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HIGHER ARITHMETIC

BY

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AND

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PREFACE

This work is intended for teachers in training in normal schools and for students in high schools, the former because they need a broader view of arithmetic than an elementary textbook can give, and the latter because they need a thorough review of the principles and the leading applications of arithmetic before they enter upon their life work. The book may be studied in any year of a high school, but it is desirable that the student should first be familiar with the meaning of algebraic expressions and with the solution of the simple equation. In case the student is not equipped with this slight knowledge of algebra, the teacher may omit the algebraic forms that enter into the work.

The primary purpose of the book being to lead the student to understand more completely the important parts of arithmetic with which he has a certain familiarity from his work in the elementary school, special attention is given to the principles of the subject rather than to that mechanical drill which necessarily characterizes the instruction in the lower grades. The applications are such as have special significance to teachers and commercial students, the technicalities of particular trades and vocations being avoided as unsuited to the work for which the book is intended.

While a somewhat conventional sequence of topics has been given, this sequence has not been permitted to govern the presentation of the subject. The book being intended for review purposes, topics have been allowed to cross and to correlate with one another as seems best for the student's understanding of the subject as a whole. Formal definitions play a small part, the student already being familiar with most of the terms he uses.

There is a growing demand that teachers of arithmetic and students entering business life should have at least some idea of the meaning of logarithms and the slide rule, and hence these subjects are treated briefly in the supplementary work, their study being thus made optional. The use of the slide rule has grown so rapidly within a few years, particularly in shop work and as a means of checking business computations, that some knowledge of the device is necessary for purposes of general information.

Since roots are usually found by means of tables or by the use of logarithms or the slide rule, special tables of roots are introduced. The subject itself is usually taught in algebra and hence is not considered in its theoretical aspect in this work.

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HIGHER ARITHMETIC

CHAPTER I

READING AND WRITING NUMBERS

To the Student. There is a large amount of work in arithmetic that is too advanced to be taught in the elementary school, and some of this cannot be taught to advantage even in a normal school or a high school. There is, however, a certain amount of advanced work that should be understood by those who expect to teach and by those who are taking work beyond the elementary school. It is the purpose of this book to present this material for advanced work, bringing out more clearly the essential features of computation and the leading applications of the subject.

Common Numerals. Because the Latin word for number is *numerus*, characters which are used to represent numbers are called *numerals*, and the reading of numbers is often called *numeration.*

The characters 0, 1, 2, 3, 4, 5, 6, 7, 8, and 9 are called *common numerals.*

They are also called *Hindu numerals,* because they are believed to have originated among the Hindus; *Hindu-Arabic numerals* or *Arabic numerals,* because they were transmitted to Europe by the Arabs; *digits,* because they are ten in number, like our fingers, the Latin name for fingers being *digiti*; *figures,* although the word has other meanings; and *ciphers,* although the name more commonly applies to zeros. The word "digit" is more commonly used for the numbers from one to nine.

Zero and Unity. The figure 0 is called by various names, such as *zero*, *naught*, and *cipher*. Recently it has come to be more commonly called *o*, the fifteenth letter in our alphabet.

The use of *aught* for *naught*, while allowed by some dictionaries, is not etymologically correct. In America it is looked upon as a vulgarism.

Because the Latin word for one is *unus*, 1 is often called *unity*, and the number or measure by which we may be counting is called a *unit*.

Thus 3 is made up of three units, and 7 qt. is made up of seven units, each unit in the latter case being 1 qt.

Notation. Because numerals were long ago called notes (in Latin, *notae*), the writing of numerals of any kind is often called *notation*.

Thus we speak of Roman notation, common notation, and Arabic notation. We also speak of algebraic notation, the algebraic letters standing for numbers.

Not much attention is any longer paid to the distinction between notation and numeration, it being much better to speak of the reading and writing of numbers.

Place Value. The reason why our common numerals supplanted the Roman numerals and other ancient systems of notation is that they may be written with a *place value*. That is, 51 means 50 + 1, the 5 having the value of 50 because it is in tens' place; but although, in the Roman notation, V also stands for 5 and I stands for 1, VI means V + I, or 6, the V having no place value.

In order that a system of notation may have the feature of place value, it is necessary to have a character like 0 to indicate that there is no value in the place that it occupies.

For example, we can write 708 for seven hundred eight, whereas in the Roman system we have to write DCCVIII for the same number, the 0 in our system making the simpler form possible.

Exercise. Notation

1. Why are our common numerals often called Arabic numerals or Hindu-Arabic numerals?

The first eight of these exercises may be considered orally.

2. Why are the characters which are used to represent numbers called numerals?

3. Why are the numbers or the figures from 1 to 9, or these together with 0, called digits?

The word "digit" is used in other senses, but in elementary arithmetic it has the meanings given on page 1.

4. What names are given to the figure 0, and where have you commonly heard it called *o*?

5. If you are measuring by inches, the unit of measure is 1 in. What is the unit of measure when you measure by feet? by yards? by miles?

6. What unit of measure is used in measuring the length of cloth? the volume of a water tank? the area of a farm? the distance traveled on a railway? your weight? the weight of a letter? the distance a French army marches?

7. Why is the writing of numbers called notation, and why is the reading of numbers called numeration?

8. In the number 706,400 what place values have the numerals 4, 6, and 7?

9. Write a number in which 5 shall have the place value five hundred, and 8 shall have the place value eight thousand.

10. Write five units of measure and tell for what purpose each of these units is used.

11. Write in Roman numerals and also in common numerals the number fifty followed by the number eight, thus: LVIII and 508. Write in words the number resulting in each case.

Number Names. The common names of numbers are too well known to the student to require discussion in a book of this kind. The recent history of the world, however, has so increased the use of large numbers as to require a brief consideration of this subject.

Our number system is based upon counting by tens, due to the fact that we have ten fingers. We commonly say that we count on a *scale of ten*, or on a *decimal scale*, the word "decimal" coming from the Latin word *decem*, meaning ten.

In order to understand the system it is necessary to consider the meaning of certain expressions, as follows: 10^2, read "10 square" or "10 to the second power," means 10×10; 10^3, read "10 cube" or "10 to the third power," means $10 \times 10 \times 10$; 10^4, read "10 to the fourth power," stands for $10 \times 10 \times 10 \times 10$; and similarly for other powers.

We may now consider the following:

1000, one thousand, is 10^3;
1,000,000, one million, is 1000 thousand or 10^6;
1,000,000,000, one billion, is 1000 million or 10^9;
1,000,000,000,000, one trillion, is 1000 billion or 10^{12}.

We may proceed in the same way, multiplying by 1000 each time, to quadrillions, quintillions, sextillions, septillions, octillions, nonillions, decillions, and so on, as far as we wish. Names beyond billions, however, have no practical value and need not be learned.

So rarely used have been the names beyond millions that the meaning of the word "billion" is not settled. In England a billion is a million million, which we call a trillion. France uses the same system that we do, but usually calls a thousand million a *milliard.*

It is coming to be a business custom to write a number like $125 "one hundred twenty-five dollars" instead of "one hundred *and* twenty-five dollars" in filling out checks.

Exercise. Number Names

1. From what two words is the word "thirteen" derived?

Such matters may be looked up in the dictionary, but there should be no trouble in guessing the answer correctly.

The first five of these exercises may be considered orally.

2. From what two words is the word "twenty" derived, and what did the word originally mean?

3. What power of 10 is 1000? 10,000? 100,000?

4. What power of 10 is one hundred million?

5. What power of 10 is 1 followed by seven zeros?

6. It is estimated that the distance to a certain star is 21,000,000,000,000 mi. Write the distance in words.

7. If we should measure the distance to the sun in inches we should find it to be over 6,000,000,000,000 in. Write this distance in words.

Such a distance would never be expressed in inches, but the result serves to give us some idea of the meaning of such a large number.

8. How much is the square of 1000? of 10,000?

9. How much is the eighth power of 10?

10. In the metric system there are 1000 millimeters in a meter and there are 1000 meters in a kilometer. How many millimeters are there in a kilometer? How many square millimeters in a square kilometer?

Write in words the following numbers:

11. 845,206.
12. 506,392.
13. 125,072,860.
14. 207,207,207.
15. 3,148,000,000.
16. 27,000,236,000.

Write in words, as on a check, the following numbers:

17. $225.
18. $250.
19. $1200.
20. $1505.
21. $1950.
22. $2775.
23. $10,500.
24. $15,708.

Significant Figures. If we say that the distance to the sun is 93,000,000 mi., we mean that this number is correct to the nearest million. The zeros do not signify any value; they simply give place values to the 9 and the 3.

The figures 1, 2, 3, 4, 5, 6, 7, 8, and 9 are always called *significant figures*, and if in a number one or more zeros occur between two significant figures, such zeros are also called significant figures.

In the numbers 0.07, 0.00065, and 200 the zeros are not considered as significant figures, but in the numbers 305 and 6003 the zeros are considered as significant.

In the quotient of 13,727 ÷ 45,000, which is 0.30504$\frac{4}{9}$, both the zeros in the decimal are significant, even when the result is written 0.3050, but in the expression $2 \div 5 = 0.400$ no zero is significant.

The distinction between significant figures and those that are not significant is a convenient one in arithmetic.

If directed to carry a result to any specified number of figures, carry it to that number of significant figures if it has so many. Let the last figure be the one nearest to accuracy.

For example, to three figures the quotient of 13,727 ÷ 45 is 305, to four figures it is 305.0, and to five figures it is 305.04.

In a case like the quotient of 1 ÷ 8, where the last figure is 5, we have this custom: $\frac{1}{8} = 0.1$ to one decimal place, 0.13 to two decimal places, and 0.125 to three decimal places; that is, we take 0.13 instead of 0.12 for 0.125 to two decimal places.

In general, a fraction of a cent is neglected in a result if it is less than $0.005, but otherwise it is called a cent. There are exceptions to this rule, however, as in the case of bills. Many stores, in making out bills, count as a whole cent any fraction of a cent.

To carry the result of a problem to a specified number of decimal places is not the same as to carry the result to that same number of significant figures. In commercial work the former is used more frequently; in physics and astronomy the latter is the more common.

Exercise. Notation

1. If we write five hundred and then write thirty after it, what number do we have when we use Roman numerals? when we use common numerals? Explain the difference in the values of the two numbers.

2. Write the quotient of $21.1 \div 300$, carrying the result to two decimal places; to two significant figures; to three decimal places; to four decimal places. In the last case indicate all the significant figures that occur in the result.

3. Explain why it is correct to write the quotient of $313 \div 4$ as $78 +$, and to write the quotient of $315 \div 4$ as $79 -$ or as $78.8 -$.

4. Multiply 0.325 by itself and write the result correct to two decimal places; to three decimal places.

To write the result correct to two decimal places requires us to find whether the third decimal place is 5 or more. If the third decimal place is 5 or more, increase the second decimal place by one.

5. Divide 0.223 by 0.733 and write the result correct to three decimal places.

6. Write the number $0.0004070050 +$, and underscore each significant figure. Explain why some of the zeros are significant figures and some are not.

7. If by careful measurement you find the length and the width of the printed part of a page of a book each to the nearest hundredth of an inch, there is no use in trying to find from the measurements the area nearer than to 0.01 of a square inch. Taking a page as measuring $3\frac{3}{4}''$ by $5\frac{15}{16}''$, how many significant figures would you have in the length? in the width? in the area? Illustrate your answer by measuring the page and then computing the area.

Index Notation. In modern scientific work, as in astronomy, physics, and chemistry, numbers with many figures are often used. This has given rise to a simple method of writing numbers known as the *index notation.*

In order to understand this method sufficiently for purposes of general information two statements must first be made:

1. In the case of 10^5, which we have learned is equal to 100,000, we may speak of 5 as the *index* of the power, although it is usually called an *exponent.*

2. We read the symbol 10^{-3}, thus: "10 to the minus 3 power." It means $\frac{1}{10^3}$, $\frac{1}{10 \times 10 \times 10}$, $\frac{1}{1000}$, or 0.001.

From these two statements it evidently follows that

7,000,000 may be written 7×10^6,
93,000,000 may be written 9.3×10^7,
2,800,000,000 may be written 2.8×10^9,
0.00000007 may be written 7×10^{-8}.

We might write 28×10^8 instead of 2.8×10^9 for 2,800,000,000, but it is more common to write 2.8×10^9, partly because of ease in computing by methods used by astronomers and others.

Grouping the Figures. For convenience in reading large numbers the figures are often grouped by threes. In newspapers and magazines this is frequently done by spacing, thus: 35 281 000. In writing numbers commas are used as in the case of 9,874,000,000.

It is not necessary to group the figures unless we have a number as large as 10,000, although it is allowable to do so with a number like 5280.

It is not customary to group the figures in decimal fractions.

The student is already familiar with such terms as "units' place," "tens' order," and "thousands' period." They have some value in elementary classes, but we shall not need them in our present work.

Exercise. Index Notation

1. The number of miles in the longest diameter of the earth's orbit about the sun is 1.85×10^8. Write this number in our common notation.

2. The orbit of Neptune is 2,793,000,000 mi. from the sun. Write this number in the index notation.

3. The population of the earth is about 1.6×10^9. Write this number in our common notation.

4. In a single year the United States has used nearly 40,000,000,000 feet of lumber. Write this number in the index notation.

5. The area of the earth's surface is 1.97×10^8 square miles. Write this number in our common notation.

Write these numbers in the index notation:

6. 4,000,000. **8.** 4,000,000,000. **10.** 35,000,000,000.

7. 7,500,000. **9.** 9,250,000,000. **11.** 89,650,000,000.

Write these numbers in our common notation:

12. 4×10^5. **13.** 1.7×10^7. **14.** 3.95×10^9. **15.** 8.3×10^{10}.

16. In the metric system 1 millimeter is 0.000001 of a kilometer. Write this number in the index notation.

17. A centigram is 10^{-5} of a kilogram. Write this number in our common notation.

Write these numbers in the index notation:

18. 0.000001. **20.** 0.00000004. **22.** 0.000000009.

19. 0.00000075. **21.** 0.000000038. **23.** 0.0000000003.

Write these numbers in our common notation:

24. 7×10^{-7}. **25.** 8.1×10^{-8}. **26.** 7.25×10^{-9}.

Roman Numerals. The chief purpose of studying Roman numerals is that we may be able to read chapter numbers in modern books, dates as they appear in old books, and dates written in modernized Roman numerals as they appear to-day. The student is already familiar with the ordinary simple forms needed for chapter numbers, and so we shall consider only the question of the larger numbers needed in reading dates.

Our method of writing the Roman numerals is quite different from the methods used by the Romans themselves or those used by their successors in the Middle Ages. This may be illustrated by a few examples.

Where we write

	M	IV	MCD	XVIII	MCM
for	1000	4	1400	18	1900

the medieval scholars wrote

	M	IIII	MCCCC	XVIII	MDCCCC
or	CIƆ	iiij	Mcccc	xviij	Mdcccc

and the Romans wrote

	∞	IIII	∞CCCC	XIIX	∞DCCCC

These forms varied. In particular, CIƆ was used for M and IƆ was used for D long after the beginning of printing.

We still have IIII on most clocks.

It is not necessary to attempt to remember any forms with which you are not already familiar. In the exercise on page 11 refer to this page whenever necessary.

The writing of Roman numerals is of little importance, but there is a value in being able to read them. In elementary classes it is not desirable to study Roman numerals except for the reading of clock time, dates, chapter numbers, and the like.

The Romans occasionally wrote $\overline{V}$ for 5000, and similarly for other thousands, but they usually wrote large numbers in words.

Exercise. Roman Numerals

1. The monument to Dr. Samuel Johnson, a great English writer, records that he lived LXXV years II months XIII days, dying in MDCCLXXXIV. Write this statement, using our common numerals instead of Roman numerals.

2. Write the numbers MCMXXV and MDCCCCXXV in our common numerals. State which is the older form and why the attempt is being made to introduce the later form.

3. The pages of several old books state that the books were printed in MDCIX, CIↃCCCCLXXXX, ∞dxi, Mdcxv, and Mcccccij. Write these dates in our common numerals.

Such dates are often seen in old books. The particular dates given in this problem would be written to-day as follows: MDCIX, MCDXC, MDXI, MDCXV, and MDII.

Write the following numbers in common numerals:

4. XIV.
5. XVI.
6. LXVI.
7. XL.
8. XC.
9. CD.
10. LX.
11. CX.
12. DC.
13. DCCCXX.
14. MDCCCL.
15. DCCCIX.

16. The corner stone of a building bears an inscription showing that it was placed there in the year MDCCCCXVI. Write this date in our common numerals.

17. A magazine cover bears the date MCMXIX. Write this date in our common numerals.

18. A medieval monument records that a man died in the year Mccclxviij at the age of lxviiij years, viij months, and xxiiij days. Write the statement, using our common numerals.

19. Write a short statement of the disadvantages of the Roman numerals as compared with the numerals which we use, and the extent to which the subject should be taught in elementary classes.

Fractions. The Latin word for entire, complete, or whole is *integer*, and the word for broken is *fractus*. From these we have the word "integer," meaning a whole number, and the word "fraction," meaning a broken number.

Since it is impossible for anything to have more than four quarters, it was formerly claimed that, properly speaking, a fraction should be less than 1, and hence arose the term *proper fraction*. Since, however, we may have five "quarters" (of a dollar), it was found necessary to consider a fraction like $\frac{4}{4}$ or $\frac{5}{4}$, and it was called an *improper fraction*.

About the time of the beginning of printing, in the fifteenth century, the fractions that people commonly used were written like $\frac{3}{4}$, and hence all fractions written in this way were called *common fractions*. An integer plus a common fraction, written like $3\frac{4}{5}$, is called a *mixed number*.

At that time the only other fractions used in Europe were *sexagesimal fractions*, those with some power of 60 for denominators. We still use these in such a case as 2° 15′ 30″, which means simply $2° + \frac{15°}{60} + \frac{30°}{60^2}$, or $\left(2 + \frac{1}{4} + \frac{1}{120}\right)$ degrees.

In the sixteenth century people began to use *decimal fractions*, so called because the denominators are powers of 10, and the Latin word for ten is, as already stated, *decem*. Thus, 2.5 means $2\frac{5}{10}$, and 2.75 means $2 + \frac{7}{10} + \frac{5}{100}$.

It is not customary to speak of $\frac{7}{10}$ as a decimal fraction, because it is written in the form of a common fraction.

The number 100.025 is read "one hundred *and* twenty-five thousandths," while 0.125 is read "one hundred twenty-five thousandths."

The *denominator* (a Latin word meaning namer) names the parts into which the unit is divided, and the *numerator* (numberer) states the number of these parts.

Such statements are merely supplementary to the definitions already known. The purpose is to set forth reasons for familiar names.

Exercise. Fractions

1. Write five common fractions, five decimal fractions, and five sexagesimal fractions.

2. Write five proper fractions, five improper fractions each equal to unity, and five improper fractions each greater than unity.

3. We have the Latin word *fractus* in the English word "fraction." Write three other English words in which the same Latin word appears.

Of course it is only the main part of the word *fractus* that appears; that is, what is known as the root of the word. A dictionary will assist in answering such questions.

4. Why is a fraction with denominator 60 called a sexagesimal fraction?

Consult a dictionary, if necessary. The denominator is not written in a case like 3° 12′, but is indicated by the prime mark (′).

5. Express 2 hr. 30 min. as 2 hr. and a common fraction of an hour; as 2 hr. and a decimal fraction of an hour; as an improper fraction of an hour.

Express the following as integers plus common fractions:

6. 2.125. **8.** 3° 45′. **10.** 2° 10′ 45″.
7. 3.875. **9.** 4 hr. 15 min. **11.** 6 hr. 15 min. 30 sec.

Express the following as integers plus decimal fractions:

12. $3\frac{5}{8}$. **14.** 7° 45′. **16.** 4° 30′ 15″.
13. $5\frac{7}{16}$. **15.** 3 hr. $7\frac{1}{2}$ min. **17.** 2 hr. 5 min. $7\frac{1}{2}$ sec.

Express the following as integers plus sexagesimal fractions:

18. $2\frac{1}{2}$ hr. **19.** 3.25°. **20.** 4.75°. **21.** $48.12\frac{1}{2}$ hr.

22. Write in words the numbers 200.037 and 0.237.

Per Cents. Even before decimal fractions were invented people had learned to recognize the advantage of computing by hundredths. Instead of speaking of a certain number of hundredths, however, they spoke of this number out of a hundred, as in the case of a man who pays as interest $6 out of every $100 that he borrows, or $6 per $100. The Italian merchants, who then set the standard in business computations, called this 6 per cento, *cento* meaning hundred. They abbreviated this to 6 per 8, which later, through rapid writing, became 6 per $\frac{0}{0}$ and finally 6%.

The expression 6% means simply $\frac{6}{100}$, or 0.06.

Similarly, $1\frac{1}{2}$% means $0.01\frac{1}{2}$, or 0.015;

$\frac{3}{8}$% means $\frac{3}{8}$ of $\frac{1}{100}$, or $\frac{3}{800}$;

0.06% means 0.06 of $\frac{1}{100}$, or 0.0006;

100% means $\frac{100}{100}$, or 1;

and 250% means $\frac{250}{100}$, or 2.5.

Conversely, we may write any fraction as per cent by first reducing it to hundredths.

From this we see that there is no mathematical reason for treating ordinary computation in percentage apart from decimals, at least in a course in higher arithmetic.

Various applications of percentage may, however, require a maturity of judgment that necessitates a later treatment of the subject.

In writing 6% as a decimal, and in writing decimals in general, careful computers usually place a zero before the decimal point, thus: 0.06. It is evident that there is much more chance for error if we write $.35 instead of $0.35. In a column of figures, however, the zero should be written only at the top and in the result.

$$\boxed{\begin{array}{r}\$0.35\\ .48\\ \hline \$0.83\end{array}}$$

Exercise. Per Cents

1. If a merchant borrows \$2500 and pays $\frac{1}{2}\%$ a month for its use, how much money does he pay per month?

In the financial world $\frac{1}{2}\%$ is usually read "$\frac{1}{2}$ of 1%," although "$\frac{1}{2}$ per cent" would be correct, and similarly for other fractions.

2. If a man whose credit is not very good borrows money at 1% a month, how much must he pay for the use of \$50 for 1 mo.? for 2 mo.?

3. How much is 0.06% of \$500? 0.6% of \$500? 6% of \$500? 60% of \$500?

Express the following as per cents, using the symbol % :

4. $\frac{1}{2}$.
5. $\frac{5}{8}$.
6. $\frac{7}{8}$.
7. $\frac{3}{16}$.
8. 0.07.
9. $0.07\frac{1}{2}$.
10. 0.8.
11. $0.2\frac{1}{2}$.
12. 10.2.
13. 375.

Express the following as decimals:

14. 2%.
15. $\frac{1}{2}\%$.
16. $1\frac{1}{2}\%$.
17. $3\frac{7}{8}\%$.
18. 0.1%.
19. 0.05%.
20. 125%.
21. 200%.
22. 1025%.
23. 10.25%.

Express the following as common fractions or mixed numbers:

24. 2%.
25. 5%.
26. $2\frac{1}{2}\%$.
27. $1\frac{5}{8}\%$.
28. 0.3%.
29. 0.5%.
30. 250%.
31. 375%.
32. 37.5%.
33. 3.75%.

34. If $\frac{1}{2}$ of a certain number is 375, what is the number?

35. If 50% of a certain sum is \$750, what is the sum?

36. If $2\frac{3}{4}$ times a certain number is 825, what is the number?

37. If 275% of a certain sum is \$3300, what is the sum?

38. A merchant's sales last month were 165% of what they were the preceding month. If they were \$33,000 last month, how much were they the preceding month?

Ratio. In ancient times, when there were no good symbols for fractions, it was often the custom of mathematicians to use *ratios* instead. If they wished to say that one number is $\frac{5}{8}$ of another number, they said that the ratio of the first number to the second is as 5 to 8.

In later times the ratio of 5 to 8 was written 5 : 8. For purposes of computation, however, there is no difference between the fraction $\frac{5}{8}$ and the ratio 5 : 8, the *value* of the latter being $\frac{5}{8}$, or 0.625.

There is one difference between a fraction and a ratio, however: the terms of a ratio must be alike, as in the case of 5 : 8, \$5 : \$9, and 7 ft. : 10 ft.; while in the case of a fraction we may have $\frac{\$7}{10}$ or $\frac{9 \text{ ft.}}{15}$. It will be found that this distinction does not affect the practical computations of arithmetic.

Since for our practical purposes a ratio is the same as a fraction, the writing of a ratio is properly included in this chapter.

We shall not consider such names as *antecedent* (the first term of a ratio) and *consequent* (the second term), since these names are not needed in our work in practical arithmetic. They are terms that were introduced into arithmetic at a time when ratios were more extensively used than at present.

An expression of the equality of two ratios is called a *proportion*; for example, $5:8=10:16$ is a proportion, read "5 is to 8 as 10 is to 16." We shall not consider proportions at this time, since in pure arithmetic work and in the solution of commercial problems we have no need for them. They were formerly extensively used in arithmetic, sometimes under the name of Rule of Three, but we now have better methods for solving problems. In geometric work and in certain parts of physics, however, proportion is still used, although commonly written as an equality of fractions. Essentially a proportion is merely a fractional equation, $x:2=3:7$ being merely another form for the equation

$$\frac{x}{2}=\frac{3}{7}.$$

When a problem asks for a ratio, this ratio should be expressed as a fraction reduced to lowest terms, like $\frac{2}{3}$ instead of $\frac{4}{6}$, or with the numbers of the fraction written in the form 2 : 3.

Exercise. Ratio

1. There are 14 boys and 16 girls in a class. What is the ratio of the number of boys to the number of girls?

2. A schoolroom is 16 ft. wide and 32 ft. long. What is the ratio of the width to the length? of the length to the width?

3. The page of a book is $4\frac{1}{4}$ in. wide and $6\frac{1}{2}$ in. long. What is the ratio of the width to the length? of the length to the width? of the length to the perimeter?

4. Which ratio is the greater, $230:1232$ or $273:1456$?

5. Express as a decimal the value of the ratio $7:8$.

6. What is the value of the ratio $96:192$? of the ratio $192:96$? of the ratio $384:192$?

7. The month of April in a recent year had 14 clear days and 16 cloudy days. Find the ratio of clear days to cloudy days; of clear days to the total number of days; of cloudy days to the total number of days.

8. The value of the ratio $x:32$ is given as $\frac{3}{4}$. What is the value of x?

9. The ratio of the circumference to the diameter of a circle is about $3\frac{1}{7}$ or 3.1416. Which of these values is the larger?

10. The ratio of the weight of a piece of metal to the weight of an equal volume of water is called the *specific gravity* of the metal. The specific gravity of copper is 8.9 and the weight of 1 cu. ft. of water is $62\frac{1}{2}$ lb. Find the weight of 1 cu. ft. of copper.

11. There are two squares, the side of one square being four times the side of the other square. Find the ratio of the perimeters and the ratio of the areas.

Between the numbers 2 and 5 there are two ratios, $2:5$ and $5:2$, but it is customary to place the smaller number first unless the contrary order is expressly demanded.

Compound Numbers. Since the ancients had no good symbols for fractions, they often avoided the latter by means of various kinds of measures. Thus, instead of speaking of 2$\frac{2}{3}$ ft. they spoke of 2 ft. 8 in., using two *denominations*, feet and inches, and stating the result as a *compound number.*

The number 2 ft. is a *denominate number*, the denomination being feet; the number 2 ft. 8 in. is a compound number, the denominations being feet and inches.

The subject of compound numbers is rapidly losing its importance because of the fact that the decimal is coming to be better understood. No one would expect to see an automobile meter recording distances in miles, rods, and yards, or even in miles and feet, it being much simpler to record the distance in miles and decimals of a mile.

Abstract and Concrete Numbers. A number that does not bear the name of any particular class of objects or measures is called an *abstract number*, and a number that is not abstract is called a *concrete number.* The numbers 3, 2$\frac{1}{2}$, 4.08, $\frac{3}{5}$, and 0.009, for example, are abstract numbers, while the numbers $3, 4 lb., 2$\frac{1}{2}$ mi., and 6 apples are concrete numbers.

It should be understood, however, that the number part of the expression $15 is 15, all number being essentially abstract. Nevertheless it is often convenient to distinguish between these two classes of numbers, and so we shall use the terms. Compound numbers and denominate numbers are special forms of concrete numbers.

Teachers may care to mention other kinds of numbers, such as cardinals and ordinals, but these are not needed for our purposes. The meaning of such terms may easily be found in a dictionary. The terms are not used in elementary classes as often as formerly.

In the solution of problems the numbers are not labeled "ft.," "in.," and so on, as often as was formerly the custom, the schools tending more and more to business usage.

Exercise. Abstract and Concrete Numbers

1. Express 2 ft. 9 in. as feet, that is, as 2 ft. and a decimal of a foot or as 2 ft. and a common fraction of a foot.

2. Express 3 lb. 10 oz. as pounds; as ounces.

3. Express 440 yd. as a common fraction of a mile; as a decimal of a mile.

4. Express 2 hr. $22\frac{1}{2}$ min. as hours and a decimal.

5. Express $3\frac{3}{4}$ hr. as hours and minutes.

6. Express $3\frac{41}{80}$ hr. as hours, minutes, and seconds.

7. Express 2.875 tons as tons and pounds.

Express the following as directed:

8. $5\frac{3}{4}$ lb. as pounds and a decimal; as pounds and ounces.

9. 2 yd. 2 ft. as yards and a common fraction.

10. 3 sq. ft. 72 sq. in. as square feet and a decimal.

11. 2 cu. ft. 1296 cu. in. as cubic feet and a decimal.

12. 3.75 gal. as gallons and quarts; as quarts; as pints.

13. The ratio of 2 lb. 8 oz. to $3\frac{3}{4}$ lb. in simplest form.

14. The ratio of 7 ft. to 4 ft. 8 in. in simplest form.

From the following select the abstract numbers, the concrete numbers, and the compound numbers:

15. 2.47.

16. 3 ft. 2 in.

17. $24,725.

18. 23.25 mi.

19. $\frac{1}{2}$ in.

20. 0.5 mi.

21. 3.25 mi.

22. 4 hr. 20 min.

23. 8345.

24. 834.5.

25. 834 ft.

26. $10,275.

Find the ratio of the following:

27. 2 hr. 40 min. 18 sec. to 8 hr. 54 sec.; to 48 min. $5\frac{2}{5}$ sec

28. $3^\circ\,15'\,15''$ to $13^\circ\,1'$; to $1^\circ\,37'\,37\frac{1}{2}''$; to $48'\,48\frac{3}{4}''$.

Algebraic Notation. In working with numbers it is often found very convenient to use a kind of shorthand known as *algebraic notation.* Thus, if we wish to state briefly that the area of a rectangle is equal to the product of the base and height, always meaning in such cases the product of the abstract numbers representing the base and height, we may do so in algebraic notation as follows:

$$a = bh.$$

Most students who use this book are already familiar with algebraic notation. Others will probably have no difficulty with this notation if they are given a little assistance by the teacher. It is possible, however, to omit all algebraic notation throughout the book if the teacher thinks it advisable to do so.

The expression bh means the same as $b \times h$. If the base is 13 in. and the height is 8 in., the area is 13 × 8 sq. in., or 104 sq. in.; that is,

$$a = bh = 13 \times 8 = 104,$$

and our common sense then tells us that the area is 104 sq. in. and not 104 in. or 104 sq. ft.

The letters which are employed in algebraic notation always represent numbers, but it is not the custom to state these as denominate numbers. We usually consider them as abstract numbers and then use our common sense to tell us the denomination of the result.

An expression like $a = bh$ is called a *formula*, but we also speak of bh as the formula for a.

When we give values to certain letters in an expression like bh, we are said to *evaluate* the expression. Thus we evaluated bh when we used 13 for b and 8 for h, the value of bh becoming 104.

Similarly, if $v = lwh$ and if $l = 10$, $w = 6$, and $h = 4$, then $v = 10 \times 6 \times 4 = 240$.

Exercise. Algebraic Notation

1. The formula for v, the volume of a box that is l inches long, w inches wide, and h inches high, is lwh. This statement may be written $v = lwh$. Evaluate the expression lwh when $l = 14$, $w = 9$, $h = 8\frac{1}{2}$.

In such a case the value of v may be given in cubic inches or as an abstract number. In algebra it is customary to leave the result abstract unless there is likely to be some uncertainty as to its meaning.

2. In the formula $a = \pi r^2$, where a represents the area of a circle, r the radius, and π (*pī*) the number $3\frac{1}{7}$, find the value of a when $r = 21$.

The expression r^2, read "r-square," means $r \times r$; the value $3\frac{1}{7}$ for π is only approximate, but is sufficiently near the correct value for our purposes. A nearer value is 3.1416.

If 21 is the number of inches in r, then the value of a will express the number of square inches in the area, and similarly for feet and square feet, yards and square yards, and so on.

3. If a rectangle is l long and w wide, the perimeter is $2l + 2w$. Find the value of this expression if $l = 7\frac{1}{2}$ and $w = 5\frac{1}{4}$; if $l = 9.3$ and $w = 7.8$.

4. If a block has the dimensions stated in Ex. 1, the total surface has the area $2lw + 2lh + 2wh$. Find the value of this expression if $l = 9$, $w = 8$, and $h = 6$.

5. The formula for the simple interest on p dollars for t years at the rate r, is $i = prt$. Find the value of i when $p = 2500$, $r = 0.06$, and $t = \frac{1}{2}$.

6. The formula for the circumference of a circle is $c = \pi d$, where c represents the circumference, d the diameter, and π the number $3\frac{1}{7}$. Write this formula in words, not using any algebraic notation.

7. From the formula in Ex. 6 find the circumference of a wheel that has a diameter of 21.7 in.

Exercise. Review of Chapter I

1. The numbers one, two, three, four, and so on are sometimes called the *natural numbers* or the *natural series of integers.* What do you understand is the reason for these names?

The first five of these exercises may be considered orally.

2. Explain the statement that ten units of any order make one unit of the next higher order.

3. Explain the statement that the moving of a figure one place to the left increases its value tenfold.

4. It is sometimes said that the Roman numerals IV, IX, XL, XC, and CD illustrate the subtractive principle of the Roman notation. What does this mean?

5. Sometimes the Romans wrote a bar over a numeral to increase its value a thousandfold. What does this statement mean? Read the numbers $\overline{\mathrm{V}}$, $\overline{\mathrm{C}}$, and $\overline{\mathrm{M}}$.

6. Express in our common notation the number of meters from the equator to the north pole, the number in the index notation being 3.937×10^8.

Write in words the following numbers:

7. 0.0000738.
8. 700.0000038.
9. 0.3008.
10. 3000.0008.
11. 200.0009.
12. 0.0209.

Express the following as sexagesimal fractions:

13. $2\frac{14}{15}$ hr.
14. 3.75°.
15. $3\frac{5}{12}$ hr.
16. 3.5′.

Express the following as mixed numbers:

17. $\frac{17}{2}$.
18. $\frac{125}{3}$.
19. $\frac{425}{20}$.
20. $\frac{375}{80}$.

Express the following as integers and decimals:

21. $3\frac{5}{8}$.
22. $2\frac{3}{4}$ hr.
23. 3° 15′.
24. $\frac{75}{8}$.

CHAPTER II

ADDITION

Purpose of the Chapter. The purpose of this chapter is to consider the subject of addition with respect to all kinds of number that are used in business and industrial arithmetic.

Accuracy of the work, including the most practical methods of checking results, will also be considered. A computer may be successful even if he is not unusually rapid in his work, but absolute accuracy is essential. Errors may occur in a computation, but such errors must be discovered and corrected before the computer submits his results.

Little attention will be paid in this book to such matters as are fully covered in the elementary school. For example, no drill should be expected in the number combinations or in rapid addition.

Language of Addition. The Latin writers, from whom we derived most of our terms in arithmetic, spoke of a *numerus addendus*, the phrase meaning "number to be added," and in the course of time this expression was changed to *addend.* They spoke of the result in addition as *summa*, and from this we have our word "sum."

The word "sum" was formerly used to mean a problem in arithmetic as well as the result in addition. It was also used to mean the result in multiplication. We use "total" quite as often as "sum" in commercial work.

The symbol + is read "plus," the word being the Latin for more. The symbol = is read "equals" or "is equal to," or is read in the plural forms of these expressions.

Addition of Integers and Decimals. The reasons underlying the common method of adding integers are readily seen in the case of $876 + 549$. For purposes of explanation we may arrange the work as follows:

$$\begin{array}{rcl}
876 = & 800 + \ 70 + \ 6 & \\
\underline{549} = & \underline{500 + \ 40 + \ 9} & \\
 & 1300 + 110 + 15 & = 1400 + 20 + 5 = 1425
\end{array}$$

It is evident that most of this work is not essential except as it explains what is done mentally. Since $9 + 6 = 15 = 10 + 5$, we write 5 under units and add 1 to tens. We then have $1 + 4 + 7 = 12$, the number of tens, and this is 1 hundred + 2 tens, so that we write 2 under tens and add 1 to hundreds. We then have $1+5+8=14$, the number of hundreds, and this is 1 thousand + 4 hundreds, so that we write 4 under hundreds and add 1 to thousands. The sum is 1425.

The theory is evidently the same if decimals are involved.

All of the above work is perfectly familiar from elementary arithmetic, but it serves to explain a general principle which is found in all kinds of addition:

In adding numbers add all the parts separately and then simplify the result as much as possible.

The application of this principle to various kinds of numbers will be considered in this chapter.

In every case our common sense will tell us what is meant by adding the parts and simplifying the result. For example, in adding $3\frac{3}{4}$ and $2\frac{7}{8}$ we naturally add $\frac{7}{8}$ and $\frac{3}{4}$, the result being $\frac{13}{8}$, or $1\frac{5}{8}$; we then add 2 and 3, the result being 5; we then simplify the expression $5 + 1\frac{5}{8}$ and the sum is $6\frac{5}{8}$. Practically we see that $\frac{7}{8} + \frac{3}{4} = \frac{13}{8} = 1\frac{5}{8}$, and $1 + 2 + 3 = 6$, so that the sum is $6\frac{5}{8}$.

Exercise. Integers and Decimals

1. Arranging the work as in the case of 876 + 549 on the opposite page, add 3968 and 2759.

2. As in Ex. 1, add 5296, 3874, 8649, and 3001.

3. Considering 0 as a number, show that there is a number which when added to itself gives a result that is equal to itself, and when added to any other number gives a result that is equal to that other number.

4. Consider the following cases, state what the computer has done in each case, and write a brief discussion of the advantages you see in the form that seems to you the best:

I	II	III	IV
$4.28	$4.28	$4.28	$4.28
3.96	3.96	3.96	3.96
.75	.75	.75	.75
8.30	8.30	8.30	8.30
9.68	9.68	9.68	9.68
27	24	27	2400
27	27	270	270
24	27	2400	27
$26.97	$26.97	$26.97	$26.97

5. In this work in addition state how the computer has proceeded, explaining the meaning of the small 2's and discussing the advantage of this method over the methods given in Ex. 4.

$$\begin{array}{r} 396 \\ 829 \\ 385 \\ \hline {\scriptstyle 22} \\ 1610 \end{array}$$

This is a common method of remembering the numbers to be added to the next column, or, to use an old expression, to be "carried."

6. Add $2.87, $32.68, $9.42, $17.26, $3.75, $340.02, $0.78, and $104, arranging the work in the best form.

Checks in Addition. A good computer checks every step of his work. He thus discovers an error as soon as it is made, and does not allow it to vitiate the rest of his work.

In the following case the vertical lines represent the ruling commonly used to indicate dollars and cents:

3	234	96	
1	038	34	
	414	96	
	83	77	4772.03
	79	80	
2	300	48	
	92	03	
	229	76	2702.07
1	345	4	
7	474	10	7474.10

In adding the numbers the computer used two checks:

1. He added each column of figures from the bottom to the top, writing the units of the sum and also writing, in smaller figures, the tens to be added to the next column. He then added from the top to the bottom, thus checking the sum in each column.

The small figures representing the numbers to be added to the next column may be written on a separate piece of paper.

2. He added the numbers in the upper part and those in the lower part separately, the partial sums being $4772.03 and $2702.07. He then added these partial sums and found that the total sum, $7474.10, agreed with the first total.

Much of the work in addition is now performed by machinery. Adding machines may be seen in any large bank. In bookkeeping, however, the figures are usually written with a pen, and the addition is performed without the aid of a machine.

Exercise. Checks in Addition

Perform the following additions, check each by two methods, and report the total amount of time taken in copying, adding, and checking, and also report the total number of figures which the checks showed to be incorrect:

1.

32	481	76
	926	30
	29	42
	107	05
123	418	26
	92	80
36	437	68
	39	49
	7	09
	512	42

2.

	420	76
103	070	82
	75	43
	6	00
	420	72
25	035	82
	628	07
4	172	83
	348	60
	526	89

3.

35	368	42
	29	87
	433	05
	125	36
26	974	08
101	598	60
	224	37
	586	37
	419	02
	267	48

4.

31	248	72
	396	87
362	008	72
	340	09
	27	62
	38	49
107	200	78
	53	86
	48	72
	40	75
	872	23
	41	68
3	400	27
	872	63
1	284	55

5.

263	608	92
39	420	70
	39	86
	400	30
	37	98
105	072	60
3	004	75
	290	00
	87	61
	86	42
1	428	75
3	000	00
	928	72
1	876	28
2	954	50

6.

415	207	82
375	728	77
	63	98
	897	25
	50	75
88	026	91
	370	00
	54	96
127	298	78
	96	51
	230	75
	33	85
	900	00
	856	66
2	744	34

Reduction of Common Fractions. Before considering the general case of the addition of common fractions, it is necessary to speak of the general nature of addends.

If we add 5 apples and 3 oranges we have 8 units of two different kinds, some apples and some oranges. We may state the result as 5 apples + 3 oranges, but this merely states the problem without really solving it. We may, however, give the addends the same name and say that 5 pieces of fruit and 3 pieces of fruit are 8 pieces of fruit.

Similarly, to add 5 ft. and 3 in. we may give the addends the same name, and this we do by *reducing* 5 ft. to inches or by reducing 3 in. to feet. We then have

$$5 \text{ ft.} + 3 \text{ in.} = 5 \times 12 \text{ in.} + 3 \text{ in.} = 60 \text{ in.} + 3 \text{ in.} = 63 \text{ in.},$$

or $$5 \text{ ft.} + 3 \text{ in.} = 5 \text{ ft.} + 3 \times \tfrac{1}{12} \text{ ft.} = 5 \text{ ft.} + \tfrac{3}{12} \text{ ft.} = 5\tfrac{1}{4} \text{ ft.}$$

The principle is the same in the case of fractions. If, for example, we have to add $\frac{5}{8}$ and $\frac{3}{4}$, we may give the addends the same name, thus:

$$\tfrac{5}{8} + \tfrac{3}{4} = \tfrac{5}{8} + \tfrac{6}{8} = \tfrac{11}{8} = 1\tfrac{3}{8},$$

the name being eighths.

The following principles of reduction are familiar from elementary arithmetic:

Both terms of a fraction may be multiplied by the same number without changing the value of the fraction.

Both terms of a fraction may be divided by the same number without changing the value of the fraction.

Such expressions as *common denominator*, *least common denominator*, *lowest terms*, and *least common multiple* are also familiar from elementary arithmetic.

Thus 24, 48, and 72 are common denominators of the fractions $\frac{3}{4}$, $\frac{5}{8}$, and $\frac{7}{12}$, but of these denominators 24 is the least common denominator, being the least common multiple of 4, 8, and 12.

Exercise. Reduction of Fractions

1. Draw a line 6 in. long and show that $\frac{1}{2}$ of the line is the same as $\frac{3}{6}$ of the line. What principle of reduction is illustrated in this case?

2. In Ex. 1 show that $\frac{4}{6}$ of the line is the same as $\frac{2}{3}$ of the line. What principle of reduction is illustrated?

3. How many halves of 1 are there in 1? How many halves of 1 are there in 3? Express 7 as halves.

4. Draw a line 3 in. long and show that $\frac{1}{4}$ of 3 in. is the same as $\frac{3}{4}$ of 1 in.; that is, show that $\frac{3}{4}$ is the same as $\frac{1}{4}$ of 3, and is the same as $3 \div 4$.

5. Give another illustration, as in Ex. 4, to show that a fraction is an expression of division, the numerator being the dividend and the denominator being the divisor.

6. Explain the reason for believing that $\frac{8}{5} = 1\frac{3}{5}$.

7. Reduce $\frac{2}{3}$, $\frac{3}{4}$, and $\frac{5}{6}$ to twelfths; to twenty-fourths.

8. What is the least common denominator of the fractions $\frac{3}{4}$, $\frac{3}{5}$, and $\frac{5}{8}$? Express each of these three fractions with this common denominator.

9. Reduce $\frac{5}{8}$ to sixteenths; to forty-eighths.

10. Reduce $\frac{5}{8}$ to a decimal by dividing 5 by 8, carrying the result to three decimal places.

11. How many fifths of 1 are there in 1? in 2? in $2\frac{3}{5}$?

12. How many sixteenths of 1 are there in 1? in 3? in $3\frac{7}{16}$? in $5\frac{11}{16}$? in $9\frac{15}{16}$? in $\frac{1}{8}$?

13. How do you find the least common denominator of several fractions? Explain the reason.

14. What is meant by reducing a fraction to lowest terms? Give an illustration.

15. Reduce $\frac{75}{125}$, $\frac{25}{125}$, $\frac{25}{250}$, and $\frac{75}{250}$ to lowest terms.

Addition of Common Fractions. As shown on page 28, in order to add fractions it is necessary to reduce them to fractions having a common denominator. We may briefly review the reason as follows:

$$\begin{aligned}
5 \text{ apples} + 3 \text{ oranges} &= 5 \text{ pieces of fruit} + 3 \text{ pieces of fruit}\\
&= 8 \text{ pieces of fruit};\\
5 \text{ ft.} + 3 \text{ in.} &= 60 \text{ in.} + 3 \text{ in.}\\
&= 63 \text{ in.};\\
\tfrac{5}{8} + \tfrac{3}{4} &= \tfrac{5}{8} + \tfrac{6}{8}\\
&= \tfrac{11}{8}\\
&= 1\tfrac{3}{8}.
\end{aligned}$$

To add fractions reduce them to fractions having the least common denominator, add the numerators for the numerator of the sum, write this numerator over the least common denominator, and then reduce this fraction to its simplest form.

The method may be understood from the problem of finding the sum of $\frac{3}{4}$, $\frac{5}{8}$, and $\frac{7}{12}$. We can easily see that 24 is the least common denominator. Then

$$\frac{3}{4} = \frac{6 \times 3}{6 \times 4} = \frac{18}{24},$$

$$\frac{5}{8} = \frac{3 \times 5}{3 \times 8} = \frac{15}{24},$$

and

$$\frac{7}{12} = \frac{2 \times 7}{2 \times 12} = \frac{14}{24}.$$

Adding, we have $\frac{47}{24}$, or $1\frac{23}{24}$.

Decimal fractions having generally taken the place of all except the simplest common fractions, we rarely have any need to add common fractions except in cases in which it is evident what denominator is to be used. The explanation of the method of finding the least common denominator is therefore no longer of much importance.

Exercise. Common Fractions

1. In adding $\frac{1}{2}$ and $\frac{3}{4}$ could we use 8 as the common denominator and then simplify the result? What advantages or disadvantages would there be in using 4 instead of 8?

2. As in Ex. 1, consider the relative advantages of using 12 and 24 as the common denominator in adding $\frac{3}{4}$ and $\frac{5}{6}$.

3. In adding $\frac{3}{8}$ and $\frac{5}{12}$ which method is shorter, reducing each fraction to 24ths or proceeding as follows?

$$\frac{3}{8}+\frac{5}{12}=\frac{3\times12+5\times8}{8\times12}=\frac{36+40}{96}=\frac{76}{96}=\frac{19}{24}.$$

In a case like $3 \times 12 + 5 \times 8$ the signs of multiplication have the precedence; that is, the multiplications are performed before the addition. Of course most of the work would be performed mentally.

The purpose of Exs. 1–3 is to show that it is often quite as simple to use the product of the two denominators as the common denominator as it is to reduce to the least common denominator.

4. Add the fractions $\frac{1}{2}$, $\frac{3}{4}$, and $\frac{5}{8}$, stating which seems to you the best number to take as the common denominator and why this seems the best.

5. In adding common fractions you reduce the fractions to fractions having a common denominator. Do you do this in adding such decimal fractions as 0.5, 0.75, and 0.375? Explain your answer.

6. In adding 2 and $\frac{3}{4}$ do you reduce to fractions having a common denominator? Explain your answer.

7. In adding $\frac{3}{8}$, $\frac{5}{12}$, and $\frac{4}{15}$ what common denominator do you use? How do you find this denominator? Could you use any other denominator to advantage? Give a full explanation of each answer.

Such a case in addition would rarely be found in practical work, and for this reason the theory is not discussed in the text.

Addition of Compound Numbers. If we compare the addition of integers, compound numbers, and fractions, we see that the fundamental principle is the same, although at first there seem to be some differences. Consider, for example, the three cases which follow:

$$\begin{array}{lll}
27 = 2 \text{ tens} + \ 7 & \quad 2 \text{ ft.} \ \ 7 \text{ in.} & \quad 2\frac{7}{15} \\
\underline{56} = \underline{5 \text{ tens} + \ 6} & \quad \underline{5 \text{ ft.} \ \ 6 \text{ in.}} & \quad \underline{5\frac{6}{15}} \\
\ 7 \text{ tens} + 13 & \quad 7 \text{ ft. } 13 \text{ in.} & \quad 7\frac{13}{15}
\end{array}$$

When the sums are arranged as shown above, each is seen to be 7 of one kind of unit and 13 of another kind. The difference now appears in simplifying the results, for in the first case we have 7 tens + 13 = 83; in the second case, 7 ft. 13 in. = 8 ft. 1 in.; while in the third case the result is evidently in its simplest form already.

In simplifying the first of these cases we have $6 + 7 = 13$, and we write 3 in units' column and add 1 to the tens.

In simplifying the second of these cases we have 6 in. + 7 in. = 13 in. = 1 ft. 1 in., and we write 1 in. in the column of inches and add 1 to the feet.

In the third of these cases we have $\frac{6}{15} + \frac{7}{15} = \frac{13}{15}$, and since this fraction is less than one unit of the next column, we write it in the column of fractions, that is, of 15ths.

In every case the fundamental principle is the one given on page 24.

Tables of compound numbers are given on pages 241–248.

It should be repeated that compound numbers have lost much of their former importance, owing to the increasing use of decimals. We still have occasion to add numbers expressed in feet and inches; occasionally in hours, minutes, and seconds; less often in degrees, minutes, and seconds; and, with decreasing frequency, in years, months, and days; but we rarely have any other need for the addition of compound numbers. For this reason the subject is given less and less attention from year to year.

Exercise. Compound Numbers

Add the following numbers, compare the figures used, and state why the results do not all have the same figures:

1. 8 ft. 8 in.	**2.** 88	**3.** 8.8	**4.** $8\frac{8}{15}$
9 ft. 8 in.	98	9.8	$9\frac{8}{15}$
3 ft. 7 in.	37	3.7	$3\frac{7}{15}$

In writing compound numbers for adding, the denominations are usually written at the tops of the columns. For the purpose here in mind it is a little better to write them as in Ex. 1.

Add the following numbers, compare the figures used, and state why the results do not all have the same figures:

5. 3 hr. 9 min. 7 sec.	**7.** 397	**9.** 39 lb. 7 oz.
9 hr. 7 min. 5 sec.	975	97 lb. 5 oz.
3 hr. 2 min. 3 sec.	323	32 lb. 3 oz.

6. 3 yr. 9 mo. 7 da.	**8.** 3.97	**10.** 39 pk. 7 qt.
9 yr. 7 mo. 5 da.	9.75	97 pk. 5 qt.
3 yr. 2 mo. 3 da.	3.23	32 pk. 3 qt.

Add the following, applying the check or checks that seem the best adapted to each case:

11. 10° 37′ 42″	**12.** 103,742	**13.** 10 mi. 3742 ft.
9° 29′ 38″	92,938	9 mi. 2938 ft.
8° 42′ 27″	84,227	8 mi. 4227 ft.
16° 52′ 42″	165,242	16 mi. 5242 ft.
12° 0′ 0″	120,000	12 mi. 0 ft.
13° 17′ 25″	131,725	13 mi. 1725 ft.

Exs. 4, 5, 6, 9, 10, 11, and 13 would rarely occur in practical life. In Ex. 13 decimals would ordinarily be used.

Short Methods in Addition. There are various short and convenient methods of performing the usual operations upon and with numbers. In the case of addition these may be described as follows:

$$\begin{array}{r} 24\\ 33\\ 46\\ 39\\ 72\\ 78\\ \hline 292 \end{array}$$

1. In adding the units' column, instead of wasting time by saying "8 plus 2 is equal to 10, 10 plus 9 is equal to 19," and so on, look at the two lowest figures and think "10." Then, looking at the rest in order, think "19, 25, 28, 32."

2. Try to add by convenient groups instead of adding by single numbers. Look at the first two figures in units' column and think "10"; then at the next two, which together make 15, and think "25"; then at the next two and think "32." Similarly, in tens' column, the addition, beginning with 3 from units' column, might proceed as follows: "10, 20, 29."

3. Good computers often add two columns at a time, thus: 78, 80 (that is, 78 + 2), 150 (that is, 80 + 70), 159, 189, 195, 235, 238, 268, 272, 292, each time adding first the units and then the tens.

4. Take advantage of number peculiarities such as appear in finding the sum of 937 and 986. Since 986 = 1000 − 14, our best way is to add 1000 to 937, making 1937, and then to subtract 14, leaving 1923. No rule can be given that will apply to all cases of this kind; it is merely a matter of common sense in taking advantage of any peculiarities of the numbers to be added.

5. In adding two numbers of two figures each it is better to begin at the left. For example, to add 37 and 49, first add 30 to 49 and then add 7, or else add 40 to 37 and then add 9; that is, think of the problem as 79 + 7 or as 77 + 9 instead of 49 + 37. This method can sometimes be used to advantage with numbers of three figures.

Exercise. Short Methods

Add the following, taking advantage of grouping:

1. 837	**2.** 248	**3.** 472	**4.** 308	**5.** 761
273	322	527	122	228
425	746	346	680	396
685	324	653	123	424
907	781	296	764	323
193	229	814	223	287
462	537	845	458	446
538	263	265	322	254

Add the following, taking advantage of the fact that one of the addends is near 100, 1000, or 10,000:

6. 826	**7.** 1792	**8.** 2497	**9.** 3569	**10.** 9689
95	998	994	988	9989

Add the following, taking first two columns at once and then the columns separately, and determining which is the easier:

11. 27	**12.** 36	**13.** 18	**14.** 24	**15.** 62	**16.** 29
43	42	72	96	86	93
39	83	36	83	93	46
86	91	54	72	81	89
21	77	54	68	48	87
33	68	36	55	77	56
58	43	72	43	65	29
92	29	18	27	57	40
87	62	57	62	46	87
13	38	13	28	44	78

Add the following, using the shortest methods:

17. 389,296	**18.** 407,862	**19.** 999,991	**20.** 508,269
999,989	99,996	909,877	490,768

Algebraic Addition. The principle of addition in algebra is the same as the principle of addition in arithmetic: we add the parts or terms of the addends and then simplify the result as much as possible.

If a student has not had algebra, pages 36 and 37 may be omitted; but the pages are helpful as showing the general nature of addition.

Consider, for example, the addition of $3x+7$, $4x+15$, $3x-9$, $7x$, and $6x-4$.

$$\boxed{\begin{array}{r}3x+7\\4x+15\\3x-9\\7x\\6x-4\\\hline 23x+9\end{array}}$$

Adding -4, -9, 15, and 7, we find that the sum of these terms is $+9$.

Adding $6x$, $7x$, $3x$, $4x$, and $3x$, we find that the sum of these terms is $23x$.

Since we cannot simplify the expression $23x+9$, we write this as the sum.

If we knew the value of x, we could simplify this sum. For example, if $x=1$ then $23x+9=32$; and if $x=10$ then $23x+9=239$.

Consider also the following additions:

$$\begin{array}{r}3h+4t+8\\5h+7t+4\\\hline 8h+11t+12\end{array}\qquad\qquad\begin{array}{r}348\\574\\\hline 922\end{array}$$

In these two cases it will be seen that the numbers in the addends are the same, but that they are not the same in the sums. If, however, we know that $h=100$ and $t=10$, then from the algebraic sum we have

$$\begin{aligned}8\times 100+11\times 10+12&=800+110+12\\&=922.\end{aligned}$$

In other words, the arithmetic addition is merely a special case of the algebraic addition, the case in which $h=100$ and $t=10$.

Similarly, the sums in the additions

$$\begin{array}{r}4T+7h+6t+5\\3T+5h+4t+9\\\hline 7T+12h+10t+14\end{array}\qquad\qquad\begin{array}{r}4765\\3549\\\hline 8314\end{array}$$

become identical if $T=1000$, $h=100$, and $t=10$.

Exercise. Algebraic Addition

1. Add $4t+3$ and $3t+5$. Then let $t=10$ in the addends and also in the sum, and perform the corresponding arithmetic addition.

2. Add $7f+6i$ and $9f+5i$. Then let $f=1$ ft. and $i=1$ in. in the addends and in the sum, and perform the corresponding arithmetic addition.

Letters used in algebra always stand for numbers. In particular, we may think of them as standing for concrete numbers, such as 1 ft. Then $7f$ means, in this case, 7×1 ft., or 7 ft.

3. Add $4h+7t+9$ and $6h+8t+5$. Then let $h=100$ and $t=10$, and perform the corresponding arithmetic addition.

4. The principle of algebraic addition is used often in commercial arithmetic. For example, taking a standard bale of cotton as 500 lb., some bales will be found overweight and some underweight. If four bales weigh respectively 517 lb., 487 lb., 524 lb., and 482 lb., it is easier to write the weights as shown in the margin at the right. Add these weights and find the total weight.

$$\boxed{\begin{array}{l} 500+17 \\ 500-13 \\ 500+24 \\ \underline{500-18} \\ \end{array}}$$

5. Add $7u+3f$, $6u+2f$, and $4u+f$. Then let $u=1$ and $f=\frac{1}{5}$ and perform the corresponding arithmetic addition.

6. Add $2T+3h+4t+3$ and $5T+2h+3t+4$. Then let $T=1000$, $h=100$, and $t=10$ and perform the corresponding arithmetic addition.

7. Add $2T+4h+9t+7$ and $6T+9h+7t+5$. Then let $T=1000$, $h=100$, and $t=10$ and perform the corresponding arithmetic addition.

8. As in Exs. 6 and 7, add $5T+9h+7t$ and $9T+8t+5$ and perform the corresponding arithmetic addition.

Exercise. Review of Chapters I and II

1. Find the sum of all the integers from 1 to 10.

In such cases both the first number and the last number are to be included. There is a short method, based on series, with which you may be familiar.

2. You have seen that all cases of addition are alike in general principle. Write this general principle in the form of a simple rule to cover various kinds of number.

3. Add $3h + 4t + 9$ and $8h + 9t + 7$. Perform the corresponding arithmetic addition when $h = 100$ and $t = 10$.

4. Add $5x + 2y + 3z$, $7x + 9y + 4z$, and $6x + 8y + 5z$. Perform the corresponding arithmetic addition when $x = 100$, $y = 10$, and $z = 1$; when $x = 10$, $y = 1$, and $z = 0.1$.

5. Add $5x + 3y$ and $8x + 4y$. Perform the corresponding arithmetic addition when $x = 1000$ and $y = 1$.

Consider the following additions and state why the figures in the sums are not all the same:

6. $\begin{array}{r} 3t + 7u \\ 2t + 3u \\ \underline{4t + 7u} \end{array}$	**7.** $\begin{array}{r} 37 \\ 23 \\ \underline{47} \end{array}$	**8.** $\begin{array}{r} 3\text{ yd. } 7\text{ in.} \\ 2\text{ yd. } 3\text{ in.} \\ \underline{4\text{ yd. } 7\text{ in.}} \end{array}$	**9.** $\begin{array}{r} 3 + \frac{7}{10} \\ 2 + \frac{3}{10} \\ \underline{4 + \frac{7}{10}} \end{array}$	**10.** $\begin{array}{r} 307 \\ 203 \\ \underline{407} \end{array}$

11. Write an explanation covering each step in the following process of adding $2\frac{7}{8}$ and $4\frac{5}{16}$: $2\frac{7}{8} = 2\frac{14}{16}$, $4\frac{5}{16} = 4\frac{5}{16}$, and hence $2\frac{7}{8} + 4\frac{5}{16} = 2\frac{14}{16} + 4\frac{5}{16} = 6\frac{19}{16} = 7\frac{3}{16}$.

12. In mental addition it is usually better to begin at the left, as stated on page 34. Thus in the case of $48 + 59$ we have $48 + 59 = 98 + 9 = 107$. Try this method in written work, using the case of $48{,}396 + 31{,}478$, and decide whether it is better to begin at the left than at the right.

13. Apply Ex. 12 to the case of $40{,}075 + 16{,}820$.

CHAPTER III

SUBTRACTION

Purpose of the Chapter. The purpose of this chapter is to consider the subject of subtraction with respect to all kinds of number used in business arithmetic, except as this is rendered unnecessary by the work of Chapter II. In order to make more clear the arithmetic process, optional work is also given in algebraic subtraction.

There are five general methods of subtraction used in this country. A pupil in the elementary school who has learned any one of the four given on page 40 should be allowed to continue its use, the difference in speed and accuracy between that method and any other being too slight to make any change desirable. For a student of higher arithmetic, however, some knowledge of the various methods is desirable.

Language of Subtraction. The Latin for "number to be diminished" is *numerus minuendus*, and that for "number to be taken from under" is *numerus subtrahendus*. From these two expressions came our words "minuend" and "subtrahend." The result in subtraction has been called by various names. Of these we still use the words "remainder" and "difference." Formerly the word "rest" was used for the result, and we often to-day use such an expression as "keep the rest." In bookkeeping the word "balance" is commonly used, as in the expression "How much is my balance?"

The symbol – is read "minus," the word being the Latin for less.

Methods of Subtraction. Before considering the five methods of subtraction in use in this country it is necessary to observe that the difference between any two numbers is the same if we increase each by 10, by 100, or by any other number; that is, $7 - 2 = 17 - 12 = 27 - 22 = 37 - 32$, and so on.

Expressed algebraically, $a - b = a + n - (b + n)$. The use of such algebraic statements is optional with the teacher.

It is also convenient to consider a special case, say that of $832 - 269$, and to write the following statements:

$$\begin{array}{rl@{\qquad}rl}
832 = & 700 + 120 + 12 & 832 + 110 = & 800 + 130 + 12 \\
\underline{269} = & \underline{200 + \ \ 60 + \ \ 9} & \underline{269 + 110} = & \underline{300 + \ \ 70 + \ \ 9} \\
563 = & 500 + \ \ 60 + \ \ 3 & 563 \qquad = & 500 + \ \ 60 + \ \ 3
\end{array}$$

Four of the five methods may now be given as follows:

1. In the first of the above statements we have $12 - 9 = 3$, $12 - 6 = 6$, $7 - 2 = 5$, and the result is 563.

2. In the second statement we have $12 - 9 = 3$, $13 - 7 = 6$, $8 - 3 = 5$, and the result is 563.

3. In the first statement we may consider what number must be added to the subtrahend to make the minuend, thus: $9 + \mathbf{3} = 12$, $6 + \mathbf{6} = 12$, $2 + \mathbf{5} = 7$, and the result is 563.

4. In the second statement we may proceed as in the third method and we have $9 + \mathbf{3} = 12$, $7 + \mathbf{6} = 13$, $3 + \mathbf{5} = 8$, and the result is 563.

Of these methods the first is slightly easier to explain, the second is slightly more rapid and accurate than the first, and the fourth is probably still more rapid and accurate. The third and fourth are often called the addition methods or the making-change methods.

The experience of the world shows that there is no great difference in the value of these various methods. The student may safely continue to use the one with which he is familiar.

Exercise. Methods of Subtraction

1. In subtracting 327 from 402 proceed as follows: $12-7=5$, $9-2=7$, $3-3=0$. Write an explanation of the work, showing where the 12, 9, and 3 are obtained in the three parts of the minuend.

2. In subtracting 428 from 613 proceed as follows: $13-8=5$, $11-3=8$, $6-5=1$. Write an explanation of the work, showing where the 3 and 5 came from in the parts of the subtrahend.

3. In subtracting 506 from 734 proceed as follows: $6+\mathbf{8}=14$, $0+\mathbf{2}=2$, $5+\mathbf{2}=7$. Write an explanation of the work, showing the reason involved in each step.

4. In subtracting 679 from 1302 proceed as follows: $9+\mathbf{3}=12$, $8+\mathbf{2}=10$, $7+\mathbf{6}=13$. Write an explanation of the work, showing the reason involved in each step.

5. Write a statement of your opinion as to the best of the four methods given on page 40 and illustrated in Exs. 1–4. Try to consider the question without reference to the method which you have always used, but chiefly with respect to speed and accuracy for beginners.

6. Apply each of the four methods of subtraction to the case of $5\frac{1}{4}-3\frac{7}{8}$; to the case of 5 ft. 3 in. – 3 ft. 8 in.

7. Apply each of the four methods of subtraction to the case of \$47,263.42 – \$29,375.08.

8. Apply each of the four methods of subtraction to the case of 11 hr. 10 min. 36 sec. – 7 hr. 50 min. 45 sec.

9. Apply each of the four methods of subtraction to the case of 100,000 – 40,726; to the case of 100,000 – 999.

10. Show that $a-b=a+n-(b+n)$, as stated on page 40, and hence that the difference is not changed by adding the same number to both minuend and subtrahend.

Complementary Method. The fifth method of subtraction used in this country is of little value in ordinary cases, but it is used when working with various kinds of adding machines.

It is also used in connection with computation by logarithms.

The *complement* of a one-figure number is the difference between that number and 10; of a two-figure number, between that number and 100; of a three-figure number, between that number and 1000; and so on.

Since

$$\begin{aligned} 7-3 &= 7+10-3-10 \\ &= 7+(\text{complement of } 3)-10 \end{aligned}$$

we see that to subtract a number of one figure we may add the complement of the subtrahend and then take away 10. That is, $7-3=7+7-10=14-10=4$.

Similarly,

$$\begin{aligned} 72-48 &= 72+(100-48)-100 \\ &= 72+52-100 \\ &= 124-100 \\ &= 24. \end{aligned}$$

The parentheses are used above merely to aid the eye in considering as a single number the expression inclosed.

It will naturally be thought that no one would ever subtract in this way. As stated above, however, the method is used in machine calculation. It is also convenient in a case like the one here shown.

$$\begin{array}{rr} 74{,}208 & 74{,}208 \\ -\,63{,}327 & 36{,}673 \\ +\,14{,}292 & 14{,}292 \\ -\,18{,}306 & 81{,}694 \\ \hline 6{,}867 & 6{,}867 \end{array}$$

The complement of 63,327 is easily found by subtracting 7 from 10 and each of the other digits from 9, and similarly for 18,306. The complement can be written easily from left to right. Since there are two complements to 100,000 in this example, we must subtract 200,000 from the sum.

Exercise. Complementary Method

1. Write the complements of 5, 7, 2, 1, and 9.

2. Write the complements of 37, 42, 68, 54, and 99.

Simply think of the left-hand digit as subtracted from 9 and then of the right-hand digit as subtracted from 10.

3. Beginning at the left, write the complements of 268, 392,470, 1,287,693, 34,298,002, and 999,999.

4. Beginning at the left, write the complements of 20, 400, and 14,237,500.

In finding the complement of 14,237,500, take each digit from 9 until the last significant figure, 5, is reached. Take that from 10 and then write zeros for the rest of the complement.

5. Write the complement of 42,037,070.

6. Find the value of $96 + 74 - 38 + 15 - 29 + 7 - 35$, first writing in a single column the numbers to be added and the complements of the numbers to be subtracted, then adding, and finally subtracting 300. State the reason for subtracting 300.

7. As in Ex. 6, find the value of $237 - 142 + 986 - 384 + 700 - 509 - 227 - 113$. What number should be subtracted from the sum, and why?

8. As in Exs. 6 and 7, find the value of $426{,}933 + 874{,}009 - 298{,}342 + 112{,}730 - 508{,}673$.

9. As in Exs. 6–8, find the value of $327{,}642 - 29 + 43{,}821 - 5287 - 6834 + 24{,}963 - 17{,}226$.

Notice that the numbers now have not all the same number of digits, and that some ingenuity must be shown in finding the complement or else in the number to be subtracted from the sum.

10. Find the value of $527.3684 + 304.5671 - 127.2643 - 342.4004 + 432.8 - 247.6$.

Checks in Subtraction. The best check in subtraction consists of adding the subtrahend and the remainder, the result being the minuend if the work is correct. In case the subtraction is performed by the addition method, the check is still valid; but it is then better to add from the bottom upwards, since the remainder was obtained by adding from the top downwards. In the example here shown we check the work by adding 22.82 and 7.6, the sum being 30.42.

$$\boxed{\begin{array}{r} 30.42 \\ \underline{7.6} \\ 22.82 \end{array}}$$

Short Methods in Subtraction. There are no short methods of subtraction that are generally usable. Occasionally a special case arises in which a good computer takes advantage of his knowledge of algebra. For example, if we have to take 99,975 from 342,703, we should observe that 99,975 = 100,000 – 25. The student of algebra should now perform the subtraction mentally, observing that

$$\begin{aligned} 342{,}703 - 99{,}975 &= 342{,}703 - (100{,}000 - 25) \\ &= 342{,}703 - 100{,}000 + 25 \\ &= 242{,}703 + 25. \end{aligned}$$

In other words, a good computer would simply add 25 to 242,703, the result being 242,728.

Algebraic Subtraction. The principle of subtraction is the same in algebra as in arithmetic. Of the methods given on page 40, the third and fourth are the best in algebra. In the case here shown we see that we must add $-3b$ to $3b$ to make 0, and $-7b$ more to make $-7b$; that is, we must add $-10b$ to $3b$ to make $-7b$. Similarly, we must add $2a$ to $-2a$ to make 0, and $9a$ more to make $9a$; that is, we must add $11a$ to $-2a$ to make $9a$.

$$\boxed{\begin{array}{r} 9a - 7b \\ \underline{-2a + 3b} \\ 11a - 10b \end{array}}$$

Exercise. Checks and Short Methods

Perform the following subtractions and check the work:

1. 138,427 38,428	**3.** 290,100 43,807	**5.** 206,380 158,456	**7.** 512,346 386,518
2. 129,372 68,796	**4.** 352,786 286,512	**6.** 400,226 254,859	**8.** 560,280 376,354

Using short methods, perform the following subtractions and check the work:

9. 64,386 992	**10.** 70,233 9,990	**11.** 428,722 99,995	**12.** 508,726 99,980

Perform the following subtractions and check the work:

13. 9 yd. 20 in. 7 yd. 32 in.	**15.** 356 ft. 7 in. 89 ft. 11 in.	**17.** 135° 42′ 25″ 102° 52′ 34″
14. 127 ft. 3 in. 68 ft. 7 in.	**16.** 113 yd. 21 in. 9 yd. 32 in.	**18.** 180° 0′ 0″ 85° 38′ 22″

19. From $5h + 2t + 3u$ subtract $2h + 5t + 7u$; from 523 subtract 257; find whether the two results have the same figures, and explain any difference in the results.

The student should see that we might use negative numbers in our system of notation, but that the result would be unsatisfactory, the negative numbers being very awkward to use.

Perform the following subtractions and check the work:

20. $3h + 4t + 7u$ $2h + 3t + 2u$	**21.** 347 232	**22.** 34.7 23.2	**23.** 30,047 20,032

Notice the similarity between Ex. 20 and Exs. 21–23.

Exercise. Review of Chapters I–III

1. Write three illustrations to show that the difference between two numbers is not changed by adding the same number to each of the two numbers.

The student should consider Ex. 10, page 41.

2. Show that $25-(12-9)=25+9-12$, and apply this principle to finding the value of $372.48-(238.56-98.42)$.

If the student is not familiar with algebra he should be informed that the operation within the parentheses is to be performed first. If he is familiar with algebra he should see that the above is a special case of $a-(b-c)=a-b+c$.

3. Using the shortest method you know, find the value of $700.023-99.987$; of $9112.05-999.9$.

4. Using the shortest method you know, find the value of $75{,}286-24{,}392+37{,}483-36{,}579$.

5. In mental subtraction it is usually easier to begin at the left. Thus, $321-146=221-46=181-6=175$. Try this method in written work, using the case of $23{,}486-12{,}378$, and decide whether it is advantageous to begin at the left.

6. We think that the Roman numerals were very awkward for computation, and so they were; but for the most common operations, addition and subtraction, they were not particularly difficult except that it took more time to write them. Add DCLXIII and CXII, and then from DCLXIII subtract CXII.

DCLXIII
C XII

7. As in Ex. 6, from the sum of CXIII and DCLXVII subtract CCCXVI.

It should be understood that Exs. 6 and 7 are inserted merely to show that the general principles of addition and subtraction might apply to such cases.

CHAPTER IV

MULTIPLICATION

Language of Multiplication. The Latin word *multus* means many and *plicare* means to fold, so that the word "multiplication" means many folding; as in our word "manifold."

The Latin expression for "number to be multiplied" is *numerus multiplicandus*, whence our word *multiplicand.*

The word *multiplier* means literally a manifolder.

The word *product* comes from the Latin word *productus* and means "carried forward." It was often used for the result in addition, and in bookkeeping it is practically so used at present, as in the expression "brought forward."

Since multiplication was originally a short method of obtaining a sum, as in $2 \times \$3 = \$3 + \$3 = \$6$, the word "times" was used to designate multiplication. In this way the sign × came to be read "times."

When multiplication came also to include a fractional multiplier the word "times" took on a new meaning and the world came to speak of $2\frac{1}{2}$ times \$8. We do not ordinarily say "$\frac{1}{2}$ times \$8," preferring to say "$\frac{1}{2}$ of \$8," but it is logically as correct as to say "$2\frac{1}{2}$ times \$8."

Formerly the multiplier was written after the multiplicand, as in the case of $\$3 \times 2$; but in this country at present we write the multiplier first, as in $2 \times \$3$, because it is read first.

In a case like $2 \times 3 = 6$, 2 and 3 are the *factors* of 6, and similarly for $2 \times 3 \times 5 = 30$, and so on.

If an integer has no factors except itself and unity, it is called a *prime number.*

Nature of Multiplication. There are several questions relating to the nature of multiplication which a student of higher arithmetic should consider. As already stated, the pupil in the elementary school learns that multiplication is a short method of finding the result in addition, as in the case of $\$3 + \$3 + \$3 + \$3 = 4 \times \$3 = \$12$. From this he is led to see that the multiplier is always abstract and that the product is concrete if the multiplicand is concrete, the unit being the same as the one used in the multiplicand. He is also taught that it is incorrect to say "$\frac{1}{2}$ times 8," but that he should say "$\frac{1}{2}$ of 8."

In higher arithmetic, however, it becomes necessary to consider the nature of multiplication from the standpoint of modern demands and of the more mature judgment of the student. This is not for the purpose of saying that the work of the elementary school should be changed, but in order to see that a more mature mind can accept that which a child mind would fail to comprehend.

The world long hesitated to say "$2\frac{1}{2}$ times 8," because it felt that it was impossible to do anything $2\frac{1}{2}$ times. For example, try touching this page with a pencil $2\frac{1}{2}$ times. Finally, however, it extended the meaning of "times" and agreed to the convenient expression. Even yet, however, it hesitates to say "$\frac{1}{2}$ times 8," preferring to say "$\frac{1}{2}$ of 8," although "$\frac{1}{2}$ times 8" is just as sensible as "$2\frac{1}{2}$ times 8."

Similarly, most writers decline to sanction the expression 2 ft. × 3 ft. = 6 sq. ft., because they believe that it is better to write 2 × 3 sq. ft. = 6 sq. ft. Some writers, however, extend the idea of multiplication to include the case of the concrete multiplier, the product then being different from the multiplicand.

The question is really one of definition; if the need arises, mathematics extends its definitions.

Exercise. Nature of Multiplication

1. Instead of saying "$2\frac{1}{2}$ times 8," why would it not be better to use "times" in the older sense and say "2 times 8 plus $\frac{1}{2}$ of 8"? Write your answer.

2. Instead of saying "$\frac{1}{2}$ of 8," would it be better to say "$\frac{1}{2}$ times 8"? Write your answer.

Considering the question first from the standpoint of a child, which expression would give the better idea of the meaning of a fraction? Considering it from your standpoint, is "times" any more convenient than "of"? Does it sound any better?

3. The amount of work necessary to lift a weight of 1 lb. to the height of 1 ft. is called a foot-pound. The amount of work necessary to lift a weight of 3 lb. to the height of 2 ft. is 6 foot-pounds. In physics this computation is often expressed thus: 2 ft. × 3 lb. = 6 foot-pounds. Is such a statement correct? State your reasons.

4. In practical construction work a computer will write 2 ft. × 3 ft. = 6 sq. ft. Is such a statement convenient? Is it correct? State your reasons.

5. State which of the following forms you would prefer to use if you were teaching children, and give the reason: 2 × 3 × 4 cu. ft. = 24 cu. ft., or 2 ft. × 3 ft. × 4 ft. = 24 cu. ft.

6. In finding the approximate area of a circle we multiply the square of the radius by $3\frac{1}{7}$. If the radius is 7 in. would you write the statement $3\frac{1}{7} \times 7 \times 7$ sq. in., $3\frac{1}{7} \times 7 \times 7 \times 1$ sq. in., $3\frac{1}{7} \times 7^2 \times 1$ sq. in., or would you say that $3\frac{1}{7} \times 7$ in. $\times$ 7 in. $= 154$ sq. in.? State your reason.

The purpose of this page is not so much to secure uniformity of answers as to cause the student to think of the advantages and disadvantages of various forms, first for an elementary pupil and next for a mature student or a practical computer.

Multiplication of Integers. In seeking reasons underlying the common method of multiplying one integer by another, we shall first consider the multiplication of 35 by 7.

We have to multiply both 30 and 5 by 7. For purposes of explanation the first three of the following plans are useful, but for practical purposes the fourth is the best.

$$\begin{array}{rr} 35 = & 30 + \;\; 5 \\ & 7 \\ \cline{2-2} & 210 + 35 \\ = & 245 \end{array} \qquad \begin{array}{rl} 35 & \\ 7 & \\ \cline{1-1} 35 & = 7 \times \;\; 5 \\ 210 & = 7 \times 30 \\ \cline{1-1} 245 & = 7 \times \overline{35} \end{array} \qquad \begin{array}{r} 35 \\ 7 \\ \hline 35 \\ 21\;\; \\ \hline 245 \end{array} \qquad \begin{array}{r} 35 \\ 7 \\ \hline 245 \end{array}$$

Let us now consider the multiplication of 35 by 27.

Evidently we have to multiply 35 by both 20 and 7, or by 2 tens and 7 units. For purposes of explanation the first of the following plans has some advantages, but for practical purposes the second one is, of course, the better.

$$\begin{array}{rl} 35 & \\ 27 & \\ \cline{1-1} 245 & = \;\; 7 \times 35 \\ 700 & = 20 \times 35 \\ \cline{1-1} 945 & = \overline{27} \times 35 \end{array} \qquad \begin{array}{r} 35 \\ 27 \\ \hline 245 \\ 70\;\; \\ \hline 945 \end{array}$$

Hence in multiplying we proceed as follows:

Multiply each part of the multiplicand by each part of the multiplier and then add the products.

We shall find that this rule applies to all kinds of multiplication, including fractions, compound numbers, and algebraic expressions.

Exercise. Multiplication of Integers

1. In multiplying 35 by 7 would it be as convenient to multiply 30 by 7 first and then to multiply 5 by 7? If you used the first plan on page 50, would it be better to multiply 30 before multiplying 5? State your reasons.

2. In multiplying 35 by 27 would it be as convenient for you first to multiply 35 by 2 (tens) and then to multiply 35 by 7? Would it be as easy for a child to get the figures in the right columns if he did this? Which plan do you think would be the better for you, and why?

We shall find that there are some cases in which it is much better to begin with the left-hand figure of the multiplier. Many computers do this in all cases. In practical computation it makes little difference which order is followed.

3. Multiply 359 by 26, first beginning to multiply by 2 (tens) and then beginning with 6. Do you find any advantage in one plan over the other?

4. Multiply 359 by 267, first beginning with 2 and then beginning with 7.

5. Multiply 425 by 300, first multiplying by each of the zeros as if they were ordinary integers, and then by the 3; then multiplying in the usual manner, annexing two zeros to the product of 3 × 425. Write a brief explanation of the second plan.

6. Multiply 527 by 206, writing a brief explanation of your method of proceeding when you come to the zero.

7. Multiply 42,000 by 350, writing a brief explanation of your method of treating the zeros.

8. Instead of multiplying by 45, could you first multiply by 9 and then multiply this product by 5? Try this in the case of 45 × 138 and write a brief explanation.

Multiplication of Fractions. In multiplying \$3 by 2 we may think that \$3 is taken twice as an addend; that is, $2 \times \$3 = \$3 + \$3 = \$6$. In multiplying $\frac{3}{4}$ by $\frac{2}{3}$, however, it does not mean anything to say that we take $\frac{3}{4}$ as an addend $\frac{2}{3}$ of a time. Evidently, therefore, we must extend our idea of multiplication if multiplying by a fraction is to mean anything. In the elementary school the pupil is not advanced enough to understand what is meant by extending a definition to include a new type, and so there is given some kind of explanation by the help of folded paper or some similar device, but all that is really done is to explain the necessity for a new definition.

In higher arithmetic we define the product of two fractions as the fraction formed by taking the product of the numerators for the numerator and the product of the denominators for the denominator. For example,

$$\frac{2}{3} \text{ of } \frac{3}{4} \text{ means } \frac{2 \times 3}{3 \times 4}, \text{ or } \frac{1}{2}.$$

Evidently in such a case we shall save time by canceling before we multiply, thus:

$$\frac{\cancel{2}}{\cancel{3}} \times \frac{\cancel{3}}{\underset{2}{\cancel{4}}} = \frac{1}{2}.$$

The sign × is usually read "of" when placed between two fractions. For students of higher arithmetic, however, the extension of the meaning of the word "times" makes it possible to use this term, although it is quite unnecessary to do so.

In a case like $2\frac{1}{2} \times 3\frac{3}{4}$ it is better to reduce to improper fractions before multiplying than to attempt to multiply the parts separately; that is, it is better to take $2\frac{1}{2} \times 3\frac{3}{4}$ as $\frac{5}{2} \times \frac{15}{4}$ than to multiply $3\frac{3}{4}$ by $\frac{1}{2}$ and then by 2.

Exercise. Multiplication of Fractions

1. Draw a line $\frac{3}{4}$ in. long, divide it into 3 equal parts, and then find the combined length of two of these parts. In other words, show that $\frac{2}{3}$ of $\frac{3}{4}$ is $\frac{1}{2}$, thus justifying the definition of the multiplication of fractions.

2. Draw a line $\frac{5}{8}$ in. long and find the length of $\frac{3}{5}$ of this line, again justifying the definition.

3. What definition do you assume for the product of three fractions? Find the value of $\frac{2}{3} \times \frac{3}{4} \times \frac{5}{8}$.

4. Draw a line $\frac{5}{8}$ in. long, find the length of $\frac{3}{4}$ of it, and then find the length of $\frac{2}{3}$ of this result, thus justifying the definition assumed by you in Ex. 3.

5. Multiply $3\frac{1}{4}$ by $2\frac{1}{2}$ by first reducing both to improper fractions.

6. Multiply $3\frac{1}{4}$ by $2\frac{1}{2}$ by multiplying each part of the multiplicand by each part of the multiplier and adding the products: $\frac{1}{2} \times \frac{1}{4} + \frac{1}{2} \times 3 + 2 \times \frac{1}{4} + 2 \times 3$. Compare the results of Exs. 5 and 6 and write a statement as to which method seems to you to be the easier.

7. Multiply $15\frac{3}{8}$ by $7\frac{2}{3}$ by the two methods suggested in Exs. 5 and 6, writing your opinion as to which method is the easier, with the reason.

8. If 1 yd. of velvet costs $2.50, how much will $\frac{3}{8}$ yd. cost? What will you do with the fraction of a cent in the answer?

While the answer as to the fraction of a cent is evident in this case, it should be said that merchants, in making out a bill, often call any fraction of a cent a whole cent.

9. If 1 yd. of velvet costs $3.20, how much will $2\frac{1}{8}$ yd. cost? How much will $4\frac{1}{4}$ yd. cost?

There is a short method of finding the second answer from the first. Consider the value of $2 \times 2\frac{1}{8}$.

Multiplication of Decimals. The only question that needs our attention in connection with the multiplication of decimals is the one relating to the position of the decimal point. For example, $0.29 \times 3.428 = \frac{29}{100} \times \frac{3428}{1000} = \frac{29 \times 3428}{100000}$, so that we have only to multiply as if the numbers were 29 and 3428, then writing the result as hundred-thousandths.

$$\boxed{\begin{array}{r} 3.428 \\ 0.29 \\ \hline 30852 \\ 6856 \\ \hline 0.99412 \end{array}}$$

The following is a convenient rule for placing the decimal point:

To find the number of decimal places in a product, add the number of decimal places in the factors.

In the above case there are three decimal places in the multiplicand and two decimal places in the multiplier, and hence there are five decimal places in the product.

This rule, while helpful to a pupil in the elementary school, could, like most rules, be dispensed with in higher arithmetic. Because we are multiplying thousandths by hundredths in the above case, our reason at once tells us that the result must be hundred-thousandths.

Similarly, the nature of the product in each of the following cases must be as stated:

1.35×2.78	ten-thousandths
12.75×16.86	ten-thousandths
3.428×5.936	millionths
4.263×9.8732	ten-millionths

For practical purposes we do not often have to multiply decimals having more than two or three places. In such cases as arise we can easily determine the number of decimal places by considering what the nature of the product must be.

Exercise. Multiplication of Decimals

1. How many decimal places are there in the square of a number of three decimal places? State the reason, not referring to the rule given on page 54.

2. How many decimal places are there in the cube of a number of two decimal places? State the reason.

3. A number having three decimal places is multiplied by a number having four decimal places. How many decimal places are there in the product? State the reason.

4. Write as a decimal the product of 3.5×10^{-5} and 75.

5. Multiply seventy-five ten-thousandths by sixty-two thousandths, first by using common fractions and then by using decimals. Show that the results agree.

6. In commercial work we frequently have to raise a number like 1.05 to various powers. How many decimal places are there in the sixth power of 1.05? State the reason.

7. A bank bids 1.0753 on a million dollars of city bonds. How much must it pay if the bid is accepted? How much more would it have had to pay if the bid had been 1.0754?

A bid of 1.0753 means that the bank will pay $1.0753 for a dollar's worth of bonds, par value, or $1075.30 for a thousand-dollar bond. The purpose of the problem is to show the significance of carrying a decimal to several places.

8. A bank is bidding on a $50,000,000 bond issue, the bid being on the basis of $1, par value. To how many decimal places should it express the bid in order to come to the nearest $10,000 in the total price?

9. A life-insurance company is bidding on a $20,000,000 bond issue, the bid being on the basis of $1, par value. What is the difference in total cost between a bid of 1.0107 and one of 1.01072?

Approximate Results. We know that the circumference of a circle may be found by multiplying the diameter by the number commonly called π (pī). By means of an instrument that measures to the nearest hundredth of an inch, the diameter of the shaft of a ship's propeller, where it revolves in a steel collar, is found to be 10.75 in., and the circumference is required. Approximate values of π are $3\frac{1}{7}$, 3.14, 3.142, 3.1416, 3.14159. Using these values as multipliers we have

$$3\tfrac{1}{7} \times 10.75 \text{ in.} = 33.786 \text{ in.}$$
$$3.14 \times 10.75 \text{ in.} = 33.7550 \text{ in.}$$
$$3.142 \times 10.75 \text{ in.} = 33.77650 \text{ in.}$$
$$3.1416 \times 10.75 \text{ in.} = 33.772200 \text{ in.}$$
$$3.14159 \times 10.75 \text{ in.} = 33.7720925 \text{ in.}$$

Since our measurement is correct only within 0.01 in., the result is 33.77 in. to the nearest hundredth.

A practical question now arises as to how we can obtain the approximate result 33.77 in. with the least unnecessary work. This does not mean that our work is to be inaccurate, but that the result is to be exact to a certain degree of approximation.

Suppose that we take 3.14159 for π and use it as the multiplicand. If we wish the result to 0.01, we should carry each multiplication to 0.001. Multiplying as usual, except that we begin at the left of the multiplier, and putting a dot in each place beyond thousandths instead of multiplying to find that place, we obtain the result 33.77. We thus save several multiplications. This process is called *contracted multiplication.*

$$\boxed{\begin{array}{rl} 3.14159 & \\ 10.75 & \\ \hline 31.416 & . \\ 2.199 & . \; . \; . \\ .157 & . \; . \; . \; . \\ \hline 33.772 & . \; . \; . \; . \end{array}}$$

In the case of a large number of decimal places the gain is important.

Exercise. Approximate Results

1. With an instrument that measures only to the nearest tenth of an inch, the diameter of a wheel is found to be 18.6 in. Find the circumference.

Contracted multiplication is to be used in all such cases.

2. A certain computation requires the value of π^2 to the nearest hundredth. Taking π as 3.14159, what is the value?

3. The dimensions of a box to the nearest tenth of an inch are 15.7 in., 12.8 in., and 8.6 in. Find the volume to the nearest tenth of a cubic inch.

4. A meter is 39.37 in., to the nearest hundredth of an inch. How many square inches are there, to the nearest hundredth, in a square meter?

5. The volume of a cylinder is found by multiplying the height by the area of the base. Expressed as an algebraic formula, $V = \pi r^2 h$, where V is the number of cubic units, r the number of units in the radius, and h the number of units in the height. If $r = 17.82$ and $h = 12.95$, each expressing the number of inches to the nearest hundredth, find the number of cubic inches in the volume.

Perform the following multiplications, carrying each product to two decimal places:

6. 17.2396 × 29.4829.
7. 28.0378 × 42.0892.
8. 125.7263 × 496.3462.
9. 57.362 × 4.38763.
10. 30.478 × 6.02976.
11. 0.08723 × 6395.47.

Perform the following multiplications, carrying each product to three decimal places:

12. 48.29683 × 50.72089.
13. 6.24387 × 5.31097.

Finding Per Cents. To find a certain per cent of a given number is merely a case of multiplying by a decimal. For example, to find $3\frac{1}{2}\%$ of \$140 we have

$$0.03\frac{1}{2} \times \$140 = \$4.90.$$

Since $3\frac{1}{2}\%$ means $0.03\frac{1}{2}$, and since

if to $1 \times$ a number
we add $0.03\frac{1}{2} \times$ the same number
we have $1.03\frac{1}{2} \times$ the number,

we see that to increase a number by $3\frac{1}{2}\%$ of itself we multiply it by $1.03\frac{1}{2}$, that is, by $1 + 3\frac{1}{2}\%$.

Similarly, to decrease a number by 5% of itself we multiply it by $1 - 5\%$, that is, by $1 - 0.05$, or by 0.95, or by 95%.

Since most of the practical problems in percentage depend upon finding a certain per cent of a given number, as in the above cases, we see that most of the work in percentage is merely a matter of multiplication.

It is convenient to know the following relations:

$\frac{1}{2} = 0.50 = 50\%$
$\frac{1}{3} = 0.33\frac{1}{3} = 33\frac{1}{3}\%$
$\frac{2}{3} = 0.66\frac{2}{3} = 66\frac{2}{3}\%$
$\frac{1}{4} = 0.25 = 25\%$
$\frac{3}{4} = 0.75 = 75\%$
$\frac{1}{5} = 0.20 = 20\%$
$\frac{2}{5} = 0.40 = 40\%$
$\frac{3}{5} = 0.60 = 60\%$

$\frac{1}{6} = 0.16\frac{2}{3} = 16\frac{2}{3}\%$
$\frac{5}{6} = 0.83\frac{1}{3} = 83\frac{1}{3}\%$
$\frac{1}{8} = 0.12\frac{1}{2} = 12\frac{1}{2}\%$
$\frac{3}{8} = 0.37\frac{1}{2} = 37\frac{1}{2}\%$
$\frac{5}{8} = 0.62\frac{1}{2} = 62\frac{1}{2}\%$
$\frac{7}{8} = 0.87\frac{1}{2} = 87\frac{1}{2}\%$
$\frac{1}{12} = 0.08\frac{1}{3} = 8\frac{1}{3}\%$
$\frac{1}{25} = 0.04 = 4\%$

In elementary arithmetic it is desirable to consider percentage as a separate topic. As already stated, however, we may consider all cases of percentage in higher arithmetic merely as multiplications or divisions.

The most important applications of percentage will be considered in connection with practical business problems in Chapters VI–XI.

Exercise. Finding Per Cents

1. If you can buy a book that is marked \$1.25 for 20% less than the marked price, how much must you pay?

2. If a dealer pays \$960 for an automobile f.o.b. Detroit, and the pro rata expense for freight, store charges, advertising, interest, and all other overhead charges is \$235, and if he sells it so as to gain 20% on the total cost, how much does he gain?

The symbol "f.o.b." means "free on board," that is, the automobile is delivered on the cars at Detroit for \$960. "Overhead charges" means such charges as should be added to the original cost to make the total cost. The term is variously used, but we shall for the present take the meaning as covering such items as those stated in Ex. 2.

3. A large concern sells a shipment of goods for \$75,250 and makes a profit of $12\frac{1}{2}$% on the selling price. How much is the profit?

In a large business the profit is commonly computed, at present, upon the selling price.

4. A merchant's business this year is 14% more than last year. What per cent is it of his business last year? If his business last year at this time had amounted to \$175,800, how much does his business amount to thus far this year?

5. A manufacturing house did a business of \$575,000 two years ago, but last year it did only 85% as much. What per cent did the business fall off? What was the amount of the business last year?

By using the simplest method find each of the following:

6. 50% of \$750.
7. $33\frac{1}{3}$% of \$822.
8. 25% of \$1260.
9. $16\frac{2}{3}$% of \$7320.
10. $112\frac{1}{2}$% of \$1600.
11. 104% of \$8000.
12. $162\frac{1}{2}$% of \$6400.
13. $108\frac{1}{3}$% of \$2400

Checks in Multiplication. We may check our work in multiplication in various ways. If we are multiplying 883 by 68, for example, we may check by multiplying 68 by 883. Instead of this we may multiply 883 by 34, which is half of 68, and then multiply the product by 2. These checks, however, take too much time, and so we rarely use them.

There is another check, much simpler than either of these, that is commonly used and that rarely fails to discover an error; it is known as the *check of casting out nines* or simply the *check of nines*. Briefly described, it is applied as follows:

$$\begin{array}{r} 883 \\ 68 \\ \hline 7064 \\ 5298 \\ \hline 60{,}044 \end{array}$$

In the multiplier, $8 + 6 = 14 = 9 + 5$; reject, or "cast out," the 9 and there remains 5. Write this 5 in the left angle of the cross as shown.

In the multiplicand, $3 + 8 + 8 = 19 = 2 \times 9 + 1$; cast out the 9's and there remains 1. Write this 1 in the angle at the right.

The product of this 5 and this 1 is 5. Write this 5 at the top. In the case of a product greater than 9, cast out the 9's and write the remainder at the top.

In the product, $4 + 4 + 6 = 14$; cast out the 9 and there remains 5. Write this 5 at the bottom.

If the figure at the top (5) agrees with the figure at the bottom (5) the work is probably correct.

We shall assume for the present that the above statements are true and shall show their application by means of various problems on page 61. On page 62 we shall explain the reasons upon which the check of casting out nines is based.

The check of nines does not detect all errors, but it detects the great majority of them. It forms one of the most helpful devices that we have for detecting errors in multiplication and division, and the student will be well repaid for acquiring the habit of using it in all computations of this kind.

Exercise. Checks in Multiplication

1. Multiply 496 by 45. Check by multiplying 496 by 9 and then multiplying this product by 5.

2. Multiply 575 by 283. Check by multiplying 283 by 575.

3. Multiply 1275 by 98. Check by multiplying 1275 by 100 and then subtracting 2×1275. Explain this check.

Perform the following multiplications and check each result by casting out nines:

4. 127×962.

5. 346×795.

6. 239×952.

7. 261×792.

8. 306×603.

9. 528×825.

10. 4.73×6.82.

11. 29.7×4.38.

12. 5.81×83.7.

13. Prove the following statement with respect to 2743:

$$\begin{array}{r@{\;=\;}l} 2000 & 2 \times 999 + 2 \\ 700 & 7 \times 99 + 7 \\ 40 & 4 \times 9 + 4 \\ 3 & 3 \\ \hline 2743 & (\text{multiples of } 9) + 2 + 7 + 4 + 3 \end{array}$$

14. In Ex. 13 show that 2743 is divisible by 9 if the sum of the digits is divisible by 9.

This shows that we can find the remainder arising from dividing a number by 9 by simply casting out nines.

By "divisible" we always mean exactly divisible; that is, divisible with no remainder.

Determine mentally which of these products are incorrect:

15. $687 \times 87{,}147 = 52{,}985{,}379$.

16. $22.2 \times 307.05 = 6836.51$.

17. $675 \times 600{,}130 = 405{,}087{,}950$.

18. $11^3 = 1341$.

19. $11^4 = 14{,}641$.

20. $121^3 = 1{,}771{,}561$.

Check of Nines Explained. The check of nines may be used without understanding the reason on which it is based. It is not possible to explain it satisfactorily in the elementary school, but those who understand algebra will readily see the reason involved.

Pages 62 and 63 may be omitted at the discretion of the teacher.

Any number may be considered a multiple of 9 plus some remainder. This remainder is usually called the *excess of nines* in the number.

For example, $35 = 3 \times 9 + 8$, the excess being 8; $72 = 8 \times 9 + 0$, the excess being 0; $6 = 0 \times 9 + 6$, the excess being 6. In general, any number may therefore be written $9x + e$, where e is the excess of nines.

Any number may be supposed to have a units, b tens, c hundreds, and so on; thus, 907 has 7 units, 0 tens, and 9 hundreds. Any number may therefore be represented by

$$a + 10b + 100c + 1000d + \cdots$$

or by $$a + 9b + b + 99c + c + 999d + d + \cdots$$

or by $$9b + 99c + 999d + \cdots + a + b + c + d + \cdots.$$

This is a multiple of 9 plus the sum of the digits. Hence

Nine is a factor of a number if it is a factor of the sum of the digits of the number, and not otherwise.

For example, 473,986 has an excess of 1, as may be seen by adding the digits and casting out nines as we add.

If we represent two numbers by $9x + a$ and $9y + b$ we can easily show that their product is $9^2xy + 9ay + 9bx + ab$, or $9(9xy + ay + bx) + ab$, a multiple of 9 plus the product of the excesses a and b. Therefore

The excess of nines in the product of two numbers is the excess of nines in the product of the excesses of the numbers.

This is the principle used on page 60.

Exercise. Check of Nines

Find the excess of nines in each of the following numbers:

1. 42,073,683. **3.** 872,361,426. **5.** 1,234,567,890.

2. 21,021,021. **4.** 433,221,100. **6.** 9,876,543,210.

Write each of the following numbers in the form $9x + e$:

7. 52. **8.** 128. **9.** 603. **10.** 3486. **11.** 9275.

12. Show that $10{,}000\,e + 1000\,d + 100\,c + 10\,b + a$ is divisible by 9 if $e + d + c + b + a$ is divisible by 9.

13. Prove that a number is divisible by 3 if the sum of the digits is divisible by 3.

On page 62 show that $9\,b + 99\,c + 999\,d + \cdots + a + b + c + d + \cdots$ is a multiple of 3 plus the sum of the digits.

14. Check the multiplication $995 \times 1474 = 1{,}466{,}630$ by casting out threes.

Proceed exactly as with casting out nines on page 60.

15. Multiply $9\,x + a$ by $9\,y + b$, proving that the product stated on page 62 is correct.

16. Explain how any number may be represented by $a + 10\,b + 100\,c + 1000\,d + 10{,}000\,e + \cdots$. Illustrate this by taking the case of 24,390.

17. Using the form given in Ex. 16, show that any number may also be represented by $a + 11\,b - b + 99\,c + c + 1001\,d - d + 9999\,e + e + \cdots$, or by $11\,b + 99\,c + 1001\,d + 9999\,e + \cdots + (a + c + e + \cdots) - (b + d + \cdots)$, which is a multiple of 11 plus the difference between the sums of the odd and the even orders of digits.

18. Apply Ex. 17 to the checking of the multiplication of 72,826 by 34,008 by the excess of elevens.

Short Methods in Multiplication. A practical computer makes use of certain short methods in multiplication. These methods are usually learned in the elementary school, but their importance is rarely appreciated at that time, and in most cases their validity is not understood. We shall state briefly the most important of these methods.

Since $5 = \frac{10}{2}$, we may multiply by 5 by first multiplying by 10 and then dividing by 2.

In the case of an integer we multiply by 10 by annexing 0; in the case of a decimal, by moving the decimal point one place to the right. The two processes are essentially the same.

Since $25 = \frac{100}{4}$, we may multiply by 25 by first multiplying by 100 and then dividing by 4.

Since $33\frac{1}{3} = \frac{100}{3}$, we may multiply by $33\frac{1}{3}$ by first multiplying by 100 and then dividing by 3.

Since $125 = \frac{1000}{8}$, we may multiply by 125 by first multiplying by 1000 and then dividing by 8.

Since $9 = 10 - 1$, we may multiply by 9 by first multiplying by 10 and then subtracting the multiplicand.

For example, $9 \times 725 = 7250 - 725 = 6525$. There are evidently similar short methods for multiplying by 99, 999, and so on.

Since $11 = 10 + 1$, we may multiply by 11 by first multiplying by 10 and then adding the multiplicand.

For example, $11 \times 725 = 7250 + 725 = 7975$. We may also multiply 725 by 11 by writing $0 + 5$ for units, $5 + 2$ for tens, $2 + 7$ for hundreds, and $7 + 0$ for thousands, but the method given above is the one commonly used.

There are various other short methods, but either they are easily inferred or are not of particular importance.

For example, it is easier to take $\frac{2}{3}$ of a number than to multiply the number by $0.66\frac{2}{3}$, and it is easier to multiply certain numbers by $1000 \times \frac{3}{4}$ than to multiply them by 750.

Exercise. Short Methods

1. Multiply 4729 by 998 and then subtract 2 × 4729 from 4,729,000. Which is the easier of the two methods, and why are the results the same?

2. Which is the easier, to take $\frac{1}{3}$ of $3423 or to take $33\frac{1}{3}\%$ of it? Why are the results the same?

3. Which is the easier, to take $\frac{1}{8}$ of $7296 or to take $12\frac{1}{2}\%$ of it? Why are the results the same?

4. By the simplest method find 25% of $1680, and explain the principle involved.

5. In multiplying 575 by 168 show that you may first multiply by 8 and then obtain the tens by multiplying this product by 20, finally adding the two products.

6. Multiply 1275 by 246 by taking advantage of a relation similar to the one in Ex. 5.

7. Multiply 98,246 by 756 by first multiplying by 7 to obtain the number of hundreds and then multiplying this product by 8 to obtain the units, finally adding the two products. Explain the process.

Perform the following multiplications, using a short method in each case:

8. 5 × $4270.

9. 5% of $3280.

10. 25 × $3620.

11. 25% of $8240.

12. $33\frac{1}{3}\%$ of $9675.

13. 125 × 16,736.

14. 1.25 × 29,672.

15. $12\frac{1}{2}\%$ of 33,616.

16. 9 × $82,375.

17. 99 × $42,380.

18. 999 × $29,670.

19. 189 × 142,380.

20. 255 × 527,360.

21. 816 × 427,750.

22. 714 × 125,250.

23. 11 × 2,375,482.

Multiplication of Compound Numbers. If we compare the multiplication of integers, fractions, and compound numbers, we see that the fundamental principle is the same, although there seem at first to be some differences. Consider, for example, the following cases of multiplication:

$$\begin{array}{r} 35 \\ 7 \\ \hline 245 \end{array} \qquad \begin{array}{r} 3\text{ tens} + \ \ 5 \\ 7 \\ \hline 21\text{ tens} + 35 = 245 \end{array} \qquad \begin{array}{l} 3\frac{5}{8} \\ 7 \\ \hline 21\frac{35}{8} = 25\frac{3}{8} \end{array} \qquad \begin{array}{r} 3\text{ ft.}\quad 5\text{ in.} \\ 7 \\ \hline 23\text{ ft.}\ \ 11\text{ in.} \end{array}$$

In each case the multiplicand is 3 something + 5 of something else, such as 3 tens + 5 units, 3 units + 5 eighths, or 3 ft. + 5 in. In each case we multiply the parts separately, and in each case we reduce the result as much as we can.

It is seen, therefore, that no new principle is involved in the case of compound numbers.

It is seldom that we need to multiply compound numbers of any kind, but in any case we rarely have more than two denominations involved. The fundamental principle is the same, however, in all cases.

Algebraic Multiplication. Similarly, there is no new principle involved in algebraic multiplication. If we multiply $t+f$ by s, the product is $st+sf$; if t is 30, f is 5, and s is 7, this becomes $7 \times 35 = 7 \times 30 + 7 \times 5$.

In the case of $(2a-4)(3a+7)$ we multiply each term of the multiplicand by each term of the multiplier and then add the results. The product is $6a^2+2a-28$.

$$\boxed{\begin{array}{rrr} & 3a + & 7 \\ & 2a - & 4 \\ \hline & -12a - & 28 \\ 6a^2 + & 14a & \\ \hline 6a^2 + & 2a - & 28 \end{array}}$$

It is seen, therefore, that no new principle is involved in the algebraic case.

In the case of polynomials of three or more terms the principle is the same as the one already stated.

Exercise. Compound and Algebraic Numbers

1. Multiply 7 lb. 9 oz. by 6; by 12; by 20.

2. Multiply 2° 38′ 45″ by 3; by 8; by 30.

3. Multiply 9 yd. 27 in. by 2; by 7; by 60.

4. Multiply 48 by 6 and 4 ft. 8 in. by 6. Why are not the figures the same in the two products?

5. In multiplying 59, $5\frac{9}{10}$, and 5 lb. 9 oz. by 7 the figures are practically the same in the multiplicands. Explain the difference in the figures in the products.

6. Multiply 9 ft. 7 in., 9 lb. 7 oz., 9 hr. 7 min., 9 yd. 7 in. and 9 min. 7 sec. each by 12. Explain the difference in the figures in the products.

7. Multiply $4t+3$ by 9 and then multiply 43 by 9. Explain the difference in the figures in the products. Consider the two products in the special case of $t=10$.

8. Multiply $7h+5t+9$ by 8 and then multiply 759 by 8. Explain the difference in the figures in the products. Consider the two products in the special case of $h=100$, $t=10$.

9. Find the product of $5x+7$ and $3x+4$; of 57 and 34; of 5.7 and 3.4. Compare the three products. Compare the products in the special case of $x=10$.

10. Find the product of $7x+3y$ and $2x+9y$ and the product of 7.3 and 2.9. Compare the products in the special case of $x=1$ and $y=0.1$.

11. From the product of $x+y$ and $x-y$ find a simple rule for stating at sight the product of two such expressions as $100+4$ and $100-4$. Apply this rule to finding the product of 104 and 96; of 105 and 95; of 2007 and 1993.

12. From the product of $x+y$ and $x+y$ find a simple rule for squaring a number like 37, that is, $30+7$.

Exercise. Review of Chapters I–IV

1. Write in common notation the number 5.7×10^{12}.

2. Write in index notation the number 27,000,000,000.

3. Write in common notation the numbers MCMXX and MDCCCCXXIV.

4. Express $3\frac{3}{4}$ as an improper fraction; as a decimal.

The term "mixed decimal" is often used in school for a number like 2.75, but we rarely use the term in practical life.

5. Express 25% as a common fraction; as a decimal.

6. Add 3 ft. 9 in. and 7 ft. 8 in.; 39 and 78; 3 lb. 9 oz. and 7 lb. 8 oz.; 3.9 and 7.8; 3.09 and 7.08. Explain the difference in the figures of the results.

7. From 9 ft. 3 in. subtract 3 ft. 7 in., and then find the values of $93 - 37$, $9\frac{3}{8} - 3\frac{7}{8}$, $9.3 - 3.7$, $9.003 - 3.007$, and 9 yd. 3 in. – 3 yd. 7 in. Explain the difference in the figures of the results.

8. Multiply 8 ft. 5 in. by 9; 85 by 9; 8.005 by 9; $8\frac{5}{8}$ by 9. Explain the difference in the figures of the results.

9. Given 735 and 627, find their sum, their difference, and their product, giving a check in each case.

10. A field is 836 yd. long and 642 yd. wide. Find the area to the nearest 100 sq. yd.; to the nearest 10 sq. yd.

11. Given that $\pi = 3.14159265$, find the value of π^2 to four figures only, performing no unnecessary work.

12. A meter being 39.37 in., find to the nearest cubic inch the number of cubic inches in a cubic meter.

Using short methods, find the following:

13. $66\frac{2}{3}\%$ of \$987.96.

14. $87\frac{1}{2}\%$ of \$1826.40.

15. 312×4876.

16. $999 \times 42{,}758$.

CHAPTER V

DIVISION

Language of Division. The Latin expression for "number to be divided" is *numerus dividendus*, and from this comes our word *dividend.* Because a certain part of the net earnings of a stock company is divided among the stockholders, this part is also called a dividend, or, more commonly, the amounts received by the stockholders are called dividends.

The Latin word for "divider" is *divisor*, whence we have our name for the divider in an example in division.

The Latin word for "how often" is *quoties*, and from this comes our word *quotient*, the quotient telling how often the divisor is contained in the dividend.

These terms are not widely used outside the school, but they are used often enough to make it worth while to know them.

Symbols in Division. There are several ways of indicating the division of, for example, 36 by 9. We may do so by the symbols $36 \div 9$, as in English-speaking countries; $36 : 9$, as is done in most other countries; $\frac{36}{9}$, as is done in all countries; or $36/9$, as is frequently done in printed mathematical works. We may also write $9\underline{)36}$, and a few writers have recently attempted to introduce the form $9\overline{)36}$ in spite of the fact that it is used neither in business nor in advanced mathematics, not being convenient in the latter field.

In practical business the symbols $\div$ and : are rarely used. They are helpful in school chiefly because they save space in writing and printing problems.

Nature of Division. The pupil in the elementary school is often taught that division is the process of finding how many times one number is contained in another. Division, however, has a more extended meaning. It may be defined as the operation by which, given the product of two numbers and one of the numbers, the other number is found. Thus,

GIVEN PRODUCT	GIVEN NUMBER	NUMBER TO BE FOUND
\$6	\$2	3
\$6	3	\$2
$\frac{5}{8}$	$\frac{1}{8}$	5
$\frac{5}{8}$	$\frac{2}{3}$	$\frac{15}{16}$

Teachers often distinguish between the first and second of these cases by calling the first one *measuring* and the second one *partition.* The reason is that we measure the length of a line by seeing how many times the line contains another given length, and similarly for area, capacity, and so on; but when we divide \$6 by 3 we separate \$6 into 3 equal parts.

When computers in practical industrial work use such forms as 2 ft. × 3 ft. = 6 sq. ft., they naturally use the inverse form, 6 sq. ft. ÷ 2 ft. = 3 ft. They also speak of dividing distance by time to get the speed. For example, if an airplane goes 60 mi. in 30 min. they obtain the rate per minute by dividing 60 mi. by 30 min., although in elementary teaching we divide 60 mi. by 30. They express the work in this form:

$$60 \text{ mi.} \div 30 \text{ min.} = 2 \text{ mi./min.},$$

reading the result "2 miles per minute."

Such forms are not necessary in the elementary school. If given, they would only confuse a pupil. Advanced students must understand, however, that these extensions of the idea of division are recognized by standard authorities. The forms given would not be correct unless the idea of division were extended so as to include them.

Exercise. Nature of Division

1. How many times can you subtract $2 from $6 before the difference becomes zero? Write a brief statement showing how one form of division is related to subtraction.

It will be recalled that one form of multiplication is similarly related to addition.

2. Taking the word "times" in the sense in which children at first use it, that is, to mean an integral number of times, which of the first four cases discussed on page 70 show how many times one number is contained in another?

3. Given the divisions indicated by 8 ft. ÷ 2 ft. = 4 and 8 ft. ÷ 4 = 2 ft., which one may be called measuring and which may be called partition? State the reason in each case.

4. A locomotive engineer finds that a drive wheel makes 2600 revolutions in 5 min. To find the number of revolutions per minute (R.P.M.) he writes

$$2600 \text{ R.} \div 5 \text{ min.} = 520 \text{ R.P.M.}$$

How would you state this so that it would be more clearly understood by an elementary pupil?

5. If an automobile goes 160 ft. in 8 sec., and you wish to express the rate per second, these two statements might be considered:

$$160 \text{ ft.} \div 8 = 20 \text{ ft., the rate per second;}$$
$$160 \text{ ft.} \div 8 \text{ sec.} = 20 \text{ ft./sec.}$$

Which is the better form for an elementary pupil, and why is it the better?

6. In elementary classes it is said that if the dividend is concrete and the divisor abstract, the quotient is concrete and like the dividend. Illustrate this statement.

Division of Integers and Decimals. Division by an integer is explained fully in elementary textbooks. For our purposes we need consider only the following cases:

$$\begin{array}{rrl}
 & 144 & \\
12) & \overline{1728} & \\
 & \underline{1200} & = 100 \times 12 \\
 & 528 & \\
 & \underline{480} & = 40 \times 12 \\
 & 48 & \\
 & \underline{48} & = \underline{4 \times 12} \\
 & 0 & 144 \times 12
\end{array}
\qquad
\begin{array}{rr}
 & 144 \\
12) & \overline{1728} \\
 & \underline{12} \\
 & 52 \\
 & \underline{48} \\
 & 48 \\
 & \underline{48}
\end{array}
\qquad
\begin{array}{rr}
 & 1.44 \\
12) & \overline{17.28} \\
 & \underline{12} \\
 & 52 \\
 & \underline{48} \\
 & 48 \\
 & \underline{48}
\end{array}$$

The first of these divisions shows a form often used in the elementary school in explaining the reasons for the steps.

The second division shows the same work simplified.

The third division shows the same work for the case of $17.28 \div 12$. We now see one reason for writing the quotient above the dividend instead of, as was formerly done, at the right. By writing it above, we see at once where the decimal point is to be placed. In short division we obtain the same advantage by placing the quotient below the dividend, as is the business custom.

$$\begin{array}{rr}
12) & \underline{17.28} \\
 & 1.44
\end{array}$$

If there is a decimal in both dividend and divisor, as in the case of $17.28 \div 1.2$, we may multiply both dividend and divisor by 10, 100, or any other number that will make the divisor an integer, in this case by 10. We then have $172.8 \div 12$.

Some teachers prefer to write $17.2\overset{\vee}{8} \div 1.2$, placing the decimal point in the quotient above the check.

Division by per cents is simply a special case of division by decimals; thus, $220 ÷ 110% = $220 ÷ 1.10 = $200.

Exercise. Integers and Decimals

1. Divide 4375 by 35, writing the work in the full form given in the first case on page 72. State how your work shows that you have subtracted 35 from 4375 the number of times stated in the quotient.

2. Divide 13.125 by 35, writing a statement of the reason why the decimal point in the quotient is placed above the decimal point in the dividend.

3. Give three illustrations showing that both dividend and divisor may be multiplied by the same number without changing the quotient.

4. State the reason why the quotient of 1.3125 divided by 3.5 is the same as the quotient of 13.125 divided by 35.

5. In dividing one decimal by another it was suggested on page 72 that both dividend and divisor should be multiplied by some number that would make the divisor an integer. In the case of 1.3125 ÷ 3.5 would it be just as well to multiply both numbers by 2, giving 2.625 ÷ 7? State the reason.

6. Why would it not be better, in the case of 1.3125 ÷ 3.5, to multiply the numbers by some number that would make both integral? For example, we might multiply each by 10,000, the problem then becoming 13,125 ÷ 35,000.

7. In the case of 1.3125 ÷ 3.5 what advantages do you see in either of the forms 13.125 ÷ 35 and $1.\overset{\vee}{3}125 \div 3.5$ over the other?

Perform the following divisions:

8. 4.356 ÷ 23.4.

9. 13.338 ÷ 0.38.

10. 1.06575 ÷ 0.725.

11. 0.015225 ÷ 0.021.

12. What is meant by the average of 33.4, 28.7, 29.8, and 31.6, and how is it found? Find this average.

Checks in Division. We may check our work in division in various ways. If we are dividing 23,936 by 68, for example, and find the quotient to be 352, we may check by multiplying 352 by 68, the product being the dividend, 23,936. That is, if there is no remainder,

The dividend is the product of the quotient and the divisor.

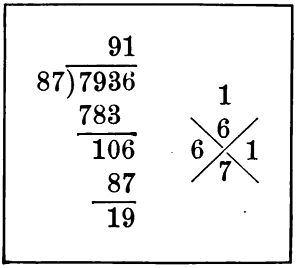
If there is a remainder,

The dividend is the product of the quotient and the divisor plus the remainder.

For example, if we find that 23,950 ÷ 352 = 68, with remainder 14, then $23{,}950 = 68 \times 352 + 14$.

This principle includes the preceding one, for if there is no remainder we simply add zero, that is, we have no remainder to add.

We may also apply the check of nines in division, and indeed this is the simplest and quickest of all checks to use. Consider, for example, the case of 7936 ÷ 87, the quotient being 91 and the remainder being 19.



Drawing a cross as shown here, and as used in multiplications, we write 6, which is the excess of nines in 87, the divisor, in the left angle of the cross; we then write 1, which is the excess of nines in 91, the quotient, at the right. We write the excess of nines in 7936, the dividend, at the bottom. Then at the top we write 1 and 6, the excess of nines in 19, the remainder, and in 6 × 1, the product of the excesses, respectively. Then 1 + 6 and 7 should be equal, as they are.

Careful computers check each step of their work so as not to vitiate the subsequent steps by any error. The check of casting out nines is particularly helpful in this respect.

Exercise. Checks in Division

1. Divide 19,224 by 54. Check the result by dividing 19,224 by 9 and then dividing this quotient by 6. Explain why this is a check.

2. Divide 32,712 by 58. Check the result by multiplying the quotient by the divisor.

3. Divide 301,190 by 64. Check by multiplying the quotient by the divisor and adding the remainder to the result.

Perform the following divisions and check each result by casting out nines:

4. $8556 \div 92$.

5. $5586 \div 57$.

6. $82.36 \div 5.8$.

7. $27{,}456 \div 32$.

8. $67{,}682 \div 4.3$.

9. $36{,}270 \div 0.39$.

10. $891{,}900 \div 99$.

11. $433{,}130 \div 31$.

12. $749{,}840 \div 86$.

13. Proceeding as in the case of checking by nines, check the division of 4599 by 73 by casting out threes.

14. Check the division of 472,932 by 302, and also the division of 587,286 by 419, by casting out threes.

Determine mentally which of the following quotients are incorrect, writing the reason in each case:

15. $2{,}137{,}512 \div 372 = 5746$.

16. $9{,}734{,}700 \div 111 = 8770$.

17. $5{,}731{,}200 \div 199 = 2880$.

18. $6{,}882{,}920 \div 110 = 6257$.

19. $65{,}668 \div 456 = 144$.

20. $221{,}340 \div 102 = 2170$.

21. $211{,}429 \div 5873 = 36$.

22. $313{,}220 \div 8465 = 37$.

23. $411{,}100 \div 5481 = 75$, with remainder 25.

24. $4{,}946{,}858 \div 872 = 5673$, with remainder 2.

25. Use the check of casting out nines to determine whether $7279 \div 91 = 79$. Then divide 7279 by 91 and state why the check is not so satisfactory in this case.

Division of Fractions. Pupils in the elementary school learn that they can divide one common fraction by another by multiplying the dividend by the reciprocal of the divisor.

For example, $\frac{3}{8} \div \frac{2}{3} = \frac{3}{2} \times \frac{3}{8} = \frac{9}{16}$.

In higher arithmetic it is desirable to make the explanation of this fact somewhat more clear by the use of algebra.

We may first consider the division of two concrete numbers, say 2 yd. by 3 ft. Since these numbers represent different units we first express them in terms of the same unit for the same reason as in the addition of fractions, pages 28 and 30. We may evidently use either the yard or the foot for this purpose, thus:

$$2 \text{ yd.} \div 3 \text{ ft.} = 6 \text{ ft.} \div 3 \text{ ft.} = 2,$$

or

$$2 \text{ yd.} \div 3 \text{ ft.} = 2 \text{ yd.} \div 1 \text{ yd.} = 2,$$

the denominations of feet or yards finally disappearing.

Similarly with any two fractions, which we may represent by $\frac{a}{b}$ and $\frac{c}{d}$. We may express them with any common denominator, but bd is evidently the best to use. We then have

$$\frac{a}{b} \div \frac{c}{d} = \frac{ad}{bd} \div \frac{bc}{bd} = \frac{ad}{bc},$$

the denominators disappearing as in the cases given above.

But
$$\frac{a}{b} \times \frac{d}{c} = \frac{ad}{bc},$$

so that
$$\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c}.$$

In dividing one fraction by another we obtain the same result by multiplying the dividend by the reciprocal of the divisor.

Cancellation should be used to shorten work whenever possible.

Exercise. Division of Fractions

Perform the following divisions and show the relations among the several cases:

1. $\frac{2}{5} \div \frac{3}{8}$. **3.** 2 ft. ÷ 3 in. **5.** \$2 ÷ 3¢.

2. $\frac{2}{3} \div \frac{3}{4}$. **4.** 2 yd. ÷ 3 in. **6.** 2 lb. ÷ 3 oz.

In each of the above cases there are 2 units of one kind divided by 3 units of another kind. How do you proceed in each case? Why are the results different?

Perform the following divisions, first using common fractions, then reducing to decimals, and finally showing that the two results have the same value in each case:

7. $\frac{3}{4} \div \frac{1}{2}$. **9.** $\frac{3}{8} \div \frac{1}{4}$. **11.** $\frac{3}{4} \div \frac{5}{8}$. **13.** $\frac{1}{2} \div \frac{3}{16}$.

8. $\frac{1}{2} \div \frac{3}{4}$. **10.** $\frac{1}{4} \div \frac{3}{8}$. **12.** $\frac{5}{8} \div \frac{3}{4}$. **14.** $\frac{3}{16} \div \frac{1}{2}$.

Perform the following divisions, first reducing the mixed numbers to improper fractions:

15. $3\frac{1}{2} \div 2\frac{1}{4}$. **16.** $4\frac{3}{8} \div 2\frac{1}{2}$. **17.** $5\frac{2}{3} \div 1\frac{1}{2}$. **18.** $8\frac{5}{8} \div 2\frac{3}{4}$.

Perform the following divisions without first reducing the mixed numbers to improper fractions:

19. $3\frac{3}{4} \div 1\frac{7}{8}$. **20.** $8\frac{1}{4} \div 2\frac{3}{4}$. **21.** $14\frac{1}{2} \div 3\frac{5}{8}$. **22.** $8\frac{1}{8} \div 4\frac{1}{16}$.

In Exs. 19–22 simply estimate each quotient and multiply the divisor by this quotient and see if the result is the dividend.

Perform the following divisions, first reducing to decimals:

23. $34\frac{7}{8} \div 3\frac{7}{8}$. **24.** $31\frac{1}{2} \div 3\frac{1}{2}$. **25.** $38 \div 4\frac{3}{4}$.

26. What is the average of $3\frac{1}{4}$ in., $3\frac{3}{8}$ in., $3\frac{1}{2}$ in., $2\frac{7}{8}$ in., and $3\frac{1}{8}$ in.?

27. What is the average of $3\frac{3}{4}$ lb., 3 lb. 11 oz., 3.8 lb., 60 oz., and 4 lb.?

Approximate Results. We know that the diameter of a circle is found by dividing the circumference by π, one of the approximate values of π being 3.14159. It is often easier to measure the circumference of a wheel than to measure the diameter, and in ordinary cases we may be able to find the circumference to about 0.01 in. If we find the circumference to be 176.71 in. and wish to compute the diameter, we cannot hope for a result that is accurate beyond 0.01 in. at most.

Since 100's divided by 100,000's gives a quotient less than 0.01 the last three figures of the dividend will not affect the quotient within two decimal places. Since, however, we wish to be sure that the third decimal place is not 5 or more, we reject only the last two figures of the dividend. Since the last figure of the divisor will come under 10's in the dividend when we begin to divide, we write 3.1416 for the divisor. Multiplying both dividend and divisor by 10,000, we then have 1767100 ÷ 31416. The process will now be understood by studying the following example of *contracted division:*

56.25	
31416)1767100.0	
157080	50 × 31416
19630	
18852	6 × 3142
778	
628	0.2 × 314
150	
155	0.05 × 31

In the last multiplication, 5 × 31 is nearer 150 than is 4 × 31.

Even if the student is not likely to make much use of contracted division, it is desirable that he should know its general nature.

Exercise. Approximate Results

1. Divide 176.71 by 3.14159 in the usual way, comparing the work with that of the contracted method on page 78.

2. In Ex. 1 use $\frac{22}{7}$ for the value of π and find by how much the result differs from the results obtained there and on page 78.

3. Divide 1 by 3.14159, giving the result correct to three decimal places.

4. Use the result of Ex. 3 to find the value of $176.71 \div \pi$. Compare the quotient with the results in Exs. 1 and 2.

Multiplying by the reciprocal of π is much easier than dividing by π.

5. Taking the estimated population of the United States in a recent year as 105,253,300 and the area as 3,026,789 sq. mi., find the average population per square mile to the nearest unit, rejecting all unnecessary figures in the division.

Perform the following operations, carrying the result in each case to the nearest hundredth and using no unnecessary figures:

6. $27.4689 \div 3.728$.

7. $39.29684 \div 48.37$.

8. $8.29863 \div 15.635$.

9. $2.3682 \div 14.936$.

10. $1.9837 \div 176.294$.

11. $273.826 \div 983.471$.

12. A meter is 39.37 in. Express 1250.8 in. as meters to the nearest tenth.

13. Divide 71.4 by 3.14159265, carrying the quotient to four significant figures.

14. Taking 1 lb. as 0.4536 kilograms, find the number of pounds in 25 kilograms, carrying the result to tenths.

15. Find the value of $3.429 \div 0.0076816$, carrying the quotient to the nearest integer.

Short Methods in Division. The student is already familiar with various short methods of division, having learned them in elementary arithmetic. Of these, some of the most common will be recalled from the following brief statements:

$$75 \div 10 = 7.5,$$

$$3575 \div 100 = 35.75,$$

$$375 \div 5 = 375 \div \tfrac{10}{2} = 2 \times 37.5 = 75,$$

$$347 \div 25 = 347 \div \tfrac{100}{4} = 4 \times 3.47 = 13.88,$$

$$1275 \div 50 = 1275 \div \tfrac{100}{2} = 2 \times 12.75 = 25.5,$$

$$3963 \div 75 = 3963 \div \tfrac{300}{4} = \tfrac{4}{3} \text{ of } 39.63 = 52.84,$$

$$29{,}375 \div 125 = 29{,}375 \div \tfrac{1000}{8} = 8 \times 29.375 = 235,$$

$$1{,}000{,}000 \div 16\tfrac{2}{3} = 1{,}000{,}000 \div \tfrac{100}{6} = 6 \times 10{,}000 = 60{,}000.$$

It is occasionally convenient to divide by factors. Thus, in the case of $306 \div 18$ we have $306 \div 9 \div 2 = 34 \div 2 = 17$. By this method we can often proceed mentally without doing any writing.

$$\boxed{\begin{array}{r} 127 \\ 144\overline{)18288} \\ 388 \\ 1008 \end{array}}$$

Computers often use a short method, here shown for the case of $18{,}288 \div 144$. Instead of writing the products of 1×144, 2×144, and 7×144, we mentally subtract 1×144 from 182, 2×144 from 388, and 7×144 from 1008. This is easily done if we subtract by the addition method. For example, in the case of 7×144, we proceed as follows:

$$7 \times 4 + 0 = 28,$$

$$7 \times 4 + 2 \text{ (from the 28)} + 0 = 30,$$

$$7 \times 1 + 3 \text{ (from the 30)} + 0 = 10.$$

This shows us that $7 \times 144 + 000$ is 1008. In other words there is no remainder if we subtract 7×144 from 1008. Expressed briefly,

$$1008 - 7 \times 144 = 0.$$

Exercise. Short Methods in Division

1. Give the short method of dividing a number by 1000 and then illustrate and explain it.

2. Give the short method of dividing a number by 7000 and then illustrate and explain it.

3. Find a short method of dividing by 125% and explain it.

Find short methods of dividing by the following numbers, illustrating and explaining each:

4. $12\frac{1}{2}$.
5. $37\frac{1}{2}$.
6. $62\frac{1}{2}$.
7. $87\frac{1}{2}$.
8. $33\frac{1}{3}$.
9. $66\frac{2}{3}$.
10. $33\frac{1}{3}\%$.
11. $66\frac{2}{3}\%$.
12. 25%.
13. 50%.
14. 75%.
15. 20%.
16. 0.5.
17. 0.05.
18. $0.12\frac{1}{2}$.
19. 6.25.

Using the short method given in the lower part of page 80, perform the following divisions:

20. 14,400 ÷ 12.
21. 384,800 ÷ 26.
22. 855,600 ÷ 46.
23. 657,200 ÷ 31.
24. 208,575 ÷ 103.
25. 870,672 ÷ 204.
26. 204,930 ÷ 495.
27. 252,150 ÷ 205.

28. We have several times used $\frac{22}{7}$ as the approximate value of π. Find a short method of dividing by this value.

The student should write half of the dividend and then perform one multiplication and an easy division.

29. A man's capital is 25% more than it was a year ago. It now appears on his books as $94,062.50. Using a short method, find how much it was a year ago.

30. The amount of coal consumed in a certain factory last year was 520,113 tons, or $37\frac{1}{2}\%$ more than it was the preceding year. How much was consumed the preceding year?

Division of Compound Numbers. If we compare the division of integers, compound numbers, and fractions by an abstract number, we see that the fundamental principle is the same. Consider, for example, the following cases:

$$\begin{array}{r}5\underline{)155}\\31\end{array}\qquad\qquad\begin{array}{r}5\underline{)15\text{ ft. }5\text{ in.}}\\3\text{ ft. }1\text{ in.}\end{array}\qquad\qquad\begin{array}{r}5\underline{)15\tfrac{5}{8}}\\3\tfrac{1}{8}\end{array}$$

In each of these cases we have for the result 3 units of the higher order and 1 of the lower order.

If, however, we have a case in which the division of the higher order is not exact, the figures in the quotient will, of course, vary according to the number of units of the lower order that make one unit of the higher order, as in the following cases:

$$\begin{array}{r}7\underline{)238}\\34\end{array}\qquad\qquad\begin{array}{r}7\underline{)23\text{ ft. }8\text{ in.}}\\3\text{ ft. }4\tfrac{4}{7}\text{ in.}\end{array}\qquad\qquad\begin{array}{r}7\underline{)23\tfrac{8}{9}}\\3\tfrac{26}{63}\end{array}$$

In the case of dividing one compound number by another the analogy to the work of division with fractions given on page 76 is apparent, thus:

$$17\text{ ft. }4\text{ in.} \div 2\text{ ft. }2\text{ in.} = 208\text{ in.} \div 26\text{ in.} = 8,$$

or

$$17\text{ ft. }4\text{ in.} \div 2\text{ ft. }2\text{ in.} = 17\tfrac{1}{3}\text{ ft.} \div 2\tfrac{1}{6}\text{ ft.} = 8.$$

Similarly, $17\frac{4}{5} \div 2\frac{2}{5} = \frac{89}{5} \div \frac{12}{5} = \frac{89}{12} = 7\frac{5}{12}$.

In each case the two numbers are reduced to the same unit; that is, to inches, to feet, or to fifths.

The reason why the results differ is that in the last case 5 units of the lower order (fifths) make 1 unit of the next higher order, whereas in the case of feet and inches it takes 12 units of the lower order to make 1 unit of the next higher order.

We rarely have to divide one compound number by another.

Exercise. Division of Compound Numbers

1. Perform the following divisions, noticing the similarity among them:

$7\overline{)28\text{ ft. }7\text{ in.}}$	$7\overline{)28\frac{7}{25}}$	$7\overline{)28\frac{7}{8}}$	$7\overline{)287}$
$7\overline{)28\text{ lb. }7\text{ oz.}}$	$7\overline{)28.07}$	$7\overline{)28.7}$	$7\overline{)28.007}$
$7\overline{)28\text{ yd. }7\text{ in.}}$	$7\overline{)28\frac{7}{32}}$	$7\overline{)28\frac{7}{16}}$	$7\overline{)28.0\frac{7}{25}}$

2. Perform the following divisions, explaining the reason why the figures differ in the results:

$5\overline{)9\text{ ft. }5\text{ in.}}$	$5\overline{)9\frac{5}{9}}$	$5\overline{)9\frac{5}{8}}$	$5\overline{)95}$
$5\overline{)9\text{ lb. }5\text{ oz.}}$	$5\overline{)9\frac{5}{32}}$	$5\overline{)9\frac{5}{24}}$	$5\overline{)9.005}$

3. Divide 9 ft. by 2 ft. 3 in., first by reducing each to inches and next by reducing each to feet. State which is the better method and why it is better.

4. Divide 10 lb. 8 oz. by 3 lb. 8 oz. and divide 10 ft. 8 in. by 3 ft. 8 in. Explain why the figures in the two results are not the same.

5. Divide 7 hr. by 2 hr. 30 min., 7 yd. by 2 yd. 30 in., and \$7 by \$2.30. Explain why the figures in the three results are not the same.

6. Divide 2 hr. 7 min. 30 sec. by 2 and explain why the figures in the quotient are not the same as in the quotient of 2730 ÷ 2.

7. Divide 4 hr. 7 min. 20 sec. by 2 hr. 3 min. 40 sec. and 4720 by 2340. Explain why the figures in the two results are not the same.

8. Find the average of 3 ft. 2 in., 3 ft. $3\frac{1}{2}$ in., and 3 ft. $2\frac{7}{8}$ in.

Algebraic Division. The general principle of algebraic division is the same as that involved in all other kinds of division. This will be seen in the following cases:

$$\begin{array}{r} 2\underline{)8t^2+6t+4} \\ 4t^2+3t+2 \end{array} \qquad\qquad \begin{array}{r} 2\underline{)864} \\ 432 \end{array}$$

$$\begin{array}{r} 2\underline{)8\text{ hr. }6\text{ min. }4\text{ sec.}} \\ 4\text{ hr. }3\text{ min. }2\text{ sec.} \end{array} \qquad\qquad \begin{array}{r} 2\underline{)86\tfrac{4}{5}} \\ 43\tfrac{2}{5} \end{array}$$

We see that we simply divide the parts of the numbers separately, the sum of the results forming the quotient. If there are any reductions necessary in fractions or in compound numbers, of course the form of the result will depend upon the number of units of one order that make one unit of the next higher order. For example, consider the following:

$$\begin{array}{r} 3\underline{)2t^2+3t+4} \\ \tfrac{2}{3}t^2+t+1\tfrac{1}{3} \end{array} \qquad\qquad \begin{array}{r} 3\underline{)234} \\ 78 \end{array}$$

In the first of these cases we do not know the value of t. If $t=10$, the first result reduces at once to the second.

Similarly, consider the following cases:

$$\begin{array}{r} 3\underline{)2h+3m+4s} \\ \tfrac{2}{3}h+m+1\tfrac{1}{3}s \end{array} \qquad\qquad \begin{array}{r} 3\underline{)2\text{ hr. }3\text{ min. }4\text{ sec.}} \\ 41\text{ min. }1\tfrac{1}{3}\text{ sec.} \end{array}$$

In the first of these cases we do not know the values of h, m, and s. If $h=60\,m.$, the two results are similar.

Compare also the following cases:

$$\begin{array}{r} a+b \\ a+b\overline{)a^2+2ab+b^2} \\ \underline{a^2+ab} \\ ab+b^2 \\ \underline{ab+b^2} \end{array} \qquad\qquad \begin{array}{r} 11 \\ 11\overline{)121} \\ \underline{11} \\ 11 \\ \underline{11} \end{array}$$

If $a=10$ and $b=1$, the first case reduces to the second.

Exercise. Algebraic Division

1. Perform the following divisions:

$2\overline{)2\,t+8}$ $\qquad 2\overline{)2\,a+8\,b}$ $\qquad 2\overline{)2\text{ ft. }8\text{ in.}}$ $\qquad 2\overline{)28}$

2. Perform the following divisions:

$3\overline{)6\,a+9\,b+15\,c}$ $\qquad 3\overline{)6\text{ hr. }9\text{ min. }15\text{ sec.}}$ $\qquad 3\overline{)69.15}$

3. Perform the following divisions and explain the difference between the two quotients:

$3\overline{)9\,a+2\,b+7\,c}$ $\qquad 3\overline{)9\text{ hr. }2\text{ min. }7\text{ sec.}}$

4. To what case in arithmetic division does the division of $a^2+4\,a+4$ by $a+2$ reduce when $a=1$? when $a=10$?

5. Divide $a^3+7\,a^2+2\,a+8$ by $a+2$. If $a=10$ this reduces to the division of what numbers? Are the quotients in the two cases the same? If not, can you reduce one to the other when $a=10$?

6. Divide $t^3+3\,t^2+3\,t+1$ by $t^2+2\,t+1$. Discuss the special case in which $t=10$.

7. Divide $t^3-3\,t^2+3\,t-1$ by $t^2-2\,t+1$. Discuss the special case in which $t=10$.

8. Divide $a^3+3\,a^2b+3ab^2+b^3$ by $a+b$. Discuss the special case in which $a=4$ and $b=1$.

9. Divide $a^2+2\,a+1$ by $a+1$. Discuss the special case in which $a=9$; in which $a=14$.

The student should see the intimate relation between algebra and arithmetic in all this kind of work.

10. What is the average of 2.4, -1.3, 1.4, and -1.8?

11. The average daily temperatures in Duluth during a certain week in January were 10°, 7.5°, $-2°$, $-5.9°$, $-12°$, 0°, 15.7°. Find the average of these daily temperatures.

Exercise. Review of Chapters I–V

1. Write in the common notation the number 4.7×10^{6}.

2. Write 29,000,000,000 in the index notation.

3. Write in the common notation the number 7×10^{-8}.

4. The date of an old book appears as CIƆDCII. Write the date in common numerals.

5. The Romans occasionally, but not commonly, wrote a bar over a numeral to indicate that it was multiplied by 1000. Write in common numerals the number $\overline{\mathrm{XV}}\mathrm{DVII}$.

6. Perform the following additions:

$$\begin{array}{rcrcrcr}
6\,t^2 + 2\,t + 5 & \qquad & 625 & \qquad & 7\,t^2 + 9\,t + 1 & \qquad & 791 \\
t^2 + \,t + 5 & & 115 & & 2\,t^2 + 9 & & 209 \\
\hline
\end{array}$$

Explain the dissimilarity in the appearance of the results in the first two cases; in the last two cases.

7. In Ex. 6 consider the results when $t = 10$.

8. Multiply 3 ft. 9 in. by 4; 3 yd. 9 in. by 4; $3\frac{9}{10}$ by 4; and 39 by 4. Explain the dissimilarity in the results.

9. Multiply 38.725 by 8.6283, carrying the product to three significant figures and performing no unnecessary work.

10. Divide 41.6238 by 6.9434, carrying the quotient to two significant figures and performing no unnecessary work.

11. Multiply 17.43 by 8.24, checking the result by division and also by casting out nines.

12. Divide 12,096 by 10.08, checking the result by multiplication and also by casting out nines.

13. Find the average of 7 lb. 9 oz., $7\frac{3}{4}$ lb., 120 oz., 7.8 lb., 7 lb. 11 oz., and 7.75 lb.

14. Find the average of $a^2 + b$, $a^2 + 2\,b$, $a^2 - b$, $a^2 + 1.5\,b$, $a^2 - \frac{1}{2}\,b$, and a^2.

CHAPTER VI

PERCENTAGE

Purpose of the Chapter. That part of arithmetic which treats of per cents is usually called *percentage.* Percentage is not, however, a separate arithmetic process like addition or multiplication; the arithmetic part is covered in the multiplication and division of decimals, and the important problems are all covered in the applications to business. Nevertheless there are a few principles that are used in the solution of problems of various kinds, and these may conveniently be considered by themselves. The consideration of these principles is the purpose of this chapter.

Terms used in Percentage. There are certain terms in percentage that were formerly used extensively in elementary arithmetic but that are now considered of little value. If the student of higher arithmetic is acquainted with algebra, however, the terms are helpful in the development of formulas, and these formulas make the subject much clearer. It is for such students that these terms are briefly considered.

The student of commercial arithmetic will find the practical applications of percentage given later. His knowledge of per cents and their uses will be much clearer if he approaches the subject assisted by the help that algebra affords.

Students who are not familiar with algebra may omit this chapter. The elementary treatment of per cents has already been given. Only enough topics are taken in this chapter to show the general nature of percentage.

Definitions. A number of which some per cent is taken is called the *base.*

The result of taking any per cent of a number is called the *percentage.*

The number by which the base is multiplied to produce the percentage is called the *rate.*

For example 6% of $200 is $12, and in this case $200 is the base, $12 the percentage, and 6%, or 0.06, the rate.

In this example the rate per cent is 6, but the rate is 6%.

It will be observed that the word "percentage" is used in arithmetic in two senses, one to mean a certain product and the other to mean a chapter like the one now being studied.

The sum of the base and percentage is called the *amount.*

This term is frequently used in interest, where the sum of the principal (which is the base) and the interest (which is the percentage) is called the amount.

Formulas. If we denote the base by b, the percentage by p, the rate by r, and the amount by a, we have, by definition,

$$br = p, \tag{1}$$

$$b + p = a. \tag{2}$$

From these two formulas we easily derive others by the aid of algebra, thus:

Dividing (1) by r, $b = p/r$,

or, dividing (1) by b, $r = p/b$.

Substituting (1) in (2), $b + br = a$;

or, factoring, $b(1 + r) = a$.

Dividing by $(1 + r)$, $b = a/(1 + r)$.

The fraction p/r may, if desired, be written $\frac{p}{r}$, and similarly for other fractions, although a form like p/r is more convenient in printing.

Exercise. Formulas

1. From the formula $b = p/r$ write a rule for finding b in terms of p and r.

It should be observed that a formula of this kind is merely a shorthand statement of a rule.

2. From the formula $r = p/b$ write a rule for finding the rate in terms of the percentage and the base.

3. If the percentage is 34.08 ft. and the rate is 8%, what is the base?

4. If the percentage is \$0.16 and the base is \$3.20, what is the rate?

5. A real-estate dealer charged \$112.50 for selling a piece of property for \$4500. What per cent did he receive on the selling price?

In such a case it is customary to ask what per cent he received, what rate he charged, or what was the rate of his commission. The language varies, but what is wanted is the rate.

The problem may be solved by using the formula given in Ex. 2, or by using the equation

$$r \times \$4500 = \$112.50,$$

or the equation

$$4500\,r = 112.50.$$

It may also be solved without algebra by observing that \$112.50 is the product of \$4500 and the rate, and hence that \$112.50 ÷ \$4500 must give the rate.

6. From the formula $a = b(1 + r)$ write a rule for finding the amount in terms of the base and the rate.

7. From the formula of Ex. 6 find the amount of \$750 at simple interest at 6% for 1 yr.

8. From the formula of Ex. 6 derive a formula for r in terms of a and b, and then write a rule for finding the rate in terms of the amount and the base.

Formulas in Commission. Although it is possible to solve the simpler business problems without using algebraic formulas, it is much easier to solve some of them if we have command of such aids. This is apparent in such problems as relate to commission.

If an agent buys a piece of land for p dollars and his rate of commission is r, then t, the total price which must be paid, is the cost (p) plus the agent's commission ($r \times p$); that is,

$$\begin{aligned} t &= p + rp \\ &= p(1+r). \end{aligned} \tag{1}$$

From the above formula for t we can easily see that

$$p = \frac{t}{1+r} \tag{2}$$

and that

$$r = \frac{t}{p} - 1. \tag{3}$$

In other words, we have formulas for t, p, and r, from which we can easily state rules, whereas by arithmetic alone it would be more difficult to find such rules.

If $p = \$4000$ and $r = 3\%$, from formula (1) we have

$$\begin{aligned} t &= \$4000\,(1.03) \\ &= \$4120. \end{aligned}$$

If $t = \$2912$ and $r = 4\%$, from formula (2) we have

$$p = \frac{\$2912}{1.04} = \$2800.$$

If $t = \$3811$ and $p = \$3700$, from formula (3) we have

$$\begin{aligned} r &= \frac{\$3811}{\$3700} - 1 \\ &= 1.03 - 1 \\ &= 0.03. \end{aligned}$$

Exercise. Formulas in Commission

1. An agent buys a piece of land for a customer and pays \$2500 for it. At $2\frac{1}{2}\%$ commission how much does the agent receive for his work? What is the total price that the customer pays?

2. An agent bought a piece of land for a customer and charged \$180 as his commission, the rate being $2\frac{1}{4}\%$. How much did the agent pay for the land? How much did it cost the customer?

3. An agent bought a piece of land on a commission of $2\frac{3}{4}\%$. The total cost to the customer was \$4932. How much did the agent pay for the land?

4. An agent buys a piece of land for \$5400 and charges his customer \$5535 for it, this sum including his commission. What is the rate of commission?

5. Write a rule derived from formula (1), page 90.

6. Write a rule derived from formula (2), page 90.

7. Show that formula (3), page 90, may be written $r = (t - p)/p$ and write a rule derived from formula (3) and one derived from this formula.

8. Make a problem to be solved by either of the formulas or either of the rules of Ex. 7, and solve it.

9. If an agent sells for me a piece of land for p dollars and his rate of commission is r, then n, the net price which I receive, is the selling price (p) less rp. Write a formula for n, using two forms as in formula (1), page 90. Write a rule derived from this formula.

10. Make a problem to be solved by the formula or rule of Ex. 9, and solve it.

11. In formula (3), if $p = \$1780$ and $t = \$1833.40$, find r.

Formulas in Discount. If a merchant offers goods at a discount, the rate of discount being r, a person buying goods marked m dollars receives a discount of rm dollars. Hence n, the net price of the goods, is given by the formula

$$\begin{aligned} n &= m - rm \\ &= m(1-r), \qquad (1) \end{aligned}$$

writing the formula either way as may be convenient.

By simple algebra we can now derive two other formulas:

$$m = \frac{n}{1-r}, \qquad (2)$$

and

$$r = 1 - \frac{n}{m}. \qquad (3)$$

The following examples illustrate the various formulas:

1. If goods marked \$56 are sold at a discount of 15%, then from (1)

$$n = \$56 - 0.15 \times \$56 = \$47.60,$$

or

$$\begin{aligned} n &= \$56 \times (1 - 0.15) \\ &= \$56 \times 0.85 = \$47.60. \end{aligned}$$

Although it is better in general to write the multiplier first, as it is read, when we substitute in a formula it is more convenient to follow the order of the letters. When the multiplier is written second, the symbol × is read "multiplied by."

2. If goods are bought at a discount of 10% and the net price is \$43.20, then from formula (2) we have

$$m = \frac{\$43.20}{1 - 0.10} = \frac{\$43.20}{0.90} = \$48.$$

3. If goods are marked \$38 but are sold for \$34.96, then from formula (3) we have

$$r = 1 - \frac{\$34.96}{\$38} = 1 - 0.92 = 0.08.$$

Exercise. Formulas in Discount

1. From formula (1), page 92, write a rule for finding the net price when the marked price and the rate of discount are given.

2. Explain how formula (2) is derived from formula (1).

3. From formula (2) write a rule for finding the marked price when the net price and the rate of discount are given.

The student may find it interesting to write this rule without the aid of algebra. It can be done, but it is not so easy.

4. Explain how formula (3) is derived from formula (1).

5. From formula (3) write a rule for finding the rate of discount when the net price and the marked price are given.

The student may try this without algebra, as in Ex. 3.

6. The net price of some goods marked $72 is $63.36. What is the rate of discount?

7. The net price of some goods is $59.80, the rate of discount being 8%. What is the marked price?

8. Some goods marked $45 are sold at a discount of 6%. How much is the discount?

9. In Ex. 8 what is the net price?

10. If d represents the discount, n the net price, and r the rate of discount, find a formula for d and then write this formula as a rule.

11. Make and solve a problem based on Ex. 10.

12. A manufacturer bills some goods to a jobber at $\frac{1}{6}$ off. The jobber remits $610.40, the net price. What is the list price of the goods?

The list price is the price before the discount is taken off. Discounts are often stated as common fractions when this method is simpler than the use of decimals. For example, it is simpler to compute with $\frac{1}{6}$ than with $16\frac{2}{3}\%$.

Formulas in Simple Interest. If a man lends p dollars, known as the *principal*, for t years, the annual rate of interest being r, the interest for 1 yr. is pr dollars, and hence the interest for t years is prt dollars. If i represents the number of dollars of interest, then

$$i = prt. \tag{1}$$

From this formula we can easily derive the formulas

$$p = i/rt, \tag{2}$$

$$r = i/pt, \tag{3}$$

and

$$t = i/pr. \tag{4}$$

The following examples illustrate the various cases represented by the four formulas:

1. If a man lends \$500 for 2 yr. at 6%, the interest is found from (1) as follows:

$$i = prt = \$500 \times 0.06 \times 2 = \$60.$$

2. If a man lends some money for $1\frac{1}{2}$ yr. at 4% and receives \$15 interest, the principal is found from (2) as follows:

$$p = \frac{i}{rt} = \frac{\$15}{0.04 \times 1\frac{1}{2}} = \frac{\$15}{0.06} = \$250.$$

3. If a man lends \$400 for 3 yr. and receives \$60 interest, the rate of interest is found from (3) as follows:

$$r = \frac{i}{pt} = \frac{\$60}{\$400 \times 3} = \frac{1}{20} = 0.05.$$

4. If a man lends \$700 for a certain length of time at 3% and receives \$26.25 interest, the number of years is found from (4) as follows:

$$t = \frac{i}{pr} = \frac{\$26.25}{\$700 \times 0.03} = \frac{\$26.25}{\$21} = 1\tfrac{1}{4}.$$

Exercise. Formulas in Simple Interest

1. From formula (1), page 94, write a rule for finding the interest when the principal, rate, and time are given.

2. Explain how formula (2) is derived from formula (1).

Not all of the formulas given on page 94 are equally important. In elementary arithmetic only the first formula demands much attention, but in higher arithmetic it is desirable to see what a powerful instrument we have in a simple formula.

3. From formula (2) write a rule for finding the principal when the interest, rate, and time are given.

4. Explain how formula (3) is derived from formula (1).

5. From formula (3) write a rule for finding the rate when the interest, principal, and time are given.

6. Explain how formula (4) is derived from formula (1).

7. From formula (4) write a rule for finding the time when the interest, principal, and rate are given.

8. How much is the interest on $275 for 9 mo. at 5%? After solving, write an explanation without directly using algebra or the formula.

9. What principal will produce $27.30 interest in 1 yr. at $3\frac{1}{2}$%? $27.30 interest in 2 yr. at $3\frac{1}{4}$%?

10. At what rate of interest will $1250 produce $75 interest in $1\frac{1}{2}$ yr.?

11. How long will it take $1325 to produce $185.50 interest at 4%?

12. Designating the amount of the principal and interest by a, find a formula for a in terms of p, r, and t.

13. From the formula in Ex. 12 write a rule for finding the amount in terms of the principal, rate, and time.

14. Make and solve a problem applying Exs. 12 and 13.

Exercise. Review of Chapters I–VI

1. Express 6 per cent as a common fraction, as a decimal, and as an integer followed by the sign of per cent.

2. Express 0.000027 in the index notation and as a common fraction.

3. Write 6%, 0.6%, 600%, and 0.06% in words and also as common fractions.

4. Write $\frac{1}{2}\%$ as a common fraction and as a decimal.

5. Write in words the numbers 200.035 and 0.235.

6. Which is the greater, $\frac{1}{5}$ or $\frac{1}{6}$? 5% or 6%? Explain your answers.

7. Which is the stronger solution of boric acid, a $\frac{1}{10}$ solution or a $\frac{1}{12}$ solution? a 10% solution or a 12% solution?

8. Multiply $3t^2+2t+1$ by 3. To what arithmetic multiplication does this reduce when $t=10$?

9. Find 125% of $840 and check the result by casting out nines.

10. If 235% of a number is $1645, what is the number? Check the result by casting out nines.

11. Multiply 2.07263 by 3.1278, carrying the product to three significant figures and performing no unnecessary work.

12. Divide 3.728 by 9.8643, carrying the quotient to three significant figures and performing no unnecessary work.

13. At what rate of interest will the interest on $5750, together with the principal, amount to $6555 in 4 yr.?

14. What principal will produce $47.50 interest at 5% in 2 yr.? in 4 yr.?

15. At what rate must $12,500 be invested to produce $593.75 interest in a year? to produce $1250 interest in 2 yr.?

CHAPTER VII

THRIFT AND INVESTMENT

Nature of the Work. The student has now reviewed from the standpoint of higher arithmetic the fundamental operations of the subject, has seen the intimate relation of all the important types of addition, subtraction, multiplication, and division, and is now ready to review the applications. There would be little value, however, in merely solving the same kinds of problems that are found in elementary arithmetic. Suitable as these may be for beginners, they are not what are needed by the advanced student. We shall therefore consider at this time a somewhat smaller number of problems than is necessary for the elementary pupil, but we shall introduce such problems of daily life as are advanced enough to be interesting as well as valuable to students in the high school and to those who are preparing to teach arithmetic.

There are certain types of problems that everyone is liable to meet. These have to do with our personal accounts, our budgets and expenses, our habits of thrift, and our daily purchases. Others of these problems relate to investments, either in the bank or in securities or real estate; to such social problems as concern community welfare; to our relation to the state and to the common measures of daily life. Not everyone who studies arithmetic may need to know the problems of corporate finance of the farm, of the machine shop, or of the mining industry, but everyone must know the arithmetic of such topics as those mentioned above.

Personal Cash Account. One of the first uses that an adult has for arithmetic is the keeping of his personal cash account. The recent extension of the income tax and the increased necessity for thrift and economy imposed by war conditions has made it important that everyone, whatever his financial standing, should keep a record of his receipts and expenditures. The following is a model personal cash account:

1924					1924				
May	1	Cash on hand	276	85	May	2	Rent	35	
	2	Wages	30			3	Groceries	8	75
	5	R. S. Jones	10			4	Meat bill	5	40
						8	Balance	267	70
			316	85				316	85
May	8	Balance	267	70					

The receipts are entered on the left or debit side of the account and the payments are entered on the right or credit side. The few items given above are sufficient to show the general nature of such an account.

The balance on the credit side is found by subtracting the sum of the expenditures (\$35 + \$8.75 + \$5.40) from the sum of the receipts (\$316.85). This balance is then entered on the credit side and is thence carried to the debit side, where it becomes the cash on hand or the new balance for the next week. Bookkeepers usually write the balance in red.

By the aid of such an account a person can tell at a glance how his balance stands from week to week or from month to month. He can also see where his money goes and where he can effect economies and increase his thrift. Every student of higher arithmetic should begin to keep a personal cash account if he has not already done so.

Exercise. Personal Cash Accounts

Given the following items, make out cash accounts and balance them, checking each operation:

1. Receipts: May 1, cash on hand, $885.95; May 2, M. T. Lothrop, $75.50; May 3, S. N. Goode, $15; May 8, wages, $24. Payments: May 1, grocery bill, $12.75; May 2, meat bill, $8.40; May 2, gas bill, $4.20; May 4, entertainment, $1; May 5, charity, $5. Balanced May 8.

2. Receipts: Feb. 8, cash on hand, $762.80; Feb. 12, salary, $50. Payments: Feb. 8, rent, $25; Feb. 9, grocery bill, $7.20; Feb. 10, meat bill, $5.30; Feb. 12, church, $3; Feb. 13, charity, $1.50; Feb. 14, book, $1.10; Feb. 14, hat, $3.50. Balanced Feb. 15.

3. Receipts: June 1, cash on hand, $1072.85; June 1, salary, $250; June 2, interest, $26.50; June 3, S. P. Jones, $25; June 5, coupons, $140. Payments: June 1, Marshall Field bill, $18.50; June 2, books, $3.20; June 3, suit, $35; June 4, theater tickets, $6; June 6, charity, $10; June 8, board, $12. Balanced June 8.

4. Receipts: Sept. 1, cash on hand, $1542.60; Sept. 1, coupons, $275; Sept. 5, sale of car, $650. Payments: Sept. 1, certificate of deposit, $1000; Sept. 2, Tiffany bill, $75; Sept. 3, books, $7.75; Sept. 4, overcoat, $40; Sept. 4, hat, $4; Sept. 5, expenses to Chicago, $17.80; Sept. 8, church and charity, $12.50; Sept. 8, incidentals for the week, $5.50. Balanced Sept. 8.

5. Receipts: Nov. 15, cash on hand, $127.80; Nov. 15, salary, $150. Expenses: Nov. 15, board, $15; Nov. 15, suit, $37.50; Nov. 15, Lord & Taylor, $32.25; Nov. 16, postage, $5; Nov. 17, express, $0.38; Nov. 20, incidentals, $2.75. Balanced Nov. 22.

Thrift. One result of the great war is the feeling of necessity on the part of everyone that our country must give more attention to the question of thrift. This does not mean that we should hoard money like a miser, but that we should use what we earn to the best advantage not merely to ourselves but to the community in which we live and to the country as a whole.

A study of thrift involves such topics as the preparing of personal or family budgets, economy in purchases, avoidance of waste, careful management, the increase of money by simple interest, the increase by compound interest on money invested in savings banks or in certain government securities, and the various kinds of safe investments. We shall consider these and similar topics in this chapter.

Budget. A systematic plan for future expenditures, based upon income, is called a *budget.*

One advantage of a budget is that a person, a family, or a corporation has a check upon any sudden desire to be extravagant in one or more lines, thus tending to live beyond the income of the year.

In the case of families experience has shown about what per cent of income may safely be assigned to various lines of expenditures, and the same is true with respect to corporations. One cause of financial difficulty in the home or in business is the fact that money is spent freely as it comes in, with no thought of future needs or of emergencies that are likely to arise.

If the student has not prepared a budget for himself, he should prepare one for the ensuing month for his personal inspection. At the end of the month he should see how closely he came to his estimates and should then try to improve upon his budget. By a little experience the budget can be made for a year in advance.

Exercise. Budgets

1. A family with an income of $1500 a year allowed 25% of the income for food, 20% for rent, 20% for clothing, 20% for higher life, including charity, education, books, and entertainment, 10% for other expenses, and the rest for investment. How much money was allowed for each of these purposes?

The student should see that the arithmetic principle involved is merely that of multiplying by a decimal. There is evidently a simple check by adding certain numbers. No solution of any problem should ever be allowed to go unchecked.

2. A family with an income of $2500 a year allowed 25% for food, 20% for rent, 15% for clothing, 25% for higher life, 10% for other expenses, and the rest for investment. How much money was allowed for each of these purposes?

The per cents stated in Exs. 1 and 2 are the results of careful studies of family expenses where the incomes range from $1500 to $4000. The changes in the cost of living do not materially affect the relative per cents of the income expended for various purposes.

3. A study of family expenditures in a manufacturing community showed that 43% of the income went for food, 19% for rent, 11% for clothing, 22% for all other expenses, and the rest for savings. If a family had an income of $1200 and made up a budget on this basis, how much money was allowed for each of these purposes.

Prepare budgets on the basis of Ex. 1, the incomes in the several cases being as follows:

4. $1800. 5. $1850. 6. $1775. 7. $1925.

Prepare budgets on the basis of Ex. 2, the incomes in the several cases being as follows:

8. $2800. 9. $3250. 10. $3750. 11. $3900.

Household Account. Having made a budget for the family expenses for the year, the question then arises as to whether it is being followed. This can be done only by keeping a careful list of household expenditures and summarizing these expenditures at stated times during the year. The following is a convenient form for keeping an account in a home:

1925			RECEIPTS		PAYMENTS	
May	1	Cash on hand	56	95		
		Allowance for week	25			
		Rent			20	
	4	Church and charity			3	
	5	Laundry			1	20
		Help			3	
	7	Entertainment			1	
		Grocery bill			5	80
		Meat bill			4	60
		Gas bill			3	20
		Incidentals for week			1	30
	8	Balance			38	85
			81	95	81	95
	8	Cash on hand	38	85		

The balance should agree with the cash on hand. The balance is found as on page 98. The sum of the receipts should agree with the sum of the payments, this agreement forming a check on the work.

Whether one is living within the limits of the budget cannot be told from day to day or even from week to week, because a bill for a month's rent and one for a month's gas may come in on the same day; but the account for a month will usually tell rather closely.

Exercise. Household Accounts

Make out and balance the following accounts:

1. Receipts: Jan. 1, cash on hand, $88.50; Jan. 1, allowance for week, $30; Jan. 2, loan repaid by Mrs. Sinclair, $5. Payments: Jan. 2, rent, $17.50; Jan. 3, grocery bill, $6.20; meat bill, $4.20; gas bill, $2.90; Jan. 4, entertainment, 75¢; Jan. 5, books, $1.75; charity, $2.50; Jan. 6, hat, $3.50; Jan. 7, church, $1; Jan. 8, contribution to hospital, $5; Jan. 8, sundries for the week, $1.35. Balanced Jan. 8.

2. Receipts: Oct. 15, cash on hand, $128.50. Payments: Oct. 15, waists, $7.50; Oct. 16, dress goods, $12.75; board, $10; Oct. 17, shoes, $4.50; umbrella, $1.50; books, $2.20; Oct. 18, dressmaker, $5.40; magazines, 60¢; Oct. 19, church and charity, $2.25; Oct. 20, gloves, $1.80; Oct. 21, trimming, $1.60. Balanced Oct. 22.

3. Receipts: July 1, cash on hand, $43.75; allowance for week, $27.50; July 3, sale of canned fruit, $12.25; July 5, sale of marmalade, $8.75; July 6, sale of jelly, $7.50. Payments: July 2, telephone bill, $2.25; July 3, grocery bill, $15.42; July 4, entertainment, $1.50; July 5, meat bill, $6.20; July 6, dressmaker, 4 da. @ $1.75; July 7, church and charity, $2.75; July 8, dress goods, $8.75. Balanced July 8.

4. Write an account setting forth the reasonable expenses for a week for a girl or boy in your class, considering the added necessity for economy imposed as a result of the war.

5. Write an account setting forth the reasonable expenses for a week for a family of two adults and four children, the prices being such as are found in your community. Include the ordinary expenses for food, light, clothing, entertainment, and charity. Take as the cash on hand $12.30 and the weekly allowance as $25.

Living within the Budget. In order to find whether one is living within one's budget it is necessary to summarize one's expenditures at stated periods, say every month or every three months. The following problem will serve to illustrate the method.

A family has found that the expenses for food should average about 27% of the income and hence has allowed 27% in its budget for this purpose. The annual income is $1680 and the bills for food were $31.80 in April, $32.75 in May, and $38.90 in June. It is desired to know if the family is living within its budget in this respect, what per cent it is expending for food, and how much has been either over-expended or saved.

Since the income is $1680 a year, for 3 mo. we have

$$\tfrac{1}{4} \text{ of } \$1680 = \$420.$$

Since 27% of $420 is allowed for food in 3 mo., we have

$$27\% \text{ of } \$420 = \$113.40, \text{ the allowance.}$$

The expenses for food are found by adding:

$$\$31.80 + \$32.75 + \$38.90 = \$103.45, \text{ expenses for food.}$$

Since some per cent of $420 is $103.45, we have the product ($103.45) of two numbers (some per cent and $420) and one of these numbers ($420) given to find the other number (some per cent). Hence we divide the product ($103.45) by the given number ($420), and we find the other number, thus:

$$\$103.45 \div \$420 = 0.246$$
$$= 24.6\% +, \text{ per cent spent for food.}$$

A much simpler solution is affected by algebra, thus:

$$420\,x = 103.45,$$

whence $x = 24.6\% +.$

Furthermore, $113.40 – $103.45 = $9.95, amount saved.

That is, the family lived within the budget in this respect. It spent 24.6% for food instead of 27%, and it saved $9.95 on its food bill in 3 mo.

Exercise. Living within the Budget

1. A family budget allows 26% of the income for food. The income is $150 a month. The food cost $36.70 last month. This was what per cent of the income? It is how many dollars less than the budget allows?

2. A family budget allows 22½% of the income for such expenses as charity, education, books, and entertainment, summarizing all these items as relating to higher life. The income is $2000 a year. The household account shows that the expenditures for higher life were $28.75 in January, $36.20 in February, and $41.75 in March. These expenditures are what per cent of the income for that period? By how much do they differ from the amount allowed in the budget? Are they within the amount allowed or do they exceed it?

3. A family budget allows 17½% of the income for clothing. The income is $125 a month. The household account shows that the expenditures for clothing in the first 6 mo. of the year were $8.75, $11.50, $28.70, $24.60, $35.75, and $25.65. These expenditures are what per cent of the income for that period? By how much do they differ from the amount allowed in the budget, and in which way?

A person's income and the expenditures for a certain purpose being respectively as stated below, find the per cent of expenditure to income:

4. $750, $87.50.
5. $925, $72.25.
6. $1250, $68.50.
7. $1400, $120.25.
8. $1500, $225.
9. $1750, $58.75.
10. $2200, $64.80.
11. $2500, $125.50.

The student should use his judgment as to how far to carry the results. A suggestion has already been made in this matter.

Economy of Purchase. Thrift is fostered not only by limiting our purchases to our needs and our reasonable comforts but by making these purchases with due attention to economy. It is never economy to buy a thing simply because it is cheap, and it is often poor economy to buy the things that we need at the cheapest price. Sometimes it is good economy to buy in quantity when we need a considerable amount, can easily store it, and can use it within a short time. If, however, the articles depreciate in value by keeping, or if we lose money in the form of interest on the cost through not using them for a long time, or if we are thereby tempted to use more than we need, it is poor economy to buy in quantity.

In the problems on page 107 we assume that the material can easily be stored and that it is all needed within a reasonable time.

Discount. In general, any allowance made on a money transaction is called a *discount.* Thus, if we buy some goods, the seller may offer us an allowance on the price if the money is paid at once or within a stated number of days.

Similarly, if we sell to a bank a note for $100 due in 30 da., the bank will pay us $100 less a certain discount.

The terms relating to discounts, marked prices, list prices, and net prices are familiar from the study of elementary arithmetic.

If we need the goods and if the quality is satisfactory, it is often economy to take advantage of discounts offered at stores. Thus, if a suit is marked $40 and can be bought at the close of the season at 15% off, for cash, we have the following:

$$15\% \text{ of } \$40 = \$6, \text{ the discount,}$$

and

$$\$40 - \$6 = \$34, \text{ the net price.}$$

We might have found the net price in one step by multiplying $40 by 100% − 15%, that is, by 0.85.

Finding discounts are merely cases of multiplying by decimals.

Exercise. Economy and Discount

1. When a cereal is quoted at 18¢ per package or $2 per dozen, what per cent does a purchaser save on the package rate if he buys by the dozen?

As stated on page 106, it is assumed that the purchaser needs the larger amount, will use it within a reasonable time, has facilities for storing it, and will not waste it through having it on hand.

In this case the purchaser saves 12 × $0.18 – $2, or $0.16. The problem, therefore, is to find what per cent 16¢ is of 12 × $0.18. In such cases carry the result to the nearest tenth of 1%.

2. A housewife can buy coffee at 35¢ a pound or 5 lb. for $1.65. What per cent does she save on the pound rate by purchasing 5 lb. at a time?

3. When olives are selling at 40¢ per bottle or $4.50 per dozen bottles, what per cent is saved on the single rate by purchasing by the dozen?

4. When fancy bacon sells at 50¢ a jar or $5.50 per dozen jars, what per cent is saved on the single rate by purchasing by the dozen?

5. What is the net price of a suit that is marked $32 but can be bought at 15% off?

6. What is the net price of a hat that is marked $3.75 but can be bought at $\frac{1}{3}$ off?

As already stated, discounts are often quoted in common fractions instead of per cents. In this case $\frac{1}{3}$ off is the same as 33$\frac{1}{3}$% off.

7. A coat marked $18 was sold for $14.40. What was the per cent of discount?

8. A cloak was bought at a discount of 30%, the net price being $15.40. What was the marked price?

Such a problem is not nearly so practical as Ex. 7, and Ex. 7 is not nearly so practical as Exs. 1–6.

Investing Money. It is the duty of everyone who has not already done so to try to save enough money for such emergencies as sickness, loss of property by accident, loss of position through hard times, unforeseen increases in taxes due to the war, or unexpected necessity for assisting others. It is also necessary to provide for old age and for those who may be dependent upon us later in life. For this reason it is necessary to know what kinds of investments offer reasonable security and fair financial return.

A very desirable way to invest money is in government bonds or their equivalent. The rate of income is not high, but the security is the best that is possible, and everyone who buys such a bond has the feeling that he is helping his government in a great cause.

Conditions resulting from the war vary so rapidly and the form of government securities is subject to such changes that it is impossible to describe all types of these securities, nor is it necessary to describe them for the purposes which we now have in view.

The national government also conducts a postal savings bank which pays a low rate of interest but is absolutely safe and very convenient.

Aside from these methods of investing, one of the best means is through the ordinary savings banks. These banks are usually supervised with great care by the state, and while the rate of interest is not high the banks are so safe as to be extensively used by those who have small sums to invest. After accumulating a sufficient sum in this way in a savings bank a depositor may withdraw his money from the bank and invest it in some of the other kinds of securities that are described in the next few pages.

The details of the method of investing money in government securities or in savings banks have nothing to do with the arithmetic questions involved and hence are not discussed in this chapter.

Exercise. Investing Money

1. If you deposit $250 on Jan. 1 in a savings bank that pays 4% interest per annum, the interest being paid semi-annually, and if you leave the money 6 mo., how much interest is due July 1?

Such simple questions in interest will be understood from the work in elementary arithmetic and from that given on page 94.

2. A boy finds that he can save on an average 15¢ a day every day in the year except Sunday. In an ordinary year that has 52 Sundays how much can he save? How much can he save in a leap year having 53 Sundays?

3. A man who spent for tobacco 75¢ a day, including Sundays, gave up this habit and deposited in a savings bank the money that he thus saved. Not counting the interest, how much money would he save in 10 yr., there being three leap years in this period?

4. If a man on Jan. 1 has $875 in a savings bank that pays 2% every 6 mo., and if he leaves the money for that period, how much interest is due July 1?

5. It is interesting to see how much a man would save in a month if he put $1 in a savings bank on the first business day of a month, $2 on the second, $4 on the third, $8 on the fourth, and so on, doubling each day the amount of the preceding day. If there were 25 business days in that month, how much would he save?

Such a problem is obviously fictitious. It is interesting, however. Such students as may have studied geometric series in algebra will know a short method of solving this problem. Other students will find it an excellent example in accurate computation. The interest in the solution is increased if a rough estimate of the result is made in advance and then written down so that the computer can compare it later with the real result.

Compound Interest. If a person invests his money in a savings bank the simple interest is added to the principal and becomes part of it whenever the interest becomes due. The principal is then said to draw *compound interest.*

Both principal and accumulated interest then draw interest for the next interest period.

The sum of the principal and compound interest is called the *compound amount.*

The theory of simple interest has been treated on page 94 as fully as necessary for our purposes. The practical questions relating to simple interest do not concern the investor so much as the borrower, and hence they will be considered in connection with banking.

If a person continually reinvests his simple interest when it becomes due, he in effect receives compound interest.

For example, if you put \$100 in a savings bank on Jan. 1, and if the bank pays 4% per annum, compounded semiannually, you will have

1.02 × \$100, or \$102, at the end of 6 mo.,
1.02 × \$102, or \$104.04, at the end of 1 yr.,
1.02 × \$104.04, or \$106.12, at the end of $1\frac{1}{2}$ yr.,
1.02 × \$106.12, or \$108.24, at the end of 2 yr.,
1.02 × \$108.24, or \$110.40, at the end of $2\frac{1}{2}$ yr.,
1.02 × \$110.40, or \$112.61, at the end of 3 yr.

Stated another way, at the end of two interest periods you will have $1.02^2 \times \$100$; at the end of three interest periods, $1.02^3 \times \$100$; and so on. Hence, if P is the principal, A the amount, r the rate per interest period, and n the number of interest periods, we have

$$A = P(1 + r)^n.$$

In the case of savings banks the interest is usually computed only on the dollars, not on any fractional parts of \$1. Sometimes no interest is allowed unless the money has been on deposit at least 3 mo. It is usually computed on Jan. 1 and July 1, or on the day preceding.

Exercise. Compound Interest

1. From the formula $A=P(1+r)^n$ derive a formula for P in terms of A, r, and n.

2. From the formula $A=P(1+r)^n$ write a rule for finding the compound amount in terms of the principal, rate per interest period, and number of interest periods.

3. Given that $1.035^4=1.148$, find the compound amount of $1000 at the end of 4 yr. at $3\frac{1}{2}\%$, compounded annually.

4. Given the principal and the compound amount, how do you find the compound interest? Illustrate by using the result of Ex. 3.

5. If i is the compound interest, write a formula for i in terms of P, r, and n. Then find the value of i when $P=\$800$, $r=4\%$, and $n=3$.

6. Find the compound amount of $1500 for 3 yr. at 5%.

Interest is to be compounded annually unless the contrary is stated.

Find the compound amount of each of the following:

7. $2500, 2 yr., 4%.

8. $1750, 3 yr., 3%.

9. $100,000, 2 yr., 5%.

10. $900,000, 2 yr., 4%.

11. A war savings stamp was bought on Jan. 1, 1918, for $4.12. On Jan. 1, 1923, it was redeemed by the government by the payment of $5. Show that this was approximately equivalent to allowing 4% interest, compounded quarterly.

At the end of 3 mo. the compound amount was $1.01 \times \$4.12$; at the end of 6 mo., $1.01^2 \times \$4.12$; at the end of 60 mo., $1.01^{20} \times \$4.12$. Unless the student has tables to assist him, the simplest way for raising 1.01 to the 20th power is to square it; then to square this result, which will give the fourth power; then to square this, giving the eighth power; then to square again, giving the 16th power; and then to multiply this by 1.01^4, giving the 20th power. He should use his judgment as to rejecting unnecessary figures.

Simple and Compound Interest Compared. Persons having small amounts of money invested usually take the interest when it is due and spend it for living expenses. If, however, they immediately invest at the same rate all the interest as it becomes due, they will thus receive compound interest. This is practically what is done by such large investors as life-insurance companies, and what most people can do if they make a serious effort.

If a person bought a war savings stamp at $4.12 and it was redeemed at $5 after 5 yr., he would be receiving substantially 4% interest compounded quarterly; but if he had invested the $4.12 at 4% simple interest for 5 yr., he would have $4.94 at the end of that time. In other words, in this case the difference is only 6¢. If, however, he had invested $4120, the difference would have been $60, which is certainly an amount that is worth saving.

Considering the case of a large investor, suppose that a life-insurance company can invest $5,000,000 safely at 5%, interest payable semiannually, reinvesting the interest at the same rate as the interest becomes due. How much more income will the company receive in 10 yr. if it invests at compound interest than it would have received had it merely taken the simple interest and not reinvested it as stated?

Since the interest is compounded semiannually the rate is 2½% per period, and there are 20 periods. We therefore have

$$A = P(1 + r)^n = \$5{,}000{,}000 \times 1.025^{20}.$$

Business men use tables to find the value of 1.025^{20}, and so we shall suppose that we know this value to be, as it is, 1.6386164. We then have

$$A = \$5{,}000{,}000 \times 1.6386164 = \$8{,}193{,}082.$$

But the amount at simple interest is $7,500,000.
Hence the gain through investing at compound interest is

$$\$8{,}193{,}082 - \$7{,}500{,}000, \text{ or } \$693{,}082.$$

Exercise. Simple and Compound Interest

1. A life-insurance company can invest $1,500,000 at an average rate of $4\frac{1}{2}\%$, payable semiannually, and can reinvest at the same rate all interest as it becomes due, thus securing compound interest. How much more interest will it receive in this way in 20 yr. than it would have received at simple interest at $4\frac{1}{2}\%$?

The company receives $2\frac{1}{4}\%$ semiannually, and there are 40 half years. We therefore have $1.0225^{40} \times \$1,500,000$ as the compound amount. Since the computation of 1.0225^{40} is too laborious, special tables are constructed giving such values. The student may take 2.4351890 as the value of 1.0225^{40} and use this in the solution.

2. A man has $2000 on deposit in a savings bank on Jan. 1. The bank pays 4% per annum, compounded semiannually. How much more interest will the man receive in 10 yr. than he would have received by investing the money for the same period at 4% simple interest?

The student may take the value of 1.02^{20} as 1.4859474.

3. In Ex. 2 how much less will the man receive than he would have received by investing the money for the same period at 6% simple interest?

4. Which will produce the more interest in 5 yr., an investment of $2500 in a savings bank that pays 4% per annum, compounded semiannually, or an investment of the same sum in government bonds paying $4\frac{1}{4}\%$, this interest not being reinvested when paid?

The government bonds pay simple interest semiannually, and the interest must be reinvested in order to compound it.

5. How long will it take a sum of money to double itself at 6% simple interest? Show that it will take less than 13 yr. for money to double itself at 6% compound interest.

Investments in Stocks and Bonds. Formerly people who had money to invest lent it to borrowers and took their notes, or else they put their money in banks, or purchased land, or became connected with some partnership. At present most great undertakings are carried on by corporations, and these corporations are allowed by law to issue stocks and bonds. Many people who have money to invest buy these *securities*, as they are called.

Difference between Stocks and Bonds. By purchasing some shares of stock a person becomes part owner of the corporation. For example, if a street railway issues stock to the extent of $1,000,000, a person who owns $1000 worth of stock owns $\frac{1000}{1000000}$, or $\frac{1}{1000}$, of the railway.

By purchasing a bond a person simply lends money to the corporation issuing the bond. For example, a man who buys a government bond for $1000 is thereby lending $1000 to the government.

A bond specifies the time when it is to be paid. It may do this by giving a precise date, like July 1, 1940, or by giving certain limits, as on any interest date between July 1, 1940, and July 1, 1950.

Bonds pay a fixed rate of interest, this interest usually being paid every 6 mo. Stocks pay *dividends* at stated times, but the amount of the dividends depends upon the amount that the corporation has earned. For example, a government $4\frac{1}{4}\%$ bond for $1000 pays $2\frac{1}{8}\%$ every 6 mo., or $21.25, while a railway company that paid a semiannual dividend of $2\frac{3}{4}\%$ last January might pay only 2% next July.

It is assumed for the present that the student is familiar with the general nature of stocks and bonds from his work in elementary arithmetic. Further details are discussed on pages 116 and 144. Owing to rapidly changing conditions due to the war, we shall not consider the question of any tax on the transfer of stocks and bonds.

Exercise. Stocks and Bonds

1. A person owns a $1000 bond that pays 5% per annum. How much is his income from this bond in 4 yr.?

2. A man owns $1000 worth of stock in a manufacturing company. The company's business is such that it pays a regular quarterly dividend of $1\frac{3}{4}\%$. How much does the man receive in dividends in 4 yr.?

3. A certain corporation issues bonds bearing 5% interest and pays regularly an annual dividend of 5% on its stock, there being no prospect of increased dividends, but some chance of a decrease. Which is the better plan for a person who has $2000 to invest, to buy the bonds or to buy the stock? What is the annual income in each case? State your reason for preferring one security to the other.

4. A man has $7000 of 5% bonds, $5000 of $4\frac{1}{2}\%$ bonds, and $8000 of stock that regularly pays 2% quarterly. Find his total annual income from these securities.

Find the annual income on the following amounts of stock, assuming the dividends to continue as stated:

5. $7500, $1\frac{3}{4}\%$ quarterly.
6. $8500, $2\frac{1}{4}\%$ quarterly.
7. $4000, $1\frac{1}{2}\%$ quarterly.
8. $15,000, 8% annually.
9. $4500, 2% semiannually.
10. $6000, 3% semiannually.

Find the annual income on the following amounts of bonds:

11. $2500, 5%.
12. $3500, $5\frac{1}{2}\%$.
13. $4500, 6%.
14. $7500, $4\frac{3}{4}\%$.
15. $25,000, $4\frac{1}{4}\%$.
16. $37,500, $4\frac{1}{2}\%$.

17. A company which has a capital of $2,500,000 divides $106,250 in dividends. How much does a man receive whose stock represents 5% of the capital of the company?

18. In Ex. 17, suppose that the man owed $7\frac{1}{2}\%$ of the stock.

Above Par and Below Par. If a stock pays uniformly a high rate of dividend, that is, more than can be received from other safe investments, people will be so anxious to buy it that they will pay more than $100 for a $100 share. The stock is then said to be *above par.* If a $100 share can be bought for just $100, the dividends are about on a par with other investments, and the stock is said to be *at par.* If the dividends are low, the public will usually not be so interested in owning the stock, and the stock will then be *below par.*

Buying Stock. A purchaser usually buys and sells stock through a *broker*, generally in a place called a *stock exchange*, a kind of auction room for such business.

Brokerage. The broker charges *brokerage* or *commission*, usually $\frac{1}{8}\%$ of the par value of the stock, that is, $12\frac{1}{2}$¢ per $100 par value of the stock.

Meaning of Stock Quotations. A quotation of $118\frac{3}{4}$ means that a share of stock, if its par value is $100, will cost the purchaser $\$118.75 + \$\frac{1}{8}$, or $12\frac{1}{2}$¢, brokerage. Thus the buyer will pay $\$118.87\frac{1}{2}$ per share, while the seller, who must also pay brokerage of $\$\frac{1}{8}$, or $12\frac{1}{2}$¢, a share to his broker, will receive $\$118.75 - \$0.12\frac{1}{2}$, or $\$118.62\frac{1}{2}$ per share.

In stock quotations fractions are usually expressed in halves, fourths, or eighths, and occasionally in sixteenths. Fractional parts of a share cannot, in general, be bought.

How much must a buyer pay for 10 shares of stock quoted at $137\frac{1}{8}$, and how much will the seller receive, allowing the usual brokerage?

One share costs $\$137\frac{1}{8} + \$\frac{1}{8}$ brokerage, or $\$137\frac{1}{4}$.
Therefore 10 shares cost $10 \times \$137\frac{1}{4}$, or $1372.50.
The seller, however, receives for each share $\$137\frac{1}{8} - \$\frac{1}{8}$, or $137.
Hence the seller receives for 10 shares $10 \times \$137$, or $1370.
That is, each man pays his broker $1.25.

Exercise. Stocks

1. A corporation in which the par value of a share is $100 declares a quarterly dividend of $1\frac{1}{4}\%$. How much does a man receive who owns 25 shares?

In all cases involving stocks $100 will be taken as the par value of a share unless the contrary is stated.

2. A manufacturing company declares a semiannual dividend of $3\frac{1}{4}\%$. How much does an investor receive who owns 15 shares?

3. A corporation with a capital of $250,000 sets aside $17,500 of its earnings this year for dividends. What is the rate of dividend?

4. If a corporation with a capital of $375,000 apportions $13,125 in dividends, how much does the owner of 25 shares of stock receive?

5. A man buys 50 shares of stock quoted at $98\frac{7}{8}$. Allowing the usual brokerage, how much does the stock cost him?

6. A man sells 30 shares of stock quoted at $107\frac{5}{8}$. Allowing the usual brokerage, how much does he receive?

7. An investor bought 50 shares of D. L. and W. R. R. stock quoted at $173\frac{1}{8}$ and sold these shares a few days later at $174\frac{1}{4}$. Not considering loss of interest through holding the stock, what was his net gain?

In actual practice the investor should consider the loss of interest and should include any dividend that might have been allowed while he held the stock.

8. A man bought 75 shares of Union Pacific at $118\frac{5}{8}$, kept it a week, and sold it at $117\frac{7}{8}$. Find his net loss.

It is customary to speak of standard stocks as in Ex. 8 instead of adding the words "railway stock." The prices are those quoted in the newspapers; brokerage must be considered.

Other Kinds of Investments. Besides depositing money in a savings bank and purchasing stocks or bonds, a person may lend money on a promissory note or on a bond and mortgage of a private individual; he may invest the money by purchasing real estate and renting this real estate to tenants; or he may invest the money in business.

For our present purposes we may assume that the student is familiar with the nature of a promissory note from his study of simple interest in elementary arithmetic. A bond given by a private individual is usually secured by a mortgage on some of the individual's property, this mortgage allowing the creditor, that is, the one who lends the money, to sell the property and thus pay the bond if the debtor fails to pay it or to pay the interest when due.

The arithmetic involved in all this work is merely that of simple interest, so that we need not elaborate the subject further. So far as the arithmetic calculations are concerned, a bond is the same as a promissory note. The chief difference is that a note ceases to be good if it runs a few years (in most states 6 yr.) after it is due or after the last payment is made upon it; but a bond is a more formal document and runs for a longer period after it is due or after the last payment is made upon it (in most states 20 yr.).

It depends largely upon the investor as to what kind of investment is the best for him. If a man is accustomed to buying and selling land, he may quite safely invest in this way. If a woman has but little money and wishes this invested with perfect safety, she should put the money in a good savings bank, buy thoroughly reliable bonds, or buy stock that does not pay an extravagant dividend and that is known to banks as having paid dividends regularly for a considerable number of years.

If one is not familiar with investments, it is desirable to select only conservative and well-tried securities.

Exercise. Other Kinds of Investments

1. A man lends $750 for 8 mo. at 6%. Find the amount of principal and interest.

2. An investor lends a farmer $2500 at 6%, taking his bond and a mortgage on his farm. How much is the annual interest on the bond?

3. Mr. Jacobs bought a house for $9000 and rented it for $56 a month. He paid annually $100 for taxes, $9 for insurance, and $90 for repairs. At the end of 3 yr. from the time when he purchased the property he sold it for $10,400. How much more did he gain than he would have gained by investing the money for 3 yr. at 6%?

Find the interest on the following amounts for the times and rates specified:

4. $750, 90 da., $5\frac{1}{2}$%.
5. $875, 6 mo., 6%.
6. $1250, 8 mo., 6%.
7. $2500, 3 yr. 2 mo., 5%.
8. $3750, 1 yr. 2 mo. 15 da., $5\frac{1}{2}$%.
9. $2400, 2 yr. 1 mo. 10 da., $4\frac{1}{2}$%.

In ordinary commercial transactions promissory notes do not run as long as stated in Exs. 7–9, but an investor may let a note run longer than is the custom with banks. Usually an investor requires that the interest be paid semiannually or annually.

10. A man having $5000 to invest bought a house with this amount. He paid 2% commission to the agent who attended to the purchase, 2% taxes on an assessed valuation of $4000, $250 for repairs, and $50 for insurance. If he had not purchased the house he could have lent the $5000 at 6%. During the year he rented the house for $75 a month. At the end of the year he found that he had spent $150 of his own time in looking after the place, and so he decided to sell it. He sold it for $4800, paying the agent 2% commission for making the sale. Find his gain or his loss on the transaction.

Life Insurance. The business of assuming the risk of financial loss by death, bodily injury, sickness, fire, tornado, flood, or other contingent event is called *insurance.*

The subject of life insurance is closely related to the subject of thrift and investment.

Policy. A contract of insurance stating the terms under which an insurance company agrees to pay a certain sum to the insured or to some other beneficiary named in the contract is called an *insurance policy*, or simply a *policy.*

Premium. The sum paid by the insured to the insurance company for assuming the risk stated in the policy is called the *premium.*

If the premium is all paid at one time, it is called a *single premium*; but if it is paid in annual installments, each installment is called an *annual premium.*

For example, if a man takes out a policy with a life-insurance company and agrees to pay $17.52 a year for life, in return for which the company agrees to pay $1000 to his wife or children upon his death, he has taken out an ordinary life policy and his annual premium is $17.52.

No company would be justified in issuing such a policy to only one person; but if a large number of such contracts are made, the deaths will ordinarily occur with sufficient regularity to enable the company to issue such policies with entire safety. The fundamental principle of insurance is that of probability based on a large number of cases.

The annual premium that must be paid for a life policy increases with the age of the insured at the time he takes out his policy. The premiums to be paid on various kinds of policy are computed by the *actuaries* of the companies and are given in tables supplied to the agents. The mathematics involved in preparing these tables is very elaborate, but the use of the tables requires little technical knowledge.

Exercise. Life Insurance

1. A man aged 22 yr. takes out a life policy for $5000, the annual premium being $18.30 per $1000. How much is the total annual premium?

2. A man aged 22 yr. takes out a life policy for $15,000, the policy allowing the premiums all to be paid in 10 payments, each of $42.74 per $1000. How much is the total annual premium, and why is the rate so much higher than in Ex. 1?

This kind of policy is known as a 10-payment policy.

3. A man aged 22 yr. takes out a life policy for $7500 on the 15-payment plan, the rate being $31.98 per $1000. How much is the total annual premium, and why is the rate higher than in Ex. 1 and lower than in Ex. 2?

4. A man aged 22 yr. takes out a 10-year endowment policy for $25,000, the rate being $101.42 per $1000. This policy insures the man's life for 10 yr., and at the end of that period the company will pay the face of the policy, $25,000, to the insured if he is then living. Find the total annual premium, and state why the rate is so much higher than the rates mentioned in Exs. 1–3.

5. If a man takes out a life policy for $12,500 at $24.48 per $1000, what is the sum of the premiums that he must pay in 20 payments?

6. A man takes out a $7000 life policy in one company at $25.80 per $1000 and a $4500 life policy in another company at $25.65 per $1000. What do the two premiums together amount to in a year?

7. A man takes out a $15,000 life policy at $28.75 per $1000. He dies just after eight annual premiums have been paid. How much does the beneficiary receive? How much does the beneficiary receive in excess of the premiums paid?

Exercise. Review of Chapters I–VII

1. Express $2\frac{7}{8}\%$ as a decimal.

2. Express 0.005 and 1.005 as per cents.

3. An ancient monument bears the date Mcccclviiii. Write the date in common numerals.

4. A modern book bears the date MCMXVII. Write the date in common numerals.

5. Multiply 427.38 by 296 and check the result by casting out nines.

6. Divide 198,614,250 by 529,638 and check the result by casting out nines.

7. Multiply $\frac{5}{8}$ by $\frac{3}{8}$ and explain each step of the process.

8. Divide $\frac{5}{8}$ by $\frac{3}{8}$ and explain each step of the process.

9. Find the value of $0.1 + 0.1$; of $0.1 - 0.1$; of 0.1×0.1; of $0.1 \div 0.1$.

10. Explain why it is necessary to reduce the fractions to a common denominator if we wish to add $\frac{2}{3}$ and $\frac{3}{4}$, but not if we wish to multiply $\frac{2}{3}$ by $\frac{3}{4}$.

11. How much is 5% of $175? 0.05% of $175?

12. If 103% of a certain number is 283.25, what is the number?

13. What number increased by 100% of itself is equal to 175? decreased by 100% of itself is equal to 175? decreased by 100% of itself is equal to zero?

Not all of these questions have simple numerical answers.

14. When a sum of money is multiplied by 2, it is increased by what per cent of itself? Give an illustration. By what per cent of itself is a sum of money increased when multiplied by 3? by 1.04? by $2\frac{1}{2}$? by 11? by $\frac{1}{2}$? by $\frac{3}{4}$?

Notice that the wording should be changed in two of these cases.

CHAPTER VIII

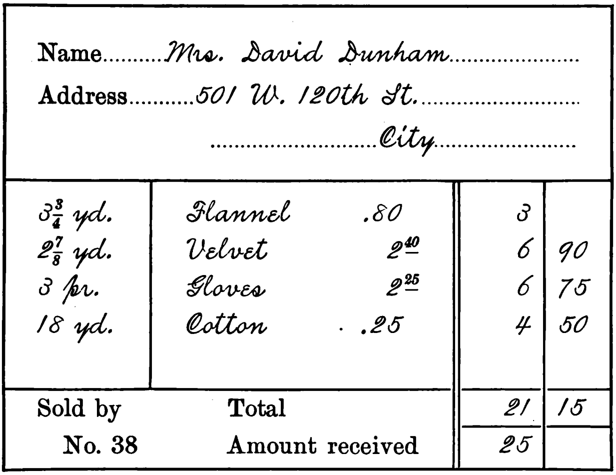
MERCANTILE ARITHMETIC

Nature of Mercantile Arithmetic. The boy or the girl seeking employment in a store needs to know the technical terms of ordinary mercantile transactions, the use of sales checks, bills, invoices, and inventories, the methods of computing profit and loss, and the general financial problem involved. These are the topics that we shall consider in this chapter. We shall not consider those subjects which are of so technical a nature that they must be studied apart from arithmetic if one wishes to acquire proficiency in the work.

Technical Terms and Abbreviations. The following abbreviations and symbols are used in commercial work:

@	at	f.o.b.	free on board
a/c	account	frt.	freight
bal.	balance	I., inv.	invoice
B/L	bill of lading	l.p.	list price
bot.	bought	M	1000
C	100	mdse.	merchandise
chg.	charge	#	number (as in #5)
chk.	check		pounds (as in 5#)
√	correct	*o/d*	on demand
c/o	care of	pc.	pieces
disc.	discount	pkg., pa.	packages
doz. or dz.	dozen	sund.	sundries
Dr.	debit, debtor	via	by way of
Ex., exp.	express	$^{1, 2, 3}$	fourths ($2^3 = 2\frac{3}{4}$)

Cash Check. In large stores the clerks are required to fill out cash checks showing the goods sold. Such checks vary in form, but the following kind is frequently used:



Name..........Mrs. David Dunham....................

Address...........501 W. 120th St...........................

..........................City.......................

$3\frac{3}{4}$ yd.	Flannel	.80	3	
$2\frac{7}{8}$ yd.	Velvet	$2^{\underline{40}}$	6	90
3 pr.	Gloves	$2^{\underline{25}}$	6	75
18 yd.	Cotton	.25	4	50
Sold by	Total		21	15
No. 38	Amount received		25	

In filling out such checks the salesman should perform all computations mentally if possible. For example, in the first of the above items he should think "$2.40 and $0.60 are $3"; in the second one, "$4.80 and $2.10 are $6.90." In all cases advantage should be taken of short methods. For example, in multiplying 25¢ by 18 he should see that he need merely divide $18 by 4, or $9 by 2; or else he should see that $5 – $0.50 = $4.50.

The cashier finds the change due by starting with $21.15 and subtracting by the making-change method, thus: "21.15 and 0.10 is 21.25, and 0.25 is 21.50, and 0.50 is 22, and 3 is 25, making $3.85," which amount is then returned to the customer as change. The convenience of making change in this way is influencing our method of subtraction at present.

Exercise. Cash Checks

Make out cash checks for the following sales and state the amount of change due, taking advantage of short methods and doing the computation mentally as far as possible:

1. 25 yd. gingham @ 24¢, $12\frac{1}{2}$ yd. linen @ \$1.20, $2\frac{1}{2}$ yd. muslin @ 32¢, $3\frac{3}{8}$ yd. velvet @ \$3.20. Amount received, \$50.

2. 15 yd. dimity @ 3 yd. for \$1, $7\frac{1}{2}$ yd. madras @ 40¢, 16 yd. silk @ \$1.25, 12 yd. cheesecloth @ 15¢. Amount received, \$30.

3. $6\frac{1}{4}$ yd. ribbon @ 16¢, $7\frac{1}{2}$ yd. suiting @ \$1.10, 16 yd. velveteen @ \$1.10, 26 yd. taffeta @ \$1.50. Amount received, \$75.

4. 8 doz. pr. hinges @ \$5.25, 6 doz. table knives @ \$10.50, 9 doz. locks @ \$5.10, $3\frac{1}{2}$ doz. bolts @ \$3. Amount received, \$175.

8 doz. pr. hinges @ \$5.25 means 8 doz. pairs at \$5.25 per dozen.

5. 3 lb. tea @ 75¢, 5 lb. coffee @ 40¢, 2 lb. cocoa @ 60¢, 10 lb. codfish @ 12¢, 5 lb. tapioca @ 10¢. Amount received, \$10.

6. 25 yd. silk @ 96¢, 8 yd. tucking @ $12\frac{1}{2}$¢, 4 doz. buttons @ $12\frac{1}{2}$¢, 16 yd. cheviot @ \$2.25. Amount received, \$75.

7. 4 pr. gloves @ \$1.75, 5 yd. crépon @ \$4.20, $12\frac{1}{2}$ yd. cheviot @ \$1.20, 12 yd. velvet @ \$2.50. Amount received, \$75.

8. 8 lb. sugar @ $12\frac{1}{2}$¢, 6 lb. butter @ 50¢, 3 lb. cheese @ 25¢, 4 lb. tea @ 75¢, 5 lb. coffee @ 40¢. Amount received, \$10.

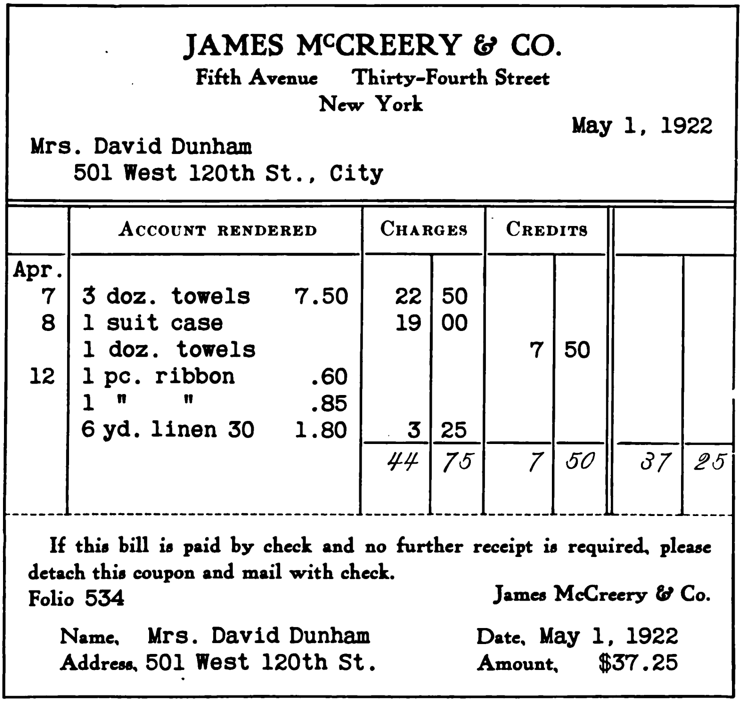
9. 2 boxes cocoa @ 28¢, 3 cans corn @ 18¢, 2 pecks potatoes @ 35¢, 4 bottles olives @ 35¢. Amount received, \$5.

10. $7\frac{1}{2}$ yd. cotton @ 28¢, 36 yd. velvet @ \$2.50, $5\frac{1}{2}$ yd. silk lining @ \$2.10. Amount received, \$110.

11. $32\frac{1}{2}$ yd. linen @ 60¢, $18\frac{1}{2}$ yd. suiting @ \$1.50, $6\frac{3}{8}$ yd. silk @ \$3.20. Amount received, \$70.

12. $8\frac{1}{2}$ yd. India linen @ 68¢, $7\frac{3}{4}$ yd. cheviot @ \$2, $16\frac{1}{2}$ yd. dimity @ 38¢. Amount received, \$30.

Bill. The following is a common form of a retail bill:



JAMES M^cCREERY & CO.
Fifth Avenue Thirty-Fourth Street
New York

May 1, 1922

Mrs. David Dunham
501 West 120th St., City

	Account rendered		Charges		Credits			
Apr.								
7	3 doz. towels	7.50	22	50				
8	1 suit case		19	00				
	1 doz. towels				7	50		
12	1 pc. ribbon	.60						
	1 " "	.85						
	6 yd. linen 30	1.80	3	25				
			44	75	7	50	37	25

If this bill is paid by check and no further receipt is required, please detach this coupon and mail with check.

Folio 534 James McCreery & Co.

Name, Mrs. David Dunham Date, May 1, 1922
Address, 501 West 120th St. Amount, $37.25

Items returned are usually typewritten in red.

It is becoming a common practice in cities not to ask for receipted bills when payment is made by check. The indorsed check is in itself a receipt.

The totals are usually found by a mechanical device attached to the billing typewriter unless the entries are made at different times.

The older form of bill, with the receipt written below, is familiar from elementary arithmetic. Either the older or the more modern form may be used on page 127.

Exercise. Bills

Make out bills for the following, using the names of firms in your community or taking fictitious names:

1. Charges: Jan. 15, 1 suit case, $11; Jan. 16, 2 kimonos @ $5.95; Jan. 17, 4 pc. braid @ 25¢. Credits: Jan. 17, kimono, $5.95. Bill rendered Feb. 1.

2. Charges: Nov. 9, 1 pa. sage, 5¢; 3 lb. rice @ 12¢; 1 pa. shredded wheat, 14 ¢; 2 pa. gelatine @ 18¢; Nov. 11, 1 pa. flour, 58¢; 1 box sugar, 26¢; 2 bot. celery salt @ 12¢; 1 qt. Lucca oil, $1.15; Nov. 12, 6 pa. matches @ 10¢; 2 doz. oranges @ 60¢. Credits: Nov. 13, 3 pa. matches @ 10¢; 1 bot. celery salt, 12¢. Bill rendered Nov. 15.

3. Charges: Dec. 10, 1 sugar canister, 85¢; 1 cake chest, $2.85; 1 silver brush, 75¢; Dec. 11, 1 chamois, $1.20; Dec. 13, 1 ice-cream freezer, $3.25; 1 coffee pot, $1.25; 2 graters @ 15¢; 1 egg beater, 45¢; 3 cake cutters @ 12¢; 1 ironing board, $1.30. Credits: Dec. 14, 1 cake chest, $2.85; 1 coffee pot, $1.25. Bill rendered Dec. 15.

4. Charges: Jan. 12, 6 handkerchiefs @ 75¢; $1\frac{1}{2}$ doz. towels @ $9; 3 spools cotton @ 15¢; $\frac{1}{2}$ doz. doilies @ $2.50; 5 yd. fringe @ 95¢; $\frac{1}{2}$ doz. towels @ $6; 8 yd. crêpe @ $3; 6 yd. linen @ 30¢; Jan. 22, 3 pc. ribbon @ 85¢; Jan. 25, 4 doz. buttons @ 18¢; Jan. 27, 5 yd. ribbon @ 15¢; 2 combs @ 55¢; 1 gown @ $3.45. Credits: Jan. 13, $\frac{1}{2}$ doz. towels @ $9; Jan. 28, 1 gown @ $3.45. Bill rendered Feb. 1.

5. Charges: May 7, $4\frac{1}{2}$ yd. cotton @ 38¢, 17 yd. gingham @ 54¢; 12 yd. linen @ 56¢; May 9, $18\frac{1}{2}$ yd. muslin @ 42¢; 1 bag, $2.80; 9 yd. velveteen @ $1.35; $14\frac{1}{2}$ yd. madras @ 64¢; May 15, 20 yd. linen suiting @ 1.40; $9\frac{3}{4}$ yd. cheviot @ $2; $3\frac{5}{8}$ yd. India linen @ 80¢; 1 parasol, $3.25. Credits: May 10, 1 bag, $2.80; May 17, 1 parasol, $3.25. Bill rendered June 1.

Trade Discount. Dealers, jobbers, and retail merchants, when they buy merchandise directly from the manufacturers or from the wholesalers, are usually allowed a discount from the catalogue or list price of goods. Such a discount allowed to the trade is called a *trade discount.*

Discount Series. If large quantities of goods are ordered, a further discount from the *first net price* is often allowed, then a discount from the *second net price*, and so on, this resulting in a *chain of discounts* or a *discount series.*

Terms. The conditions of payment for a bill of goods are called the *terms*, and these are usually stated on the bill. For example, "Terms: 3% 10, net 30," or "Terms: 3/10, N/30," means that a discount of 3% is allowed if the bill is paid within 10 da., and that the bill must be paid within 30 da.

It often happens that a large dealer wishes goods in the autumn for the holiday trade and is willing to store them during the summer and thus relieve the manufacturer of this demand upon his floor space. The manufacturer may then date his bill Nov. 1, "Terms: 30 da." If the buyer wishes to pay the bill, say on July 1, the manufacturer makes an allowance known as *anticipation.* The usual anticipation is from $\frac{1}{2}\%$ a month to 1% a month.

Discount Formulas. If the list price is l and the net price is n, and if there is a single rate of discount r, then

$$n = l - rl = l(1 - r).$$

If there are two discounts, r and r', then

$$n = l(1 - r)(1 - r').$$

Since we can interchange the factors $1 - r$ and $1 - r'$, it makes no difference which discount is taken first.

In the last result discard any fraction less than $\frac{1}{2}$¢.

Exercise. Trade Discount

1. If the list price is \$100 and there is a chain of discounts 10, 5, 2, what is the net price?

The discounts 10%, 5%, 2% are usually written simply 10, 5, 2, and similarly for other discounts.

2. In Ex. 1 show that the net price is the same if the discounts are either 2, 5, 10 or 5, 10, 2.

3. If a bill for \$1000 is dated Oct. 1, terms cash, and the payment is anticipated on Aug. 1, how much is the anticipation at $\frac{3}{4}$% a month?

4. On a bill for \$728.60 discounts of 12%, 8% are allowed. Find the net price.

5. A bill for \$1275.80 reads "Terms: Cash, 10, 8, 3." Find the net price.

6. A bill for \$1682.90 reads "Terms: 5/10, N/30." If the bill is paid at once, how much is the discount?

7. The Harrison Hardware Company sells the following bill of goods: 5 doz. #18 bolts @ \$1.75; 8 doz. #25 E chain bolts @ \$10.50; 6 doz. #10 K locks @ \$22.50; $4\frac{1}{2}$ doz. #36 A hinges @ \$4.80; 16 doz. #48 brass plates @ \$3.60. Allowing discounts of 25, 10, 8, find the net price.

8. A city jobber buys the following bill of goods: 750 yd. worsted suitings @ \$2.10 less 18, 10, 5; 800 yd. woolens @ \$1.85 less 10, 10; 1800 yd. lining @ 72¢ less 8, 8, 2; 1200 yd. buckram @ 36¢ less 10, 8, 3. Find the net price.

9. Show that the net price of a bill of goods on which the discounts are 10, 5 can be found by multiplying the list price by 0.855.

In commercial work, $(1-r)(1-r')$, in this case 0.855, is called the *net-cost-rate factor*, but we shall not need to consider this technical feature further in a general course of this kind.

Invoice. An itemized statement of goods bought is called an *invoice.*

An invoice is arithmetically the same as a bill. It is a term that is often used in wholesale transactions.

The following is a common form of invoice of goods bought at wholesale from a jobber:

THE BRONSON IMPORTING COMPANY
2487 Boston Avenue
Chicago

May 17, 1924

Sold to J. M. Bryan Co.,
La Salle, Ill.

Terms: 2/10,N/30

May	17	7 bx. Castile soap	4	20	29	40		
		750# Java coffee		30	225	00		
		16 hf.cht. Oolong tea	56	25	900	00		
							1154	40

Statement. An invoice usually accompanies the goods when they are shipped, and on the first of each month a *statement* of the following nature is usually sent out by the jobber:

STATEMENT
THE BRONSON IMPORTING COMPANY
2487 Boston Avenue
Chicago

Account No. 2968

June 1, 1924

All bills due on the first of the month following purchase

J. M. Bryan Co.,
La Salle, Ill.

June	1	To bal. as per statement May 1	127	60		
May	17	To mdse. as per invoice	1154	40		
	20	" " " " "	386	90		
					1668	90
	22	By mdse. returned	82	50	82	50
		Balance			1586	40

Exercise. Invoices and Statements

1. If the invoice on page 130 is paid within 10 da., what is the net price paid by the J. M. Bryan Co.?

2. A statement of $1546.98 worth of goods bought on July 5, July 10, and July 17, but all billed as of Oct. 20, is accompanied by a letter stating that an anticipation of $\frac{1}{2}$% a month will be allowed if the amount is paid on July 20. If paid on that date, how much is the anticipation and what is the net price of the goods?

Find the net amounts of the following bills:

3. 600 bags coffee @ $32.50 less 33$\frac{1}{3}$, 10; 200 bags coffee @ $42 less 25, 10, 5; 400 bags coffee @ $48.50 less 40, 15, 5; 300 bags coffee @ $53.80 less 30, 20, 10.

4. 120 bbl. sugar @ $14.50 less 6, 2$\frac{1}{2}$; 140 bbl. sugar @ $15.20 less 5, 5; 75 bbl. sugar @ $16.50 less 5, 3, 2; 125 bbl. sugar @ $18.75 less 6, 6.

5. Using fictitious names for the buyer and seller, write an invoice of goods as follows: 200 bags Java coffee @ $38; 50 bags Mocha coffee @ $52; 100 bbl. sugar @ $14.80; 350 bbl. flour @ $10.50. Discounts 20, 8, 5.

6. Using fictitious names for the buyer and seller, write an invoice of goods as follows: 720 yd. taffeta @ $1.30; 640 yd. lawn @ 36¢; 240 yd. hangings @ $1.62; 8 gross pompons @ $188; 540 yd. silk @ $2.20; 820 yd. velvet @ $2.35. Discounts 12, 4, 2.

7. The Hardcastle Hardware Company sold to the J. M. Brown Company an invoice of goods as follows: 8 doz. locks @ $16.75 less 20, 10, 2; 2$\frac{1}{2}$ doz. bolts @ $4.20 less 20, 5, 5; 16 doz. galvanized iron pails @ $3.20 less 30, 10, 5. Write the invoice and find the net price.

Profit or Loss on Cost. A merchant frequently reckons his profit or loss at some rate per cent of the cost of goods, of the money invested, or, as we shall see on page 134, at some rate per cent of the selling price of the goods.

The difference between the selling price and the total cost price is called the *gross profit* or the *gross loss.*

Cost of doing Business. In computing the cost of goods it is customary to add to the wholesale price, known as the *prime cost,* the freight, drayage, salaries of buyers, and all other expenses incurred in getting the goods ready to sell. These expenses are called *buying expenses.*

Such charges as rent, insurance, light, bookkeeping, and interest on the money invested are called *overhead charges* or simply *overhead.*

The expenses incurred in selling an article are called *selling expenses.*

The total of buying expenses, overhead, and selling expenses is called the *cost of doing business.*

The sum of the prime cost and the cost of doing business is called the *total cost.*

Illustrative Problem. The buying expenses for a certain line of goods are $7400, the share of overhead expenses of buying and selling the goods is $9800, the selling expenses are $11,250, the prime cost is $142,250. What per cent of the prime cost must be added to the cost of each article to cover each of the first three items?

Since $7400 is a certain per cent of $142,250, we have the product, $7400, and one factor, $142,250, to find the other factor. Hence

$$\$7400 \div \$142{,}250 = 0.052 = 5.2\%, \text{ added for buying;}$$

$$\$9800 \div \$142{,}250 = 0.069 = 6.9\%, \text{ added for overhead;}$$

$$\$11{,}250 \div \$142{,}250 = 0.079 = 7.9\%, \text{ added for selling.}$$

The sum, 20%, is to be added for the cost of doing business.

Exercise. Profit or Loss on Cost

1. A manufacturer determines his selling price by adding $22\frac{1}{2}\%$ to the manufacturing cost. If it costs $72.50 to manufacture a certain machine, find the price at which the manufacturer must sell it.

In all such cases state the answer to the nearest cent.

2. A merchant buys 1920 yd. of linen at $1.05 a yard. His buying and selling expenses and overhead are $21\frac{1}{8}\%$ of the prime cost. At what price must he sell the linen in order to make a profit of 12% on the total cost?

3. A hardware dealer buys 75 stoves at $32.40. He pays $87.30 freight on the entire shipment, charges up $15 for other buying expenses, allows $2.30 for overhead and selling expenses on each stove, and sells the stoves for $45 each. What is his per cent of profit on the total cost?

4. A dealer adds $33\frac{1}{3}\%$ of the prime cost of a certain article to cover its share of the cost of doing business. He then sells the article at a profit of 15% on the total cost. If the prime cost is $33.75, what is the selling price?

5. A furniture dealer buys some tables at $16.75 each. His buying and selling expenses and overhead are 24% of the prime cost. If he sells the tables at $25 each, what per cent does he make on the prime cost? on the total cost?

6. A dealer buys some glassware for $578.50, the cost of doing business being estimated at $126.75. Through his fault some of the glassware is broken, and he sells the rest of the lot for $600. What per cent of the total cost does he gain or lose on the transaction?

7. A manufacturer determines his selling price by adding 32% to the manufacturing cost. Find the cost of manufacturing an article which he sells for $3.96.

Profit or Loss on Selling Price. Profit or loss was formerly always computed on the prime cost of goods. As the cost of doing business became more of an item, the profit or loss was computed on the prime cost plus the cost of doing business, that is, on the total cost.

At present, however, in large business houses it is the custom to estimate the cost of the goods, the various items of the cost of doing business, and the profit, each as a per cent of the selling price.

By keeping careful records of their business, including all leading items of the cost of doing business, large dealers are able so to standardize their expenses as to determine in advance the per cent of the selling price that should be allowed to the items of the cost of doing business in order to allow them to realize the expected profit. This profit is then computed on the selling price instead of, as formerly, on the prime cost or the total cost.

Illustrative Problem. A large department store pays the manufacturer \$6.85 for a certain article. Experience has shown that the cost of doing business is $19\frac{1}{2}\%$ of the selling price. If the desired profit is to be 12% of the selling price, what should be the selling price?

We have prime cost = \$6.85.

If we let s represent the selling price, we have

$$s = \$6.85 + 0.19\tfrac{1}{2}\,s + 0.12\,s,$$

or

$$s - 0.19\tfrac{1}{2}\,s - 0.12\,s = \$6.85,$$

$$0.68\tfrac{1}{2}\,s = \$6.85.$$

Dividing by $0.68\frac{1}{2}$, $s = \$10.$

That is, the selling price is \$10.

We may solve without using algebra if we wish. Since we must deduct $19\frac{1}{2}\%$ and 12% from the selling price, we must deduct $31\frac{1}{2}\%$ in all. Hence the prime cost is $1 - 31\frac{1}{2}\%$, or $68\frac{1}{2}\%$, of the selling price. Hence the selling price is $\$6.85 \div 68\frac{1}{2}\%$, or \$10.

Exercise. Profit or Loss on Selling Price

1. A gain of 10% on the selling price of an article is equivalent to what per cent of gain on the cost?

2. By selling an article for \$2.40 a dealer gains $\frac{1}{3}$ on the selling price. If he sells the article for \$2.50, what per cent does he gain on the selling price?

3. A house finds that the average cost of doing business is 18%. If it pays \$2670 for some goods and its profit basis is 14%, at what price must it sell the goods?

In all problems on this page both the cost of doing business and the profit are reckoned on the selling price. Such an expression as "profit basis" will easily be understood without formal definition.

4. In Ex. 3 explain why the result can be found by multiplying \$2670 by 1.4706.

Instead of dividing by $1 - (0.18 + 0.14)$, computers multiply by the reciprocal of 0.68. They have tables giving such reciprocals.

5. A dealer buys some kitchen chairs at \$2.24 each, discounted at 25, 10. The cost of doing business being 16% and the profit basis being 12%, find to the next larger multiple of 5¢ the price at which each chair must be sold.

6. A hardware dealer buys stoves at \$27.50. His cost of doing business is on a 28% basis and his profit basis is 15%. Find to the next larger multiple of 25¢ the price at which each stove must be sold.

7. A merchant buys a line of carpeting at \$1.84 a yard. His cost of doing business is on a 14% basis and his profit basis is 10%. Find to the next larger multiple of 10¢ the price per yard at which the carpeting must be sold.

8. A dealer pays \$975.40 for a lot of goods. The cost of doing business is 20% and the profit basis is 12%. Find the profit on the sale of the entire lot.

Borrowing Money. While it is poor policy for an individual to borrow money to pay for his ordinary living expenses, and particularly for the purchase of luxuries, it is often very good policy for a merchant to resort to borrowing for the purpose of taking advantage of such discounts as may be allowed or of leaving his investments undisturbed.

For example, if a merchant receives a bill for $1200 worth of goods, the terms being 2/10, N/60, and if he is reasonably sure of selling all the goods within 60 da. at a profit of $100, he can well afford to borrow the money at 6% and take advantage of the discount.

For the discount is 2% of $1200, or $24. Suppose that this is paid, for example, at the end of 8 da., and the merchant borrows the money for 50 da., which probably is as long as he needs it, the interest on $1200 for 50 da. at 6% is $10, which is $14 less than the discount he gains.

Furthermore, if the merchant can make $400 during the holidays by buying $4000 worth of goods in November, he can well afford to pay the interest on the money for a couple of months. Whether he can safely depend on making $400 in this way is a matter of judgment based upon experience.

Suppose, for example, that the net cost of the goods, less all discounts, is $4000 on Nov. 15, and that he can borrow the money for 2 mo. at 5% and can safely depend on clearing $400 above all cost of doing business. The interest on $4000 for 2 mo. at 5% is only $33.33⅓, or $33.34, as the bank will probably compute it, and this is a small amount compared with the $400 that he will make.

In the case of large corporations the amount saved by careful borrowing is often very great. For example, if a corporation is reasonably certain of making $1,250,000 on a certain contract, and if it needs $3,500,000 for 60 da. in order to undertake the work, the interest that it would have to pay would be negligible compared with the gain that would accrue from undertaking the contract.

Exercise. Borrowing Money

1. A merchant receives an invoice of goods amounting to $1487.50 less 20, 10, 5 if paid within 10 da. His experience shows that he can sell the goods within 90 da. after the date of payment at a net profit of $250 above all expenses except the interest that he may pay on the money that he borrows. If he needs to borrow $1400 for 90 da. at 6% to make the purchase, show that it will pay him to do so.

2. A merchant receives an invoice of goods amounting to $2875 less 15, 10, 10 if paid within 10 da. His experience shows that he can sell the goods within 60 da. after the date of payment at a net profit of $375 above all expenses except the interest that he may pay on the money that he borrows. If he borrows for 60 da. at 5% the money to the nearest $100 necessary to pay the bill, how much is his net profit?

3. A wholesale dealer buys from the manufacturer a bill of goods amounting to $38,750 less 20, 10, 5 if paid within 3 da. The dealer takes advantage of the discounts and borrows the money to the nearest $100 for 60 da. at 5%. His total cost of doing business, exclusive of interest on the bill, is on a 15% basis and his profit basis is 10%, both reckoned on the selling price. How much does he gain on the transaction?

4. A dealer buys from a wholesale house some goods amounting to $4270 less 10, 5, 5. He has the cash for half of the bill and borrows the money to pay the other half, giving his note for the amount to the next higher $50 for 60 da. at 5%. How much interest does he pay?

5. A corporation can take a contract for $4,500,000 and be reasonably certain to make 30%. In order to do this it must borrow $2,000,000 for 90 da. If the money can be had at $5\frac{1}{2}\%$, how much will the corporation gain by borrowing the money and taking the contract?

Simple Interest. If the merchant borrows money as described on page 136 he must pay interest for its use. While the computing of interest is familiar from the work in elementary arithmetic, there are a few features that may properly be considered at this time.

It seldom happens that interest has to be computed on odd amounts for years, months, and days, and at unusual rates. For example, it would be very unusual to be called upon to find the interest on $1287.57 for 9 yr..8 mo. 22 da. at 7.35%. Such problems give some test in computation, but they give such a false idea of practical business as to be objectionable for school work.

Ordinarily interest is computed by the use of interest tables. These tables of *common interest* are based on 360 da. to the year, and they give a little more than the *exact interest* which is based on 365 da. to the year. In paying interest banks often use the 365-day table, but in charging interest they use the 360-day table.

All methods of reckoning interest are fundamentally the same, depending upon this principle:

To find the interest, given the principal, rate, and time, multiply the principal by the rate, and the product by the number that expresses the time in years.

This rule may be expressed algebraically, thus:

$$i = prt.$$

From this formula we readily derive certain formulas which we have already considered briefly, and we may now restate them, by way of review, as follows:

$$p = \frac{i}{rt}, \qquad r = \frac{i}{pt}, \qquad t = \frac{i}{pr}.$$

The form i/rt, used on page 94, is often convenient in printing, but for written work the forms given above are more common.

Exercise. Simple Interest

1. Given the interest, rate, and time, write a rule for finding the principal and explain how the rule is found.

2. Given the interest, principal, and time, write a rule for finding the rate and explain how the rule is found.

3. Given the interest, principal, and rate, write a rule for finding the time and explain how the rule is found.

4. Find the interest on $2500 for 90 da. at 6%, first using 360 da. to the year and then using 365 da. to the year.

5. There is a rule, known as the Six Per Cent Method, which is sometimes stated as follows: To find the interest for 60 da. at 6% move the decimal point in the principal two places to the left; for 6 da., three places to the left. Explain the validity of this rule.

6. Using the rule in Ex. 5, find the interest at 6% on $1750 for 60 da.; for 30 da.; for 90 da.; for 15 da.; for 10 da.; for 45 da.

7. In large city banks it is often necessary to compute the interest for 1 da. at certain rates. Among the rules in common use is the following: To find the interest for 1 da. at $4\frac{1}{2}$% point off three places and divide by 8. Apply this rule to finding the interest on $14,800 for 1 da. at $4\frac{1}{2}$% and explain the validity of the rule.

8. Derive a rule, similar to the one in Ex. 7, for finding the interest for 1 da. at 4%.

9. Prove that the common interest is equal to the exact interest increased by $\frac{1}{72}$ of itself, and that the exact interest is equal to the common interest decreased by $\frac{1}{73}$ of itself.

The common interest, i, for 1 da. is found from $i = pr \cdot \frac{1}{360}$, while the exact interest, i', is found from $i' = pr \cdot \frac{1}{365}$. Now find i in terms of i'; then find i' in terms of i.

Property Insurance. The careful merchant will insure his property against damage or loss from one or more causes. The ordinary terms used in insurance have already been considered on page 120.

Those who do the business of insuring the owner against loss of property are often called *underwriters.*

For mercantile purposes there are several kinds of insurance. Of these the most common is fire insurance, some of the others being burglar insurance, plate-glass insurance, tornado insurance, insurance against accidents to employees, elevator insurance, live-stock insurance, automobile insurance, crop insurance, transportation insurance, and, if the merchant is engaged in sending goods by sea, marine insurance.

In case of loss the company which issues the insurance pays the amount of the policy, if the loss has been as great as this amount, subject to such conditions as the policy specifies.

The practical question for the merchant, however, is the same in all these forms of insurance, and it relates to the cost. If the merchant wishes to carry $10,000 of insurance, he must know the rate per $100, or the rate per cent, so as to find the premium that he must pay. The rate is ordinarily stated as so much per $100 of the policy.

For example, find the premium on a $4500 policy at $1.20.

At $1.20 per $100, the cost for $4500 is 45 × $1.20, or $54.

A person wishing to take out insurance should first make a careful estimate of the value of the property to be insured. He should then decide upon the amount of insurance. Since a fire usually destroys only a part of the property covered, an insurer might safely take out insurance for only part of the value. This is the reason why insurance companies often require that the policy must cover 80% of this value if it is to be paid in full in case of total loss, as stated on page 141.

Exercise. Property Insurance

1. A merchant insures his stock of goods worth $80,000 for 80% of its value at $1.30. Find the premium.

Many companies require that the property be insured for at least 80% of its value if the full amount of the policy is to be paid in case the loss is as great as this amount. The rate varies with the nature of the risk.

2. A frame building worth $7000 is insured for 80% of its value, at 95¢. Find the premium.

3. A village insures a schoolhouse worth $75,000 for 80% of its value, at 60¢. Find the premium.

4. A merchant insures his building worth $25,000 for 80% of its value, at $1.50. Find the premium.

5. Merchandise worth $20,000 is insured for 85% of its value, at $1.50. What is the premium?

6. A man insures his house and contents for $7500, at $1.35. What is the premium?

7. When policies are written for 2 yr., 3 yr., 4 yr., 5 yr., the rates are often respectively $1\frac{1}{2}$, 2, $2\frac{1}{2}$, 3 times the rate for 1 yr. A merchant takes out a policy of $75,000 on his stock at $1.10 a year. Find the premium if he takes out the policy for 1 yr.; for 2 yr.; for 3 yr.; for 4 yr.; for 5 yr.

8. A merchant insures his stock, valued at $48,000, for 85% of its value, at $1.60. Find the premium.

9. A man insures his house worth $9000 for $\frac{7}{8}$ of its value, at $1.20. Find the premium.

10. A man pays $1040 for insuring a building for $65,000. Find the rate per $100.

11. A merchant insures his stock for $\frac{7}{8}$ of its value, at $1.50. The premium is $420. Find the value of the stock.

Such an example is not practical but is introduced to illustrate the type. The merchant knows the value of his stock before he insures it.

Exercise. Review of Chapters I–VIII

1. Write in common numerals one hundred and seventy-five thousandths; one hundred seventy-five thousandths.

2. The first year of this century being 1901, find the sum of the years of the century up to and including the present year. Check the addition in two ways.

3. Multiply $17,275 by 1.04^2 and check the result by casting out nines.

4. In Ex. 3 check the result by dividing it by 1.04 and then dividing by 1.04 the quotient thus obtained.

5. Check each division in Ex. 4 by casting out nines.

6. Multiply 17 ft. 7 in. by 8; 17 lb. 7 oz. by 8; $17.07 by 8; $17\frac{7}{8}$ by 8; 17 yd. 7 in. by 8; 177 by 8. Explain the difference in the figures in the results.

7. Which is the stronger, a 6% solution of a medicine or a 10% solution? a $\frac{1}{6}$ solution or a $\frac{1}{10}$ solution? State the reason in each case.

8. How long will it take a sum of money to double itself at 6% simple interest? at 5%? at 4%?

9. Divide $8t^2 + 4t + 2$ by 2; 8 hr. 4 min. 2 sec. by 2; 842 by 2; 84.2 by 2; $84\frac{2}{3}$ by 2; 8.42 by 2; 8.0042 by 2; and $8x + 4y + 2z$ by 2. Explain the difference in results.

10. If 107% of some number is 2033, what is the number? Write a brief explanation of how you proceeded to find the answer.

11. The product of two numbers is 911,784 and one of the numbers is 8. What is the other number? Prove that the answer is correct.

12. Find the net price of a bill of goods listed at $7842.50 and discounted at 10, 8, 5, 2.

CHAPTER IX

CORPORATION ARITHMETIC

Nature of a Corporation. In early times an individual who went into business did so by himself, the amount of the capital invested, whether in goods or in money, being small. This form of business enterprise is still a common one and probably will continue to be so.

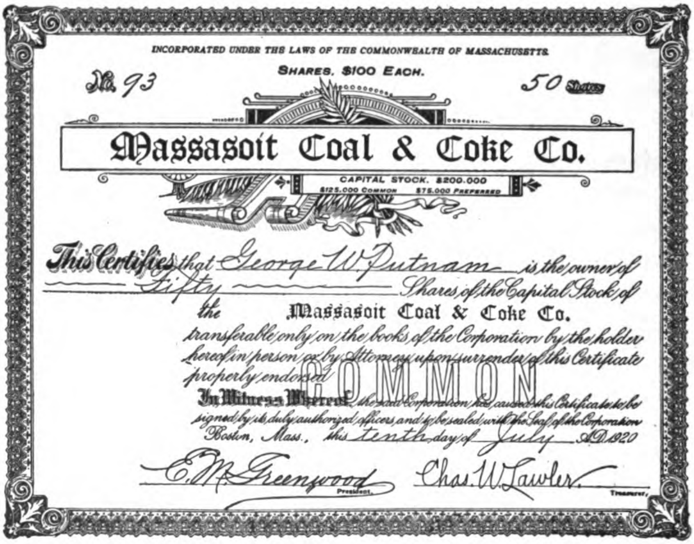
Later, when business became more complex and when larger capital was required, men formed partnerships, and this method of carrying on business is still common, although it is not relatively as important as was formerly the case.

As business became still more complicated and larger amounts of capital were needed, the number of persons engaged in the various enterprises naturally increased, a few of them being delegated to carry on the work. In modern times this method of operation has led to the establishing of groups of persons authorized by law to act as one body. Such a group of persons is called a *corporation.* Corporations are often called *companies* or *joint stock companies.*

A person who owns a certain part of the capital of a corporation is called a *stockholder*, the amount that he owns being evidenced by a *certificate of stock* showing how many *shares* he owns of the *stock* or *capital stock* of the company. The stockholders elect a *board of directors* and this board elects officers to manage the company.

That portion of the income which is divided among the stockholders makes up what are known as the *dividends* paid by the company.

Certificate of Stock. The following is an example of a certificate of stock in a corporation in Massachusetts:



INCORPORATED UNDER THE LAWS OF THE COMMONWEALTH OF MASSACHUSETTS.

No. 93 SHARES, $100 EACH. 50 Shares

Massasoit Coal & Coke Co.

CAPITAL STOCK, $200.000
$125.000 COMMON $75.000 PREFERRED

This Certifies that George W. Putnam is the owner of Fifty Shares of the Capital Stock of the Massasoit Coal & Coke Co. transferable only on the books of the Corporation by the holder hereof in person or by Attorney upon surrender of this Certificate properly endorsed

COMMON

In Witness Whereof, the said Corporation has caused this Certificate to be signed by its duly authorized officers and to be sealed with the Seal of the Corporation Boston, Mass., this tenth day of July A.D. 1920

E. M. Greenwood President.

Chas. W. Lawler Treasurer.

A CERTIFICATE OF STOCK

There are often two kinds of stock: *preferred stock*, which pays a fixed rate of dividend if enough money is earned; and *common stock*, which takes whatever dividends may be paid after other claims have been satisfied.

The above certificate represents common stock. It is signed by the president and treasurer of the company. These and other necessary officers are elected by the directors, these directors having been elected by the stockholders. Each share owned by a stockholder entitles him to one vote, and hence a few stockholders may control the business.

The par value of a share, referred to on page 116, varies with different companies. We shall take it to be $100.

Exercise. Stock

1. A company organizes with a capital of $1,500,000 and issues all its stock at a par value of $100 a share. How many shares does it issue?

2. In Ex. 1 the company issues $500,000 in 6% preferred stock and $1,000,000 in common stock. If it has $100,000 available for dividends at the end of a year, what rate of dividend can it pay on the common stock?

3. If a company with a capital of $3,000,000, all represented by common stock of par value $100 a share, has $70,000 available for semiannual dividends, what is the semiannual rate of dividend that it can declare and how much will a person receive who owns 50 shares of stock?

4. A company having a capital of $2,750,000 declares a dividend of $2\frac{3}{4}$%. How much money does it thereby distribute among its stockholders?

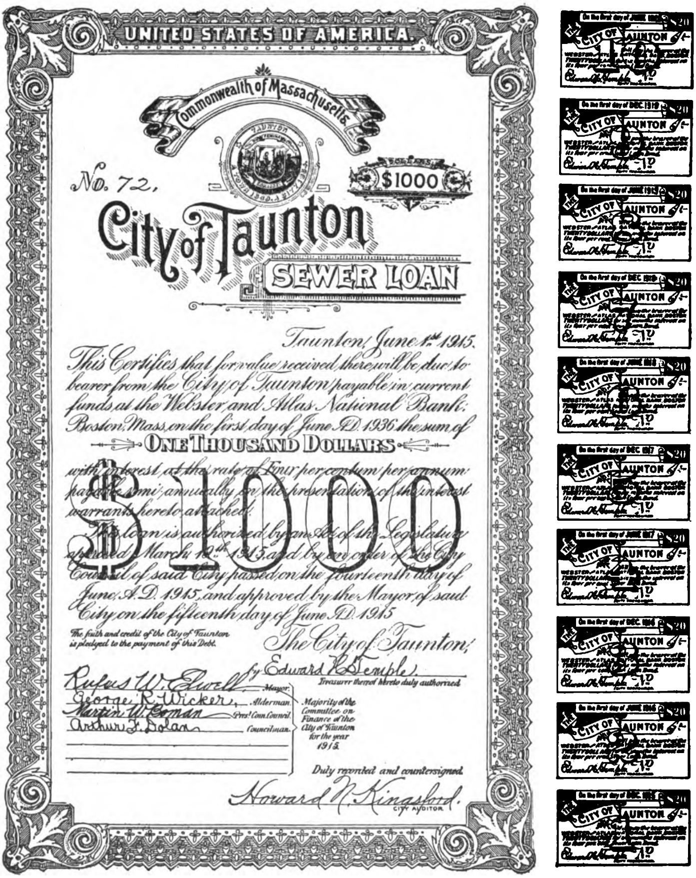
5. How much does a holder of 125 shares of railway stock of par value $50 a share receive from the company when a $3\frac{1}{4}$% dividend is declared?

6. How much capital has a company which declares a 4% dividend of $20,000?

7. If the earnings of a company have been such as to allow of the same dividends for a long period of years, and the outlook is for maintaining these dividends, and if the preferred stock pays 7% and the common stock 10%, which kind of stock will command the higher price? If the preferred stock is quoted at 110 and the common stock is considered equally safe as to its rate of dividends, at about what price will the common stock be quoted?

There are other factors to be considered, but for our purposes an approximate result may be based upon the rate of dividends.

Bond. When governments or corporations borrow any considerable amount of money, they often issue *bonds.* Bonds usually have *coupons* annexed providing for the payment of the interest when it is due. The following is an illustration of a *coupon bond* with a few of the coupons annexed.



UNITED STATES OF AMERICA.

Commonwealth of Massachusetts.

No. 72,

$1000

City of Taunton

SEWER LOAN

Taunton, June 1st 1915.

This Certifies that for value received there will be due to bearer from the City of Taunton payable in current funds at the Webster and Atlas National Bank, Boston, Mass. on the first day of June A.D. 1936 the sum of

ONE THOUSAND DOLLARS

with interest at the rate of Four per centum per annum payable semi-annually on the presentation of the interest warrants hereto attached.

This loan is authorized by an Act of the Legislature approved March 19th 1915 and by an order of the City Council of said City passed on the fourteenth day of June, A.D. 1915, and approved by the Mayor of said City on the fifteenth day of June A.D. 1915

$1000

The faith and credit of the City of Taunton is pledged to the payment of this Debt.

The City of Taunton,

by Edward H. Temple.

Treasurer thereof hereto duly authorized

Rufus W. Elwell Mayor.

George R. Wicker, Alderman.

Martin W. Coman Pres! Com. Council.

Arthur F. Dolan Councilman.

Majority of the Committee on Finance of the City of Taunton for the year 1915.

Duly recorded and countersigned.

Howard N. Kingsford.

CITY AUDITOR.

A COUPON BOND

Exercise. Bonds

1. State briefly the difference between stock and a bond.

2. A company borrows $125,000 in order to improve its plant. It issues bonds bearing $5\frac{1}{2}\%$ interest. How much must be set aside each year for interest?

In addition to paying the interest the company should set aside each year an amount large enough so that the sum of these amounts with interest shall pay the debt when due. Such a fund is called a *sinking fund.* Sinking funds are studied in commercial algebra.

3. A company has a capital of $2,500,000 and regularly pays a quarterly dividend of $1\frac{1}{4}\%$. It also has outstanding bonds to the amount of $1,350,000 bearing 6% interest. How much must the company set aside each year for paying the dividends on the stock and the interest on the bonds?

4. A board of education recommends to a city that it issue bonds to the amount of $75,000 for a new schoolhouse. The city issues these bonds, the rate of interest being 5%. How much is the annual interest charge?

A city is a special kind of corporation and is empowered by law to issue bonds under certain restrictions. The law often limits the amount of bonds that any corporation may issue.

5. A manufacturing company issues bonds to the amount of $750,000 for the purpose of obtaining money to improve its plant. The interest charges on these bonds is $39,375. What rate of interest do the bonds pay?

Conditions are less real in Exs. 5 and 6 than in Exs. 1–4. We usually have the amount of the bonds given, together with the rate, to find the interest charge.

6. A company issues bonds bearing $5\frac{1}{2}\%$ interest. The annual interest charge is $13,200. Find the amount of the bonds issued.

Wages. The scale of wages, the allowance for overtime, the method of payment for piecework, and the question of profit sharing are connected with economics more closely than with higher arithmetic. If the wages are at a fixed rate per hour, say 60¢, the working day consisting of 8 hr. and the working week of 5½ working days, the wages for the week are conveniently computed by such a table as the following:

HOURS	0 DA.	1 DA.	2 DA.	3 DA.	4 DA.	5 DA.
0		$4.80	$9.60	$14.40	$19.20	$24.00
1	$0.60	5.40	10.20	15.00	19.80	24.60
2	1.20	6.00	10.80	15.60	20.40	25.20
3	1.80	6.60	11.40	16.20	21.00	25.80
4	2.40	7.20	12.00	16.80	21.60	26.40
5	3.00	7.80	12.60	17.40	22.20	¼ hr. $0.15
6	3.60	8.40	13.20	18.00	22.80	½ hr. .30
7	4.20	9.00	13.80	18.60	23.40	¾ hr. .45

If a man worked 8 hr. a day for 3 da., 7 hr. on the fourth day, and 3½ hr. on Saturday, his wages would be $18.60 (under 3 da. and opposite 7 hr.) + $1.80 (under 0 da. and opposite 3 hr.) + $0.30 (after ½ hr.), or $20.70 in all.

Overtime is often paid on the basis of time and a half; that is, the pay for 1 hr. overtime is equal to that for 1½ hr. regular time. Where payment is made for both time and overtime, two tables are used.

Many large corporations are studying the question of profit sharing; that is, of not only paying regular wages but apportioning the profits so that the laborers as well as the stockholders receive a share.

Although the results of the war are such as to modify somewhat the question of wages, the arithmetic of the subject, which relates chiefly to addition and multiplication and to the use of tables, does not vary materially.

Exercise. Wages

1. Make a wage table similar to the one on page 148, but based upon a wage of 48¢ an hour.

2. Make a wage table for overtime from 1 hr. to 6 hr. on the basis of 72¢ an hour for overtime.

3. From the tables of Exs. 1 and 2 find the wages due a man who works in a shop as follows: Monday, 8 hr.; Tuesday, 7 hr.; Wednesday, 12 hr.; Thursday, 8 hr.; Friday, 10½ hr.; Saturday, 4 hr.

4. Instead of making a table by days and hours, make one by hours from 1 to 44, the wages being 52¢ an hour. From this table fill in the following section of a pay roll:

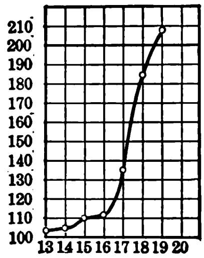
Men's Numbers	Hours	Amount Due
3	24½	
7	30	
11	44	
12	42½	
21	38¾	

5. The profit-sharing plan of a manufacturing company with a capital of $800,000 allows a dividend of 10% on the capital, and divides half the balance of the profits among the employees according to their wages. Find the share of a workman whose wages are $1252 in a year when the pay roll is $412,500 and the profits are $138,740.

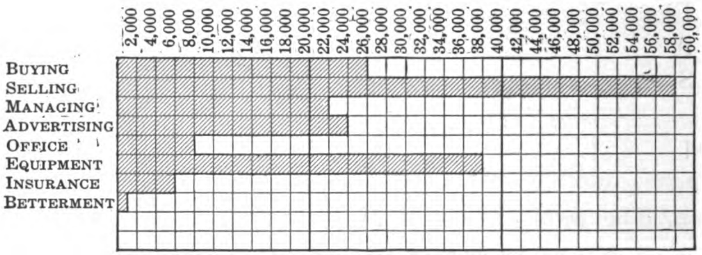
6. In Ex. 5 how much will a workman receive whose wages for the year are $1095.50?

7. A manufacturing company pays 80¢ an hour for a certain class of work, with time and a half for overtime. Find the wages due for 17½ hr. of time and 4¾ hr. of overtime.

Graph. In all those lines of modern business in which statistics are extensively used the *graph* plays an important part. For example, a company that did a business of $103,000 in 1913, $105,000 in 1914, $110,000 in 1915, and $112,000 in 1916 found that its business increased to $135,000 in 1917, $185,000 in 1918, and $208,000 in 1919. While these figures show a decided increase, we appreciate the sudden change more easily if we look at the graph. Such graphs are familiar to all readers of the newspapers and magazines of the present day, having become very common of late years. Graphs used in business are frequently called *pictograms.*



The following kind of graph is known as a *bar pictogram*:



When the statistics were first presented to the directors in this manner they saw much more clearly than ever before that the selling expenses were excessive, that the amount paid for advertising the goods was relatively too small, that the expenditure for equipment needed investigating, and that the amount assigned to the social betterment of their employees was altogether too small to produce good results.

Squared paper like the above is very convenient for drawing graphs. It may be purchased at a stationer's shop.

Exercise. Graphs

1. Using a line graph, as in the first illustration on page 150, represent the change in the average price of lake copper for six consecutive years, as follows: year A, 12.7¢; year B, 16.7¢; year C, 16.7¢; year D, 13.5¢; year E, 28¢; year F, 32¢.

2. Using a line graph, represent the change in the annual business of a company whose total business for seven consecutive years was as follows: 1914, $28,000; 1915, $32,000; 1916, $78,000; 1917, $114,000; 1918, $120,000; 1919, $132,000; 1920, $124,000.

3. The net income of a certain railway for a recent year was applied as follows:

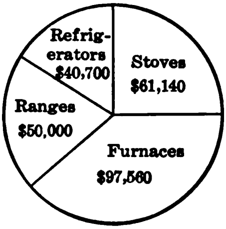
Sinking-fund requirements	$600,000
Dividends on preferred stock	5,000,000
Dividends on common stock	20,000,000
Additions and betterments	1,200,000
Surplus to profit-and-loss account . . .	6,500,000

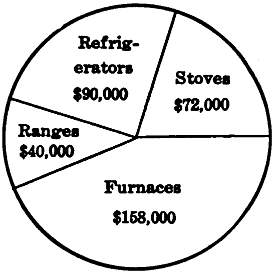
Represent these statistics by a bar pictogram, using $\frac{1}{4}$ in. to represent $1,000,000.

4. The total annual business of a certain company for the last ten years was as follows: $175,000, $185,000, $195,000, $225,000, $210,000, $230,000, $260,000, $240,000, $275,000, $300,000. Draw a graph, either by the use of a line or by the use of bars, to show the trend of business.

5. In a certain large city market it was found that when the best eggs sold at 75¢ a dozen, 150 doz. were sold a day; when they were 70¢ a dozen, 225 doz.; when 65¢ a dozen, 260 doz.; when 60¢, 300 doz.; when 55¢, 375 doz.; when 50¢, 400 doz.; when 45¢, 450 doz.; and when 35¢, 600 doz. Draw a graph to show the relation of price to sales.

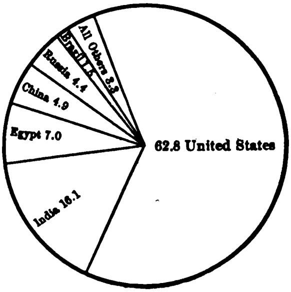
Circular Pictogram. It has become common in connection with the financial statistics of corporations to make use of the circular pictogram. For example, these circles represent the comparative amounts of business done by a certain house in two consecutive years. They aid the eye in grasping the following statistics:





	FIRST YEAR	SECOND YEAR
Stoves	$61,140	$72,000
Furnaces	97,560	158,000
Ranges	50,000	40,000
Refrigerators.	40,700	90,000

By looking at the above graphs we see that the total business has increased, the second circle being considerably larger than the first. Although more stoves were sold in the second year, we see that there was a relative shrinkage. The sale of ranges has fallen off actually and relatively.



A circular pictogram is particularly convenient when we are dealing with per cents. For example, this pictogram shows the per cents of the world's mill supply of cotton contributed by each country in a recent year. It tells us at a glance that the United States contributes considerably more than half of the supply of the whole world.

Exercise. Circular Pictograms

1. A drug store finds that 35% of its income is derived from the sale of candies, cigars, and soda water; 40% from patent medicines; 15% from sundries; and the balance from prescriptions. Represent these facts by a circular pictogram.

2. Of the six most important textile fibers, cotton is 54% in quantity; wool, $14\frac{1}{2}$%; jute, 14%; flax, 10%; hemp, 7%; and silk $\frac{1}{2}$%. Represent these facts by a circular pictogram.

3. Of the pupils who attend a certain school 11% are in the first grade, 17% in the second, 16% in the third, 14% in the fourth, 13% in the fifth, 12% in the sixth, 10% in the seventh, and 7% in the eighth. Represent these facts by a circular pictogram.

4. A man's various investments are as follows: 8% of his capital is in land, 47% in bonds, 30% in stocks, 5% in banks, and the balance in notes. Represent these facts by a circular pictogram.

5. A manufacturing company devotes 62% of its gross income to wages, 8% to office expenses, 12% to advertising, 10% to selling expenses, 5% to dividends, 2% to interest, and the balance to incidental expenses. Represent these facts by a circular pictogram.

6. A company finds that the buying expenses are 16% of its gross income; office expenses, 4%; management, 10%; other overhead, 25%; selling expenses, 30%; interest, 10%; dividends, 3%; incidentals, 2%. Represent these facts by a circular pictogram.

7. A company's inventory shows that 9% of its capital is invested in land, 48% in buildings, 32% in equipment, and the balance in patents. Represent these facts by a circular pictogram.

Exercise. Review of Chapters I–IX

1. Write the number 0.00075 as a common fraction in lowest terms; using the per cent sign; in the index notation.

2. Express $\frac{5}{16}$ as a decimal; as a number with the per cent sign; as a common fraction with denominator 64.

3. Express the ratio of 7 to 16 using the ratio symbol; as a common fraction with denominator 32; as a common fraction with some other denominator; using the per cent sign.

4. Express 3 hr. $11\frac{1}{4}$ min. as hours and a common fraction; as hours and a decimal; as minutes and a fraction; as seconds.

5. Find $133\frac{1}{3}\%$ of \$7500 and check the result by division.

6. Find 125% of \$8400 and check the result by casting out nines.

7. If 2 ft. 3 in. is 5% of a certain length, what is the length? Write a brief explanation.

8. What per cent of \$375 is \$125? What per cent of \$125 is \$375? Write a brief explanation.

9. Show that a number is divisible by three if the sum of its digits is divisible by three.

10. Find a short method of dividing by $62\frac{1}{2}$, and give an explanation.

11. A bill of goods amounting to \$1728.75 is discounted at 10, 10, 5. Find the net amount.

12. In the formula $A = P(1+r)^n$, it is given that $r = 0.04$, $n = 3$, and $P = \$5000$. Find the value of A.

13. Solve Ex. 12 when $r = 0.04\frac{1}{4}$, $n = 3$, and $P = \$1000$.

14. A manufacturing company has a capital of \$500,000, and has \$12,500 available for a semiannual dividend. Find the rate of dividend.

CHAPTER X

INDUSTRIAL ARITHMETIC

Nature of the Work. The arithmetic needed in industrial work relates almost entirely to the use of measurements and the applications of formulas. The measurements that apply to industrial work in general, that is, to all kinds of such work, are relatively few and are easily understood from the study of elementary arithmetic and from such intuitive geometry as is commonly taught in arithmetic or in the course in mathematics in the junior high school.

In addition to the work mentioned above there is, of course, the arithmetic that must be studied in the special trades, but this consists of understanding the technical terms of the particular trade rather than of learning any new principles of arithmetic.

In this chapter we shall consider only such measurements and formulas as everyone should know as matters of general information relating to the industrial life about us. Ability to use these measurements and formulas will be found to be a sufficient preparation for anyone who takes up the technical work in any industry.

It should be added that a convenient device known as the slide rule has recently come into extended use in the computations of industry. It is impossible to give a satisfactory explanation of this instrument unless the student has a slide rule in hand, but a brief statement relating to it will be found on page 237. The slide rule is based upon logarithms, another convenient device used in computing.

Common Measures. Industry in general makes use of a relatively small number of the measures which are usually taught in the schools. In many of the various special industries there are measures adapted to the particular line of goods manufactured, and these measures are best learned when the individual enters upon the work. They are not necessary for purposes of general information.

In measuring lengths the inch, foot, and yard are the most common units. While the abbreviations *ft.* and *in.* are used for feet and inches respectively, the symbols ′ and ″ are more common in the shop.

In practical measuring, fractional parts of an inch are usually expressed with denominators 2, 4, 8, 16, 32, or 64; less often with denominators 3, 6, 12, 24, or 48; and with growing frequency as decimals.

In measuring areas the square inch, square foot, and square yard, with their decimal subdivisions or with the simplest common fractions, are used.

In measuring volumes the cubic inch, cubic foot, and cubic yard, with their decimal subdivisions or with the simplest common fractions, are used. The perch, of $24\frac{3}{4}$ cu. ft. or 25 cu. ft., is still used, but with diminishing frequency.

The common tables of liquid and dry measure, familiar from elementary arithmetic, are used in industry, but there is a growing custom of measuring by weight many substances that were formerly sold by the gallon or the bushel.

In measuring weight the ounce, pound, and ton of 2000 lb. are generally used, although in some industries the long ton of 2240 lb. is still found.

The other measures are generally used, if they are used at all, only in special industries.

The metric measures are considered on pages 170–177.

The tables on pages 241–248 should be consulted if necessary.

Exercise. Common Measures

1. Make a table for the conversion into decimals of an inch of the following fractions of an inch: $\frac{1}{2}$, $\frac{1}{4}$, $\frac{3}{4}$, $\frac{1}{8}$, $\frac{3}{8}$, $\frac{5}{8}$, $\frac{7}{8}$, $\frac{1}{16}$, $\frac{3}{16}$, $\frac{5}{16}$, $\frac{7}{16}$, $\frac{9}{16}$, $\frac{11}{16}$, $\frac{13}{16}$, and $\frac{15}{16}$.

Such tables are extensively used in industrial work, particularly when specifications are given in one set of fractions and must be worked out in another set of fractions.

2. Continue the table in Ex. 1 for the necessary thirty-seconds and sixty-fourths.

3. Make a table for the conversion to the nearest sixty-fourths of an inch, but expressed in lowest terms, of the following decimals of an inch: 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 0.15, 0.25, 0.35, 0.45, 0.55, 0.65, 0.75, 0.85, and 0.95.

4. Express 3′ $8\frac{5}{8}$″ as inches and a decimal of an inch; as feet and a decimal of a foot.

5. Express 3 sq. ft. 18 sq. in. as square feet and a common fraction of a square foot; as square feet and a decimal of a square foot; as square inches.

The decimal is coming to be the most common form.

6. Express 2 cu. ft. 108 cu. in. as cubic feet and a common fraction of a cubic foot; as cubic feet and a decimal of a cubic foot; as cubic inches.

7. Taking 2150.42 cu. in. to a bushel, find the number of cubic inches in 1 pk.; in 1 qt.; in 1 pt.

8. Taking 231 cu. in. to a gallon, find the number of cubic inches in 1 qt. and in 1 pt.

By comparing the results found in solving Exs. 7 and 8 it will be seen that the liquid quart and the liquid pint are not the same as the dry quart and the dry pint.

Accuracy of Measurement. No measurement that the student will ever make will be absolutely accurate. We compute with absolute accuracy; we measure accurately only within certain limits. If the length of this page is measured, the result may be correct to the nearest hundredth of an inch; if great care is taken, the length of the school building may be found to the nearest eighth of an inch; the distance between two railway stations might be found to the nearest foot; but with the best measurements we are well satisfied if we compute the distance to the sun within 100,000 mi. Merely to know that a result is correct within 1 in. or within 1 mi. does not, therefore, tell us anything about the skill of the measurer.

An error of 4 in. in the length of a room that is really 33 ft. 4 in. long is an error of 4 in. in 400 in., or 1%; but an error of 4 in. in the length of a building that is really 333 ft. 4 in. long is an error of only 0.1%. The *absolute error* is 4 in. in each case, but the *relative error* is ten times as great in the first one as it is in the second.

Since the ratio of absolute error to true result is nearly the same as the ratio of absolute error to false result, and since the latter ratio is more easily found, the following definitions are used in industrial work:

The ratio of absolute error to false result is called the *relative error.*

The relative error expressed in per cent is called the *percentage error.*

If a room is measured as 16 ft. 8 in. long, and an error of 2 in. may have been made, the possible relative error is $\frac{2}{200}$ and the possible percentage error is 1%.

If a distance is measured as 166 ft. 8 in., and an error of 2 in. may have been made, the possible relative error is $\frac{2}{2000}$ and the possible percentage error is 0.1%.

Exercise. Accuracy of Measurement

1. A physician prescribes $\frac{1}{20}$ gr. of strychnine as a tonic. If a nurse should make a mistake and give 1 gr., what would be the absolute error? the relative error? the percentage error?

Although the absolute error is slight the percentage error is so high as to be fatal. In industrial work we do not actually know the error, for if we did we would correct it. We simply know from experience that there may be an error one way or the other of a certain amount.

2. The length of a rectangle is measured as 17 ft. $3\frac{1}{2}$ in., and experience has shown that there is a possible error of $\frac{1}{8}$ in. What is the percentage error?

3. If an error in measuring the side of a square is 2%, show that the error in computing the area is about 4%.

If l is the computed length, the true length is $l \pm 0.02\,l$, or $l(1 \pm 0.02)$. Hence the area is $l^2(1 \pm 0.02)^2$. Show that this is nearly $l^2(1 \pm 0.04)$.

4. A carpenter measures the floor of a rectangular hall and finds it to be 112 ft. long and 64 ft. wide with a possible error in each case of 2 in. Find the percentage error in computing the area.

5. A workman measured the diameter of a wheel and found it to be 50 in., the real diameter being 4 ft. Calculate the circumference from each of these data and then find the percentage error in the diameter and in the circumference.

The circumference may be taken as $3\frac{1}{7}$ times the diameter.

6. A coal dealer allows and claims an error of 5% in selling coal. If he sells 2 T. at 2050 lb. and 1975 lb. respectively, what is the percentage error in each case? Does it come within the allowance?

7. A shell is loaded with 300 lb. of explosive. If a variation of 1 oz. is allowed in the weight, what is the percentage error allowed?

Plane Rectilinear Figure. A figure which lies wholly in one plane and which is bounded by straight lines is called a *plane rectilinear figure.*

The rectilinear figures used in industry and of which measurements are the most often required are the rectangle and the triangle. To these may be added the trapezoid and the parallelogram as next in importance. The square is, of course, merely a special kind of rectangle.

The following rules for measurement, already familiar from elementary arithmetic, are proved in geometry:

The area of a rectangle is the product of the length and width.

If the length is 8 in. and the width is 6 in., the area is 6 × 8 sq. in., or 48 sq. in. In industry this is practically always written as

$$6'' \times 8'' = 48 \text{ sq. in.}$$

Expressed as a formula, $A = lw$.

The area of a triangle is half the product of the base and height.

Expressed as a formula, $A = \frac{1}{2} bh$.

There is also a convenient formula for the area of a triangle in terms of the three sides a, b, and c, in which we also use s for $\frac{1}{2}(a+b+c)$. The formula is $A = \sqrt{s(s-a)(s-b)(s-c)}$.

If $a = 3$, $b = 4$, $c = 5$, then $A = \sqrt{6 \times 3 \times 2 \times 1} = 6$.

The area of a trapezoid is half the product of the height and the sum of the bases.

Expressed as a formula, $A = \frac{1}{2} h(b + b')$.

The area of a parallelogram is the product of the base and height.

Expressed as a formula, $A = bh$.

Such inverse cases as $b = A/h$ do not occur often enough in practical industry to necessitate a review in a work of this nature.

Exercise. Plane Rectilinear Figures

1. A rectangle is $5\frac{7}{8}''$ long and $3\frac{1}{4}''$ wide. Find the area correct to the nearest eighth of a square inch.

2. The base of a triangle is 4.7″ long and the height of the triangle is 2.4″. Find the area correct to the nearest tenth of a square inch.

3. The bases of a trapezoid are $7\frac{1}{4}''$ and $5\frac{3}{8}''$ respectively and the height is $4\frac{1}{16}''$. Find the area correct to the nearest fourth of a square inch.

4. Draw any plane rectilinear figure of seven sides and show that its area can be found by cutting it into figures whose areas can be found by one or more rules given on page 160.

5. Draw a plane rectilinear figure of five sides, that is, a pentagon. Take the necessary measurements with a ruler and thus find the area.

As a check it is desirable to draw the figure on squared paper and estimate the area by counting the squares, including any square of which at least half lies within the figure and excluding the others.

6. The sides of a rectangle are given as 3″ and 3.8″, where the width is exact and the length is correct to the nearest tenth of an inch. The length therefore lies between 3.75″ and 3.85″. Draw the figure as accurately as you can for each of these lengths and show that the greatest possible error arising from taking 3.8″ as the length is 0.15 sq. in.

7. The base of a triangle is given as 4.6″ exactly, and the height is measured as 3.4″ correct to the nearest 0.1 in. Between what limits does the area of the triangle lie?

8. A workman measures the side of a square and finds it to be 3.24″, but the last figure is uncertain, being possibly either 3 or 5. If he uses 3.24″ for finding the area of the square, what is the greatest possible error that can arise?

Measurement of Polyhedrons. A solid figure which is bounded by planes is called a *polyhedron.* A cube or a brick is an example of a polyhedron.

The polyhedron most frequently measured in industrial work is the *rectangular solid*, a polyhedron of six faces, every edge being perpendicular to two other edges as in the case of an ordinary brick.

Next in importance to the rectangular solid is the *prism*, a polyhedron in which two opposite faces, called the *bases*, are equal and parallel, the other faces being parallelograms.

Of much less importance in industrial work is the *pyramid*, a polyhedron in which one face, called the *base*, is a polygon of any number of sides and the other faces are triangles having a common vertex.

All the polyhedrons mentioned above are already familiar to the student from elementary arithmetic or from the mathematics already studied in the high school.

The following rules for measurement are already familiar:

The volume of a rectangular solid is the product of the length, width, and height.

If the dimensions of such a solid are 8 in., 6 in., and 4 in., the volume is 8 × 6 × 4 cu. in., or 192 cu. in. In industry this is practically always written as $8'' \times 6'' \times 4'' = 192$ cu. in.

Expressed as a formula, $V = lwh$.

The volume of a prism is the product of the base and height.

Expressed as a formula, $V = bh$.

The height is the perpendicular distance between the bases.

The volume of a pyramid is one third the product of the base and height.

Expressed as a formula, $V = \frac{1}{3}bh$.

The height is the perpendicular distance from the vertex to the base.

Exercise. Polyhedrons

1. Draw rough sketches of a rectangular solid, a prism, and a pyramid, writing under each the name of the polyhedron and the formula for the volume.

2. A box is $9\frac{1}{2}''$ long, $6\frac{3}{4}''$ wide, and $5\frac{7}{8}''$ high. Find the volume to the nearest half of a cubic inch.

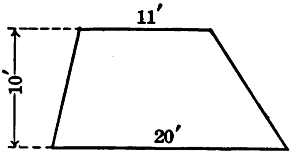
3. The base of a prism has an area of 17.35 sq. in., and the height of the prism is 6.18″. Find the volume to four significant figures.

4. The base of a pyramid has an area of 17.3 sq. in. and the height of the pyramid is 9.7″. Find the volume to three significant figures.

5. A rectangular water reservoir is to have a capacity of 50,000 gallons, and its base is to be 20′ by 30′. Allowing 231 cu. in. to a gallon, find to the next larger foot the depth of water that must be allowed for the reservoir.

6. A railway embankment extends 700′ along a river bank. A cross section of the prism is a trapezoid 10′ high, with lower base 20′ and upper base 11′. Find the number of cubic yards of earth in the embankment.

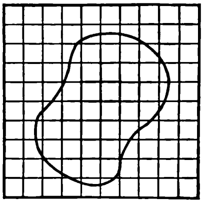
The figure illustrates a nonsymmetric trapezoid, the wall along the river being steeper than the other side.



7. A pyramidal tent 7′6″ high has for its base a square whose side is 9′6″. Find the volume.

8. Wishing to find the volume of an irregular piece of metal a workman lowered it into a barrel of water that was brimful, allowing the water to run over into a cubic tank that was 18″ on an edge. The water filled the tank to a depth of $3\frac{1}{2}''$. Find the volume of the piece of metal.

Measurement of Curvilinear Figures. Of the curvilinear figures, that is, plane figures bounded by curve lines, the one used almost exclusively in general industrial lines is the circle. The area of an irregular curvilinear figure may be found approximately by drawing the figure, as here shown, on squared paper, the size of each square being known, and then counting the squares inclosed.



In estimating the area of a figure like the one here shown, the student should use his judgment as to the counting of the parts of squares. It is usually sufficient to include only each fraction which is at least half of a square.

As shown in elementary arithmetic or in geometry,

The circumference of a circle is $3\frac{1}{7}$ *times the diameter.*

It is shown in geometry that the circumference is πd, where π (pī) is a symbol that stands for 3.14159+, a number that is approximately $3\frac{1}{7}$.

Expressed as a formula, $c = 3\frac{1}{7}\,d$, or $c = \pi d$.

Since $d = 2\,r$, we may write $c = 2\,\pi r$.

Since it is generally easier to measure a diameter than a radius, unless a circle is drawn with a known radius, the formula $c = \pi d$ is a better one to use than $c = 2\,\pi r$.

Since $c = \pi d$, it follows that $d = c/\pi$, or $c = \frac{7}{22}\,d$.

The area of a circle is $3\frac{1}{7}$ *times the square of the radius.*

Expressed as a formula, $a = 3\frac{1}{7}\,r^2$, or $a = \pi r^2$.

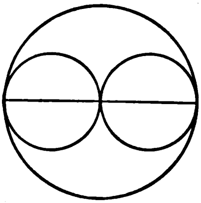
From this formula we see that $r = \sqrt{a/\pi}$.

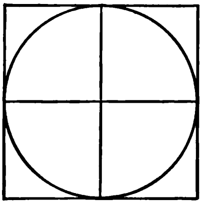
The area of a circle is required chiefly in connection with the volume of a cylinder or the cross section of a cylindric shaft of some kind.

The approximate value of π is easily found by winding a piece of paper around a cylinder, pricking through the paper where it overlaps, unwrapping the paper, measuring the distance between the pinholes, and then dividing the result by the diameter of the cylinder.

Exercise. Curvilinear Figures

1. Examine these two figures and write statements of what they tell as to the area of a circle compared with the area of a circle of twice the diameter; the area of a circle compared with the square on the radius; the area of a circle compared with the square on the diameter.





2. By wrapping paper about a cylinder as suggested at the bottom of page 164, find the approximate value of π.

Carry the result to two decimal places, but do not expect the second place to be accurate.

3. A circular race track is $\frac{1}{2}$ mi. round. Find the number of yards in the diameter.

4. If a bundle of asparagus 10 in. round costs 50¢, how much should a bundle cost that is 15 in. round?

5. A workman finds the circumference of a shaft to be 11 in. In order to find the strength of the shaft he must know the area of a cross section. Find this area.

First find the diameter or the radius of the shaft.

6. A draftsman is required to draw a circle with circumference 6.6 in. What radius shall he use?

7. If the cost varies as the weight, and if a solid steel cylindric pillar 4 in. in diameter costs $17.50, how much will a pillar of the same length cost if the diameter is 6 in.?

Measurement of Round Bodies. Of the round bodies, generally taken to include the cylinder, cone, and sphere, the one most frequently used in industry is the cylinder. Water tanks; cans for fruit, sirup, and oil; metal columns, rods, and shafts; pillars of stone, brick, or concrete; air shafts and the interior of chimneys; pipes and wires of various kinds; electric conduits; and many other examples of a similar kind show the uses to which a workman applies the cylinder.

The measurements of these various forms of cylinders usually refer to the cross section, and therefore involve the circle; to finding the volume, as in the case of the capacity of a cylindric tank; or to finding the curve surface, as in the case of knowing the surface of a silo that is to be painted.

It is shown in elementary arithmetic and in geometry that the following statements are correct:

The volume of a cylinder is the product of the base and height.

Expressed as a formula, $V = bh.$

If the base of the cylinder is 14 sq. in. and the height is $4\frac{1}{2}$ in., the volume is $4\frac{1}{2} \times 14$ cu. in., or 63 cu. in.

The curve surface of a cylinder is the product of the circumference and height.

Expressed as a formula, $S = ch.$

Since $c = \pi d$, we have $S = \pi dh.$

The volume of a cone is one third the product of the base and height.

Expressed as a formula, $V = \frac{1}{3}bh.$

The curve surface of a cone is one half the product of the circumference of the base and the slant height.

Expressed as a formula, $S = \frac{1}{2}cs.$

Exercise. Round Bodies

1. A workman has a cylindric can 14″ in diameter and about 15″ high. He knows that it is either a 5-gallon or a 10-gallon can, but he has forgotten which. There being 231 cu. in. in a gallon, find which it is.

2. A rule for finding the approximate number of gallons that a water main will hold per yard of length is to divide the square of the diameter by 8. Check the accuracy for a 14-inch water main and find the percentage error.

3. A boiler has 240 tubes, each 16′ long and 2.1″ in diameter. Find the total heating surface of the tubes.

4. A casting when submerged in a cylindric tank of diameter 14″ causes the water to rise 9″. Find the weight of the casting at 0.28 lb./cu. in.

The expression 0.28 lb./cu. in. means 0.28 lb. per cubic inch. In general, the symbol / means either division or the expression "per."

5. Find the cost of lining the curve surface and base of a cylindric tank 33″ in diameter and 28″ high, the lining being sheet lead worth 16¢/sq. ft. and the labor costing $1.25.

6. Find the number of cubic inches of metal per mile in a wire $\frac{1}{8}''$ in diameter.

7. A can has a cover in the shape of a cone. The slant height is 5″ and the diameter of the base is 9″. Find the number of square inches in the curve surface.

8. A pile of sand is in the form of a cone. The circumference of the base is 33′ and the height of the pile is 5′. How many cubic yards of sand are there?

9. The formulas for the volume and surface of a sphere are respectively $\frac{4}{3}\pi r^3$ and $4\pi r^2$. Find the volume and the surface of a sphere of radius 4.9″.

Measurement of Land. The mathematics of the farming industry relates chiefly to the keeping of accounts, the making of inventories, and the measuring of land. The subject of accounts has been sufficiently considered on pages 98–105, and the subject of inventories is too technical as to language and too elementary as to the arithmetic involved to have place in a work of this nature. The measuring of land, however, should be considered.

The measures of length used on farms may be summarized as follows:

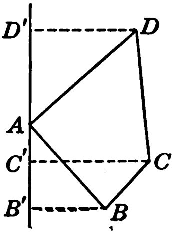
7.92 inches (in.) = 1 link (li.)
12 in. = 1 foot (ft.)
3 ft. = 1 yard (yd.)
25 li. = $5\frac{1}{2}$ yd. = $16\frac{1}{2}$ ft. = 1 rod (rd.)
4 rd. = 66 ft. = 100 li. = 1 chain (ch.)
320 rd. = 5280 ft. = 80 ch. = 1 mi.

The following measures of area are used:

9 square feet (sq. ft.) = 1 square yard (sq. yd.)
$30\frac{1}{4}$ sq. yd. = $272\frac{1}{4}$ sq. ft. = 625 square links (sq. li.)
= 1 square rod (sq. rd.)
16 sq. rd. = 1 square chain (sq. ch.)
10 sq. ch. = 160 sq. rd. = 43,560 sq. ft. = 1 acre.

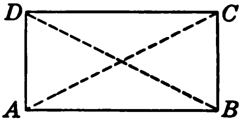
These tables need not be memorized, the most important facts being already familiar to the student.

A knowledge of the methods of measuring triangles and trapezoids is sufficient for any ordinary case of the measuring of land, as will be seen from an examination of the annexed figure. The area of the field *ABCD* is the sum of the areas of two trapezoids minus the areas of two triangles.

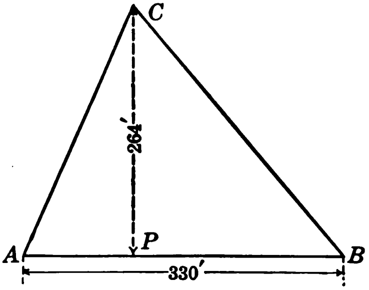


Exercise. Measurement of Land

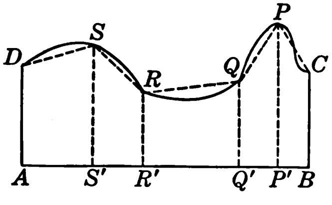
1. In order to find whether a field $ABCD$ is a rectangle a man measures and finds that $AB = DC$. Is this sufficient to determine the fact? He also finds that $BC = AD$. Is this added measurement necessary? Is it sufficient? He also finds that $AC = BD$. Is this added measurement necessary? Is it sufficient? If AB is 40 rd. and AD is 20 rd., what is the area?



2. In order to find the area of this field a man revolved a cord about C, lengthening and shortening it until its farther extremity just grazed AB. He thus found the perpendicular CP. By measurement he then found that $CP = 264'$ and that $AB = 330'$. Find the number of acres in the area of the field.



3. A field $ABCD$ is bounded on one side $CPQRSD$ by a river, A and B being right angles. From P, Q, R, and S perpendiculars are drawn as in the figure. It is found that $AD = 8$ rd., $SS' = 10$ rd., $RR' = 6$ rd., $QQ' = 7$ rd., $PP' = 12$ rd., $BC = 8$ rd. It is also found that $AS' = 6$ rd., $S'R' = 4$ rd., $R'Q' = 8$ rd., $Q'P' = 3$ rd., $P'B = 3$ rd. Find the approximate area of the field.



4. To find the area of the field $ABCD$ shown on page 168 a surveyor ran a north-and-south line through A and then drew the perpendiculars BB', CC', and DD' as shown. If $BB' = 6$ rd., $CC' = 10$ rd., $DD' = 9$ rd., $B'C' = 4$ rd., $C'A = 3$ rd., and $AD' = 8$ rd., what is the area?

Metric Measures. The measures that were formerly used by the world proved to be so unsystematic and inconvenient, varying so much in different countries, that most of the civilized nations adopted a new system in the nineteenth century. This set of measures was devised in France about the year 1800 and is known as the *metric system.*

The metric system is now obligatory in more than thirty countries and is in partial use in most of the others. Before the European war we had little need for the metric system except in scientific work, where practically all the world uses it. Our recent great expansion of foreign trade, however, has necessitated the use of this system in describing goods intended for export and in making machinery of various kinds. Furthermore, the educated person needs to know the principal units of the system for the reason that he reads about them in the newspapers and magazines. We read of 75-millimeter guns, of distances of 8 kilometers and of 1200 meters, or of shells that weigh 250 kilograms, and it is convenient to know what these measures mean.

It is not for the schools to attempt to decide whether the metric system will probably replace our common system, but it is important that they should give to every pupil a fairly good knowledge of the general nature of a system of weights and measures that is so widely used.

It is easy to understand the metric system, for there are only six prefixes and six important measures to learn. The prefixes and their meanings are as follows:

milli-	0.001	deka-	10
centi-	0.01	hekto-	100
deci-	0.1	kilo-	1000

In order to have the metric system seem more familiar, we shall select problems from other than mere industrial sources.

Exercise. Metric System

1. A mill is what part of $1? A millimeter is what part of a meter? If a meter is about 40 in., a millimeter is about what part of an inch? Draw a line about a millimeter long.

The meter is 39.37 in. long, but in the problems on this page we may think of it as about 40 in.

2. A cent is what part of $1? A centimeter is what part of a meter? A centimeter is about what part of an inch? Draw a line about a centimeter long.

3. What is the meaning of deci-? A decimeter is what part of a meter? A decimeter is about how many inches long? Draw a line about a decimeter long.

4. A French 75 is a French gun that fires a projectile 75 millimeters in diameter. Draw a line representing this diameter approximately.

5. What is the meaning of kilo-? A kilometer is how many meters? Taking a meter as $3\frac{1}{4}$ ft., how many feet in a kilometer? This is about how many tenths of a mile?

6. A gram is about $\frac{1}{500}$ lb., and so a kilogram is about how many pounds?

A kilogram is more nearly 2.2 lb.

7. A regiment protected by a battery of 125-millimeter guns advanced 3 kilometers, captured a hill 160 meters high, and bombed the enemy's trenches with bombs weighing a kilogram apiece. Write the sentence, translating the metric measures into our common measures.

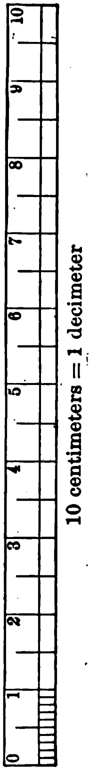
8. A liter is about a quart, and so a hektoliter is about how many quarts? about how many gallons?

The word *liter* is pronounced "lee-ter."

By answering the questions on this page the student has learned some of the more important measures of the metric system.

Metric Length. The table of length is as follows:

A **kilometer** (km.)	= 1000 meters	
A hektometer	= 100 meters	
A dekameter	= 10 meters	
Meter (m.)		
A decimeter (dm.) =	0.1	of a meter
A **centimeter** (cm.) =	0.01	of a meter
A **millimeter** (mm.) =	0.001	of a meter



The meter is 39.37 in., or about $3\frac{1}{4}$ ft., or a little over a yard; the kilometer is about 0.62 of a mile. When the metric system was invented in France over a century ago, the meter was intended to be one ten-millionth of the distance on the surface of the earth from the equator to the pole, but it varies slightly from this standard.

As in United States money we seldom speak of anything except dollars and cents, so in the metric system only those measures printed in black letters in the tables of this chapter are in common use.

All the units of the system are derived from the meter. Every compound name is accented on the first syllable; thus, *mil'limeter.*

The teacher should be supplied with a meter stick, a liter, and a cubic centimeter, and these can easily be made in school if necessary. The work should be as practical as possible.

The abbreviations in this book are in common use. Some, however, use Km., Dm., and dm. for kilometer, dekameter, and decimeter.

Any one of these measures may be expressed in terms of another measure by *simply moving the decimal point to the right or left.*

Thus, as 245¢ = 24.5 d. = $2.45,

so 2475 mm. = 247.5 cm. = 24.75 dm. = 2.475 m.

Exercise. Metric Length

1. The distance from Chicago to New York by one route is about 1500 km. Express this distance in miles.

2. The distance from New York to Albany is 229 km. Express this distance in miles.

The purpose of Exs. 1 and 2 is to allow us to form some fair estimate of distances expressed in the metric system. Think of some building about 0.6 mi. from the school so as to visualize 1 km.

3. In a gymnasium where scientific records are kept it is found that a certain boy is 144 cm. tall. Express this in inches; in feet and inches; in feet and a decimal.

Practically we work in the metric system or in our common system, rarely having any occasion to transfer from one to the other. As in Exs. 1 and 2, the purpose here is merely to visualize the measures.

4. Measure the length of this page in centimeters and tenths; in decimeters and hundredths; in millimeters. Measure the width of the printed portion in the same way.

If not provided with a ruler marked in millimeters, transfer upon a strip of paper the measure shown on page 172.

5. Measure the diameter of a 5-cent piece and the diameter of a dime, stating the results in the metric system.

6. Draw a line *AX* and from one end *A* mark off *AC* equal to 17 mm.; then *CD* equal to 9 mm.; then *DE* equal to 26 mm. Measure *AE*, write the result, and check by adding the separate lengths.

7. Make a fine dot with ink on the rim of a silver "quarter." Before the ink dries roll the coin along a piece of paper until the ink has marked the paper twice, thus giving a line equal to the circumference. Measure in millimeters the diameter and this line which is equal to the circumference. Check the work by the formula $c = \pi d$.

Metric Area. The table of area is as follows:

A square kilometer (sq. km.) = 1,000,000 square meters
A square hektometer = 10,000 square meters
A square dekameter = 100 square meters
Square meter (sq. m.)
A square decimeter = 0.01 of a square meter
A **square centimeter** (sq. cm.) = 0.0001 of a square meter
A **square millimeter** (sq. mm.) = 0.000001 of a square meter

There is no generally recognized set of abbreviations for square measure. Instead of using sq. m., scientific writers often use m^2, and so for the other measures.

In measuring land the square dekameter is called an *are* (ăr); the square hektometer is called a *hektare* (ha.). The hektare is equal to 2.47 acres, or nearly 2½ acres. The student will have no present use for these measures and so they need not be considered further.

Metric Volume. The table of volume is as follows:

Cubic meter (cu. m.)
A **cubic decimeter** (cu. dm.) = 0.001 of a cubic meter
A **cubic centimeter** (cu. cm.) = 0.000001 of a cubic meter
A cubic millimeter (cu. mm.) = 0.000000001 of a cubic meter

There is no generally recognized set of abbreviations for cubic measure. Instead of using cu. m., scientific writers often use m^3, and so for the other measures.

We may also have cubic hektometers, cubic kilometers, and cubic dekameters. These terms are seldom used, however, and need not be considered by us at this time.

In measuring wood the cubic meter is called a *stere* (st.). The word *stere* is pronounced "stair." The student will have no real use for this measure, however, and so it will not be considered further.

It is evident that the above tables can at once be obtained from the table of length by merely squaring or cubing. They need not be specially learned, since they can always be obtained in this way, as is evident from Exs. 2 and 3 on page 175.

Exercise. Metric Area and Volume

1. How many feet in 1 yd.? How many square feet in 1 sq. yd.? How many cubic feet in 1 cu. yd.? How many cubic feet in a cube that is $1\frac{1}{2}$ yd. on an edge?

2. How many centimeters in 1 m.? How many square centimeters in 1 sq. m.? How many cubic centimeters in 1 cu. m.? How many square centimeters in a square that is 1.2 m. on a side?

3. How many inches in 1 yd.? How many square inches in 1 sq. yd.? How many square millimeters in 1 sq. cm.? How many cubic millimeters in 1 cu. cm.?

4. Find the area of a rectangle 17 mm. by 28 mm.

5. Find the area of a triangle whose base is 5 cm., and height 7 cm. Express the result both as square centimeters and as square millimeters.

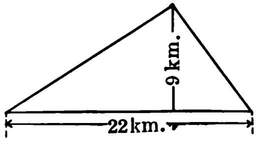
6. Find the area of the cross section of a French 75.

This means the area of the circular cross section of the bore of a gun that has a diameter of 75 mm.

7. In Ex. 6 find the circumference of the bore.

8. During the war an army advanced on a straight front of 22 km. and occupied a triangular piece of territory 9 km. in depth, as shown by the figure. How many square kilometers did the army gain?

9. In Ex. 8, in order to grasp the situation in terms of our common units, express the lengths in miles and find the area in square miles.



10. The specifications for a piece of machinery require a cylinder 0.48 m. long and 0.2 m. in diameter. Find the volume of the cylinder.

Metric Capacity. The table of capacity is as follows:

A **hektoliter** (hl.) = 100 liters
A dekaliter = 10 liters
Liter (l.)
A deciliter (dl.) = 0.1 of a liter

A liter is the volume of a cube 1 dm. on an edge, and so it is possible to express this table in terms of cubic measures. For this reason the centiliter and milliliter are not often used.

The liter is practically the same as our quart.

A liter contains about 61.024 cu. in. and is equivalent to 1.0567 liquid quarts or 0.908 of a dry quart. These details need not be memorized.

Metric Weight. The table of weight is as follows:

A **metric ton** (t.) = 1,000,000 grams
A **kilogram** (kg.) = 1,000 grams
Gram (g.)

The quintal, 100 kg., is also used; but, like the dekagram and hektogram, the term is not employed frequently enough to demand our attention. The meaning of the words "decigram," "centigram," and "milligram" is evident, and the student will commonly find these expressed merely as decimals of a gram.

1 cu. cm. of water weighs 1 g.
1 l. of water weighs 1 kg.
1 cu. m. of water weighs 1 t.

The standard units vary slightly from these theoretical units, but for practical purposes these statements are commonly used.

A kilogram is about 2.2 lb.; and a metric ton is about 2204.6 lb. A kilogram is usually called a *kilo* (kē-lo).

A gram is equivalent to 15.432 grains, or about 0.035 avoirdupois ounces. A pound is equivalent to about 0.4536 kg. Our common ton is about 0.907 of a metric ton. The metric ton is about the same as the long ton, which is sometimes used in this country.

Exercise. Metric Capacity and Weight

1. A manufacturer has a demand from a South American country for some cylinders with capacity of 125 l. He may roughly estimate this as how many gallons per cylinder?

2. An exporting house receives an order for 25 metric tons of copper plates, and needs to know this weight in pounds. How many pounds are there?

3. A tank 3 m. long, 1.8 m. wide, and 1.2 m. deep is filled with water. What is the weight of the water?

4. A gas cylinder is 1.8 m. long and has a diameter of 0.3 m. Find the number of liters in the cylinder.

5. An exporting house receives an order for a shipment of liter measures, each measure to be a cylinder with diameter 0.1 m. Find the height of each measure.

6. A liter measure in the form of a cylinder is 0.2 m. high. Find the diameter of the cylinder.

7. A manufacturer is required to supply a tank 3.4 m. long and 2.8 m. wide, with capacity 19,000 l. Find the height of the tank to the nearest 0.1 m.

8. An order is received by an exporting house for a shipment of cotton amounting to 225 metric tons. It is required to know, to the nearest bale, the number of bales of 500 lb. each that will meet the requirements.

9. Find in kilograms the weight of the water a tank 4 m. by 3 m. by 6 m. will hold. Find also in pounds the weight of the water a tank 4 ft. by 3 ft. by 6 ft. will hold, taking 62.5 lb. as the weight of 1 cu. ft. of water.

10. Estimate in meters the length, width, and height of your schoolroom, and from these estimates find the number of cubic meters in the room.

Specific Gravity. In various lines of industry as well as in science we need to know the weight of a substance compared with the weight of water. For example, a piece of silver is 10.5 times as heavy as an equal volume of water. If we have a piece of metal that resembles silver in color but is only 8.7 times as heavy as an equal volume of water, we know that the metal is not pure silver.

The ratio of the weight of a given substance to the weight of an equal volume of water is called the *specific gravity* of the substance.

Thus, if a cubic decimeter of copper weighs 8.9 kg., the specific gravity of copper is 8.9, because 1 cu. dm. is the same as 1 l., and 1 l. of water weighs 1 kg. Therefore copper is 8.9 times as heavy as water.

Specific gravity is also known as *relative density.*

It is usually easy to find the volume of any substance by immersing it in a vessel full of water and measuring the amount of water that runs over. There are other and better methods of finding the volumes of certain solids, but they need not be considered at this time. The student who wishes to know them may consult a work on physics.

In the work on pages 179 and 180 the weight of 1 cu. ft. of water may be taken as 62.5 lb. or as 1000 oz.

This approximate value may be used in finding specific gravities. For example, if 27 cu. in. of metal weighs 250 oz., the specific gravity of the metal is $250 \div (27 \times \frac{1000}{1728})$, or 16.

By means of specific gravity the purity of metals and other substances may be determined.

In the case of gases the weight of the gas is usually compared with that of air or hydrogen instead of water.

The letters s.g. are often used to mean "specific gravity." They may be used in the solutions of the problems on page 179.

Exercise. Specific Gravity

1. Freshly fallen loose snow weighs 12 lb./cu. ft. What is the specific gravity of the snow?

2. A steel forging measures 12 ft. by 10 in. by 6 in. If its specific gravity is 7.8, how much does it weigh?

3. If the specific gravity of gold is 19.3, what is the weight of a cube of gold 2 mm. on an edge?

4. A tunnel 625 yd. long having a cross section of 64 sq. yd. is excavated through rock of specific gravity 2.7. Find the weight of the rock removed.

5. How much space will be filled by 14 tons of wrought iron of specific gravity 7.7?

6. Find the average specific gravity of a piece of brick construction weighing 114 lb. per cubic foot.

7. If 3 l. of alcohol weigh 2.37 kg., what is the specific gravity of alcohol?

8. If 13 l. of milk weigh 13.39 kg., what is the specific gravity of milk?

9. The specific gravity of silver is 10.5. Some spoons weighing 743.4 g., thought to be silver, are immersed in a jar full of water and 70.8 cu. cm. of water overflow. Do the spoons stand the specific-gravity test?

10. A body immersed in a liquid always weighs less than it does in air by the weight of the liquid that it displaces. If a piece of gold weighs 38.6 g. in air and 36.6 g. in water, what is its specific gravity?

11. A brass model made on a scale of 1 in. to a foot is to be constructed of steel. The specific gravity of brass is 8.4 and that of steel is 7.8. A piece of the model weighs 14 oz. What will the corresponding piece of the machine weigh?

Exercise. Review of Chapters I–X

1. The specific gravity of steel being 7.8, find the weight of a steel rod 2 m. long and 4 cm. in diameter.

2. If a ton of coal on board a steamer occupies 41 cu. ft. of space, and if the specific gravity of a lump of coal is 1.53, find the fractional part of the space that is lost in the bunker owing to the irregular shapes of the lumps of coal.

3. Find the depth of the water in a reservoir 6 ft. by 8 ft., there being 800 gal. of water in the reservoir.

4. A pattern for an iron casting is made of pine weighing 40 lb. a cubic foot. The pattern weighs 28 lb. and the specific gravity of cast iron is 7.7. Find the weight of the casting.

5. A rectangle is 8 cm. long and 6 cm. wide. If you make a mistake in measuring and find each side 1 mm. too long, what is the percentage error in the area?

6. Multiply 14.788 by 9.832, finding the product correct to four significant figures.

7. Divide 17.293 by 8.384, finding the quotient correct to three significant figures.

8. Find the amount of $1750 for 1 yr. 6 mo. at $5\frac{1}{2}\%$.

9. A man bought 100 shares of stock, par value $100 a share, at $107\frac{1}{2}$, brokerage as usual. Find the cost.

10. During a shower a rain gauge registers a fall of 0.59″. Find to the nearest 100 tons the weight of water that fell on 1 sq. mi. of land.

11. Find the area of a triangle with base 72 ft. 8 in. and height 45 ft.

12. Find the volume of a sphere with diameter 1 ft. 9 in.

13. The specific gravity of mercury being 13.57, find the weight of 1 cu. ft. of mercury.

CHAPTER XI

ARITHMETIC OF THE BANK

Nature of Banking. Banks exist for three reasons, so far as most people are concerned. First, they afford a place where money may be safely kept, to be drawn out as needed. Most banks pay little or no interest on the money deposited with them, but they have the use of the money in return for the accommodation they give the depositors and for the security they offer.

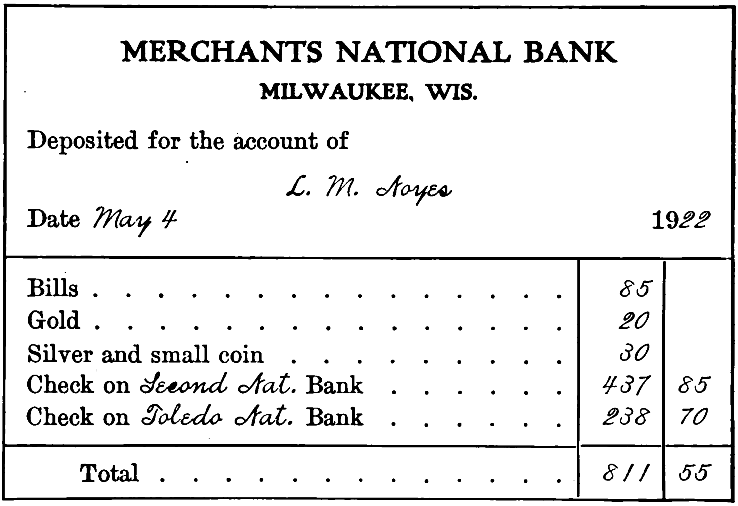
In the second place, banks exist for the purpose of lending money. People formerly borrowed from one another, but now the borrowing is done chiefly from banks. The banks charge interest for the money that they lend, and their profits come chiefly from this source.

The third reason for the existence of banks is that they attend to various financial matters for their customers, such as those relating to exchange. These include the paying of money that some customer may owe in a distant place or the collecting of money due.

Banks are of two general kinds, the savings banks, which pay compound interest on all deposits, and the commercial banks which, in general, do not pay interest on ordinary accounts, although they allow interest on certificates showing that money is to be left for at least 30, 60, or 90 da.

We shall not need to discuss trust companies, since the arithmetic questions involved are the same as those relating to banks.

Depositing Money. When a reliable person wishes to open an account with a bank, he begins by depositing some money and receiving a bank book. In depositing money a printed *deposit slip* is filled out by him each time, as here shown.



MERCHANTS NATIONAL BANK
MILWAUKEE, WIS.

Deposited for the account of

L. M. Noyes

Date May 4 1922

Bills	85	
Gold	20	
Silver and small coin	30	
Check on Second Nat. Bank	437	85
Check on Toledo Nat. Bank	238	70
Total	811	55

The money and the deposit slip are then presented to the *receiving teller* or to the *cashier*, together with the customer's bank book in which the teller or cashier enters the amount deposited. This bank book is left with the bank, usually about once a month in the case of commercial banks, so that the customer's account may be verified and any errors in bookkeeping or in the customer's check book corrected.

Many banks do not insist on having inserted in a deposit slip the name of the bank on which a check is drawn, but it is desirable for the depositor to insert the name in the case of each of the checks so as more easily to trace any error in his account.

Exercise. Depositing Money

1. On June 7 M. S. Randall had on hand bills to the amount of $765 and silver and small coin to the amount of $78.52. He also had checks as follows: Liberty National Bank, $34.85; Miners Bank, $128.75; Third National Bank, $86.50; Lincoln Trust Co., $52.60; First National Bank, $326.70; Merchants National Bank, $208.43. For the day's use he reserved bills to the amount of $250 and silver and small coin to the amount of $50, and deposited the rest in the First National Bank. Write a deposit slip such as Mr. Randall would give to the receiving teller when he deposits the money.

Make out deposit slips for the following deposits, naming some bank in your vicinity:

2. Bills, $482; silver, $45; checks on Farmers Bank, $375.80; First National, $226.68.

3. Bills, $525; gold, $30; silver, $80; checks on Jefferson Trust, $75.50; Manufacturers Bank, $528.70.

4. Bills, $250; gold, $10; silver, etc., $65; checks, $58, $75, $325.50, $648.75.

5. Bills, $575; silver, etc., $35; checks, $57.25, $28.50, $96.80, $325.75, $128.30.

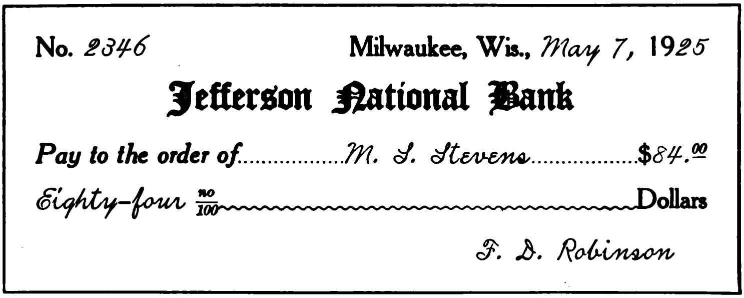
6. Bills, $250; silver, etc., $125; checks, $35.75, $75.35, $82.80, $63.25, $325.60.

7. Bills, $600; silver, etc., $75; checks, $45.50, $65.80, $38.75, $45, $92.90, $83.20, $126.40, $37.82.

8. Bills, $750; silver, etc., $45; checks, $81.25, $72.68, $51.80, $28.75, $68.38, $248.35, $129.50.

9. Bills, $1450; silver, etc., $25; checks, $48.75, $36.90, $43.45, $75.60, $47.50, $135.50, $235.25, $65.80, $240.75, $21.80, $342.75, $298.62.

Check. As learned in elementary arithmetic, when a person has an account in a bank of deposit and wishes to draw some money for himself or to direct the bank to pay some money to another person, he writes a *check* for the amount. The following is a common form of a check payable to order:



No. 2346 Milwaukee, Wis., May 7, 1925

Jefferson National Bank

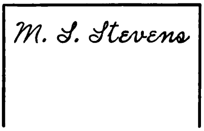
Pay to the order of……………M. S. Stevens……………\$84.00

Eighty-four $\frac{no}{100}$～～～～～～～～～～Dollars

F. D. Robinson

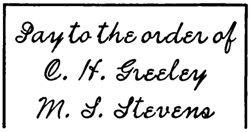
In the case here considered F. D. Robinson is the *drawer* of the check and M. S. Stevens is the *payee.*

A check may be made payable to "Self," in which case the drawer alone can collect it; or to the order of the payee, as in the above check, in which case the payee must *indorse* it, that is, he must write his name across the back; or to the payee or "bearer," or to "Cash," in which case anyone can collect it. Various common forms of indorsement are as follows:



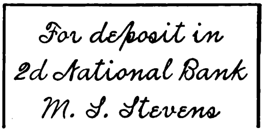
M. S. Stevens

BLANK INDORSEMENT



Pay to the order of
C. H. Greeley
M. S. Stevens

INDORSEMENT TO ANOTHER PERSON



For deposit in
2d National Bank
M. S. Stevens

INDORSEMENT FOR DEPOSIT

Indorsements for deposit in a bank where the indorser has an account are often made with a rubber stamp.

Exercise. Drawing Money

1. Write a check for $25.75 on some fictitious bank, the drawer being John Doe and the payee being Richard Roe or order. Indorse the check properly.

2. Write a check for $36, the drawer being Richard Roe and the payee being John Doe or bearer.

3. Write a check for $17.05, the drawer being Peter Peters and the payee being John Jones or order. Indorse the check in full, payable to the order of Samuel Simpson.

Such an indorsement to another person is called an *indorsement in full.* In order to collect the money Samuel Simpson must now indorse the check by writing his name under that of John Jones.

4. Write a check for $16.25 payable to yourself.

5. If your deposits in a bank have been $25.50, $38.05, $56.30, $15.10, $12.42, $15.60, and $7.50, and you have drawn checks for $18.02, $6.35, $7.25, $18.75, and $14.30, what is then your balance in the bank?

6. A man having $1426.50 in the bank deposits $475.40, $396.85, $572.86, $392.80, $287.96, $342.80, $228.75, and $235.75, and draws checks for $326.42, $187.96, $273.49, $61.72, and $8.20. He also pays by check a bill for $182.60 less 10, 5, and another bill for $147.50 less 8, 3. What is now his balance?

7. If your last balance in the bank was $75.80 and you have since then deposited $33.48, $51.73, $27, $14.37, $22, $85, $35.75, and $21.80, and have drawn checks for $71.20, $96.80, $125, and $57.20, what is your present balance?

8. A man having $1685.75 in the bank deposits $287.90, $368.73, $496.37, $128.75, and $425, and draws a check to pay a note of $350 plus interest for 3 mo. at 6%. What is now his balance?

Bank Book. After a depositor has opened an account at a commercial bank he makes deposits from time to time. He hands the money, checks, and deposit slip with his bank book to the receiving teller, who writes on the left-hand page the amount deposited. This book is left at the bank every few weeks to be *balanced.* The following is an extract from a depositor's book, showing deposits on the left-hand page and withdrawals on the right-hand page:

LEFT-HAND PAGE					RIGHT-HAND PAGE		
May	2	Balance	4276	83			
	5	E. K. R.	423	40			
	9	C. H. T.	142	86	VOUCHERS RETURNED AS PER LIST DATED JUNE 1, 1925	3486	90
	15	L. P.	368				
	18	E. A. S.	43	08	Balance	1767	27
			5254	17		5254	17
June	1	Balance	1767	27			

The letters after the dates of deposit on the left-hand page are the initials of the names of the receiving tellers. The *vouchers* mentioned on the right-hand page are the checks which have been paid and are now returned.

The book is balanced in the same way that accounts are balanced, as described on page 98; that is, the balance is found by subtracting the sum of the vouchers from the sum of the deposits including as a deposit the balance brought over from the preceding period.

The vouchers are not listed on the right-hand page, but a separate list is returned with the checks. This list is usually made on a machine that types the items and adds them, thus making the chance of error very small.

Exercise. Bank Books

1. A merchant having a balance of $9872.58 in a bank deposits $1768.40, $3942.80, $1986.42, $3928.77, $826.87, and $2873.30 on the successive days of a week. During the same period he gives checks for $3286.75, $142.80, $38.76, $923.48, $4.20, $753.64, $250, $424.36, $297.50, $1286.49, $38.75, $92.80, $120, and $42.36. What is then his balance?

2. In Ex. 1 write two pages like those given on page 186, showing how the book appears when balanced.

3. A company having a balance of $14,287.75 in a bank on May 1 deposited daily for six days, beginning on that date, the following amounts: $3876.50, $1275.80, $2307.85, $2136.90, $2882.75, $3007.80. During the same period checks were paid by the bank amounting in all to $12,982.75. Write two pages like those given on page 186, showing how the book appeared when balanced.

Write two pages as in Exs. 2 and 3, using the following items, and balance the account:

4. Nov. 1, balance, $1287.64; deposited Nov. 2, $398.64; Nov. 3, $128.76; Nov. 5, $298.78; Nov. 7, $342.68; Nov. 10, $537.86; Nov. 15, $676.80; Nov. 20, $1126.75; Nov. 25, $3284.85; Nov. 28, $1296.35; Nov. 29, $836.87. Vouchers returned, Dec. 1, $3986.42.

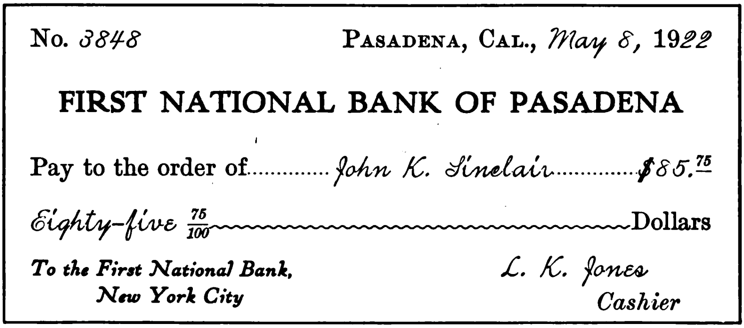
5. May 1, balance, $2643.87; deposited May 3, $968.42; May 4, $75.40; May 6, $287.75; May 9, $536.83; May 11, $1287.93; May 15, $782.64; May 18, $530; May 23, $683.40; May 26, $1126.38; May 28, $876.35; May 29, $575.50. Vouchers returned, June 1, $5264.75.

6. Make a problem similar to Exs. 4 and 5 and write the two pages as directed.

Transmitting Money. If a person owes money to some one living at a distance, he may send or carry the money; he may send a postal, express, or telegraph money order; or he may send his check or a bank draft.

Bank Draft. A check drawn by one bank on another is called a *bank draft*, a *cashier's check*, or simply a *draft*.

The following illustrates a common form of bank draft:



No. 3848 PASADENA, CAL., May 8, 1922

FIRST NATIONAL BANK OF PASADENA

Pay to the order of............John K. Sinclair............$\$85.^{\underline{75}}$

Eighty-five $\frac{75}{100}$~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~Dollars

To the First National Bank,
New York City

L. K. Jones
Cashier

If John K. Sinclair of Pasadena owed B. Altman & Co. of New York $85.75, he might purchase a draft like the above and indorse it:

Pay to the order of B. Altman & Co., New York
John K. Sinclair

He would then send the draft to B. Altman & Co., who would indorse it and deposit it in their bank for collection.

Banks usually charge from 0.1% to $\frac{1}{4}\%$ *premium* on the face of a draft to pay for the trouble and expense.

Sometimes banks are so much in need of money that is on deposit in a city like Chicago that they will sell a draft on that city at less than the face; that is, at a *discount*. A draft sold for exactly its face value is sold at *par*. A draft sold for more than its face value is sold at a *premium*.

Exercise. Transmitting Money

1. A retail dealer bought from a jobber a bill of goods to the amount of $1765.75 less 10, 10, 5. He sent a check for the balance. What was the amount of the check?

2. A man owing $1250 sent his check in payment. The creditor deposited the check in his bank, and the bank charged him 0.1% for collecting. What was the net amount received by the creditor?

3. A man deposited a check for $3000 in a bank, and the net proceeds after the bank had deducted its charge for collecting were $2997. What was the rate that the bank charged for collecting?

4. A man wishes to send a check to a jobber for an amount such that after the bank has deducted 0.1% for collecting there will remain exactly $6493.50. What shall he make the face of the check?

5. A manufacturer has received checks for $2700, $875, $3250, and $3750. His bank charges him 0.1% for collecting. How much does he pay in all for the collecting of the four checks?

6. A draft for $55,000 was bought for $54,972.50. Was exchange at a premium or at a discount? Find the rate.

7. A draft cost $2952.95, including 1% premium. Find the face and the premium.

8. At 0.1% premium find the cost of a draft to cover bills for $250, $300, and $450.

Find the cost of the following drafts:

9. $2500, 0.1% premium.

10. $3500, 0.2% premium.

11 $4200, 0.1% premium.

12. $5600, $\frac{1}{4}$% discount.

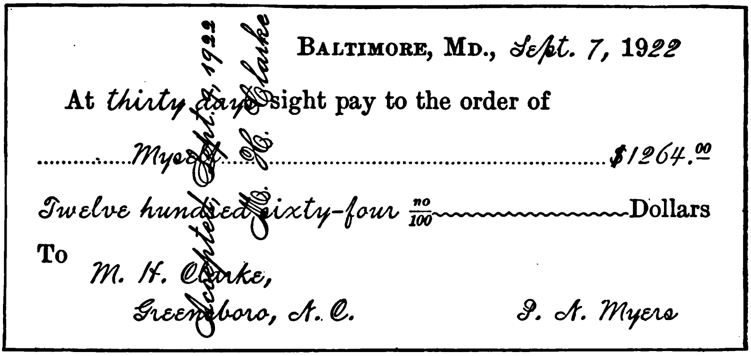
13. $3700, 0.2% discount.

14. $35,850, 0.2% premium.

Collecting by Draft. It is possible to collect a debt due in another place by *drawing* on the one who owes the money. The one to whom the money is due writes a draft on the debtor and sends it through a bank. Such a draft is called a *commercial draft* to distinguish it from a bank draft.

A commercial draft may be drawn "at sight," in which case the debtor must pay it as soon as presented; or it may be a *time draft*, say "at thirty days sight" or, what is the same thing, "thirty days after sight," in which case the debtor has 30 da. before he need pay it.

The following is a common form of commercial draft:



BALTIMORE, MD., Sept. 7, 1922

At thirty days sight pay to the order of

..................Myers & Co. $1264.00

Twelve hundred sixty-four no/100 ~~~~~~~~ Dollars

To M. H. Clarke,
Greensboro, N. C.

P. N. Myers

Accepted Sept. 9, 1922
M. H. Clarke

Other forms need not be considered at the present time.

In this case P. N. Myers has sold M. H. Clarke $1264 worth of goods on 30 days' credit. When he sends the bill he also sends to his bank in Baltimore this draft, indorsed, payable to the order of the bank "For collection only," and this bank sends it to a bank in Greensboro. Then the draft is sent to Mr. Clarke, who writes across the face of the draft the word "Accepted," dating and signing it as shown. The draft must then be paid on Oct. 9, the Greensboro bank remitting the money to the Baltimore bank.

Exercise. Collecting by Draft

1. A manufacturer in Chicago sells to a dealer in Memphis a bill of goods amounting to $1725.50. The terms are "6/10 or draft at 30 da." The dealer not being financially in position to avail himself of the discount, the manufacturer draws upon him at 30 days sight. Write the draft, using fictitious names, and write the acceptance.

2. Suppose that you have bought a bill of goods from a jobber and have received an invoice of $685.75, the terms being "Sight draft 10 da. less 2%." Suppose also that the invoice is dated Feb. 1 and that you do not take advantage of the discount. Write the draft that the jobber might draw upon you; write the jobber's indorsement "For collection only" and the indorsement of his bank so that it can be collected by a bank in your city or village; and, finally, write your acceptance.

3. Suppose that you purchased on May 4 a bill of goods of $875, terms 30 da., and that the seller sent a draft directly to you instead of sending it through the bank. Write the draft and your acceptance.

The manufacturer can now indorse this draft and sell it to his bank for a little less than its face value, thus getting his money at once. The bank will be repaid by receiving, in 30 da., the face of the draft, which is more than the bank paid the manufacturer. Drafts of this kind are salable; that is, they form part of those obligations that are negotiable and are known as *commercial paper.*

There has recently developed a custom of sending out with a bill of goods a draft to cover the cost. This draft may be for 30 da. or for some other definite time agreed upon. The debtor is expected to accept this at once and return it. The draft is then deposited, and the bank collects it. Such paper is called by the general name of *acceptances.* The subject of acceptances is too technical for detailed consideration in a work of this nature.

Borrowing Money from a Bank. Business men often have to borrow money from banks, giving their notes, by which they promise to repay the money on demand or in a certain time, usually 30 da., 60 da., or 90 da. The interest on time notes is paid in advance and is called *discount.* The interest on demand notes is paid when the note is paid. Banks usually reckon discount by days, using tables based on 360 da. to the year.

The face of a note less the discount is called the *proceeds.*

The following is a common form of a note:

NEW YORK, *June 17,* 19*23*

Ten days after date, for value received, *I* promise to pay
to the order of.................*Myself*.....................................*$725.^{00}*
Seven hundred twenty-five $\frac{no}{100}$~~~~~~~~~~~~~~~~Dollars
at the **Second National Bank, New York.**

John J. Peters

The forms of these notes vary. Some of them have a number of printed conditions, but these conditions have nothing to do with the arithmetic principle. City banks frequently lend money to men of high financial credit with no indorsers except themselves. If there is any question as to the man's credit he is usually required to put some valuable security in care of the bank to be sold if the note is not paid when due. Such security is called *collateral.*

In large cities demand notes are usually for funds needed for a day to settle stock transactions. Such a note usually draws a low rate of interest, this interest being paid when the note is paid.

Exercise. Borrowing from a Bank

1. Suppose that you borrow $750 from a bank, giving a note at 5% payable on demand, and that you pay the note in 10 da. Find the amount due when the 10 da. expire.

2. Suppose that you borrow $425 from a bank, giving a note payable in 30 da. If the bank discounts the note at 6%, how much is the discount and how much are the proceeds?

3. A manufacturer sells to a jobber a bill of goods amounting to $1250 and takes the jobber's note due in 90 da. with interest at 6%. The manufacturer keeps the note for 30 da. and then discounts it at a bank at 5%. Find the proceeds.

Sixty days later the note will be worth $1250 plus the interest at 6% for 90 da. The bank will give the manufacturer this amount less the discount at 5% for 60 da.

4. A dealer buys some goods listed at $12,750. If he pays for these goods within 30 da., he will, by the terms of the invoice, be allowed discounts of 10, 5. In order to do this he finds that he must borrow the money for 60 da., when he will have enough to meet the bill. If he can borrow the money at 5%, how much will he gain by doing so?

5. Write a note for $725, due in 90 da., with interest at 6%. If this note is discounted 30 da. later at $5\frac{1}{2}$%, what are the proceeds?

6. Write a note for $725, due in 90 da., with interest at $5\frac{1}{2}$%. If this note is discounted 30 da. later at 6%, what are the proceeds?

7. Write a note for $975, due in 60 da., with interest at $5\frac{1}{4}$%. If this note is discounted 30 da. later at 6%, what are the proceeds?

8. Write a demand note for $500 with interest at 3%. If the note is paid 5 da. later, how much is the amount?

Foreign Money. If an importing house or a private individual receives a bill of goods from abroad, this bill must be paid in the money of the country from which it came. This may be done by postal money order, up to a certain amount; by express money order, issued by an express company; by telegraphic money order; or, as is usually the case, by bank draft. Foreign drafts are often called *bills of exchange.*

Bills of exchange, like domestic drafts, may be above par (when exchange is at a premium), at par, or below par (when exchange is at a discount).

The most important units of value in countries whose currency, like that of the United States, may be called international are as follows:

In the British system

1 pound (£) = 20 shillings (s. or /) = 240 pence (d.) = $4.8665 = about $5.

In the French, Swiss, and Belgian system

1 franc (fr.) = 100 centimes = 19.3¢ = about 20¢.

The Italian system is the same except that the franc is called a lira (lē′rä), and the centime is called by its Italian name. The same system of values is used in Spain and Greece, the names being changed.

1 German mark (M.) = 100 pfennigs (pf.) = 23.8¢ = about 25¢;
1 Japanese yen (¥) = 1 Mexican peso (₱) = 49.8¢ = about 50¢.

British exchange, often called sterling exchange, is quoted at dollars to the pound (4.88 meaning that $4.88 will buy a bill of exchange for £1). If English exchange is above 4.8665, exchange is at a premium; if below, it is at a discount.

French exchange is quoted either at francs to a dollar (5.15 meaning that $1 will buy a bill of exchange for 5.15 fr.) or at cents to the franc (19.7 meaning that 19.7¢ will buy a bill of exchange for 1 fr.).

War conditions have seriously affected all relative values, and it may be some years before they will again be normal.

Exercise. Foreign Money

1. An importing house buys a bill of goods in Sheffield, England, amounting to £180 10s. Exchange being 4.80, how much will a draft cost for this amount?

Take £180.5 for £180 10 s., and notice that a draft for £1 costs $4.80.

2. A merchant buys a quantity of silk in Lyons, France, amounting to 7850 fr. 20c. Exchange being 5.05, how much will a draft cost for this amount?

Take 7850.2 fr. for 7850 fr. 20c., and notice that at the rate of exchange given, a draft for 5.05 fr. can be bought for $1.

3. How large a draft on London can be bought for $1205 when exchange is 4.82?

4. How large a draft on Paris can be bought for $12,500 when exchange is 5.28?

5. A merchant imports goods costing £348 6 s. What is the cost of the bill of exchange at 4.82?

6. An importer buys a shipment of Lyons silk to the value of 54,750 fr. What will the bill of exchange cost at 5.20?

Taking 25¢ to the shilling, express approximately in our money:

7. 15s. 6d. **8.** 17s. 9d. **9.** 35s. 8d. **10.** 12s. 4d.

Taking $5 to the pound, express approximately in our money:

11. £17 8s. **12.** £24 9s. **13.** £62 6s. **14.** £125 10s.

Taking 5 fr. to the dollar, express approximately in our money:

15. 75 fr. 50c. **16.** 19 fr. 20c. **17.** 72 fr. 75c. **18.** 34,870 fr.

Taking 2 ¥ to the dollar, express approximately in our money:

19. 350 ¥. **20.** 7500 ¥. **21.** 37,280 ¥. **22.** 52,725 ¥.

Exercise. Review of Chapters I–XI

1. Write 1927 in Roman numerals.

2. Write 5,900,000,000 in the index notation.

3. Multiply 237,286 by 98,368 and check the result by casting out nines.

4. Multiply 74,248 by 12,500, using the shortest method, and check the result by casting out nines.

5. Divide 61,389 by 125, using the shortest method, and check the result by casting out nines.

6. Find the discount on a bill amounting to $4287.60, the rates being stated as 10, 8, 2.

7. Find the amount of cash that will settle promptly a bill for $172.80, the terms being stated as 5/10, N/30.

8. Find the interest on $375 for 90 da. at 5½%.

9. Find the interest on $1725 for 174 da. at 6%.

10. A man discounts a note of $1250, due in 60 da., at 5%. Find the discount and the proceeds.

11. The British crown is 5s. and the florin is 2s. A half crown is what part of £1? A florin is what part of £1? A florin is what part of a crown?

12. If we use 2¢, 25¢, and $5 respectively as the approximate values of 1 d., 1 s., and £1, what is the equivalent of £4 8 s. 9 d. in our money?

13. If you buy some books in London for £2 8 s. 6 d., how much will a draft cost to pay this bill, exchange being 4.84?

14. If you receive from Paris a bill for 125 fr. 50 c., and exchange is 5.40, how much will a bill of exchange cost to cover the amount?

15. If a man buys 100 shares of stock quoted at 89¼ and sells it the same day at 91½, how much does he gain?

CHAPTER XII

ARITHMETIC OF CIVIC LIFE

Purpose of the Chapter. We are all members of various communities that we call by such names as family, village, town, city, county, or state, and we all belong to a much larger community that we call the United States of America. To all the communities to which we belong we owe many duties, partly in return for the protection and comforts that they give us, but in a larger sense because of a feeling on the part of all of us that we do not live for ourselves alone but to make the world better for those who now live and for those who are to follow us.

A part of our duty as citizens is the obligation that we feel to help support the commonwealth. For this reason we are careful of our household expenses so that we shall not waste the common property of the family; we are careful of our public property, since destruction or damage means restoration for which we must all pay our share; and we pay our taxes, directly or indirectly, for the support of our schools, roads, police and fire departments, and the like, since all these make for human betterment.

Every student takes part in the support of the government. When taxes become higher, there is naturally a corresponding increase in rents and prices, so that everyone pays more for the necessities of life.

It is the purpose of this chapter to show the relation of arithmetic to a few of the typical duties which every citizen owes to his community.

Expenses of the United States Government. The expenses of the United States government vary from year to year, but before the European war they averaged about $2,000,000 a day. Some items of our annual income and expenditures were then approximately as follows:

INCOME

Customs (duties on imported goods)	$213,000,000
Internal revenue (tobacco, etc.)	388,000,000
Income and corporation taxes	200,000,000
Miscellaneous	58,000,000

EXPENDITURES

War Department	$225,000,000
Navy Department	250,000,000
Pensions	159,000,000
Indians	18,000,000
Interest on public debt	23,000,000
Salaries, diplomatic service, etc.	204,000,000

Tariff. The United States collects a large part of its income by a tax on goods brought into the country. This income is called *customs revenue*, *tariff*, or *duty*.

Customs revenue is collected at *customhouses*. These are situated at *ports of entry*.

Goods imported may be on the *free list* and not subject to duty, as raw silk; subject to *ad valorem* (on the value) duty, a certain per cent on the value at the place of purchase, as 15% ad valorem on books; subject to *specific* duty, a certain amount per bushel, or other measure, as 10¢ per bushel of 50 lb. on apples; subject to both *ad valorem* and *specific* duty, as 60% ad valorem and 40¢ per pound on a certain kind of perfumery.

Exercise. Tariff

1. Find the duty at 45% on an importation of silk valued at $2775.

2. The duty on a certain kind of perfumery is 40¢ a pound and 60% ad valorem. Find the total duty on an importation of 450 lb. of this perfumery valued at $3375.

3. Find the specific duty at 35¢ a pound on 1275 lb. of a certain product.

4. An importer pays $561.30 ad valorem duty on goods valued at $3742. What is the rate of duty?

5. The duty on a lot of goods purchased in England is $906.25, the rate being 12½%. Find the valuation.

6. An importing house pays 48,500 fr. for some decorated porcelain in Paris. If the government rates a franc as 19.3¢, and if the duty is 40%, how much is the duty?

7. A tobacco dealer paid a duty of $2.50 a pound on 3280 lb. of tobacco wrapper. Find the total duty, using a short method and checking the result by casting out nines.

8. A fruit dealer imported from Cuba 12,250 pineapples, the duty being $5 per 1000. Find the total duty, using a short method and checking as in Ex. 7.

9. An importing house bought a bill of lace in England amounting to £257 10s. 6d. If the government rates £1 as $4.8665, and if the duty is 45%, find the total duty.

10. The duty at 25% paid on a certain invoice of goods bought in France was $2175. Find the valuation on which the duty was computed.

Although the war has caused the rates of duty, the national income and expenses, and the cost of commodities to fluctuate, the general principles of this subject remain unchanged.

State and Local Taxes. State and local expenses are met in general by *direct taxes*, as distinguished from such an *indirect tax* as the tariff which the United States government *levies.* State and local taxes are usually a certain per cent levied on the value of the land, money, and other property of individuals, business concerns, and corporations.

The property to be taxed is valued by officers known as *assessors*, the value which they place upon the property being called the *assessed valuation.*

Upon the assessed valuation a certain *tax rate* is fixed, sometimes stated as a certain per cent of the value, sometimes as the number of mills of tax on each dollar of valuation, and sometimes as the amount per $100 of valuation. Thus the tax rates $0.005\frac{1}{2}$, $5\frac{1}{2}$ mills, and 55¢ all mean substantially the same.

The tax rate is found by dividing the amount to be raised by the assessed valuation. Thus, if a large city needs to raise $30,000,000 in taxes, and if the assessed valuation of the property is $3,000,000,000, the tax rate is 1%, 10 mills, or $1, since

$$\$30,000,000 \div \$3,000,000,000 = 0.01,$$

$$0.01 \times \$1 = 10 \text{ mills,}$$

and

$$0.01 \times \$100 = \$1.$$

If a man's property is assessed at $17,500 and the tax rate is 10 mills, he must pay 10 mills, or 1¢, on every dollar, or $1 on every $100. That is, on $17,500 he must pay 1%, or $175. This is paid to an official known as the *collector.*

Practically the tax is found from a tax table, which the collector consults.

The technical details of collecting taxes vary so much and the arithmetic work involved is so slight, that no further treatment of the subject is desirable in a work of this kind. The same is true of such special taxes as the inheritance tax, taxes on automobiles, and the poll tax.

Exercise. State and Local Taxes

1. A man's property is worth $45,000 and is assessed at $\frac{2}{3}$ of its value. The tax rate is 12 mills. What is the tax?

2. If the tax rate is 14 mills, what tax must be paid by a man whose property is assessed at $12,500?

3. The assessed valuation of a certain village is $475,000 and the total tax is $3800. What is the tax rate?

4. If the tax rate is 13 mills, how much tax must be paid on property assessed at $42,500?

5. If a man's tax is $195 and the rate is 13 mills, what is the assessed valuation of his property?

6. If a man's property is assessed at $13,000 and his tax is $156, what is the rate of tax?

7. If the assessed valuation of the property in a village is $973,000 and the total amount which is to be raised by taxation is $11,676, what is the tax rate?

8. A man died leaving $28,750, of which $20,000 went to his wife, $8000 to his child, and $750 to a nephew. Under the laws of his state there is a tax of 1% on the amount above $5000 that his wife receives, 1% on the amount above $5000 that his child receives, and 5% on the amount above $500 that his nephew receives. Find the tax paid by each.

9. In the borough of Manhattan, New York City, the assessed valuation in a recent year was $5,427,451,103 and the tax rate for city purposes was computed as 0.019365647347. Find to the nearest $100,000 the amount of tax that this would bring, supposing that all the tax is collected.

The problem is given principally to show a concrete case in which the large sums involved required the carrying of a tax rate to a considerable number of decimal places. From the various partial tax rates the tax rate for the borough was finally taken as 20.2 mills.

Income Tax. Part of the money needed to run the United States government is raised by a tax on incomes. Every citizen of the United States, whether residing at home or abroad, and every person residing in the United States must pay the government a certain per cent of his income above a certain amount. This is called the *normal rate.* Persons whose incomes are large have to pay excess taxes, the rate of taxation increasing as the amount of income increases.

The European war has made such enormous demands upon the resources of our government that the rate of taxation has increased very rapidly within a few years, and it is not probable that it will become standardized again for a number of years to come. All that the schools can be expected to do in connection with the matter is to explain the general nature of the income tax, it being understood that the rate and other details vary from year to year.

If a man has an income of $7500 a year, of which $2000 is exempt from taxation on account of his being the head of a family, and if the normal rate is 2%, his normal income tax is 2% of $5500, or $110.

Internal Revenue. A considerable part of our national revenue comes from taxes on tobacco. This, together with the income tax and certain other taxes, constitutes the *internal revenue* of the government. In addition to this a large amount comes from the sale of postage stamps, but this is all used to pay the expenses of carrying the mails and maintaining the post offices. The recent growth of prohibition of the manufacture and sale of liquor has a tendency to reduce the amount of internal revenue from this source, so that no statistics can be given that are likely to be of value in the study of higher arithmetic. The general principle of taxation is, however, the same in any case.

Exercise. Internal Revenue

1. If a man has an income of $8475.50, of which $1000 is exempt from taxation, find his normal income tax at 2%.

2. In a certain year a special government tax dropped from $89,000,000 to $62,000,000. Find the per cent of decrease.

3. Owing to the European war the income tax rose in one year from about $360,000,000 to about $4,000,000,000. Find the per cent of increase.

4. If a corporation has a net income of $1,374,285 in a certain year, and if the normal rate of income tax was 2%, how much was the normal tax paid by the corporation?

5. A man had a net income of $23,000 in a year in which the law required him to pay 2% as a normal income tax on his income above $4000, 2% excess tax on his income above $2000, and surtaxes as follows: 1% on his income from $5000 to $7500, 2% from $7501 to $10,000, 3% from $10,001 to $12,500, 4% from $12,501 to $15,000, 5% from $15,001 to $20,000, and 8% from $20,001 to $40,000. Compute his total income tax.

Owing to the rapid changes in the tax laws, no such problem should be taken as a standard. This is merely a type of problem that is likely to arise in the computing of income taxes.

6. A certain quality of cigar is taxed $3 per 1000, and an extra war tax of $7 per 1000 is levied. Find the total tax on an output of 275,000 cigars.

7. A certain quality of cigarette is taxed $3.60 per 1000, and an extra war tax of $1.20 per 1000 is levied. Find the total tax on an output of 3,250,000 cigarettes.

8. When tobacco is taxed 8¢ per pound with an extra war tax of 5¢ per pound, what is the total tax on an output of 2,758,500 lb. of tobacco?

Exercise. Review of Chapters I-XII

1. Find the interest on $7550 for 60 da. at 6%, and from the result find the interest on the same sum for 60 da. at 5%; at $4\frac{1}{2}$%; at 4%; at 3%.

2. Find the interest on $12,500 for 30 da. at 6%, and from the result find the interest on the same sum for 30 da. at $5\frac{1}{2}$%; at 5%; at $4\frac{1}{2}$%; at 4%.

3. The last chapter in a book is numbered XXXII. What per cent of the total number of chapters has a person read when he has finished reading through Chapter XXIV?

4. Write a statement of the shortest way you know of finding $\frac{7}{8}$ of a number; $\frac{4}{5}$ of a number; $\frac{3}{4}$ of a number. State the reason in each case.

5. A draft for $7250 is bought at 0.1% discount. How much does the draft cost?

6. A man having some money to invest buys 10 shares of railway stock at $104\frac{3}{4}$. Adding the usual brokerage, how much does he pay for the stock?

7. A large department store has sales in a certain year amounting to $78,256,400 and computes its selling price so as to allow a profit of 12% on the sales. Find the profit.

8. A clothing manufacturer is taxed on the value of his plant. The assessed valuation is $137,500 and the tax rate is $6\frac{1}{2}$ mills. How much is his tax?

9. Write 6,300,000,000 in the index notation.

10. A gun has a diameter of 82 mm. Express the diameter in inches to the nearest hundredth of an inch.

11. A liter of a certain liquid weighs 0.976 kg. Find the specific gravity of the liquid and explain the process by which you obtained the result.

CHAPTER XIII

ADVANCED PROBLEMS

Purpose of the Review. The purpose of this chapter is to afford to students who expect to enter into commercial and industrial work a review of industrial and other general types of business problems of a somewhat more difficult nature than is found in the preceding chapters. This work may be omitted by students who have, in the opinion of the teacher, no special need for this kind of exercise, or selections may be made from the problems to meet individual needs.

Exercise. Miscellaneous Problems

1. From a piece of silk that contains $35\frac{7}{8}$ yd. there have been sold at different times $12\frac{3}{4}$ yd., $2\frac{1}{2}$ yd., $2\frac{3}{16}$ yd., and $8\frac{5}{8}$ yd. How many yards remain?

2. If a certain kind of gun metal is composed of $90\frac{1}{2}$ parts by weight of copper to $9\frac{1}{2}$ parts of tin, how many ounces of tin are there in a pound of the gun metal? How many ounces of copper are there in a pound of the gun metal?

3. A square plot of land 127 yd. on a side has a path 1 yd. wide running round the inside of it. Find the cost of graveling this path at 35¢ per square yard.

4. A street $\frac{3}{4}$ mi. long has on each side a sidewalk $7\frac{1}{2}$ ft. wide. How much will it cost to pave the sidewalks with stones, each measuring 2 ft. 9 in. by 1 ft. 8 in., if the stones cost, including the laying, $1.50 each?

5. How many planks 11 ft. long and 9 in. wide are needed to cover a platform 27 ft. 6 in. long and 8 yd. wide? What will be the cost at 40¢ a square foot?

6. How many tiles 9 in. long and 4 in. wide will be required to pave a walk 8 ft. wide that surrounds a rectangular court 60 ft. long and 36 ft. wide?

7. The length and width of a map are $4\frac{1}{2}$ ft. and $3\frac{1}{3}$ ft. respectively. If the map represents 77,760 sq. mi. of country, how many square miles are there to a square inch?

8. Allowing 1000 shingles for 120 sq. ft., how many thousand will be required for the pitched roof of a house 60 ft. long if the width of each side of the roof is $24\frac{1}{2}$ ft.?

9. Allowing 1000 shingles for 110 sq. ft., how many thousand will be required for the pitched roof of a barn 40 ft. long if the width of each side of the roof is 24 ft.?

10. How many bricks will be required to build a wall 75 ft. long, 6 ft. high, and 16 in. thick if each brick is 8 in. long, 4 in. wide, and $2\frac{1}{4}$ in. thick?

11. The ceiling of a room 27 ft. long, 24 ft. wide, and 10 ft. high is to be raised so as to increase the space by 84 cu. yd. What will then be the height of the room?

12. A manufacturer has three machines, A, B, C, for making buttons. A and B together can turn out a required number of buttons in 48 hr.; A and C together in 30 hr.; B and C together in $26\frac{2}{3}$ hr. How long will it take each machine alone to turn out the required number?

13. If a wheel turns 17° 30′ in 0.2 sec., how long does it take to make a complete revolution?

14. If a wheel makes a complete revolution in 11 sec., through how many degrees does the wheel turn per second? Through how many degrees does it turn per minute?

Make out receipted bills for these accounts, supplying dates:

15. R. K. Seeley bought of M. R. Russell ½ doz. cups and saucers @ $6; 4 doz. plates @ $4.25; 2½ doz. tumblers @ 95¢; 3 doz. butter plates @ 85¢; 4½ doz. bowls @ $1.10; 5 doz. dinner plates @ $3.85; 1½ doz. pitchers @ $6.25; 6 platters @ $1.08; 9 platters @ 95¢.

16. M. S. Rogers bought of Roberts & Co. 4 doz. No. 7 teakettles @ $1.60 each; 2 safety ash barrels @ $3.50; 3 doz. common scrapers @ 90¢; 8 eagle shovels @ 40¢; 6 black registers @ $2.50; 8 spice boxes @ 75¢; ½ doz. dish pans @ $11; 2 doz. stove lifters @ 90¢; ½ doz. dripping pans @ $8; ½ gross teaspoons @ 60¢ a dozen; 1 doz. ash sifters @ $14; 1½ doz. coal hods @ $10.50.

17. P. S. Tyler bought of Johnson & Co. 2 bbl. flour @ $9.75; 25 lb. coffee @ 43¢; 3 lb. Oolong tea @ 70¢; 15 bottles olives @ 35¢; 2 boxes graham wafers @ 80¢; ½ doz. cans tomatoes @ $1.20; ½ doz. cans peaches @ $3.50; 4 hams, 48 lb., @ 22½¢ a pound; 6 strips bacon, 19 lb. 9 oz., @ 33¢ a pound; 3 lb. rice @ 9¢; 3 lb. tapioca @ 8¢; 40 lb. rye meal @ 6¢; 5 lb. boneless codfish @ 24¢; ½ doz. cans plums @ $2.90; ½ doz. packages seedless raisins @ $1.90; 30 lb. yellow corn meal @ 6½¢; 4 cans canned milk @ 19¢.

18. M. D. Keyes bought of R. M. Kane 1 range @ $65; 1 Rockford heater @ $30; 4 lb. English stovepipe @ 25¢; 3 lb. Russian stovepipe @ 35¢; 8 lb. sheet zinc @ 16¢; 1 set kitchen knives and forks @ $2.50; 2 washtubs @ $1.65; 1 washboard @ 45¢; 1 set nickel sadirons @ $1.10; 2 milk cans @ 75¢; 1 hand lamp @ 80¢; 1 stand lamp @ $3.50; 1 granite iron washbowl @ 80¢; 1 tea canister and 1 coffee canister @ 40¢ each; 1 carving knife and fork @ $2; 1 corn popper @ 48¢; 1 rolling-pin @ 35¢; 2 porcelain kettles @ $1.30; 1 granite iron coffeepot @ $1.40.

19. If 2 lb. of rosin is melted with 5 oz. of mutton tallow to make a grafting wax, how many ounces of tallow will 20 oz. of the wax contain?

20. Two men purchase some property together, one paying $1250 and the other $1000. If the property rises in value and is sold for $3600, what is the share of each?

21. A certain kind of gun metal is composed of 3 parts by weight of tin to 100 parts of copper. What weight of each of these metals is there in a cannon weighing 721 lb.?

22. Plumbers' solder contains 2 parts by weight of lead and 1 part of tin. How many pounds of each metal are required to make 100 lb. of solder?

23. The average age of the boys in the four classes of a school is 18.4 yr., 17.9 yr., 16.8 yr., and 15.7 yr. The classes contain 29, 33, 34, and 33 boys respectively. What is the average age of the boys in the school?

24. If a certain grade of gunpowder contains by weight 75% of saltpeter, 10% of sulphur, and 15% of charcoal, how many pounds of each are there in a ton of gunpowder?

25. Air is composed of 20.0265%, by volume, of oxygen and 79.9735% of nitrogen. How many cubic feet of each are there in 1750 cu. ft. of air?

26. If 7 lb. of a certain article loses 3 oz. in weight by drying, what per cent of its original weight is water?

27. If 7 lb. of a dry article has lost 3 oz. by drying, what per cent of its original weight was water?

28. If a dry article exposed to damp air absorbed 3 oz. of water, and then weighed 7 lb., what per cent of its present weight is water?

29. If $9000 yields $675 interest in 1 yr. 6 mo., what sum will yield $72 interest at the same rate in the same time?

30. A merchant, in selling goods, deducts from the marked price 5% for cash. What is the marked price of goods for which he receives $14.25?

31. If an ore loses $41\frac{1}{2}$% of its weight in roasting, and $43\frac{3}{4}$% of the remainder in smelting, how much ore is required to yield 1000 tons of metal?

32. An agent bought 25 sewing machines with 15, 10, and 5 off the list price of $80 each, and sold them at a discount of 10% off the list price. What was the net amount received for the sewing machines and what was the profit?

33. The Eagle Manufacturing Co. fixes the price of its shell crushers so as to realize 40% profit on the cost of production. The company submits a bid to furnish 300 machines for $3827, f.o.b. the point of delivery. The freight would be $47. Find the cost of producing one machine.

Remember that f.o.b. means "free on board"; that is, charges prepaid until the goods are on the train or boat at the place specified.

34. The Acme Boot Company buys boots from the manufacturers at $30 per dozen pairs. If the discounts from the list price amount to 23% on the average, and the firm wishes to realize a profit of 20% on the cost, what list price must it quote on the boots?

35. A retailer has found that the average cost of doing business is 20% of his gross sales for the year. His profits have averaged 25% of the gross sales. At what price must he sell stoves that cost him $18 each to cover these items at the rates mentioned?

36. How much water must be added to a barrel of vinegar, 85% pure, to reduce it to a 50% solution?

37. How much pure alcohol must be added to a gallon of alcohol, 20% pure, so that the mixture shall be 30% pure?

38. In 90 oz. of an alloy of silver and copper there is 6 oz. of silver. How much copper must be added so that 50 oz. of the new alloy may contain 2 oz. of silver?

39. In 3 oz. of a mixture of water and listerine there is 1 oz. of listerine. How much listerine must be added to make the new solution contain 75% listerine?

40. Air consists of four parts, by volume, of nitrogen and one part of oxygen. How many cubic feet of oxygen are there in a schoolroom 30 ft. by 40 ft. by 10 ft.? How high must a room 30 ft. by 40 ft. be to contain 3600 cu. ft. of oxygen?

41. If the duty on a certain perfumery is 40¢ a pound and 60% ad valorem, what is the total duty on an importation of 32 lb. of this perfumery valued at $240?

42. The records of a business show gross sales of $1,250,000 for the year. The cost of doing business for the year was $200,000, and the net profit was $300,000. What must be the selling price of articles that cost $300 in order to cover the cost of doing business and profit?

43. The McGraw Hardware Company sold the following bill of goods: 2 doz. locks @ $18 less 20, 10, 5; $1\frac{1}{2}$ doz. locks @ $20 less 20, 5; 5 doz. $2\frac{1}{2}''$ bolts @ $1.50 less 30, 5. Find the amount of the bill.

44. The Roberts Woolen Company bought the following: 1500 yd. lining @ $0.35 less 10, 10, 5; 850 yd. worsted suitings @ $1.70 less 20, 10, 10; 500 yd. woolens @ $1.50 less 10, 10. Find the amount of the bill.

45. James Wilson of Denver purchased from a dealer in Chicago crockery amounting to $1690. Owing to careless packing, $30 worth of crockery was broken, and Wilson deducted $30 from the bill. He paid the balance by draft, costing 0.1% premium. How much did he pay for the draft?

46. The St. Cloud Trading Co. receives two bids: one offers to supply 1200 lb. butter @ $0.30 less 20, 10; the other 1200 lb. @ $0.31 less 10, 10, 10 for the same quality of butter. Which is the better for the company, and how much better?

47. A dealer remembers that the net price of an article was $2.70 after a discount of 25, 10 had been allowed, but he cannot find the list price. What was the list price?

48. A dealer buys galvanized iron pails in lots of 100, thereby being allowed discounts of 45, 20, 5 from the list price of 20¢ each. If he sells them at the list price, what is his rate of profit on the net cost?

49. On a bill of goods listed at $2875 with discounts 25, 10, 10, 5, what difference does it make in the net price if the factor is taken to four decimal places instead of five? How is it with a bill of $250? with a bill of $75?

50. A clerk receives $15 a week and $1\frac{1}{2}\%$ commission on goods sold. If he sells $492 worth of goods in a week, how much is his income for the week? Write a formula.

51. The Sinclair Drygoods Company bought 2500 yd. of linings, listed at 30¢ per yard, discounts 20, 10, 5. The goods were sold so as to cover overhead of 18% of the selling price, and a profit of 20% of the selling price. Find the selling price of the 2500 yd. and the price per yard.

52. F. P. Jacobs discounted at a bank at $4\frac{1}{2}\%$ his 90-day note for $10,000. With the proceeds he purchased cloth at $2 per yard, investing all but $7.50. During the 90 da. he disposed of all but 250 yd. of the cloth at an average price of $3.25, depositing the proceeds of the sales in the bank. At the end of the 90 da. Mr. Jacobs paid the bank $10,000 from the amount he had deposited. How much was his profit for the 90 da.?

53. Which is the lower price, articles at $3.50 each, less 2%, or the same articles at $4 each, less 7%? How much is saved by buying 6 doz. at the cheaper price?

54. An agent buys eggs for a St. Louis dealer. His commission, including delivery at the railway station, is 1½% of the price he pays. If he buys on Monday 24 cases of 30 doz. each, paying 33¢ per dozen, and on Tuesday 32 cases of the same capacity, paying 34¢ per dozen, how much is his commission? How much does the St. Louis dealer pay, not including railway charges?

55. If J. M. Kane sells R. S. Miller a bill of goods amounting to $4960 less 20%, 25%, 10%, terms half by a 60-day note at 6% and half on account due in 60 da., what is the amount of the note at the end of the 60 da.?

56. A fruit dealer bought a lot of oranges at $4.50 per C and marked them so as to gain 33⅓% and still to allow for a 10% loss through bad debts. Find to the nearest 5¢ the selling price per dozen.

57. On July 25 a dealer buys 6000 lb. raisins @ 22¢, less 25%, 20%, 10%, terms 2/10, N/30. He pays $6.50 for freight and pays the bill for the raisins on July 27. Find the total cost of the raisins and the cost per pound.

58. A school orders 1500 programs for a concert, the size being 6¼″ by 9″. The programs were printed from stock weighing 120 lb. per 500 sheets, and the sheets were 25″ by 36″. Find the cost of the paper at 18¢ per pound.

59. A commission house pays its accounts to manufacturers by drafts on the following cities for these amounts and at these rates: Chicago, $8050, 0.1% premium; St. Louis, $2400, 0.1% premium; Pittsburgh, $1720, 0.1% premium; New York, $2300, 0.2% discount; Boston, $3200, 0.3% discount. What is the net gain or loss on exchange?

SUPPLEMENTARY WORK

ADVANCED THEORY AND PRACTICE

Nature and Use of Roots. The student has now finished the part of higher arithmetic that is commonly used by the large majority of people in their daily work. The supplementary work here offered may therefore be considered as optional and may be taken, if the time allows, by those who have special need for it or who wish to extend their knowledge to cover the topics here presented.

In engineering and in various lines of industry it is necessary to use the roots of numbers, particularly the square root.

If a number is the product of two equal factors, either factor is called the *square root* of the number. For example, because $25 = 5 \times 5$, we say that 5 is the square root of 25, and we indicate this by the symbols $5 = \sqrt{25}$.

According to this definition 3 would have no square root, because 3 has not two equal factors. We therefore extend the idea of root to include approximate factors, and say that $\sqrt{3} = 1.732$ to four significant figures or to three decimal places, because $1.732^2 = 3$, approximately.

If a number is the product of three equal factors, any one of these factors is called the *cube root* of the number; one of four equal factors, the *fourth root*, and so on, approximate roots being allowed as in the case of the square root. Thus, the cube root of 8, written $\sqrt[3]{8}$, is 2, the *index of the root* being 3, and $\sqrt[4]{81}$ is 3, the index of the root being 4.

In practice all roots are found by means of tables, and hence we shall give but little attention to the older methods.

Square Root. It is easily seen by algebraic multiplication that

$$(t+u)^2 = t^2 + 2\,tu + u^2.$$

The square of a number contains the square of the tens, plus twice the product of the tens and units, plus the square of the units.

Since $1 = 1^2$, $100 = 10^2$, $10{,}000 = 100^2$, and so on, it is evident that the square root of any number between 1 and 100 lies between 1 and 10; the square root of any number between 100 and 10,000 lies between 10 and 100. In other words, the square root of any integral number expressed by *one* or *two* figures is a number of *one* figure; expressed by *three* or *four* figures is a number of *two* figures; and so on.

If, therefore, an integral number is separated into periods of two figures each, from the right to the left, the number of figures in the square root is equal to the number of the periods of figures. The last period at the left may have one figure or two figures; for example, 22 09 and 7 89 04 81.

Required to find the square root of 3481.

The first period, 34, contains the square of the tens' number of the root. Since the greatest square in 34 is 25, then 5, the square root of 25, is the tens' figure of the root.

Subtracting the square of the tens, the remainder contains twice the tens × the units, plus the square of the units. If we divide by twice the tens (that is, by 100, which is 2 × 5 tens) we shall find approximately the units' figure. Dividing 981 by 100 we have 9 as the units' figure.

$$\begin{array}{r|l} \multicolumn{2}{r}{34\ 81\,(59} \\ \multicolumn{2}{r}{25\quad\ \ } \\ \hline 100 & 9\ 81 \\ 109 & 9\ 81 \\ \hline \multicolumn{2}{r}{\cdot} \end{array}$$

Since twice the tens × the units, plus the square of the units, is equal to (twice the tens + the units) × the units, that is, since $2 \times 50 \times 9 + 9^2 = (2 \times 50 + 9) \times 9$, we add 9 to 100 and multiply the sum by 9. The product is 981, thus completing the square of 59. Checking the work, $59^2 = 3481$.

If there are more figures in the root, we simply consider the part already found as tens and the next figure as units, and proceed as before

Table of Square Roots. The time has gone when people can afford to find square root by the method given on page 214. For practical purposes tables are always used. A convenient table is given on pages 216–219.

In this table the first two figures of the number whose root we seek are given at the left, in the column marked **N**, and the third figure is given at the top. For example, the square root of 1.00 is 1.000, this being opposite the number 1.0 and under 0; the square root of 1.01 is 1.005, this being opposite the number 1.0 and under 1; the square root of 1.02 is 1.010; and so on.

Furthermore, since $\sqrt{100}=10$, multiplying a number by 100 multiplies the square root by 10. Therefore we have

$$\sqrt{1.76}=1.327 \qquad \sqrt{8.38}=2.895$$
$$\sqrt{176}=13.27 \qquad \sqrt{838}=28.95$$
$$\sqrt{17600}=132.7 \qquad \sqrt{83800}=289.5$$

In the right-hand part of each page of the table are some columns headed 1, 2, 3, · · ·, 9. In these are found the numbers to be added for a fourth figure. For example, $\sqrt{2.68}=1.637$; but if we wish to find $\sqrt{2.687}$ we look along the line from 2.6 until we come to the right-hand column marked 7, and there we find 2. This shows that we should add 0.002 to 1.637, giving $\sqrt{2.687}=1.639$. Similarly,

$$\sqrt{7.653}=2.766+0.001=2.767$$
$$\sqrt{934.9}=30.56+0.01=30.57$$
$$\sqrt{81.55}=9.028+0.003=9.031$$
$$\sqrt{8155}=90.28+0.03=90.31$$

the last two roots being found on page 219.

Thus the table gives the square roots of all numbers from 1 to 9999, which is as far as we often need to go.

SQUARE ROOTS FROM 1 TO 9.999

N	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
1.0	1.000	1.005	1.010	1.015	1.020	1.025	1.030	1.034	1.039	1.044	0	1	1	2	2	3	3	4	4
1.1	1.049	1.054	1.058	1.063	1.068	1.072	1.077	1.082	1.086	1.091	0	1	1	2	2	3	3	4	4
1.2	1.095	1.100	1.105	1.109	1.114	1.118	1.122	1.127	1.131	1.136	0	1	1	2	2	3	3	4	4
1.3	1.140	1.145	1.149	1.153	1.158	1.162	1.166	1.170	1.175	1.179	0	1	1	2	2	3	3	3	4
1.4	1.183	1.187	1.192	1.196	1.200	1.204	1.208	1.212	1.217	1.221	0	1	1	2	2	2	3	3	4
1.5	1.225	1.229	1.233	1.237	1.241	1.245	1.249	1.253	1.257	1.261	0	1	1	2	2	2	3	3	4
1.6	1.265	1.269	1.273	1.277	1.281	1.285	1.288	1.292	1.296	1.300	0	1	1	2	2	2	3	3	4
1.7	1.304	1.308	1.311	1.315	1.319	1.323	1.327	1.330	1.334	1.338	0	1	1	2	2	2	3	3	3
1.8	1.342	1.345	1.349	1.353	1.356	1.360	1.364	1.367	1.371	1.375	0	1	1	1	2	2	3	3	3
1.9	1.378	1.382	1.386	1.389	1.393	1.396	1.400	1.404	1.407	1.411	0	1	1	1	2	2	3	3	3
2.0	1.414	1.418	1.421	1.425	1.428	1.432	1.435	1.439	1.442	1.446	0	1	1	1	2	2	2	3	3
2.1	1.449	1.453	1.456	1.459	1.463	1.466	1.470	1.473	1.476	1.480	0	1	1	1	2	2	2	3	3
2.2	1.483	1.487	1.490	1.493	1.497	1.500	1.503	1.507	1.510	1.513	0	1	1	1	2	2	2	3	3
2.3	1.517	1.520	1.523	1.526	1.530	1.533	1.536	1.539	1.543	1.546	0	1	1	1	2	2	2	3	3
2.4	1.549	1.552	1.556	1.559	1.562	1.565	1.568	1.572	1.575	1.578	0	1	1	1	2	2	2	3	3
2.5	1.581	1.584	1.587	1.591	1.594	1.597	1.600	1.603	1.606	1.609	0	1	1	1	2	2	2	3	3
2.6	1.612	1.616	1.619	1.622	1.625	1.628	1.631	1.634	1.637	1.640	0	1	1	1	2	2	2	2	3
2.7	1.643	1.646	1.649	1.652	1.655	1.658	1.661	1.664	1.667	1.670	0	1	1	1	2	2	2	2	3
2.8	1.673	1.676	1.679	1.682	1.685	1.688	1.691	1.694	1.697	1.700	0	1	1	1	1	2	2	2	3
2.9	1.703	1.706	1.709	1.712	1.715	1.718	1.720	1.723	1.726	1.729	0	1	1	1	1	2	2	2	3
3.0	1.732	1.735	1.738	1.741	1.744	1.746	1.749	1.752	1.755	1.758	0	1	1	1	1	2	2	2	3
3.1	1.761	1.764	1.766	1.769	1.772	1.775	1.778	1.780	1.783	1.786	0	1	1	1	1	2	2	2	3
3.2	1.789	1.792	1.794	1.797	1.800	1.803	1.806	1.808	1.811	1.814	0	1	1	1	1	2	2	2	2
3.3	1.817	1.819	1.822	1.825	1.828	1.830	1.833	1.836	1.838	1.841	0	1	1	1	1	2	2	2	2
3.4	1.844	1.847	1.849	1.852	1.855	1.857	1.860	1.863	1.865	1.868	0	1	1	1	1	2	2	2	2
3.5	1.871	1.873	1.876	1.879	1.881	1.884	1.887	1.889	1.892	1.895	0	1	1	1	1	2	2	2	2
3.6	1.897	1.900	1.903	1.905	1.908	1.910	1.913	1.916	1.918	1.921	0	1	1	1	1	2	2	2	2
3.7	1.924	1.926	1.929	1.931	1.934	1.936	1.939	1.942	1.944	1.947	0	1	1	1	1	2	2	2	2
3.8	1.949	1.952	1.954	1.957	1.960	1.962	1.965	1.967	1.970	1.972	0	1	1	1	1	2	2	2	2
3.9	1.975	1.977	1.980	1.982	1.985	1.987	1.990	1.992	1.995	1.997	0	1	1	1	1	2	2	2	2
4.0	2.000	2.002	2.005	2.007	2.010	2.012	2.015	2.017	2.020	2.022	0	0	1	1	1	1	2	2	2
4.1	2.025	2.027	2.030	2 032	2.035	2.037	2.040	2.042	2.045	2.047	0	0	1	1	1	1	2	2	2
4.2	2.049	2.052	2.054	2.057	2.059	2.062	2.064	2.066	2.069	2.071	0	0	1	1	1	1	2	2	2
4.3	2.074	2.076	2.078	2.081	2.083	2.086	2.088	2.090	2.093	2.095	0	0	1	1	1	1	2	2	2
4.4	2.098	2.100	2.102	2.105	2.107	2.110	2.112	2.114	2.117	2.119	0	0	1	1	1	1	2	2	2
4.5	2.121	2.124	2.126	2.128	2.131	2.133	2.135	2.138	2.140	2.142	0	0	1	1	1	1	2	2	2
4.6	2.145	2.147	2.149	2.152	2.154	2.156	2.159	2.161	2.163	2.166	0	0	1	1	1	1	2	2	2
4.7	2.168	2.170	2.173	2.175	2.177	2.179	2.182	2.184	2.186	2.189	0	0	1	1	1	1	2	2	2
4.8	2.191	2.193	2.195	2.198	2.200	2.202	2.205	2.207	2.209	2.211	0	0	1	1	1	1	2	2	2
4.9	2.214	2.216	2.218	2.220	2.223	2.225	2.227	2.229	2.232	2.234	0	0	1	1	1	1	2	2	2
5.0	2.236	2.238	2.241	2.243	2.245	2.247	2.249	2.252	2.254	2.256	0	0	1	1	1	1	2	2	2
5.1	2.258	2.261	2.263	2.265	2.267	2.269	2.272	2.274	2.276	2.278	0	0	1	1	1	1	2	2	2
5.2	2.280	2.283	2.285	2.287	2.289	2.291	2.293	2.296	2.298	2.300	0	0	1	1	1	1	2	2	2
5.3	2.302	2.304	2.307	2.309	2.311	2.313	2.315	2.317	2.319	2.322	0	0	1	1	1	1	2	2	2
5.4	2.324	2.326	2.328	2.330	2.332	2.335	2.337	2.339	2.341	2.343	0	0	1	1	1	1	1	2	2

SQUARE ROOTS FROM 1 TO 9.999

N	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
5.5	2.345	2.347	2.349	2.352	2.354	2.356	2.358	2.360	2.362	2.364	0	0	1	1	1	1	1	2	2
5.6	2.366	2.369	2.371	2.373	2.375	2.377	2.379	2.381	2.383	2.385	0	0	1	1	1	1	1	2	2
5.7	2.387	2.390	2.392	2.394	2.396	2.398	2.400	2.402	2.404	2.406	0	0	1	1	1	1	1	2	2
5.8	2.408	2.410	2.412	2.415	2.417	2.419	2.421	2.423	2.425	2.427	0	0	1	1	1	1	1	2	2
5.9	2.429	2.431	2.433	2.435	2.437	2.439	2.441	2.443	2.445	2.447	0	0	1	1	1	1	1	2	2
6.0	2.449	2.452	2.454	2.056	2.458	2.460	2.462	2.464	2.466	2.468	0	0	1	1	1	1	1	2	2
6.1	2.470	2.472	2.474	2.476	2.478	2.480	2.482	2.484	2.486	2.488	0	0	1	1	1	1	1	2	2
6.2	2.490	2.492	2.494	2.496	2.498	2.500	2.502	2.504	2.506	2.508	0	0	1	1	1	1	1	2	2
6.3	2.510	2.512	2.514	2.516	2.518	2.520	2.522	2.524	2.526	2.528	0	0	1	1	1	1	1	2	2
6.4	2.530	2.532	2.534	2.536	2.538	2.540	2.542	2.544	2.546	2.548	0	0	1	1	1	1	1	2	2
6.5	2.550	2.551	2.553	2.555	2.557	2.559	2.561	2.563	2.565	2.567	0	0	1	1	1	1	1	2	2
6.6	2.569	2.571	2.573	2.575	2.577	2.579	2.581	2.583	2.585	2.587	0	0	1	1	1	1	1	2	2
6.7	2.588	2.590	2.592	2.594	2.596	2.598	2.600	2.602	2.604	2.606	0	0	1	1	1	1	1	2	2
6.8	2.608	2.610	2.612	2.613	2.615	2.617	2.619	2.621	2.623	2.625	0	0	1	1	1	1	1	2	2
6.9	2.627	2.629	2.631	2.632	2.634	2.636	2.638	2.640	2.642	2.644	0	0	1	1	1	1	1	2	2
7.0	2.646	2.648	2.650	2.651	2.653	2.655	2.657	2.659	2.661	2.663	0	0	1	1	1	1	1	2	2
7.1	2.665	2.666	2.668	2.670	2.672	2.674	2.676	2.678	2.680	2.681	0	0	1	1	1	1	1	1	2
7.2	2.683	2.685	2.687	2.689	2.691	2.693	2.694	2.696	2.698	2.700	0	0	1	1	1	1	1	1	2
7.3	2.702	2.704	2.706	2.707	2.709	2.711	2.713	2.715	2.717	2.718	0	0	1	1	1	1	1	1	2
7.4	2.720	2.722	2.724	2.726	2.728	2.729	2.731	2.733	2.735	2.737	0	0	1	1	1	1	1	1	2
7.5	2.739	2.740	2.742	2.744	2.746	2.748	2.750	2.751	2.753	2.755	0	0	1	1	1	1	1	1	2
7.6	2.757	2.759	2.760	2.762	2.764	2.766	2.768	2.769	2.771	2.773	0	0	1	1	1	1	1	1	2
7.7	2.775	2.777	2.778	2.780	2.782	2.784	2.786	2.787	2.789	2.791	0	0	1	1	1	1	1	1	2
7.8	2.793	2.795	2.796	2.798	2.800	2.802	2.804	2.805	2.807	2.809	0	0	1	1	1	1	1	1	2
7.9	2.811	2.812	2.814	2.816	2.818	2.820	2.821	2.823	2.825	2.827	0	0	1	1	1	1	1	1	2
8.0	2.828	2.830	2.832	2.834	2.835	2.837	2.839	2.841	2.843	2.844	0	0	1	1	1	1	1	1	2
8.1	2.846	2.848	2.850	2.851	2.853	2.855	2.857	2.858	2.860	2.862	0	0	1	1	1	1	1	1	2
8.2	2.864	2.865	2.867	2.869	2.871	2.872	2.874	2.876	2.877	2.879	0	0	1	1	1	1	1	1	2
8.3	2.881	2.883	2.884	2.886	2.888	2.890	2.891	2.893	2.895	2.897	0	0	1	1	1	1	1	1	2
8.4	2.898	2.900	2.902	2.903	2.905	2.907	2.909	2.910	2.912	2.914	0	0	1	1	1	1	1	1	2
8.5	2.915	2.917	2.919	2.921	2.922	2.924	2.926	2.927	2.929	2.931	0	0	1	1	1	1	1	1	2
8.6	2.933	2.934	2.936	2.938	2.939	2.941	2.943	2.944	2.946	2.948	0	0	1	1	1	1	1	1	2
8.7	2.950	2.951	2.953	2.955	2.956	2.958	2.960	2.961	2.963	2.965	0	0	1	1	1	1	1	1	2
8.8	2.966	2.968	2.970	2.972	2.973	2.975	2.977	2.978	2.980	2.982	0	0	1	1	1	1	1	1	2
8.9	2.983	2.985	2.987	2.988	2.990	2.992	2.993	2.995	2.997	2.998	0	0	1	1	1	1	1	1	2
9.0	3.000	3.002	3.003	3.005	3.007	3.008	3.010	3.012	3.013	3.015	0	0	0	1	1	1	1	1	1
9.1	3.017	3.018	3.020	3.022	3.023	3.025	3.027	3.028	3.030	3.032	0	0	0	1	1	1	1	1	1
9.2	3.033	3.035	3.036	3.038	3.040	3.041	3.043	3.045	3.046	3.048	0	0	0	1	1	1	1	1	1
9.3	3.050	3.051	3.053	3.055	3.056	3.058	3.059	3.061	3.063	3.064	0	0	0	1	1	1	1	1	1
9.4	3.066	3.068	3.069	3.071	3.072	3.074	3.076	3.077	3.079	3.081	0	0	0	1	1	1	1	1	1
9.5	3.082	3.084	3.085	3.087	3.089	3.090	3.092	3.094	3.095	3.097	0	0	0	1	1	1	1	1	1
9.6	3.098	3.100	3.102	3.103	3.105	3.106	3.108	3.110	3.111	3.113	0	0	0	1	1	1	1	1	1
9.7	3.114	3.116	3.118	3.119	3.121	3.122	3.124	3.126	3.127	3.129	0	0	0	1	1	1	1	1	1
9.8	3.130	3.132	3.134	3.135	3.137	3.138	3.140	3.142	3.143	3.145	0	0	0	1	1	1	1	1	1
9.9	3.146	3.148	3.150	3.151	3.153	3.154	3.156	3.158	3.159	3.161	0	0	0	1	1	1	1	1	1

SQUARE ROOTS FROM 10 TO 99.99

N	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
10	3.162	3.178	3.194	3.209	3.225	3.240	3.256	3.271	3.286	3.302	2	3	5	6	8	9	11	12	14
11	3.317	3.332	3.347	3.362	3.376	3.391	3.406	3.421	3.435	3.450	1	3	4	6	7	9	10	12	13
12	3.464	3.479	3.493	3.507	3.521	3.536	3.550	3.564	3.578	3.592	1	3	4	6	7	8	10	11	13
13	3.606	3.619	3.633	3.647	3.661	3.674	3.688	3.701	3.715	3.728	1	3	4	5	7	8	10	11	12
14	3.742	3.755	3.768	3.782	3.795	3.808	3.821	3.834	3.847	3.860	1	3	4	5	7	8	9	11	12
15	3.873	3.886	3.899	3.912	3.924	3.937	3.950	3.962	3.975	3.987	1	3	4	5	6	8	9	10	11
16	4.000	4.012	4.025	4.037	4.050	4.062	4.074	4.087	4.099	4.111	1	2	4	5	6	7	9	10	11
17	4.123	4.135	4.147	4.159	4.171	4.183	4.195	4.207	4.219	4.231	1	2	4	5	6	7	8	10	11
18	4.243	4.254	4.266	4.278	4.290	4.301	4.313	4.324	4.336	4.347	1	2	3	5	6	7	8	9	10
19	4.359	4.370	4.382	4.393	4.405	4.416	4.427	4.438	4.450	4.461	1	2	3	5	6	7	8	9	10
20	4.472	4.483	4.494	4.506	4.517	4.528	4.539	4.550	4.561	4.572	1	2	3	4	6	7	8	9	10
21	4.583	4.593	4.604	4.615	4.626	4.637	4.648	4.658	4.669	4.680	1	2	3	4	5	6	8	9	10
22	4.690	4.701	4.712	4.722	4.733	4.743	4.754	4.764	4.775	4.785	1	2	3	4	5	6	7	8	9
23	4.796	4.806	4.817	4.827	4.837	4.848	4.858	4.868	4.879	4.889	1	2	3	4	5	6	7	8	9
24	4.899	4.909	4.919	4.930	4.940	4.950	4.960	4.970	4.980	4.990	1	2	3	4	5	6	7	8	9
25	5.000	5.010	5.020	5.030	5.040	5.050	5.060	5.070	5.079	5.089	1	2	3	4	5	6	7	8	9
26	5.099	5.109	5.119	5.128	5.138	5.148	5.158	5.167	5.177	5.187	1	2	3	4	5	6	7	8	9
27	5.196	5.206	5.215	5.225	5.235	5.244	5.254	5.263	5.273	5.282	1	2	3	4	5	6	7	8	9
28	5.292	5.301	5.310	5.320	5.329	5.339	5.348	5.357	5.367	5.376	1	2	3	4	5	6	7	7	8
29	5.385	5.394	5.404	5.413	5.422	5.431	5.441	5.450	5.459	5.468	1	2	3	4	5	5	6	7	8
30	5.477	5.486	5.495	5.505	5.514	5.523	5.532	5.541	5.550	5.559	1	2	3	4	4	5	6	7	8
31	5.568	5.577	5.586	5.595	5.604	5.612	5.621	5.630	5.639	5.648	1	2	3	3	4	5	6	7	8
32	5.657	5.666	5.675	5.683	5.692	5.701	5.710	5.718	5.727	5.736	1	2	3	3	4	5	6	7	8
33	5.745	5.753	5.762	5.771	5.779	5.788	5.797	5.805	5.814	5.822	1	2	3	3	4	5	6	7	8
34	5.831	5.840	5.848	5.857	5.865	5.874	5.882	5.891	5.899	5.908	1	2	3	3	4	5	6	7	8
35	5.916	5.925	5.933	5.941	5.950	5.958	5.967	5.975	5.983	5.992	1	2	2	3	4	5	6	7	8
36	6.000	6.008	6.017	6.025	6.033	6.042	6.050	6.058	6.066	6.075	1	2	2	3	4	5	6	7	7
37	6.083	6.091	6.099	6.107	6.116	6.124	6.132	6.140	6.148	6.156	1	2	2	3	4	5	6	7	7
38	6.164	6.173	6.181	6.189	6.197	6.205	6.213	6.221	6.229	6.237	1	2	2	3	4	5	6	6	7
39	6.245	6.253	6.261	6.269	6.277	6.285	6.293	6.301	6.309	6.317	1	2	2	3	4	5	6	6	7
40	6.325	6.332	6.340	6.348	6.356	6.364	6.372	6.380	6.387	6.395	1	2	2	3	4	5	6	6	7
41	6.403	6.411	6.419	6.427	6.434	6.442	6.450	6.458	6.465	6.473	1	2	2	3	4	5	5	6	7
42	6.481	6.488	6.496	6.504	6.512	6.519	6.527	6.535	6.542	6.550	1	2	2	3	4	5	5	6	7
43	6.557	6.565	6.573	6.580	6.588	6.595	6.603	6.611	6.618	6.626	1	2	2	3	4	5	5	6	7
44	6.633	6.641	6.648	6.656	6.663	6.671	6.678	6.686	6.693	6.701	1	2	2	3	4	5	5	6	7
45	6.708	6.716	6.723	6.731	6.738	6.745	6.753	6.760	6.768	6.775	1	1	2	3	4	4	5	6	7
46	6.782	6.790	6.797	6.804	6.812	6.819	6.826	6.834	6.841	6.848	1	1	2	3	4	4	5	6	7
47	6.856	6.863	6.870	6.877	6.885	6.892	6.899	6.907	6.914	6.921	1	1	2	3	4	4	5	6	7
48	6.928	6.935	6.943	6.950	6.957	6.964	6.971	6.979	6.986	6.993	1	1	2	3	4	4	5	6	6
49	7.000	7.007	7.014	7.021	7.029	7.036	7.043	7.050	7.057	7.064	1	1	2	3	4	4	5	6	6
50	7.071	7.078	7.085	7.092	7.099	7.106	7.113	7.120	7.127	7.134	1	1	2	3	4	4	5	6	6
51	7.141	7.148	7.155	7.162	7.169	7.176	7.183	7.190	7.197	7.204	1	1	2	3	4	4	5	6	6
52	7.211	7.218	7.225	7.232	7.239	7.246	7.253	7.259	7.266	7.273	1	1	2	3	3	4	5	6	6
53	7.280	7.287	7.294	7.301	7.308	7.314	7.321	7.328	7.335	7.342	1	1	2	3	3	4	5	5	6
54	7.348	7.355	7.362	7.369	7.376	7.382	7.389	7.396	7.403	7.409	1	1	2	3	3	4	5	5	6

SQUARE ROOTS FROM 10 TO 99.99

N	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
55	7.416	7.423	7.430	7.436	7.443	7.450	7.457	7.463	7.470	7.477	1	1	2	3	3	4	5	5	6
56	7.483	7.490	7.497	7.503	7.510	7.517	7.523	7.530	7.537	7.543	1	1	2	3	3	4	5	5	6
57	7.550	7.556	7.563	7.570	7.576	7.583	7.589	7.596	7.603	7.609	1	1	2	3	3	4	5	5	6
58	7.616	7.622	7.629	7.635	7.642	7.649	7.655	7.662	7.668	7.675	1	1	2	3	3	4	5	5	6
59	7.681	7.688	7.694	7.701	7.707	7.714	7.720	7.727	7.733	7.740	1	1	2	3	3	4	4	5	6
60	7.746	7.752	7.759	7.765	7.772	7.778	7.785	7.791	7.797	7.804	1	1	2	3	3	4	4	5	6
61	7.810	7.817	7.823	7.829	7.836	7.842	7.849	7.855	7.861	7.868	1	1	2	3	3	4	4	5	6
62	7.874	7.880	7.887	7.893	7.899	7.906	7.912	7.918	7.925	7.931	1	1	2	3	3	4	4	5	6
63	7.937	7.944	7.950	7.956	7.962	7.969	7.975	7.981	7.987	7.994	1	1	2	3	3	4	4	5	6
64	8.000	8.006	8.012	8.019	8.025	8.031	8.037	8.044	8.050	8.056	1	1	2	2	3	4	4	5	6
65	8.062	8.068	8.075	8.081	8.087	8.093	8.099	8.106	8.112	8.118	1	1	2	2	3	4	4	5	5
66	8.124	8.130	8.136	8.142	8.149	8.155	8.161	8.167	8.173	8.179	1	1	2	2	3	4	4	5	5
67	8.185	8.191	8.198	8.204	8.210	8.216	8.222	8.228	8.234	8.240	1	1	2	2	3	4	4	5	5
68	8.246	8.252	8.258	8.264	8.270	8.276	8.283	8.289	8.295	8.301	1	1	2	2	3	4	4	5	5
69	8.307	8.313	8.319	8.325	8.331	8.337	8.343	8.349	8.355	8.361	1	1	2	2	3	4	4	5	5
70	8.367	8.373	8.379	8.385	8.390	8.396	8.402	8.408	8.414	8.420	1	1	2	2	3	4	4	5	5
71	8.426	8.432	8.438	8.444	8.450	8.456	8.462	8.468	8.473	8.479	1	1	2	2	3	4	4	5	5
72	8.485	8.491	8.497	8.503	8.509	8.515	8.521	8.526	8.532	8.538	1	1	2	2	3	3	4	5	5
73	8.544	8.550	8.556	8.562	8.567	8.573	8.579	8.585	8.591	8.597	1	1	2	2	3	3	4	5	5
74	8.602	8.608	8.614	8.620	8.626	8.631	8.637	8.643	8.649	8.654	1	1	2	2	3	3	4	5	5
75	8.660	8.666	8.672	8.678	8.683	8.689	8.695	8.701	8.706	8.712	1	1	2	2	3	3	4	5	5
76	8.718	8.724	8.729	8.735	8.741	8.746	8.752	8.758	8.764	8.769	1	1	2	2	3	3	4	5	5
77	8.775	8.781	8.786	8.792	8.798	8.803	8.809	8.815	8.820	8.826	1	1	2	2	3	3	4	4	5
78	8.832	8.837	8.843	8.849	8.854	8.860	8.866	8.871	8.877	8.883	1	1	2	2	3	3	4	4	5
79	8.888	8.894	8.899	8.905	8.911	8.916	8.922	8.927	8.933	8.939	1	1	2	2	3	3	4	4	5
80	8.944	8.950	8.955	8.961	8.967	8.972	8.978	8.983	8.989	8.994	1	1	2	2	3	3	4	4	5
81	9.000	9.006	9.011	9.017	9.022	9.028	9.033	9.039	9.044	9.050	1	1	2	2	3	3	4	4	5
82	9.055	9.061	9.066	9.072	9.077	9.083	9.088	9.094	9.099	9.105	1	1	2	2	3	3	4	4	5
83	9.110	9.116	9.121	9.127	9.132	9.138	9.143	9.149	9.154	9.160	1	1	2	2	3	3	4	4	5
84	9.165	9.171	9.176	9.182	9.187	9.192	9.198	9.203	9.209	9.214	1	1	2	2	3	3	4	4	5
85	9.220	9.225	9.230	9.236	9.241	9.247	9.252	9.257	9.263	9.268	1	1	2	2	3	3	4	4	5
86	9.274	9.279	9.284	9.290	9.295	9.301	9.306	9.311	9.317	9.322	1	1	2	2	3	3	4	4	5
87	9.327	9.333	9.338	9.343	9.349	9.354	9.359	9.365	9.370	9.375	1	1	2	2	3	3	4	4	5
88	9.381	9.386	9.391	9.397	9.402	9.407	9.413	9.418	9.423	9.429	1	1	2	2	3	3	4	4	5
89	9.434	9.439	9.445	9.450	9.455	9.460	9.466	9.471	9.476	9.482	1	1	2	2	3	3	4	4	5
90	9.487	9.492	9.497	9.503	9.508	9.513	9.518	9.524	9.529	9.534	1	1	2	2	3	3	4	4	5
91	9.539	9.545	9.550	9.555	9.560	9.566	9.571	9.576	9.581	9.586	1	1	2	2	3	3	4	4	5
92	9.592	9.597	9.602	9.607	9.612	9.618	9.623	9.628	9.633	9.638	1	1	2	2	3	3	4	4	5
93	9.644	9.649	9.654	9.659	9.664	9.670	9.675	9.680	9.685	9.690	1	1	2	2	3	3	4	4	5
94	9.695	9.701	9.706	9.711	9.716	9.721	9.726	9.731	9.737	9.742	1	1	2	2	3	3	4	4	5
95	9.747	9.752	9.757	9.762	9.767	9.772	9.778	9.783	9.788	9.793	1	1	2	2	3	3	4	4	5
96	9.798	9.803	9.808	9.813	9.818	9.823	9.829	9.834	9.839	9.844	1	1	2	2	3	3	4	4	5
97	9.849	9.854	9.859	9.864	9.869	9.874	9.879	9.884	9.889	9.894	1	1	2	2	3	3	4	4	5
98	9.899	9.905	9.910	9.915	9.920	9.925	9.930	9.935	9.940	9.945	0	1	1	2	2	3	3	4	4
99	9.950	9.955	9.960	9.965	9.970	9.975	9.980	9.985	9.990	9.995	0	1	1	2	2	3	3	4	4

POWERS AND ROOTS

No.	Squares	Cubes	Square Roots	Cube Roots	No.	Squares	Cubes	Square Roots	Cube Roots
1	1	1	1.000	1.000	**51**	2 601	132 651	7.141	3.708
2	4	8	1.414	1.260	52	2 704	140 608	7.211	3.733
3	9	27	1.732	1.442	53	2 809	148 877	7.280	3.756
4	16	64	2.000	1.587	54	2 916	157 464	7.348	3.780
5	25	125	2.236	1.710	55	3 025	166 375	7.416	3.803
6	36	216	2.449	1.817	56	3 136	175 616	7.483	3.826
7	49	343	2.646	1.913	57	3 249	185 193	7.550	3.849
8	64	512	2.828	2.000	58	3 364	195 112	7.616	3.871
9	81	729	3.000	2.080	59	3 481	205 379	7.681	3.893
10	100	1 000	3.162	2.154	**60**	3 600	216 000	7.746	3.915
11	121	1 331	3.317	2.224	61	3 721	226 981	7.810	3.936
12	144	1 728	3.464	2.289	62	3 844	238 328	7.874	3.958
13	169	2 197	3.606	2.351	63	3 969	250 047	7.937	3.979
14	196	2 744	3.742	2.410	64	4 096	262 144	8.000	4.000
15	225	3 375	3.873	2.466	65	4 225	274 625	8.062	4.021
16	256	4 096	4.000	2.520	66	4 356	287 496	8.124	4.041
17	289	4 913	4.123	2.571	67	4 489	300 763	8.185	4.062
18	324	5 832	4.243	2.621	68	4 624	314 432	8.246	4.082
19	361	6 859	4.359	2.668	69	4 761	328 509	8.307	4.102
20	400	8 000	4.472	2.714	**70**	4 900	343 000	8.367	4.121
21	441	9 261	4.583	2.759	71	5 041	357 911	8.426	4.141
22	484	10 648	4.690	2.802	72	5 184	373 248	8.485	4.160
23	529	12 167	4.796	2.844	73	5 329	389 017	8.544	4.179
24	576	13 824	4.899	2.884	74	5 476	405 224	8.602	4.198
25	625	15 625	5.000	2.924	75	5 625	421 875	8.660	4.217
26	676	17 576	5.099	2.962	76	5 776	438 976	8.718	4.236
27	729	19 683	5.196	3.000	77	5 929	456 533	8.775	4.254
28	784	21 952	5.292	3.037	78	6 084	474 552	8.832	4.273
29	841	24 389	5.385	3.072	79	6 241	493 039	8.888	4.291
30	900	27 000	5.477	3.107	**80**	6 400	512 000	8.944	4.309
31	961	29 791	5.568	3.141	81	6 561	531 441	9.000	4.327
32	1 024	32 768	5.657	3.175	82	6 724	551 368	9.055	4.344
33	1 089	35 937	5.745	3.208	83	6 889	571 787	9.110	4.362
34	1 156	39 304	5.831	3.240	84	7 056	592 704	9.165	4.380
35	1 225	42 875	5.916	3.271	85	7 225	614 125	9.220	4.397
36	1 296	46 656	6.000	3.302	86	7 396	636 056	9.274	4.414
37	1 369	50 653	6.083	3.332	87	7 569	658 503	9.327	4.431
38	1 444	54 872	6.164	3.362	88	7 744	681 472	9.381	4.448
39	1 521	59 319	6.245	3.391	89	7 921	704 969	9.434	4.465
40	1 600	64 000	6.325	3.420	**90**	8 100	729 000	9.487	4.481
41	1 681	68 921	6.403	3.448	91	8 281	753 571	9.539	4.498
42	1 764	74 088	6.481	3.476	92	8 464	778 688	9.592	4.514
43	1 849	79 507	6.557	3.503	93	8 649	804 357	9.644	4.531
44	1 936	85 184	6.633	3.530	94	8 836	830 584	9.695	4.547
45	2 025	91 125	6.708	3.557	95	9 025	857 375	9.747	4.563
46	2 116	97 336	6.782	3.583	96	9 216	884 736	9.798	4.579
47	2 209	103 823	6.856	3.609	97	9 409	912 673	9.849	4.595
48	2 304	110 592	6.928	3.634	98	9 604	941 192	9.899	4.610
49	2 401	117 649	7.000	3.659	99	9 801	970 299	9.950	4.626
50	2 500	125 000	7.071	3.684	**100**	10 000	1 000 000	10.000	4.642

Exercise. Square Root

1. Recalling the fact that the square on the hypotenuse of a right triangle is equal to the sum of the squares on the two sides, find the hypotenuse of a right triangle whose sides are 51 in. and 68 in.

By the *sides* of a right triangle are understood the two shorter sides, those which form the right angle.

Find the square root of each of the following numbers:

2. 3364. **3.** 3844. **4.** 6084. **5.** 54,756.

Find, to three significant figures, the square root of each of the following numbers:

6. 135. **7.** 2875. **8.** 10.68. **9.** 5.342.

In case a number contains a decimal, separate the figures into periods, proceeding both ways from the decimal point.

10. Find to three significant figures the diagonal of a floor that is 64 ft. long and 38 ft. wide.

11. A telegraph pole is set perpendicular to the ground, and a taut wire, fastened to it 18 ft. above the ground, leads to a stake 12 ft. from the foot of the pole. Find to the nearest hundredth of a foot the length of the wire.

12. The formula for the area of a circle being $a = \pi r^2$, we can easily find a formula for r. Find this formula, and then, using $3\frac{1}{7}$ for π, find to two decimal places the radius of a circle whose area is 22 sq. in.

13. From the formula of Ex. 12 find to two significant figures the radius that a tinsmith should use in cutting a circular hole for a pipe, the area of the opening being 159 sq. in.

14. Find the diagonal of a square lot of land of which the area is 20,736 sq. ft.

Using the table, find the square root of each of the following:

15. 7.5.
16. 2.8.
17. 34.
18. 46.
19. 1.25.
20. 3.46.
21. 789.
22. 987.
23. 2.346.
24. 3.549.
25. 487.9.
26. 632.4.
27. 84.7.
28. 85.73.
29. 68.41.
30. 9872.

31. The foot of a 39-foot ladder is 14 ft. from the wall of a building against which the top rests. How high does the ladder reach on the wall?

32. How far from the wall of a house must the foot of a 60-foot ladder be placed in order that the top of the ladder may touch a window sill 52 ft. from the ground?

33. A rope stretched from the top of a 35-foot pole just reaches the ground 26 ft. from the foot of the pole. Assuming the rope to be straight, how long is it?

34. What must be the diameter of a water pipe in order that the area of the cross section shall be 3 sq. in.?

35. What must be the diameter of a water main in order that the area of the cross section shall be 5 sq. ft.?

36. What must be the diameter of the piston head of an engine in order that the area may be 128 sq. in.?

37. A cylindric water tank is 24 ft. high and has a capacity of 32,000 cu. ft. What is the diameter?

38. A tinsmith wishes to make some cylindric gallon cans. They are to be 9 in. high. What must be the area of the base? What radius must he use to draw the circle?

A gallon contains 231 cu. in.

39. A square lot has an area of 17,500 sq. ft. How far is it around the lot? How far is it around a lot of four times this area?

Logarithms. There is a simple device, easily learned and readily applied, for finding products, quotients, powers, and roots. This device makes use of numbers known as *logarithms*.

We shall first learn how to use logarithms without attempting to explain the reasons involved. After having seen the value of logarithms we shall consider these reasons.

On page 215 we found that $\sqrt{1.76} = 1.327$. We shall now find $\sqrt{1.76}$ by means of logarithms, thus checking this answer. Without considering for the present what is meant by the "logarithm of 1.76," we shall write $\log 1.76$ to stand for this expression.

On page 224 look in column **N** for 1.7, then look to the right of this and in column **6** find .2455. Then write

$$\log 1.76 = 0.2455.$$

Then $$\tfrac{1}{2}\log 1.76 = 0.1228.$$

Now see if you can find 0.1228 among the logarithms. The nearest we come to it is 0.1206, which is 0.0022 too small. Now look on the same line with 0.1206 in the columns at the right, and we see 23, the number nearest 22, under **7**, which means that we must annex 7 to the number of which 0.1206 is the logarithm in order to obtain the number of which $0.1206 + 0.0023$ is the logarithm.

But, from the table,

$$\log 1.32 = 0.1206,$$

and so $$\begin{aligned}\log 1.327 &= 0.1229\\ &= 0.1228, \text{ approximately.}\end{aligned}$$

That is, 1.327 is approximately the number that has 0.1228 for its logarithm.

We have thus found $\sqrt{1.76}$ by simply dividing $\log 1.76$ by 2 and using the tables. That is, finding the square root of a number has been reduced to dividing by 2.

LOGARITHMS

N	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
1.0	.0000	.0043	.0086	.0128	.0170	.0212	.0253	.0294	.0334	.0374	4	8	12	17	21	25	29	33	37
1.1	.0414	.0453	.0492	.0531	.0569	.0607	.0645	.0682	.0719	.0755	4	8	11	15	19	23	26	30	34
1.2	.0792	.0828	.0864	.0899	.0934	.0969	.1004	.1038	.1072	.1106	3	7	10	14	17	21	24	28	31
1.3	.1139	.1173	.1206	.1239	.1271	.1303	.1335	.1367	.1399	.1430	3	6	10	13	16	19	23	26	29
1.4	.1461	.1492	.1523	.1553	.1584	.1614	.1644	.1673	.1703	.1732	3	6	9	12	15	18	21	24	27
1.5	.1761	.1790	.1818	.1847	.1875	.1903	.1931	.1959	.1987	.2014	3	6	8	11	14	17	20	22	25
1.6	.2041	.2068	.2095	.2122	.2148	.2175	.2201	.2227	.2253	.2279	3	5	8	11	13	16	18	21	24
1.7	.2304	.2330	.2355	.2380	.2405	.2430	.2455	.2480	.2504	.2529	2	5	7	10	12	15	17	20	22
1.8	.2553	.2577	.2601	.2625	.2648	.2672	.2695	.2718	.2742	.2765	2	5	7	9	12	14	16	19	21
1.9	.2788	.2810	.2833	.2856	.2878	.2900	.2923	.2945	.2967	.2989	2	4	7	9	11	13	16	18	20
2.0	.3010	.3032	.3054	.3075	.3096	.3118	.3139	.3160	.3181	.3201	2	4	6	8	11	13	15	17	19
2.1	.3222	.3243	.3263	.3284	.3304	.3324	.3345	.3365	.3385	.3404	2	4	6	8	10	12	14	16	18
2.2	.3424	.3444	.3464	.3483	.3502	.3522	.3541	.3560	.3579	.3598	2	4	6	8	10	12	14	15	17
2.3	.3617	.3636	.3655	.3674	.3692	.3711	.3729	.3747	.3766	.3784	2	4	6	7	9	11	13	15	17
2.4	.3802	.3820	.3838	.3856	.3874	.3892	.3909	.3927	.3945	.3962	2	4	5	7	9	11	12	14	16
2.5	.3979	.3997	.4014	.4031	.4048	.4065	.4082	.4099	.4116	.4133	2	3	5	7	9	10	12	14	15
2.6	.4150	.4166	.4183	.4200	.4216	.4232	.4249	.4265	.4281	.4298	2	3	5	7	8	10	11	13	15
2.7	.4314	.4330	.4346	.4362	.4378	.4393	.4409	.4425	.4440	.4456	2	3	5	6	8	9	11	13	14
2.8	.4472	.4487	.4502	.4518	.4533	.4548	.4564	.4579	.4594	.4609	2	3	5	6	8	9	11	12	14
2.9	.4624	.4639	.4654	.4669	.4683	.4698	.4713	.4728	.4742	.4757	1	3	4	6	7	9	10	12	13
3.0	.4771	.4786	.4800	.4814	.4829	.4843	.4857	.4871	.4886	.4900	1	3	4	6	7	9	10	11	13
3.1	.4914	.4928	.4942	.4955	.4969	.4983	.4997	.5011	.5024	.5038	1	3	4	6	7	8	10	11	12
3.2	.5051	.5065	.5079	.5092	.5105	.5119	.5132	.5145	.5159	.5172	1	3	4	5	7	8	9	11	12
3.3	.5185	.5198	.5211	.5224	.5237	.5250	.5263	.5276	.5289	.5302	1	3	4	5	6	8	9	10	12
3.4	.5315	.5328	.5340	.5353	.5366	.5378	.5391	.5403	.5416	.5428	1	3	4	5	6	8	9	10	11
3.5	.5441	.5453	.5465	.5478	.5490	.5502	.5514	.5527	.5539	.5551	1	2	4	5	6	7	9	10	11
3.6	.5563	.5575	.5587	.5599	.5611	.5623	.5635	.5647	.5658	.5670	1	2	4	5	6	7	8	10	11
3.7	.5682	.5694	.5705	.5717	.5729	.5740	.5752	.5763	.5775	.5786	1	2	3	5	6	7	8	9	10
3.8	.5798	.5809	.5821	.5832	.5843	.5855	.5866	.5877	.5888	.5899	1	2	3	5	6	7	8	9	10
3.9	.5911	.5922	.5933	.5944	.5955	.5966	.5977	.5988	.5999	.6010	1	2	3	4	5	7	8	9	10
4.0	.6021	.6031	.6042	.6053	.6064	.6075	.6085	.6096	.6107	.6117	1	2	3	4	5	6	8	9	10
4.1	.6128	.6138	.6149	.6160	.6170	.6180	.6191	.6201	.6212	.6222	1	2	3	4	5	6	7	8	9
4.2	.6232	.6243	.6253	.6263	.6274	.6284	.6294	.6304	.6314	.6325	1	2	3	4	5	6	7	8	9
4.3	.6335	.6345	.6355	.6365	.6375	.6385	.6395	.6405	.6415	.6425	1	2	3	4	5	6	7	8	9
4.4	.6435	.6444	.6454	.6464	.6474	.6484	.6493	.6503	.6513	.6522	1	2	3	4	5	6	7	8	9
4.5	.6532	.6542	.6551	.6561	.6571	.6580	.6590	.6599	.6609	.6618	1	2	3	4	5	6	7	8	9
4.6	.6628	.6637	.6646	.6656	.6665	.6675	.6684	.6693	.6702	.6712	1	2	3	4	5	6	7	7	8
4.7	.6721	.6730	.6739	.6749	.6758	.6767	.6776	.6785	.6794	.6803	1	2	3	4	5	5	6	7	8
4.8	.6812	.6821	.6830	.6839	.6848	.6857	.6866	.6875	.6884	.6893	1	2	3	4	4	5	6	7	8
4.9	.6902	.6911	.6920	.6928	.6937	.6946	.6955	.6964	.6972	.6981	1	2	3	4	4	5	6	7	8
5.0	.6990	.6998	.7007	.7016	.7024	.7033	.7042	.7050	.7059	.7067	1	2	3	3	4	5	6	7	8
5.1	.7076	.7084	.7093	.7101	.7110	.7118	.7126	.7135	.7143	.7152	1	2	3	3	4	5	6	7	8
5.2	.7160	.7168	.7177	.7185	.7193	.7202	.7210	.7218	.7226	.7235	1	2	2	3	4	5	6	7	7
5.3	.7243	.7251	.7259	.7267	.7275	.7284	.7292	.7300	.7308	.7316	1	2	2	3	4	5	6	6	7
5.4	.7324	.7332	.7340	.7348	.7356	.7364	.7372	.7380	.7388	.7396	1	2	2	3	4	5	6	6	7

LOGARITHMS

N	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
5.5	.7404	.7412	.7419	.7427	.7435	.7443	.7451	.7459	.7466	.7474	1	2	2	3	4	5	5	6	7
5.6	.7482	.7490	.7497	.7505	.7513	.7520	.7528	.7536	.7543	.7551	1	2	2	3	4	5	5	6	7
5.7	.7559	.7566	.7574	.7582	.7589	.7597	.7604	.7612	.7619	.7627	1	2	2	3	4	5	5	6	7
5.8	.7634	.7642	.7649	.7657	.7664	.7672	.7679	.7686	.7694	.7701	1	1	2	3	4	4	5	6	7
5.9	.7709	.7716	.7723	.7731	.7738	.7745	.7752	.7760	.7767	.7774	1	1	2	3	4	4	5	6	7
6.0	.7782	.7789	.7796	.7803	.7810	.7818	.7825	.7832	.7839	.7846	1	1	2	3	4	4	5	6	6
6.1	.7853	.7860	.7868	.7875	.7882	.7889	.7896	.7903	.7910	.7917	1	1	2	3	4	4	5	6	6
6.2	.7924	.7931	.7938	.7945	.7952	.7959	.7966	.7973	.7980	.7987	1	1	2	3	3	4	5	6	6
6.3	.7993	.8000	.8007	.8014	.8021	.8028	.8035	.8041	.8048	.8055	1	1	2	3	3	4	5	5	6
6.4	.8062	.8069	.8075	.8082	.8089	.8096	.8102	.8109	.8116	.8122	1	1	2	3	3	4	5	5	6
6.5	.8129	.8136	.8142	.8149	.8156	.8162	.8169	.8176	.8182	.8189	1	1	2	3	3	4	5	5	6
6.6	.8195	.8202	.8209	.8215	.8222	.8228	.8235	.8241	.8248	.8254	1	1	2	3	3	4	5	5	6
6.7	.8261	.8267	.8274	.8280	.8287	.8293	.8299	.8306	.8312	.8319	1	1	2	3	3	4	5	5	6
6.8	.8325	.8331	.8338	.8344	.8351	.8357	.8363	.8370	.8376	.8382	1	1	2	3	3	4	4	5	6
6.9	.8388	.8395	.8401	.8407	.8414	.8420	.8426	.8432	.8439	.8445	1	1	2	2	3	4	4	5	6
7.0	.8451	.8457	.8463	.8470	.8476	.8482	.8488	.8494	.8500	.8506	1	1	2	2	3	4	4	5	6
7.1	.8513	.8519	.8525	.8531	.8537	.8543	.8549	.8555	.8561	.8567	1	1	2	2	3	4	4	5	5
7.2	.8573	.8579	.8585	.8591	.8597	.8603	.8609	.8615	.8621	.8627	1	1	2	2	3	4	4	5	5
7.3	.8633	.8639	.8645	.8651	.8657	.8663	.8669	.8675	.8681	.8686	1	1	2	2	3	4	4	5	5
7.4	.8692	.8698	.8704	.8710	.8716	.8722	.8727	.8733	.8739	.8745	1	1	2	2	3	4	4	5	5
7.5	.8751	.8756	.8762	.8768	.8774	.8779	.8785	.8791	.8797	.8802	1	1	2	2	3	3	4	5	5
7.6	.8808	.8814	.8820	.8825	.8831	.8837	.8842	.8848	.8854	.8859	1	1	2	2	3	3	4	5	5
7.7	.8865	.8871	.8876	.8882	.8887	.8893	.8899	.8904	.8910	.8915	1	1	2	2	3	3	4	4	5
7.8	.8921	.8927	.8932	.8938	.8943	.8949	.8954	.8960	.8965	.8971	1	1	2	2	3	3	4	4	5
7.9	.8976	.8982	.8987	.8993	.8998	.9004	.9009	.9015	.9020	.9025	1	1	2	2	3	3	4	4	5
8.0	.9031	.9036	.9042	.9047	.9053	.9058	.9063	.9069	.9074	.9079	1	1	2	2	3	3	4	4	5
8.1	.9085	.9090	.9096	.9101	.9106	.9112	.9117	.9122	.9128	.9133	1	1	2	2	3	3	4	4	5
8.2	.9138	.9143	.9149	.9154	.9159	.9165	.9170	.9175	.9180	.9186	1	1	2	2	3	3	4	4	5
8.3	.9191	.9196	.9201	.9206	.9212	.9217	.9222	.9227	.9232	.9238	1	1	2	2	3	3	4	4	5
8.4	.9243	.9248	.9253	.9258	.9263	.9269	.9274	.9279	.9284	.9289	1	1	2	2	3	3	4	4	5
8.5	.9294	.9299	.9304	.9309	.9315	.9320	.9325	.9330	.9335	.9340	1	1	2	2	3	3	4	4	5
8.6	.9345	.9350	.9355	.9360	.9365	.9370	.9375	.9380	.9385	.9390	1	1	2	2	3	3	4	4	5
8.7	.9395	.9400	.9405	.9410	.9415	.9420	.9425	.9430	.9435	.9440	0	1	1	2	2	3	3	4	4
8.8	.9445	.9450	.9455	.9460	.9465	.9469	.9474	.9479	.9484	.9489	0	1	1	2	2	3	3	4	4
8.9	.9494	.9499	.9504	.9509	.9513	.9518	.9523	.9528	.9533	.9538	0	1	1	2	2	3	3	4	4
9.0	.9542	.9547	.9552	.9557	.9562	.9566	.9571	.9576	.9581	.9586	0	1	1	2	2	3	3	4	4
9.1	.9590	.9595	.9600	.9605	.9609	.9614	.9619	.9624	.9628	.9633	0	1	1	2	2	3	3	4	4
9.2	.9638	.9643	.9647	.9652	.9657	.9661	.9666	.9671	.9675	.9680	0	1	1	2	2	3	3	4	4
9.3	.9685	.9689	.9694	.9699	.9703	.9708	.9713	.9717	.9722	.9727	0	1	1	2	2	3	3	4	4
9.4	.9731	.9736	.9741	.9745	.9750	.9754	.9759	.9763	.9768	.9773	0	1	1	2	2	3	3	4	4
9.5	.9777	.9782	.9786	.9791	.9795	.9800	.9805	.9809	.9814	.9818	0	1	1	2	2	3	3	4	4
9.6	.9823	.9827	.9832	.9836	.9841	.9845	.9850	.9854	.9859	.9863	0	1	1	2	2	3	3	4	4
9.7	.9868	.9872	.9877	.9881	.9886	.9890	.9894	.9899	.9903	.9908	0	1	1	2	2	3	3	4	4
9.8	.9912	.9917	.9921	.9926	.9930	.9934	.9939	.9943	.9948	.9952	0	1	1	2	2	3	3	4	4
9.9	.9956	.9961	.9965	.9969	.9974	.9978	.9983	.9987	.9991	.9996	0	1	1	2	2	3	3	3	4

Finding a Logarithm. Whatever may be the reason for the work on page 223, it is certain that if we wish to use logarithms we must know how to find them from the table which is given on pages 224 and 225. We shall therefore first learn how to use this table.

In general, a logarithm consists of two parts, the integral part being called the *characteristic* of the logarithm and the decimal part being called the *mantissa.*

Thus if log 2353 = 3.3717, the characteristic of the logarithm is 3 and the mantissa is 0.3717.

Finding the Characteristic. For reasons given hereafter, the characteristic of the logarithm of a number greater than 1 is always one less than the number of integral places.

Hence the logarithm of a number between 1000 and 10,000 lies between 3 and 4, and is therefore 3 plus some fraction.

The same rule shows us that the logarithm of a number between 10,000 and 100,000 lies between 4 and 5, and is therefore 4 plus some fraction.

For reasons given later, the characteristic of the logarithm of a decimal fraction is one more than the number of zeros between the decimal point and the first significant figure.

Hence the logarithm of a number between 0.001 and 0.01 lies between -3 and -2, and is therefore -3 plus some fraction. Hence the characteristic of the logarithm of a number like 0.0075 is -3.

Of course, instead of saying that log 1475 is 3 plus a fraction, we might say that it is 4 minus a fraction; and instead of saying that log 0.007 is -3 plus a fraction, we might say that it is -2 minus a fraction. This however would require a separate table of negative mantissas and would therefore be inconvenient.

For convenience, *the mantissa is always taken as positive,* although the characteristic may be either positive or negative.

Rule for the Characteristic. We may now summarize the directions given on page 226 by stating the following rule:

1. *The characteristic of a number greater than 1 is positive and is one less than the number of integral places in the number.*

For example, $\log 75 = 1 +$ some mantissa,

$\log 472.8 = 2 +$ some mantissa,

and $\log 14{,}800.75 = 4 +$ some mantissa.

2. *The characteristic of a number between 0 and 1 is negative and is one more than the number of zeros between the decimal point and the first significant figure in the decimal fraction.*

For example, $\log 0.02 = -2 +$ some mantissa,

and $\log 0.00076 = -4 +$ some mantissa.

The logarithm of a negative number is an imaginary number, and hence such logarithms are not used in computation.

The Negative Characteristic. If $\log 0.02 = -2 + 0.3010$, we cannot write the logarithm -2.3010, because this would mean that both the mantissa and characteristic were negative. It is therefore customary to write $\log 0.02 = \bar{2}.3010$, which means that only the characteristic 2 is negative.

That is, $\bar{2}.3010 = -2 + 0.3010$. We may also write it $0.3010 - 2$, or $8.3010 - 10$, or in any similar manner that will show clearly that the characteristic is -2 and the mantissa 0.3010.

Exercise. Characteristics

Write the characteristics of the logarithms of:

1. 64.

2. 64.8.

3. 648.

4. 6.48.

5. 6480.

6. 8235.

7. 823.5.

8. 82.35.

9. 8.235.

10. 82,350.

11. 0.9.

12. 0.09.

13. 0.99.

14. 0.095.

15. 0.0005.

16. 0.4004.

17. 0.0044.

18. 0.0444.

19. 0.4444.

20. 0.0004.

Mantissa Independent of Decimal Point. For reasons that may be inferred from the explanation given on page 235 and that are fully explained by algebra, the mantissa of a logarithm is independent of the decimal point; that is, the mantissas are the same for log 2350, log 235, log 0.235, log 0.000235, and so on, wherever the decimal point is placed.

The mantissa of the logarithm of a number is unchanged by any change in the position of the decimal point of the number.

This is a fact of great importance, for if the table of logarithms, which we shall soon describe, gives us the mantissa of log 235, we know that we may use the same mantissa for log 0.00235, log 2.35, and so on.

Exercise. Mantissas and Characteristics

Given that log 725 = 2.8603, find:

1. log 72.5.
2. log 7.25.
3. log 0.725.
4. log 7250.
5. log 72,500.
6. log 0.0725.
7. log 725,000.
8. log 0.00725.
9. log 7,250,000.

Given that log 26,630 = 4.4254, find:

10. log 2.663.
11. log 26.63.
12. log 266.3.
13. log 0.2663.
14. log 0.02663.
15. log 0.002663.
16. log 266,300.
17. log 2,663,000.
18. log 26,630,000

Given that log 7.154 = 0.8545, find:

19. log 71.54.
20. log 715.4.
21. log 0.7154.
22. log 0.07154.
23. log 7154.
24. log 715,400.

Given that log 36.07 = 1.5571, find:

25. log 360.7.
26. log 3.607.
27. log 36,070.
28. log 360,700.
29. log 0.3607.
30. log 0.03607.

Table of Logarithms. A table of logarithms to four decimal places is given on pages 224 and 225. It gives the mantissas of the logarithms of all numbers less than 1000.

Tables in which the mantissas are given to more than four decimal places are used when a greater degree of accuracy is required, but for ordinary computations a four-place table is usually sufficient.

The columns at the right of each page of the tables are called *difference columns,* and their further use is explained below.

In the table the numbers are given under **N** and the tenths under the columns headed 0, 1, 2, ⋯ 9.

Since only the mantissas are given in the table, *always write the characteristic before looking up the mantissa, so that it shall not be forgotten.*

Finding the Logarithm of a Number. The following examples explain the use of the table:

1. Find the logarithm of 73.4.

First write the characteristic, 1.

In column **N** look for the first two figures, 7.3.

Then look to the right of 7.3 and in column **4**. Here the mantissa is found to be 0.8657.

Hence log 73.4 = 1.8657.

2. Find the logarithm of 3534.

First write the characteristic, 3.

As in Ex. 1, in column **N** look for the first two figures, 35.

Then look to the right of 3.5 and in column 3. Here the mantissa is found to be 0.5478. In the difference columns at the right we find, on the same line, the difference 5 under the column headed **4**.

We then proceed thus:

$$\log 3530 = 3.5478.$$

Adding 0.0005, found in the difference columns, we have

$$\log 3534 = 3.5483.$$

This plan of finding the logarithm of a number of more figures than those given directly in the tables is called *interpolation.*

3. Find the logarithm of 0.0002703.

First write the characteristic, − 4.

Instead of proceeding as in Ex. 2, we may proceed as follows:

$$\begin{aligned} \log 0.000271 &= 0.4330 - 4 \\ \log 0.000270 &= \underline{0.4313} - 4 \\ 0.3 \text{ of } \overline{0.0017} &= 0.0005 \end{aligned}$$

Adding 0.0005 to 0.4313 − 4, we have 0.4318 − 4.

We may write log 0.0002703 with the − 4 at the left, thus: $\bar{4}.4318$. When we have subtractions to perform, however, it is less confusing to place the negative characteristic at the right, as shown above. It is also convenient to write the negative characteristic at the right in performing other operations on logarithms.

4. Find the logarithm of 7.

Since the mantissa of log 7 is the same as that of log 7.0, we look for 7.0 in column **N** and under column **0**. Hence log 7 = 0.8451.

Exercise. Finding Logarithms

Using the table on pages 224 and 225, find the logarithms of:

1. 34.
2. 35.
3. 86.
4. 49.
5. 70.
6. 200.
7. 300.
8. 370.
9. 375.
10. 3756.
11. 37.56.
12. 375.6.
13. 8.
14. 6.
15. 0.9.
16. 0.5.
17. 0.43.
18. 0.87.
19. 2.44.
20. 3.77.
21. 47.7.
22. 577.
23. 6770.
24. 7775.
25. 32.
26. 322.
27. 322.2.
28. 3222.
29. 0.32.
30. 0.032.
31. 0.322.
32. 0.477.
33. 4.270.
34. 0.6000.
35. 0.0065.
36. 0.6550.
37. 282.
38. 282.3.
39. 282.9.
40. 28.29.
41. 2.829.
42. 327.
43. 3275.
44. 32.75.
45. 3.275.
46. 327.5.
47. 32,750.
48. 63,750.

Antilogarithm. The number corresponding to a given logarithm is called an *antilogarithm.*

Thus if log 676 is 2.8299, the antilogarithm of 2.8299 is 676.

Finding Antilogarithms. Antilogarithms are found from the table of logarithms by looking for the number corresponding to the given mantissa, and locating the decimal point according to the characteristic.

1. Find the antilogarithm of 3.4265.

Looking in the table for the mantissa 0.4265, we find that it is opposite 2.6 in column **N** and under column **7**. It is therefore the mantissa of the logarithm of 2.67.

Since the characteristic is 3, there must be four integral places in the antilogarithm. Hence the antilogarithm must be 2670.

Therefore the antilogarithm of 3.4265 is 2670.

2. Find the antilogarithm of $\overline{2}.8404$.

Looking for the mantissa 0.8404, we do not find it in the table. The nearest we come to it is 0.8401, which is 0.0003 too small. Now look in the same line with 0.8401, and in the column of differences at the right, and notice that 3 is given under **5**. This means that we must annex 5 to the antilogarithm of 0.8401 in order to find the antilogarithm of 0.8401 + 0.0003.

From the table we then have

$$\log 0.0692 = \overline{2}.8401$$

and so

$$\log 0.06925 = \overline{2}.8404.$$

That is,

$$\text{antilog}\, \overline{2}.8404 = 0.06925.$$

3. Find the antilogarithm of 0.3664.

Instead of proceeding as in Ex. 2, we may find the antilogarithm of 0.3664 as follows: Looking in the table for the mantissa 0.3664, we find that it lies between 0.3655 and 0.3674, whose difference is 0.0019. Since $0.3664 - 0.3655 = 0.0009$, the given mantissa is $\frac{9}{19}$ of the way from 0.3655 to 0.3674. But the antilogarithm of 0.3655 is 2.32, and the antilogarithm of 0.3674 is 2.33. Adding $\frac{9}{19}$ of the difference of the antilogarithms to 2.32, we have 2.325, the antilogarithm required.

Exercise. Antilogarithms

Using the table, find the antilogarithms of:

1. 0.7803.
2. 1.8363.
3. 2.8451.
4. $\bar{1}.5988$.
5. 0.5065.
6. 3.5211.
7. 2.5977.
8. 2.8785.
9. $\bar{2}.6415$.
10. $\bar{3}.6964$.
11. 0.1761.
12. 1.4082.
13. 1.8212.
14. $\bar{2}.5403$.
15. $\bar{4}.5999$.
16. 4.7882.
17. 1.8845.
18. 2.8844.
19. $\bar{2}.8846$.
20. 3.8457.
21. 0.0000.
22. 3.0000.
23. 3.7952.
24. 2.7406.
25. $0.7410 - 1$.
26. 3.7735.
27. 2.2620.
28. $0.4210 - 2$.
29. $\bar{1}.7280$.
30. $\bar{2}.8506$.
31. $0.4099 - 1$.
32. $0.5403 - 3$.

33. If the logarithm of the product of two numbers is 3.8615, what is the product of the numbers?

34. If the logarithm of the quotient of two numbers is 2.8069, what is the quotient of the numbers?

35. If the logarithm of the square of a certain number is 2.4594, what is the square of the number? What is the number and how is it most easily found?

36. If the logarithm of the square root of a certain number is 1.4472, what is the square root of the number? What is the number and how is it most easily found?

37. If we wish to multiply 236 by 27.9, what logarithms do we need? Find these logarithms from the table.

38. If we know that the logarithm of a certain result that we are seeking is 3.6222, what is the result?

39. There is a certain number such that the logarithm of its square is 3.3979. What is the number?

40. If the logarithm of the cube root of a certain number is 0.4314, what is the logarithm of the number? What is the cube root of the number?

Multiplication by Logarithms. Without at present explaining the reason involved, we shall now show how to find by logarithms the product of several numbers.

Find the product of 2.73, 4.867, 51.2, and 731.4.

$$\begin{array}{rl}
\log 2.73 & = 0.4362 \\
\log 4.867 & = 0.6872 \\
\log 51.2 & = 1.7093 \\
\log 731.4 & = \underline{2.8641} \\
 & \ 5.6968 = \log 497{,}500.
\end{array}$$

That is, the product is 497,500, certainly correct to three figures and probably correct to four figures.

The logarithm of a product is the sum of the logarithms of the factors.

Division by Logarithms. Divide 67.39 by 1.994, giving the result to four significant figures.

$$\begin{array}{rl}
\log 67.39 & = 1.8286 \\
\log 1.994 & = \underline{0.2998} \\
 & \ 1.5288 = \log 33.79.
\end{array}$$

That is, the quotient is 33.79.

The logarithm of a quotient is the logarithm of the dividend minus the logarithm of the divisor.

In case the subtraction gives a negative mantissa, we proceed as in the following example:

Divide 1.994 by 67.39.

As above, $\log 1.994 = 0.2998,$

$\log 67.39 = 1.8286.$

We now add 10 to the logarithm of 1.994 and also subtract 10, the work appearing as follows:

$$\begin{array}{rl}
\log 1.994 & = 10.2998 - 10 \\
\log 67.39 & = \underline{\ \ 1.8286 \qquad\ } \\
\end{array}$$

Subtracting, we have $8.4712 - 10 = \bar{2}.4712.$

From the table, the antilogarithm of $\bar{2}.4712$ is 0.02959.

Therefore $1.994 \div 6.739 = 0.02959.$

Finding a Power by Logarithms. In finding compound interest, a subject of the greatest importance in large investments, we frequently have to find the value of an expression like 1.04^{15}. Using logarithms, find this value.

$$\log 1.04 = 0.0170,$$

$$15 \log 1.04 = 0.2550 = \log 1.799.$$

That is, $$1.04^{15} = 1.799.$$

The logarithm of a power of a number is the logarithm of the number multiplied by the exponent of the power.

Finding a Root by Logarithms. Find the cube root of 42.83.

$$\log 42.83 = 1.6317,$$

$$\tfrac{1}{3} \log 42.83 = 0.5439 = \log 3.499.$$

That is, $$\sqrt[3]{42.83} = 3.499.$$

The logarithm of a root of a number is the logarithm of the number divided by the index of the root.

Exercise. Computation by Logarithms

Multiply as indicated, using logarithms:

1. 26×35.
2. 82×98.
3. 27.6×31.4.
4. 34.8×526.9.
5. 421.3×872.8.
6. 687.5×9436.

Divide as indicated, using logarithms:

7. $144 \div 6$.
8. $250 \div 5$.
9. $42.5 \div 2.5$.
10. $625 \div 2.5$.
11. $2969 \div 314.2$.
12. $12.73 \div 823.4$.

Find the value of each of the following:

13. 1.04^5.
14. 1.05^8.
15. 1.035^6.
16. 1.045^{12}.
17. 25^3.
18. 68^5.
19. 529^7.
20. 83.9^8.
21. $\sqrt{27.4}$.
22. $\sqrt{4.396}$.
23. $\sqrt[3]{2.434}$.
24. $\sqrt[3]{74.86}$.
25. $\sqrt[4]{5}$.
26. $\sqrt[5]{1.6}$.
27. $\sqrt[6]{28.3}$.
28. $\sqrt[7]{928.8}$.

Reasons in Logarithmic Work. As the student has probably learned from algebra, $10^{\frac{1}{2}}$ means $\sqrt{10}$, $10^{\frac{1}{3}}$ means $\sqrt[3]{10}$, $10^{\frac{2}{3}}$ means $\sqrt[3]{10^2}$, and $10^{0.23}$ means $10^{\frac{23}{100}}$, or $\sqrt[100]{10^{23}}$.

That is, *a fractional exponent has a meaning.*

We know that $10^2 \times 10^3 = 10 \times 10 \times 10 \times 10 \times 10 = 10^5$.

Now logarithms are merely exponents of 10.

Thus, because $10^2 = 100$, we evidently have $\log 100 = 2$, and because $10^{\frac{1}{4}} = 1.779$, we have $\log 1.779 = 0.25$.

That is, $\log (10^2 \times 10^3) = \log 10^{2+3} = \log 10^5$.

This explains why we add the logarithms of the factors in order to find the logarithm of the product.

Similarly,
$$10^5 \div 10^3 = \frac{10 \times 10 \times 10 \times 10 \times 10}{10 \times 10 \times 10}$$
$$= 10^{5-3} = 10^2.$$

That is, $\log (10^5 \div 10^3) = \log 10^{5-3} = \log 10^2$.

This explains why we subtract logarithms in division.

In the case of powers, $(10^3)^2 = 10^3 \times 10^3 = 10^6 = 10^{3\times 2}$.

This explains why we multiply the logarithm by the exponent of the power in the case of finding a power.

In the case of roots, $\sqrt{10^6} = \sqrt{10^3 \times 10^3} = 10^3 = 10^{6\div 2}$.

This explains why we divide the logarithm by the index of the root in the case of finding a root.

If $10^{3.3711} = 2350$, then $\log 2350 = 3.3711$.

Dividing by 10, we have
$$10^{3.3711-1} = 2350; \text{ whence } \log 235 = 2.3711.$$

This explains why the mantissa is independent of the position of the decimal point in a number.

Thus, by the aid of logarithms, multiplication is performed by addition, division is performed by subtraction, powers are found by a single multiplication, and roots are found by a single division.

Exercise. Computations by Logarithms

Perform the following computations by logarithms, carrying each result to four significant figures:

1. 4.67×38.4.
2. 3.57×526.
3. 60.7×80.5.
4. 409×209.
5. 37×4765.
6. 79×389.7.
7. 58×48.99.
8. 33×4.176.
9. 9×0.1728.
10. 7×0.0146.
11. $3.9 \div 2.7$.
12. $35.7 \div 4.3$.
13. $62.8 \div 0.31$.
14. $0.007 \div 0.85$.
15. $0.072 \div 0.09$.
16. $62.83 \div 0.7$.
17. $3.009 \div 9.8$.
18. $4 \div 3.142$.
19. $8 \div 31.47$.
20. $0.7 \div 3.14$.
21. $\sqrt{3}$.
22. $\sqrt[3]{3}$.
23. $\sqrt[4]{7}$.
24. $\sqrt[5]{6}$.
25. $\sqrt[6]{128}$.
26. $\sqrt[7]{347.6}$.
27. $\sqrt[8]{0.0009}$.
28. 72.37^7.
29. 3.107^{13}.
30. 0.46^{20}.
31. Find the value of $\sqrt{3.74 \times 22.95 \times 517.8}$.
32. Find the value of $\sqrt[3]{0.6 \times 0.0783 \times 328.5}$.
33. Find the value of $2.76 \times 39.35 \times 72.86 \times 0.04$.

Perform the following multiplications by logarithms:

34. 3.389×0.000625.
35. 49.76×0.000046.
36. 0.372×0.004233.
37. 4.216×5.734.
38. 39.7×67.64.
39. 37.82×2.74.
40. 4.675×0.0083.
41. 618.4×843.9.

Perform the following divisions by logarithms:

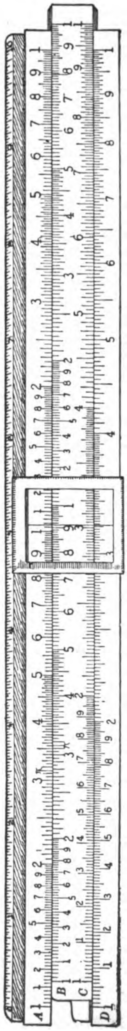
42. $\dfrac{67.73}{32.81}$.
43. $\dfrac{0.5987}{2.427}$.
44. $\dfrac{476.8}{0.006342}$.
45. $\dfrac{0.06193}{37.99}$.
46. $\dfrac{0.3398}{0.3926}$.
47. $\dfrac{0.0004872}{0.123}$.

Slide Rule. There has lately come into common use in this country, especially among shop foremen and others who wish to do rapid calculation with results to three significant figures, a simple instrument known as the *slide rule.*

It is impossible to give a satisfactory explanation of a slide rule unless the student has the instrument in his hand at the time. All that we shall attempt is to show how such a rule looks and to give some idea of its use. Unless the school supplies the students with slide rules or they purchase them for their own use, pages 237–240 should be omitted as a class exercise.

Slide rules are usually accompanied by pamphlets giving directions for their use.

A common form of slide rule is here shown, reduced to about two thirds the actual size.

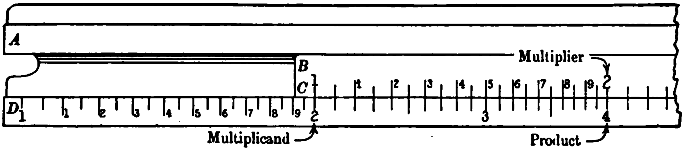


The rule in the middle, marked *BC*, slides along the two outer rules. The glass plate in the middle, known as the *runner* or *cursor*, slides either way, a hair line being ruled upon it to facilitate reading the results.

In reality the lengths on the slide rule are the mantissas of the logarithms of the numbers shown, so that in working with this instrument we are really using logarithms mechanically.

Teachers will recognize that the only value that a student will derive from merely reading pages 237–240 is that he will know that there is such an instrument as a slide rule and will have a general idea of its nature. The slide rule is coming into common use for many simple calculations and also for the purpose of checking more extensive computations.

Operations with the Slide Rule. If we wish to multiply 2 by 2 on the slide rule we simply set 1 on the scale marked *C* exactly over 2 (the multiplicand) on the scale marked *D*, and under 2 (the multiplier) on *C* we read 4 (the product) on *D*.

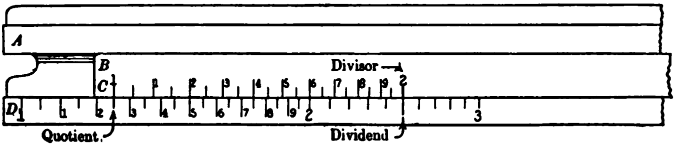


Similarly, to multiply 45 by 6, we set 1 on *C* over 45 on *D*, and under 6 on *C* we read 270 on *D*. In the case of decimals it is usually evident where to place the decimal point.

That is, *place the 1 on C over the multiplicand on D, and read the product on D below the multiplier on C.*

With the ordinary slide rule it is possible to use factors of three significant figures each and to find the product of these factors correct to three significant figures.

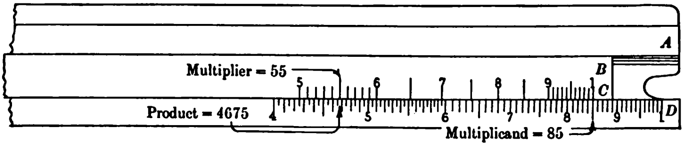
If we wish to divide 25 by 2 we set 2 (the divisor) on *C* over 25 (the dividend) on *D*, and under 1 on *C* we read 125 (the quotient figures) on *D*. Since it is evident where the decimal point should be placed, we write 12.5 as the quotient.



Similarly, to divide 270 by 6, we set 6 on *C* over 270 on *D*, and below the 1 on *C* we read 45 on *D*.

That is, *place the divisor on C over the dividend on D, and read the quotient on D below the 1 on C, placing the decimal point at the proper place.*

Number of Integral Places in the Product. We have thus far left the student to determine by his common sense the number of integral places in a product. It is desirable, however, to have a definite method of determining this number. The number of integral places manifestly depends on the characteristic of the logarithm of the result, and this is equal to the sum of the characteristics of the logarithms of the numbers, plus anything that is carried over in adding the mantissas. For example, consider the product of 55 × 85.



If we set the unit on *C* over 55 on *D*, or over 85 on *D*, the 85 on *C* or 55 on *C* will be off the *D*-scale to the right. This means that the two mantissas together are equal to more than a unit, and the characteristic of the logarithm of the product will be (1+1)+1 carried from the mantissas. It is thus apparent that there will be four integral places in the result, and we place the decimal point accordingly.

Hence, *each time that it is necessary to move the slide to the left instead of to the right, add 1 to the characteristic.*

Thus, in the case of 250 × 285 the sum of the characteristics is 4. Setting the slide, we can read the result with the slide projecting to the right, and so the characteristic is 4 and there are five integral places in the result. We read the result on *D* under 285 on *C* to be 71,250.

But in the case of 35 × 850, in which the characteristics are 1 and 2, by setting the left unit on *C* over 35 on *D* we find that the sum of the mantissas of 35 and 850 is more than the length of the rule, or more than 1. The characteristic of the result is therefore 1 + 2 + 1 = 4, and the result will have five integral places. Placing the right-hand unit on *C* over 850 on *D*, going to 35 on *C*, and dropping down to *D* we find the result to be 29,750.

Number of Integral Places in the Quotient. When the slide projects to the right in division, we subtract the characteristic of the divisor from that of the dividend to find the characteristic of the quotient. When the slide projects to the left, we reduce this characteristic by 1.

Required to divide 47.5 by 0.675.

Placing 675 on *C* above 475 on *D*, under the right-hand unit on *C* we read the quotient 704. The difference of the characteristics is $1-(-1)$, or 2. Since the slide projected to the left we have $2-1=1$, the characteristic of the quotient. Hence the result is 70.4.

Exercise. Multiplication and Division

1. A cubic foot of water weighs 62.5 lb., and the specific gravity of a certain grade of steel is 8. Find by the slide rule the weight of 1 cu. ft. of this grade of steel.

The problem requires the multiplication of 62.5 by 8. Place the right-hand 1 on *C* over 6.25 on *D* and read the result 500 on *D* just below the 8 on *C*, making due allowance for the decimal point.

2. The specific gravity of a certain grade of cast iron is 7.2. Find the weight of 1 cu. ft. of this grade of cast iron.

Using the slide rule, perform these multiplications:

3. 4×7.

4. 4×73.

5. 4×7.3.

6. 2.1×4.7.

7. 2.4×8.7.

8. 3.8×6.8.

9. 1.25×24.

10. 2.35×5.65.

11. 3.25×7.45.

Using the slide rule, perform these divisions:

12. $8 \div 4$.

13. $4 \div 8$.

14. $75 \div 1.5$.

15. $4995 \div 45$.

16. $49.95 \div 4.5$.

17. $1.572 \div 12$.

18. $52.43 \div 4.9$.

19. $5.292 \div 4.9$.

20. $92.15 \div 9.7$.

21. If 12 workmen together receive $52, find by the slide rule the average amount received by each.

TABLES FOR REFERENCE

Length

12 inches (in.) = 1 foot (ft.)
3 feet = 1 yard (yd.)
5½ yards, or 16½ feet = 1 rod (rd.)
320 rods, or 5280 feet = 1 mile (mi.)

mi.	rd.	yd.	ft.	in.
			1	12
		1	3	36
	1	5½	16½	198
1	320	1760	5280	63,360

A *hand* (4 in.) is used in measuring the height of horses; a *fathom* (6 ft.), in measuring depths of water at sea; a *knot* (nautical mile, 1.152 common, or statute, miles, or 6080.27 ft.), in measuring distances at sea. Carpenters, mechanics, and others usually write 2′ 6″ for 2 ft. 6 in.

Square Measure

144 square inches (sq. in.) = 1 square foot (sq. ft.)
9 square feet = 1 square yard (sq. yd.)
30¼ square yards = 1 square rod (sq. rd.)
160 square rods = 1 acre (A.)
640 acres = 1 square mile (sq. mi.)

sq. mi.	A.	sq. rd.	sq. yd.	sq. ft.	sq. in.
				1	144
			1	9	1296
		1	30¼	272¼	39,204
	1	160	4840	43,560	6,272,640
1	640	102,400	3,097,600	27,878,400	4,014,489,600

CUBIC MEASURE

1728 cubic inches (cu. in.) = 1 cubic foot (cu. ft.)
27 cubic feet = 1 cubic yard (cu. yd.)
128 cubic feet = 1 cord (cd.)

cu. yd.	cu. ft.	cu. in.
	1 =	1728
1 =	27 =	46,656

A perch of stone or masonry is usually 1 rd. long, 1 ft. high, and 1½ ft. thick, and contains 24¾ cu. ft., but this varies in different parts of the country. A cubic yard of earth is considered a load.

AVOIRDUPOIS WEIGHT

16 ounces (oz.) = 1 pound (lb.)
2000 pounds = 1 ton (T.)

T.	lb.	oz.
	1 =	16
1 =	2000 =	32,000

100 lb. is sometimes called a *hundredweight* (cwt.). The ton of 2000 lb. is sometimes called the *short ton*, there being an old ton of 2240 lb., known as the *long ton*, which is used in the customhouse and in some wholesale transactions in mining products. There is also the *long hundredweight* of 112 lb., but it is rarely used except in English trade.

TROY WEIGHT

24 grains (gr.) = 1 pennyweight (pwt. or dwt.)
20 pennyweights = 1 ounce (oz.)
12 ounces = 1 pound (lb.)

The avoirdupois pound contains 7000 gr., the troy pound 5760 gr. A *carat* weight is 200 milligrams, or 3.08647 gr.

APOTHECARIES' WEIGHT

20 grains (gr.) = 1 scruple (sc. or ℈)
3 scruples = 1 dram (dr. or ʒ)
8 drams = 1 ounce (oz. or ℥)
12 ounces = 1 pound (lb.)

LIQUID MEASURE

4 gills (gi.) = 1 pint (pt.)
2 pints = 1 quart (qt.)
4 quarts = 1 gallon (gal.)
31½ gallons = 1 barrel (bbl.)
63 gallons = 1 hogshead (hhd.)

pt. gi.
qt. 1 = 4
gal. 1 = 2 = 8
bbl. 1 = 4 = 8 = 32
hhd. 1 = 31½ = 126 = 252 = 1008
1 = 2 = 63 = 252 = 504 = 2016

A gallon contains 231 cu. in.

Casks holding from 28 gal. to 43 gal. are called barrels, and casks holding from 54 gal. to 63 gal. are called hogsheads. Whenever barrels or hogsheads are used as *measures*, a barrel means 31½ gal., and a hogshead means 63 gal. A barrel contains about 4⅕ cu. ft.

APOTHECARIES' LIQUID MEASURE

60 minims (♏) = 1 fluid dram (♏ lx. or f. ʒ)
8 fluid drams = 1 ounce (fl. drm. viij. or f. ℥)
16 ounces = 1 pint (fl. oz. xvj. or O)
8 pints = 1 gallon (cong.)

DRY MEASURE

2 pints (pt.) = 1 quart (qt.)
8 quarts = 1 peck (pk.)
4 pecks = 1 bushel (bu.)

qt. pt.
pk. 1 = 2
bu. 1 = 8 = 16
1 = 4 = 32 = 64

A bushel contains 2150.42 cu. in., or about 1¼ cu. ft. A dry quart contains 67.2 cu. in., while the liquid quart contains only 57.75 cu. in. In measuring grain, seeds, and small fruits the measure must be *even* full. In measuring apples, potatoes, and other large articles the measure must be *heaping* full.

Surveyors' Linear Measure

7.92 inches (in.) = 1 link (li.)
25 links = 1 rod (rd.)
100 links = 4 rods = 1 chain (ch.)
80 chains = 1 mile (mi.)

Surveyors' Square Measure

16 square rods (sq. rd.) = 1 square chain (sq. ch.)
10 square chains = 1 acre (A.)
640 acres = 1 section (sec.)
36 sections = 1 township (T.)

A section is a square mile.

Time Measure

60 seconds (sec.) = 1 minute (min.)
60 minutes = 1 hour (hr.)
24 hours = 1 day (da.)
7 days = 1 week (wk.)
About 30 days = 1 month (mo.)
365 days = 1 common year (yr.)
366 days = 1 leap year

yr.	mo.	wk.	da.	hr.	min.	sec.
					1 =	60
				1 =	60 =	3600
			1 =	24 =	1440 =	86,400
		1 =	7 =	168 =	10,080 =	604,800
1 =	12 =		365 =	8760 =	525,600 =	31,536,000
			366 =	8784 =	527,040 =	31,622,400

Angle Measure

60 seconds (″) = 1 minute (′)
60 minutes = 1 degree (°)
360 degrees = 4 right angles

Arc Measure

60 seconds (″) = 1 minute (′)
60 minutes = 1 degree (°)
360 degrees = 1 circumference

Numbers

12 units = 1 dozen (doz.)
12 dozen = 1 gross (gr.)
12 gross = 1 great gross
20 units = 1 score

Paper

24 sheets = 1 quire
20 quires = 1 ream
2 reams = 1 bundle
5 bundles = 1 bale

The great gross and score are becoming obsolete. For convenience in counting, 500 sheets are now commonly called a ream.

United States Money

10 mills = 1 cent (ct. or ¢)
10 cents = 1 dime (d.)
10 dimes = 1 dollar ($)
10 dollars = 1 eagle (E.)

The term "eagle" is not used. The mill is not coined, although the name is frequently used in speaking of fractional parts of a cent.

British Money

4 farthings = 1 penny (d.)
12 pence = 1 shilling (s. or /)
20 shillings = 1 pound (£) or sovereign

Notes on the Common Units of Measure. The smaller units of length were originally derived from the measure of the human body. The *inch* was the length from the end of the thumb to the first joint. Four inches were taken as the *palm* or *handbreadth,* and three handbreadths as the *foot.* The entire length of the arm was the *yard.* The distance to which a man's two hands can extend, across the shoulders, was the *fathom.* The ancient *pace* was the distance covered in two steps, and was taken to be six feet, and a thousand paces (*mille passus,* in Latin) was the *mile.* The *pound* is a very ancient weight. The Roman *gallon* of water weighed ten pounds. The Roman pound was divided into twelve *ounces* (*unciæ*). In 1266 the English law decided that the penny should weigh as much as 32 grains of wheat "in the midst of the ear," and that 20 pennies should regulate the weight of the ounce, 12 ounces should make a pound, 8 pounds a gallon, and 8 gallons a bushel. Our word *quart* means quarter of a gallon. *Acre* is an old name for a cultivated field. The word *yard* is from an old word meaning stick. *Pint* means a mark, there having been a mark made halfway up the old quart measures.

METRIC LENGTH

A **kilometer** (km.) = 1,000 meters
A hektometer = 100 meters
A dekameter = 10 meters
Meter (m.)
A decimeter (dm.) = 0.1 of a meter
A **centimeter** (cm.) = 0.01 of a meter
A **millimeter** (mm.) = 0.001 of a meter

METRIC SQUARE MEASURE

A square kilometer (sq. km.) = 1,000,000 square meters
A square hektometer = 10,000 square meters
A square dekameter = 100 square meters
Square meter (sq. m.)
A square decimeter = 0.01 of a square meter
A **square centimeter** (sq. cm.) = 0.0001 of a square meter
A **square millimeter** (sq. mm.) = 0.000001 of a square meter

METRIC CUBIC MEASURE

A cubic hektometer = 1,000,000 cubic meters
A cubic dekameter = 1,000 cubic meters
Cubic meter (cu. m.)
A **cubic decimeter** (cu. dm.) = 0.001 of a cubic meter
A **cubic centimeter** (cu. cm.) = 0.000001 of a cubic meter
A cubic millimeter (cu. mm.) = 0.000000001 of a cubic meter

METRIC CAPACITY

A **hektoliter** (hl.) = 100 liters
A dekaliter = 10 liters
Liter (l.)
A deciliter (dl.) = 0.1 of a liter
A centiliter (cl.) = 0.01 of a liter
A milliliter (ml.) = 0.001 of a liter

METRIC WEIGHT

A **kilogram** (kg.) = 1000 grams
Gram (g.)
A decigram (dg.) = 0.1 of a gram
A centigram (cg.) = 0.01 of a gram
A milligram (mg.) = 0.001 of a gram

METRIC LENGTH EQUIVALENTS

Meter	= 39.37 in.	Inch	= 0.02540 m.
	= 3.28083 ft.	Foot	= 0.30480 m.
	= 1.09361 yd.	Yard	= 0.91440 m.
	= about 1.1 yd.		= about 0.9 m.
Millimeter	= 0.03937 in.	Inch	= 25.4001 mm.
Centimeter	= 0.3937 in.		= 2.54001 cm.
Kilometer	= 0.62137 mi.	Mile	= 1.60935 km.
	= about $\frac{3}{5}$ mi.		= about 1.6 km.

METRIC SURFACE EQUIVALENTS

Sq. meter	= 10.76387 sq. ft.	Sq. foot	= 0.09290 sq. m.
	= 1.1960 sq. yd.	Sq. yard	= 0.83613 sq. m.
Hektare	= 2.471 A.	Acre	= 0.40469 ha.
	= about $2\frac{1}{2}$ A.		= about $\frac{2}{5}$ ha.
Sq. kilometer	= 0.38610 sq. mi.	Sq. mile	= 2.59000 sq. km.

METRIC VOLUME EQUIVALENTS

Cu. centimeter	= 0.06102 cu. in.	Cu. inch	= 16.38716 cu. cm
Cu. meter	= 35.31388 cu. ft.	Cu. foot	= 0.02832 cu. m.
	= 1.30792 cu. yd.	Cu. yard	= 0.76456 cu. m.
Stere	= 0.27589 cd.	Cord	= 3.62458 st.

METRIC CAPACITY EQUIVALENTS

Liter	= 1.05668 liq. qt.	Liq. quart	= 0.94636 l.
	= 0.90808 dry qt.	Dry quart	= 1.10123 l.
	= 0.26417 gal.	Gallon	= 3.78543 l.
	= 0.11351 pk.	Peck	= 8.80982 l.
Hektoliter	= 2.83774 bu.	Bushel	= 0.35239 hl.

METRIC WEIGHT EQUIVALENTS

Gram	= 15.4324 gr.	Grain	= 0.06480 g.
	= about $15\frac{1}{2}$ gr.		= about $\frac{3}{50}$ g.
	= 0.03527 oz. av.	Ounce av.	= 28.34953 g.
Kilogram	= 2.20462 lb. av.	Pound av.	= 0.45359 kg.
	= about $2\frac{1}{5}$ lb.		= 0.00045 t.
Metric ton	= 2204.62 lb. av.	Short ton	= 0.90718 t.

Spanish Measures of Length

1 vara = $33\frac{1}{3}$ in.
3 varas = 100 in.
36 varas = 100 ft.
108 varas = 100 yd.
1900.8 varas = 1 mi.

In those sections of the United States that formerly belonged to Mexico, the Spanish measures are still used in land surveying and the student may be called upon to know them if he lives there.

The *vara* is taken as the unit of measure.

In Mexico the length of the vara is 32.9927 in., in California 33 in., in Texas $33\frac{1}{3}$ in. In solving local problems its length should be taken as $33\frac{1}{3}$ in. in Texas and 33 in. in California. In other states where the vara is used, follow the local custom.

Spanish Measures of Area

1,000,000 square varas = 1 labor
= 177.136 acres
25 labors = 1 square league
= 4428.4 acres
5645.376 square varas = 1 acre

A *labor* is a square each side of which is 1000 varas.

A *square league* is a square each side of which is 5000 varas.

Cooking

4 teaspoons (t.) = 1 tablespoon (T.)
16 tablespoons = 1 cup (C.)
2 cups = 1 pint (pt.)

This table is commonly used in classes in domestic science, but the abbreviations are not yet standardized.

The tables given on pages 241–248 are not to be memorized except as local conditions require certain isolated facts in them to be known. They are given solely for reference purposes and include all the information relative to denominate numbers that a student is likely to need. They will be found helpful for subsequent reference after leaving school, and students should refer to them as to a dictionary or encyclopedia.

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