second and third is 24.

REMARK.—In the following examples, it is required to find only that value of the unknown quantity which satisfies the conditions of the question in an arithmetical sense. It forms, however, a good exercise to determine the negative value, and modify the question, as above.

6. Find a number, such that if its square be diminished by 6 times the number itself, the remainder shall be 7.

7. Find a number, such that twice its square, plus 3 times the number itself, shall be 65.

8. Find a number, such that if its square be diminished by 1, and $\frac{2}{3}$ of the remainder be taken, the result shall be equal to 5 times the number divided by 2.

9. Find two numbers whose difference is 8, and whose product is 240.

10. A person bought a number of sheep for $80; if he had bought 4 more for the same money, he would have paid $1 less for each; how many did he buy?

11. There are two numbers, whose difference is 10, and if 600 be divided by each, the difference of the quotients is also 10; what are the numbers?

12. A pedestrian, having to walk 45 mi., finds that if he increases his speed $\frac{1}{2}$ mi. an hr., he will perform his task $1\frac{1}{4}$ hr. sooner than if he walked at his usual rate; what is that rate?

13. Divide the number 14 into two parts, the sum of whose squares shall be 100.

14. In an orchard of 204 trees, there are 5 more trees in a row than there are rows; required the number of rows, and of trees in a row.

15. A and B start at the same time to travel 150 mi.; A travels 3 mi. an hr. faster than B, and finishes his journey $8\frac{1}{3}$ hr. before him; at what rate per hr. did each travel?

16. A company at a tavern had $1 and 75 cts. to pay; but before the bill was paid two of them went away, when those who remained had each 10 cts. more to pay; how many were in the company at first?

17. The product of two numbers is 100, and if 1 be taken from the greater and added to the less, the product of the resulting numbers is 120; what are the numbers?

18. If 4 be subtracted from a father's age, the remainder will be thrice the age of the son; and if 1 be taken from the son's age, half the remainder will be the square root of the father's age. Required the age of each.

19. A young lady being asked her age, answered, "If you add the square root of my age to $\frac{3}{8}$ of my age, the sum will be 10." Required her age.

20. Bought a horse, which I afterward sold for $24, thus losing as much per cent. upon the price of my purchase as the horse cost; what did I pay for him?

PROPERTIES OF THE ROOTS OF AN AFFECTED QUADRATIC EQUATION.

215. The preceding examples show that in a quadratic equation, the unknown quantity has two values. This principle may be proved directly, as follows:

Take the general form, $x^2+2px=q$, in which $2p$ and q may be either both positive or both negative, or one positive and the other negative. Completing the square,

We have $x^2+2px+p^2=q+p^2$.

Assume $q+p^2=m^2$.

That is, $\sqrt{q+p^2}=m$;

Then, $(x+p)^2=m^2$.

Transposing, $(x+p)^2-m^2=0$.

Resolving into factors, $(x+p+m)(x+p-m)=0$.

Now, this equation can be satisfied in two ways, and in *only* two; that is, by making either of the factors equal to 0, Art. 210.

If we make the second factor equal to zero,

We have $x+p-m=0$.

Or, by transposing, $x=-p+m=-p+\sqrt{q+p^2}$.

If we make the first factor equal to zero,

We have $x+p+m=0$.

Or, by transposing, $x=-p-m=-p-\sqrt{q+p^2}$. Hence,

1. *Every quadratic equation has two roots, and only two.*

2. *Every affected quadratic equation, reduced to the form $x^2+2px=q$ may be decomposed into two binomial factors, of which the first term in each is x, and the second, the two roots with their signs changed.*

Thus, the two roots of the equation $x^2-5x=-6$, or $x^2-5x+6=0$, are $x=2$ and $x=3$; hence, $x^2-5x+6=(x-2)(x-3)$.

From this, it is evident that the direct method of resolving a quadratic trinomial into its factors, is to place it equal to zero, and then find the roots of the equation.

In this manner, let the learner solve the questions in Art. 95.

By reversing the operation, we can readily form an equation, whose roots shall have any given values. Thus,

Let it be required to form an equation whose roots shall be 4 and -6.

We must have $x=4$ or $x-4=0$.

And $x=-6$ or $x+6=0$.

Hence, $(x-4)(x+6)=x^2+2x-24=0$.

Or, $x^2+2x=24$.

Which is an equation whose roots are $+4$ and -6.

1. Find an equation whose roots are 7 and 10.
Ans. $x^2-17x=-70$.

2. Whose roots are -3 and -1. Ans. $x^2+4x=-3$.

3. Whose roots are $+2$ and -1. Ans. $x^2-x=2$.

216. Resuming the equation $x^2+2px=q$.

The first value of x is $-p+\sqrt{q+p^2}$.

The second value of x is $-p-\sqrt{q+p^2}$.

Their sum is $-2p$, which is the coëfficient of x, taken with a contrary sign. Hence, *The sum of the roots of a quadratic equation reduced to the form $x^2+2px=q$, is equal to the coëfficient of the first power of x taken with a contrary sign.*

If we take the product of the roots, we have

First root, $=-p+\sqrt{q+p^2}$

Second root, $=-p-\sqrt{q+p^2}$

$$\begin{aligned} &p^2-p\sqrt{q+p^2} \\ &\quad +p\sqrt{q+p^2}-(q+p^2) \\ &p^2 \;\ldots\ldots\; -(q+p^2)=-q. \end{aligned}$$

But $-q$ is the known term of the equation, taken with a contrary sign. Hence,

The product of the two roots of a quadratic equation, reduced to the form $x^2+2px=q$, is equal to the known term taken with a contrary sign.

REMARK.—In the preceding demonstrations, we have regarded $2p$ and q as positive; the same course of reasoning will apply in each of the four different forms.

217. In the equation $x^2+2px=q$, or first form, the two values of x are

$$-p+\sqrt{q+p^2}$$

And $$-p-\sqrt{q+p^2}.$$

The value of the part $\sqrt{q+p^2}$ must be a quantity greater than p, since the square root of p^2 alone is p. Hence,

The first root is equal to $-p$ plus a quantity greater than p; therefore, it is *essentially positive.*

The second root being equal to the sum of two negative quantities, p, and a quantity greater than p, is *essentially negative.*

It is also obvious, that the second or negative root is numerically greater than the first, or positive root. See problems 7, 8, 9, Art. 212.

In the equation $x^2-2px=q$, or second form, the two values of x are

$$+p+\sqrt{q+p^2}$$

And $$+p-\sqrt{q+p^2}.$$

Reasoning as before, we find that the first root is *essentially positive*, and greater than $2p$.

The second root is equal to p, minus a quantity greater than p; therefore, it is *essentially negative*.

The first, or positive root, is evidently greater than the second, or negative root. See problems 10 and 11, Art. 212.

In the equation $x^2+2px=-q$, or third form, the two values of x are

$$-p+\sqrt{-q+p^2}$$

And $$-p-\sqrt{-q+p^2}.$$

Here, the value of $\sqrt{-q+p^2}$, is less than p; hence, the first root is $-p$, plus a quantity less than p; therefore, it is *essentially negative*.

It is plain that the second root is *essentially negative*.

Hence, in the third form, both roots are negative. See problems 12 and 13, Art. 212.

In the equation $x^2-2px=-q$, or fourth form, the two values of x are

$$+p+\sqrt{-q+p^2}$$

And $$+p-\sqrt{-q+p^2}.$$

The value of the radical part, as in the preceding form, is less than p. Hence, the first root is *essentially positive*.

The second root, being equal to p, minus a quantity less than p, is *essentially positive*.

Hence, in the fourth form, both roots are positive. See problems 14 and 15, Art. 212.

REVIEW.—215. Show that every quadratic equation has two roots, and only two. 216. To what is the sum of the roots equal? To what is the product equal?

217. Show that in the first form one of the roots is positive, and the other negative; and that the negative root is numerically greater than the positive.

218. In the third and fourth forms, the radical part, $\sqrt{-q+p^2}$, will be further considered.

If q is greater than p^2, we are required to extract the square root of a negative quantity, which is impossible. See Art. 195. Therefore,

In the third and fourth forms, when q is greater than p^2, that is, when the known term is negative, and greater than the square of half the coëfficient of the first power of x, both values of the unknown quantity are impossible.

To explain the cause of this impossibility, we must first solve the following problem:

Into what two parts must a number be divided, so that the product of the parts shall be the greatest possible?

Let $2p$ represent any number, and let the parts, into which it is supposed to be divided, be $p+z$ and $p-z$. The product of these parts is $(p+z)(p-z)=p^2-z^2$.

Now, this product increases as z diminishes, and is greatest when z^2 is least; that is, when z^2 or z is 0. But, when z is 0, the parts are p and p. Hence,

When a number is divided into two equal parts, their product is greater than that of any other two parts into which it can be divided.

Or, as the same principle may be otherwise expressed,

The product of any two unequal numbers is less than the square of half their sum.

As an illustration of this principle, take the number 10, and divide it into different parts.

$10=9+1$, and $9\times1=9$
$10=8+2$, and $8\times2=16$
$10=7+3$, and $7\times3=21$
$10=6+4$, and $6\times4=24$
$10=5+5$, and $5\times5=25$

We see that the product of the parts becomes greater as they approach to equality, and greatest when they are equal.

Now, in Art. 216, it has been shown that $2p$ is equal to the sum of the two values of x, and that q is equal to their product. But, when q is greater than p^2, we have the product of two numbers greater than the square of half their sum, which is impossible.

If, then, any problem furnishes an equation in which the known term is negative, and greater than the square of half the coëfficient of x, we may infer that the conditions of the problem are incompatible with each other. The following is an example:

Let it be required to divide the number 12 into two such parts, that their product shall be 40.

Let x and $12-x$ represent the parts.

Then, $$x(12-x)=40.$$

Or, $$x^2-12x=-40.$$

$$x^2-12x+36=-4.$$

$$x-6=\pm\sqrt{-4}.$$

And $$x=6\pm\sqrt{-4}.$$

These values show that the problem is impossible. This we also know, from the preceding theorem, since 12 can not be divided into any two parts, whose product will be greater than 36.

Remarks.—1. When the coëfficient of x^2 is negative, as in the equation $-x^2+mx=n$, the pupil may not perceive that it is embraced in the four general forms. This apparent difficulty is obviated, by multiplying both sides of the equation by -1, or by changing the signs of all the terms.

2. Since the sign of the square root of x^2, or of $(x+p)^2$, is $\pm$, it might seem, that when $x^2=m^2$, we should have $\pm x=\pm m$, that is, $+x=\pm m$, and $-x=\pm m$; such is really the case, but $-x=+m$ is the same as $+x=-m$, and $-x=-m$ is the same as $+x=+m$.

Hence, $+x=\pm m$, embraces all the values of x. In the same manner it is necessary to take only the plus sign of the square root of $(x+p)^2$.

Review.—217. Show that in the 2d form, one root is positive and the other negative; and that the positive root is greater than the negative. That in the 3d, both are negative. That in the 4th, both are positive.

QUADRATIC EQUATIONS CONTAINING TWO UNKNOWN QUANTITIES.

NOTE.—A full discussion of equations of this class does not properly belong to an elementary treatise. The following examples embrace such only as are capable of solution by simple methods. See RAY'S ALGEBRA, SECOND BOOK.

219. In solving equations of this kind, the first step is to eliminate one of the unknown quantities. This may be performed by methods already given. See Arts. 158, 159, 160.

1. Given $x-y=2$ and $x^2+y^2=100$, to find x and y.

By the first equation, $x=y+2$.

Substituting this value of x, in the second,

$$(y+2)^2+y^2=100.$$

From which we readily find, $y=6$, or -8.

Hence, $x=y+2=8$, or -6.

2. Given $x+y=8$, and $xy=15$, to find x and y.

From the first equation, $x=8-y$.

Substituting this value of x, in the second,

$$y(8-y)=15.$$

Or, $y^2-8y=-15$.

From which y is found to be 5 or 3.

Hence, $x=3$, or 5.

There is a general method of solving questions of this form, without completing the square, which should be well understood. To explain it, suppose we have the equations,

$$x+y=a.$$
$$xy=b.$$

Squaring the first, $x^2+2xy+y^2=a^2$.

Multiplying the second by 4, $4xy=4b$.

Subtracting, $x^2-2xy+y^2=a^2-4b$.

Extracting the square root, $x-y=\pm\sqrt{a^2-4b}$.
But $x+y=a$.
Adding, $2x=a\pm\sqrt{a^2-4b}$.
Or, $x=\frac{1}{2}a\pm\frac{1}{2}\sqrt{a^2-4b}$.
Subtracting, $2y=a\mp\sqrt{a^2-4b}$.
Or, $y=\frac{1}{2}a\mp\frac{1}{2}\sqrt{a^2-4b}$.

If we have the equations $x-y=a$ and $xy=b$, we may find the values of x and y, in like manuer, by squaring each member of the first equation, and *adding* to each side 4 times the second.

Extracting the square root, we obtain the value of

$$x+y=\pm\sqrt{a^2+4b}.$$

From this, and $x-y=a$;
We find $x=\frac{1}{2}a\pm\frac{1}{2}\sqrt{a^2+4b}$.
$y=\frac{1}{2}a\mp\frac{1}{2}\sqrt{a^2+4b}$.

3. Given $x+y=a$ and $x^2+y^2=b$, to find x and y.

Squaring the first, $x^2+2xy+y^2=a^2$ (3)
But $x^2+y^2=b$ (2)
Subtracting, $2xy=a^2-b$ (4)
Take (4) from (2), $x^2-2xy+y^2=2b-a^2$.
Extracting the root, $x-y=\pm\sqrt{2b-a^2}$.
$x+y=a$.
Adding and dividing, $x=\frac{1}{2}a\pm\frac{1}{2}\sqrt{2b-a^2}$.
Subtracting and dividing, $y=\frac{1}{2}a\mp\frac{1}{2}\sqrt{2b-a^2}$.

4. Given $x^2+y^2=a$ and $xy=b$, to find x and y.

Adding twice the second to the first,

$$x^2+2xy+y^2=a+2b.$$

Extracting the square root, $x+y=\pm\sqrt{a+2b}$.

Subtracting twice the second from the first,

$$x^2-2xy+y^2=a-2b.$$

Extracting the square root, $x-y=\pm\sqrt{a-2b}$.
Whence, $x=\pm\frac{1}{2}\sqrt{a+2b}\pm\frac{1}{2}\sqrt{a-2b}$.
And $y=\pm\frac{1}{2}\sqrt{a+2b}\mp\frac{1}{2}\sqrt{a-2b}$.

5. Given $x^3+y^3=a$ and $x+y=b$, to find x and y.

Dividing the first by the second,

$$x^2-xy+y^2=\frac{a}{b} \qquad (3)$$

Squaring the second, $x^2+2xy+y^2=b^2. \qquad (4)$

Subtracting (3) from (4), $3xy=\frac{b^3-a}{b}$

Or, $xy=\frac{b^3-a}{3b} \qquad (5)$

Take (5) from (3), $x^2-2xy+y^2=\frac{4a-b^3}{3b}$.

Extracting the root, $x-y=\pm\sqrt{\left(\frac{4a-b^3}{3b}\right)}$

But $x+y=b$.

Whence, $x=\frac{1}{2}b\pm\frac{1}{2}\sqrt{\left(\frac{4a-b^3}{3b}\right)}$

And $y=\frac{1}{2}b\mp\frac{1}{2}\sqrt{\left(\frac{4a-b^3}{3b}\right)}$

In a similar manner, if we have $x^3-y^3=a$ and $x-y=b$, we find $x=\pm\frac{1}{2}\sqrt{\left(\frac{4a-b^3}{3b}\right)}+\frac{1}{2}b$, and $y=\pm\frac{1}{2}\sqrt{\left(\frac{4a-b^3}{3b}\right)}-\frac{1}{2}b$.

6. $\left.\begin{array}{l} x^2+y^2=34 \\ x^2-y^2=16 \end{array}\right\}$ . . . .

7. $\left.\begin{array}{l} x+y=16 \\ xy=63 \end{array}\right\}$ . . . .

8. $\left.\begin{array}{l} x-y=5 \\ xy=36 \end{array}\right\}$ . . . .

9. $\left.\begin{array}{l} x+y=9 \\ x^2+y^2=53 \end{array}\right\}$ . . . .

10. $\left.\begin{array}{l} x-y=5 \\ x^2+y^2=73 \end{array}\right\}$ . . . .

11. $\left.\begin{array}{l} x^3+y^3=152 \\ x+y=8 \end{array}\right\}$ . . . .

12. $\left.\begin{array}{l} x^3-y^3=208 \\ x-y=\ \ 4 \end{array}\right\}$ . . . .

13. $\left.\begin{array}{l} x^3+y^3=19(x+y) \\ x-y=3 \end{array}\right\}$ . . . . .

14. $\left.\begin{array}{l} x+y=11 \\ x^2-y^2=11 \end{array}\right\}$ . . . .

15. $\left.\begin{array}{r} (x-3)(y+2)=12 \\ xy=12 \end{array}\right\}$ . . . .

16. $\left.\begin{array}{l} y-x=2 \\ 3xy=10x+y \end{array}\right\}$ . . . . . .

17. $\left.\begin{array}{l} 3x^2+2xy=24 \\ 5x-3y=1 \end{array}\right\}$ . . . . . .

18. $\left.\begin{array}{l} \frac{1}{x}+\frac{1}{y}=\frac{5}{6} \\ \frac{1}{x^2}+\frac{1}{y^2}=\frac{13}{36} \end{array}\right\}$ . . . .

19. $\left.\begin{array}{l} x-y=2 \\ x^2y^2=21-4xy \end{array}\right\}$ . . . . . .

In solving question 18, let $\frac{1}{x}=v$, and $\frac{1}{y}=z$; the question then becomes similar to the 9th. In question 19, find the value of xy from the second equation, as if it were a single unknown quantity.

PROBLEMS PRODUCING QUADRATIC EQUATIONS CONTAINING TWO UNKNOWN QUANTITIES.

1. The sum of two numbers is 10, and the sum of their squares 52; what are the numbers?

2. The difference of two numbers is 3, and the difference of their squares 39; required the numbers.

3. Divide the number 25 into two such parts, that the sum of their square roots shall be 7.

4. The product of a number, consisting of two places, by the sum of its digits, is 160; divided by 4 times the digit in unit's place, the quotient is 4; what is the number?

5. The difference of two numbers, multiplied by the greater, =16, but by the less, =12; required the numbers.

6. Divide 10 into two such parts, that their product shall exceed their difference by 22.

7. The sum of two numbers is 10, and the sum of their cubes is 370; required the numbers.

8. The difference of two numbers is 2, and the difference of their cubes is 98; required the numbers.

9. If a number, consisting of two places, is divided by the product of its digits, the quotient will be 2; if 27 is added to it, the digits will be inverted; what is the number?

10. Find three such quantities, that the quotients arising from dividing the products of every two of them, by the one remaining, are a, b, and c.

11. Find two numbers, the sum of whose squares exceeds twice their product, by 4, and the difference of whose squares exceeds half their product, by 4.

12. The fore wheel of a carriage makes 6 revolutions more than the hind wheel, in going 120 yd.; but if the circumference of each wheel is increased 1 yd., it will make only 4 revolutions more than the hind wheel, in the same distance; required the circumference of each wheel.

13. A and B depart from the same place, and travel in the same direction; A starts 2 hr. before B, and after traveling 30 mi., B overtakes A; had each traveled half a mi. more per hr., B would have traveled 42 mi. before overtaking A. At what rate did they travel?

14. A and B started at the same time, from two different points, toward each other; when they met on the road, it appeared that A had traveled 30 mi. more than B. It also appeared that it would take A 4 da. to travel the road that B had come, and B 9 da. to travel the road that A had come. Find the distance of A from B at starting.

GENERAL REVIEW.

Define algebra. Unit of measure. Difference between an abstract and concrete number. Between a theorem and a problem. Define power of a quantity. Coëfficient. Exponent. Root. Reciprocal of a quantity. Subtraction. How does algebraic differ from arithmetical subtraction? Illustrate.

Difference between a prime and a composite number. When are two quantities prime to each other? Define the G.C.D. of two or more numbers. The L.C.M. A fraction. Terms of a fraction. How add fractions? Multiply? Divide?

Define an equation. How many classes of quantities in an equation? Define a quadratic equation. A literal equation. How clear an equation of fractions? Rule for the solution of simple equations. Define elimination. Rule for elimination by substitution. By addition and subtraction. By comparison.

Define transposition. How are signs affected by transposition? How many methods of elimination? Illustrate each by an example. Rule for elimination in three or more unknown quantities. What is meant by generalization? Illustrate by an example. When is the answer to a problem termed a formula?

What is a negative solution? When is an equation independent? When redundant? Define evolution. What is the square root of a number? Rule for extracting square root. Why can not a binomial be a perfect square? Define a radical of the second degree. A surd. What are similar radicals?

Rule for addition of radicals. Division. State difference between a pure and an affected quadratic equation. Rule for solution of an affected quadratic. Show that every quadratic equation has two roots, and only two.

VIII. PROGRESSIONS AND PROPORTION.

ARITHMETICAL PROGRESSION.

220. A **Series** is a succession of quantities or numbers, connected together by the signs + or —, in which succeeding terms may be derived from those which precede them, by a rule deducible from the *law* of the series.

Thus, $1+3+5+7+9+$, etc.,
$2+6+18+54+$, etc., are series.

In the former, any term may be derived from that which precedes it, by adding 2; and, in the latter, any term may be found by multiplying the preceding term by 3.

221. An **Arithmetical Progression** is a series of quantities which increase or decrease, by a *common difference*.

Thus, the numbers 1, 3, 5, 7, 9, etc., form an *increasing* arithmetical progression, in which the common difference is 2.

The numbers 30, 27, 24, 21, 18, 15, etc., form a *decreasing* arithmetical progression, in which the common difference is 3.

REMARK.—An arithmetical progression is termed, by some writers, an *equidifferent series*, or a *progression by difference*.

Again, a, $a+d$, $a+2d$, $a+3d$, $a+4d$, $a+5d$, etc., is an *increasing* arithmetical progression, whose first term is a, and common difference d.

If d be negative, it becomes a, $a-d$, $a-2d$, $a-3d$, $a-4d$, $a-5d$, etc., which is a *decreasing* arithmetical progression.

222. If we take an arithmetical series, of which the first term is a, and common difference d, we have

1st term $= \ldots\ldots\ldots a$;
2d term $=$1st term $+d=a+d$;
3d term $=$2d term $+d=a+2d$;
4th term $=$3d term $+d=a+3d$; and so on.

Hence, the coëfficient of d in any term is less by unity, than the number of that term in the series; therefore, the nth term $=a+(n-1)d$.

If we designate the nth term by l, we have $l=a+(n-1)d$.

For a decreasing series we also have $l=a-(n-1)d$. Hence,

TO FIND ANY TERM OF AN ARITHMETICAL SERIES,

Rule.—1. FOR AN INCREASING SERIES.—*Multiply the common difference by the number of terms less one, and add the product to the first term.*

2. FOR A DECREASING SERIES.—*Multiply the common difference by the number of terms less one, and subtract the product from the first term.*

1. The first term of an increasing arithmetical series is 3, and common difference 5; required the 8th term.

Here l, or 8th term, $=3+(8-1)5=3+35=38$, Ans.

2. The first term of a decreasing arithmetical series is 50, and common difference 3; required the 10th term.

Here l, or 10th term, $=50-(10-1)3=50-27=23$, Ans.

REVIEW.—220. What is a series? 221. What is an arithmetical progression? Give an example of an increasing series. Of a decreasing series.

222. Rule for finding the last term of an increasing arithmetical series. Of a decreasing series. Prove these rules.

In the following examples, a denotes the first term, and d the common difference of an arithmetical series; d being *plus* when the series is *increasing*, and *minus* when it is *decreasing*.

3. $a=3$, and $d=5$; required the 6th term.
4. $a=7$, and $d=\frac{1}{4}$; required the 16th term.
5. $a=2\frac{1}{2}$, and $d=\frac{1}{3}$; required the 100th term.
6. $a=0$, and $d=\frac{1}{2}$; required the 11th term.
7. $a=30$, and $d=-2$; required the 8th term.
8. $a=-10$, and $d=-2$; required the 6th term.

9. If a body falls during 20 sec., descending $16\frac{1}{12}$ ft. the first sec., $48\frac{1}{4}$ ft. the next, and so on, how far will it fall the twentieth sec.?

223. Given, the first term a, the common difference d, and the number of terms n, to find s, the sum of the series.

If we take an arithmetical series, of which the first term is 3, common difference 2, and number of terms 5, it may be written in the following forms:

$$\begin{array}{ccccc} 3, & 5, & 7, & 9, & 11, \\ 11, & 9, & 7, & 5, & 3. \end{array}$$

It is obvious that the sum of all the terms in either of these lines will represent the sum of the series; that is,

$$s=3+5+7+9+11$$

And $$s=11+9+7+5+3$$

Adding, $$2s=14+14+14+14+14$$

$$=14\times5, \text{ the number of terms, }=70.$$

Whence, $$s=\tfrac{1}{2} \text{ of } 70=35.$$

Now, let $l=$ the last term, and $n=$ the number of terms. Writing the series as before,

$$s=a+(a+d)+(a+2d)+(a+3d)+ \ldots +l$$

And $$s=l+(l-d)+(l-2d)+(l-3d)+ \ldots +a$$

Adding, $$2s=(l+a)+(l+a)+(l+a)+(l+a) \ldots +(l+a)$$

Hence, $$2s=(l+a)n, \text{ and}$$

$$s=(l+a)\frac{n}{2}=\left(\frac{l+a}{2}\right)n. \quad \text{Hence,}$$

TO FIND THE SUM OF AN ARITHMETICAL SERIES,

Rule.—*Multiply half the sum of the two extremes by the number of terms.*

From the preceding, it appears that the sum of the extremes is equal to the sum of any other two terms equally distant from the extremes.

Since $l=a+(n-1)d$, if we substitute this in the place of l in the formula $s=(l+a)\frac{n}{2}$, it becomes $s=(2a+(n-1)d)\frac{n}{2}$. Hence,

TO FIND THE SUM OF AN ARITHMETICAL SERIES,

Rule.—*To twice the first term, add the product of the number of terms less one, by the common difference, and multiply the sum by half the number of terms.*

1. Find the sum of an arithmetical series, of which the first term is 3, last term 17, and number of terms 8.

$$s=\left(\frac{3+17}{2}\right)8=80, \text{ Ans.}$$

2. Find the sum of an arithmetical series, whose first term is 1, last term 12, and number of terms 12.

3. Find the sum of an arithmetical series, whose first term is 0, common difference 1, and number of terms 20.

4. Find the sum of an arithmetical series, whose first term is 3, common difference 2, and number of terms 21.

Review.—223. What is the rule for finding the sum of an arithmetical series? Prove the rule.

5. Find the sum of an arithmetical series, whose first term is 10, common difference -3, and number of terms 10.

224. The equations, $l=a+(n-1)d$

$$s=(a+l)\frac{n}{2},$$

furnish the means of solving this general problem:

Knowing any three of the five quantities a, d, n, l, s, which enter into an arithmetical series, to determine the other two.

This question furnishes ten cases, for the solution of which we have always two equations, with only two unknown quantities.

1. Let it be required to find a in terms of l, n, d.

From the first formula, by transposing, we have $a=l-(n-1)d$; that is,

The first term of an increasing arithmetical series is equal to the last term diminished by the product of the common difference into the number of terms less one.

From the same formula, we find $d=\frac{l-a}{n-1}$; that is,

In any arithmetical series, the common difference is equal to the difference of the extremes, divided by the number of terms less one.

225. By means of the preceding rules, we are enabled to solve such problems as the following:

REVIEW.—224. What are the fundamental equations of arithmetical progression, and to what general problem do they give rise?
224. To what is the first term of an increasing arithmetical series equal? To what is the common difference of an arithmetical series equal?

Let it be required to insert five arithmetical means between 3 and 15.

Here, the two given terms with the five to be inserted make seven. Hence, $n=7$, $a=3$, $l=15$, from which we find $d=2$. Adding the common difference to 3 and the succeeding terms, we have for the series 3, 5, 7, 9, 11, 13, 15.

If we insert the same number of means between the consecutive terms of a series, the result will form a new progression. Thus,

If we insert 3 terms between the terms in 1, 9, 17, etc., the new series will be **1**, 3, 5, 7, **9**, 11, 13, 15, **17**, etc.

1. Find the sum of the natural series of numbers 1, 2, 3, 4, ... carried to 1000 terms.

2. Required the last term, and the sum of the series, 1, 3, 5, 7, ... to 101 terms.

3. How many times does a common clock strike in a week?

4. Find the nth term, and the sum of n terms of the natural series of numbers 1, 2, 3, 4, ...

5. The first and last terms of an arithmetical series are 2 and 29, and the common difference is 3; required the number of terms and the sum of the series.

6. The first and last terms of a decreasing arithmetical series are 10 and 6, and the number of terms 9; required the common difference, and the sum of the series.

7. The first term of a decreasing arithmetical series is 10, the number of terms 10, and the sum of the series 85; required the last term and the common difference.

8. Required the series obtained from inserting four arithmetical means between each of the two terms of the series 1, 16, 31, etc.

9. The sum of an arithmetical progression is 72, the first term is 24, and the common difference is —4; required the number of terms.

This question presents the equation $n^2-13n=-36$, which has two roots, 9 and 4. These give rise to the two following series, in each of which the sum is 72.

First series, 24, 20, 16, 12, 8, 4, 0, —4, —8.
Second series, 24, 20, 16, 12.

10. A man bought a farm, paying for the first A. $1, for the second $2, for the third $3, and so on; when he came to settle, he had to pay $12880; how many A. did the farm contain, and what was the average price per A.?

11. If A start from a certain place, traveling a mi. the first da., $2a$ the second, $3a$ the third, and so on; and at the end of 4 da., B start after him from the same place, traveling uniformly $9a$ mi. a da.; when will B overtake A?

Let $x=$ the number of da. required; then, the distance traveled by A in x da. $=a+2a+3a$, etc., to x terms, $=\frac{1}{2}ax(x+1)$; and the distance traveled by B in $(x-4)$ da. $=9a(x-4)$.

Whence, $\frac{1}{2}ax(x+1)=9a(x-4)$. From which $x=8$, or 9.

Hence, B overtakes A at the end of 8 da.; and since, on the ninth da., A travels $9a$ mi., which is B's uniform rate, they will be together at the end of the ninth da.

12. A sets out 3 hr. and 20 min. before B, and travels at the rate of 6 mi. an hr.; in how many hr. will B overtake A, if he travel 5 mi. the first hr., 6 the second, 7 the third, and so on?

13. A and B set out from the same place, at the same time. A travels at the constant rate of 3 mi. an hr., but B's rate of traveling is 4 mi. the first hr., $3\frac{1}{2}$ the second, 3 the third, and so on; in how many hr. will A overtake B?

REVIEW.—225. How do you insert any number of arithmetical means between two given numbers?

GEOMETRICAL PROGRESSION.

226. A **Geometrical Progression** is a series of terms, each of which is derived from the preceding, by multiplying it by a constant quantity, termed the *ratio.*

Thus, 1, 2, 4, 8, 16, etc., is an *increasing* geometrical series, whose common ratio is 2.

Also, 54, 18, 6, 2, etc., is a *decreasing* geometrical series, whose common ratio is $\frac{1}{3}$.

Generally, a, ar, ar^2, ar^3, etc., is a geometrical progression, whose common ratio is r, and which is an *increasing* or *decreasing* series, according as r is *greater* or *less* than 1.

It is obvious that the common ratio will be ascertained by dividing any term of the series by that which precedes it.

227. To find the last term of the series.

Let a denote the 1st term, r the ratio, l the nth term, and s the sum of n terms; then, the respective terms of the series will be

$$\begin{array}{cccccccccc} 1, & 2, & 3, & 4, & 5 & \ldots & n-3, & n-2, & n-1, & n \\ a, & ar, & ar^2, & ar^3, & ar^4 & \ldots & ar^{n-4}, & ar^{n-3}, & ar^{n-2}, & ar^{n-1}. \end{array}$$

That is, the exponent of r in the *second* term is 1, in the *third* 2, in the *fourth* 3, and so on; the nth term of the series will be, $l=ar^{n-1}$. Hence,

TO FIND ANY TERM OF A GEOMETRICAL SERIES,

Rule.—*Multiply the first term by the ratio raised to a power, whose exponent is one less than the number of terms.*

1. Find the 5th term of the geometric progression, whose first term is 4, and common ratio 3.

$$l=4\times3^4=4\times81=324, \text{ the fifth term.}$$

2. Find the 6th term of the progression 2, 8, 32, etc.

3. Given the 1st term 1, and ratio 2, to find the 7th term.

4. Given the 1st term 4, and ratio 3, to find the 10th term.

5. Find the 9th term of the series, 2, 10, 50, etc.

6. Given the first term 8, and ratio $\frac{1}{2}$, to find the 15th term.

7. A man purchased 9 horses, agreeing to pay for the whole what the last would cost, at $2 for the first, $6 for the second, etc.; what was the average price of each?

228. To find the sum of all the terms of the series.

Let a, ar, ar^2, ar^3, etc., be any geometrical series, and s its sum; then,

$$s=a+ar+ar^2+ar^3 \quad . \quad . \quad . \quad . \quad . \quad +ar^{n-2}+ar^{n-1}$$

Multiplying this equation by r, we have

$$rs=ar+ar^2+ar^3+ar^4 \quad . \quad . \quad . \quad . \quad +ar^{n-1}+ar^n.$$

The terms of the two series are identical, except the *first* term of the first series, and the *last* term of the second series. Subtracting the first equation from the second, we have

$$rs-s=ar^n-a$$

Or, $$(r-1)s=a(r^n-1)$$

Hence, $$s=\frac{a(r^n-1)}{r-1}$$

Since $l=ar^{n-1}$, we have $rl=ar^n$

Therefore, $$s=\frac{ar^n-a}{r-1}=\frac{rl-a}{r-1}.$$ Hence,

Review.—226. What is a geometrical progression? Give example of an increasing geometrical series. Of a decreasing. How may the common ratio in any geometrical series be found?

227. How is any term of a geometrical series found? Explain the principle of this rule.

TO FIND THE SUM OF A GEOMETRICAL SERIES,

Rule.—*Multiply the last term by the ratio, from the product subtract the first term, and divide the remainder by the ratio less one.*

1. Find the sum of 10 terms of the progression 2, 6, 18, 54, etc.

The last term $=2\times3^9=2\times19683=39366$.

$$s=\frac{lr-a}{r-1}=\frac{118098-2}{3-1}=59048, \text{ Ans.}$$

2. Find the sum of 7 terms of the progression 1, 2, 4, etc.

3. Find the sum of 10 terms of the progression 4, 12, 36, etc.

4. Find the sum of 8 terms of the series, whose first term is $6\frac{1}{4}$, and ratio $\frac{3}{2}$.

5. Find the sum of $3+4\frac{1}{2}+6\frac{3}{4}+$, etc., to 5 terms.

If the ratio r is less than 1, the progression is decreasing, and the last term lr is less than a. To render both terms of the fraction positive, change the signs of the terms, Art. 132, and we have $s=\frac{a-rl}{1-r}$, for the sum of the series when the progression is decreasing.

6. Find the sum of 15 terms of the series 8, 4, 2, 1, etc.

7. Find the sum of 6 terms of the series 6, $4\frac{1}{2}$, $3\frac{3}{8}$, etc.

REVIEW.—228. Rule for the sum of the terms of a geometrical series. Prove this rule. When the series is decreasing, how may the formula be written so that both terms of the fraction may be positive?

229. In a decreasing geometrical series, when the number of terms is infinite, the last term becomes *infinitely small*, that is, 0. Therefore, $rl=0$, and the formula $s=\frac{a-rl}{1-r}$ becomes $s=\frac{a}{1-r}$. Hence,

TO FIND THE SUM OF AN INFINITE DECREASING SERIES,

Rule.—*Divide the first term by one minus the ratio.*

1. Find the sum of the infinite series $1+\frac{1}{3}+\frac{1}{9}+$, etc.

Here, $a=1$, $r=\frac{1}{3}$, and $s=\frac{a}{1-r}=\frac{1}{1-\frac{1}{3}}=\frac{1}{\frac{2}{3}}=\frac{3}{2}$, Ans.

2. Find the sum of the infinite series $1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+$, etc.

3. Find the sum of the infinite series $9+6+4+$, etc.

4. Find the sum of the infinite series $1-\frac{1}{3}+\frac{1}{9}-\frac{1}{27}+$, etc.

5. Find the sum of the infinite series $1+\frac{1}{x^2}+\frac{1}{x^4}+\frac{1}{x^6}+$, etc.

6. Find the sum of the infinite geometrical progression $a-b+\frac{b^2}{a}-\frac{b^3}{a^2}+\frac{b^4}{a^3}-$, etc., in which the ratio is $-\frac{b}{a}$.

7. A body moves 10 ft. the first sec., 5 the next, $2\frac{1}{2}$ the next, and so on forever; how many ft. would it move over?

REVIEW.—229. Rule for the sum of a decreasing geometrical series, when the number of terms is infinite. Prove this rule.

230. The equations, $l=ar^{n-1}$, and $s=\frac{ar^n-a}{r-1}$ furnish this general problem:

Knowing three of the five quantities a, r, n, l, and s, of a geometrical progression, to find the other two.

This problem embraces ten different questions, as in arithmetical progression. Some of these, however, involve the extraction of high roots, the application of logarithms, and the solution of higher equations than those treated of in the preceding pages.

The following is one of the most simple and useful of these cases:

Having given the first and last terms, and the number of terms of a geometrical progression, to find the ratio.

Here, $l=ar^{n-1}$, or $r^{n-1}=\frac{l}{a}$. Hence, $r=\sqrt[n-1]{\left(\frac{l}{a}\right)}$.

1. The first and last terms of a geometrical series are 3 and 48, the number of terms 5; required the intermediate terms.

Here, $l=48$, $a=3$, $n-1=5-1=4$.
Hence, $r=\sqrt[4]{\frac{48}{3}}=\sqrt[4]{16}=2$.

2. In a geometrical series of three terms, the first and last terms are 4 and 16; required the middle term.

In a geometrical progression of three terms, the middle term is called a *mean proportional* between the other two.

3. Find a mean proportional between 8 and 32.

4. The first and last terms of a geometrical series are 2 and 162, and the number of terms 5; required the ratio.

RATIO AND PROPORTION.

231. Two quantities of the same kind may be compared in two ways:

1st. By finding *how much* the one *exceeds* the other.

2d. By finding *how many times* the one *contains* the other.

If we compare the numbers 2 and 6, by the first method, we say that 2 is 4 *less* than 6, or that 6 is 4 *greater* than 2.

If we compare 2 and 6 by the second method, we say that 6 is equal to *three times* 2, or that 2 is *one third* of 6.

The second method of comparison gives rise to proportion.

232. **Ratio** is the quotient which arises from dividing one quantity by another of the *same kind*.

Thus, the ratio of 2 to 6 is 3; the ratio of a to ma is m.

REMARKS.—1st. In comparing two numbers or quantities by their quotient, the number expressing the ratio will depend on which is made the *standard* of comparison. Some writers make the *first* of the two numbers the standard of comparison, and say the ratio of 2 to 6 is 3; others make the *last* the standard, and say the ratio of 2 to 6 is $\frac{1}{3}$. The former method is adopted in this work.

2d. In order that two quantities may have a ratio to each other, they must be of the same kind. Thus, 2 *yd.* has a ratio to 6 *yd.*, because the latter is *three times* the former; but 2 *yd.* has no ratio to $6, since the one can not be either greater, less, or any number of times the other.

233. When two numbers, as 2 and 6, are compared, the *first* is called the *antecedent*, the *second* the *consequent*.

REVIEW.—231. In how many ways may two quantities of the same kind be compared? Compare 2 and 6 by the first method. By the second.

232. What is ratio? Give an illustration. 233. When two numbers are compared, what is the first called? The second? Example.

An antecedent and consequent, when spoken of as *one*, are called a *couplet;* when spoken of as *two*, the *terms* of the ratio.

Thus, when the ratio of 2 to 6 is spoken of, 2 and 6 together, form a couplet, of which 2 is the first term, and 6 the second.

234. Ratio is expressed in two ways:

1st. In the form of a fraction, of which the *antecedent* is the *denominator*, and the *consequent* the *numerator*.

Thus, the ratio of 2 to 6, is expressed by $\frac{6}{2}$; the ratio of 3 to 12, by $\frac{12}{3}$, etc.

2d. By placing two points (:) between the terms.

Thus, the ratio of 2 to 6, is written 2 : 6; the ratio of 3 to 8, 3 : 8, etc.

235. The ratio of two quantities may be either a whole number, a common fraction, or an *interminate* decimal.

Thus, the ratio of 2 to 6 is $\frac{6}{2}$, or 3.

The ratio of 10 to 4 is $\frac{4}{10}$, or $\frac{2}{5}$.

The ratio of 2 to $\sqrt{5}$ is $\frac{\sqrt{5}}{2}$, or $\frac{2.236+}{2}$, or 1.118+.

From the last illustration, it is obvious that the ratio of two quantities can not always be expressed exactly, except by symbols; but, by employing decimals, we may find the *approximate* ratio to any required degree of exactness.

236. Since the ratio of two numbers is expressed by a fraction, it follows that whatever is true of a fraction, is true of the terms of a ratio. Hence,

REVIEW.—234. When are the antecedent and consequent of a ratio called a couplet? When called terms? By what two methods is ratio expressed? Example. 235. What forms may the ratio of two quantities have?

1st. *To multiply the consequent, or divide the antecedent by any number, multiplies the ratio by that number.* Arts. 122, 125.

Thus, the ratio of 4 to 12, is 3.
The ratio of 4 to 12×5, is 3×5.
The ratio of $4\div2$ to 12, is 6, which is equal to 3×2.

2d. *To divide the consequent, or multiply the antecedent by any number, divides the ratio by that number.* Arts. 123, 124.

Thus, the ratio of 3 to 24, is 8.
The ratio of 3 to $24\div2$, is 4, which is equal to $8\div2$.
The ratio of 3×2 to 24, is 4, which is equal to $8\div2$.

3d. *To multiply or divide both the antecedent and consequent by any number, does not alter the ratio.* Arts. 126, 127.

Thus, the ratio of 6 to 18, is 3.
The ratio of 6×2 to 18×2, is 3.
The ratio of $6\div2$ to $18\div2$, is 3.

237. When the two numbers are *equal*, the ratio is called a ratio of *equality;* when the second is greater than the first, a ratio of *greater inequality;* when less, a ratio of *less inequality.*

Thus, the ratio of 4 to 4, is a ratio of equality.
The ratio of 4 to 8, is a ratio of greater inequality.
The ratio of 4 to 2, is a ratio of less inequality.

We see, from this, that a ratio of equality may be expressed by 1; a ratio of greater inequality, by a number

REVIEW.—236. How is a ratio affected by multiplying the consequent, or dividing the antecedent? Why? By dividing the consequent, or multiplying the antecedent? Why? By multiplying or dividing both antecedent and consequent by the same number? Why?

237. What is a ratio of equality? Of greater inequality? Of less inequality? Examples.

greater than **1**; and a ratio of less inequality, by a number less than **1**.

238. A **Compound Ratio** is the product of two or more ratios.

Thus, the ratio $\frac{10}{3}$, compounded with the ratio $\frac{6}{5}$, is $\frac{10}{3}\times\frac{6}{5}=\frac{60}{15}=4$. In this case, 3 multiplied by 5, is said to have to 10 multiplied by 6, the ratio compounded of the ratios of 3 to 10 and 5 to 6.

239. Ratios may be compared by reducing the fractions which represent them to a common denominator.

Thus, the ratio of 2 to 5 is less than the ratio of 3 to 8, for $\frac{5}{2}$ or $\frac{15}{6}$ is less than $\frac{8}{3}$ or $\frac{16}{6}$.

PROPORTION.

240. Proportion is an equality of ratios.

Thus, if a, b, c, d are four quantities, such that $\frac{b}{a}$ is equal to $\frac{d}{c}$, then a, b, c, d form a proportion, and we say that a is to b, as c is to d; or, that a has the same ratio to b, that c has to d.

Proportion is written in two ways, by using,

1st. The colon and double colon; thus, $a : b :: c : d$.

2d. The sign of equality; thus, $a : b = c : d$.

The first is read, a is to b as c is to d; the second is read, the ratio of a to b equals the ratio of c to d.

From the preceding definition, it follows, that when four quantities are in proportion, the second divided by the first gives the same quotient as the fourth divided by the third.

REVIEW.—238. When are two or more ratios said to be compounded? Examples.

239. How may ratios be compared to each other? 240. What is proportion? Example. How are four quantities in proportion written? How read? Examples.

This is the *test* of the proportionality of four quantities. Thus, if a, b, c, d are the four terms of a true proportion, so that $a : b :: c : d$, we must have $\frac{b}{a}=\frac{d}{c}$.

If these fractions are equal to each other, the proportion is *true;* if they are not equal, it is *false.*

Let it be required to find whether $3 : 8 :: 2 : 5$.

Since $\frac{8}{3}=\frac{5}{2}$ is not a true equation, the proportion is false.

REMARK.—The words *ratio* and *proportion* are often misapplied. Thus, two quantities are said to be in the *proportion* of 3 to 4, instead of, in the *ratio* of 3 to 4.

A ratio subsists between *two* quantities, a proportion only between *four*. It requires *two equal ratios* to form a proportion.

241. In the proportion $a : b :: c : d$, each of the quantities a, b, c, d is called a *term.* The first and last terms are called the *extremes;* the second and third, the *means.*

242. Of four proportional quantities, the first and third are called *antecedents;* and the second and fourth, *consequents*, Art. 233. The last is said to be a fourth proportional to the other three, taken in their order.

243. Three quantities are in proportion, when the first has the same ratio to the second that the second has to the third. In this case, the *middle* term is called a *mean proportional* between the other two. Thus, if we have the proportion

$$a : b :: b : c$$

then b is called a *mean proportional* between a and c, and c is called a *third proportional* to a and b.

REVIEW.—240. Give examples of a true and false proportion. What is a test of the proportionality of four quantities? 241. What are the first and last terms of a proportion called? The second and third?

242. What are the first and third terms of a proportion called? The second and fourth? 243. When are three quantities in proportion? Example. What is the second term called? The third?

244. Proposition I.—*In every proportion, the product of the means is equal to the product of the extremes.*

Let $a : b :: c : d.$

Then, since this is a true proportion, we must have

$$\frac{b}{a}=\frac{d}{c}.$$

Clearing of fractions, we have $ad=bc$.

Illustration by numbers, $3 : 6 :: 5 : 10$, and $6\times5=3\times10$.

From the equation $bc=ad$, we have $d=\frac{bc}{a}$, $c=\frac{ad}{b}$, $b=\frac{ad}{c}$, and $a=\frac{bc}{d}$, from which we see, that if any three terms of a proportion are given, the fourth may be readily found.

The first three terms of a proportion, are ac, bd, and $acxy$; what is the fourth? Ans. $bdxy$.

REMARK.—This proposition furnishes a more convenient *test* of the proportionality of four quantities, than the method given in Article 240. Thus, $3 : 8 :: 2 : 5$ is a *false* proportion, since 3×5 is not equal to 8×2.

245. Proposition II.—Conversely, *If the product of two quantities is equal to the product of two others, two of them may be made the means, and the other two the extremes of a proportion.*

Let $bc=ad.$

Dividing each of these equals by ac, we have

$$\frac{bc}{ac}=\frac{ad}{ac}; \text{ or, } \frac{b}{a}=\frac{d}{c}.$$

That is, $a : b :: c : d.$

Illustration, $5\times8=4\times10$, and $4 : 5 :: 8 : 10$.

In applying this PROP., *take either factor on either side of the equation for the first term of the proportion, pass to the other side of the equation* *for the mean terms, and return for the fourth term.* Eight proportions may be written from each of the above equations. Thus:

$bc=ad$	$5\times8=4\times10$
$b:a::d:c$	$5:4::10:8$
$b:d::a:c$	$5:10::4:8$
$c:a::d:b$	$8:4::10:5$
$c:d::a:b$	$8:10::4:5$
$a:b::c:d$	$4:5::8:10$
$a:c::b:d$	$4:8::5:10$
$d:b::c:a$	$10:5::8:4$
$d:c::b:a$	$10:8::5:4$

246. Proposition III.—*If three quantities are in continued proportion, the product of the extremes is equal to the square of the mean.*

If $a:b::b:c$,

Then, by Art. 244, $ac=bb=b^2$.

It follows, from Art. 245, that the converse of this proposition is also trne. Thus, if

$$ac=b^2,$$

Then, $a:b::b:c$; that is,

If the product of the first and third of two quantities is equal to the square of a second, the first is to the second as the second is to the third.

Illustration: If $4:6::6:9$, then, $4\times9=6^2=36$.

If $2\times8=16$, then, $2:\sqrt{16}::\sqrt{16}:8$.

Or, $2:4::4:8$.

247. Proposition IV.—*If four quantities are in proportion, they will be in proportion by* ALTERNATION; *that is, the first will have the same ratio to the third that the second has to the fourth.*

Let $a:b::c:d$.

This gives $\frac{b}{a}=\frac{d}{c}$.

Multiplying both sides by $\frac{c}{b}$, $\frac{bc}{ab}=\frac{cd}{bc}$, or $\frac{c}{a}=\frac{d}{b}$.

That is, $a:c::b:d$.

Illustration, $2:7::6:21$, and $2:6::7:21$.

248. Proposition V.—*If four quantities are in proportion, they will be in proportion by* INVERSION; *that is, the second will be to the first as the fourth to the third.*

Let $a : b :: c : d$.

This gives $\frac{b}{a} = \frac{d}{c}$.

Inverting the fractions, we have $\frac{a}{b} = \frac{c}{d}$.

That is, $b : a :: d : c$.

Illustration, $2 : 5 :: 6 : 15$, and $5 : 2 :: 15 : 6$.

249. Proposition VI.—*If two sets of proportions have an antecedent and consequent in the one, equal to an antecedent and consequent in the other, the remaining terms will be proportional.*

Let $a : b :: c : d$. (1)

And $a : b :: e : f$. (2)

Then will $c : d :: e : f$.

For, from (1), $\frac{b}{a} = \frac{d}{c}$; and from (2), $\frac{b}{a} = \frac{f}{e}$.

Hence, $\frac{d}{c} = \frac{f}{e}$.

This gives $c : d :: e : f$.

Illustration, $3 : 5 :: 6 : 10$.

$3 : 5 :: 9 : 15$.

And $6 : 10 :: 9 : 15$.

250. Proposition VII.—*If four quantities are in proportion, they will be in proportion by* COMPOSITION; *that is, the sum of the first and second will be to the second, as the sum of the third and fourth is to the fourth.*

Let $a : b :: c : d$.

Then will $a+b : b :: c+d : d$.

From the 1st proportion, $\frac{b}{a} = \frac{d}{c}$.

Inverting the fractions, $\frac{a}{b} = \frac{c}{d}$.

Adding unity to each, $\frac{a}{b}+1=\frac{c}{d}+1.$

Therefore, $\frac{a+b}{b}=\frac{c+d}{d}.$

This gives $b:a+b::d:c+d;$

Or, by inversion, $a+b:b::c+d:d.$

Illustration, $3:4::6:8$

$3+4:4::6+8:8;$

Or, $7:4::14:8.$

REMARK.—In a similar manner, it may be proved, that the sum of the first and second terms will be to the *first*, as the sum of the third and fourth is to the *third*.

251. Proposition VIII.—*If four quantities are in proportion, they will be in proportion by* DIVISION; *that is, the difference of the first and second will be to the second, as the difference of the third and fourth is to the fourth.*

Let $a:b::c:d.$

Then will $a-b:b::c-d:d.$

From the 1st proportion, $\frac{b}{a}=\frac{d}{c}.$

Inverting the fractions, $\frac{a}{b}=\frac{c}{d}.$

Subtracting unity from each, $\frac{a}{b}-1=\frac{c}{d}-1.$

Therefore, $\frac{a-b}{b}=\frac{c-d}{d}.$

This gives $b:a-b::d:c-d;$

Or, by inversion, $a-b:b::c-d:d.$

Illustration, $8:5::16:10$

$8-5:5::16-10:10;$

Or, $3:5::6:10.$

REMARK.—In a similar manner, it may be proved that the difference of the first and second will be to the *first*, as the difference of the third and fourth is to the *third*.

252. Proposition IX.—*If four quantities are in proportion, the sum of the first and second will be to their difference, as the sum of the third and fourth is to their difference.*

Let	$a : b :: c : d.$ (1)
Then will	$a+b : a-b :: c+d : c-d.$
From (1), by composition,	$a+b : b :: c+d : d.$ (2)
From (1), by division,	$a-b : b :: c-d : d.$ (3)
By alternation, (2) and (3) become	$a+b : c+d :: b : d.$
	$a-b : c-d :: b : d.$
Therefore, Art. 249,	$a+b : c+d :: a-b : c-d;$
Or, by alternation,	$a+b : a-b :: c+d : c-d.$
Illustration,	$5 : 3 :: 10 : 6.$
	$5+3 : 5-3 :: 10+6 : 10\ \ 6.$
Or,	$8 : 2 :: 16 : 4.$

253. Proposition X.—*If four quantities are in proportion, like powers or roots of those quantities will also be in proportion.*

Let	$a : b :: c : d.$
Then will	$a^n : b^n :: c^n : d^n.$
For, since	$\frac{b}{a}=\frac{d}{c}$
Raising to the nth power,	$\frac{b^n}{a^n}=\frac{d^n}{c^n}.$
That is,	$a^n : b^n :: c^n : d^n,$

Where n may either be a whole number or a fraction.

Illustration,	$2 : 3 :: 4 : 6.$
	$2^2 : 3^2 :: 4^2 : 6^2.$
Or,	$4 : 9 :: 16 : 36.$
Also,	$a^2 : b^2 :: m^2a^2 : m^2b^2$
And	$\sqrt{a^2} : \sqrt{b^2} :: \sqrt{m^2a^2} : \sqrt{m^2b^2}.$
Or,	$a : b :: ma : mb.$

254. Proposition XI.—*If two sets of quantities are in proportion, the products of the corresponding terms will also be in proportion.*

Let $a : b :: c : d.$ (1)

And $m : n :: r : s.$ (2)

Then will $am : bn :: cr : ds.$

For, from the 1st, $\frac{b}{a}=\frac{d}{c};$

And, from the 2d, $\frac{n}{m}=\frac{s}{r}.$

Multiplying these together, $\frac{bn}{am}=\frac{ds}{cr}.$

This gives $am : bn :: cr : ds.$

Illustration,
$$3 : 5 :: 6 : 10.$$
$$4 : 3 :: 8 : 6.$$
$$12 : 15 :: 48 : 60.$$

255. Proposition XII.—*In any continued proportion, that is, any number of proportions having the same ratio, any one antecedent is to its consequent, as the sum of all the antecedents is to the sum of all the consequents.*

Let $a : b :: c : d :: m : n$, etc.

Then will $a : b :: a+c+m : b+d+n.$

Since $a : b :: c : d,$

And $a : b :: m : n,$

We have $bc=ad.$

And $bm=an.$

Add ab to each, or put $ab=ab.$

The sums of these equalities give
$$ab+bc+bm=ab+ad+an.$$

Or, $b(a+c+m)=a(b+d+n).$

Therefore, Art. 245, $a : b :: a+c+m : b+d+n.$

Illustration,
$$3 : 4 :: 6 : 8 :: 9 : 12.$$
$$3 : 4 :: 3+6+9 : 4+8+12.$$

Or, $3 : 4 :: 18 : 24.$

REMARK.—There are several other propositions in Proportion, that may be easily demonstrated, in a manner similar to the preceding, but they are of little use.

GENERAL REVIEW.

Define mathematics. State points of difference between arithmetic and algebra. What the unit of measure of 17 bushels? Of 5 feet? Define a power. How express a known and unknown quantity? Write the principal signs used in algebra. Define a residual quantity. The reciprocal of a quantity. Of a fraction. What is an algebraic expression?

Define addition. Is addition the same process in algebra and arithmetic? State the general rule for addition. Define subtraction. In algebra, does the term *difference* denote a number less than the minuend? State the rule for subtraction. Define multiplication. State the rule for the signs. For the exponents. Give general rule for multiplication.

Define division. What is a prime number? When are two quantities prime to each other? Define the greatest common divisor. The least common multiple. A fraction. State and illustrate Proposition I.; IV.; VI. Rule for the signs of fractions in multiplication. How resolve a fraction into an infinite series? Rule for dividing a fraction by a fraction.

What is an equation? Define a quadratic equation. A numerical equation. Literal equation. How is every equation to be regarded? What is solving an equation? State the six axioms. Define an axiom. Transposition. How clear an equation of fractions? In how many ways may the unknown quantity be connected with the known, and how separated in each case? Rule for solution of simple equations.

In the solution of a problem, what are explicit conditions? What are implied conditions? Define elimination. How many and what methods of elimination? Define and give illustration of elimination by substitution. State the rule. By comparison. Give rule. By addition and subtraction. Rule. How form equations when the problem contains three unknown quantities?

What is a negative solution? What does it indicate?. State difference between a formula and a rule. What is meant by generalization? Define an independent equation. Illustrate by example. Define an indeterminate equation. Illustrate by example. What are redundant conditions? Show that a simple equation has but one root.

Define power. Root. Exponent. Coëfficient. State rule for raising a monomial to any given power. A polynomial. A fraction. State the four laws found by examining the different powers of a binomial. What the law of the number of terms in any power of a binomial? Of the signs of the terms? Of the exponents of the letters? Of the coefficients of the terms? State the uses of the binomial theorem.

Define evolution. What relation exists between the number of places of figures in any number, and the number of places in its square? Of what is every number composed? State the rule for the extraction of the square root of numbers. What is the difference between the squares of two consecutive numbers?

Define a perfect square. An imperfect square. A surd. How extract the square root of a decimal? Of a fraction, when both terms are not perfect squares? Rule for extraction of the square root of a monomial. According to what law is the square of a polynomial formed? State rule for extracting the square root of a polynomial. What are radicals of the second degree? What are similar radicals?

Rule for the addition of radicals of the second degree. For subtraction. Multiplication. Division. Of a fraction whose denominator contains a radical. Rule for the solution of a pure quadratic equation. State the difference between a pure and an affected quadratic equation. Rule for the solution of an affected quadratic equation. Give the Hindoo rule.

What is a series? An arithmetical progression? Give example of an increasing arithmetical progression. Of a decreasing. State rules for both increasing and decreasing arithmetical series. Define geometrical progression. Illustrate both an increasing and decreasing geometrical progression. Rule for finding the sum of a geometrical series.

Define ratio. In how many ways may quantities of the same kind be compared? Illustrate by examples. What are the terms of a ratio? By what two methods is ratio expressed? Define compound ratio. Proportion. How is proportion indicated? State the difference between a ratio and a proportion. Give the terms of a proportion. State Proposition I.; IV.; V.

THE END.