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H. SPENCER JONES

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ELEMENTARY MATHEMATICAL ASTRONOMY

BY

C. W. C. BARLOW, M.A., B.Sc.

GOLD MEDALLIST IN MATHEMATICS AT M.A. LONDON
SIXTH WRANGLER AND FIRST CLASS FIRST DIVISION PART II. MATHEMATICAL TRIPOS, CAMBRIDGE

AND

G. H. BRYAN, Sc.D., M.A., F.R.S.

HONORARY FELLOW OF ST. PETER'S COLLEGE, CAMBRIDGE
LATE PROFESSOR OF MATHEMATICS IN THE UNIVERSITY COLLEGE OF NORTH WALES

REVISED BY

SIR HAROLD SPENCER JONES, Sc.D., M.A., F.R.S.

ASTRONOMER ROYAL
HONORARY FELLOW OF JESUS COLLEGE, CAMBRIDGE

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PREFACE TO THE FIRST EDITION

For some time past it has been felt that a gap existed between the many excellent popular and non-mathematical works on Astronomy, and the standard treatises on the subject, which involve high mathematics. The present volume has been compiled with the view of filling this gap, and of providing a suitable text-book for such examinations as those for the B.A. degree of the University of London.

It has not been assumed that the reader's knowledge of mathematics extends beyond the more fundamental portions of Geometry, Algebra, and Trigonometry. A knowledge of elementary Dynamics will, however, be required in reading the last three chapters.

The principal properties of the Sphere required in Astronomy have been collected in the first chapter; and, as it is impossible to understand Kepler's Laws without some knowledge of the properties of the Ellipse, the more important of these have been collected in the Appendix for the benefit of students who have not read Conic Sections.

Articles marked with an asterisk are of special difficulty or of relatively small importance and may be omitted at discretion.

PREFACE TO THE FIFTH EDITION

For half a century " Barlow and Bryan " has remained one of the standard textbooks on elementary mathematical astronomy. During this time there have been great advances in physical astronomy, or astrophysics, as this branch of astronomy is commonly called. This subject is not treated in this book, which is concerned with the foundations of astronomy—the general mathematical and dynamical structure upon which everything else depends.

For the present edition the work has been completely revised and, though the original plan has been adhered to, there has been considerable rearrangement of the chapters. The conceptions of apparent and mean sidereal time, made necessary by the precision of modern time-keepers, have been introduced. The definition of the equation of time has been brought into accordance with the " Nautical Almanac " and common usage of to-day. The chapters on Refraction, Parallax and Aberration—the phenomena that affect the observed position of a celestial body—have been brought together. A chapter on Precession and Nutation, including the reduction from apparent to mean place of a star, has been introduced. A brief description of the bubble sextant, for observations in aircraft, is given and the section dealing with the position line method of determining the position of a ship or aircraft has been considerably expanded. An account is given of the arrangement of data in the "Air Almanac." The sections dealing with the obsolete method of finding longitude by the method of lunar distances and various other sections of little interest have been omitted.

It is hoped that in this new edition the work will be found of increased usefulness, both as a textbook and for reference purposes.

H. SPENCER JONES.

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MATHEMATICAL ASTRONOMY

CHAPTER I

SPHERICAL GEOMETRY

1. The Sphere; Great Circles; Small Circles

A knowledge of the properties of the sphere is required in mathematical Astronomy. It is convenient, therefore, to summarise the properties that will be referred to in the course of this volume.

A Sphere may be defined as a surface, all points on which are at the same distance from a certain fixed point. This point is the *Centre,* and the constant distance is the *Radius.*

The surface formed by the revolution of a semicircle about its diameter is a sphere. For the centre of the semicircle is kept fixed, and its distance from any point on the surface generated will be equal to the radius of the semicircle.

In Fig. 1 let *PqQP′* be any position of the revolving semicircle whose diameter *PP′* is fixed. Let *OQ* be the radius perpendicular to *PP′*, *Cq* any other line perpendicular to *PP′*, meeting the semicircle in *q*. (We may suppose these lines to be marked on a semicircular disc of cardboard.) As the semicircle revolves, the lines *OQ*, *Cq* will sweep out planes perpendicular to *PP′*, and the points *Q*, *q* will trace out in these planes circles *HQRK*, *hqrk,* of radii *OQ*, *Cq* respectively. From this it may readily be seen that *every plane section of a sphere is a circle.*

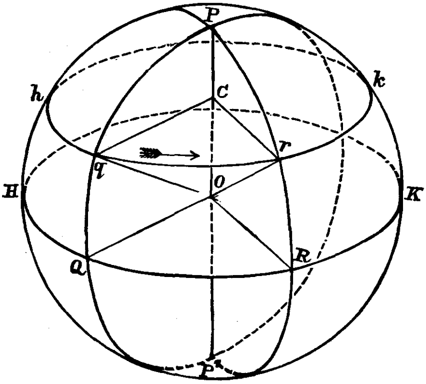

FIG. 1.

A *great circle* of a sphere is the circle in which it is cut by any plane passing through the centre (*e.g. HQRK*, *PqQP′* or *PrRP′*). A *small circle* is the circle in which the sphere is cut by any plane *not* passing through the centre (*e.g. hqrk*).

The *axis* of a great or small circle is the diameter of the sphere perpendicular to the plane of the circle. The *poles* of the circle are the extremities of this diameter. (Thus, the line *PP′* is the axis, and *P*, *P′* are the poles of the circles *HQK* and *hqk*.)

Secondaries to a circle of the sphere are great circles passing through its poles. (Thus, PQP' and PRP' are secondaries of the circles HQK, hqk.)

A great circle on a sphere is analogous to a straight line in a plane. A straight line joining two points in a plane is the shorter distance between those points; so also the shortest distance between two points on a sphere is the arc of the great circle through those points. Thus, in Fig. 1, the shortest distance between the points QR is the arc QR of the great circle $HQRK$. If the radius of the sphere is R and the angle QOR, subtended by the arc QR at the centre of the circle, is denoted by O, the length of the arc QR is RO, where O is expressed in circular measure, or radians. It is convenient to take the radius of the sphere as unity; the length of the arc QR is then equal to O. Thus the angular distance between two points on a sphere is measured by the arc of the great circle joining them or by the angle which this arc subtends at the centre of the sphere. If the angle QOR is 60°, we can say that the length of the great circle arc QR is $\pi/3$ radians, or that it is 60°.

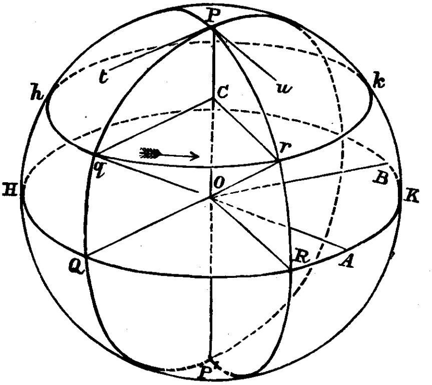

FIG. 2.

Small circles on a sphere are analogous in their general properties to circles in a plane. Secondaries to a great circle are analogous to perpendiculars on a straight line. The distance of a point from any great circle is the length of the arc of a secondary drawn from the point to the circle. Thus rR is the distance of the point r from the great circle $HQRK$.

2. Spherical Angles

The angle between two great circles is the angle between their planes. Thus the angles between the circles PQ, PR in Fig. 2 is the angle between the planes PQP', PRP'. It is called the *spherical angle* QPR.

The angle between two great circles is equal to—

(i) The angle between the tangents to them at their points of intersection.

(ii) The arc which they intercept on a great circle to which they are both secondaries.

(iii) The angular distance between their poles.

Let Pt, Pu be the tangents at P to the circles PQ, PR, and let A, B be the poles of the circles. These tangents are perpendicular to OP and therefore parallel to OQ, OR. If we suppose the semicircle PQP' to revolve about PP' into the position PRP', the tangent at P will revolve from Pt to Pu, the radius perpendicular to OP will revolve from OQ to OR, and the axis will revolve from OA to OB. All these lines will revolve through an angle equal to the angle between the planes PQP' PRP', and this is the angle QPR between the circles. Hence,

$$\text{Angle between circles } PQ,\ PR = \angle tPu = \angle QOR = \angle AOB.$$

3. Spherical Triangles

A *spherical triangle* is a portion of the spherical surface bounded by three arcs of great circles. Thus, in Fig. 2, PQR is a spherical triangle, but Pqr is not a spherical triangle, because qr is not an arc of a great circle. We may, however, draw a great circle passing through q and r, and thus form a spherical triangle Pqr.

A spherical triangle, like a plane triangle, has six parts, viz. its three sides and its three angles. The sides are generally measured by the angles they subtend at the centre of the sphere, so that the six parts are all expressed as *angles*. No part is supposed to exceed two right angles or 180°. The circumference of a great circle is 360°. Two points on the sphere, such as Q and R in Fig. 2, can be joined by two arcs of a great circle, one of which, QR, is less than or equal to 180°, whilst the other, $QHKAR$, is greater than or equal to 180°. PQR is a spherical triangle, but the figure $PQHKARP$ is not a spherical triangle. Similarly, no angle of a spherical triangle can exceed two right angles.

A spherical triangle has the property, in common with plane triangles, that the sum of any two sides is greater than the third side. But whereas the sum of the three angles of a plane triangle is equal to two right angles, the sum of the three angles of a spherical triangle is always greater than two right angles. The amount by which the sum of the three angles exceeds two right angles is termed the *spherical excess*.

A plane triangle may have one angle a right angle. A spherical triangle, on the other hand, may have one, two, or three angles that are right angles. Thus, in the triangle PQR (Fig. 2), the angles PQR, PRQ are both right angles because, PP' being perpendicular to the plane, $HQRK$, any plane through PP', such as PQP', is necessarily perpendicular to the plane $HQRK$. If the planes PQP', PRP', are at right angles to one another, the three angles of the triangle PQR will be right angles.

It may be noted that in a triangle such as PQR, in which the angles PQR, PRQ are right angles, the two sides, PQ, PR are quadrants and therefore right angles. The third side QR is equal to the opposite angle QPR.

If, in addition, the angle QPR is a right angle (Fig. 3), QR will be a quadrant. The triangle PQR will, therefore, have all its angles right angles, and all its sides quadrants, and each vertex will be the pole of the opposite side.

The planes of the great circles forming the sides, are three planes through the centre O mutually at right angles, and they divide the surface of the sphere into eight of these triangles ; thus the area of each triangle is one-eighth of the surface of the sphere.

If the sides of a spherical triangle, when expressed as angles, are *very small*, so that its linear dimensions are very small compared with the radius of the sphere, the triangle is very approximately a plane triangle.

Thus, although the Earth's surface is spherical, a triangle whose sides are a few yards in length, if traced on the Earth, will not be distinguishable from a plane triangle. If the sides are several miles in length, the triangle will still be very nearly plane.

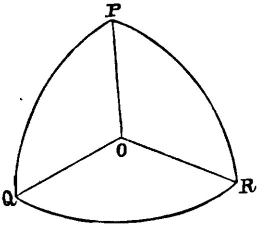

FIG. 3.

4. Small Circles

All points on a great circle are at a constant (angular) distance from its pole. For, as the generating semicircle revolves about PP' (Fig. 2), carrying q along the small circle hk to r, the arc Pq is equal to the arc Pr and the angle POq is equal to the angle POr. The constant angular distance Pq is called the spherical, or angular radius of the small circle. The pole P is analogous to the centre of a circle in plane geometry.

Circles which have the same axis and poles lie in parallel planes. For the planes HQK, hqk are parallel, both being perpendicular to the axis PP'. Such circles are called *parallels*.

5. Length of Small Circle Arc

The arc of a small circle subtending a given angle at the pole is proportional to the sine of the angular radius.

Let qr be the arc of the small circle $hqrk$, subtending $\angle qPr$ at P, and let C be the centre of the circle. Evidently $\angle qCr = \angle QOR$ (since Cq, Cr are parallel to OQ, OR). Hence, the arcs qr, QR are proportional to the radii Cq, OQ, therefore

$$\frac{\text{arc } qr}{\text{arc } QR} = \frac{Cq}{OQ} = \frac{Cq}{Oq} = \sin POq = \sin Pq.$$

But QR is the arc of a great circle subtending the same angle at the pole P, hence the arc qr is proportional to $\sin Pq$, as was to be shown. Since $qQ = 90° - Pq$, therefore $\sin Pq = \cos qQ$, so that the arc qr is proportional to the cosine of the angular distance of the small circle qr from the parallel great circle QR.

6. A Useful Result

The following result is of great use in astronomy:—

Let $AHBK$ (Fig. 4) be any given great or small circle whose pole is P, Z any other given point on the sphere, and let the great circle ZP meet the given circle in the points A, B. Then A, B are the two points on the given circle whose distances from Z are greatest and least respectively.

For let H be any other point on the circle. Join ZH, HP.

Then, in spherical $\triangle\, ZPH$, $ZP + PH > ZH$. But $PH = PA$;

$$\text{therefore}\quad ZP + PA > ZH,$$

i.e. $$ZA > ZH.$$

Also, if Z is on the opposite side of the circle to P, then

$$ZH + PH > PZ;$$
$$ZH + PB > PZ;$$
$$ZH > PZ - PB,$$

i.e. $$ZH > ZB.$$

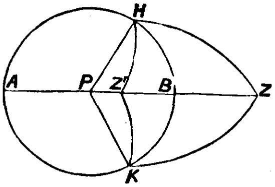

FIG. 4.

If Z' be a point on the same side of the circle as P, then

$$PZ' + Z'H > PH. \quad \text{But as } PH = PB,\ PZ' + Z'H > PB$$
$$\text{or } Z'H > PB - PZ',$$

i.e. $$Z'H > Z'B, \text{ as before.}$$

Hence, A is further from Z, Z', and B is nearer to Z, Z', than any other point on the circle.

If H, K are the two points on the circle equidistant from Z, the spherical triangles ZPH, ZPK have ZP common, $ZH = ZK$ (by hypothesis), and $PH = PK$, hence they are equal in all respects; thus

$$\angle ZPH = \angle ZPK, \text{ and } \angle PZH = \angle PZK.$$

Hence PH, PK are equally inclined to PB, as are also ZH, ZK.

Similar properties hold in the case of the point Z'. These properties are of frequent use.

7. Application to the Earth

The results of the preceding sections can be illustrated by the specification of positions on the surface of the Earth. We shall see later that the Earth is very nearly spherical in form. For the present we assume that it is actually a sphere. It rotates about an axis, called the *polar axis*, which meets its surface in two points, *P*, *P′* (Fig. 5). *P* is called the *North Pole* and *P′* the *South Pole*. The *terrestrial equator*, *HQRK*, is the great circle on the Earth whose plane is perpendicular to the Earth's axis. A *terrestrial meridian*, *PqQP′*, is the section of the Earth's surface by a plane passing through its axis. The Earth being assumed to be a sphere, a meridian will be a great circle passing through the terrestrial poles.

8. Terrestrial Longitude

The *Longitude* of a place on the Earth is the angle between the terrestrial meridian through that place, and a certain meridian fixed on the Earth, and called the *Prime Meridian*.

Thus, in Fig. 5, if *PRP′* represents the prime meridian, the longitude of any place *q* is measured by the angle *RPq*.

The longitude of *q* is also measured by *RQ*, the arc of the equator intercepted between the meridian of the place and the prime meridian.

By international agreement, the prime meridian from which the longitudes of all places on the Earth are measured is defined as the meridian passing through the Airy transit instrument at the Royal Observatory, Greenwich.

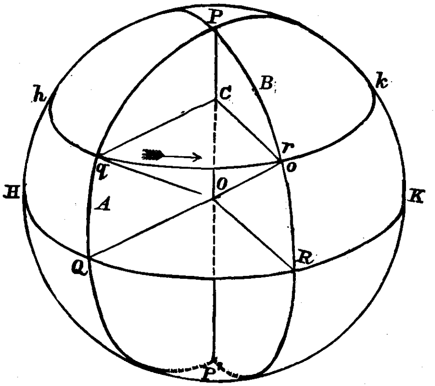

FIG. 5.

As all places on a given meridian have the same longitude, the terrestrial meridians are often called *meridians of longitude*.

Longitudes are measured both eastwards and westwards from the prime meridian, from 0° to 180°. Thus, if, in Fig. 5, *PQP′* denotes the prime meridian, the longitude of *r* is measured by the arc *QR* and is east of Greenwich ; if *PRP′* denotes the prime meridian, the longitude of *q* is measured by the arc *RQ* and is west of Greenwich.

The plane passing through the polar axis and Greenwich divides the Earth into two hemispheres. All places whose longitudes are between 0° and 180° E. lie in the *eastern hemisphere*; those whose longitudes are between 0° and 180° W. lie in the *western hemisphere.*

9. Terrestrial Latitude

The latitude of a place on the Earth is its angular distance from the equator, measured along the meridian. Thus, in Fig. 5, the latitude of q is the arc Qq, or the angle QOq. The equator, $HQRK$, divides the Earth into two hemispheres, the *northern hemisphere*, which contains the north pole, P, and the *southern hemisphere*, which contains the south pole, P'. A place in the northern hemisphere, such as q, is said to have a *north latitude*; a place in the southern hemisphere is said to have a *south latitude*. Latitude is measured in degrees, from 0° to 90° N. or S.

All points on a small circle, $hqrk$, parallel to the equator, have the same latitude. For this reason, parallels to the equator are usually termed *parallels of latitude*. The angular radius of the parallel of latitude, Pq, is equal to $90°-qQ$ or, in other words, is the complement of the latitude.

The complement of the latitude is called the *colatitude*. The colatitude is the angular radius q of the parallel of latitude.

If we now consider two points, pq, on the same parallel of latitude, the length of the small circle arc, qr, is, from Art. 5, equal to $QR \cos QOq$. But the great circle arc, QR, is the difference of longitude between the two places, whilst the angle QOq is the latitude, which we denote by ϕ. We thus have the distance, measured along the parallel of latitude, between two places on the same parallel is equal to (difference of longitude) × cos ϕ. If the difference of longitude is measured in degrees, the distance is also given in degrees.

It should be noted that, in forming the difference of longitude, regard must be paid to whether the longitudes are east or west. Thus the difference of longitude between two places whose longitudes are 120° W. and 30° W. is 90°; between two places whose longitudes are 120° W. and 30° E. is 150°. Similarly, in forming the difference of latitude between two places, regard must be paid to whether the latitudes are north or south.

10. Principal Formulae for Solving Spherical Triangles

Any three given parts suffice to determine a spherical triangle, but there are certain "ambiguous cases" when the problem admits of more than one solution.

The formulae required in solving spherical triangles form the subject of Spherical Trigonometry, and are in every case different from the analogous formulae in Plane Trigonometry. There is this further difference, that a spherical triangle is completely determined if its three angles are given.

Thus, two spherical triangles will, in general, be equal if they have the following parts equal:—

(i) Three sides.
(ii) Two sides and included angle.
(iii) Two sides and one opposite angle.
(iv) Three angles.
(v) Two angles and adjacent side.
(vi) Two angles and one opposite side.

Cases (iii) and (vi) may be ambiguous.

A, B, C denote the angles of the triangle; a, b, c the sides opposite to these.

The principal formulae, by the aid of which any spherical triangle may be solved, are as follows :—

$$\cos a = \cos b \cos c + \sin b \sin c \cos A \quad \ldots\ldots\ldots\ldots\ldots\ldots(1)$$

$$\cos A = -\cos B \cos C + \sin B \sin C \cos a \quad \ldots\ldots\ldots\ldots(2)$$

$$\sin a \cos B = \cos b \sin c - \sin b \cos c \cos A \quad \ldots\ldots\ldots\ldots(3)$$

$$\sin a \cos C = \cos c \sin b - \sin c \cos b \cos A \quad \ldots\ldots\ldots(3')$$

$$\cot a \sin b = \cos b \cos C + \cot A \sin C \quad \ldots\ldots\ldots\ldots\ldots\ldots(4)$$

$$\frac{\sin a}{\sin A} = \frac{\sin b}{\sin B} = \frac{\sin c}{\sin C} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(5)$$

Proofs of these formulae may be found in textbooks on spherical trigonometry.

By the cyclic interchange of the three sides and of the three angles, two other formulae corresponding to each of the formulae (1) to (4) may be written down. The above formulae are not all independent; thus (3) may be readily deduced from two formulae of type (1); formulae of type (3) may be deduced from formulae (4) and (5).

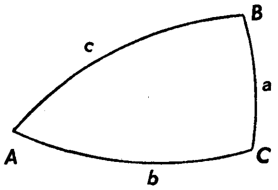

FIG. 6.

Which of these formulae are used depends upon which parts of the spherical triangle are known. If, for instance, two sides and the included angle are given, the third side can be obtained from (1). Three sides and one angle being then known, the remaining angles can be found from (5). If two angles and the adjacent side are known, the third angle can be found from (2), and so on.

The formulae take a specially simple form in the case of right-angled triangles. The various formulae applicable in this case can be conveniently summarised in the following manner :—

RIGHT-ANGLED TRIANGLES.—C being the right angle, there are five other parts, which come in the following order, $b\ A\ c\ B\ a$. There is a convenient general rule which embraces all the formulae of right-angled spherical triangles.

Arrange the following five quantities in order round a circle: b, $90° - A$, $90° - c$, $90° - B$, a. Then:—

(i) The sine of any of these quantities is equal to the product of the tangents of the two adjacent quantities,

(ii) and also it is equal to the product of the cosines of the two opposite quantities.

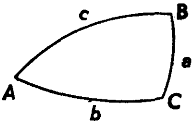

FIG. 7*a*.

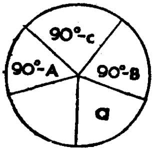

FIG. 7*b*.

In order to solve the triangle, we must know two of the five quantities, and one or other of these two formulae will enable us to find any of the remaining three quantities.

In many cases the simplest way to solve a general triangle is to draw a great circle through one of the angles at right angles to the opposite side, and then apply the formulae applicable to right-angled triangles.

CHAPTER II

THE CELESTIAL SPHERE

I—DEFINITIONS—SYSTEMS OF COORDINATES

11. Astronomy—Descriptive, Gravitational, Physical

Astronomy is the science which deals with the celestial bodies. These comprise all the various bodies distributed throughout the universe, such as the Earth (considered as a whole), the Moon, the Sun, the planets, the comets, the fixed stars, and the nebulae. It is convenient to divide Astronomy into three different branches.

The first may be called *Descriptive Astronomy*. It is concerned with observing and recording the motions of the various celestial bodies, and with applying the results of such observations to predict their positions at any subsequent time. It includes the determination of the distances, and the measurement of the dimensions of the celestial bodies.

The second, or *Gravitational Astronomy*, is an application of the principles of dynamics to account for the motions of the celestial bodies. It includes the determination of their masses.

The third, called *Physical Astronomy*, is concerned with determining the nature, physical condition, temperature, and chemical constitution of the celestial bodies.

The first branch has occupied the attention of astronomers in all ages. The second owes its origin to the discoveries of Sir Isaac Newton in the seventeenth century; while the third branch has been almost entirely built up in the last and the present centuries.

In this book we shall treat almost exclusively of Descriptive and Gravitational Astronomy.

12. The Celestial Sphere

On observing the stars it is not difficult to imagine that they are bright points dotted about on the inside of a hollow spherical dome, whose centre is at the eye of the observer. It is impossible to form any direct conception of the distances of such remote bodies; all we can see is their relative directions. Moreover, most astronomical instruments are constructed to determine only the directions of the celestial bodies. Hence it is important to have a convenient mode of representing directions.

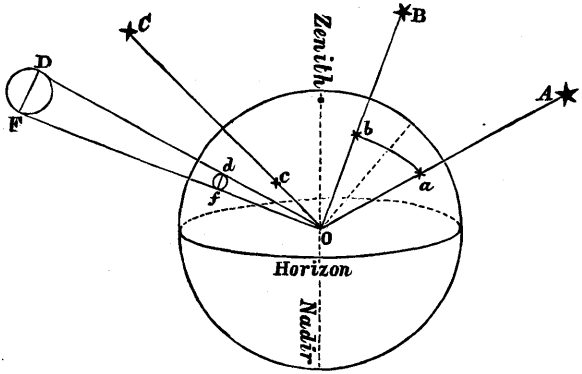

FIG. 8.

The way in which this is done is shown in Figure 8. Let O be the position of any observer, A, B, C, etc. any stars or other celestial bodies. About O, as centre, describe a sphere with any convenient length as radius, and let the lines joining O to the stars A, B, C meet this sphere in a, b, c respectively. Then the points, a, b, c will represent, on the sphere, the directions of the stars A, B, C, for the lines joining these points to O will pass through the stars themselves. In this manner we obtain, on the sphere, an exact representation of the appearance of the heavens as seen from O. Such a sphere is called the *Celestial Sphere*.

This sphere may be taken as the dome upon which the stars *appear* to lie. But it must be carefully borne in mind that the stars *do not actually lie on a sphere at all,* and that they are only so represented for the sake of convenience.

13. Angular Distances and Angular Magnitudes

Any plane through the observer will be represented on the celestial sphere by a great circle. The arc of the great circle *ab* (Fig. 8) represents the angle *aOb* or *AOB* which the stars *A*, *B* subtend at *O*. This angle is generally measured in degrees, minutes, and seconds, and is called the *angular distance* between the stars. This angular distance must not be confused with their actual distance *AB*. In the same way, when we are dealing with a body of perceptible dimensions, such as the Sun or Moon (*DF*, Fig. 8), we shall define its *angular diameter* as the angle *DOF*, subtended by a diameter at the observer's eye. This angular diameter is measured by the arc *df* of the celestial sphere, that is, by the diameter of the projection of the body on the celestial sphere. From the figure it is evident that

$$\frac{df}{Od}=\frac{DF}{OD}.$$

Since *DF* is the actual linear diameter of the body, measured in units of length, the last relation shows us that the angular diameter (*df*) of a body varies directly as its linear diameter *DF*, and inversely as *OD*, the distance of the body from the observer's eye.

As the eye can only judge of the dimensions of a body from its angular magnitude, this result is illustrated by the fact that the nearer an object is to the eye the larger it looks, and *vice versa*. Thus, if the distance of the object be doubled, it will only look half as broad and half as high. This assumes the angle subtended by the body is so small that its sine equals its circular measure.

14. The Directions of the Stars are very approximately Independent of the Observer's Position on the Earth

This is simply a consequence of the enormously great distances of all the stars from the Earth. Thus, let x (Fig. 9) denote any star or other celestial body, S, E two different positions of the observer. If the distance SE be only a very small fraction of the distance Sx, the angle ExS will be very small, and this angle measures the difference between the directions of x as seen from E and from S.

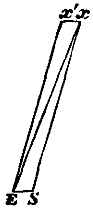

FIG. 9.

In illustration, if we see a group of objects a mile or two off, then move an inch or two in any direction, we shall observe no perceptible change in the apparent directions or relative positions of the objects.

If Ex' be drawn parallel to Sx, the angle xEx' will be equal to ExS, and will therefore be very small indeed. Hence, Ex will very nearly coincide in direction with Ex'. Thus, considering the vast distances, we see that *the lines joining a star to different points of the Earth may be considered as parallel* (not true for Sun, Moon, planets).

The stars will, therefore, always be represented by the *same* points on a star globe, or celestial sphere, no matter what be the position of the observer. The great use of the celestial sphere in astronomy depends on this fact.

15. Motion of Meteors

The projection of bodies on the celestial sphere is well illustrated by the motion, relative to the earth, of a swarm of meteors. Where such a swarm is moving uniformly, all the meteors describe (approximately) parallel straight lines. If we draw planes through these lines and the observer, they will intersect in a common line, namely, the line through the observer parallel to the direction of the common motion of the meteors. The planes will, therefore, cut the celestial sphere in great circles, having this line as their common diameter. These great circles represent the apparent paths of the meteors on the celestial sphere. The paths appear, therefore, to radiate from a common point, namely, one of the extremities of this diameter.

This point is called the *Radiant*, and by observing its position the direction of relative motion of the meteors is found.

16. Zenith and Nadir.—Horizon

If, through the observer, a line be drawn in the direction in which gravity acts (*i.e.* the direction indicated by a plumb-line), it will meet the celestial sphere in two points. One of these is vertically above the observer, and is called the *Zenith*; the other is vertically below the observer, and is called the *Nadir*. (Fig. 8, and Z, N, Fig. 10.)

The plane through the observer perpendicular to the direction to the zenith will cut the celestial sphere in a great circle. This great circle is called the *Celestial Horizon*. (Fig. 8, and $sEnW$, Fig. 10.) Its poles are the zenith and the nadir.

17. Diurnal Motion of the Stars

If we observe the sky at different intervals during the night, we shall find that the stars always maintain the same configurations relative to one another, but that their actual situations in the sky, relative to the horizon, are continually changing. Some stars will set in the west, others will rise in the east. One star which is situated in the constellation called the " Little Bear," remains almost fixed. This star is called Polaris, or the Pole Star. All the other stars describe on the celestial sphere small circles (Fig. 10) having a common pole P very near the Pole Star, and the revolutions are performed in the same period of time, namely, about 23 hours 56 minutes of our ordinary time.

18 Celestial Poles, Equators, and Meridian

The common motion of the stars may most easily be conceived by imagining them to be attached to the surface of a sphere which is made to revolve uniformly about the diameter PP'.

The extremities of this diameter are called the *Celestial Poles.* That pole, P, which is above the horizon in northern latitudes is called the *North Pole,* the other, P', is called the *South Pole.*

The great circle, $EQRW$, having these two points for its poles, is called the *Celestial Equator.* It is, therefore, the circle which would be traced out by the diurnal path of a star distant 90° from either pole.

The *Meridian* is the great circle ($PZP'N$, Fig. 11) passing through the zenith and nadir and the celestial poles. It cuts both the horizon and equator at right angles, since it passes through their poles.

19. The Cardinal Points. Verticals

CARDINAL POINTS.—The *East* and *West Points* (E, W, Fig. 11), are the points of intersection of the equator and horizon. The *North* and *South Points* (N, S) are the intersections of the meridian with the horizon.

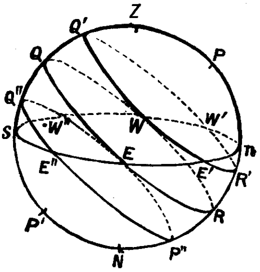

FIG. 10.

VERTICALS.—Secondaries to the horizon, *i.e.* great circles through the zenith and nadir, are called *Vertical Circles,* or, briefly, *Verticals.* Thus, the meridian is a vertical. The *Prime Vertical* is the vertical circle ($ZENW$) passing through the east and west points.

Since P is the pole of the circle $QERW$, and Z is the pole of $nEsW$, therefore E, W are the poles of the meridian $PZP'N$. Hence the horizon, equator, and prime vertical which pass through E, W, are all secondaries to the meridian; they therefore all cut the meridian at right angles.

20. Annual Motion of the Sun.—The Ecliptic

The Sun, while participating in the general diurnal rotation of the heavens, possesses, in addition, an independent motion of its own relative to the stars. This motion of the Sun relative to the stars is an apparent motion caused, as we shall see in Chapter XVII by the motion of the Earth round the Sun. The Earth completes one revolution in its orbit round the Sun in the course of a year, so that the Sun returns to its same apparent position in the heavens, relative to the stars, after a year.

Imagine a star globe worked by clockwork so as to revolve about an axis pointing to the celestial pole in the same periodic time as the stars. On such a moving globe the directions of the stars will always be represented by the same points. During the daytime let the direction of the Sun be marked on the globe, and let this process be repeated every day for a year. We shall thus obtain on the globe a representation of the Sun's path relative to the stars, and it will be found that—

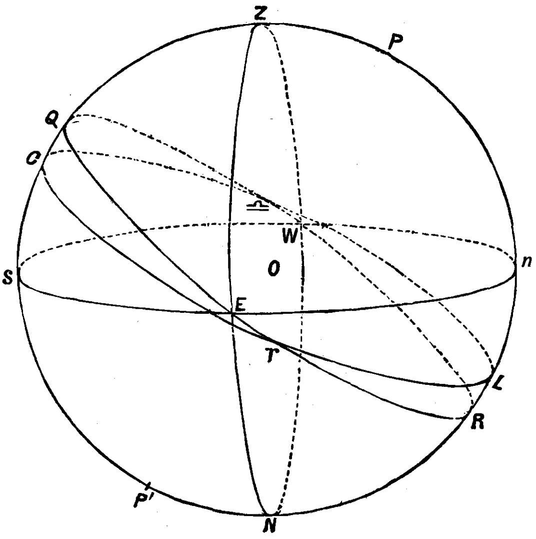

Fig. 11.

(i) The Sun moves from west to east, and returns to the same position among the stars in the period called a year;

(ii) The relative path on the celestial sphere is a great circle, inclined to the equator at an angle of about 23° 27′.

This great circle (*C* ♈ *L* ♎, Fig. 11) is called the *Ecliptic*. We may, therefore, briefly define the ecliptic as the great circle which is the trace, on the celestial sphere, of the Sun's annual path relative to the stars.

The intersections of the ecliptic and equator are called *Equinoctial Points*. One of them is called the *First Point of Aries*; this is the point through which the Sun passes when crossing from south to north of the equator, and it is usually denoted by the symbol ♈. The other is called the *First Point of Libra*, and is denoted by the symbol ♎. The inclination of the ecliptic to the equator is called the *Obliquity*. In Fig. 11, $Q\Upsilon C$ is the obliquity. This angle, which is about 23° 27′, we shall denote by ϵ.

21. Coordinates

In Analytical Geometry, the position of a point in a plane is defined by two coordinates. In like manner, the position of a point on a sphere may be defined by means of two coordinates. Thus, the position of a place on the Earth is defined by the two coordinates, latitude and longitude. For fixing the positions of celestial bodies, the following different systems of coordinates are used :—

(i) altitude (or zenith distance) and azimuth (horizontal system);

(ii) right ascension and declination (equatorial system);

(iii) celestial latitude and longitude (ecliptic system).

These three systems of coordinates will be considered in order.

22. Altitude or Zenith Distances and Azimuth

Let Fig. 12 represent the celestial sphere, Z, N being the zenith and nadir; n, s the north and south points and the great circle sXn the horizon. P is the celestial pole and $sZPn$ is the meridian.

Let x be any star. Draw the vertical circle ZxX. Then the position of x may be defined by either of the following pairs of coordinates, which are analogous to the Cartesian and polar coordinates of a point in a plane respectively :—

(*a*) The arc nX and the arc Xx.

(*b*) The arc Zx and the angle nZx.

Practically, however, the two systems are equivalent; for, since Z is the pole of sX, $ZX = 90°$, therefore

$$Zx = 90° - xX, \quad \text{and} \quad \text{angle } nZx = \text{arc } nX.$$

The *Altitude* of a star (Xx) is its angular distance from the horizon, measured along a vertical.

The *Zenith Distance* (abbreviation, Z.D.) is its angular distance from the zenith (Zx), or the complement of the altitude.

The *Azimuth* (nX or nZx) is the arc of the horizon intercepted between the north point and the vertical of the star, or the angle which the star's vertical makes with the meridian. The azimuth is measured from 0° to 180° eastwards or westwards.

In Fig. 12, QwR represents the equator, w being the west point. It is a great circle whose pole is P. Hence $PQ = 90°$.

Now ZQ, the angular distance of the zenith from the equator, is the terrestrial latitude of the observer. But $ZQ = Pn$ because $ZQ + PZ = PZ + Pn = 90°$.

But Pn is the altitude of the pole, P. Hence we have the result that *the altitude of the pole is equal to the latitude of the observer.*

The diurnal motion causes the star x to describe the small circle UxV, parallel to the celestial equator, QwR, with P as its pole. Both the altitude and azimuth of the star change continually during the course of the diurnal motion. From Art. 6, it follows that ZU is the least, and ZV the greatest distance of Z from the small circle UxV. Hence the star has its smallest zenith distance and its greatest altitude when it is on the meridian south of the zenith. At the instant when the star is at U, it is said to *south.* The star has its greatest zenith distance when it is on the meridian north of the zenith.

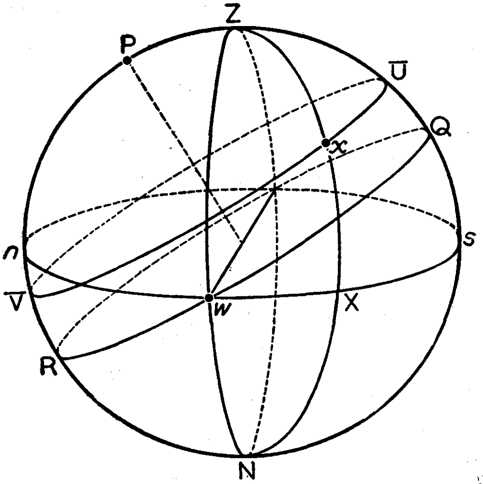

FIG. 12.

A celestial body is said to *culminate* when its zenith distance is least or greatest. At *upper culmination* the zenith distance is least; at *lower culmination* the zenith distance is greatest.

23. Polar Distance, or Declination, and Hour Angle

From the pole, P, draw through x the great circle PxM; this circle is a secondary to the equator QWR. (Fig. 13).

Then we may take for the coordinates of z the arc Px and the angle sPx. Or we may take the arc xM, which is the complement of Px, and the arc QM, which = angle QPx.

The *North Polar Distance* of a star (abbreviation, N.P.D.) is its angular distance (Px) from the celestial pole.

The *Declination* (abbreviation, Decl.) is the angular distance from the equator (xM), measured along a secondary, and is, therefore, the complement of the N.P.D.

The great circle PxM through the pole and the star is called the star's *Declination Circle.*

The *Hour Angle* of the star (ZPx) is the angle which the star's declination circle makes with the meridian.

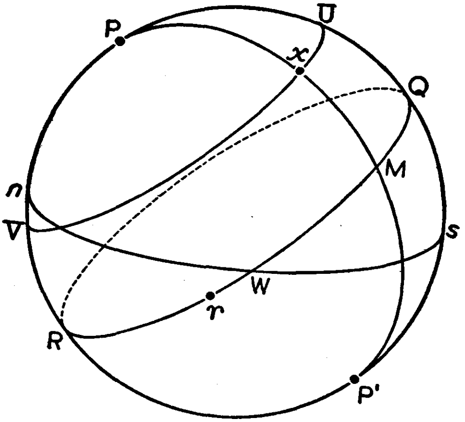

FIG. 13.

The declination may be considered positive or negative, according as the star is to the north or south of the equator ; it is also customary to specify this by the letter N. or S., as the case may be, and this is called the *name* of the declination. South declinations are always to be regarded as negative.

The hour angle is generally measured from the meridian towards the west, and is reckoned from 0° to 360°.

Either the declination and hour angle or the N.P.D. and hour angle may be taken as the two coordinates of a star.

24. Declination and Right Ascension

The position of a celestial body is, however, more frequently defined by its declination and right ascension.

The declination has been already defined, in Art. 23, as the angular distance of the star from the equator, measured along a secondary. (xM, Fig. 13.)

The *Right Ascension* (R.A.) is the arc of the equator intercepted between the foot of this secondary and the First Point of Aries. Thus, ΥM, Fig. 13, is the R.A. of the star x.

The R.A. of a star is always measured from ♈ *eastwards* reckoning from 0° to 360°. Thus the star *ω Piscium,* whose declination circle cuts the equator 1° 34′ 18″ *west* of ♈, has the R.A. 360° — 1° 34′ 18″ or 358° 25′ 42″.

The Diurnal motion of the star x is along the small circle UxV, parallel to the equator QMR. The declination, xM, and the north polar distance, xP, therefore remain constant during the diurnal motion. The hour angle ZPx or QM, however, increases at a uniform rate.

The First Point of Aries, ♈, partakes in the common diurnal motion of the stars and its hour angle increases at the same rate as the hour angles of the stars. The difference of hour angles of the First Point of Aries and of a star therefore remains constant during the diurnal motion. This difference of hour angle is ΥM, the right ascension of the star; or, otherwise expressed,

$$H.A.\Upsilon - H.A.\star = R.A.\star.$$

The right ascension and declination of a star thus both remain constant during the diurnal motion. It will be seen later that they are subject to slow changes caused by the phenomenon known as the precession of the equinoxes.

25. Celestial Latitude and Longitude

The position of a celestial body may also be referred to the eclptiic instead of the equator.

The *Celestial Latitude* is the angular distance of the body from the ecliptic, measured along a secondary to the ecliptic. (Hx, Fig. 14.)

The *Celestial Longitude* is the arc of the ecliptic intercepted between this secondary and the first point of Aries, measured eastwards from ♈. (ΥH, Fig. 14.)

The clestial latitude and longitude of a celestial body are unaffected by the diurnal motion, because this motion does not alter the relative positions of x, ♈ and H. These coordinates are most useful in defining the positions of the Sun, Moon and planets, because the Sun always moves in the ecliptic, ΥH ♎, whilst the paths described by the Moon and planets are always very near the ecliptic.

26. Relation between Horizontal and Equatorial Systems of Coordinates

If the hour angle and declination of a star are given, its altitude and azimuth may be found, and *vice versa*, by the formulae of spherical trigonometry, the latitude of the observer being assumed to be known.

We denote the hour angle of the celestial body x by h, and its declination by δ. We also denote its azimuth by a and its zenith distance by z. The latitude is denoted by ϕ. (See Fig. 15.)

In the spherical triangle PZx, the side PZ is $90^\circ - \phi$; the side Px is $90^\circ - \delta$; the side Zx is z. The angle PZx is a. The angle ZPx is the angle between the meridian and the star's declination circle. Two cases have to be distinguished; (i) if the star is west of the meridian, the angle ZPx is the hour angle, h; (ii) if the star is east of the meridian, its hour angle is greater than 180°, since the hour angle is measured from the meridian towards the west, from 0° to 360°; but no angle of a spherical triangle can be greater than 180° and the angle ZPx is therefore in this case $360^\circ - h$. The spherical triangle ZPx is shown at the side of Fig. 15 for the two cases when the star is respectively west and east of the meridian.

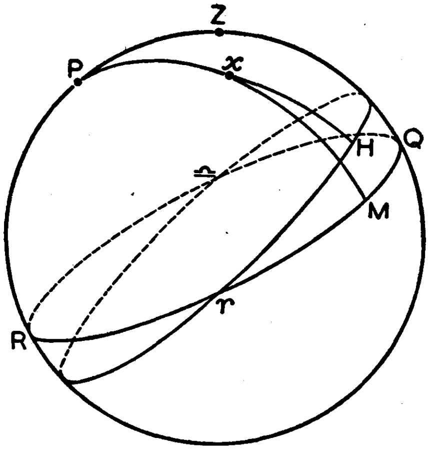

FIG. 14.

(i) We suppose first that we know the azimuth and zenith distance, and require to find the hour angle and declination. We use the formulae given in § 10. From formulae (1) taking Px as the side a, we have

$$\sin \delta = \sin \phi \cos z + \cos \phi \sin z \cos a$$

which gives δ, since ϕ, z, a are assumed known. δ lies between 0° and 90°; if $\sin \delta$ is positive, δ is positive and the declination is north; if $\sin \delta$ is negative, δ is negative and the declination is south.

The side Px of the triangle now being known, h can be obtained by a further application of formula (1), Art. 10, in the form

$$\cos \delta \cos \phi \cos h = \cos z - \sin \delta \sin \phi.$$

This formula holds whether the star is east or west of the meridian. There is only one value of h between 0° and 180° which satisfies this equation ; the hour angle of the star is h, if the star is west of the meridian and $360° - h$, if it is east of the meridian.

(ii) We suppose next that we know the hour angle and declination and require to find the azimuth and zenith distance.

We again use the formula (1) to find z, in the form

$$\cos z = \sin \delta \sin \phi + \cos \delta \cos \phi \cos h.$$

The zenith distance z must lie between 0° and 180°. If $\cos z$ is positive, z is less than 90° ; if $\cos z$ is negative, z is greater than 90° and the star is below the horizon.

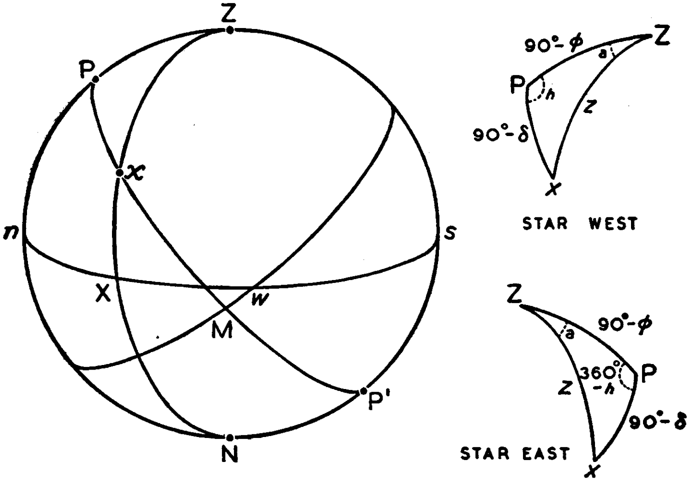

FIG. 15.

Having found z, a is determined from

$$\cos \phi \sin z \cos a = \sin \delta - \sin \phi \cos z$$

a lies between 0° and 180°. If h is less than 180°, the azimuth is westerly ; if h is greater than 180°, the azimuth is easterly.

27. Recapitulation

Below is a list of all the definitions of this chapter, with references to Fig. 16.

GREAT CIRCLES.	THEIR POLES.
Horizon, *nEsW*.	Zenith, *Z*; Nadir, *Z′*.
Equator, *EQWR*.	North Pole, *P*; South Pole, *P′*.
Meridian, *ZsZ′n*.	East Point, *E*; West Point, *W*.
Prime Vertical, *ZEZ′W*.	North Point, *n*; South Point, *s*.

Ecliptic, ♈ *C* ♎ *L*; Equinoctial Points, ♈, ♎ viz :—First Point of Aries, ♈, and First Point of Libra, ♎; Vertical of Star, *ZxX*; Declination Circle of Star, *PxM*.

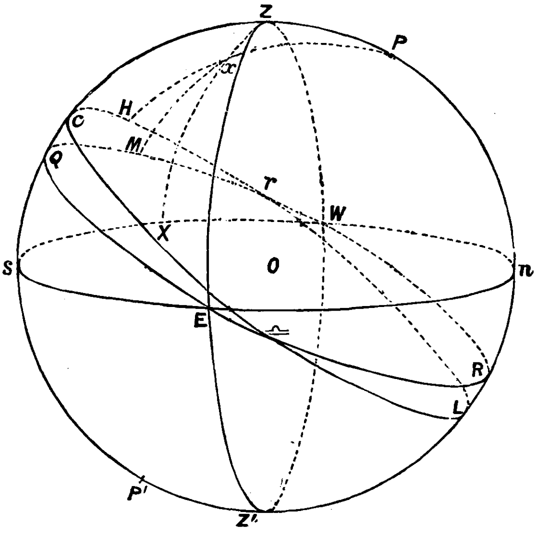

FIG. 16.

COORDINATES.

Altitude, *Xx*; or Zenith Distance, *Zx*.	Azimuth, *sX* = *sZx*.
North Polar Distance, *Px*.	Hour Angle, *QM* = *ZPx*.
Declination, *Mx*.	Right Ascension, ♈ *M*.
Celestial Latitude, *Hx*.	Celestial Longitude, ♈ *H*.

OTHER ANGLES.—Obliquity of Ecliptic $(\epsilon) = C$ ♈ Q. Observer's Latitude $(\phi) = ZQ = nP$. Colatitude $= PZ$.

Notice that the circles on the remote side of the celestial sphere are *dotted*.

II. THE DIURNAL ROTATION OF THE STARS

28. Sidereal Day and Sidereal Time

The rotation of the Earth causes the stars to transit in succession across any given meridian. The interval between two successive passages of a fixed star over the meridian of any place is called a *sidereal* day. The sidereal day is the true period of the Earth's rotation. Like the civil day, it is divided into 24 hours (*h.*), and these are subdivided into 60 minutes (*m.*) of 60 seconds (*s.*) each. From the facts stated in Art. 17, it appears that the sidereal day is about four minutes shorter than the civil or mean solar day (see Art. 58); its actual length is 23h. 56m. 4·100s. of mean solar time. In actual practice, however, transits of the First Point of Aries and not of a star are used to define the sidereal day. We shall see, in Art. 468, that the position of the First Point of Aries is not fixed but that it has a slow retrograde motion along the ecliptic, amounting to about 50″ a year, due to the phenomenon known as precession. The conventionally adopted sidereal day is in consequence 0·009s shorter than the true period of rotation of the Earth and is equal to 23h. 56m. 4·091s. of mean solar time.

The beginning of the sidereal day, corresponding to 0h. 0m. 0s. sidereal time, is taken as the instant when the first point of Aries crosses the meridian. Sidereal clocks, showing *sidereal time,* are used in observatories. The hands should indicate 0h. 0m. 0s. when the first point of Aries crosses the meridian. The hours are reckoned from 0h. up to 24h. when ♈ again comes to the meridian and a new day begins.

The rotation of the Earth is subject to very small irregularities, which can not be detected except by observations of very great refinement. We therefore assume that the earth revolves at a perfectly uniform rate, so that the angles described by any star about the pole are proportional to the times of describing them. Thus, the *hour angle* of a star (measured towards the west) is proportional to the interval of sidereal time that has elapsed since the star was on the meridian.

Now, in 24 sidereal hours the star comes round again to the meridian, after a complete revolution, the hour angle having increased from 0° to 360°. Hence the hour angle increases at the rate of 15° per hour. Hence, also, it increases 15′ per minute, or 15″ per second.

The hour angle of a star is, for this reason, generally measured by the number of hours, minutes, and seconds of sidereal time taken to describe it. It is then said to be *expressed in time.* Thus—

The hour angle of a star, when expressed in time, is the interval of sidereal time that has elapsed since the star was on the meridian.

In particular, since the instant when ♈ is on the meridian is the commencement of the sidereal day, we see that

The sidereal time is the hour angle of the first point of Aries when expressed in time.

It should be noted that at any instant the sidereal time is different for two different meridians, because the hour angle of the first point of Aries is different for two different meridians. Thus the sidereal time at any instant is related to the observer's meridian and for this reason it is often called *local sidereal time.*

28a. Changing the Measure of an Angle

To reduce to angular measure any angle expressed in time. Multiply by 15. The hours, minutes, and seconds of time will thus be reduced to degrees, minutes, and seconds of angle. Conversely :—

To reduce to time from angular measure. Divide by 15, and for degrees, minutes, and seconds write hours, minutes, and seconds.

Examples.—1. *To find, in angular measure, the hour angle of a star at 15h. 21m. 50s. of sidereal time after its transit.*

The process stands thus—

$$\begin{array}{rrr} 15 & 21 & 50 \\ & & 15 \\ \hline 230 & 27 & 30 \end{array}$$

The angular measure of the hour angle is **230° 27′ 30″.**

2. *To find the sidereal time required to describe* 230° 27′ 30″ (*converse of Ex.* 1)·

$$\begin{array}{r|rrr} 15) & 230 & 27 & 30 \\ \hline & 15 & 21 & 50 \end{array}$$

; Required time = **15h. 21m. 50s.**

29. Transits

The passage of the star across the meridian is called its *Transit.*

Let x be the position of any star in transit (Fig. 17). In Art. 24, the right ascension of a star was defined as the arc of the equator intercepted between the First Point of Aries and the foot of the secondary through the Pole and the star, measured eastwards from ♈.

$$\begin{aligned} \text{The star's R.A.} &= \text{♈}Q \text{ or } \text{♈}PQ = \text{hour angle of ♈} \\ &= \text{sidereal time expressed in angle.} \end{aligned}$$

Hence, *the right ascension of a star, when expressed in time, is equal to the sidereal time of its transit.*

When the Sun is at ♈, its right ascension is zero. As the sun moves eastwards along the ecliptic, its right ascension increases. It is for this reason that right ascension is measured in the eastward direction.

In practice the R.A. of a star is usually expressed in time. Thus, the R.A. of α *Lyrae* is given in the tables as 18h. 34m. 54·3s. and not as 278° 43′ 34·5″.

Again, let z be the meridian zenith distance Zx, considered positive if the star transits north of the zenith, δ the star's north declination Qx, and ϕ the north latitude QZ. We have evidently

$$Qx = QZ + Zx\,;$$
$$\text{or}\quad \delta = \phi + z$$

or (*star's N. decl.*) = (*lat. of observer*) + (*star's meridian Z.D. north*).

This formula will hold universally if declination, latitude, and zenith distance are considered negative when south.

Hence the R.A. and decl. of a star may be found by observing its sidereal time of transit and its meridian Z.D., the latitude of the observatory being known.

Conversely, if the R.A. and decl. of a star are known, we can, by observing its time of transit and meridian Z.D., determine the sidereal time and the latitude of the observatory.

By finding the sidereal time we may set the astronomical clock. It is impossible, however, to construct a clock that will keep time with perfect accuracy. What is required is to know the *clock-error*, the amount by which the clock is fast or slow, and the *clock-rate*, the rate at which the clock-error is increasing or decreasing. If the error of the clock at a certain time is known and also the rate of the clock, the error at some subsequent time can be estimated. The time given by the clock at this instant, when corrected for the error, will be the sidereal time at that instant, it being supposed that the clock-rate is uniform.

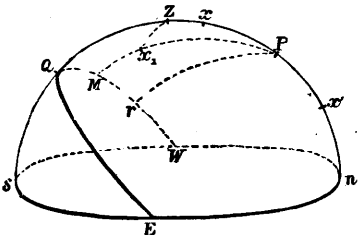

FIG. 17.

30. General Relation between R.A. and Hour Angle

Let x_1 (Fig. 17) be any star not on the meridian. Then

$$\angle\, QPx_1 = \angle\, QP\Upsilon - \angle\, \Upsilon\, Px_1 = \angle\, QP\Upsilon - \Upsilon M\,;$$

hence, if angles are expressed in time :—

(*star's hour angle*) = (*sidereal time*) — (*star's R.A.*).

Hence, given the R.A. of a star, we can find its hour angle at any given sidereal time ; if we are also given the declination or its equivalent, the north polar distance, we can determine the star's position on the observer's celestial sphere, because the hour-angle determines the declination circle on which the star lies, whilst its north polar distance fixes the position of the star on that declination circle. Or we can construct the star's position thus: On the equator, in the *westward* direction from Q, measure off $Q\Upsilon$ equal to the sidereal time (reckoning 15° to the hour). From Υ *eastwards,* measure ΥM equal to the star's R.A.; and from M, in the direction of the pole, measure off Mx_1 equal to the star's declination. We thus find the star x_1.

31. Circumpolar Stars

A *Circumpolar Star* at any place is a star whose polar distance is less than the latitude of the place. Its declination must, therefore, be greater than the colatitude. We have seen, in Art. 22, that the altitude of the pole is equal to the latitude of the place. At lower culmination, when the star is on the meridian north of the zenith, the distance of the star from the pole is less than the distance of the horizon from the pole, since this is equal to the latitude. Hence at lower culmination, when the zenith distance of the star is greatest, the star is above the horizon. In other words, a circumpolar star never reaches the horizon and therefore never rises or sets. The whole of its diurnal path in the sky is above the horizon; it is for this reason that such stars are called circumpolar stars.

On the meridian let Px and Px' be measured, each equal to the N.P.D. of such a star (Fig. 18). Then x and x' will be the positions of the star at its transits. Both x' and x will be above n. Hence, during a sidereal day a circumpolar star will transit twice, once above the pole (at x) and once below the pole (at x'), and both transits will be visible. They succeed one another at intervals of 12 sidereal hours (since $xPx' = 180°$). Now (Fig. 18):—

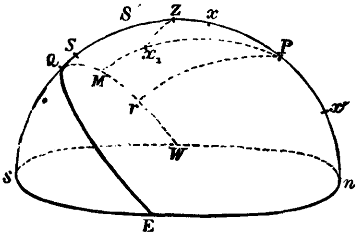

FIG. 18.

$$nx - nP = Px = Px' = nP - nx';$$

$$\text{or } nP = \tfrac{1}{2}(nx + nx');$$

that is, *the observer's latitude is half the sum of the altitudes of a circumpolar star at upper and lower culminations.*

$$\text{Also, } Px = \tfrac{1}{2}(nx - nx');$$

that is, *the Star's N.P.D. is half the difference of its two meridian altitudes.*

We have here neglected the effects of refraction by the atmosphere of the Earth. The altitudes should be corrected for refraction as explained later in Chapter VI.

These results will require modification if the upper culmination takes place south of the zenith as at S. The meridian altitude will then be measured by sS, and not nS. Here, $nS = 180° - sS$, and we shall, therefore, have to replace the altitude at upper culmination by its *supplement.*

SOUTH CIRCUMPOLAR STARS.—If the *south* polar distance of a star is less than the north latitude of the observer, the star will always remain *below* the horizon, and will, therefore, be invisible. Such a star is called a *South Circumpolar Star.*

EXAMPLE.—The constellation of the Southern Cross (*Crux*) is invisible in Europe, for the declination of its principal star, which forms the base of the cross, is 62° 46′ *S*; therefore its south polar distance is 27° 14′, and it will not be visible in north latitudes higher than 27° 14′.

32. Rising, Southing, and Setting of Stars

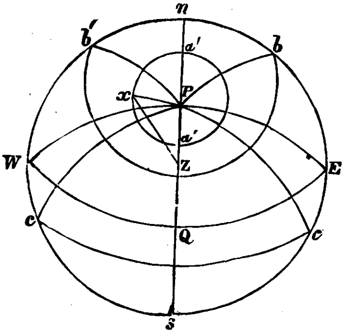

FIG. 19.

If the N. and S. polar distances of a star are both greater than the latitude, it will transit alternately above and below the horizon. This shows that the star will be invisible during a certain portion of its diurnal course. Astronomically, the star is said to *rise* and *set* when it crosses the celestial horizon.

Let *b*, *b′* be the positions of any star when rising and setting respectively.

The spherical triangles *Pnb*, *Pnb′* are equal, since the sides *Pb* and *Pb′* are equal, each being the star's N.P.D., the side *Pn* is common and the angles at *n* are right angles. Therefore

$$\angle nPb = \angle nPb',$$

and the supplements of these angles are also equal, that is,

$$\angle sPb = \angle sPb'.$$

But the angle *sPb*, when reduced to time, measures the interval of time taken by the star to get from *b* to the meridian, and *sPb′* measures the time taken from the meridian to *b′*. Hence :—

The interval of time between rising and southing is equal to the interval between southing and setting.

Thus, if t, t' are the times of rising and setting, and T the time of transit, we have

$$T - t = t' - T \text{ giving } T = \tfrac{1}{2}(t + t'), \text{ or :—}$$

The time of transit is the arithmetic mean between the times of rising and setting.

If the star is on the equator, it will rise at *E* and set at *W*. Since *EQW* is a semicircle, exactly half the diurnal path will be above the horizon, and the interval between rising and setting will be 12 sidereal hours. If the star is to the north of the equator, it will rise at some point *b* between *E* and *n*, so that

$$\angle bPs > \angle EPs,$$

i.e.

$$\angle bPs > 90^\circ,$$

and the star will be above the horizon for more than 12 hours. Similarly, if the star is south of the equator, it will rise at a point *c* between *E* and *s*, and will be above the horizon for less than 12 hours.

From the equality of the triangles *bPn*, *b'Pn* (Fig. 19), we also see that

$$nb = nb', \quad \text{and} \quad sb = sb'.$$

Hence the diameter (*ns*) of the celestial sphere, joining the north and south points, bisects the arc (*bb'*) between the directions of a star at rising and setting.

This gives us an easy method of roughly determining, by observation, the directions of the cardinal points; but, owing to the usual irregularities in the visible horizon, the method is not very exact.

33. Hour-Angle and Azimuth of Rising and Setting

When the latitude of the observer, ϕ, and the declination of the star, δ, are known, the hour-angles and azimuths at rising and setting are easily found. If h_1, h_2 are the hour-angles at rising and setting and a is the azimuth east of north at rising or west of north at setting, we have in the triangles *Pbn*, *Pb'n* (Fig. 19) $Pb = Pb' = 90^\circ - \delta$; $Pn = \phi$; $bn = b'n = a$; $\angle bPn = h_1 - 180^\circ$; $\angle b'Pn = 180^\circ - h_2$ and the angles *Pnb*, *Pnb'* are right angles.

From the rules for right-angled triangles in Art. 10 we obtain :—

$$\cos bPn = \tan Pn \cot Pb\,; \quad \cos b'Pn = \tan Pn \cot Pb'$$

or

$$\cos h_1 = -\tan\phi \tan\delta = \cos h_2,$$

h_1, h_2 are thus the two values of h between 0° and 360° which satisfy the equation $\cos h = -\tan\phi\tan\delta$. h_1 is the larger of the two values, whose sum is 360°.

The azimuth a is given by $\cos Pb = \cos Pn \cos bn$ or

$$\sin\delta = \cos\phi\cos a.$$

From these formulae for h and a, it readily follows that when the star is on the equator, so that $\delta = 0$, $\cos h$ and $\cos a$ are both zero. Therefore $h_1 = 270^\circ = 18h.$, $h_2 = 90^\circ = 6h.$ and $a = 90^\circ$. Thus there is an interval of 12 hours between rising and setting, which occur at the east and west points respectively.

If t, t' are the times of rising and setting and T is the time of transit, we have

$$T - t = 360° - h_1 ; \quad t' - T = h_2$$

The sidereal line of transit T is equal to the star's R.A. If this is known, the sidereal times of rising and setting can be found when h_1, h_2 have been computed.

We have neglected the effect of refraction on the times of rising and setting. This will be considered in Chapter VI.

III.—The Sun's Annual Motion in the Ecliptic—Practical Applications

34. The Sun's Motion in Longitude, Right Ascension and Declination

In Art. 20, we briefly described the Sun's apparent motion in the heavens relative to the fixed stars. We defined a *Year* as the period of a complete revolution, starting from and returning to any fixed point on the celestial sphere. The *Ecliptic* was defined as the great circle traced out by the Sun's path, and its points of intersection with the Equator were termed the *First Point of Aries* and *First Point of Libra*, or together, the *Equinoctial Points*.

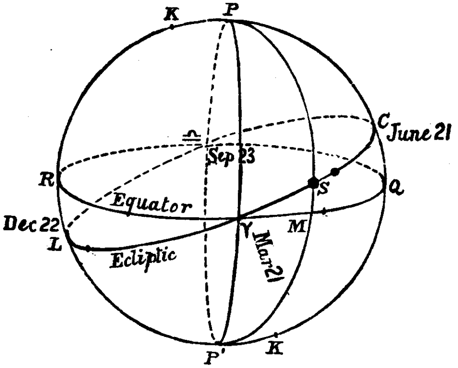

Fig. 20.

We shall now trace, by the aid of Fig. 20, the variations in the Sun's coordinates during the course of a year, starting with March 21st,* when the Sun is in the first point of Aries. We shall, as usual, denote the obliquity by ϵ so that $\epsilon = 23° 27'$ nearly.

On March 21st the Sun crosses the equator, passing through the first point of Aries (♈). This is the *Vernal Equinox*, and it is evident from the figure that

$$\textit{Sun's longitude} = 0°, \ \textit{R.A.} = 0\text{h}, \ \textit{Decl.} = 0°.$$

From March 21st to June 21st the Sun's declination is north, and is increasing.

---

* Owing mainly to the fact that the year is not an integral number of days, such dates vary somewhat from year to year.

On June 21st the Sun has described an arc of 90° from ♈ on the ecliptic, and is at C (Fig. 20). This is called the *Summer Solstice.* The Sun's polar distance CP is then a minimum and therefore its decl. a maximum.

$$\text{Also}\quad \Upsilon Q = 90^\circ \text{ and } CQ = \angle C\Upsilon Q = \epsilon. \quad \text{Hence}$$

$$\textit{Sun's longitude} = \mathbf{90^\circ},\ \ R.A. = \mathbf{90^\circ} = \mathbf{6h.},$$

$$N.\ Decl. = \epsilon\ (a\ maximum).$$

From June 21st to September 23rd the Sun's declination is still north, but is decreasing.

On September 23rd the Sun has described 180°, and is at the first point of Libra (♎), the other extremity of the common diameter of the ecliptic and equator. This is the *Autumnal Equinox,* and we have

$$\textit{Sun's long.} = \mathbf{180^\circ},\ \ R.A. = \mathbf{180^\circ} = \mathbf{12h.},\ Decl. = \mathbf{0^\circ}.$$

From Sept. 23rd to Dec. 22nd the Sun is south of the equator, and its south declination is increasing.

On December 22nd the Sun has described 270° from ♈, and is at L. This is called the *Winter Solstice.* The Sun's polar distance LP is then a maximum, and

$$\simeq R = \simeq L = 90^\circ,\ LR = \angle L \simeq R = \epsilon. \quad \text{Hence}$$

$$\textit{Sun's longitude} = \mathbf{270^\circ},\ \ R.A. = \mathbf{270^\circ} = \mathbf{18h.},$$

$$S.\ Decl. = \epsilon\ (a\ maximum).$$

From December 22nd to March 21st the Sun's declination is still south, but is decreasing.

Finally, on March 21st, when the Sun has performed a complete circuit of the ecliptic, we have

$$\textit{Sun's long.} = \mathbf{360^\circ},\ \ R.A. = \mathbf{360^\circ} = \mathbf{24h.},\ \ Decl. = \mathbf{0^\circ}.$$

The longitude and R.A. are again reckoned as zero, and they, together with the declination, undergo the same cycle of changes in the following year.

35. Sun's Variable Motion in R.A.

We observe that the Sun's right ascension is equal to its longitude four times in the year, viz. at the two equinoxes and the two solstices.

At other times this is not the case.

For example, between the vernal equinox and summer solstice we have $\Upsilon M < \Upsilon S$, or Sun's R.A. < longitude.

Hence, even if the Sun's motion in longitude be supposed uniform, its R.A. will not increase quite uniformly. There is a further cause of the want of uniformity, namely, that the Sun's motion in longitude is not quite uniform; but this need not be considered in the present chapter.

36. Direct and Retrograde Motions

The direction of the Sun's annual revolution relative to the stars, *i.e.* motion from west through south to east, is called *direct*. The opposite direction, that of the diurnal apparent motions of the stars or revolution from east to west, is called *retrograde.*

The real revolutions of all bodies forming the solar system, with the exception of some comets and a few satellites are direct, but as seen from the Earth the planets frequently appear to have retrograde motion.

The apparent *retrograde* diurnal motion is accounted for by the *direct* rotation of the Earth about its polar axis.

37. Equinoctial and Solstitial Points—Colures

From Art. 34 it appears that the *Summer* and *Winter Solstices* may be defined as the times of the year when the Sun attains its greatest north and south declinations respectively. At these times the declination is therefore practically stationary for a few days. The meridian zenith distance of the Sun is least at the summer solstice and greatest at the winter solstice. At the summer solstice the sun halts in its northern motion in the sky before beginning to move southwards again; at the winter solstice it halts in its southern motion before beginning to move northwards again. This is the reason why C and L are called solstices (meaning standing still).

The corresponding positions of the Sun in the ecliptic (C, L, Fig. 20) are called the *Solstitial Points.* In the same way the *Equinoctial Points* (♈, ♎) are the positions of the Sun at the *Vernal and Autumnal Equinoxes* when its declination is zero. We have seen in Art. 33 that when a celestial body is on the equator, the interval between rising and setting is 12 hours. Then when the Sun is at ♈ or ♎, the lengths of day and night are equal, whence the term equinoctial points.

The declination circle P♈P'♎, passing through the equinoctial points, is called the *Equinoctial Colure.* The declination circle $PCP'L$, passing through the solstitial points, is called the *Solstitial Colure.* The latter passes through the poles of the ecliptic (K, K').

38. To find the Sun's Right Ascension and Declination

In the *Nautical Almanac,* the Sun's R.A. and Dec. are tabulated for midnight (0h.) for every day of the year, together with the change in 24 hours. To find their values at any time of the day we have only to multiply the daily variation by the fraction of the day from the nearest midnight and add this quantity to the value at that midnight, if the preceding midnight has been used, or subtract it, if the succeeding midnight has been used.

Example.—*To find the Sun's R.A. and Dec. on March 30th,* 1940, *at* 7*h.* 56*m. in the afternoon.*

We find from the Almanac for 1940,
(i) Sun's R.A. at March 31·0 = 0h. 37m. 13·8s.
Daily variation = + 218·4s.
7h. 56m. in the afternoon is 4h. 04m. before midnight = 0·1694 days.
R.A. at March 31·0 = 0h. 37m. 13·8s.
— ·1694 × 218·4s. = — 37·0
Required R.A. = 0h. 36m. 36·8s.

(ii) Sun's Dec. at March 31·0 = + 4° 00′ 44″
Daily variation = + 1395″
Sun's Dec. at March 31·0 = + 4° 00′ 44″
— ·1694 × 1395″ = — 3′ 56″
Required Dec. = + 3°. 56′ 48″

39. Rough Determination of the Sun's R.A.

We can, without the *Nautical Almanac* find to within a degree or two, the Sun's R.A. on any given date, as follows :—

A year contains 365¼ days. In this period the Sun's R.A. increases by 360°. Hence its average rate of increase is very nearly 30° per month, or 1° per day.

Knowing the Sun's R.A. at the nearest equinox or solstice, (0° on March 21st, 90° on June 21st, 180° on September 23rd, and 270° on December 22nd, approximately) we add 1° for every day later, or subtract 1° for every day before that epoch. If the R.A. is required in *time*, we allow for the increase at the rate of 2h. per month, or 4m. per day.

40. The Gnomon.—Determination of Obliquity of Ecliptic

The Greek astronomers observed the Sun's motion by means of the Gnomon, an instrument consisting essentially of a vertical rod standing in the centre of a horizontal floor. The direction of the shadow cast by the Sun determined the Sun's azimuth, while the length of the shadow, divided by the height of the rod, gave the tangent of the Sun's zenith distance. To find the meridian line, a circle was described about the rod as centre, and the directions of the shadow were noted when its extremity just touched the circle before and after noon. The sun's Z.D.'s at these two instants being equal, their azimuths were evidently equal and opposite, and the bisector of the angle between the two directions was therefore the meridian line.

The Sun's meridian zenith distances were then observed both at the summer solstice, when the Sun's N. decl. is ϵ and meridian Z.D. least, and at the winter solstice, when the Sun's S. decl. is ϵ and meridian Z.D. greatest. Let these Z.D.'s be z_1 and z_2 respectively, and let ϕ

be the latitude of the place of observation. From Art. 29, we readily see that

$$z_1 = \phi - \epsilon, z_2 = \phi + \epsilon$$

$$\text{so that } \phi = \tfrac{1}{2}(z_2 + z_1), \epsilon = \tfrac{1}{2}(z_2 - z_1);$$

thus determining both the latitude and the obliquity.

41. The Zodiac

The position of the ecliptic was defined by the ancients by means of the constellations of the Zodiac, which are twelve groups of stars, distributed at about equal distances round a belt or zone, and extending about 8° on each side of the ecliptic. The Sun and planets were observed to remain always within this belt. The vernal and autumnal equinoctial points were formerly situated in the constellations of Aries and Libra, whence they were called the First Point of Aries and the First Point of Libra. Their positions are *very slowly* varying, but the old names are still retained. Thus, the " First Point of Aries " is now situated in the constellation Pisces.

The early astronomers probably determined the Sun's annual path by noting the morning and evening stars. After a year the same stars would be seen, and it would be concluded that the Sun performed a revolution in a year. We learn from Egyptian records that the *heliacal* rising of Sirius, *i.e.* the first occasion the star was seen in morning twilight, was noted with special care.

42. Astronomical Diagrams and Practical Applications

We can now solve many problems connected with the motion of the celestial bodies, such as determining the direction in which a given star will be seen from a given place, at a given time, on a given date, or finding the time of day at which a given star souths at a given time of year.

We have, on the celestial sphere, certain circles, such as the meridian, horizon, and prime vertical, also certain points, such as the zenith and cardinal points, whose positions relative to terrestrial objects always remain the same. Besides these, we have the poles and equator, which remain fixed, with reference *both* to terrestrial objects and to the fixed stars. We have also certain points, such as the equinoctial points, and certain circles, such as the ecliptic, which partake of the diurnal motion of the stars, performing a retrograde revolution about the pole once in a sidereal day. Lastly, we have the Sun, which moves in the ecliptic, performing one retrograde revolution relative to the meridian in a solar day, or one direct revolution relative to the stars in a year. We can assume in these problems that the Sun is on the meridian at noon and that its hour-angle increases at the rate of 15° per hour.

In drawing a diagram of the celestial sphere, the positions of the meridian, horizon, zenith, and cardinal points should first be represented, usually in the positions shown in Fig. 21. Knowing the latitude *nP* of the place, we find the pole *P*. The points *Q*, *R*, where the equator cuts the meridian, are found by making $PQ = PR = 90°$; and the points *Q*, *R*, with *E*, *W*, enable us to draw the equator.

We now have to find the equinoctial points. How to do this depends on the data of the problem. Thus we may have given—

(i) The sidereal time;

(ii) The hour angle of a star of known R.A. and decl.;

(iii) The time of day and time of year.

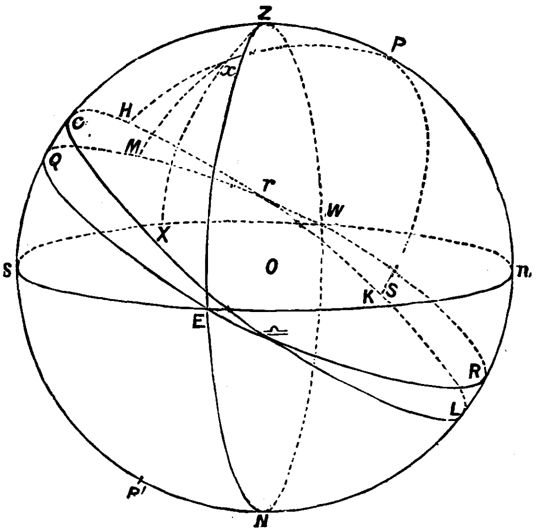

FIG. 21.

In case (i), the sidereal time multiplied by 15 gives, in degrees, the hour angle (*Q*♈) of the first point of Aries. Measuring this angle from the meridian westwards, we find Aries, and take Libra opposite to it. Any star of known decl. and R.A. can be found by taking on the equator ♈*M* = star's R.A., and taking on *MP*, *Mx* = star's decl.

The ecliptic may be drawn passing through Aries and Libra, and inclined to the equator at an angle of about 23½° (just over ¼ right angle). As we go round from west to east, or in the *direct* sense, the ecliptic passes from south to north of the equator at Aries; this shows on which side to represent the ecliptic. Knowing the time of year, we now find the Sun (roughly) by supposing it to travel to or from the nearest equinox or solstice about 1° per day from west to east.

In case (ii), we either know the hour angle, *QM* or *QPM* of a known star (*x*), or, what is the same thing, the sidereal interval since its transit; or, in particular, it is given that the star is on the meridian. Each of these data determines *M*, the foot of the star's declination circle.

From M we measure $M\Upsilon$ westwards equal to the star's R.A. This finds Aries.

In case (iii), the solar time multiplied by 15 gives the Sun's hour angle QPS in degrees. From the time of year we can find the Sun's R.A., ΥPS. From these we find $QP\Upsilon$ and obtain the position of Aries just as in case (ii).

It will be convenient to remember that hour angle is measured from the southern meridian westwards, that azimuth is measured from the north point eastwards or westwards, while right ascension and celestial longitude are measured from the first point of Aries eastwards. Thus, since the Sun's diurnal motion is retrograde, and its annual motion direct, the Sun's hour angle, R.A., and longitude are all increasing.

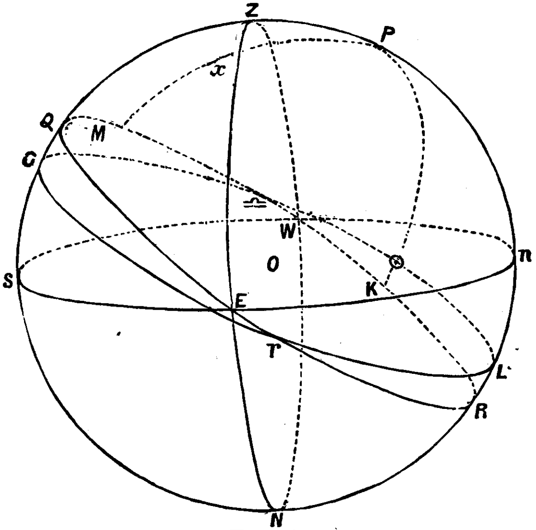

FIG. 22.

Most problems of this class depend for their solution chiefly on the consideration of arcs measured along the equator, or (what amounts to the same) angles measured at the pole.

In another class of problems depending on the relation between the latitude, a star's decl. and meridian latitude (Art. 29), we have to deal with arcs measured along the meridian. These two classes include nearly all problems on the celestial sphere which do not require spherical trigonometry.

Examples.—1. *Represent, in a diagram, the positions of the Sun and the star* ζ *Herculis as seen by an observer in London on Aug.* 19*th*, 1940, *at* 8 *p.m.*,[*] *the following data being given:—Latitude of London* = 51°, *R.A. of* ζ *Herculis* = 16*h*. 39*m*., *decl.* = 31° 43′ *N*.

The construction must be performed in the following order:—

(i) Draw the observer's celestial sphere, putting in the meridian, horizon, zenith Z, and four cardinal points n, E, s, W.

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[*] The times in these examples are assumed to be the times as given by a sundial. Summer time is not taken into consideration.

(ii) Indicate the position of the pole and equator. The observer's latitude is 51°. Make, therefore, $nP = 51^\circ$. P will be the pole. Take $PQ = PR = 90^\circ$, and thus draw the equator, $QERW$.

(iii) Find the declination circle passing through the Sun. The time of day is 8 p.m. Therefore the Sun's hour angle is $8 \times 15^\circ$, or 120°. On the equator measure $QK = 120^\circ$ westwards from the meridian. Then the Sun ⊙ will lie on the declination circle PK. Since $QW = 90^\circ$, we may find K by taking $WK = 30^\circ = \frac{1}{3}\,WR$.

(iv) Find the first points of Aries and Libra. The date of observation is August 19th. Now, on September 23rd the Sun is at ♎. Also from August 19th to September 23rd is 1 month 4 days. In this interval the Sun travels about 34° from west to east. Hence the Sun is 34° west of ♎. And we must measure $K♎ = 34^\circ$ eastwards from K, and thus find ♎.

The first point of Aries (♈) is the opposite point on the equator.

(v) We may now draw the ecliptic $C♈L♎$ passing through the first points of Aries and Libra, and inclined to the equator at an angle of about $23\frac{1}{2}^\circ$ (*i.e.* slightly over $\frac{1}{4}$ of a right angle). The Sun is above the equator on August 19th; hence the ecliptic cuts PK above K. This shows on which side of the equator the ecliptic is to be drawn; we might settle this by remembering that the ecliptic rises above the equator in the direction of increasing longitude from ♈.

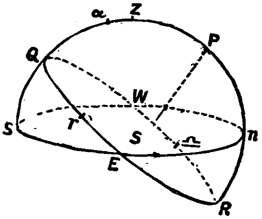

FIG. 23.

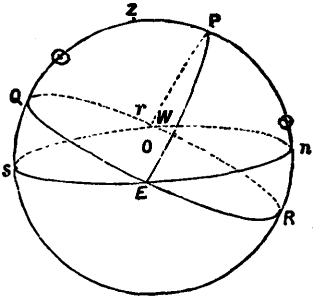

FIG. 24.

The intersection of the ecliptic with PK determines ⊙, the position of the Sun.

(vi) Having found ♈, we can now find ζ *Herculis*. Its right ascension is 16h. 39m., in time, = 249° 45′ in angular measure. On the equator measure off $♈M = 259^\circ\,45'$ in the direction west to east (*i.e.* the direction of direct motion) from ♈; we must, therefore, take $♎M = 69^\circ\,45'$. On the declination circle MP, measure off $Mx = 31^\circ\,43'$ towards P. Then x is the required position of ζ *Herculis*.

2. *Find (roughly) at what time of the year the Star α Cygni (R.A. = 20h. 39m., decl. = 45° 04′ N.) souths at 7 p.m.*

Let α be the position of the star on the meridian (Fig. 23). At 7 p.m. the Sun's western hour angle (QS or QPS) = 7h. = 105°.

Also ♈ RQ, the Star's R.A. = 20h. 39m. Hence ♈RS, the Sun's R.A. = 20h. 39m. — 7h. = 13h. 39m.; or, in angular measure, Sun's R.A. = 204° 45′.

Now, on September 23rd, Sun's R.A. = 180°, and it increases at about 1° per day. Hence the Sun's R.A. will be 205° about 25 days later, *i.e.* about **October 18th.**

3. *At noon on the longest day* (*June 25th*) *a vertical rod casts on a horizontal Plane a shadow whose length is equal to the height of the rod. Find the latitude of the place and the Sun's altitude at midnight.* (See Fig. 24).

From the data, the Sun's Z.D. at noon, $Z\odot$, evidently $= 45°$.
Also, if QR be the equator, $\odot Q =$ Sun's decl. $= \epsilon = 23° 27'$ (approx.);
Therefore latitude of place $= ZQ = 45° + 23° 27' =$ **68° 27′.**

If $\odot'$ be the Sun's position at midnight,

$$P\odot' = P\odot = 90° - 23° 27' = 66° 33'.$$
$$\text{But } Pn = \text{lat.} = 68° 27'.$$
$$\text{So that } \odot' n = 68° 27' - 66° 33' = 1° 54';$$

and the Sun will be above the horizon at an alt. of **1° 54′** at midnight.

EXAMPLES

1. Why are the following definitions alone insufficient?—The zenith and nadir are the poles of the horizon. The horizon is the great circle of the celestial sphere whose plane is perpendicular to the line joining the zenith and nadir.

2. The R.A. of an equatorial star is 270°; determine approximately the times at which this star rises and sets on the 21st June. In what quarter of the heavens should we look for the star at midnight?

3. Explain how to determine the position of the ecliptic relatively to an observer at a given hour on a given day. Indicate the position of the ecliptic relatively to an observer at Cambridge at 10 p.m. at the autumnal equinox. (Lat. of Cambridge = 52° 12′ 51·6″.)

4. Prove geometrically that the least of the angles subtended at an observer by a given star and different points of the horizon is that which measures the star's altitude.

5. Show that in latitude 52° 13′ N. no circumpolar star when southing can be within 75° 34′ of the horizon.

6. Represent in a figure the position of the ecliptic at sunrise on March 21st as seen by an observer in latitude 45°. Also in latitude 67½°.

7. If the ecliptic were visible in the first part of the preceding question, describe the variations which would take place during the day in the positions of its points of intersection with the horizon.

8. Determine when the star whose declination is 30° N. and whose R.A. is 356° will cross the meridian at midnight.

9. The declination and R.A. of a given star are 22° N. and 6h. 20m. respectively. At what period of the year will it be (i) a morning, (ii) an evening star? In what part of the sky would you then look for it?

10. Find the Sun's R.A. (roughly) on January 25th, and thus determine about what time *Aldebaran* (R.A. 4h. 33m.) will cross the meridian that night.

11. Where and at what time of the year would you look for *Fomalhaut*? (R.A. 22h. 54m., decl. 29°. 56′ S.)

12. At the summer solstice the meridian altitude of the Sun is 75°. What is the latitude of the place? What will be the meridian altitude of the Sun at the equinoxes and at the winter solstice?

EXAMINATION PAPER

1. Explain how the directions of stars can be represented by means of points on a sphere. Explain why the configurations of the constellations do not depend on the position of the observer, and why the angular distance of two different bodies on the celestial sphere gives no idea of the actual distance between them.

2. Define the terms—*horizon, meridian, zenith, nadir, equator, ecliptic, vertical, prime vertical,* and represent their positions in a figure.

3. Explain the use of coordinates in fixing the position of a body on the celestial sphere, and define the terms—*altitude, azimuth, polar distance, hour angle, right ascension, declination, longitude, latitude.* Which of these coordinates always remain constant for the same star?

4. Define the *obliquity of the ecliptic* and the *latitude of the observer.* Give (roughly) the value of the obliquity, and of the latitude of London. Indicate in a diagram of the celestial sphere twelve different arcs and angles which are equal to the latitude of the observer.

5. What is meant by a *sidereal day* and a *sidereal hour*? How could you find the length of a sidereal day without using a telescope? Why is sidereal time of such great use in connection with astronomical observations?

6. Show that the declination and right ascension of a celestial body can be determined by meridian observations alone.

7. What is meant by a *circumpolar star*? What is the limit of declination for stars which are circumpolar in latitude 60° N.? Indicate in a diagram the belt of the celestial sphere containing all the stars which rise and set.

8. Define the terms—*year, equinoxes, solstices, equinoctial and solstitial points, equinoctial and solstitial colures.* What are the dates of the equinoxes and solstices, and what are the corresponding values of the Sun's declination, longitude, and right ascension? Find the Sun's greatest and least meridian altitudes at London.

9. Why is it that the interval between two transits of the Sun is rather greater than a sidereal day? Show how the Sun's R.A. may be found (roughly) on any given date, and find it on July 2nd, expressed in hours, minutes, and seconds.

10. Indicate (roughly) in a diagram the positions of the following stars as seen in latitude 51° on July 2nd at 10 p.m. :—*Capella* (R.A. 5h. 12m. 15s., decl. 45° 56′ 21″ N.); *α Lyrae* (R.A. 18h. 34m. 54s., decl. 38° 43′ 36″ N.); *α Scorpii* (R.A. 16h. 25m. 43s., decl. 26° 18′ 2″ S.); *α Ursae Majoris* (R.A. 11h. 0m. 2s., decl. 62° 4′ 31″ N.).

CHAPTER III

ON TIME

I.—The Mean Sun and Equations of Time

43. Disadvantage of Sidereal Time

In Chapter II, Section II, we explained how the rotation of the Earth with respect to the stars could be used to provide a means of reckoning time. This time, called sidereal time, was defined by the diurnal motion of the first point of Aries. We shall show that this measure of time is not suitable for everyday use.

The sidereal time of the Sun's transit across the meridian, which occurs at midday, is equal to the Sun's R.A. (Art. 29). But we have seen (Section II, Art. 34) that the Sun's R.A. increases throughout the year at the rate of approximately 1° per day. The time of the Sun's transit, in the sidereal time system, therefore gets later and later day by day, and by a total amount of 24h. in the course of the year.

Thus, *e.g.* the time of noon would be 0h. on March 21st, 6h. on June 21st, 12h. on September 23rd, and 18h. on December 22nd, and the phenomena of day and night would bear no constant relation to that time. This makes it impossible to use sidereal time for everyday purposes. The Sun is the heavenly body which is most closely related to human activities, because it is the apparent diurnal motion of the Sun that controls the hours of darkness and light. It is therefore necessary, for the purposes of everyday life, to choose a system of time that is closely related to the Sun and such that the middle of each day comes at or near the time when the Sun transits across the meridian. The most natural system of time to choose to meet this requirement is *apparent solar time*, which is the time as indicated by a sun-dial.

44. Apparent Solar Time

Apparent Noon is the time of the Sun's upper transit across the meridian, that is, in north latitudes, the time when the Sun *souths.* *Apparent Midnight* is the time of the Sun's transit across the meridian below the pole (and usually below the horizon).

An *Apparent Solar Day* is the interval between two consecutive apparent noons, or two consecutive midnights.

Like the sidereal day, the solar day is divided into 24 hours, which are again divided into 60 minutes of 60 seconds each. For ordinary purposes the day is divided into two portions: the morning, lasting from midnight to noon; the evening, from noon till midnight; and in each portion times are reckoned from 0h. (usually called 12h.) up to 12h. For *astronomical* purposes the practice up to the end of the year 1924 was to measure the solar time by the number of solar hours that have elapsed since the preceding *noon.* Thus, 6.30 A.M. on January 2nd was reckoned, astronomically, as 18h. 30m. on January 1st. On the other hand, 12.53 P.M. was reckoned as 0h. 53m., being 53 minutes past noon. Since the beginning of the year 1925 the astronomical day has been considered to begin at midnight, like the civil day. But the former astronomical method occurs so frequently in astronomical works that the student should be acquainted with it.

During a solar day the Sun's hour angle increases from 0° to 360° at the rate of 15° per hour. Hence :—

Apparent solar time = Sun's hour angle expressed in time.

At noon the Sun is on the meridian. The sidereal time, being the hour angle of ♈, is the same as the Sun's R.A., *i.e.*

Sidereal time of apparent noon = Sun's R.A. at noon.

At any other time, the difference between the sidereal and solar times, being the difference between the hour angles of ♈ and the Sun, is equal to the Sun's R.A. Hence, as in Art. 30, we have

(Sidereal time) — (apparent solar time) = Sun's R.A.

If a and $a + x$ are the right ascensions of the Sun at two consecutive noons, then, since a whole day has elapsed between the transits, the total sidereal interval is 24h. $+ x$, and exceeds a sidereal day by the amount x. But the interval is a solar day.

Hence, *the apparent solar day is longer than the sidereal day, and the difference is equal to the sun's daily motion in R.A.*

45. Disadvantage of Apparent Solar Time

Apparent solar time, as defined in the preceding section, avoids the disadvantage of sidereal time mentioned in Art. 43. In apparent solar time, 12h. is the instant of apparent noon. But this system of reckoning time entails an inconvenience of a different nature. We have seen that the difference in length between the apparent solar day and the sidereal day is equal to the Sun's daily increase in R.A. In Art. 35, we showed that this increase takes place at a rate which is not quite the same at different times of the year. Hence, the difference between a solar and a sidereal day is not quite constant. But the length of a sidereal day is constant (Art. 28). Hence the apparent solar day is not quite constant in length, and apparent solar time cannot be measured by a clock whose rate is uniform. If a clock, whose rate was perfectly uniform, were rated so that 24h. by the clock corresponded exactly to the average length of the mean solar day, then the extreme differences between the clock and apparent solar time would amount to about a quarter of an hour in either direction. A good pendulum clock will keep a uniform time to an accuracy of about one second a year. To keep apparent solar time, it would be necessary for clocks to be altered day by day and by amounts that varied progressively through the course of the year. To avoid such inconveniences we adopt a system of time that is called *mean solar time.*

46. The Mean Sun.—Definitions

Mean solar time is defined by means of what is called the *Mean Sun.* This is not really a Sun at all, but simply a point, which is imagined to move round the equator on the celestial sphere.* The hour angle of this moving point measures mean time, just as the hour angle of ♈ measures sidereal time ; and the mean Sun has to satisfy the following requirements :—

1st. It must never be very far from the Sun in hour angle.

2nd. Its R.A. must increase uniformly during the year.

Now the inequalities in the motion in R.A. which render the true Sun unsuitable as a timekeeper, are due to two causes.

1st. The Sun does not move uniformly in the ecliptic, its longitude increasing less rapidly in summer than in winter. This is a consequence of the distance from the Sun to the Earth not being constant through the year.

2nd. Since the Sun moves in the ecliptic, and not in the equator, its celestial longitude is in general different from its R.A. Hence, even if the Sun were to revolve uniformly, its R.A. would not increase uniformly.

In defining the mean Sun, or moving point which measures mean time, these two causes of irregularity are obviated separately as follows :—

The *Dynamical Mean Sun* is defined to be a point which coincides with the true Sun at *perigee,* when the distance between the Sun and the Earth is least, and which moves round the *ecliptic* in the same period (a year) as the true Sun, but at a uniform rate.

Thus, in the dynamical mean Sun, irregularities due to the Sun's unequal motion in longitude are removed, but those due to the obliquity of the ecliptic still remain.

The *Astronomical Mean Sun* is defined to be a point which moves round the *equator* in such a way that its R.A. is always equal to the longitude of the dynamical mean Sun.

Since the longitude of the dynamical mean Sun increases uniformly, the R.A. of the astronomical mean Sun increases uniformly. Hence the motion of the latter point *does* give us a uniform measure of time.

The astronomical mean Sun is, therefore, the moving point chosen in defining mean time. It is usually called simply the *Mean Sun.*

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* The conception of the mean Sun as a *moving point* is important. It would be physically impossible for a *body* to move in this manner.

47. Mean Noon and Mean Solar Time.—Equation of Time

Mean Noon is defined as the time of transit of the mean Sun.

A *Mean Solar Day* is the interval between two successive mean noons. Like the apparent and sidereal days, it is divided into 24 mean solar hours. During this interval, the hour angle of the mean Sun increases from 0° to 360°. Hence the *mean solar time* at any instant is measured by the mean Sun's hour angle, converted into time at the rate of 1h. per 15°, or 4m. per 1°.

The Sun itself is frequently spoken of as the *True Sun*, or *Apparent Sun*, to distinguish it from the mean Sun. As explained in Art. 44 the hour angle of the true Sun measures the apparent solar time, and its time of transit is called apparent noon.

The *Equation of Time** is the name given to the amount which must be added to the mean time to obtain the apparent time.

Thus, the time indicated by a sun-dial is determined by the position of the shadow thrown by the true Sun, and is the apparent solar time; while a clock, which should go at a uniform rate, is regulated to keep mean time. The equation of time will then be defined by the relation,

$$(\textit{Time by clock}) + (\textit{Equation of time}) = (\textit{Time by dial}).$$

The equation of time is positive if the Sun is "before the clock," or the true Sun transits before the mean Sun. If the Sun is "after the clock," or the mean Sun transits first, the equation of time is negative. The value of the equation of time for 0h. of every day in the year is given in the *Nautical Almanac*.

48. The Two Components of the Equation of Time

The equation of time is divided into two parts. The first, which is called the *equation of time due to the eccentricity*, or to the *unequal motion*, is measured by the difference between the hour angles of the true and dynamical mean Suns. The second, or the *equation due to the obliquity*, is measured by the difference of hour angle between the dynamical and astronomical mean Suns.

49. Equation of Time due to Unequal Motion

We shall now trace the variations during the year of that portion of the equation of time which is due to the Sun's unequal motion in the ecliptic. We shall denote this portion by E_1.

We shall see in Art. 129 that the apparent path of the Sun round the Earth is an ellipse. The distance between the Sun and the Earth is least on December 31st, when the Sun is said to be in *perigee*, and greatest on July 1st, when the Sun is said to be in *apogee*. The Sun's longitude changes more rapidly the nearer the Earth is to the Sun. Hence its rate of change is greatest at perigee and least at apogee.

---

* Thus, "equation of time" is not an *equation* at all in the generally accepted sense of the word, but an interval of time (positive or negative).

Let the true Sun be denoted by S, and the dynamical mean Sun (which moves in the ecliptic) by S_1. If angles are measured in time, then (Fig. 25).

$$E_1 = (\text{hour angle of } S) — (\text{hour angle of } S_1) = \angle SPS_1 ;$$

$$\text{or } E_1 = (\text{R.A. of } S_1) — (\text{R.A. of } S) ;$$

since R.A. and hour angle are measured in opposite directions.

When the Sun is in perigee (p) (on December 31st), S_1 coincides with S by definition ; so that $E_1 = 0$.

From perigee (p) to apogee (a), the Sun has described 180°, and the time taken is half that of a complete revolution. Hence, S_1 will also have described 180°, thus at apogee (July 1st), E_1 *is again 0.*

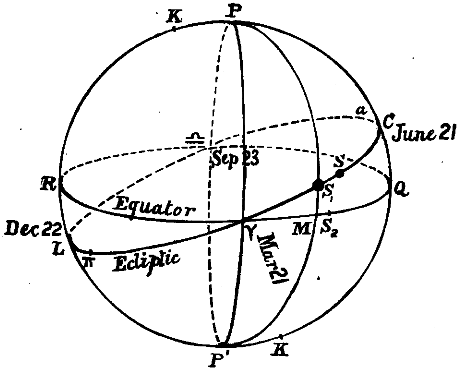


FIG. 25.

Now since S is moving most rapidly at perigee, and most slowly at apogee, S will move ahead of S_1 after perigee and S will lag behind S_1 after apogee.

Thus: From perigee to apogee E_1 *is negative,*

From apogee to perigee E_1 *is positive,*

and E_1 *vanishes twice a year, viz. at perigee and apogee.*

50. Equation of Time due to Obliquity

Let the portion of the equation of time due to the obliquity be denoted by E_2.

Take S_2 on the equator so that $\Upsilon S_2 = \Upsilon S_1$. Then S_2 will be the astronomical mean Sun. Draw PS_1M, the secondary to the equator through S_1. Then

$$\begin{aligned} E_2 &= \text{hour angle of } S_1 — \text{hour angle of } S_2 \\ &= \angle S_1PS_2 \text{ (taken positive if } S_1 \text{ is west of } S_2) \\ &= \angle \Upsilon PS_2 — \angle \Upsilon PS_1 = \Upsilon S_2 - \Upsilon M = \Upsilon S_1 - \Upsilon M, \end{aligned}$$

all angles being supposed converted into time at the rate of 15° to the hour.

At the vernal equinox,* when S_1 is at Υ, S_2 will also be at Υ; so that

$$E_2 = 0.$$

---

* The vernal and autumnal equinoxes are, strictly, the times when S, and not S_1, coincides with the equinoctial points, but, as S_1 is always near S, the distinction need not be considered here. The same remarks apply to the solstices.

Between the vernal equinox and summer solstice, the angle $♈S_1M$ will be $< 90°$, and, therefore, $< ♈MS_1$; hence, $♈M < ♈S_1$; therefore $♈M < S♈_2$; and E_2 *is positive.*

At the summer solstice, S_1 is at C, and S_2 at Q, where $♈Q = ♈C = 90°$. Hence $♈QC = 90°$; and M is also at Q; so that $E_2 = 0$.

Between the summer solstice and autumnal equinox we shall have $M♎ < S_1♎$. But $♈M♎ = ♈S_1♎ = 180°$; therefore $♈M > ♈S_1$; $♈M > ♈S_2$; and E_2 *is negative.*

At the autumnal equinox, since $♈C♎ = ♈Q♎ = 180°$, S_1, S_2 will both coincide with ♎; so that $E_2 = 0$.

In a similar manner we may show that:

From the autumnal equinox to the winter solstice, E_2 *is positive.*

At the winter solstice, $E_2 = 0$.

From the winter solstice to the vernal equinox, E_2 *is negative.*

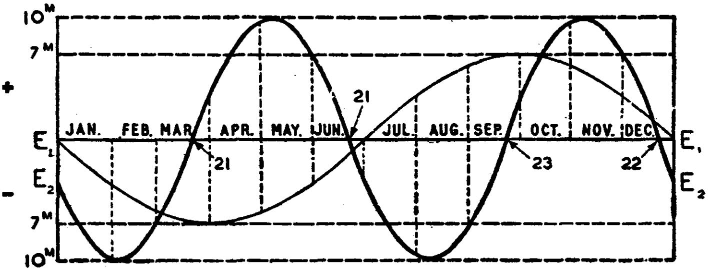


FIG. 26.

Collecting these results, we see that

(i) From equinox to solstice E_2 *is positive.*

(ii) From solstice to equinox E_2 *is negative.*

(iii) E_2 *vanishes four times a year, viz. at the equinoxes and solstices.*

51. Graphic Representation of Equation of Time

The values of the equation of time at different seasons may now be represented graphically by means of a curved line, in which the abscissa of any point represents the time of year, and the ordinate represents the corresponding value of the equation of time.

In the accompanying figure (Fig. 26) the horizontal line or axis from E_1 to E_1' represents a year, the twelve divisions representing the different months as indicated. The thin curve represents the values of E_1, the portion of the equation of time due to the unequal motion; this curve is obtained by drawing ordinates perpendicular to the horizontal axis and proportional to E_1. Where the curve is below the horizontal line E_1 is negative.

The thick curved line is drawn in a similar manner, and represents, on the same scale, the values of E_2, the equation of time due to the obliquity.

In drawing the diagrams to scale, it is necessary to know the maximum values of E_1, E_2.

We can calculate E_1 with more than sufficient accuracy by the following method. In Fig. 27 $ABA'B'$ represents the apparent orbit of the Sun around the Earth E, which is an ellipse of small eccentricity, such that EC is about $\frac{1}{60}$th of CA. H is the second focus of the ellipse (see Appendix, § 2), ACA' is the major axis and EL, CB, HK are perpendicular to AA'.

If we neglect the slight curvature of BK (which is really much less than in the figure, EC and HC being exaggerated) the triangle BKM is equal to ECM. To each add the figure $ALBME$, then the sector $ALBKME = ALBC = \frac{1}{4}$ of the whole ellipse. Hence the time of describing the arc $ALBK$ is $\frac{1}{4}$ of the periodic time; therefore an imaginary Sun whose angular velocity about S is uniform has moved through 90° from A, and is at L when the true Sun is at K.

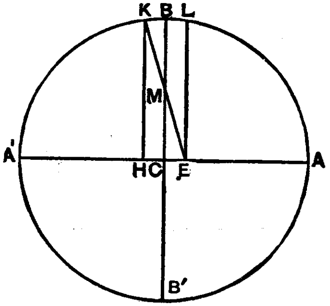


FIG. 27.

The angle KEL has KL/LE for tangent: now $KL = HE = \frac{1}{30}$ of CA; and EL is nearly equal to CA. Hence

$$\tan KEL = \tfrac{1}{30} = \cdot 0333.$$

From the tables $KEL = 1° 54'$. This is the difference of R.A. between the true Sun and the dynamical mean Sun. In time it is equivalent to 7m. 36s. or 7·6m.

Since this is the difference if R.A. at the middle of the period from perihelion to aphelion, we assume that it is the maximum difference.

E_2 can be calculated approximately by using the formulae for right-angled triangles given in Art. 10. In the spherical triangle ΥMS_1 (Fig. 25), the angle at M is a right-angle. The angle at Υ is ϵ, the obliquity of the ecliptic, 23° 27'. We have

$$\cos \epsilon = \cot \Upsilon S_1 \tan \Upsilon M.$$

Since E_2 vanishes for longitudes 0° and 90°, we may assume it to have its maximum value for a longitude of about 45°. Putting $\Upsilon S_1 = 45°$, we obtain

$$\Upsilon M = 42° 32'.$$

Hence E_2, which is measured by the difference between ΥS_1 and ΥM, has a maximum value of about $2\frac{1}{2}° =$ 10m. in time.

From the above we have then :—

The greatest value of E_1 is about 7 minutes.

„ „ „ „ E_2 „ 10 „

Hence the greatest distances of the thin and thick curves from the horizontal axis should be taken to be about 7 and 10 units of length respectively.

We may now draw the diagram representing E, the total equation of time. We have

$$E = E_1 + E_2.$$

Hence, at every point of the horizontal line we must erect an ordinate whose length is equal to the algebraic sum of the ordinates (taken with their proper sign) of the two curves which represent E_1 and E_2. The extremities of these ordinates will determine a new curve which represents E.

This curve is drawn separately in Fig. 28.

It cuts the horizontal axis in four points. At these points the ordinate vanishes, and E is zero. Hence—

The Equation of Time vanishes four times a year.

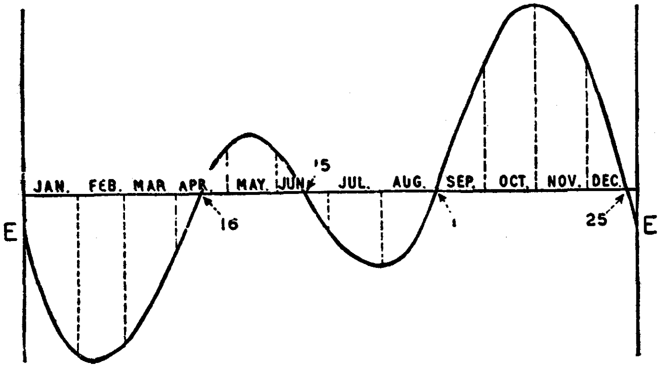


FIG. 28.

52. Miscellaneous Remarks

From Fig. 28 it will be seen that the largest fluctuations in the equation of time occur in the autumn and winter months; during spring and summer they are much smaller.

The days on which the equation of time vanishes are about April 16th, June 14th, September 1st, and December 25th.

Between these days E increases numerically, and then decreases, attaining a positive or negative value at some intermediate time. These maxima are:

— 14m. 21s. on or about Feb. 12th; + 3m. 45s. on or about May 15th.
— 6m. 22s. on or about July 27th; — 16m. 22s. on or about Nov. 3rd.

53. Inequality in the Lengths of Morning and Afternoon

If we neglect the small change in the Sun's declination during the day, the interval from sunrise to apparent noon is equal to the interval from apparent noon to sunset. But by morning and afternoon are meant the intervals between sunrise and *mean* noon, and between *mean* noon and sunset respectively. Hence, unless mean and apparent noon coincide, *i.e.* unless the equation of time vanishes, the morning and afternoon will not be equal in length.

Let r, s be the mean times of sunrise and sunset, E the equation of time. Then:—

$$12\text{h.} - r = \text{interval from sunrise to mean noon.}$$

But mean noon occurs later than apparent noon by E; thus:—

$$12\text{h.} - r - E = \text{interval from sunrise to apparent noon.}$$

Similarly, $s + E =$ interval from apparent noon to sunset; therefore:—

$$12\text{h.} - r - E = s + E,$$

or

$$r + s = 12\text{h.} - 2E,$$

so that *the sum of the times of sunrise and sunset is less than* 12 *hours by twice the equation of time.*

The length of the morning is 12h. — r, and that of the afternoon is s. Now the last relation gives

$$2E = (12 - r) - s$$

or 2 (*equation of time*) = (*length of morning*) — (*length of afternoon*).

About the shortest day (December 22nd) the curve representing the equation of time is going downwards, hence E is decreasing. But the length of day is changing very slowly (because it is a minimum), hence, for a few days, the half length, $s + E$, may be regarded as constant. Hence, s must increase, and, therefore, the mean time of sunset is later each day. Similarly, it may be shown that sunrise is also later. The afternoons, therefore, begin to lengthen, while the mornings continue to shorten.

Similarly, about June 21st, the afternoons continue to lengthen after the longest day, although the mornings are already shortening.

Example.—*On Nov. 1, the sundial is 16m. 20s. before the clock. Given that the Sun rose at 6h. 54m., find the time of sunset.*

Time from sunrise to mean noon = 12h. — 6h. 54m. = 5h. 6m.
Time from apparent noon to mean noon = 0h. 16m. 20s.
So that time from sunrise to apparent noon = 4h. 49m. 40s.
And time from apparent noon to sunset = 4h. 49m. 40s.
And time from mean noon to sunset = 4h. 49m. 40s. — 16m. 20s.
= 4h. 33m. 20s.

Hence, the time of sunset was 4h. 33m. correct to the nearest minute.

II.—THE SUNDIAL

54. The Appliance

The *Sundial* consists essentially of a rod or flat blade, called a *gnomon* or *style* (*OA*, Fig. 29), which is fixed with its edge parallel to the Earth's axis, and therefore pointing in the direction of the celestial pole. The shadow from *OA* is thrown on the dial-plate, which is usually either horizontal or on a wall facing south. The direction of the edge of the shadow determines the hour angle of the Sun, and therefore the apparent time.

The plane through *OA*, the edge of the style, and through the edge of the shadow, evidently passes through the Sun ; also it passes through the celestial pole, therefore it will meet the celestial sphere in the Sun's hour or declination circle.

Let *OA*xii be the meridian plane, which is the plane of the shadow at apparent noon, and whose position is supposed known. Then, in order to graduate the plate for the times 1, 2, 3... o'clock, it is only necessary to determine the positions of the planes *OA*i, *OA*ii, *OA*iii, etc., which make angles of 15°, 30°, 45°, etc., with the meridian plane. Since the Sun's hour angle increases 15° per hour, these planes will be the planes bounding the shadow at 1, 2, 3... o'clock respectively.

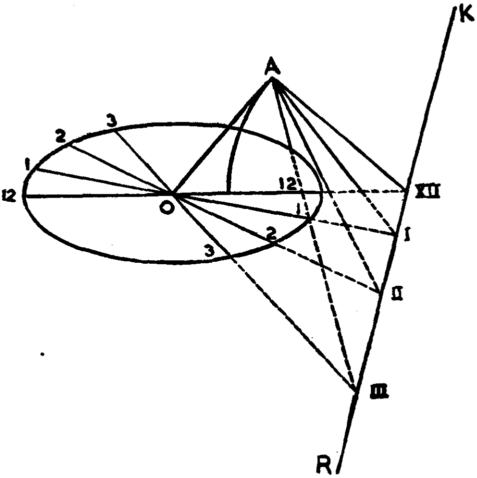


FIG. 29.

If we join the points *O*i, *O*ii, *O*iii, etc., these will be the corresponding lines of shadow in the plane of the gnomon, and will meet the circumference of the dial-plate (which is usually circular) at the required points of graduation 1, 2, 3, etc.

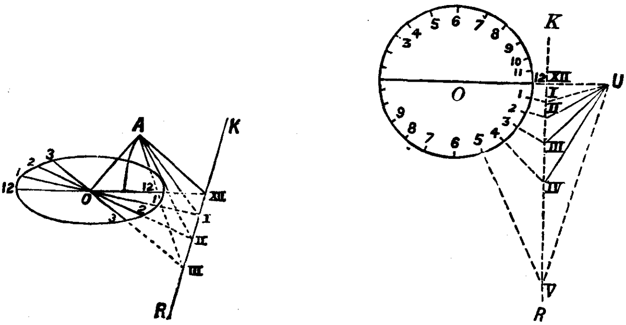
55. Geometrical Method of Graduating the Dial-plate

To find the planes *OA*i, *OA*ii, *OA*iii, etc., suppose a plane *AKR* drawn through *A* perpendicular to *OA*, meeting the plane of the dial-plate in *KR* and the meridian plane in *A*xii. If, in this plane, we take the angles xii*A*i, i*A*ii, ii*A*iii, etc., each = 15°, the points i, ii, iii...., etc., will evidently determine the directions of the shadow at 1, 2, 3,... o'clock respectively.

But in practice it is much more convenient to perform the construction *in the plane of the dial itself*. Imagine the plane AKR of Fig. 30 (*a*) turned about the line KR till it is brought into the plane of the dial, the point A of the plane being brought to U [Fig. 30 (*b*)]. Then, by making the angles xii$U$i, iUii, ii$U$iii, etc., each $= 15°$, we shall obtain the same series of points i, ii, iii as before.

If the dial-plate is *horizontal*, and ϕ is the latitude of the place (xiiOA), we have evidently therefore the following construction :—

On the meridian line, measure Oxii $= OA \sec \phi$, and xii$U =$ xi$A$ $= O$xii $\sin \phi$. Draw K xii R perpendicular to OU. Make the angles xii$U$i, iUii, ii$U$iii, etc., each $= 15°$, taking i, ii, iii, etc., on KR. Join Oi, Oii, Oiii, etc., and let the joining lines meet the circumference of the dial in 1, 2, 3, etc. These will be the required points of graduation for 1, 2, 3,...o'clock respectively.



(*a*) FIG. 30. (*b*)

III.—APPARENT AND MEAN SIDEREAL TIME

56. Apparent Sidereal Time

In Art. 28, we have mentioned that the position of the first point of Aries is not fixed, but that it has a slow retrograde motion along the ecliptic, amounting to about 50″ per year. This motion of the first point of Aries is caused by the phenomenon of *precession*, which is explained in Chapter XVIII, Sect. III. It is sufficient at this stage to state that it is due to the Earth being a spheroid and not a true sphere. The Earth is flattened towards the poles and bulges slightly along the equator; the gravitational attractions of the Sun and the Moon on the equational bulges tend to tilt the axis of the Earth and give rise, as we shall see, to the precession of the equinoxes—the first points of Aries and Libra.

A complication is introduced by the fact that the precession of the equinoxes is not uniform. The Moon's orbital plane does not coincide with the ecliptic and is not constant with respect to the background of the stars. In consequence the action of the Moon, which contributes to the precession is not uniform. The variable part of the Moon's action, together with a similar but smaller variable part of the Sun's action, give rise to irregularities in the precession of the equinoxes, which are known as *nutation.* A fuller explanation of nutation is given in Chapter XVIII, Art. 469.

Now since the right ascension of a celestial body is equal to the sidereal time of its transit or meridian passage (Art. 29) and since mean noon is, by definition, the instant of transit of the mean Sun (Art. 47), it follows that *the sidereal time at mean noon is equal to the right ascension of the mean Sun.* The right ascension of the mean Sun is measured from the true equinox; if the equinox were fixed or if it had a uniform motion, the right ascension of the mean Sun would increase at a uniform rate. But, on account of the irregularity in the motion of the equinox, known as nutation, the right ascension of the mean Sun does not increase quite uniformly. Hence also the sidereal time at mean noon does not increase uniformly. It follows that the sidereal days, which are measured by transits of the true equinox, are not of equal length.

Sidereal time, as defined in Art. 28, such that 0h. 0m. 0s. is the instant of transit of the true vernal equinox or first point of Aries is therefore not uniform. In this respect it is analogous to apparent or true solar time. For this reason the sidereal time so defined is termed *apparent sidereal time or true sidereal time.*

The time that is determined from observations of the stars is apparent sidereal time, just as the time that is given by observations of the actual Sun is apparent solar time.

57. Mean Sidereal Time

We imagine a *mean equinox* which has a uniform motion along the equator and which is so chosen that the extreme irregularities in position of the true equinox with respect to the mean equinox are of equal amount in both directions. The right ascension of the mean equinox measured from the true equinox is termed *nutation in right ascension.* The extreme values of the nutation in R.A. are $\pm 1^{s}{\cdot}2$.

The right ascension of the mean sun when measured from the mean equinox will increase uniformly and sidereal days, measured by transits of the mean equinox, will be exactly equal in length. The time kept by a perfect clock, which always reads 0h. 0m. 0s. at the instant of transit of the mean equinox, is termed *mean sidereal time.* It is analogous to mean solar time.

It will be noted that the extreme differences between mean sidereal time and apparent sidereal time ($\pm 1^s{\cdot}2$) are very much smaller than the extreme differences between mean solar time and apparent solar time. The introduction of clocks led to the necessity for mean solar time, for it would be an extremely bad clock that gave a time as non-uniform as apparent solar time. The improvement in the performance of precision clocks, used as standards of time in observatories, has reached a stage when a good clock will give a time that is more uniform than apparent sidereal time. The introduction of mean sidereal time has thus become necessary.

From the definition of the nutation in R.A., it follows that the right ascension of a star measured from the true equinox is equal to the sum of its right ascension measured from the mean equinox and the nutation in R.A. But the right ascension of a celestial body is equal to the sidereal time of its transit. It follows that :—

$$\textit{Apparent sidereal time} = \textit{Mean sidereal time} + \textit{nutation}$$

or $$\textit{Mean sidereal time} = \textit{Apparent sidereal time} - \textit{nutation.}$$

comparing with

$$\text{Apparent solar time} = \text{Mean solar time} + \text{equation of time.}$$

We see that the nutation in R.A. is analogous to the equation of time. The analogy between solar and sidereal times can be expressed thus :—

	SOLAR	SIDEREAL
Observations determine	Apparent solar time	Apparent sidereal time.
Clocks keep	Mean ,, ,,	Mean ,, ,,
The difference is	Equation of time	Nutation in R.A.

IV.—COMPARISON OF MEAN AND SIDEREAL TIMES

58. Relation between Units

One of the most important problems in practical astronomy is to find the sidereal time at any given instant of mean solar time, and conversely, to find the mean time at any given instant of sidereal time. Before doing this it is necessary to compare the lengths of the mean and sidereal days.

The length of the tropical year, the period between two successive vernal equinoxes on which the return of the seasons depends, is about

365¼ mean solar days. In this period both the true and mean Sun describe one complete revolution, or 360° from west to east relative to ♈ ; or, what is the same thing, ♈ describes one revolution from east to west relative to the mean Sun. But the mean Sun performs 365¼ revolutions from east to west relative to the meridian at any place. Therefore ♈ performs one more revolution, *i.e.* 366¼ revolutions, relative to the meridian.

Now, a sidereal day and a mean solar day have been defined (Arts. 28, 47) as the periods of revolution of ♈ and of the mean Sun relative to the meridian ; so that

$$365\tfrac{1}{4}\ \textit{mean solar days} = 366\tfrac{1}{4}\ \textit{sidereal days}.$$

From this relation we have :—

$$\begin{aligned}
\text{One mean solar day} &= \left(1 + \frac{1}{365\frac{1}{4}}\right) \text{ sidereal days} \\
&= (1 + \cdot002738) \text{ sidereal days} \\
&= 24\text{h. } 3\text{m. } 56{\cdot}5\text{s. sidereal time} \\
&= 1 \text{ sidereal day} + 4\text{m.} - 4\text{s. nearly;} \\
\text{or one mean solar hour} &= 1\text{h.} + 10\text{s.} - \tfrac{1}{6}\text{s. sidereal time,} \\
\text{and 6m. of mean solar time} &= 6\text{m.} + 1\text{s. sidereal time nearly.}
\end{aligned}$$

In like manner we have

$$\begin{aligned}
\text{One sidereal day} &= \left(1 - \frac{1}{366\frac{1}{4}}\right) \text{ mean solar days} \\
&= (1 - \cdot002730) \text{ mean days} \\
&= 23\text{h. } 56\text{m. } 4{\cdot}1\text{s. mean time} \\
&= 1 \text{ mean day} - 4\text{m.} + 4\text{s. nearly;} \\
\text{or one sidereal hour} &= 1\text{h.} - 10\text{s.} + \tfrac{1}{6}\text{s. of mean time,} \\
\text{and 6m. sidereal time} &= 6\text{m.} - 1\text{s. mean solar time nearly.}
\end{aligned}$$

59. Approximate Rules

From the results of the last paragraph we have the following approximate rules :—

(i) *To reduce a given interval of mean time to sidereal time, add* 10s. for every hour, and 1s. for every 6m. in the given interval. For every minute so added, *subtract* 1s.

(ii) *To reduce a given interval of sidereal time to mean time, subtract* 10s. for every hour, and 1s. for every 6m. in the given interval. Then *add* 1s. for every minute so subtracted.

Examples.—*Express in sidereal time an interval of 13h. 23m. 25s. mean time.*

The calculation stands as follows :—		H.	M.	S.
Mean solar interval	=	13	23	25
Add 10s. per hour on 13h.			2	10
„ 1s. per 6m. on 23m.				4
		13	25	39
Subtract 1s. per 1m. on 2m. 13·8s.				2
Required sidereal interval	=	13	25	37

2.—*Find the mean solar interval corresponding to 14h. 45m. 53s. of sidereal time.*

The calculation stands as follows :—		H.	M.	S.
Given sidereal interval	=	14	45	53
Subtract 10s. per hour on 14h. = 2m. 20s. „ 1s. per 6m. on 46m. (nearly) = 8s.			2	28
		14	43	25
Add 1s. per 1m. on 2m. 28s.				3
Required interval of mean time	=	14	43	28

If accuracy to within a few seconds is not required, the second correction of 1s. per 1m. may be omitted. On the other hand, if the interval consists of a considerable number of *days*, or if accuracy to the decimal of a second is needed, the results found by the rules will no longer be correct. We must, instead, add 1/365¼ of the given mean solar interval to get the sidereal interval, or subtract 1/366¼ of the given sidereal to get the mean solar interval.

The *Nautical Almanac* contains tables for converting intervals of mean solar time into equivalent intervals of mean sidereal time and conversely. These enable the conversions to be done without any calculation.

60. To find the Sidereal Time at a given instant of Mean Solar Time on a given date at Greenwich

The *Nautical Almanac*[*] gives under the heading "Sidereal Time" the sidereal time of mean midnight at Greenwich on every day of the year. Before 1931 it was given for mean noon.

Now the given mean time represents the number of hours, minutes, and seconds which have elapsed since mean midnight,[†] expressed in mean time. Convert this interval into sidereal time; we then have the sidereal interval which has elapsed since mean midnight. Add this to the sidereal time of mean midnight; the result is the sidereal time required.

---

[*] Or *Whitaker's Almanack*, which may be used if the *Nautical Almanac* is not at hand.

[†] Since the beginning of the year 1925, mean time has been reckoned from midnight, not from noon.

Thus, let m be the mean time at the given instant, measured from the preceding mean midnight, s_0 the sidereal time of mean midnight from the *Nautical Almanac*, and let $k = 1/365\frac{1}{4}$; so that $1 + k$ is the ratio of a mean solar unit to the corresponding sidereal unit.

Then, from mean midnight to given instant :—

Interval in mean time $= m$;
So that interval in sidereal time $= m + km$.
But, at mean midnight, sidereal time $= s_0$; therefore, at given instant :—

$$\textit{required sidereal time, } s = s_0 + m + km.$$

If the result be greater than 24h., we must subtract 24h., for times are always measured from 0h. up to 24h.

Example.—*Find the sidereal time corresponding to mean time 8h. 15m. 40s.* A.M. *on Dec. 20th,* 1940, *given that the sidereal time of mean midnight was 5h. 53m. 42s.*

From mean midnight to the given instant, the interval in mean time is

8h. 15m. 40s.

Converting this interval to sidereal time, by the method of Art. 59, we have

Mean solar interval	=	8h. 15m. 40s.
Add 10s. per hour on 8h.	=	1m. 20s.
Add 1s. per 6m. on 15m. 40s.	=	3s.
		8h. 17m. 3s.
Subtract 1s. per 1m. on 1m. 23s.		1s.
Thus, sidereal interval since mean midnight	=	8h. 17m. 2s.
But sidereal time of mean midnight	=	5h. 53m. 42s.
Sidereal time at instant required	=	14h. 10m. 44s.

61. To find the Sidereal Time at a given instant of Mean Solar Time on a given date at Greenwich (alternative method)

Under the heading " Transit of First Point of Aries," the *Nautical Almanac* gives for each day the mean time when ♈ is on the meridian, *i.e.* the mean time corresponding to sidereal time 0h. 0m. 0s This is called *sidereal noon.*

Let the given mean solar time $= m$
Let the mean time of preceding sidereal noon (*i.e.* 0h.) $= m_0$.

Then from sidereal noon to given instant :—

Interval in mean time $= m - m_0$;
so that interval in sidereal time $= (m - m_0) + k\,(m - m_0)$
and *required sidereal time* $s = (m - m_0) + k\,(m - m_0)$

Example.—*Find the sidereal time corresponding to mean time 8h. 15m. 40s. on Dec. 20th, 1940, given that mean time at 0h. sidereal on Dec. 20th was 18h. 3m. 20s.*

As the mean time instant for which sidereal time is required is earlier than the mean time at 0h. sidereal, we have interval in mean time = — 9h. 47m. 40s.

Converting this interval to sidereal time, by the method of Art. 59, we find interval in sidereal time = — 9h. 49m. 16s.

As this interval is measured from 0h. = 24h. 0m. 0s., the required sidereal time is 14h. 10m. 44s., as before.

62. To find the Mean Solar Time corresponding to a given instant of Sidereal Time at Greenwich

Subtract the sideral time of mean midnight from the given sidereal time ; this gives the interval which has elapsed since mean midnight, expressed in sidereal time. Convert this interval into mean time ; the result is the mean time required.

Let $k' = 1/366\frac{1}{4}$; so that $1 - k'$ is the ratio of a sidereal to a mean solar unit.

Let the given sidereal time $= s$, and let the sidereal time of the preceding mean midnight $= s_0$;

Then from mean midnight to given instant :—

Interval in sidereal time $= s - s_0$;

so that interval in mean time $= (s - s_0) - k'(s - s_0)$.

and *required mean time m* $= (s - s_0) - k'(s - s_0)$.

If s be less than s_0, we must add 24h. to s in order that the times s, s_0 may be reckoned from the *same* transit of ♈.

Example.—*Find the solar time corresponding to 16h. 6m. 57s. sidereal time on May 5th, 1940, sidereal time at mean midnight being 14h. 50m. 51s.*

Sidereal interval since mean midnight
= 16h. 5m. 57s. — 14h. 50m. 51s. = 1h. 16m. 6s.

Mean solar interval (Art. 59)
= 1h. 16m. 6s. — 10s. — 3s. = 1h. 15m. 53s.

Hence 1h. 15m. 53s. is the mean time. The sidereal time was also 16h. 6m. 57s. a sidereal day or 23h. 56m. 4s. previously, *i.e.* 1h. 19m. 49s. a.m. on the morning of May 4th.

63. To find the Mean Time corresponding to a given instant of Sidereal Time at Greenwich (alternative method)

The *Nautical Almanac* also contains, under the heading " Transit of First Point of Aries " the mean time when ♈ is on the meridian, or when the sidereal time is 0h. 0m. 0s., *i.e.* the mean time of sidereal noon. Let this be m_0, and let s be the given sidereal time, k' the factor $1/366\frac{1}{4}$ as before. Then

From sidereal noon to given instant, sidereal interval $= s$;
and from sidereal noon to given instant, mean solar ,, $= s - k's$
But, at sidereal noon. mean time $= m_0$;

Therefore, at given instant :—

The required mean time $= m_0 + s - k's$.

Example.—*Find the solar time corresponding to* 16*h.* 6*m.* 57*s. sidereal time on May* 5*th,* 1940. *Mean time at sidereal noon on May* 4*th is* 9*h.* 11*m.* 35*s. and on May* 5*th,* 9*h.* 7*m.* 39*s.*

From sidereal noon on May 4th to given instant
= 16h. 6m. 57s. sidereal time
= 16h. 4m. 18s. mean solar time.
Hence required mean time = 9h. 11m. 35s. + 16h. 4m. 18s.
= 25h. 15m. 53s. on May 4th.
= 1h. 15m. 53s. on May 5th.

or, alternatively,

From sidereal noon on May 5th to given instant
= — 7h. 53m. 3s. sidereal time
= — 7h. 51m. 46s. mean solar time.
Hence required mean time = 9h. 7m. 39s. — 7h. 51m. 46s. on May 5th.
= 1h. 15m. 53s. on May 5th.

64. Effect of Difference of Longitude

If A, B (Fig. 31) be two places whose difference of longitude is $L^\circ : \frac{1}{15}L$ hours, the transits of stars at A and B will take place when the meridian planes PAP' and PBP' (which are evidently also the planes of the celestial meridians of A, B respectively), pass through the direction of the star. Hence the transits will occur $\frac{1}{15}L$ hours earlier at B than they will occur at A.

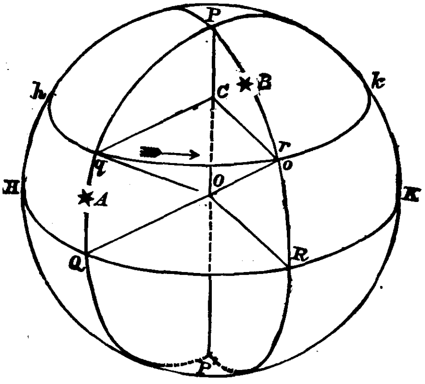


FIG. 31.

Now an observer at B will set his sidereal clock to indicate 0h. 0m. 0s. when ♈ crosses the meridian of B. When ♈ transits at A, the clock at B will mark $\frac{1}{15}L$ h., but an observer at A will *then* set his clock at 0h. 0m. 0s. Hence, if the two clocks be brought together and compared, the clock from B will be $\frac{1}{15}L$ h. faster than the clock from A. This fact may be expressed briefly by saying that the "*local*" *sidereal time* at B is $\frac{1}{15}L$ h. faster than the local sidereal time at A.

Since the **Earth** makes one revolution relative to the Sun in a *solar* day, in like manner the *local solar time* at B will be $\frac{1}{15}L$ *solar* hours faster than the local solar time at A.

Therefore, whether the local times be sidereal or solar, we have :—

Longitude of A west of B = long. of B east of A = 15 {(*local time at B*) — (*local time at A*)}.

In particular, *Longitude west of Greenwich*

$$= 15 \{(\textit{Greenwich time}) - (\textit{local time})\}$$
$$\mp 15 \ (\textit{Greenwich time of local midnight}).$$

65. To find the Sidereal Time from the Mean Solar, or the Mean Time from the Sidereal, in any given Longitude

If the longitude is not that of Greenwich, the above methods will require a slight modification, because the sidereal time of mean midnight, and mean time of transit of ♈ are tabulated for Greenwich.

In such cases, the safest plan is as follows :—Find the Greenwich time corresponding to the given local time (Art. 64). Convert this Greenwich time from mean to sidereal, or sidereal to mean, as the case may be, and then find the corresponding local time again.

Let the longitude be $L°$ west of Greenwich (L being negative if the longitude is east),

Let m_1 be the mean and s_1 the sidereal local time, and m, s the corresponding times at Greenwich,

Let k, k', m_0, s_0 have the same meanings as in Arts. 58–63.

By Art. 64 we have, whether the times be local or sidereal, (Greenwich time) — (local time in long. $L°$ W.) $= \frac{1}{15} L$ h. $= 4L$ m.

Therefore $$s - s_1 = \tfrac{1}{15} L = m - m_1.$$

(i) If m_1 is given and s_1 is required, we have (in hours),

$$m = m_1 + \tfrac{1}{15} L.$$

By Arts. 60, 61, $s = s_0 + m + km$ or $= (m - m_0) + k(m - m_0)$;

$$\begin{aligned} \textit{i.e.} \qquad s &= s_0 + m_1 + km_1 + \tfrac{1}{15} L + \tfrac{1}{15} kL \\ &= (m_1 - m_0) + k(m_1 - m_0) + \tfrac{1}{15} L + \tfrac{1}{15} kL, \end{aligned}$$

$$\begin{aligned} \text{so that} \qquad s_1 = s - \tfrac{1}{15} L &= (m_1 - m_0) + k(m_1 - m_0) + \tfrac{1}{15} kL \\ &= s_0 + m_1 + km_1 + \tfrac{1}{15} kL. \end{aligned}$$

(ii) If s_1 is given and m_1 is required, we have

$$s = s_1 + \tfrac{1}{15} L.$$

By Arts. 62, 63, $m = (s - s_0) - k'(s - s_0)$ or $= m_0 + s - k's$;

$$\begin{aligned} \textit{i.e.} \qquad m &= (s_1 - s_0) - k'(s_1 - s_0) + \tfrac{1}{15}L - \tfrac{1}{15}k'L \\ &= m_0 + s_1 - k's_1 + \tfrac{1}{15}L - \tfrac{1}{15}k'L; \end{aligned}$$

$$\begin{aligned} \text{so that} \qquad m_1 = m - \tfrac{1}{15} L &= (s_1 - s_0) - k'(s_1 - s_0) - \tfrac{1}{15}k'L \\ &= m_0 + s_1 - k's_1 - \tfrac{1}{15} k'L. \end{aligned}$$

Example.—*Find the solar time when the local sidereal time is* 5*h.* 17*m.* 32*s. on March* 21*st, the place of observation being Moscow* (*long.* 37° 34′ 15″ *E.*); *given that sidereal time of mean midnight was* 11*h.* 50*m.* 12*s. at Greenwich.*

Reduced to time 37° 34′ 15″ is 2h. 30m. 17s.

Therefore Greenwich sidereal time at instant required
= 5h. 17m. 32s. — 2h. 30m. 17s. = 2h. 47m. 15s.

Sidereal interval since Greenwich midnight
= 2h. 47m. 15s. + 24h. — 11h. 50m. 12s. = 14h. 57m. 3s.

So that Greenwich mean time = 14h. 57m. 3s. — 2m. 27s. = 14h. 54m. 36s.

and Moscow mean time = 14h. 54m. 36s. + 2h. 30m. 17s. = 17h. 24m. 53s.

66. Practical Applications

In Art. 42 we showed how to determine roughly the time of night at which a given star would transit on a given day of the year. With the introduction of mean time, in the present chapter, we are in a position to obtain a more accurate solution of the problem.

For the R.A. of any star (expressed in time) is its sidereal time of transit. If this be given, we only have to find the corresponding mean time; this will be the required time of transit, as indicated by an ordinary clock.

In the calculations required in converting the time from one measure to the other, it is advisable *not* to quote the formulae of Arts. 60–65, but to go through the various steps one by one.

If neither the sidereal time of mean midnight nor the mean time of sidereal noon is given, we must fall back on the rough method of Art. 39.

The disadvantages of using local time are obviated in Great Britain by the universal use of "Greenwich Mean Time."

67. Worked Examples

Examples.—1. *As an example of the method of reckoning time before* 1925, *find the solar time at* 5*h.* 29*m.* 28*s. sidereal time on July* 1*st,* 1891; *mean time of sidereal noon being* 17*h.* 20*m.* 8*s.*

Sidereal interval from sidereal noon to the given instant = 5h. 29m. 28s.

Mean solar interval = 5h. 29m. 28s. — 50s. — 5s. + 1s. = 5h. 28m. 34s.,

i.e. Mean solar time = 5h. 28m. 34s. + 17h. 20m. 8s. = 22h. 48m. 42s. or, 10h. 48m. 42s. A.M., July 2nd.

It was also 5h. 29m. 28s., a sidereal day or 23h. 56m. 4s. previously, *i.e.* 10h. 52m. 38s. A.M., July 1st.

2. *To find the mean time of transit of Aldebaran at Greenwich on December* 12th, 1940. *Given* :—

		H.	M.	S.
R.A. of *Aldebaran*	=	4	32	33
Sidereal time of midnight, December 12th, 1931	=	5	22	10

Since the star's R.A. is less than the sidereal time of midnight, we must increase the former by 24h., in order that both may be measured from the same "sidereal noon."

		H.	M.	S.
Sidereal time of transit + 24h.	=	28	32	33
Subtract „ „ midnight	=	5	22	10
Sidereal interval from midnight to transit	=	23	10	23
To convert into mean solar units, subtract		0	3	48
Mean Solar interval from midnight to transit	=	23	6	35

So that *Aldebaran* transits at 23h. 6m. 35s. mean time.

3. *To find the (local) sidereal time at New York at 9h. 25m. 31s. (local mean time) on the morning of September 1st,* 1940.

Longitude of New York = 74° W.
Sidereal time of mean midnight at Greenwich, Sept. 1st = 22h. 40m. 1s.

		H.	M.	S.
Local mean time at New York		9	25	31
Add for 74° west longitude reduced to time	=	4	56	0
Therefore Greenwich mean time is, September 1st ...		14	21	31
To convert this interval to sidereal units, add		0	2	22
Whence sidereal time elapsed since Greenwich midnight ...	=	14	23	53
But at Greenwich midnight sidereal time (by data) ...	=	22	40	1
Sidereal time at Greenwich is therefore		13	3	54
Subtract for 74° west longitude		4	56	0

Sidereal time at New York = 8h. 7m. 54s.

4. *To find the Paris mean time of transit of Regulus at Nice on December 26th,* 1940.

		H.	M.	S.
Longitude at Paris = 2° 21′ E. R.A. of *Regulus*	=	10	5	15
„ Nice = 7° 18′ E.				
Sidereal time at Greenwich midnight	=	6	17	21
Here local sidereal time of transit at Nice	=	10	5	15
Subtract *east* longitude of Nice, 7° 18′, in time		0	29	12
Therefore, Greenwich sidereal time of transit at Nice ...	=	9	36	3
Subtract Greenwich sidereal time at midnight		6	17	21
Sidereal interval since Greenwich midnight	=	3	18	42
To convert to mean solar units, subtract	=	0	0	33
Greenwich mean time is therefore	=	3	18	9
Add *east* longitude of Paris, expressed in time	=	0	9	24
Paris mean time of transit	=	3	27	33

That is, 3h. 27m. 33s. in the morning on December 26th.

5. This example is given as an example of the old method of time reckoning.

Find the R.A. of the Sun at true noon on October 8th, 1891, *given that the equation of time for that day is* + 12*m.* 24*s., and that the sidereal time of mean noon on March* 21*st was* 23*h.* 54*m.* 52*s.*

Mean solar interval from mean noon March 21st to mean noon Oct. 8th = 201 days.

Mean solar interval from mean noon to apparent noon on Oct. 8th = — 12m. 24s.

Hence, interval from mean noon on March 21st to apparent noon on Oct. 8th = 201d. — 12m. 24s.

Now, in 365¼ days the mean Sun's R.A. increases 24h., and the increase takes place *uniformly*.

		H.	M.	S.
Therefore, increase in mean Sun's R.A. in 201 days = 24h. × 201 ÷ 365¼	=	13	12	27
Add mean Sun's R.A. on March 21st (= sidereal time of mean noon)	=	23	54	52
Mean Sun's R.A. at mean noon Oct. 8th	=	37	7	19
or, subtracting 24h.	=	13	7	19
Subtract change of R.A. in 12m. 24s.	=			2
Mean Sun's R.A. at apparent noon Oct. 8th	=	13	7	17
But true Sun's R.A. — mean Sun's R.A. = — equation of time	=		— 12	24
So that true Sun's R.A. at apparent noon Oct. 8th ...	=	12h.	54m.	53s.

V.—THE CIVIL DAY: TIME ZONES: STANDARD TIMES

68. The Civil Day

Up to the present we have been concerned with local mean time and local apparent time. Local mean time at any place is the moment of passage of the mean Sun across its meridian. At two places which differ in longitude, the local mean times will be different, as we have seen in Art. 64. There would obviously be great inconveniences if local times were kept throughout a country. A person travelling in an easterly or westerly direction would need to be continually adjusting his watch. It is therefore customary for a country to adopt as the legal time of the country the mean time of a standard meridian. In Great Britain the standard meridian adopted for this purpose is the meridian of Greenwich and the time kept throughout Great Britain is based on Greenwich Mean Time (G.M.T.), often denoted *Universal Time* (U.T.)

The *civil day* is the day beginning at mean midnight and ending at the following mean midnight, according to the system of time adopted in the particular country.

69. Time Zones

In the case of a ship at sea, it is convenient to keep a time that does not differ greatly from the local mean time or, as it is usually called at sea, the *ship mean time*. In the merchant and passenger services, it is usual to alter the clocks each night by a number of minutes such that, at local noon the following day, the time shown by the clocks shall be approximately 12h. The time shown by the ship's clocks is called *ship time*. In the British Navy, a system of *zone times*, each of which differs by an integral number of hours from Greenwich mean time is used. The zones are bounded by meridians of longitude at intervals of 15°, or 1 hour, apart; within each such time zone, the mean time appropriate to the central meridian is kept. Thus, in the zone between longitudes 7½° E. and 7½° W., Greenwich mean time is kept. This zone is designated zone 0. The next zone to the westward lies between longitudes 7½° W. and 22½° W (or 0h. 30m. W. and 1h. 30m. W.); the time kept is one hour slow on Greenwich Mean Time and the zone is designated zone + 1. The next zone to the westward lies between longitudes 22½° W. and 37½° W. (or 1h. 30m. W. and 2h. 30m. W.): the time kept is two hours slow on Greenwich Mean Time and the zone is designated zone + 2, and so on. Similarly, the first zone eastwards from the central zone is designated zone — 1; it is bounded by longitudes 7½° E. and 22½° E. and the time kept in it is one hour fast on Greenwich. The successive zones eastwards are designated zones — 2, — 3, — 4, ...

It should be noted that in each zone, the designation of the zone (in hours) is to be *added* to the zone time to obtain Greenwich mean time.

70. The Date Line

Proceeding eastwards from Greenwich the time in the twelfth zone (which is bounded by the meridians 172½° E. and 172½° W., or 11h. 30m. E. and 11h. 30m. W.) will be 12 hours fast on Greenwich mean time, whilst proceeding westwards the time in the same zone will be 12 hours slow on Greenwich mean time. Zones — 12 and + 12 are identical; in order to avoid confusion, the half of this zone lying in the eastern hemisphere and bounded by the meridians 11h. 30m. E. and 12h. 0m. E. is called zone — 12, whilst the other half, which lies in the western hemisphere and is bounded by the meridians 11h. 30m. W. and 12h. 0m. W. is called zone + 12. The 180th meridian from Greenwich is called the *date line.*

If, at Greenwich, it is midnight on the night of say, December 14–15, the time carried by a ship approaching at that instant the date line from the west will be noon on December 15th, but on one approaching it from the east it will be noon on December 14th. On crossing the date line, the date on the former ship will be changed to December 14th, one day being thus repeated; the date on the latter ship will be changed to December 15th, and one day will be missed out.

The actual date line is slightly different from the 180th meridian from Greenwich, where this runs over or is adjacent to land areas.

The date line is slightly distorted in order to avoid inconvenient differences of date in contiguous portions of eastern Siberia, or in the groups of Pacific Islands which form geographical units.

71. Standard Times

Each country formerly adopted its own zero or prime meridian from which longitudes were measured. This resulted in many inconveniences; for instance, the longitude of the same place was different on maps prepared in different countries. In 1884 a Prime Meridian Conference met in Washington and it was recommended that the meridian through Greenwich should be universally adopted as the prime meridian from which all longitudes should be measured. The Conference further recommended that a system of standard times, differing by an integral number of hours from Greenwich mean time, should be introduced and that each country should use the particular standard time that was most appropriate to its longitude. This recommendation was not at once implemented by all countries, but most countries have now adopted standard times based on Greenwich mean time, with the modification that some of the standard times in use differ from Greenwich time by an integral number of hours + half an hour.

In countries that extend over a wide range of longitude, several standard times may be used. Thus, for instance, in the United States, times 4h, 5h, 6h, 7h and 8h. slow on Greenwich are used, according to the longitude; these times are known respectively as Atlantic, Eastern, Central, Mountain and Pacific times. A full list of standard times in use is given in the *Nautical Almanac*.

72. The Greenwich Date

In the *Nautical Almanac* the data for the Sun and planets are given for 0h. G.M.T. for each day of the year; the data for the Moon are given for each hour of G.M.T. throughout the year. When data have to be extracted from the *Almanac* it is important that the correct date should be used. Thus, suppose in zone — 8, an observation of the Moon has been secured at zone time 6h. 14m. on April 1st. The G.M.T. of the observation is obtained as follows :—

April 1d. 6h. 14m. =	March 32d. 6h. 14m.
Add zone — 8	— 8 0
	March 31d. 22h. 14m.

The position of the Moon at the time of observation will be obtained by interpolating between the positions for G.M.T. 22h. and 23h. on March 31st.

The Greenwich Date and Time is always obtained by adding the zone number in hours to the Zone Date and Time.

73. Summer Time

In many countries, it has become customary to advance the clocks during a certain portion of the year (around the summer months) by an amount that is usually one hour, but which in some cases is 30 minutes. This time, as so altered, is termed *Summer Time.*

Thus, in Great Britain, the clocks are normally advanced by one hour on a date in April and put back by one hour on a date in October. To cause the minimum of inconvenience, the change is made at 2 a.m. on a Sunday morning. During the period of summer time, this time becomes the legal time of the country.

British summer time, being one hour fast on G.M.T. is simply the time for the Zone — 1.

As a war measure, British summer time (B.S.T.) was extended to apply to the whole of the year in 1940 and subsequently. The clocks were advanced an additional hour between a date in May and a date in August, fixed by Order in Council, in 1941 and subsequent years. This time, known as Double Summer Time, is the zone time for Zone — 2.

VI.—UNITS OF TIME—THE CALENDAR

74. Tropical, Sidereal, and Anomalistic Years

Hitherto we have defined a year as the period of a complete revolution of the Sun in the ecliptic. In order to give a more accurate definition, however, it is necessary to specify the starting point from which the revolution is measured. We are thus led to three different kinds of years.

A *Tropical Year* is the period between two successive vernal equinoxes, or the time taken by the Sun to perform a complete revolution relative to the first point of Aries. This year is the natural unit marked out for the use of man, because the seasons recur after the interval of a tropical year. If, therefore, the civil year is adjusted to agree in the mean with the tropical year in length, the seasons will always recur at about the same dates in each year.

The length of the tropical year in mean solar time is very approximately 365d. 5h. 48m. 45·98s. at the present time. For many purposes it may be taken as 365¼ days.

A *Sidereal Year* is the period of a complete revolution of the Sun, starting from and returning to the secondary to the ecliptic through some fixed star. Thus, after a sidereal year the Sun will have returned to exactly the same position among the constellations.

If ♈ were a fixed point among the stars, the sidereal and tropical years would be exactly of the same length. But ♈ has an annual retrograde motion of 50·26″ among the stars. Consequently, the tropical year is rather shorter than the sidereal.

An *Anomalistic Year* is the period of the Sun's revolution relative to the apse line, or major axis of its orbit—in other words, the interval between successive passages through perigee or apogee.

Owing to a progressive motion of the apse of line, the positions of perigee and apogee move forward in the ecliptic at the rate of 11·25″ per annum (Art. 137). Hence the anomalistic year is rather longer than the sidereal.

It is easy to compare the lengths of the sidereal, tropical, and anomalistic years. For, relative to the stars—

In the sidereal year the Sun describes	360°,
In the tropical year it describes	360° — 50·26″,
In the anomalistic year it describes	360° + 11·25″;

whence :—(*Sidereal year*) : (*Tropical year*) : *Anomalistic year*)

$$= 360° : (360° - 50{\cdot}26'') : (360 + 11{\cdot}25'').$$

From this proportion it will be found that the sidereal year is about 20m. longer than the tropical, and 4½m. shorter than the anomalistic.

75. The Civil Year

For ordinary purposes, it is important that the year shall possess the following qualifications :—

1st. It must contain an exact (not a fractional) number of days.

2nd. It must mark the recurrence of the seasons.

Now the tropical year marks the recurrence of the seasons, but its length is not an exact number of days, being, as we have seen, about 365d. 5h. 48m. 45·98s. To obviate this disadvantage, the *civil year* has been introduced. Its length is sometimes 365, and sometimes 366 days, but its *average* length is almost exactly equal to that of the tropical year.

Taking an ordinary civil year as 365d., four such years will be less than four tropical years by 23h. 15m. 3·92s., or nearly a day. To compensate for this difference, every fourth civil year is made to contain 366 days, instead of 365, and is called a *leap year*. For convenience, the leap years are chosen to be those years the number of which is divisible by 4, such as 1892, 1896.

The introduction of a leap year once in every four years was due to Julius Caesar, and the calendar constructed on this principle is called the *Julian Calendar*.

Now three ordinary years and one leap year exceed four tropical years by 24h.—23h. 15m. 3·92s., *i.e.* 44m. 56·08s. Thus, 400 years of the Julian Calendar will exceed 400 tropical years by

(44m. 56·08s.) × 100, *i.e.* by 3d. 2h. 53m. 30s.

To compensate for this difference the calendar now in use was introduced by Pope Gregory XIII in 1582. In the *Gregorian Calendar* it is arranged that three days shall be omitted in every 400 years. This correction is called the *Gregorian correction* and is made as follows: Every year whose number is a multiple of 100 is taken to be an ordinary year of 365 days, instead of being a leap year of 366, unless the number of the century is divisible by 4; in that case the year is a leap year.

EXAMPLES.—(i) 1892 is divisible by 4, and the year 1892 is a leap year. (ii) 1900 is a multiple of 100, and 19 is not divisible by 4, so that 1900 is not a leap year. (iii) 2000: the number of the century is 20, and is divisible by 4, and therefore 2000 is a leap year.

The Gregorian correction still leaves a small difference between the tropical year and the average length of the civil year, amounting to only 1d. 4h. 55m. in 4000 years.

76. The Julian Day

The change of style, combined with the change of sign in B.C. years, and the two methods of expressing these years, all make it desirable to have some mode of reckoning time that goes on continuously, without any change either of sign or of method. Such a system was devised by Joseph Scaliger (1540–1609). He simply made a count of mean solar days, his zero point being the year B.C. 4713: his reasons for choosing this date need not be given; the important point is that it is earlier than any events to which accurate dates can be assigned, so that the reckoning is always positive. His father's Christian name was Julius, so he called it the system of Julian days. He was mainly occupied with dating early events, so he chose the longitude of Alexandria by which to reckon the beginning of his days; but in modern times the Julian Day has been considered to begin at Greenwich Mean Noon.

When, at the commencement of the year 1925, the beginning of the astronomical day was changed from Greenwich noon to midnight, it was decided by international agreement that the Julian day should continue to begin at Greenwich noon. The reason was that the argument in favour of the change in the beginning of the astronomical day did not apply to the Julian system of reckoning. Observations of variable stars are usually recorded in Julian days, thereby enabling the time interval between any two observations to be at once obtained; it would be inconvenient to introduce a discontinuity of half a day into the system.

The *Nautical Almanac* has for many years given tables for reducing calendar dates to Julian days. It will suffice to give here the equivalents for noon on January 1st in the years 1900, 1940, 2000, respectively:

they are 2415021, 2429630, 2451545. The student will find the Julian system useful in calculating the dates of future eclipses by the cycles given in Chapter IX. We can also use it for finding the day of the week; divide the Julian number by 7; then remainder 0 is Monday, 1 is Tuesday, 2 is Wednesday, 3 is Thursday, 4 is Friday, 5 is Saturday, 6 is Sunday.

EXAMPLES

1. To what angles do Sidereal Time, Solar Time, and Mean Time correspond on the celestial sphere? Are these angles measured direct or retrograde?

2. Draw a diagram of the Equation of Time, on the supposition that perihelion coincides with the vernal equinox.

3. On May 14th the morning is 7·8 minutes longer than the afternoon: find the equation of time on that day.

4. On a sundial placed on a vertical wall facing south, the position of the end of the shadow of a gnomon at mean noon is marked on every day of the year. Show that the curve passing through these points is something like an inverted figure of eight.

5. Why are not the graduations of a level dial uniform? Show that they will be so if the dial be fixed perpendicular to the index.

6. Show that if every 5th year were to contain 366 days, every 25th year 367 days, and every 450th year 368 days, the average length of the civil year would be almost exactly equal to that of the tropical year. How many centuries would have to elapse before the difference would amount to a day?

7. Give explicit directions for pointing an equatorial telescope to a star of R.A. 22h., declination 37° N., in latitude 50° N., longitude 25° E., at 10h. Greenwich mean time, when the true Sun's R.A. is 14h. 47m. 17s., and the equation of time is + 16m. 14s.

8. If the mean time of transit of the first point of Aries be 21h. 41m. 24·4s., find the time of the year, and the sidereal time of an observation on the same day at 13h. 22m. 13·5s.

9. At Greenwich, the equation of time at apparent noon to-day is + 3m. 39·42s., and at apparent noon to-morrow it will be + 3m. 35·39s. Prove that the mean solar time at New York corresponding to apparent time 9 A.M. there this morning is 8h. 56m. 20·9s., having given that the longitude of New York is 74° 1′ W.

10. Find the sidereal time at apparent noon on Sept. 30th, 1931, at Louisville (long. 85° 30′ W.) having given the following from the *Nautical Almanac*:—

At mean midnight.

	Sun's apparent right ascension.	Equation of time to be added to mean time.
Sept. 30	12h. 21m. 32·60s.	9m. 34·26s.
Oct. 1	12h. 25m. 9·38s.	9m. 54·03s.

EXAMINATION PAPER

1. Define the *dynamical mean* Sun and the *mean Sun*, stating at what points they have the same R.A., and when the former coincides with the true Sun. Show that the mean Sun has a uniform diurnal motion, and state how it measures mean time.

2. Define the *equation of time*. Of what two parts is it generally taken to consist? State when each of these parts vanishes, is positive, or negative. Give roughly their maximum values, and sketch curves showing their variations graphically.

3. Show that the equation of time vanishes four times a year.

4. If, on a certain day, the sundial be 10 minutes before the clock, what is the value of the equation of time on that day? Will the forenoon of that day or the afternoon be longer, and by how much?

5. Define the terms *solar day, mean solar day, sidereal day*. What is the approximate difference and the exact ratio of the second and third?

6. Define the terms *civil year, tropical year, Julian day*. Why was this last introduced?

7. Show how to express mean solar time in terms of sidereal time, and *vice versa*.

8. If the mean Sun's R.A. at mean midnight at Greenwich on June 1st be 4h. 36m. 54s., find the sidereal time corresponding to 2h. 35m. 45s. mean time (1) at Greenwich, (2) at a place in longitude 25° E.

9. On what day of the year will a sidereal clock indicate 10h. 20m. at 4 P.M.?

10. In what years between 1800 and 2100 are there five Sundays in February?

CHAPTER IV

THE EARTH

I.—PHENOMENA DEPENDING ON CHANGE OF POSITION ON THE EARTH

77. Early Observations of the Earth's Form

One of the first facts ascertained by the early Greek astronomers was that the Earth's surface is globular in form. Even Homer (B.C. 850 *circ.*) speaks of the sea as convex, and Aristotle (B.C. 320) gives many reasons for believing the Earth to be a sphere. Among these may be mentioned the appearances presented when a ship disappears from view. If the surface of the ocean were a plane, any person situated above this plane would (if the air were sufficiently clear) see the whole expanse of ocean extending to the furthermost shores, with all the ships sailing on its surface. Instead of this, it is observed that as a ship begins to sail away its lowest part will, after a time, begin to sink below the apparent boundary of the surface of the sea; this sinking will continue till only the masts are visible, and finally, these will disappear below the convex surface of the water between the ship and the observer.

Another reason is suggested, by observing the stars. If the Earth's surface were a plane, any star situated above the plane would be seen simultaneously from all points of the Earth, except where concealed by mountains or other obstacles, and any star below the plane would be everywhere simultaneously invisible. In reality, stars may be visible from one place which are invisible from another; and all the appearances presented were found by the Greeks to agree with what might be expected on a spherical Earth. Eratosthenes even made a calculation of the Earth's size from the distance between Alexandria and Assouan and their latitudes deduced from the Sun's greatest meridian altitudes. He found the circumference to be 250,000 stadia, or furlongs.

Lastly, the Earth's spherical form will account for the circular form of the Earth's shadow in a lunar eclipse.

78. General Effects of Change of Position

In Art. 14 we showed that, owing to the great distance of the stars, they are seen in the same direction whatever be the position of the observer. In confirmation of this fact, it is found by observation that the angular distance between any two stars (after allowing for refraction) is observed to be independent of the place of observation.

But the directions of the zenith and horizon vary with the position of the observer. If we suppose the Earth spherical, the vertical at any point on it will be the radius drawn from the Earth's centre, while the plane of the horizon will be a tangent plane to the Earth's surface; both will depend on the place. This circumstance accounts for the difference in appearance of the heavens as seen simultaneously from different places.

79. Earth's Rotation

The apparent rotation of the heavens is accounted for by supposing that the stars are at rest, and that the Earth rotates once in a sidereal day, from west to east, about an axis parallel to the direction of the celestial pole. The observer's zenith, horizon and meridian turn about the pole from west to east, relatively to the stars, and this causes the hour angles of the stars to increase by 360° in a sidereal day, in accordance with observation.

It is impossible to decide from observations of the stars alone whether it is the Earth or the stars which rotate, just as when two railway trains are side by side it is very difficult for a passenger in one train, when observing the other, to decide which train is in motion. That the Earth rotates has, however, been conclusively proved by means of experiments, which will be described when we come to treat of dynamical astronomy.

The *terrestrial poles* are the two points in which the axis of rotation meets the Earth's surface. The terrestrial poles, equator and meridians (Art. 7) should be distinguished from the celestial poles, equator and meridians (Art. 18). Whilst the terrestrial poles, equator and meridians are on the surface of the Earth, the celestial poles, equator and meridians lie on the imaginary celestial sphere.

80. Phenomena depending on Change of Latitude

Assuming the Earth to be spherical, let $pOqp'r$ be a meridian section, C being the Earth's centre, p, p' the poles, q, r points on the equator. Then, if an observer is situated on the meridian at O, the direction of his celestial pole P will be found by drawing OP parallel to the Earth's axis $p'Cp$, while his zenith Z will lie in CO produced, if gravity acts towards the centre.

Since OP is parallel to CpP_1, therefore: angle $ZOP = OCp$.

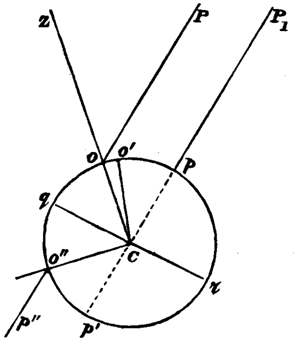


FIG. 32.

The altitude of the pole at $O = 90° - ZOP = 90° - OCp = qCO$ or, in other words, the altitude of the pole at O is equal to the latitude of O. This result has already been obtained in Art. 22.

Since the angle qCO is proportional to the arc qO, the latitude of a place is proportional to its distance from the equator.

Suppose the observer to go northwards along the meridian from O to O', then, from what has just been shown, the altitude of the pole increases from $\angle qCO$ to $\angle qCO'$, hence the increase in the altitude of the pole ($= \angle OCO'$) is proportional to the arc OO', *i.e.* to the distance travelled northwards.

81. Southern Latitudes

To an observer situated in the southern hemisphere of the Earth, as at O'', the North Pole of the heavens is below, and the South Pole, p'' is above the horizon. The *South Latitude* of the place is measured by the altitude of the South Pole, p'', and is equal to the angle qCO''.

At the terrestrial equator, the altitude of the pole is zero; hence the pole is on the horizon. At the terrestrial North Pole p, the altitude of the celestial pole is 90°, therefore the celestial pole coincides with the zenith.

At the Earth's North Pole, those stars are only visible which are north of the equator, and they always remain above the horizon. On travelling southwards, other stars, whose declination is south, are seen in the south parts of the celestial sphere, and on reaching the Earth's equator all the stars will be above the horizon at some time or other, but the Pole Star will only just rise above the horizon, near the north point. After passing the equator, the Pole Star and other stars near the North Pole disappear.

82. Radius of the Earth

The Earth's radius may be found by measuring *the distance between two places on the same meridian* and finding their *difference of latitude.*

Let the places of observation be O, O' (Fig. 32). Let the latitudes qCO, qCO' be ϕ and ϕ' respectively, expressed in degrees, and let the length $OO' = s$. We have, supposing the Earth spherical,

$$\frac{\text{angle } OCO'}{360^\circ} = \frac{\text{arc } OO'}{\text{circumference of Earth}};$$

$$\text{So that Earth's circumference} = s \times \frac{360}{\phi' - \phi};$$

$$\text{and Earth's radius} = \frac{\text{circumference}}{2\pi} = \frac{180}{\pi}\,\frac{s}{\phi' - \phi},$$

which determines the Earth's radius in terms of the data.

By observations of this kind the Earth's radius is found to be very nearly 3960 miles. For many purposes it will be sufficiently approximate to take the radius as 4000 miles. Its circumference is found by multiplying the radius by 2π, and is about 24,855 miles, or, roughly, 25,000 miles.

Conversely, knowing the Earth's radius, we can find the length of the arc of the meridian corresponding to any given difference of latitude.

83. Metre, Nautical Mile, Geographical Mile, Fathom

The French *Metre* was originally defined as the ten-millionth part of the length of a quadrant of the Earth's meridian through Paris. Owing to an error in the estimation of the quadrant, the length of the metre defined in this way is not exactly equal to the length of the standard metre. The definition is sufficiently accurate, however, for general purposes.

A *Nautical Mile* is defined as the length of a minute of arc of the *meridian.* Thus a quadrant of the meridian contains 90 × 60, or 5400 nautical miles, and the Earth's circumference contains 21,600 nautical miles. The nautical mile is equivalent to about 6080 feet or about 1·15 statute miles.

A *Nautical Fathom* is $\frac{1}{1000}$ of a nautical mile = 6ft. 1 in. nearly. A *fathom* is commonly taken as 6 feet.

A *Geographical Mile* is defined as the length of a minute of arc measured on the Earth's *equator.* Taking the Earth as a sphere, the nautical mile and geographical mile are equal.

84. The Knot—Use of the Log Line in Navigation

The *Knot* is the unit of velocity used in navigation, being a velocity of one nautical mile per hour. Thus, a ship sailing 12 knots travels at 12 nautical miles an hour.

The velocity of a ship is measured by means of the *Log Line*. This consists of a "log," or float, attached to a cord which can unwind freely from a small windlass. The log is "heaved" or dropped into the sea, and allowed to remain at rest, the cord being "paid out" as the ship moves away. By measuring the length paid out in a given interval of time (usually half a minute), the velocity of the ship may be found. To facilitate the measurement, the line has knots tied in it at such a distance apart that the number of knots paid out in the interval of time is equal to the number of nautical miles per hour at which the ship is sailing. It is from these that the unit of velocity derives the name of knot.

Now one nautical mile per hour $= \frac{1}{120}$ nautical mile per half-minute. Hence, for this interval, the knots should be tied on the line at intervals of $\frac{1}{120}$ of a nautical mile apart.

Examples.—1. *To find the number of miles in an arc of* 1°.

$$\text{An arc of } 1^\circ = \frac{\text{circumference of Earth}}{360} = \frac{24855}{360} \text{ miles} = 69\tfrac{1}{24} \text{ miles.}$$

2. *To find the number of feet in one fathom.*

By Ex. 1, 60 nautical miles $= 69\frac{1}{24}$ ordinary miles; *i.e.* 60,000 fathoms $= 69\frac{1}{24} \times 5280$ feet;

$$\text{and } 1 \text{ fathom} = \frac{69\frac{1}{24} \times 5280}{60000} \text{ feet} = 6{\cdot}077 \text{ feet.}$$

3. *To express a metre in terms of a yard.*

By definition, 40,000,000 metres = Earth's circumference = 24,855 miles;

$$\text{and } 1 \text{ metre} = \frac{24855 \times 1760}{40{,}000{,}000} \text{ yards} = 1{\cdot}0936 \text{ yards.}$$

85. The Departure

It was shown in Art. 9 that the distance, measured along the parallel of latitude, between two places on the Earth on the same parallel is equal to the product of their difference of longitude by the cosine of the latitude. If the difference of longitude is expressed in degrees, the distance is also obtained in degrees on the Earth's surface. If, however, the difference of longitude is expressed in minutes of arc, the distance will be obtained in nautical miles, because a distance of one nautical mile subtends an angle of one minute of arc at the earth's centre. The distance between the two points is then called the *departure*.

Example.—*A ship steams along the parallel of latitude* 38° 54′, *from a point with longitude* 12° 15′ *W. to a point with longitude* 55° 36′ *W. Find the departure between the two points.*

The difference of longitude = 55° 36′ — 12° 15′
= 43° 21′ = 2601′.

Hence the departure = 2601 × cos 38° 54′
= 2601 × ·7782
= 2024 nautical miles.

II.—DIP OF THE HORIZON

86. Definitions

Let *O* be an observer situated above the surface of the land or sea. Draw *OT*, *OT′* tangents to the surface. Then it is evident, from the figure, that only those portions of the Earth's surface will be visible whose distance from the observer *O* is less than the length of the tangents *OT*, *OT′*.

The boundary of the portion of the Earth's surface visible from any point is called the *Offing or Visible Horizon.* Hence, if *OACB* be the Earth's diameter through *O*, and the Earth be supposed spherical, the offing at *O* is the small circle *TtT′*, formed by the revolution of *T* about *OB*, and having for its pole the point *A* vertically underneath *O*. If, however, the Earth be not supposed spherical, the form of the offing will, in general, be more or less oval, instead of circular.

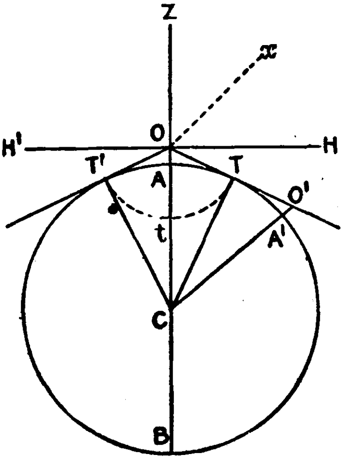


FIG. 33.

Conversely, since it is observed that the "offing" at sea is very approximately circular, whatever be the position of the observer, it may be inferred that the Earth is approximately spherical.

The *Dip of the Horizon* at *O* is the inclination to the horizontal plane of a tangent from *O* to the Earth's surface.

Hence, if *HOH′* be drawn horizontally (*i.e.* perpendicular to *OC*), the dip of the horizon will be the angle *HOT*.

87. To Determine the Distance and Dip of the Visible Horizon at a given Height above the Earth

Let $h = AO =$ given height of observer;
$a = CA =$ Earth's radius;
$d = OT =$ required distance of horizon;
$D = \angle HOT =$ required dip in *circular measure*;
$D'' =$ the number of seconds in the dip D.

(i) Since OT is tangent to the circle at T, we have :—

$$OT^2 = OA,\ OB$$
$$\text{or}\quad d^2 = h\,(2a + h) = 2ah + h^2.$$

This determines d accurately. But in practical applications h is always very small compared with $2a$; therefore h^2 may be neglected in comparison with $2ah$, and we have the approximate formula

$$d^2 = 2ah, \quad \text{or} \quad d = \sqrt{(2ah)}.$$

(ii) Since CTO is a right angle :—

$$\angle OCT = \text{complement of } \angle COT = \angle TOH = D.$$

Therefore, D being expressed in circular measure, we have :—

$$D = \frac{\text{arc } AT}{\text{radius } CT}.$$

Now, in practical cases, where the dip is small, the arc AT will not differ perceptibly in length from the straight line OT. We may, therefore, take arc $AT = d$;

$$\text{whence,}\quad D = \frac{d}{a} = \frac{\sqrt{(2ah)}}{a} = \sqrt{\frac{2h}{a}}.$$

To reduce to seconds, we must multiply by $180 \times 60 \times 60/\pi$, the number of seconds in a unit of circular measurement, and we have

$$D'' = \frac{180 \times 60 \times 60}{\pi}\sqrt{\frac{2h}{a}}.$$

COROLLARY 1.—Let a, h, d be measured in miles, and let h' be the number of feet in the height h.

Then $h' = 5280h$, and taking the Earth's radius a as 3960 miles, we have

$$d = \sqrt{\frac{2 \times 3960 \times h'}{5280}} = \sqrt{\left(\frac{3h'}{2}\right)},$$

a very useful formula.

COROLLARY 2.—Since the offing is a circle whose radius is very approximately equal to OT or d, we have

Area of Earth's surface visible from $O = \pi d^2 = 2\pi ah = \frac{3}{2}\pi h'$ in square miles.

*88. Accurate Determination of Dip

The use of approximations can be avoided by the exact formula :

$$\tan D = \frac{TO}{CT} = \frac{\sqrt{(2ah + h^2)}}{a} = \sqrt{\frac{h(2a + h)}{a^2}},$$

which is adapted to logarithmic computation.

In this, as in the preceding formulae, no account has been taken of the effect of refraction due to the atmosphere.

For this reason it is important to determine dip of the horizon by practical observations. An instrument called the Dip Sector is constructed for this purpose.

Tables have also been constructed, giving the dip of the horizon as seen from different heights. They are of great use at sea, where the altitude of a star is usually found by observing its angular distances from the offing.

89. Disappearance of a Ship at Sea

When a ship has passed the offing, the lower part will be the first to disappear. Let $A'O'$ (Fig. 33) be the position of the ship ; let its distance OO' be s, and let $k = A'O'$ be the height above sea-level of the lowest portion just visible from O. By the approximate formula we have $OT = \sqrt{(2ah)}$, $O'T = \sqrt{(2ak)}$, so that

$$s = \sqrt{(2ah)} + \sqrt{(2ak)}.$$

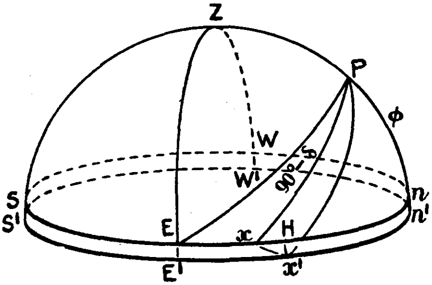


FIG. 34.

This formula determines the distance s at which an object of given height k disappears below the horizon.

90. Effect of Dip on the Times of Rising and Setting

To an observer on land, the offing is generally more or less broken by irregularities of the Earth's surface. At sea, however, the offing is well defined, and if the dip of the horizon in seconds be D'', the visible horizon, which bounds the observer's view of the heavens, is represented on the celestial sphere by a small circle parallel to the celestial horizon, and at a distance D'' below it ($n'E's'$, Fig. 34).

Hence the stars appear to rise and set when they are at an angular distance D'' below the celestial horizon. Thus they will rise sooner and set later than they would if there were no dip.

Taking the observer's latitude to be ϕ, let x', x be the positions of a star of declination δ, when rising across the visible horizon $n'E's'$ and the celestial horizon nEs respectively. Draw $x'H$ perpendicular to nEs, then $x'\ H = D''$.

Then, if the star rise t seconds earlier at x' than at x, we have :—

$$15\,t = \angle x'Px \text{ (in seconds of angle)}$$

$$= \frac{\text{arc } xx'}{\sin\ xP} = \frac{\text{arc } xx'}{\cos\ \delta} \text{ (by Art. 5).}$$

But treating the small triangle $x'x\ H$ as plane and remembering that $\angle Pxx' = 90°$, we have :—

$$xx' = \frac{x'\text{H}}{\sin\ x'xH} = \frac{D''}{\cos\ nxP};$$

$$\text{therefore, } t = \frac{1}{15} D'' \sec\delta \ \sec nxP.$$

Evidently the acceleration at rising = retardation at setting.

COROLLARY 1.—*To an observer at the Equator*, P coincides with n, so that $\angle nxP = 0$, it follows that the time of rising is accelerated by $\frac{1}{15}D''$ sec δ seconds.

COROLLARY 2.—*If the star is on the equator*, $\delta = 0$, x coincides with E, and $\angle nEP = nP = \phi$, and the acceleration $= \frac{1}{15}D''$ sec ϕ seconds.

III.—GEODETIC MEASUREMENTS—FIGURE OF THE EARTH

91. Geodesy

Geodesy is the science connected with the accurate measurement of arcs on the surface of the Earth. Such measurements may be performed with either of the two following objects :—

(i) The construction of maps.

(ii) The determination of the Earth's form and magnitude. Only the second application falls within the scope of this book.

92. A Simple Approximate Method of Finding the Earth's Radius

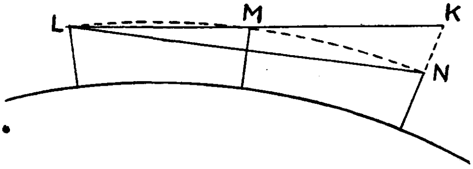


FIG. 35.

An approximate measure of the Earth's radius can be readily found by means of the following simple experiment.

Let L, M, N (Fig. 35) be the tops of three posts of the same height set up in a line along the side of a straight canal. Owing to the Earth's curvature the straight line LM will, if produced, pass a little above N. Hence, in order to see L, M in a straight line, an observer at the post N will have to place his eye at a point K, a little above N, and the height KN may be measured. Let KL, KM be also measured.

Since the posts are of equal height, L, M, N will lie on a circle concentric with, and almost coinciding with, the Earth's surface. Let the vertical KN meet this circle again in n. Then, by the geometry of the circle,

$$KL \,.\, KM = KN \,.\, Kn\,; \quad \text{or} \quad Kn = KL \,.\, KM/KN,$$

and Radius of Earth $= \frac{1}{2}Kn$ (very approximately) $= \dfrac{KL \,.\, KM}{2KN}$.

This method cannot be relied on where accuracy is required, for the small height KN is very difficult to measure, and a very slight error in its measurement would affect the final result considerably. Moreover the observations are considerably affected by refraction.

If the distances LM, MN are each one mile, the height KN is about 16 inches. If the distances LM, MN are each half a mile, the height KN is about 4 inches.

93. Ordinary Methods of Finding the Earth's Radius

Where greater accuracy is required, the radius of the Earth is obtained by measuring the length of an arc of the meridian and determining the difference of latitude of its extremities; the radius may then be calculated as in Art. 82. The instruments required for the observations include—

(i) Measuring rods or tapes;,
(ii) A theodolite, for measuring angles;
(iii) A zenith sector.

94. Measurement of a Base Line

The first step is to measure, with extreme accuracy, the length of the arc joining two selected points, several miles apart, on a level tract of country; this line is called a *Base Line*. A series of short upright posts are placed at equal distances apart along the base line, and they are adjusted till their tops are seen exactly in the same vertical plane, and are on the same level as shown by a spirit-level. Across these posts are laid measuring rods of metal, whose length is very accurately known, and these are also adjusted in a line, and made level by the spirit-level. These rods are not allowed to touch, but the small distances between their ends are measured with reading microscopes. In this way, a base line several miles long can be measured correctly to within a small fraction of an inch.

The length of the rods will depend upon their temperature, which must be noted at frequent intervals during the observations. It is now customary to use flexible tapes, instead of rods, made of a special alloy, called *invar*, whose coefficient of expansion is practically zero.

95. Triangulation

When once a base line has been measured, the distance between any two points on the Earth can be determined by the measurement of *angles* alone. For, calling the base line AB, let C be any object visible from both A and B. If the angles CAB, CBA be observed, we can solve the triangle ABC and determine the lengths of the sides CA, CB. Either of these sides, say CA, may now be taken as the base of a new triangle, whose vertex is another point, D. Thus, by observing the angles of the triangle ACD we can determine DA, DC in terms of the known length of AC. Proceeding in this way, we may divide any country into a network of triangles connecting different places of observation A, B, C, D, and the distance between any two of the places calculated, as well as the direction of the line joining them. Finally, two stations C, G are taken, which lie nearly on the same meridian. A perpendicular GH is let fall on the meridian, then the distance CH is calculated; in this way, it is possible to measure an arc of the meridian.

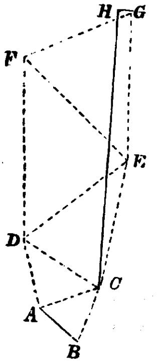


FIG. 36.

96. The Theodolite

The measurement of the angles is far easier in practice than the measurement of a base line. The instrument used for measuring angles is called a *Theodolite*. This consists of a small telescope, mounted so that it can be moved in altitude or azimuth, by turning about horizontal and vertical axes. The horizontal azimuth circle is accurately divided and provided with two verniers and reading microscopes, at a distance of 180° apart, to enable the setting to be read with accuracy. The vertical circle is usually limited to a small arc, sufficient for measuring the altitude of one terrestrial object as seen from another. The instrument is provided with sensitive spirit-levels, by means of which it can be adjusted so that the horizontal circle is truly horizontal and the vertical axis, therefore, truly vertical. A compass needle is usually provided, as an approximate guide to the direction of the north point.

By reading the horizontal circle of the theodolite, the azimuths of B, C, as seen from A, are found. By using the difference of azimuth instead of the angle *CAB*, it becomes unnecessary to take account of the height of the various stations above the Earth. For if *A*, *B*, *C* are replaced by any other points, *A′*, *B′*, *C′*, at the sea level, and vertically above or below *A*, *B*, *C*, the vertical planes joining them will be unaltered in position, and therefore the azimuths will also be unaffected.

97. The Zenith Telescope

Having thus found, with great accuracy, the length of the arc joining two stations on the same meridian, it only remains now to observe their difference of latitude.

The *Zenith Telescope* is the most useful instrument for this purpose. It is essentially similar to a theodolite, being provided with movements in altitude and azimuth, but the eye-end is provided with a movable wire and a micrometer. Observations are made by the Talcott method, which depends upon the observation of a pair of stars, situated at approximately equal distances to the north and south of the zenith respectively.

If δ_N, δ_S are respectively the declinations of the north and south stars, Z_N, Z_S their distances north and south of the zenith respectively, and ϕ the latitude, it is readily seen that :—

$$Z_N = \delta_N - \phi, \quad Z_S = \phi - \delta_S$$

$$\text{so that} \quad Z_N - Z_N = \tfrac{1}{2}(\delta_S + \delta_S) - \phi.$$

The telescope is adjusted so that the axis about which it turns is truly vertical and so that the telescope lies in the meridian. It is set to observe the transit of, say, the northern star and the horizontal micrometer wire is moved to bisect the star image, as it crosses the centre of the field at transit. Without altering the setting of the telescope, the instrument is then turned through 180° and a similar observation is made of the south star. The difference in the zenith distances (which are nearly equal, because of the choice of stars) is given at once by the difference of the readings of the micrometer at the two observations. The declinations of the two stars being known, the latitude can be at once determined.

A great advantage of this method of observation is that it is practically independent of atmospheric refraction. As we shall see in Chapter VI, the effect of refraction is to lift a star towards the zenith by an amount that increases with the zenith distance. As the two stars observed with the zenith telescope are at almost the same distance from the zenith, they are equally affected by refraction and the difference of the zenith distances is independent of the refraction.

The observations are repeated at the second station. It should be noted that if the same stars are observed at both stations, the *difference* in the observed latitudes is independent of any errors in the assumed declinations of the stars.

If s is the measured length of the arc of the meridian joining the stations, whose latitudes are supposed to be ϕ_1 and ϕ_2 (expressed in degrees) and if r is the radius of the earth, then Art. 82 gives :—

$$r = \frac{180}{\pi} \cdot \frac{s}{\phi_1 - \phi_2}.$$

98. Exact Figure of the Earth

If the Earth were an exact sphere, the same value would be found for the radius r in whatever latitude the observations were made. But in reality the length of a degree of latitude, and therefore also r, is found to be larger when the observation is made near the poles than when made near the equator, and hence it is inferred that the meridian curve is somewhat oval.

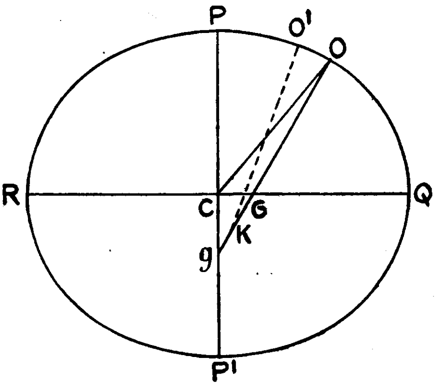


FIG. 37.

Let $PQP'R$ (Fig. 37) represent the meridian curve, OO' two near places of observation on it. Then, if OK and $O'K$ be drawn normal (*i.e.* perpendicular) to the Earth's surface at O, O', they will be the directions of the plumb-lines of the zenith sectors at O, O'. Hence the observed difference of latitudes or meridian altitudes at O, O' will give the angle OKO'.

Regarding the small arc OO' as an arc of a circle whose centre is K, we shall have approximately,

$$\text{Circular measure of } OKO' = \text{arc } OO' \div OK,$$

$$\text{or} \quad OK = \frac{\text{arc } OO'}{\text{circ. measure of } OKO'} = \frac{180}{\pi} \cdot \frac{s}{\phi_1 - \phi_2},$$

and hence r, calculated as in Art. 97, is the length OK.

The length OK is called the *radius of curvature* of the arc, and K is called the *centre of curvature*; they are respectively the radius and centre of the circle whose form most nearly coincides with the meridian along the arc OO'.

This radius of curvature OK is not, in general, equal to OC, the distance from the centre of the Earth, owing to the Earth not being quite spherical.

As the result of numerous observations, the meridian curve is found to be an *ellipse* (see Appendix), whose greatest and least diameters, called the *major* and *minor axes*, are the Earth's equatorial and polar diameters respectively. The Earth's surface is the figure formed by making the ellipse revolve about its minor axis PCP'. This figure is called an *oblate spheroid.*

99. To find the Equatorial and Polar Radii of Curvature of the Meridian Curve, supposing it to be an Ellipse

Let $PQP'R$ be the ellipse. Let $2a$, $2b$ be the lengths of its equatorial and polar diameters QCR, PCP'. Let r_1, r_2 be the required radii of curvature at Q and P respectively.

Take any point O on the ellipse, and let the normal at O meet the two axes in G and g respectively.

It is proved in treatises on Conic Sections* that

$$OG : Og = CP^2 : CQ^2 = b^2 : a^2.$$

First take O very near to Q. Then OG will become equal to the radius of curvature r_1; also Og will evidently become ultimately equal to CQ or a. Therefore :—

$$r_1 : a = b^2 : a^2 ; \quad \text{or} \quad r_1 = b^2/a.$$

Next take O very near to P. Then OG will become equal to b and Og to r_2. Therefore :—

$$b : r_2 = b^2 : a^2 ; \quad \text{or} \quad r_2 = a^2/b.$$

Thus r_1, r_2 are found in terms of a, b.

Conversely, if r_1 and r_2 are known, a and b may be found; for, by solving, we find

$$a = \sqrt[3]{(r_2{}^2 r_1)},\ b = \sqrt[3]{(r_1{}^2 r_2)}.$$

We notice that, since $a > b$, $r_1 < r_2$.

That the equatorial radius of curvature is less than the polar is also evident from the shape of the curve. This, as the figure shows, is most rounded at Q, R, and flattest or least rounded at P, P'. Hence it will require a smaller circle to fit the shape of the curve at the equator than at the poles.

100. Exact Dimensions of the Earth

The lengths of the Earth's equatorial and polar semi-diameters, a, b, are, according to Hayford (1909),

$$a = 3963{\cdot}35 \text{ miles}, \quad b = 3950{\cdot}01 \text{ miles}.$$

Thus, the Earth's equatorial semi-diameter exceeds its polar semi-diameter by 13·34 miles.

---

* Appendix, Ellipse (9).

The *mean radius* of an oblate spheroid is the radius of a sphere of equal volume, and is equal to $\sqrt[3]{(a^2b)}$. Thus, the Earth's mean radius is approximately 3958·9 miles.

The *ellipticity* or *compression* (c) is the fraction

$$c = \frac{a - b}{a}.\quad \text{For the Earth, } c = \frac{1}{297} \text{ nearly.}$$

The *eccentricity* (e) is given by the relation :—

$$e^2 = \frac{a^2 - b^2}{a^2}.$$

$$\text{Hence}\quad b^2 = a^2(1 - e^2) = a^2(1 - c)^2;$$

$$\text{and}\quad 1 - e^2 = (1 - c)^2 = 1 - 2c + c^2;$$

$$\text{or}\quad e^2 = 2c - c^2 = c(2 - c).$$

Since c is small, $2 - c = 2$, approx.; therefore $e^2 = 2c$, approx., which gives the Earth's eccentricity $e = ·0820$.

101. Geographical and Geocentric Latitude

The *Geographical Latitude* of a place is the angle which the normal to the Earth's surface at that place makes with the plane of the equator. It is the latitude determined by astronomical observations. Thus, $\angle QGO$ (Fig. 37) is the geographical latitude of O.

The *Geocentric Latitude* is the angle subtended at the Earth's centre by the arc of the terrestrial meridian between the place and the equator. Thus, $\angle QCO$ is the geocentric latitude of O.

102. Relations between the Geocentric and Geographical Latitudes

Let $\angle QGO = \phi$, $\angle QCO = \phi'$. Draw ON perp. to CQ. Then

$$GN : CN = OG : Og = b^2 : a^2;\ \text{so that } NO/CN = (NO/GN) \times (b^2/a^2);$$

$$\text{or}\quad \tan\phi' = \tan\phi \times b^2/a^2 = (1 - e^2)\tan\phi.$$

We deduce also $\tan(\phi - \phi') = \dfrac{e^2 \sin 2\phi}{2(1 - e^2\sin^2\phi)} = \frac{1}{2}e^2 \sin 2\phi$ (approx.), since e^2 is small. Hence the difference between ϕ and ϕ' is a maximum at latitude 45°, where it amounts to 11′ 36″.

EXAMPLES

1. Show that the locus of points on the Earth's surface at which the Sun rises at the same instant is half a great circle; and state the corresponding property possessed by the other half.

2. Find the least height of a mountain in Corsica in order that it may be visible from the sea-level at Mentone, at a distance of 80 miles.

3. At the equator, in longitude $L°$, a given vertical plane declines $a°$ from the north towards the west; find the latitude and longitude of the places to whose horizon the given plane is parallel.

4. Prove that, at either equinox, in latitude l, a mountain whose height is $1/n$ of the Earth's radius will catch the Sun's rays in the morning $\frac{12}{\pi \cos l}\sqrt{\frac{2}{n}}$ hours before he rises on the plain at the base.

5. Estimate to the nearest minute the value of this expression for a mountain three miles high in latitude 45°.

6. Find the distance of the horizon as seen from the top of a hill 1056 feet high.

7. Find, to the nearest mile, the radius of the Earth, supposing the visual line of a telescope from the top of one post to the top of another post two miles off, cuts a post, half way between, 8 inches below the top, the posts standing at equal heights above the water in a canal.

8. In Question 7, what would be the length of a nautical mile, adopting the usual definition?

9. Supposing the Earth spherical, and of radius r, and neglecting the refraction of the air, show that, if from the top of a mountain of height a above the level of the sea, the summit of another mountain is seen beyond the horizon of the sea, and at an elevation e above the horizon, and if its distance be known to be D, its height is approximately given by

$$a + eD + D\left(\frac{D}{2r} - \sqrt{\frac{2a}{r}}\right).$$

10. A railway train is moving north-east at 40 miles an hour in latitude 60°; find approximately, in numbers, the rate at which it is changing its longitude.

MISCELLANEOUS QUESTIONS

1. Explain the different systems of coordinates by which a star's position is fixed in the heavens.

2. Show, by a figure, where a star will be found at 9 p.m. on the 5th of June in latitude 50° N., if the star's right ascension is 12 hours and its declination 5° south.

3. Define *dip, azimuth, culmination, circumpolar, zenith.* Why would it be insufficient to define the declination of a star as its distance from the equator measured along a declination circle?

4. Three stars, A, B, C, are on the same meridian at noon, B being on the equator, and A and C equidistant from B on either side. Prove that the intervals between the setting-times of A and B and B and C are equal.

5. Show how to find approximately the Sun's R.A. at a given date. Obtain its approximate value for March 1st, August 10th, October 23rd, and January 15th.

6. Explain the terms apparent sidereal time and mean sidereal time.

7. Define a morning and evening star. Show that on the 1st of September a star, whose declination is 0°, and R.A. 11h. 28m., is an evening star, but that it is a morning star three weeks later.

8. Assuming the Earth to be a sphere, show how its radius may be practically measured.

9. Explain clearly the nature and uses of the zenith telescope.

10. A, B, C are the tops of the masts of three ships in a line, and are at equal heights above the sea-level, and O is the centre of the Earth. If the distance BC

be x miles, and r is the Earth's radius in miles, show that $\angle BAC = \frac{1}{2} \angle BOC$; and hence deduce that

$$\angle BAC = \frac{180 \times 60 \times 60}{\pi} \frac{x}{2r} \text{ seconds.}$$

Find this angle, having given $x = 2$, $r = 3960$, $\pi = 3\frac{1}{7}$.

EXAMINATION PAPER

1. Assuming the Earth to be a sphere, show that, as we travel from the equator due north, our astronomical latitude (*i.e.* the altitude of the Pole) will increase. Taking this increase as 1° for every 69 miles, find the circumference and the radius of the Earth.

2. Define the *metre*, the *nautical mile*, and the *knot*, and calculate their values in feet and feet per second respectively, taking the Earth's radius as 3960 miles.

3. How is the speed of a ship estimated? Find, in feet, the distance apart of the knots on a log line, so constructed that the number run out in half a minute measures the ship's velocity in nautical miles per hour.

4. What are the difficulties in measuring an arc of the meridian and how are they met?

5. Find the Earth's radius in fathoms, and in metres. Express the nautical mile in French units of length.

6. Obtain formulae for the distance of the visible horizon from a place whose height is given. Deduce that, if the height h be measured in inches, the distance in miles will be $\sqrt{\frac{h}{8}}$, taking the Earth's radius as 3960 miles.

7. Define the *dip of the horizon*, and show how to find it. Prove that the number of seconds in the dip is nearly 52 times the distance in miles of the offing.

8. If A, B, and C be the tops of three equal posts arranged in order two miles apart along a straight canal, show that the straight line AB passes 5 feet 4 inches above C, and that AC passes 2 feet 8 inches below B.

9. Find the length of a given parallel of latitude intercepted between two given circles of longitude.

10. Is the Earth an exact sphere? Show that a degree of latitude increases in length as we go northward. Distinguish a *nautical* from a *geographical mile*.

CHAPTER V

THE SUN'S APPARENT MOTION IN THE ECLIPTIC

I.—The Seasons

103. Introduction

In Section III of Chapter II* we described the Sun's annual motion among the stars, and showed how, in consequence of this motion, the Sun's right ascension increases at an average rate of nearly 1° per day, while his declination fluctuates between the values 23° 27′ north, and 23° 27′ south of the equator. We shall now show how this annual motion, combined with the diurnal rotation about the poles, gives rise to the variations, both in the relative lengths of day and night, and in the Sun's meridian altitude, during the course of the year; how these variations are modified by the observer's position on the Earth; and how they produce the phenomena of summer and winter.

Although both the diurnal and annual apparent motions of the Sun are known to be *really* due to the Earth's motion, it will be convenient in this section to imagine the Earth to be fixed, while the Sun and stars are moving; thus the zenith, pole, horizon, meridian, and equator will be considered fixed, as they actually appear to be to an observer on the Earth.

As the change in the Sun's declination during a single day is very small, the Sun's apparent path in the heavens from morning till night is very approximately a small circle parallel to the equator, and may be regarded as such for purposes of explanation. The effects of the variation in the declination will, however, become very apparent when we compare the Sun's diurnal paths at different seasons of the year.

Throughout this section we shall denote the obliquity of the ecliptic by ϵ, the Sun's declination at any time by δ, his zenith distance at noon by z, and the observer's latitude by ϕ.

104. Zones of the Earth.—Definitions

From Art. 30 it is evident that if the Sun passes through the zenith at noon, δ must be equal to ϕ. But δ lies between ϵ (north) and ϵ (south). Therefore ϕ must lie between the limits ϵ N. and ϵ S.

Thus, if the Sun be vertically overhead at some time in the year, the latitude must not be greater than 23° 27′ N. or S.

Again, from Art. 31 we see that the Sun, like a circumpolar star, will remain above the horizon during the whole of its revolution provided that $90° - \delta < \phi$. This requires that $\phi > 90° - \epsilon$.

Thus, if the Sun be visible all day long during a certain period of the year, the latitude must be greater than 66° 33′ N. or S.

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* The student will do well to revise Chapter II, Section III, before proceeding further.

These circumstances have led to the following definitions :—

The *Tropics* are the two parallels to the Earth's equator in north and south latitude ϵ, or 23° 27′. The northern tropic is called the *Tropic of Cancer*, the southern the *Tropic of Capricorn.*

The *Arctic* and *Antarctic Circles* are respectively the parallels of north and south latitude $90° - \epsilon$, or 66° 33′.

These four parallels divide the Earth's surface into five regions or *zones.*

The portion between the tropics is called the *Torrid Zone.*

The portion between the tropic of Cancer and the arctic circle is called the *North Temperate Zone.* The portion between the tropic of Capricorn and the antarctic circle is called the *South Temperate Zone.*

The portions north of the arctic circle, and south of the antarctic circle are called the *Frigid Zones,* and are distinguished as the *Arctic* and *Antarctic Zones.*

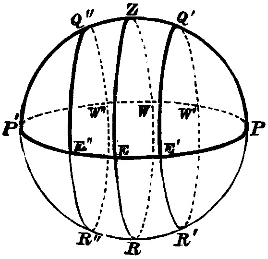

FIG. 38.

105. Sun's Diurnal Path at Different Seasons and Places

We shall now describe the various appearances presented by the Sun's diurnal motion at different times of the year, beginning in each case with the vernal equinox. We shall first suppose the observer at the Earth's equator, and shall then describe how the phenomena are modified as he travels northward towards the pole.

106. At the Earth's equator

At the Earth's equator $\phi = 0$, and the poles of the celestial sphere are on the horizon (P, P', Fig. 38). Hence, between sunrise and sunset, the Sun has always to revolve about the poles through an angle 180°, and the days and nights are always equal, each being 12 hours long.

On March 21st the Sun is on the celestial equator, and it describes the circle EZW, rising at the east point, passing through the zenith at noon, and setting at the west point.

Between March 21st and Sept. 23rd, the Sun is north of the celestial equator ; it therefore rises north of $E.$, transits north of the zenith Z, and sets north of W. Its N. meridian zenith distance z is always equal to its N. declination δ (since by Art. 30, $z = \delta - \phi$ and $\phi = 0$).

Hence, from March 21st to June 21st, z increases from 0 to ϵ N. On June 21st, z has its greatest N. value ϵ, and the Sun describes the circle $E'Q'W'$, where $ZQ' = \epsilon$.

From June 21st to Sept. 23rd, z decreases from ϵ to 0.

On Sept. 23rd, the Sun again describes the great circle EQW.

Between Sept. 23rd and March 21st, the Sun is south of the equator, and therefore it transits south of the zenith. We now have $z = \delta$, both being S.

From Sept. 23rd to Dec. 22nd, the Sun's south Z.D. at noon, z, increases from 0 to ϵ.

On Dec. 22nd, z has its greatest value ϵ (south) and the Sun describes the circle $E''Q''W''$ where $ZQ'' = \epsilon$.

From Dec. 22nd to March 21st, z diminishes again from ϵ to 0.

On March 21st, the Sun again describes the circle EQW, and the same cycle of changes is repeated the following year.

107. In the Torrid Zone North of the Equator

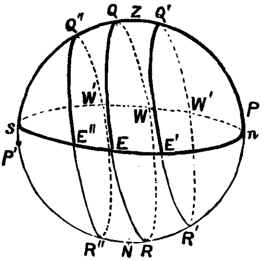

FIG. 39.

On March 21st, the Sun describes the equator EQW (Fig. 39), rising at E and setting at W. Here $\angle ZPE = \angle ZPW = 90°$, and the day and night are each 12h. long. The Sun transits S. of the zenith at Q, where $ZQ = z = \phi$.

From March 21st to June 21st, δ increases from 0 to ϵ, and the Sun's diurnal path changes from EQW to $E'Q'W'$.

The hour angles at rising and setting increase from ZPE and ZPW to ZPE' and ZPW', respectively; hence the days increase and the nights decrease in length. The day is longest on June 21st, when the hour angle ZPE' is greatest. The increase in the day is proportional to the angle EPE', and is greater the greater the latitude ϕ.

At first the Sun transits S. of the zenith, and $z = \phi - \delta$.

When $\delta = \phi$, $z = 0$, and the Sun is directly overhead at noon.

After this, the Sun transits N. of the zenith, and $z = \delta - \phi$.

On June 21st, z attains its maximum N. value $ZQ' = \epsilon - \phi$.

From June 21st to Sept. 23rd, the phenomena occur in the reverse order. The diurnal path changes gradually back to EQW. The day diminishes to 12h. The Sun, which at first continues to transit N. of the zenith, becomes once more vertical at noon when δ again $= \phi$, and than transits S. of the zenith.

From Sept. 23rd to Dec. 22nd, the Sun's path changes from EQW to $E''Q''W''$.

The eastern hour angle at sunrise decreases to ZPE''; thus the days shorten and the nights lengthen. The day is shortest on Dec. 22nd.

Also z increases from ϕ to $\phi + \epsilon$.

On Dec. 22nd, z attains the maximum value $ZQ'' = \phi + \epsilon$, and the Sun is then furthest from the zenith at noon.

From Dec. 22nd to March 21st, the length of the day increases again to 12 hours, and the Sun's meridian zenith distance decreases to $z = \phi$.

108. On the Tropic of Cancer

Here $\phi = \epsilon$. The variations in the lengths of day and night partake of the same general character as in the Torrid Zone. But the Sun only just reaches the zenith at noon once a year, namely, on the longest day, June 21st. At other times the Sun is south of the zenith at noon, and z attains the maximum value 2ϵ on December 22nd.

109. In the North Temperate Zone

In the North Temperate Zone $\phi > \epsilon$ but $< 90° - \epsilon$. Here the variations in the lengths of day and night are similar, but more marked, owing to the greater latitude.

On March 21st, the Sun describes the equator *EQWR* (Fig. 40), which is bisected by the horizon; hence the day is 12h. long.

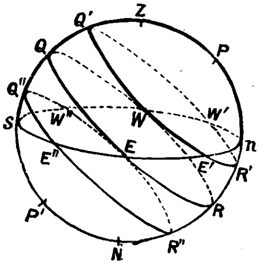

FIG. 40.

The length of the day increases from March 21st to June 21st. The day is longest on June 21st, when the Sun describes *E'Q'W'R*, and the hour angles *ZPE'*, *ZPW'* are greatest.

The days diminish to 12h. on Sept. 23rd, when the Sun again describes *EQWR*. The day is shortest on Dec. 22nd, when the Sun describes *E''Q''W''R''*.

From Dec. 22nd to March 21st, the days increase in length, and on March 21st the day is again 12 hours long.

The difference between the longest and shortest days is the time taken by the Sun to describe the angles *E'PE''*, *W''PW'*, and is therefore:

$$= \tfrac{1}{15}\,(\angle E'PE'' + \angle W''PW') = \tfrac{2}{15} \cdot \angle E'PE'',$$

the time being measured in hours and the angles in degrees.

It will be seen that $\angle E'PE''$ is greater in Fig. 40 than in Fig. 39, thus the variations are more marked in the temperate zone than in the torrid zone. The variations increase as the latitude increases.

The Sun never reaches the zenith in the temperate zone, but always transits south of the zenith. The Sun's zenith distance at noon is least on June 21st, when $z = ZQ' = \phi - \epsilon$, and is greatest on Dec. 22nd, when $z = ZQ'' = \phi + \epsilon$. At the equinoxes (March 21st and Sept. 23rd), $z = ZQ = \phi$.

110. On the Arctic Circle

Here $\phi = 90° - \epsilon$. Hence on June 21st, when the Sun's N.P.D. $= 90° - \epsilon$, the Sun at midnight will only just graze the horizon at the north point without actually setting. On Dec. 22nd at noon, the Sun's Z.D. = 90°, and the Sun will just graze the horizon without actually rising. As in the preceding case, the days increase from Dec. 22nd to June 21st, and decrease from June 21st to Dec. 22nd; on March 21st and Sept. 23rd, the day and night are each 12h. long.

111. In the Arctic Zone

In the Arctic Zone we have $\phi > 90° - \epsilon$, and the variations are somewhat different (Fig. 41).

On March 21st, the Sun describes the circle *EQW*, and the day is 12h. long.

As δ increases, the days increase and the nights decrease, and this continues until $\delta = 90° - \phi$. When this happens, the Sun at midnight only grazes the horizon at *n*.

Subsequently, while $\delta > 90° - \phi$, the Sun remains above the horizon during the whole of the day, circling about the pole like a circumpolar star. This period is called the *Perpetual Day*.

During the perpetual day, the Sun's path continues to rise higher in the heavens every twenty-four hours until June 21st, when the Sun traces out the circle *R'Q'*. The Sun's least and greatest zenith distances will then be $ZQ' = \phi - \epsilon$, and $ZR' = 180° - \epsilon - \phi$ respectively.

After June 21st, the Sun's path will sink lower and lower.

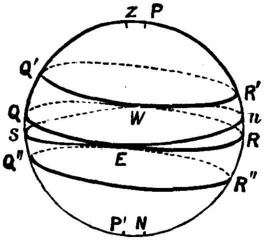

FIG. 41.

When δ is again $= 90° - \phi$ the perpetual day will end. Subsequently, the Sun will be below the horizon during part of each day. The days will then gradually shorten and the nights lengthen.

On Sept. 23rd, the Sun will again describe the circle *EQW*, and the day and night will each be 12 hours long.

The days will continue to diminish till the Sun's south declination $\delta' = 90° - \phi$. When this happens the Sun at noon will only just graze the horizon at *s*.

While $\delta' > 90° - \phi$, the Sun remains continually below the horizon. This period is called the *Perpetual Night*.

On Dec. 22nd the Sun traces out the circle *R''Q''* below the horizon.

When δ' is again $= 90° - \phi$, the perpetual night will end.

Subsequently, the day will gradually lengthen until March 21st, when it will again be 12 hours long.

112. At the North Pole

Here (Fig. 42) the phenomena are much simpler. The celestial equator coincides with the horizon. Hence, from March 21st to

Sept. 23rd, the Sun will be above the horizon, and there will be perpetual day. The Sun's altitude will attain its greatest value ϵ on June 21st, when the Sun will trace out the circle $Q'R'$.

From Sept. 23rd to March 21st there will be perpetual night. The Sun will be at its greatest depth below the horizon on Dec. 22nd, when it will trace out the circle $Q''R''$.

113. Phenomena in the Southern Hemisphere

At a place south of the equator, the variations will partake of the same general character as those in the corresponding north latitude, but the seasons will be reversed. The south pole will be above the horizon, instead of the north pole, and the days will increase in length as the Sun passes to the south of the equator. In fact, if we consider two antipodal points or places at opposite ends of a diameter of the Earth, the day at one place will coincide with the night at the other.

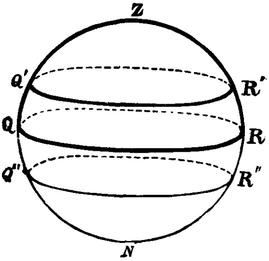

FIG. 42.

Hence, at any place between the equator and antarctic circle, Dec. 22nd will be the longest day, and June 21st the shortest.

Within the antarctic circle there will be perpetual day for a certain period before and after Dec. 22nd, and perpetual night for a certain period before and after June 21st.

The variations in the Sun's north zenith distance at noon will be the same as the variations in the south zenith distance in the corresponding north latitude six months earlier.*

114. The Seasons

Having thus described the variations in the Sun's daily path at different times and places, we shall now show how these variations account for the alternations of heat and cold on the Earth.

Astronomically, the four seasons are defined as the portions into which the year is divided by the equinoxes and the solstices. Thus, in northern latitudes,

Spring	commences	at the	Vernal Equinox	(March 21st),
Summer	,,	,,	Summer Solstice	(June 21st),
Autumn	,,	,,	Autumnal Equinox	(Sept. 23rd),
Winter	,,	,,	Winter Solstice	(Dec. 22nd).

It is obvious that the temperature at any place will depend in a great measure upon the length of the day. The portion of the Earth's surface for which the Sun is above the horizon is receiving a considerable portion of the heat of his rays, the remaining portion being absorbed by the Earth's atmosphere through which the rays have to pass. From the portion of the surface for which the Sun is below the horizon, heat is radiating away into space, although the heated atmosphere retards this radiation to a considerable extent. Thus, on the whole, the Earth is most heated when the days are longest, and conversely.

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* The student will find it instructive to trace out *fully* the variations in S. latitudes corresponding to those described in Arts. 106–112.

The variations in the Sun's meridian altitude have a still greater influence on the temperature. When the Sun's rays strike the surface of the Earth nearly perpendicularly, the same pencil of rays will be spread over a smaller portion of the surface than when the rays strike the surface at a considerable angle; hence the quantity of heat received on a square foot of the surface will be greatest when the Sun is most nearly vertical. The Sun's heating power when above the horizon is proportional to the cosine of the Sun's zenith distance or the sine of its altitude.

In this statement, however, the absorption of heat by the Earth's atmosphere has been neglected. But when the Sun's rays reach the Earth obliquely, they will have to pass through a greater extent of the Earth's atmosphere, and will, therefore, lose more heat than when they are nearly vertical. This cause will still further increase the effect of variations in the Sun's altitude in producing variations in the temperature.

115. Further Details

BETWEEN THE TROPICS.—Here the combination of the two causes above described tends to produce high temperatures, subject only to small variations during the year. The Sun's meridian altitude is always very great, and the variations in the lengths of day and night are small. If the latitude be north, the Sun's heating power is greatest while the Sun transits north of the zenith. During this period the Sun's meridian altitude is least when the days are longest. Thus the effects of the two causes in producing variations in the Sun's heat counteract one another, to a certain extent, and give rise to a period of nearly uniform but intense heat.

IN THE NORTH TEMPERATE ZONE.—In the North Temperate Zone the Sun is highest at noon when the days are longest, and therefore both causes combine to make the spring and summer seasons warmer than autumn and winter. But the highest average temperatures occur some time *after* the summer solstice; for the Earth is gaining heat most rapidly about the summer solstice, and it continues to gain heat, but less rapidly, for some time afterwards. Similarly, the Earth is losing heat most rapidly at the winter solstice, and it continues to lose heat, but less rapidly, for some time afterwards. For this reason, summer is warmer than spring, and winter is colder than autumn.

As we go northwards, the Sun's altitude at noon becomes generally lower throughout the year, and the climate therefore becomes colder. At the same time, the variations in the length of the day become more marked, causing a greater fluctuation of temperature between summer and winter.

WITHIN THE ARCTIC CIRCLE.—Here there is a warm period during the perpetual day, but the Sun's altitude is never sufficiently great to cause very intense heat. During the perpetual night the cold is extreme; and the low altitude of the Sun, when above the horizon at intermediate times, gives rise to a very low average temperature during the year.

IN THE SOUTHERN HEMISPHERE.—In this Hemisphere the seasons are reversed; for, in south latitude ϕ, when the Sun's south declination is δ, the same amount of heat will be received from the Sun as in north latitude ϕ, when his north declination is δ. Hence, the seasons corresponding to our spring, summer, autumn and winter will begin respectively on September 23rd, December 22nd, March 21st, and June 21st, and will be separated from the corresponding seasons in north latitude by six months.

116. Other Causes affecting the Seasons and Climate

It is found (as will be explained in section III) that the Sun's distance from the Earth is not quite constant during the year. The Sun is nearest the Earth about January 3rd, and furthest away on July 4th (these are the dates of perigee and apogee respectively). But the heat radiation received from the Sun varies inversely as the square of the Sun's distance. Hence the Earth receives, on the whole, more heat from the Sun after the winter solstice than after the summer solstice. This cause tends to make the winter milder and the summer cooler in the northern hemisphere, and to make the summer hotter, and the winter colder in the southern hemisphere.

The variations in the Sun's distance are, however, small, and their effect on the seasons is more than counteracted by purely terrestrial causes arising from the unequal distribution of land and water on the Earth. The sea has a much greater capacity for heat than the rocks forming the land; it is not so readily heated or cooled. In the southern hemisphere the sea greatly preponderates, the largest land-surfaces being in the northern hemisphere. Hence, the climate of the southern hemisphere is generally more equable, and the seasons are not so marked as in the northern hemisphere, quite in contradiction to what we should expect from the astronomical causes.

117. Times of Sunrise and Sunset

The *Nautical Almanac* gives each year a table of the local mean times of sunrise and sunset for every day of the year, and for 13 selected values of latitude, extending from the equator to 60° North. It also gives a table by which the times for a southern latitude can be inferred from those for the corresponding northern latitude by taking the times for the northern latitude for a different day of the year, about six months away from the actual day, and applying a tabulated correction which depends upon the difference of the equations of time for the two dates.

The times given are for the Sun's *upper* limit, so that sunrise corresponds to the moment when the Sun, affected by refraction (Chapter VI), is about to begin to appear above the horizon and sunset corresponds to the moment when the Sun, also affected by refraction, has just disappeared entirely below the horizon.

The Sun's declination very nearly repeats itself on the same day after four years, and the times of sunrise and sunset do so likewise.

The times found in this manner will be the *local solar times.* To reduce to Greenwich mean time we must add or subtract 4m. for each degree of longitude, according as the place is W. or E. of Greenwich.

Examples.—(1) *Find the times of sunrise and sunset at Glasgow (Long.* 4° 18′ *W, lat.* 55° 53′ *N.) on Dec.* 1*st,* 1940.

The *Nautical Almanac* tables give the local mean times of sunrise and sunset for latitudes + 54° and + 56°. Interpolating for the latitude of Glasgow, we obtain

	Sunrise	Sunset
Local mean time	8h. 5m.	15h. 32m.
Add longitude (4° 10′ = 17m.) ...	+ 17	+ 17
G.M.T.	8h. 22m.	15h. 49m.

The Sun therefore rises at 8.22 a.m. and sets at 3.49 p.m.

(2) *Find the South African Standard Times of sunrise and sunset at Capetown (Long.* 18° 44′ *E, lat.* 33° 56′ *S.) on April* 16*th,* 1940.

The *Nautical Almanac* table for southern latitudes gives October 19th as the date for northern latitudes corresponding to April 16th for southern latitudes.

Interpolating the table for northern latitudes, under date October 19th, for the appropriate latitude, we obtain for the local mean times of sunrise and sunset respectively 6h. 8m. and 17h. 22m. respectively. The correction for date April 16th is + 15m. We thus obtain

	Sunrise	Sunset
Local mean time	6h. 23m.	17h. 37m.
Subtract longitude (= 1h. 15m.)	−1 15	−1 15
G.M.T.	5h. 8m.	16h. 22m.

Hence South African Standard Time (2 hours fast on G.M.T.) of sunrise is 7.8 a.m. and of sunset is 6.22 p.m.

118. To find the length of the Perpetual Day and Night at places within the Arctic or Antarctic Circles

The perpetual day lasts while the Sun's declination at local midnight is greater than the colatitude (or complement of the latitude), during spring and summer. The perpetual night lasts while the Sun's S. decl. at local noon is greater than the colat. during autumn and winter. The Sun's decl. at Greenwich midnight being given for every day of the year, in the *Nautical Almanac*, it is easy to find approximately the durations of the perpetual day and night in any given latitude greater than 66° 33′.

119. To find the time the Sun takes to Rise or Set

Let D'' be the Sun's angular diameter, measured in seconds. When the Sun begins to rise, his upper limb just touches the horizon, and his centre is at a depth $\frac{1}{2}D''$ below the horizon. When the Sun has just finished rising, his lower limb touches the horizon, and his centre is at an altitude $\frac{1}{2}D''$ above the horizon. During the sunrise, the centre rises through a vertical height D''. The problem is closely similar to that of Art. 90, where the effect of dip is considered. Hence if t seconds be the time taken in rising, δ the declination of the Sun's centre, and x the inclination to the vertical of the Sun's path at rising ($Hx'x$ or nxP, Fig. 34) we have

$$\begin{aligned} t &= \tfrac{1}{15}\, D'' \sec\delta \sec x, \\ &= 4 \sec\delta \sec x \times (\odot\text{'s angular diameter in minutes}). \end{aligned}$$

As in Art. 90, this gives, for a place on the equator,

$$t = \tfrac{1}{15}\, D'' \sec\delta,$$

and at an equinox in latitude ϕ,

$$t = \tfrac{1}{15}\, D'' \sec\phi.$$

EXAMPLE.—At an equinox in latitude 60°, the ⊙'s angular diameter being 32′, the time taken to rise will be $= 4 \times 32 \times \sec 60°$ seconds $= 256$s. $=$ 4m. 16s.

120. Note

It may be mentioned that, owing to atmospheric refraction, the Sun really appears to rise earlier and set later than the times calculated by theory. As the phenomena of refraction will be discussed more fully in Chapter VI, it will be sufficient to mention here that the rays of light from the Sun are bent to such an extent by the Earth's atmosphere that the *whole* of the Sun's disc is visible when it would just be entirely below the horizon if there were no atmosphere.

Moreover, there is daylight, or rather twilight, for some time after the Sun has vanished, so that what is commonly called night does not begin for some time after sunset.

For the same reasons, the perpetual day at a place in the arctic circle is lengthened, and the perpetual night shortened, by several days.

The time taken in rising and setting is, however, practically unaffected in moderate latitudes.

II.—THE ECLIPTIC

121. The First Point of Aries

In determining the right ascensions of stars, the first step must necessarily be to find accurately the position of the first point of Aries, since this point is taken as the origin from which R.A. is measured. Observations of the stars will only enable *differences* of R.A. to be determined. Thus, for instance, the sidereal time at which a star transits across the meridian is equal to the R.A. of the star. We can determine with the transit circle, by the methods that will be explained in Chapter XIII, the times of transit of stars across the meridian; we thereby obtain the differences of their R.A.'s. If the R.A. of one star is known, the R.A.'s of the others are then obtained. But since the origin from which R.A.'s are measured is defined as the point of intersection of the ecliptic and equator and the ecliptic is determined by the apparent path of the Sun in the sky, observations of the Sun are required for finding the position of the first point of Aries. Two methods by which this position may be found will be described.

122. First Method for finding the First Point of Aries

The position of ♈ may be found thus :—At the vernal equinox the Sun's declination changes from south to north, or from negative to positive. Let the Sun's declination be observed by the transit circle at the preceding and following noons, and let the observed values be $-\delta_1$ and $+\delta_2$ (*i.e.* δ_1 S., and δ_2 N.). Let t_1, t_2 be the corresponding times of transit of the Sun's centre, as observed by the astronomical clock, and let T, the time of transit of any star, be also observed. Then :

$T - t_1 =$ difference of R.A. of Star and Sun at first noon.

$T - t_2 =$,, ,, ,, at second noon.

Let $T - t_1 = a_1$ and $T - t_2 = a_2$. We have :—

Increase in Sun's decl. in the day $= \delta_2 - (-\delta_1) = \delta_2 + \delta_1$,

,, ,, R.A. ,, $= t_2 - t_1 = a_1 - a_2$,

and both coordinates increase at an approximately uniform rate during the day.

Therefore the ☉'s decl. will have increased from $-\delta_1$ to 0 in a time $\delta_1/(\delta_1 + \delta_2)$ of a day, and the corresponding increase in R.A. will be :—

$$(a_1 - a_2) \times \delta_1/(\delta_1 + \delta_2)$$

The Sun is now at ♈, so that ☉'s R.A. is now = 0. Hence :—

$$\text{The star's R.A.} = \alpha_1 - \frac{(\alpha_1 - \alpha_2)\delta_1}{\delta_1 + \delta_2} = \frac{\alpha_1\delta_2 + \alpha_2\delta_1}{\delta_1 + \delta_2}.$$

123. Second Method for finding the First Point of Aries

The principle of the method now to be described is as follows :— Let S_1, S be two positions of the Sun shortly after the vernal and before the autumnal equinox respectively, and such that the declinations S_1M_1 and SM are equal. Then the right-angled triangles ♈M_1S_1 and ♎ MS will be equal in all respects, and we shall therefore have ♈M_1 = ♎ M.

At noon, some day shortly after March 21st, the Sun is observed with the transit circle, say when at S_1. We thus determine its meridian zenith distance z_1, and also the difference between the times of transit of the Sun and some fixed star x, whose R.A. is required. This difference, which is the difference of R.A. of the Sun and star, we shall call α_1. If δ_1 be the Sun's declination, and ϕ the observer's latitude, we shall have

$$S_1M_1 = \delta_1 = \phi - z_1,$$
$$M_1N = \alpha_1.$$

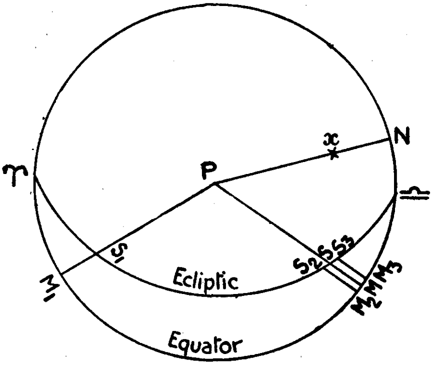

FIG. 43.

We now have to determine MN, the difference of R.A. of the Sun and star shortly before September 23rd, when the Sun's declination SM is again equal to δ_1. But the Sun can only be observed with the transit circle at noon, and it is highly improbable that the Sun's declination will again be exactly equal to δ_1 at noon on any day. We shall, however, find two consecutive days in September on which the declinations at noon, S_2M_2 and S_3M_3, are respectively greater and less than δ_1.

Let z_2 and z_3 be the observed meridian zenith distances at S_2 and S_3 ; δ_2 and δ_3 the corresponding declinations S_2M_2, S_3M_3 ; α_2 and α_3 the observed arcs M_2N and M_3N, being the differences of R.A. of the Sun and star on the two days.

During the day which elapses between the observations at S_2, S_3, we may assume that the Sun's decl. and R.A. both vary at a uniform rate, so that the change in the decl. is always proportional to the corresponding change in R.A.[*] Therefore :—

$$\frac{M_2M}{M_2M_3} = \frac{S_2M_2 - SM}{S_2M_2 - S_3M_3} = \frac{\delta_2 - \delta_1}{\delta_2 - \delta_3}$$

$$\text{or} \quad M_2M = \frac{\delta_2 - \delta_1}{\delta_2 - \delta_3} M_2M_3 = \frac{\delta_2 - \delta_1}{\delta_2 - \delta_3} (\alpha_2 - \alpha_3),$$

$$\text{and} \quad MN = M_2N - M_2M = \alpha_2 - \frac{\delta_2 - \delta_1}{\delta_2 - \delta_3} (\alpha_2 - \alpha_3).$$

Now we have shown that :—

$$\aries M_1 = M\libra$$

i.e. $\aries N - M_1N = MN - \libra N$

or $MN + M_1N = \aries N + \libra N = 2\aries N - 180^\circ = 2\aries N - 12 \text{ hours};$

$$\text{therefore} \quad \aries N = 6\text{h.} + \tfrac{1}{2}(M_1N + MN)$$

$$\text{or} \quad \aries N = 6\text{h.} + \tfrac{1}{2}\left\{\alpha_1 + \alpha_2 - \frac{\delta_2 - \delta_1}{\delta_2 - \delta_3}(\alpha_2 - \alpha_3)\right\}.$$

This determines $\aries N$, the star's R.A., in terms of α_1, α_2, α_3, the observed differences between the times of transit of the Sun and star, and δ_1, δ_2, δ_3, the Sun's declinations at the three observations. But we need not even find the declinations, for

$$\delta_1 = \phi - z_1, \quad \delta_2 = \phi - z_2, \quad \delta_3 = \phi - z_3;$$

therefore, substituting, we have:—

$$\text{The star's R.A., } \aries N = 6\text{h.} + \tfrac{1}{2}\left\{\alpha_1 + \alpha_2 - \frac{z_1 - z_2}{z_3 - z_2}(\alpha_2 - \alpha_3)\right\}.$$

This method was first used in the seventeenth century by Flamsteed, the first Astronomer Royal.

In applying either of the above methods to the numerical calculation of the right ascension of any star, it is advisable to follow the various steps as we have described them, instead of merely substituting the numerical values of the data in the final formulae.

*124. The Advantages of Flamsteed's Method

Among these the following may be mentioned:—

1st. The method does not require a knowledge of the latitude, for we do not require to find the Sun's declination. Hence, errors arising from inaccurate determination of the latitude are avoided.

2nd. One great source of error in determining Z.D.'s is the refraction of the Earth's atmosphere. Since the Sun is observed each time in the same part of the sky, z_1, z_2, z_3 will be nearly equally affected by refraction. Hence, the "principle of proportional parts" will hold so that the small differences in the true Z.D.'s are proportional to the differences in the observed Z.D.'s. Hence we may use the observed Z.D.'s *uncorrected* for refraction. Since, however, refraction varies with the barometer and thermometer, it must not be neglected where an accurate result is desired.

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[*] In other words, we assume, as in trigonometry, that the "principle of proportional parts" holds for the small variations in decl. and R.A. during the day.

Example.—*Find the Right Ascension of Sirius and the clock errors in March and Sept.* 1940, *from the following data, the rate of the clock being supposed correct.* (*Decl. of Sirius* = 16° 37′ 57″ *S.*)

1940.	Mar. 24.	Sept. 18.	Sept. 19.
Dec. of Sun at noon	1° 28′ 25″	1° 49′ 40″	1° 26′ 24″
Time of transit of Sun	0h. 13m. 48s.	11h. 43m. 4s.	11h. 46m. 39s.
Time of transit of *Sirius*	6h. 42m. 43s.	6h. 42m. 29s.	6h. 42m. 29s.

On Mar. 24th (R.A. of *Sirius*) — (Sun's R.A.) = 6h. 42m. 43s. — 0h. 13m. 48s.
= 6h. 28m. 55s.

Hence, in angular measure, the difference of R.A. is about 97°.

Draw the diagram as in Fig. 49, but make the angle $S_1PN = 97°$; N will therefore lie *between* M_1 and M_2, instead of where represented.

Also, since *Sirius* is south of the equator, it should be represented at a point x on PN produced through N. In this figure we shall have :—

$S_1M_1 = 1° 28' 25''$; $M_1N =$ 6h. 42m. 43s. — 0h. 13m. 48s. = 6h. 28m. 55s.
$S_2M_2 = 1° 49' 40''$; $NM_2 =$ 11h. 43m. 4s. — 6h. 42m. 29s. = 5h. 0m. 35s.
$S_3M_3 = 1° 26' 24''$; $NM_3 =$ 11h. 46m. 39s. — 6h. 42m. 29s. = 5h. 4m. 10s.

Also, SM is by construction equal to S_1M_1. Hence, applying the principle of proportional parts, we have

$$\frac{M_2M}{M_2M_3} = \frac{S_2M_2 - S_1M_1}{S_2M_2 - S_3M_3} = \frac{21' 15''}{23' 16''} = \frac{1275}{1396},$$

and $M_2M_3 =$ 3m. 35s. = 215s.;

therefore $M_2M = 215 \times 1275/1396 =$ 196 seconds;

and $NM =$ 5h. 0m. 35s. + 3m. 16s. = 5h. 3m. 51s.

Now, $NM_1 - NM = N\Upsilon - N\Libra = 2N\Upsilon - 12\text{h}.$

Hence, $\Upsilon N =$ 6h. + $\frac{1}{2}(NM_1 - NM)$ = 6h. + $\frac{1}{2}$(6h. 28m. 55s. — 5h. 3m. 51s.)
= 6h. + $\frac{1}{2}$(1h. 25m. 4s.) = 6h. 42m. 32s.

Thus the right ascension of *Sirius* = 6h. 42m. 32s.

Also, clock error in March = 6h. 42m. 32s. — 6h. 42m. 43s. = — 11s.
" " " " Sept. = 6h. 42m. 32s. — 6h. 42m. 29s. = + 3s.

125. Precession of the Equinoxes

Thus far we have treated the first point of Aries as being fixed, and this will evidently be the case if the equator and ecliptic are fixed in direction. But if the right ascensions of various stars are observed over an interval of several years, it will be found that the position of the first point of Aries is slowly changing, and that it moves along the ecliptic in the retrograde direction at the rate of about 50·26″ in a year.

This motion is called Precession of the Equinoxes, or, briefly, *Precession* (see Art. 28).

Precession is the consequence almost entirely of gradual changes in the direction of the plane of the equator, the ecliptic remaining almost fixed among the stars. Its effect is to produce a yearly increase of 50·26″ in the celestial longitudes of all stars, their latitudes being constant.

In a large number of years the effect of precession will be considerable. Thus, ♈ will perform a complete revolution in the period $\frac{360 \times 60 \times 60}{50{\cdot}26}$ years, *i.e.* about 25,800 years. Actually the period of a complete revolution is variable by many years; for precession varies with the amount of obliquity, and this fluctuates, according to Newcomb, between the limits 24° 13′, attained 9100 years ago, and 22° 35′, which will be attained 9600 years hence.

About the year 60 B.C. the vernal equinoctial point moved out of the constellation Aries into the adjoining constellation Pisces. It still, however, retains the old name of "First Point of Aries."

It will move into the constellation Aquarius about the year 2740 A.D.

126. Determination of Obliquity of Ecliptic

The method now used for finding the obliquity of the ecliptic is similar in principle to that of Art. 40, but the Sun's meridian zenith distance is observed by means of the transit circle instead of the gnomon.

The obliquity is equal to the Sun's greatest declination at one of the solstices. Since observations of the Sun with the transit circle can only be performed at apparent noon, while the maximum declination will probably occur at some intermediate hour of the day, it will be necessary to apply a correction for the change of declination in the interval.

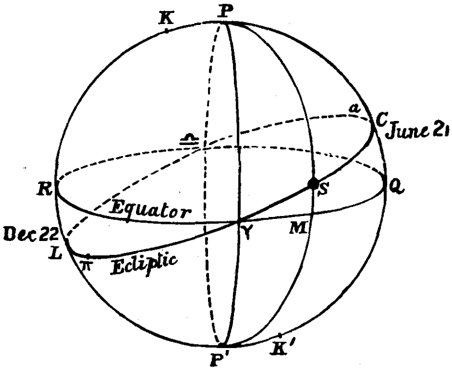

FIG. 44.

The correction required can be derived as follows: in the triangle ♈MS, the sides ♈M, MS are α, δ, the right ascension and declination of the Sun. The angle S♈M is ϵ, the obliquity and the angle at M is a right angle. From the formulae for a right-angled triangle (Art. 10), we have :—

$$\tan \epsilon \sin \alpha = \tan \delta.$$

Suppose that when α becomes $(\alpha + s)$, δ becomes $(\delta - x)$, then :—

$$\tan \epsilon \sin (\alpha + s) = \tan (\delta - x).$$

$$\text{Therefore } \tan \epsilon \{\sin (\alpha + s) - \sin \alpha\} = \tan (\delta - x) - \tan \delta$$

$$\text{or} \quad 2 \tan \epsilon \sin \frac{s}{2} \cos \left(\alpha + \frac{s}{2}\right) = - \frac{\sin x}{\cos \delta \cos (\delta - x)}.$$

Now suppose that α, δ refer to the solstices and that s and x are small. α is 90° when the Sun is at C and 270° when it is at L; δ is $+\epsilon$ at C and $-\epsilon$ at L. Then :—

$$\sin \frac{s}{2} = \frac{s}{2}; \quad \cos \left(\alpha + \frac{s}{2}\right) = \pm \frac{s}{2}; \quad \sin x = x; \quad \cos (\delta - x) = \cos \delta$$

$$\text{so that} \quad x = \pm \frac{s^2}{2} \tan \epsilon \cos^2 \delta$$

$$= \pm \frac{s^2}{2} \sin \epsilon \cos \epsilon.$$

x and s are here expressed in circular measure. If s is expressed in seconds of time and x in seconds of arc, we obtain

$$x = \pm \cdot 000273 \sin 2\epsilon \,.\, s^2.$$

We can put $\sin 2\epsilon = \cdot 730$, since the value of ϵ is known approximately, giving

$$x = \pm \cdot 000199'' . s^2$$

where s is the number of seconds of time by which the Sun's observed R.A. differs from 6h. or 18h. The observed declination must always be increased numerically by the correction x in order to give the obliquity.

As an example of the magnitude of the correction, if $s = 200$ seconds of time, the correction is 7·96″.

126a. Obliquity and Longitude

When the position of ♈ has been determined, the obliquity can also be found by a single observation of the Sun's R.A. and decl. For we thus find the two sides ♈*M*, *MS* of the spherical triangle ♈*MS*, and these data are sufficient to determine both the obliquity *M*♈*S*, and the Sun's longitude ♈*S*.

III.—THE EARTH'S ORBIT ABOUT THE SUN

127. Observations of the Sun's Relative Orbit

By daily observations with the transit circle, the decl. and R.A. of the Sun's centre at noon are found for every day of the year. From these data the Sun's longitude is calculated, as in Art. 126*a*, by solving the spherical triangle ♈*SM* (Fig. 44). If the obliquity of the ecliptic is also known, we have three data, any two of which suffice to determine the longitude ΥS. Thus the accuracy of the observations can be tested, and the Sun's motion at various times of the year can be accurately determined.

Although the determination of the Sun's actual distance from the Earth in miles is an operation of great difficulty, it is easy to *compare* the Sun's distance from the Earth at different times of the year, for this distance is always inversely proportional to the Sun's angular diameter. This property is proved in Art. 13, but numerous simple illustrations may also be used to show that the angular diameter of any object varies inversely with its distance.*

The Sun's angular diameter may be observed by means of any form of micrometer. The Sun's distances at two different observations will be in the reciprocal ratio of the corresponding angular diameters. Thus, by daily observation, the changes in the Sun's distance may be investigated.

If the circular measure of the Sun's angular diameter is $2r$, then πr^2 is called the Sun's *apparent area*. In fact, this is the area of a disc which would look the same size as the Sun if placed at unit distance from the eye.

Example.—*The Sun's angular diameter is* 31′ 32″ *at midsummer, and* 32′ 36″ *at midwinter. Find the ratio of its distances from the Earth at these times.*

The distances being inversely proportional to the angular diameters, we have

$$\frac{\text{Dist. at midsummer}}{\text{Dist. at midwinter}} = \frac{32' \, 36''}{31' \, 32''} = \frac{1956}{1892} = \frac{489}{473} = 1\tfrac{1}{30} \text{ nearly.}$$

Hence the Sun is further at midsummer than at midwinter, in the proportion of very nearly 31 to 30.

128. Kepler's First and Second Laws

We may now construct a diagram of the Sun's relative orbit. Let E represent the position of the Earth, $E\Upsilon$ the direction of the first point of Aries. Then, by making the angle ΥES equal to the Sun's longitude at noon, and ES proportional to the Sun's distance, we obtain a series of points $S, S' \ldots, S_1 \ldots$, representing the Sun's position in the plane of the ecliptic, as seen from the Earth at noon on different days of the year. Draw the curve passing through the points $S, S' \ldots, S_1 \ldots$; this curve will represent the Sun's orbit relative to the Earth, and it will be found that

I. *The Sun's annual path is an ellipse, of which the Earth is one focus,*

II. *The rate of motion is everywhere such that the radius vector* (*i.e. the line joining the Earth to the Sun*) *sweeps out equal areas in equal intervals of time.*

---

* This law only holds when the object subtends so small an angle that its sine and its circular measure are appreciably equal.

These laws were discovered by Kepler for the motion of Mars about the Sun, and he subsequently generalised them by showing that the orbits of all the other planets, including the Earth, obeyed the same laws. In their general form they are known as Kepler's First and Second Laws. (See p. 199).

129. Perigee and Apogee

When the Sun's distance from the Earth is least, the Sun is said to be in *perigee*. When the distance is greatest, the Sun is said to be in *apogee*.

The positions of perigee and apogee are called the two *Apses* of the orbit; they are indicated at p, a in Fig. 45. The line pEa joining them is the major axis of the ellipse (Ellipse, 4), and is sometimes also called the *apse line*.

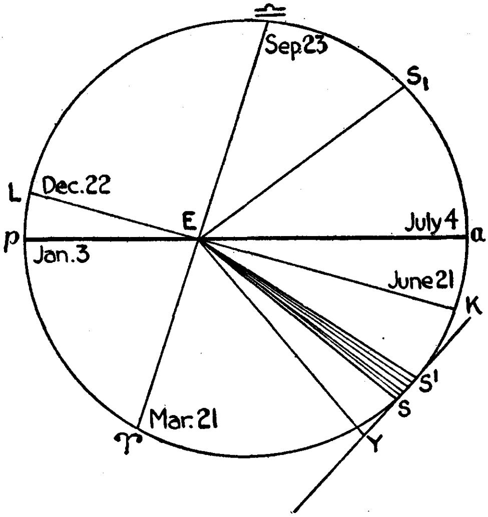

FIG. 45.

130. Verification of Kepler's First Law

The Sun's angular diameter is observed to be greatest about Jan. 3rd, and least about July 4th; we therefore conclude that these are the days on which the Sun passes through perigee and apogee respectively. The positions of perigee and apogee being thus found, the angle between $E♈$ and Ep is known, which is the longitude of perigee, when measured anticlockwise from ♈ to p.

From the winter solstice to perigee is about 12 days. Hence, during this interval the Sun will have moved through an angle of about 12°; thus:—

$$\text{longitude of perigee} = 270^\circ + 12^\circ = 282^\circ \text{ roughly.}$$

To verify that the orbit is an ellipse, it is now only necessary to show that the relation connecting ES and the angle pES is the same as that which holds in the case of the ellipse. If the orbit is an ellipse of eccentricity e, we must have :—

$$ES \times (1 + e \cos pES) = l \text{ (a constant).} \quad \text{(p. 373).}$$

Therefore the Sun's angular diameter must be always proportional to $1 + e \cos pES$.

As the result of numerous observations, it is found that this is actually the case, and the truth of Kepler's First Law for the Sun's orbit relative to the Earth is confirmed.

131. To find *e*, the Eccentricity of the Ellipse

A simple plan is to compare the greatest and least angular diameters of the Sun, *i.e.* the diameters at perigee and apogee. Since at these positions pES becomes 0° and 180° respectively, we have, from the above—

$$\begin{aligned}\text{Angular diameter at } p : \text{ang. diam. at } a &= 1/Ep : 1/Ea \\ &= 1 + e \cos 0^\circ : 1 + e \cos 180^\circ \\ &= 1 + e : 1 - e,\end{aligned}$$

from which proportion e can be found.

The greatest angular diameter of the Sun is 32′35″, about January 3rd; the least is 31′ 31″, about July 4th. We have, therefore—

$$\frac{1+e}{1-e} = \frac{32'\,35''}{31'\,31''} = \frac{1955}{1891}.$$

So that $e = 64/3846 = 1/60$, approximately.

The eccentricity can, however, be found far more accurately by studying the rate of the Sun's motion in longitude at different times of the year. Using the method of Art. 51, we find that early in April the Sun is 1° 54′ ahead of its mean place, while early in October it is the same amount behind its mean place. We have thus an arc of 3° 48′ from which to deduce e, instead of an arc of 64″, using the method of diameters.

Owing to the smallness of e, the orbit is very nearly circular, being, really, much more nearly so than is shown in Fig. 45. However, the Sun is quite appreciably out of the centre of the orbit, being 3 million miles nearer to the Earth in perigee than in apogee.

132. Verification of Kepler's Second Law

It is found, as the result of observation, that the Sun's increase in longitude in a day, at different times of year, is always proportional to the square of the angular diameter, and is, therefore, inversely proportional to the square of the Sun's distance. The following figures illustrate this relationship:—

Date	Daily motion in longitude	Apparent Diameter
Jan. 3rd	3670″	32′ 35″ = 1955″
April 3rd	3547″	32′ 2″ = 1922″
July 4th	3434″	31′ 31″ = 1891″
Oct. 3rd	3547″	32′ 2″ = 1922″

If the apparent diameters are squared and divided by the daily motion in longitude, it will be found that the result is in each case the same, viz. 1041·4.

From this relationship it may be readily deduced, in the following way, that the area described by the radius vector in one day is the same in any part of the orbit:—

Let SS' (Fig. 45) represent the small arc described by the Sun in a day in any part of the orbit. Then the sector ESS' is the area swept out by the radius vector. This sector does not differ perceptibly from the triangle ESS'; therefore, by trigonometry,

$$\text{area } ESS' = \tfrac{1}{2}ES.ES'.\sin SES'.$$

Since the change in the Sun's distance in one day is imperceptible, we may write ES for ES' in the above formula without materially affecting the result; also, since the angle SES' is small, the sine of SES' is equal to the circular measure of the angle SES'. Therefore:—

$$\text{area } ESS' = \tfrac{1}{2}ES^2 \times \angle SES'.$$

But, by hypothesis, the change of longitude SES' varies inversely as ES^2, so that $ES^2 \times \angle SES'$ is constant; it follows that the area ESS' is constant, that is, the area described by the radius vector in a day is constant. Thus, the area described in any number of days is proportional to the number of days, and generally the areas described in equal intervals of time are equal.

133. Deduction from Kepler's Second Law

(i) If the circular measure of the Sun's angular diameter is $2r$, then πr^2 is the Sun's *apparent area*. Hence the Sun's daily rate of change of longitude is proportional to the apparent area of its disc.

(ii) Since the intensity of the Sun's heat and its rate of motion in longitude both vary as the inverse square of its distance, they are proportional to one another. Hence the Earth, as a whole, receives equal amounts of heat while the Sun describes equal angles. In particular, the total quantities of heat received in the four seasons are equal.

(iii) The Sun's longitude changes most rapidly on January 3rd, and least rapidly on July 4th.

(iv) Since the apse line, or major axis, pSa, bisects the ellipse, the time from perigee to apogee is equal to the time from apogee to perigee.

134. The Lengths of the Seasons

If ♈, K, ♎, L represent the Sun's positions at the equinoxes and solstices, we have:—

$$\angle \text{♈} EK = \angle KE\text{♎} = \angle \text{♎} EL = \angle LE\text{♈} = 90^\circ,$$

and it is readily seen from figure 45 that :—

area LE♈ $<$ area ♎ EL $<$ area ♈EK $<$ area KE ♎,

and the lengths of the seasons, being proportional to these areas, are unequal, their ascending order of magnitude being

Winter, Autumn, Spring, Summer.

The actual lengths of the seasons may be readily determined from the motion of the mean Sun.

The equation of time, as defined in Art. 47, is the quantity that must be added to mean time to obtain apparent time. Since, however, apparent solar time is measured by the Sun's hour angle and mean solar time is measured by the hour angle of the mean Sun, the equation of time is the quantity that must be added to the hour angle of the mean Sun to obtain the hour angle of the true Sun. But since hour angle and right ascension are measured in opposite directions, the equation of time must be subtracted from the right ascension of the mean Sun to obtain the right ascension of the true Sun. Since, by the definition of the mean Sun, its right ascension increases uniformly, the lengths of the seasons can be compared.

At the vernal equinox, Sun's R.A. = 0h., equation of time = − 7m. 29s.
„ „ summer solstice „ = 6h., „ „ − 1m. 37s.
„ „ autumnal equinox „ = 12h., „ „ + 7m. 33s.
„ „ winter solstice „ = 18h., „ „ + 13m. 4s.

Thus, in spring, from the vernal equinox to the summer solstice, whilst the R.A. of the true Sun increases from 0h. to 6m. the R.A. of the mean Sun increases from 0h. — (7m. 29s.) = 23h. 52m. 31s. to 6h. — (1m. 37s.) = 5h. 58m. 23s., or by 6h. 5m. 52s. But in a year, the R.A. of the mean Sun increases by 360° = 24h. Hence, since the motion of the mean Sun is uniform, the length of spring is the fraction 6h. 5m. 52s./24h. of the year. Similarly for the other seasons. In this way, the lengths of the seasons are found to be

Spring	92d.	19·2h.
Summer	93d.	15·2h.
Autumn	89d.	19·0h.
Winter	89d.	0·4h.

135. To find the Position of the Apse Line

The Sun's distance remains very nearly constant for a short time before and after perigee and apogee, hence it is difficult to tell the exact instant when this distance is greatest or least. For this reason, the following method is generally used :—

The Sun's longitude is observed at two points, S, S_1, before and after the apse, when its angular diameters, or its rates of motion in longitude, are found to be equal. Then $ES = ES_1$, and the symmetry of the ellipse shows that $\angle pES = \angle pES_1$ and $\angle aES = \angle aES_1$. Hence the longitude of the apse is the arithmetic mean of the Sun's longitudes at the two observations.

PROGRESSIVE MOTION OF APSE LINE

From such observations, extending over a long period of years, it is found that the apse line is not fixed, but has a forward or direct motion in the ecliptic plane of 11·25″ in a year.

This is referred to a fixed direction in space; hence the longitude of the apse increases 11·25″ + 50·26″ = 61·51″ in a year.

136. The Sun's Apparent Annual Motion may be accounted for by supposing the Earth to revolve round the Sun

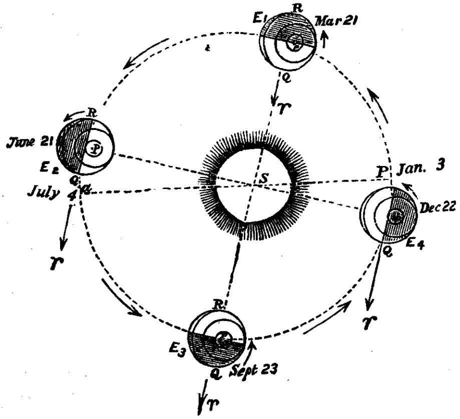

FIG. 46.

The annexed diagram will show how the Sun's annual motion in the ecliptic, as well as the changes in the Seasons, may be accounted for on the theory that the Sun remains at rest while the Earth describes an ellipse round it in the course of the year in a plane inclined at an angle 23° 27′ to the plane of the Earth's equator.

The distance of the nearest of the fixed stars is known to be over 200,000 times as great as the Earth's distance from the Sun. Hence, Art. 14 shows that the directions of the fixed stars will not change to any considerable extent, as the Earth's position varies. We shall, therefore, in the present description, consider the directions of the stars to be fixed. The directions of the various points and circles of the celestial sphere, such as the first point of Aries, will also be fixed.

On March 21st, the Earth is at E_1, and the Sun's direction E_1S determines of ♈, the first point of Aries. The Sun is vertical at a point Q on the equator, and as the Earth revolves about its axis through P, all points on the equator will come vertically under the Sun. There is night all over the shaded portion of the Earth, day over the rest. The great circle bounding the illuminated part passes through the pole P, and, therefore, bisects the small circle traced out by the daily rotation of any point on the Earth; thus, the day and night are everywhere equal. At the pole P the Sun is just on the horizon.

On June 21st, the Earth is at E_2, and the Sun's longitude $\Upsilon E_2 S =$ 90°. The Sun is vertical at a point on the tropic of Cancer. Since the arctic circle is entirely in the illuminated part there is perpetual day over the whole arctic zone.

On September 23rd, the Earth is at E_3, and the Sun's longitude $\Upsilon E_3 S$ is 180°. The Sun is once more vertical at a point R on the equator, and the day and night are everywhere 12 hours long, as they are at E_1.

On December 22nd, the Earth is at E_4, and the Sun's longitude $\Upsilon E_4 S$ (measured in the direction of the arrow) is 270°. The Sun is now at its greatest angular distance south of the equator, and overhead at a point on the tropic of Capricorn; this tropic is not represented, being on the under side of the sphere. Since the arctic circle is entirely within the shaded part there is perpetual night over the whole arctic zone.

NEW DEFINITIONS AND FACTS

According to the theory of the Earth's orbital motion, Kepler's First and Second Laws can be re-stated thus for the Earth.

I. *The Earth describes an ellipse, having the Sun in one focus.*

II. *The radius vector joining the Earth and Sun traces out equal areas in equal times about the Sun.*

The *ecliptic* is now defined as the great circle of the celestial sphere, whose plane is parallel to that of the Earth's orbit.

The Earth is nearest the Sun on January 3rd, and is then said to be in *perihelion*. The Earth is furthest from the Sun on July 4th, and is then said to be in *aphelion*. Thus, when the Sun is in perigee the Earth is in perihelion, when the Sun is in apogee the Earth is in aphelion. The positions of perihelion and aphelion are indicated by the letters p, a in Fig. 46. The line joining them is the *apse line*.

137. Geocentric and Heliocentric Latitude and Longitude

Hitherto we have been dealing only with the directions of the celestial bodies as seen from the Earth.

In dealing with the motion of the planets, it is more convenient, as a rule, to define their positions by the directions in which they would be seen by an observer situated at the centre of the Sun.

In every case, the direction of a celestial body may be specified by the two coordinates, celestial latitude and longitude, which measure respectively the arc of a secondary from the body to the ecliptic and the arc of the ecliptic between this secondary and the first point of Aries (Art. 25).

These coordinates are called the *Geocentric Latitude* and *Longitude* when employed to define the body's *geocentric* position, or position relative to the centre of the Earth. The names *Heliocentric Latitude* and *Longitude* are given to the corresponding coordinates when employed to define the body's *heliocentric* position, or position relative to the Sun's centre.

When the distance of a fixed star is immeasurably great compared with the radius of the Earth's orbit, its geocentric and heliocentric directions coincide, and there is no difference between the two sets of coordinates. There is a slight difference between the geocentric and heliocentric positions of a few of the nearest fixed stars. There are perhaps some 200 stars for which the difference exceeds a tenth of a second of arc. But, in the case of the planets, and of comets, the heliocentric latitude and longitude differ entirely from the geocentric, and laborious calculations are required to transform from one system of coordinates to the other.

One fact may, however, be noted. The direction of the Earth as seen from the Sun is always opposite to the direction of the Sun as seen from the Earth. Hence :—

The Earth's heliocentric longitude differs from the Sun's geocentric longitude by 180°.

This may be illustrated by referring to Fig. 46. We see that

$$♈SE_3 = 0°,\ ♈SE_4 = 90°,\ ♈SE_1 = 180°,\ ♈SE_2 = 270°\,;$$

thus, the Earth's longitude is 0° on September 23rd, 90° on December 22nd, 180° on March 21st, and 270° on June 21st.

EXAMPLES

1. Describe the phenomena of day and night at a pole of the Earth.

2. Show how to find how long the midwinter Moon when full is above the horizon at a place within the arctic circle of given latitude.

3. Show that the ecliptic can never be perpendicular to the horizon except at places between the tropics.

4. Show that for a place *on* the arctic circle the Sun always rises at 18h. sidereal time from December 21st to June 20th, and sets at the same sidereal time from June 20th to December 21st.

5. Find the angle between the ecliptic and the equator in order that there should be no temperate zone, the torrid zone and the frigid zone being contiguous.

6. Show how, by observations on the Sun, taken at an interval of nearly six months, the astronomical clock may be set to indicate 0h. 0m. 0s. when ♈ is on the meridian.

7. On March 24th, at noon, the Sun's declination was 1° 29′ 5·1″, and the difference of right ascension of the Sun and a star 6h. 1m. 34·45s. On September 18th, at noon, the Sun's declination was 1° 49′ 30·2″, and it was distant from the star 5h. 27m. 32·97s. in right ascension. On September 19th, at noon, the Sun's

declination was 1° 26′ 12·8″, and it was distant from the star 5h. 31m. 8·3s. in right ascension. Find the right ascension of the star and that of the Sun at the first observation.

8. Describe the appearance presented to an observer in the Sun of the parallels of latitude and the meridians of the Earth, any day (i) between the vernal equinox and the summer solstice, (ii) between the autumnal equinox and the winter solstice.

9. If a sunspot be situated near the edge of the Sun's disc, describe how its position, relative to the horizon, will change between sunrise and sunset.

10. Describe how the Sun's apparent velocity in the ecliptic varies throughout the year; and give the dates of apogee and perigee. Compare the daily motion in longitude at these dates, having given that the eccentricity of the Earth's orbit is $\frac{1}{60}$.

EXAMINATION PAPER

1. What is the astronomical reason for the Earth being divided into torrid, temperate, and frigid zones?

2. Assuming your latitude to be 52°, show by a figure the daily path of the Sun as seen by you on June 21st, December 22nd, and March 21st respectively.

3. Explain the causes of variation in the length of the day on the Earth. Give the dates at which each season begins, and calculate their lengths in days.

4. Discuss the variations in the length of the day at points within the arctic circle; and show how to find, by the *Nautical Almanac*, the length of the perpetual day.

5. Prove that, in the course of the year, the Sun is as long above the horizon at any place as below it (neglect refraction).

6. Explain how it is that winter is colder than summer, although the Sun is nearer.

7. Investigate Flamsteed's method of determining the first point of Aries.

8. From the following observations calculate the Sun's R.A. at transit over the meridian on March 30th, 1872 :—

	Sun's declination.	Sun crossed meridian.	*a Serpentis* crossed meridian.
March 30th, 1872	4° 0′ 8·1″	0h. 1m. 4·47s.	15h. 1m. 54·76s.
Sept. 11th, 1872	4° 20′ 58·8″	0h. 1m. 4·09s.	4h. 19m. 11·38s.
Sept. 12th, 1872	3° 58′ 3·0″	0h. 1m. 4·07s.	4h. 15m. 49·33s.

9. State Kepler's First Law for the orbit of the Earth relative to the Sun, and explain how the eccentricity of the orbit can be found by observations of the Sun's angular diameter.

10. State Kepler's Second Law, and find the relation between the Sun's angular velocity and its apparent area.

CHAPTER VI

ATMOSPHERICAL REFRACTION AND TWILIGHT

138. Laws of Refraction

It is a fundamental principle of Optics that a ray of light travels in a straight line, so long as its course lies in the same homogeneous medium; but when a ray passes from one medium into another, or from one stratum of a medium into another stratum of different density, it, in general, undergoes a change of direction at their surface of separation. This change of direction is called *Refraction.*

Let a ray of light SO (Fig. 47) pass at O from one medium into another, the two media being separated by the plane surface AB, and let OT be the direction of the ray after refraction in the second medium. Draw ZOZ' the normal or perpendicular to the plane AB at O. Then the three laws of refraction may be stated as follows :—

LAW I. *The incident and refracted rays SO, OT and the normal ZOZ' all lie in one plane.*

LAW II. *The ratio $\dfrac{\sin ZOS}{\sin Z'OT}$ is a constant quantity, being the same for all directions of the rays, so long as the two media are the same.**

This constant ratio of Law II is called the *relative index of refraction* of the two media, and is usually denoted by the Greek letter μ.

Thus, if TO be produced backwards to S', we have :—

$$\sin ZOS = \mu \sin Z'OT = \mu \sin ZOS',$$

The angles ZOS and $Z'OT$ are usually called the *angle of incidence* and the *angle of refraction* respectively.

LAW III. *When light passes from a rarer to a denser medium, the angle of incidence is greater than the angle of refraction.*

$$\text{Since} \quad \angle ZOS > \angle Z'OT, \ \sin ZOS > \sin Z'OT \text{ and } \mu > 1.$$

139. General Description of Atmospherical Refraction

If the Earth had no atmosphere, the rays of light proceeding from a celestial body would travel in straight lines right up to the observer's eye or telescope, and we should see the body in its actual direction.

But when a ray Sa (Fig. 48) meets the uppermost layer AA' of the Earth's atmosphere, it is refracted or bent out of its course, and its direction changed to ab. On passing into a denser stratum of air at BB', it is further bent into the direction bc, and so on; thus, on reaching the observer, the ray is travelling in a direction OT, different from its original direction, but (by Law I) in the same vertical plane.

---

* The value of the ratio varies slightly for rays of different colours but with this we are not concerned in the present chapter.

The body is, therefore, seen in the direction OS', although its real direction is aS or OS. Also, since the successive horizontal layers of air AA', BB', CC', . . . are of increasing density, the effect of refraction is to bend the ray *towards* the perpendicular to the surfaces of separation, that is, towards the vertical.

Hence *the apparent altitudes of the stars are increased by refraction.*

In reality, the density of the atmosphere increases *gradually* as we approach the Earth, instead of changing abruptly at the planes AA', BB'. . . . Consequently, the ray, instead of describing the polygonal path $SabcO$, describes a curved path, but the general effect is the same.

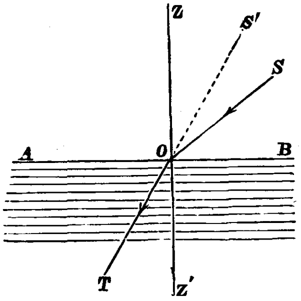

FIG. 47.

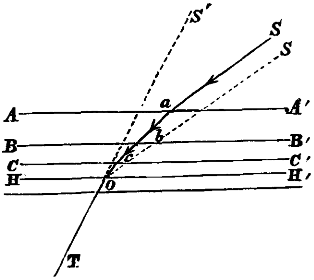

FIG. 48.

140. Law of Successive Refractions

Let there be any number of different media, separated by *parallel planes* AA', BB', CC', HH' (Fig. 48), and let $SabcOT$ represent the path of a ray as refracted at the various surfaces.

Since the media are separated by parallel planes, it is evident that the angle of refraction at the surface AA is equal to the angle of incidence at the surface BB; the angle of refraction at the surface BB is equal to the angle of incidence at the surface CC, and so on.

Now experiment shows that if μ_A, μ_B are the refractive indices of two media, A and B, relative to a vacuum, the relative refractive index for refraction from medium A to medium B is μ_B/μ_A. The law of refraction from medium A to medium B can therefore be written as :—

$$\sin ZOS = \frac{\mu_B}{\mu_A} \sin ZOS'$$

$$\text{or} \quad \mu_A \sin Z_A = \mu_B \sin Z_B$$

where Z_A, Z are respectively the angles that the directions of the ray in mediums A and B make with the zenith.

Since the interfaces of the media are parallel, this result will hold at each refraction and we shall have :—

$$\mu_A \sin Z_A = \mu_B \sin Z_B = \mu_C \sin Z_C = \ldots = \mu_N \sin Z_N$$

if N denotes the last medium. Thus :—

$$\sin Z_A = \frac{\mu_N}{\mu_A} \sin Z_N,$$

But μ_N/μ_A is the relative refractive index for refraction direct from the first medium to the last. It follows, that the final direction $S'T$ of the ray is parallel to what it would have been if the ray had been refracted directly from the first into the last medium without traversing the intervening media.

Thus, if a ray SO, drawn parallel to Sa, were to pass directly from the first medium to the last by a single refraction at O, its refracted direction would be the same as that actually taken by the ray Sa, and would coincide with OT.

141. The Formula for Astronomical Refraction

We shall now apply the above laws to determine the change in the apparent direction of a star produced by refraction.

Since the height of the atmosphere is only a small fraction of the Earth's radius, it is sufficient for most purposes of approximation to regard the Earth as flat, and the surfaces of equal density in the atmosphere as parallel planes. With this assumption, the effect of refraction is exactly the same (Art. 140) as if the rays were refracted directly into the lowest stratum of the atmosphere, without traversing the intervening strata.

Let OS (Fig. 47) be the true direction of a star or other celestial body. Then, before reaching the atmosphere, the rays from the star travel in the direction SO. Let their direction after refraction be $S'OT$, then OS' is the *apparent* direction in which the star will be seen, and the angle SOS' is the apparent change in direction due to refraction. The normal OZ points towards the zenith. Hence ZOS is the star's true zenith distance, and ZOS' or $Z'OT$ is its apparent zenith distance, and the first and third laws of refraction show that the star's apparent direction is displaced towards the zenith.

Let $\angle ZOS' = z$, $\angle S'OS = u$, so that $\angle ZOS = z + u$; and let μ be the index of refraction of the atmosphere at O. By the second law of refraction.

$$\sin (z + u) = \mu \sin z.$$

$$\text{or} \quad \sin z \cos u + \cos z \sin u = \mu \sin z.$$

Now the refraction u is in general very small. Hence, if u be measured in circular measure, $\sin u = u$, and $\cos u = 1$ very approximately. Therefore we have:

$$\sin z + u \cos z = \mu \sin z;$$

$$\text{or} \quad u = (\mu - 1) \tan z.$$

Let U be the amount of refraction in circular measure when the zenith distance is 45°. Putting $z = 45°$, we have

$$U = \mu - 1.$$

$$\text{so that} \quad u = U \tan z.$$

Thus the amount of refraction is proportional to the tangent of the apparent zenith distance.

The last result does not depend on the fact that the refraction is measured in circular measure. Hence, if u'', U'' be the numbers of seconds in u, U, we have

$$u'' = U'' \tan z,$$

The quantity U'' is called the *coefficient of refraction.* Since U is the circular measure of U'', we have :—

$$U'' = \frac{180 \times 60 \times 60}{\mu} \cdot U = 206265\,(\mu - 1),$$

whence, if U'' is known, μ can be found, and conversely.

142. Observations on the preceding Formula

In the last formula u'' represents the correction which must be added to the apparent or observed zenith distance in order to obtain the true zenith distance. By the first law, the azimuth of a celestial body is unaltered by refraction.

Thus the time of transit of a star across the meridian, or across any other vertical circle, is unaltered by refraction.

In using the transit circle, there will, therefore, be no correction for observations of right ascension, but in finding the declination the observed meridian Z.D. will require to be increased by $U'' \tan z$.

A star in the zenith is unaffected by refraction, and the correction increases as the zenith distance increases. When a star is near the horizon, the formula $u'' = U'' \tan z$ fails, since it makes $u'' = \infty$, when $z = 90°$. In this case u is no longer a small angle, so that we are not justified in putting $\sin u = u$ and $\cos u = 1$. But there is a more important reason why the formula fails at low altitudes, namely, that the rays of light have to traverse such a length of the Earth's atmosphere that we can no longer regard the strata of equal density as bounded by parallel planes. In this case, it is necessary to take into account the curvature of the Earth in order to obtain any approach to accurate results.

For zenith distances less than 75°, the formula gives satisfactory results; for greater distances the correction is too large.

The coefficient of refraction U'' is found to be about 58·2″ when the height of the barometer is 30 inches and the temperature is 50°. But the index of refraction depends on the density of the air, and this again depends on the pressure and temperature. Hence, where accurate corrections for refraction are required, the height of the barometer and the thermometer must be read. Any want of uniformity in the strata of equal density, or any uncertainty in determining the temperature, will introduce a source of error; hence it is desirable that the corrections shall be as small as possible. Observations near the zenith are the most reliable.

It is useful to note that since the circular measure of 1° is 1/57·3, the refraction at 1° from the zenith is almost exactly 1″, at 2° it is 2″, and so on, so long as the tangent can be taken as equal to its circular measure.

143. Effect of Barometric Height and Temperature

As mentioned in the preceding section, the constant of refraction depends on the pressure or barometric height and on the temperature. If U_1 denotes the constant refraction when the height of the barometer is B inches and the temperature is T° Fahrenheit and U denotes the constant when the height of the barometer is 30 inches and the temperature is 50°F., which are taken as standard conditions, the relationship between U_1 and U is given by

$$\frac{U_1}{U} = \frac{17B}{460 + T}$$

so that under these conditions the refraction becomes

$$u'' = \frac{17B}{460 + T} U'' \tan z.$$

*144. Cassini's Formula

The law of refraction was also investigated by Dominique Cassini on the hypothesis that the atmosphere is *spherical* but *homogeneous throughout*; in this way he obtained the approximate formula

$$u = (\mu - 1) \tan z\, (1 - n \sec^2 z),$$

where n is the ratio of the height of the homogeneous atmosphere to the radius of the Earth.

Cassini's formula may be proved as follows:—Let $SO'O$ be the path of a ray of light from a star S. By hypothesis this ray undergoes a single refraction on entering the homogeneous atmosphere at O'. Let O be the position of the observer, C the centre of the Earth. Produce OO' to S', CO to Z, and CO' to Z'. Let $u = \angle SOS'$ (in circular measure), $z = \angle ZOS'$, $z' = \angle Z'OS'$.

Then, by Art. 183, if u is small, we have

$$u = (\mu - 1) \tan z';$$

but here z' is not the apparent zenith distance, so that we must express $\tan z'$ in terms of $\tan z$.

Draw CT perpendicular to $O'O$ produced, and $O'N$ perpendicular to COZ. Then :—

$$O'T \tan z' = TC = OT \tan z;$$

$$\text{or} \quad \frac{\tan z}{\tan z'} = \frac{O'T}{OT} = 1 + \frac{O'O}{OT} = 1 + \frac{ON \sec z}{OC \cos z} = 1 + \frac{ON}{OC} \sec^2 z.$$

But ON is very approximately the height of the homogeneous atmosphere OH, and is therefore $= n \,.\, OC$; so that

$$\frac{\tan z}{\tan z'} = 1 + n \sec^2 z; \quad \text{or} \quad \tan z' = \frac{\tan z}{1 + n \sec^2 z};$$

whence, by substituting in the formula, we have

$$u = (\mu - 1) \frac{\tan z}{1 + n \sec^2 z}$$

$$\text{or} \quad u = (\mu - 1) \tan z \,\{1 - n \sec^2 z + n^2 \sec^4 z - n^3 \sec^6 z, \text{ etc.}\}.$$

Now n is very small; we may therefore neglect its square and higher powers; hence we obtain approximately

$$u = (\mu - 1) \tan z \,(1 - n \sec^2 z),$$

which is Cassini's formula. If the value of n be properly chosen, Cassini's formula is found to give very good results for all zenith distances up to 80°.

Since $\sec^2 z = 1 + \tan^2 z$, Cassini's formula has the form

$$u = A \tan z + B \tan^3 z$$

By determining the constants A and B from observations, instead of using their theoretical values, a much improved representation of the true refraction by the formula can be obtained.

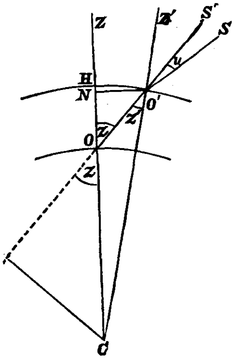

FIG. 49.

145. To Determine the Coefficient of Refraction from Meridian Observations

Assuming the "tangent law," $u = U \tan z$, the coefficient of refraction U may be found from observations of circumpolar stars as follows.

Let z_1, z_2, the apparent zenith distances of a circumpolar star, be observed at upper and lower culminations respectively. Then the true zenith distances will be

$$z_1 + U \tan z_1 \text{ and } z_2 + U \tan z_2.$$

Now, the observer's latitude is half the sum of the meridian altitudes at the two culminations (Art. 31), hence if ϕ be the latitude, we have :—

$$\phi = \tfrac{1}{2}\{(90° - z_1 - U \tan z_1) + (90° - z_2 - U \tan z_2)\},$$

$$\text{or} \quad 90° - \phi = \tfrac{1}{2}(z_1 + z_2) + \tfrac{1}{2}U(\tan z_1 + \tan z_2) \text{ ....(i)}$$

Now let a second circumpolar star be observed. Let its apparent zenith distances at upper and lower culminations be z' and z''. Then we obtain in like manner

$$90° - \phi = \tfrac{1}{2}(z' + z'') + \tfrac{1}{2}U(\tan z' + \tan z'') \text{ ....(ii)}$$

Eliminating ϕ from (i) and (ii) by subtraction, we have

$$U = \frac{(z_1 + z_2) - (z' + z'')}{(\tan z_1 + \tan z_2) - (\tan z' + \tan z'')}.$$

If the two stars have the same declination, we shall have $z_1 = z'$ and $z_2 = z''$, and the above formula will fail. Hence it is important that the two observed stars should differ considerably in declination; the best results are obtained by selecting one star very near the pole (*e.g.* the Pole Star), and the other about 30° from the pole.

146. Alternative Method (Bradley's)

Instead of using a second circumpolar star, Bradley observed the Sun's apparent Z.D.'s at noon at the two solstices. Let these be Z_1, Z_2. Now by Art. 40 since the true Z.D.'s are

$$Z_1 + U \tan Z_1 \text{ and } Z_2 + U \tan Z_2,$$

$$Z_1 + U \tan Z_1 = \phi - \epsilon,\ Z_2 + U \tan Z_2 = \phi + \epsilon;\ (\epsilon = \text{obliquity.})$$

$$\text{or}\quad 2\phi = Z_1 + Z_2 + U(\tan Z_1 + \tan Z_2) \;\ldots\ldots\ldots\ldots\; \text{(iii)}$$

Eliminating ϕ from (i), (iii), we have :—

$$U(\tan z_1 + \tan z_2 + \tan Z_1 + \tan Z_2) = 180° - (z_1 + z_2 + Z_1 + Z_2),$$

whence U is found.

147. Other Methods of finding the Refraction

Suppose that at a station on the Earth's equator, either a star on the celestial equator, or the Sun at an equinox, is observed during the day. Its diurnal path from east to west passes through the zenith, and during the course of the day its true zenith distance will change uniformly at the rate of 15° per hour. Thus the true Z.D. at any time is known. Let the apparent Z.D. be observed with an altazimuth. The difference between the observed and the calculated Z.D. is the displacement of the body due to refraction.

By this method we find the corrections for refraction at different zenith distances without making any assumptions regarding the law of refraction.

Except at stations on the Earth's equator, it is not possible to observe the refraction at different zenith distances in such a simple manner. Nevertheless, methods more or less similar can be employed. For this purpose the zenith distances of a known star are observed at different times. The true zenith distance at the time of each observation can be calculated from the known R.A. and declination. Hence the refraction for different zenith distances of the star can be determined.

This method is very useful for verifying the law of refraction after the star's declination and the observer's latitude have been found with tolerable accuracy. Moreover, it can be employed to find the corrections for refraction at low altitudes when the "tangent law" ceases to give approximate results.

148. Tables of Mean Refraction

From the results of such observations tables of mean refraction have been constructed. The tables prepared by Bessel are calculated for temperature 50° and height of barometer 29·6 inches; they give the refraction for every 5′ of altitude up to 10°, for larger intervals at altitudes between 10° and 54°, and for every 1° at altitudes varying from 54° to 90°. Subsidiary tables give the corrections, which must be added to or subtracted from the mean refraction given in the first table in allowing for differences in the temperature and barometric pressure. The corrections for temperature and pressure are applied separately.

Still more refined tables, published by the Pulkova Observatory, have now been generally adopted.

149. Effects of Refraction on Rising and Setting

At the horizon the mean refraction is about 33′; consequently a celestial body appears to rise or set when it is 33′ below the horizon. Thus, the effect of refraction is to accelerate the time of rising, and to retard, by an equal amount, the time of setting of a celestial body. In particular, the Sun, whose angular diameter is 32′, appears to be just above the horizon when it is really just below.

The acceleration in the time of rising due to refraction can be investigated in exactly the same way as the acceleration due to dip (Art. 90). If u'' denotes the refraction at the horizon in seconds, δ the declination, x the inclination to the vertical of the direction in which the body rises, the acceleration in the time of rising in seconds is :—

$$\frac{1}{15} u'' \sec x \sec \delta.$$

Taking the horizontal refraction as 33′, or 1980″, and putting $x = 0$, $\delta = 0$, we see that at the Earth's equator at an equinox, the time of sunrise is accelerated by about 2m. 12s. owing to refraction. The time of sunset is retarded by an equal amount.

When the Sun or Moon is near the horizon, it appears distorted into a somewhat oval shape. This effect is due to refraction. The whole disc is raised by refraction, but the refraction increases as the altitude diminishes; so that the lower limb is raised more than the upper limb, and the vertical diameter appears contracted. The horizontal diameter is almost unaffected by refraction. Hence, the disc appears somewhat flattened or elliptical, instead of truly circular. According to the tables of mean refraction, the refraction on the horizon is 33′, while at an altitude 30′, the refraction is only 28′ 23″, and at 35′

it is 27′ 41″. Hence, taking the Sun's or Moon's diameter as 32′, the lower limb when on the horizon is raised about 5′ more than the upper. The contraction of the vertical diameter, therefore, amounts to 5′, *i.e.* about one-sixth of the diameter itself, so that the apparent vertical and horizontal angular diameters are approximately in the ratio of 5 to 6.

Even the horizontal diameter is slightly reduced, for its two extremities are moved towards the zenith on converging great circles. The amount is nearly constant at all altitudes, about 0·5″.

150. Illusory Variations in Size of Sun and Moon

The Sun and Moon generally seem to look larger when low down than when high up in the sky. This is not an effect of refraction. It is merely a false impression formed by the observer, and is not in accordance with measurements of the angular diameter made with a micrometer. When near the horizon, the eye is apt to estimate the size and distance of the Sun and Moon by comparing them with the neighbouring terrestrial objects (trees, hills, etc.). When the bodies are at a considerable altitude no such comparison is possible, and a different estimate of their size is instinctively formed.

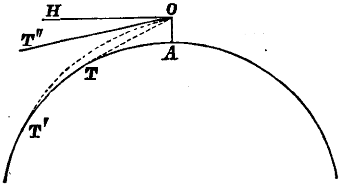

FIG. 50.

151. Effect of Refraction on Dip, and Distance of the Horizon

Since refraction increases as we approach the Earth, its effect is always to bend the path of a ray of light into a curve which is concave downwards (Fig. 50).

Let O be any point above the Earth's surface, and let $T'O$ be the curved path of the ray of light which touches the Earth at T' and passes through O. Then OT' is the distance of the visible horizon. Draw the straight tangent OT, then OT would be the distance of the visible horizon if there were no refraction; hence, it is evident from the figure that—

The Distance of the horizon is increased by refraction.

Draw OT'', the tangent at O to the curved path OT', then OT'' is the apparent direction of the horizon. Hence, from the figure we see that—

The Dip of the horizon is diminished by refraction.

Both dip and distance are still approximately proportional to the square root of the height of the observer.

152. Effect of Refraction on Lunar Eclipses and on Lunar Occultations

In a total eclipse the Moon's disc is never perfectly dark, but appears of a dull red colour. This effect is due to refraction. The Earth coming between the Sun and Moon prevents the Sun's *direct* rays from reaching the Moon, but those rays which nearly graze the Earth's surface are bent round by the refraction of the Earth's atmosphere, and thus reach the Moon's disc. The red colour is due to the same cause that makes the setting sun look red. The long red light waves have more penetrating power than the short violet ones.

From observing the " occultations " of stars when the unilluminated portion of the Moon passes in front of them, we are enabled to infer that the Moon does not possess an atmosphere similar to that of our Earth. For the directions of stars would be displaced by the refraction of such an atmosphere just before disappearing behind the disc, and just after the occultation; and no such effect has been observed.

153. Twilight

The phenomenon of *twilight* is also due to the Earth's atmosphere, and is explained as follows :—After the Sun has set, its rays still continue to fall on the atmosphere above the Earth, and of the light thus received a considerable portion is reflected or scattered in various directions. This scattered light is what we call *twilight,* and it illuminates the Earth for a considerable time after sunset. Moreover, some of the scattered light is transmitted to other particles of the atmosphere further away from the Sun, and these reflect the rays a second time ; the result of these second reflections is to increase further the duration of twilight. Twilight is said to end when this scattered light has entirely disappeared, or has at least become imperceptible. From numerous observations, twilight is found to end when the Sun is at a depth of about 18° below the horizon.

The duration of twilight, especially in high latitudes, varies with the season of the year and may even last all night. But any given degree of indirect illumination will be associated with the same depression of the Sun below the horizon. It is therefore convenient to subdivide the interval between sunrise and complete darkness into three periods, indicating times that have equal degrees of illumination.

The name *civil twilight* is applied to the time when the centre of the Sun is 6° below the horizon. It corresponds approximately to the time when ordinary outdoor civil occupations are impracticable without artificial light. The name *nautical twilight* is applied to the time when the centre of the Sun is 12° below the horizon. The name *astronomical twilight* is applied to the time when the centre of the Sun is 18° below the horizon : it is usually called simply twilight.

The *Nautical Almanac* gives tables of the times of beginning of civil, nautical and astronomical twilights in the morning and of their ending in the evening for a sufficient number of different latitudes between the equator and 60° N. to permit the times for any other latitude to be interpolated readily. The times for southern latitudes are found with the aid of an auxiliary table.

154. Times of Beginning and Ending of Twilight

In Fig. 51, Z denotes the zenith, P the pole, nXs the horizon. X denotes the position of the Sun when on the horizon and Y its position when 18° below the horizon. We denote the angle ZPX by H, the hour angle of the Sun at setting, and the angle XPY by h, so that $H + h$ is the hour angle of the Sun at the end of astronomical twilight. h is then a measure of the duration of twilight.

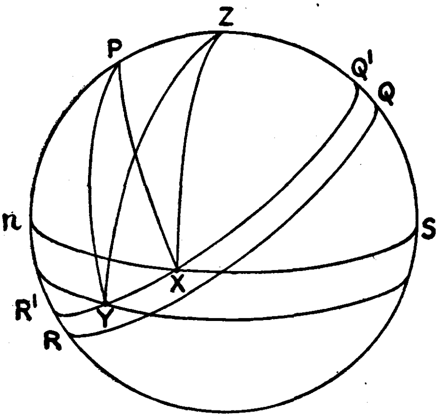

FIG. 51.

In the triangle ZPX, the side ZX is 90°; the side PZ is $90° - \phi$, where ϕ denotes the latitude; the side PX is $90° - \delta$, where δ is the declination of the Sun.

From Art. 10, formula 1, we have

$$\sin\delta\sin\phi + \cos\delta\cos\phi\cos H = 0.$$

In the triangle ZPY, the side ZY is 108°; the side PY is $90° - \delta$; the side PZ is $90° - \phi$. The same formula (Art. 10) gives

$$\sin\delta\sin\phi + \cos\delta\cos\phi\cos(H + h) = \cos 108°.$$

The first of these formulae gives H, when δ, ϕ are known; the second gives $(H + h)$. The duration of twilight, and also the hour-angle of the Sun at its beginning (in the morning) or ending (in the evening) are thus obtained.

At any point on the equator, $\phi = 0$. The first formula gives $\cos H = 0$, so that $H = 90°$ or $270°$. The second formula gives :—

$$\cos\delta\cos(H + h) = \cos 108°$$

$$\text{or} \qquad \cos\delta\sin h = \pm\cos 108°.$$

The value of sin h, and therefore h, is greatest when cos δ has its smallest value, *i.e.* at the summer and winter solstices. Also sin h has the same value for the two values $+\delta$, $-\delta$ of the declination. Thus, at a place on the equator, the duration of twilight is greatest at the two solstices and least at the equinoxes. At two times symmetrically placed on either side of an equinox, the durations of twilight are equal.

The variations in duration of twilight on the equator are not large, however. The duration is 1h. 19m. at the summer and winter solstices and 1h. 12m. at the equinoxes. The effect of refraction has been neglected; it delays the time of sunset and sunrise and shortens the duration of twilight by a few minutes.

The investigation of the duration of twilight for any latitude and declination of the Sun is a little complicated. The general effect of change of latitude can be seen by taking the case of $\delta = 0$, when the Sun is on the equator. H is then 90° or 270°, so that :—

$$\begin{aligned} \cos 108^\circ &= \cos \phi \cos (H + h) \\ &= \pm \cos \phi \sin h. \end{aligned}$$

Thus h is greater, the higher the latitude.

If the Sun, when it is 18° below the horizon is at R' on the meridian, the end of evening twilight will coincide with the beginning of morning twilight. When this is the case, it is seen from Fig. 51, that

$$\begin{aligned} RR' + R'n + nP &= 90^\circ \\ \text{or} \quad \delta + 18^\circ + \phi &= 90^\circ \\ \text{so that} \quad \phi &= 72^\circ - \delta. \end{aligned}$$

If $R'n = 90^\circ - \phi - \delta < 18^\circ$ or if $\phi > 72^\circ - \delta$ the Sun's depth below the horizon never gets as great as 18° and twilight lasts all night. But the greatest value of δ is nearly $23\frac{1}{2}^\circ$, the obliquity of the ecliptic, and occurs at midsummer. Hence there is twilight all the night about midsummer at any place whose latitude ϕ is greater than $72^\circ - 23\frac{1}{2}^\circ$ or $48\frac{1}{2}^\circ$. This includes the whole of the British Isles.

EXAMPLES

1. What would be the effect of refraction on terrestrial objects as seen by a fish under water?

2. For stars near the zenith show that the refraction is approximately proportional to the zenith distance, and that the number of seconds in the refraction is equal to the number of degrees in the zenith distance. (Take coefficient of refraction = 57″).

3. From the summit of a mountain 2,400 feet above the level of the sea, it is just possible to see the summit of another, of height 3,450 feet, at a distance of 143 miles. Find approximately the radius of the Earth, assuming that the effect of refraction is to alter the distance of the visible horizon in the ratio 12 : 13.

4. Trace the changes in the apparent declination of a star due to refraction in the course of a day, at a place in latitude 45° N., the actual declination being 50° N.

5. Prove that if the declination of a star observed off the meridian is unaffected by refraction, the star culminates between the pole and the zenith, and that the azimuth of the star from the north is a maximum at the instant considered.

6. Show how the duration of twilight gives a measure of the height of the atmosphere.

7. What is the lowest latitude in the arctic circle at which there is no twilight at midwinter, and what is the corresponding distance from the North Pole in miles?

EXAMINATION PAPER

1. What effect has refraction on the apparent position of star? Show that the greater the altitude of the star the less it is displaced by refraction, and that a star in the zenith is not displaced at all.

2. Prove (stating what optical laws are assumed) that, if the Earth and the layers of the atmosphere be supposed flat, the amount of refraction depends solely on the temperature and pressure at the Earth's surface.

3. Prove the formula for refraction, $r = (\mu - 1) \tan z$. Is this formula universally applicable? Give the reason for your answer.

4. Given that the optical coefficient of refraction of air $(\mu) = 1{\cdot}0003$, find the astronomical coefficient of refraction (U) in seconds.

5. What is the *refraction error*? How may we approximately determine the correction for refraction from observations made on the transits of circumpolar stars?

6. Show how the constant of refraction (on the usual assumption that the refraction is proportional to the tangent of the zenith distance) might be determined by observing the two meridian altitudes of a circumpolar star whose declination is known.

7. Assuming the tangent formulae for refraction, find the latitude of a place at which the upper and lower meridian altitudes of a circumpolar star were 30° and 60° ($\sqrt{3} = 1{\cdot}732$), the coefficient of refraction being $57''$.

8. Why is the Moon seen throughout a total eclipse?

9. It has been stated that "The atmosphere by its refraction acts as a lens, producing an apparent increase in the diameter (of the Sun and Moon) near the horizon. When we consider that the atmosphere, as seen from the surface of the globe, is a section of a vast lens whose radius is the semi-diameter of the Earth, it is reasonable to assume a small increase in the size of the objects seen through it, and a still greater increase when seen in the obliquity of the horizon." Why is the above statement altogether incorrect?

10. Find the duration of twilight at the equator at an equinox.

CHAPTER VII

GEOCENTRIC AND ANNUAL PARALLAX

I.—GEOCENTRIC PARALLAX

155. Definitions

By the *Parallax* of a celestial body is meant the angle between the straight lines joining it to two different places of observation.

In Art. 14 we stated that the fixed stars are seen in the same direction from all parts on the Earth; hence such stars have no appreciable parallax. The Moon, Sun, and planets, on the other hand, are at a (comparatively) much smaller distance from the Earth, and their parallax is a measurable quantity.

To avoid the necessity of specifying the place of observation, the direction of the Moon or any other celestial body is always referred to the *centre of the Earth.* The direction of a line joining the body to the Earth's centre is called the body's *geocentric* direction. The angle between the geocentric direction and the direction of the body relative to any given observatory is called the body's *Geocentric Parallax,* or more shortly, its *Parallax.* Thus the geocentric parallax is the angle subtended at the body by the radius of the Earth through the point of observation.

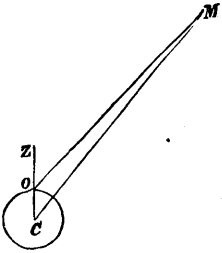

FIG. 52.

The *Horizontal Parallax* is the geocentric parallax of a body when on the horizon of the place of observation.

156. General Effects of Geocentric Parallax

Assuming the Earth to be spherical, let C (Fig. 52) be the Earth's centre, O the place of observation, and M the centre of the Moon or other observed body. Then the angle OMC is the geocentric parallax of M.

Produce CO to Z; then OZ is the direction of the zenith at O, and ZOM is therefore the zenith distance of M as seen from O (corrected of course for refraction). Now

$$\angle ZOM = \angle ZCM + \angle OMC;$$

therefore the apparent zenith distance of M is increased by the amount of the geocentric parallax. Conversely to find $\angle ZCM$ we must subtract the parallax OMC from the observed zenith distance ZOM.

The azimuth is unaltered by parallax, because OM, CM lie in the same plane through OZ.

157. To find the Correction for Geocentric Parallax

In Fig. 52, let

$a = CO =$ Earth's Radius,
$d = CM =$ Moon's (or other body's) geocentric distance,
$z = ZOM =$ observed zenith distance of M,
$p = OMC =$ parallax of M.

Since the sides of $\triangle OMC$ are proportional to the sines of the opposite angles,

$$\frac{\sin CMO}{\sin COM} = \frac{CO}{CM}, \quad \text{that is} \quad \frac{\sin p}{\sin z} = \frac{a}{d}.$$

Therefore

$$\sin p = \frac{a}{d} \sin z.$$

Let P be the horizontal parallax of M. Then, when $z = 90°$, $p = P$, and therefore the last formula gives

$$\sin P = \frac{a}{d} \sin 90° = \frac{a}{d}.$$

Hence, by substitution,

$$\sin p = \sin P \,.\, \sin z.$$

This formula is exact. But the angles p and P are in every case very small, and therefore their sines are very approximately equal to their circular measures: for the Moon this assumption involves an error of $0{\cdot}15''$; for all other bodies it is insensible. Hence we have the approximate formula

$$p = P \,.\, \sin z,$$

or, *The parallax of a celestial body varies as the sine of its apparent zenith distance.*

The last formula holds good no matter what be the unit of angular measurement. Thus, if p'', P'' denote the numbers of seconds in p, P respectively, we have, by reducing to seconds,

$$p'' = P'' \sin z.$$

Examples.—1. *Supposing the Sun's horizontal parallax to be* $8{\cdot}8''$, *find the correction for parallax when the Sun's altitude is* $60°$.

Here $z = 90° - 60° = 30°$, $P'' = 8{\cdot}8''$, and therefore

$$p'' = P'' \sin 30° = 8{\cdot}8'' \times \tfrac{1}{2} = 4{\cdot}4''.$$

2. *Find the corrections for the Moon's parallax for altitudes of* $30°$ *and* $45°$, *the Moon's horizontal parallax being* $57'$.

In the two cases we have respectively $z = 60°$ and $z = 45°$, and the corresponding corrections are

$$\begin{aligned} p'' &= 57' \sin 60° = 57' \times \tfrac{1}{2}\sqrt{3} = 28'\,30'' \times \sqrt{3} \\ &= 1710'' \times 1{\cdot}7320 = 2961{\cdot}7'' = 49'\,21{\cdot}7'' \end{aligned}$$

and

$$\begin{aligned} p'' &= 57' \sin 45° = 57' \times \tfrac{1}{2}\sqrt{2} = 28'\,30'' \times \sqrt{2} \\ &= 1710'' \times 1{\cdot}4142 = 2418{\cdot}3'' = 40'\,18{\cdot}3'' \end{aligned}$$

158. Relation between the Horizontal Parallax and Distance of a Celestial Body

In Art. 157 we showed that sin $P = a/d$. This formula may be proved independently by drawing MA to touch the Earth at A. M is on the horizon at A; the $\angle CMA$ is therefore the horizontal parallax P, and we have immediately

$$\sin P = \sin CMA = CA/CM = a/d.$$

Since P is small, we have approximately

$$\text{Circular measure of } P = a/d,$$

and therefore in seconds

$$P'' = \frac{180 \times 60 \times 60}{\pi} \frac{a}{d} = 206265 \frac{a}{d},$$

which shows that *the horizontal parallax of a body varies inversely as its distance from the Earth.*

If we know the Earth's radius a and the distance d, the last formula enables us to calculate the horizontal parallax P''. Conversely, if we know the horizontal parallax of a body we can calculate its distance.

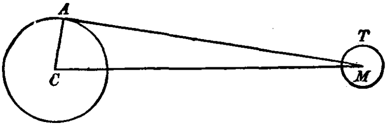

FIG. 53.

Examples.—1. *Given that the Moon's distance is* 60 *times the Earth's radius, find the Moon's horizontal parallax.*

We have $\frac{a}{d} = \frac{1}{60}$ so that :—

$$\text{circular measure of } P = \frac{1}{60} \text{ approximately.}$$

Now the unit of circular measure $= 57°{\cdot}2957$; so that :—

$$P \text{ (in angular measure)} = \frac{1}{60} \times 57°{\cdot}2957 = 57'{\cdot}2957 = 57'\ 17{\cdot}7'',$$

and this is the required horizontal parallax.

2.—*Given that the Sun's parallax is* $8{\cdot}8''$. *find the Sun's distance, the Earth's radius being* 3,960 *miles.*

$$\text{The circular measure of } 8{\cdot}8'' \text{ is } = \frac{8{\cdot}8 \times \pi}{180 \times 60 \times 60},$$

and, by the formula, we have, for the Sun's distance in miles,

$$d = \frac{a}{\text{circ. meas. of } P} = \frac{3960 \times 180 \times 60 \times 60}{8{\cdot}8 \times \pi}.$$

Taking $\pi = 3\frac{1}{7}$, and calculating the result correct to the first three significant figures, we find the Sun's distance $d = 92{,}800{,}000$ miles approximately.

159. Comparison between Parallax and Refraction

It will be noticed that while parallax and refraction both produce displacements of the apparent position of a body along a vertical circle, the displacement due to parallax is directed *away* from the zenith, and is always proportional to the *sine* of the zenith distance, while that due to refraction is directed *towards* the zenith, and is proportional to the *tangent* of the zenith distance, provided the altitude is not small. Also the correction for parallax is inversely proportional to the distance of the body, and is imperceptible, except in the case of members of our solar system; while the correction for refraction is independent of the body's distance, and depends only on the condition of the atmosphere.

The Moon's horizontal parallax is about 57′, while the horizontal refraction is only 33′. Hence, by the combined effects of parallax and refraction, the Moon's apparent altitude is diminished, or its Z.D. increased. The time of rising is, therefore, on the whole retarded, and the time of setting accelerated. The effect of parallax on the times of rising and setting may be investigated by the methods of Arts. 90, 149.

For other bodies, including the nearest planets, but with the occasional exception of comets, the correction for refraction far outweighs that due to parallax. The comet Pons-Winnecke in June 1927 passed within $3\frac{1}{2}$ million miles of the Earth. Its parallax at altitude 60°, was 2′, whereas the refraction was 33″. Lexell's comet in July 1770 passed within $1\frac{1}{2}$ million miles of the Earth, the nearest cometary approach on record.

160. Effect of Parallax on Right Ascension and Declination

When determining the right ascension or declination of a body with sensible geocentric parallax, it is necessary to correct for the parallactic displacement in order to obtain the right ascension and declination as seen from the centre of the Earth and therefore freed from the effect of the observer's position.

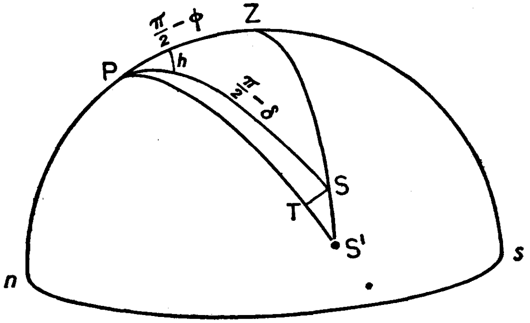

Fig. 54.

If S is the position of the body as seen from the centre of the Earth, S' the position as seen by the observer, S' lies on the vertical circle through S and $SS' = p \sin z$.

From S, draw ST perpendicular to the great circle PS'. The triangle $SS'T$ being small can be treated as plane. The displacement SS' can be resolved into the components

$$ST = SS' \sin q \text{ and } S'T = SS' \cos q,$$

where q is the angle $ZS'P$ = angle ZSP approximately, which is termed the parallactic angle.

In the triangle, PZS, the three sides PZ, ZS, SP are respectively $\frac{\pi}{2} - \phi$, z, and $\frac{\pi}{2} - \delta$, where ϕ, δ are the latitude of the place of observation and the declination of S. The angle ZPS is h, the hour angle, and the angle ZSP is q.

$$\text{Then} \quad ST = SS' \sin q = p \sin Z \sin q$$

$$\text{But since} \quad \frac{\sin q}{\cos \phi} = \frac{\sin h}{\sin z}, \quad \sin z \sin q = \sin h \cos \phi,$$

$$\text{Hence} \quad ST = p \sin h \cos \phi.$$

The displacement of S to S' decreases its right ascension by the angle $SPS' = ST/\cos \delta = p \cos \phi \sec \delta \sin h$.

The effect of parallax is therefore to decrease the right ascension of the body by the amount $p \cos \phi \sec \delta \sin h$.

When the body is to the west of the meridian, as in Fig. 54, h is $< 180°$. $\sin h$ is positive and the R.A. is, in fact, decreased; when the body is to the east of the meridian, $180° < h < 360°$. $\sin h$ is then negative and the R.A. is increased by parallax.

Thus parallax increases the R.A. when east of the meridian and decreases it when west of the meridian.

$$\text{Also} \quad S'T = SS' \cos q = p \sin z \cos q.$$

By using formula (3) of Art. 10, we find that :—

$$S'T = p \sin z \cos q = p (\sin \phi \cos \delta - \cos \phi \sin \delta \cos h).$$

Parallax decreases the declination of S by this quantity, which may be numerically positive or negative, according to the values of the quantities involved.

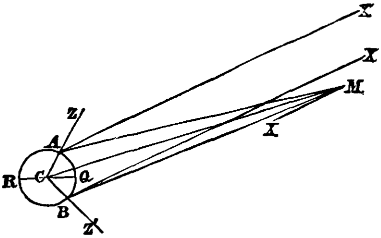

FIG. 55.

161. To find the Moon's Parallax by Meridian Observations

The Moon's parallax may be conveniently determined as follows. Let A and B be two observatories situated on the same meridian, one north, the other south, of the equator. Let M denote the Moon's centre, and let x be a star having no appreciable parallax, whose R.A. is approximately equal to that of the Moon, their declinations being also nearly equal.

Let the Moon's meridian zenith distances ZAM and $Z'BM$ be observed with the transit circles at A and B, and let xAM and xBM, the differences of the meridian Z.D.'s of the Moon and star at the two stations, be also observed

Let $z_1 = \angle ZAM, \qquad z_2 = \angle Z'BM.$

$a_1 = \angle xAM, \qquad a_2 = \angle xBM.$

$P =$ Moon's required horizontal parallax.

By Art. 157, we have, approximately,

$$\angle AMC = P \sin z_1, \qquad \angle BMC = P \sin z_2.$$

so that $$\angle AMB = P (\sin z_1 + \sin z_2) \quad \text{(i)}$$

Moreover, if MX be drawn parallel to Ax or Bx :—

$$\angle XMA = \angle MAx = a_1 ; \qquad \angle XMB = \angle MBx = a_2 ;$$

and $$\angle AMB = a_1 - a_2 \quad \text{(ii)}$$

From (i) and (ii), we have :—

$$P (\sin z_1 + \sin z_2) = a_1 - a_2 ;$$

$$P = \frac{a_1 - a_2}{\sin z_1 + \sin z_2} ;$$

whence the Moon's parallax, P, may be found.

162. Practical Details

If the two observatories are not on the same meridian, allowance must be made for the change in the Moon's declination between the two observations. Let the stations be denoted by A, B, and let B' be the place on the meridian of A, which has the same latitude as B. Then, if the Moon's meridian Z.D. be observed at B, we can, by adding or subtracting the change of declination during the interval, find what would be the meridian Z.D. if observed from B'. Moreover, the star's meridian Z.D. is the same both at B and at B'. Hence it is easy to calculate what would be the angles at B' corresponding to the observed angles at B. From the former, and the observed angles at A, we find the parallax P, as before.

To ensure the greatest accuracy, it is advisable that the difference of longitude of the two stations should be so small that the correction for the Moon's motion in declination is trifling. It is necessary, however, that $a_1 - a_2$ should be large ; for this reason the stations should be chosen one as far north and the other as far south of the equator as possible. The observatories at Greenwich and the Cape of Good Hope have been found most suitable.

The principal advantage of the above method is that the probable errors arising from any uncertainty in the corrections for refraction are diminished as far as possible.

For, since the Moon and observed star have nearly the same declination, the corrections for refraction to be applied to a_1, a_2, their small *differences* of Z.D., are very small indeed. The errors are not of so much moment in the denominator sin z_1 + sin z_2, as the latter is not itself a small quantity.

From such observations, the mean horizontal parallax of the Moon has been found to be 57′ 2·63″.

This value corresponds to a mean distance of 60·27 times the equatorial radius of the Earth, or 238,862 miles. The distance and parallax of the Moon are not, however, quite constant; their greatest and least values are in the ratio of (roughly) 19 : 17. For rough calculations, the Moon's distance may be taken as 60 times the Earth's radius.

This method can also be used to determine the parallax of a planet. The observations can be conveniently made by photography, the planet and neighbouring stars being photographed on the same plate.

163. To find the Parallax of a Planet from Observations made at a Single Observatory

The parallax of Mars, when nearest the Earth, and of some of the minor planets have also been determined by the following method, depending on the Earth's rotation.

It has been shown in Art. 160 that parallax increases the R.A. of a planet when it is east of the meridian and decreases it when it is west of the meridian, the effect in each case being proportional to sin h. The planet's right ascension, relative to certain fixed stars, is observed soon after rising, so as to obtain as large a value of sin h as is conveniently possible, and again shortly before setting. The observations are made by photography or visually, using either an equatorial provided with a micrometer or a heliometer. By comparing the observations east and west of the meridian, a larger relative parallactic displacement is available for the determination of the parallax.

The observed change of position is due partly to parallax and partly to the planet's motion relative to the Earth's centre during the interval between the observations, which produces displacements far greater than those due to parallax. But by repeating the observations on successive days, the planet's rate of motion can be accurately determined, and the displacements due to parallax can thus be separated from those due to relative motion. We must apply the small differences of refraction between the planet and stars, since the parallax problem is one of refinement.

This method can be used for the Moon, but the motion of the Moon is so rapid that the calculations are complicated. Also different stars would have to be used in the two positions, as the Moon would have moved through several degrees.

Neither this method nor the method in Art. 162 gives accurate results for the Sun, for the brilliancy of the rays renders all stars in its neighbourhood invisible.

*164. Effect of the Earth's Ellipticity

The effect of parallax is made rather more complicated by the spheroidal form of the Earth. For, by Art. 157, the *magnitude* of the horizontal parallax at any place depends on its distance from the Earth's centre, and since this distance is not the same for all places on the Earth, the horizontal parallax is not everywhere the same. Again, the *direction* in which the body is displaced is away from the line (produced) joining the centre of the Earth with the observer (Art. 156). But this line does not pass exactly through the zenith (Art. 101). Hence the displacement is not in general along a vertical, so that the azimuth as well as altitude is very slightly altered by parallax.

165. Equatorial Horizontal Parallax. Relation between Parallax and Angular Diameter

The *equatorial horizontal parallax* is the geocentric parallax of a body seen on the horizon of a phase at the Earth's equator. It is generally adopted as the measure of the parallax of a celestial body. Its sine is equal to

$$\frac{\text{Earth's } \textit{equatorial} \text{ radius}}{\text{body's geocentric distance}}.$$

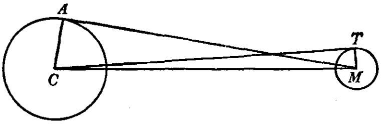

FIG. 56.

In Fig. 56 it will be seen that the angle CMA, which measures the parallax of M, also measures the Earth's angular semi-diameter as it would appear from M. Thus, *the Moon's parallax is the angular semi-diameter of the Earth as it would appear if observed from the Moon.*

166. To Find the Moon's Diameter

Let a, c be the radii of the Earth and Moon respectively, measured in miles, d the distance between their centres, P the Moon's horizontal parallax, m the Moon's angular semi-diameter as it would appear if seen from the Earth's centre. Then, from Fig. 56

$$\sin P = \frac{a}{d}, \qquad \sin m = \sin TCM = \frac{TM}{CM} = \frac{c}{d};$$

and $c : a = \sin m : \sin P = m : P$ approximately; *i.e.*

$$\frac{\text{rad. of Moon}}{\text{rad. of Earth}} = \frac{☾\text{'s angular semi-diam.}}{☾\text{'s horizontal parallax}}.$$

Hence, knowing the Moon's horizontal parallax and its angular diameter, the Moon's radius can be found.

The Moon's mean angular diameter $2m$ is observed to be about 31′ 5″. From this the Moon's actual diameter is readily found to be about 2160 miles, or $\frac{3}{11}$ of the Earth's diameter.

The surfaces of spheres are proportional to the squares, and the volumes to the cubes of their radii. Hence the Moon's superficial area is about $\frac{9}{121}$, or $\frac{3}{40}$, and its volume about $\frac{27}{1331}$, or $\frac{1}{50}$ of that of the Earth.

Example.—*Find the Moon's diameter in miles, given that* :—

$$☾\text{'s angular diameter} = 31'\,7'',$$
$$☾\text{'s equatorial horizontal parallax} = 57'\,2'',$$
$$\text{Earth's equatorial radius} = 3963 \text{ miles.}$$
$$☾\text{'s diameter } 2c = a \times \frac{2m}{P} = 3963 \times \frac{31'\,7''}{57'\,2''} = 3963 \times \frac{1867}{3422} = 2162.$$

Thus the Moon's diameter is 2162 miles.

II.—ANNUAL PARALLAX

167. Annual Parallax, Definition

By *Annual Parallax* is meant the angle between the directions of a star as seen from different positions of the Earth in its annual orbit round the Sun.

We have several times (Arts. 14, 155) mentioned that the fixed stars have no appreciable *geocentric parallax*. Their distances from the Earth are so great that the angle subtended at one of them by a diameter of the Earth is far too small to be observable even with the most accurately constructed instruments. But the diameter of the Earth's annual orbit is about 23,400 times as great as the Earth's diameter, or about 186 million miles (twice the Sun's distance), and this diameter subtends, at certain of the nearest fixed stars, an angle which is measurable—approaching 1″ in the case of the nearest stars.

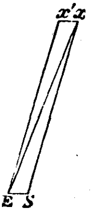

FIG. 57.

Now, the Earth, by its annual motion, passes in six months from one end to the other of a diameter of its orbit; hence, by observing the same star at an interval of six months, its displacement due to annual parallax can be measured.

Since the Sun is fixed, the position of a star on the celestial sphere is corrected for annual parallax by referring its direction to the *centre of the Sun*; this is called the star's *heliocentric* direction, as in Art. 137.

The *correction for annual parallax* is the angle between the geocentric and heliocentric directions of a star. Let S be the Sun, E the Earth, x the star (Fig. 57). Then Ex is the apparent or geocentric direction of the star, Sx its heliocentric direction, and $\angle ExS$ is the correction for annual parallax. This angle is also equal to xEx' where Ex' is parallel to Sx.

We notice that *the correction for annual parallax* (ExS) *is the angular distance of the Earth from the Sun as they would appear if seen by an observer on the star.*

168. To find the Correction for Annual Parallax

Let $r = ES =$ radius of Earth's orbit.
$d = Sx =$ distance of star.
$E = \angle SEx =$ angular distance of star from Sun.
$p = \angle ExS =$ annual parallax of star.

We have in the triangle SEx :—

$$\frac{\sin ExS}{\sin SEx} = \frac{ES}{Sx}; \qquad \text{whence* } \sin p = \frac{r}{d} \sin E \text{ .... (i)}$$

Hence the parallactic correction p is greatest when $E = 90°$. This happens twice a year, and the corresponding positions of the Earth in its orbit are evidently the intersections of the ecliptic with a plane drawn through S perpendicular to Sx. Let this greatest value of p be denoted by P, then P is called the *star's annual parallax,* or simply the star's *parallax.*

Putting $E = 90°$ in (i), we have :—

$$\sin P = \frac{r}{d}; \text{ and therefore } \sin p = \sin P . \sin E.$$

But the angles P, p are always very small; therefore their sines are very approximately equal to their circular measures. Thus we have approximately

$$P \text{ (in circular measure} = \frac{r}{d}, \quad p = P \sin E;$$

and, if P', p'' denote the numbers of seconds in P, p,

$$P'' = \frac{180 \times 60 \times 60}{\pi} \frac{r}{d} = 206{,}265 \frac{r}{d} \text{ (approximately)}.$$

$$\text{and} \quad p'' = P'' \sin E.$$

169. Relation between the Parallax and Distance of a Star

If a star's parallax be known, its distance from the Sun is given by the formula

$$P'' = \frac{180 \times 60 \times 60}{\pi} \frac{r}{d} = 206{,}265 \frac{r}{d};$$

$$\text{whence} \quad d = \frac{180 \times 60 \times 60}{\pi \times P''} r = 206{,}265 \frac{r}{P''},$$

where r is the Sun's distance from the Earth.

---

* Notice the close similarity between the present investigation and that of geocentric parallax in Art. 157.

The distances of the stars are so great (the distance of the nearest known star being about 25 million million miles) that it is inconvenient to express them in miles. A larger unit of distance is needed. The unit commonly used is the *parsec*, which, as the term implies, is the distance corresponding to a parallax of one second of arc. From the above relationship between d and P'', it is evident that the parsec is 206,265 r where r, the distance of the Earth from the Sun, may be taken as 93 million miles. Expressed in miles, the parsec is approximately 19·18 million million miles or $19{\cdot}18 \times 10^{12}$ miles.

If the distance d is expressed in parsecs, the relationship between distance and parallax becomes simply :—

$$d = \frac{1}{P''}.$$

The distance, in parsecs, is therefore the reciprocal of the parallax in seconds of arc.

Stellar distances are sometimes expressed also in terms of light-year. A *light-year* is the distance travelled by light in the course of a year. The velocity of light has been found to be 186,285 miles a second. The light-year is therefore :—

$$186{,}285 \times 60 \times 60 \times 24 \times 365\tfrac{1}{4} = 5{\cdot}89 \times 10^{12} \text{ miles.}$$

It follows that one parsec is equal to 19·18/5·89 light-years, or to 3·26 light-years,

Examples. 1. *The parallax of Castor is* 0·063″; *find its distance.*

$$\text{We have} \quad d = 206{,}265 \frac{r}{P''} = \frac{206{,}265 \times 93{,}000{,}000}{0{\cdot}063}$$

$$= 304{,}000{,}000{,}000{,}000, \quad \text{or} \quad 304 \times 10^{12} \text{ miles}$$

approximately. It would be useless to attempt to calculate more figures of the result with the given data, which are only approximate. It is most convenient (besides being shorter) to write the result in the second form.

Alternatively, the distance may be given as $1/{\cdot}063 = 15{\cdot}9$ parsecs or $15{\cdot}9 \times 3{\cdot}26 = 51{\cdot}8$ light-years. It follows that the light by which Castor is seen left the star about 52 years previously.

2. *Find the distance of α Centauri* (i) *in terms of the Sun's distance,* (ii) *in miles,* (iii) *in parsecs,* (iv) *in light-years, taking its parallax to be* 0·750″.

$$\text{Here} \quad d = \frac{206{,}265r}{{\cdot}75} = 275{,}000r$$

$$= 275 \times 10^3 \times 93 \times 10^6 = 25{,}575 \times 10^9$$

$$= 256 \times 10^{11} \text{ miles approximately.}$$

The distance in parsecs = $1/{\cdot}750 = 1{\cdot}33$ parsec.

The distance in light-years = $1{\cdot}33 \times 3{\cdot}26 = 4{\cdot}35$ light years.

170. General Effects of Parallax

Since Ex' is parallel to Sx, it is in the same plane as ES and Ex. Hence the lines ES, Ex, Ex' cut the celestial sphere of E at points S, x, x_0, lying in one great circle, and we have the two following laws :—

(i) *Parallax displaces the apparent position of a star from its heliocentric position towards the direction of the Sun.*

(ii) *The parallactic displacement of any star at different times varies as the sine of its angular distance from the Sun.*

Let Fig. 59 represent the observer's celestial sphere, S the Sun. Let x be the apparent or geocentric position of the star, whose parallax is P. Draw the great circle Sx and produce it to x_0, making

$$xx_0 = P \sin Sx.$$

Then x_0 represents the star's heliocentric position, and this is its position as corrected for annual parallax.

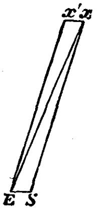

FIG. 58.

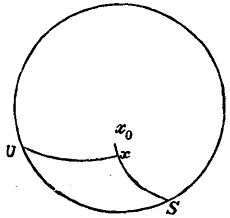

FIG. 59.

Conversely, if the star's heliocentric position x_0 is given, we may obtain its geocentric or apparent position x by joining x_0S, and on it taking

$$x_0x = P \sin Sx = P \sin Sx_0$$

very approximately (for the difference between $P \sin Sx$ and $P \sin Sx_0$ is exceedingly small, and may be neglected).

The terms *Parallax in Latitude* and *Parallax in Longitude* are used to designate the corrections for parallax which must be applied to the celestial latitude and longitude of a star respectively. Similarly, the *parallax in decl.* and *parallax in R.A.* denote the corresponding corrections for the decl. and R.A.

171. To show that any Star, owing to Parallax, appears to describe an Ellipse

In Fig. 58, Ex' is parallel to the star's heliocentric direction; therefore, x' is fixed, relative to the Earth. Moreover, $x'x = ES$. Hence, as the Sun S appears to revolve about the Earth in a year, the star x will appear as though it revolved in an equal orbit about its heliocentric position x', in a plane parallel to the ecliptic.

Let the circle MN (Fig. 60) represent this path, which the star x appears to describe in consequence of parallax. This circle is viewed obliquely, owing to its plane not being in general perpendicular to Ex'; hence, if mn denote its projection on the celestial sphere, the laws of perspective show that *mn* is an ellipse. (Appendix, 12.) This small ellipse is the curve described by the star on the celestial sphere during the year.

PARTICULAR CASES.— *A star in the ecliptic* moves as if it revolved about its mean position in a circle in the ecliptic plane, hence its projection on the celestial sphere oscillates to and fro in *a straight line* (more accurately a small arc of a great circle) of length $2P$.

For a star in the pole of the ecliptic the circle MN is perpendicular to Ex', hence Ex' describes a right cone, and the projection x describes on the celestial sphere a *circle*, of angular radius P, about the pole K.

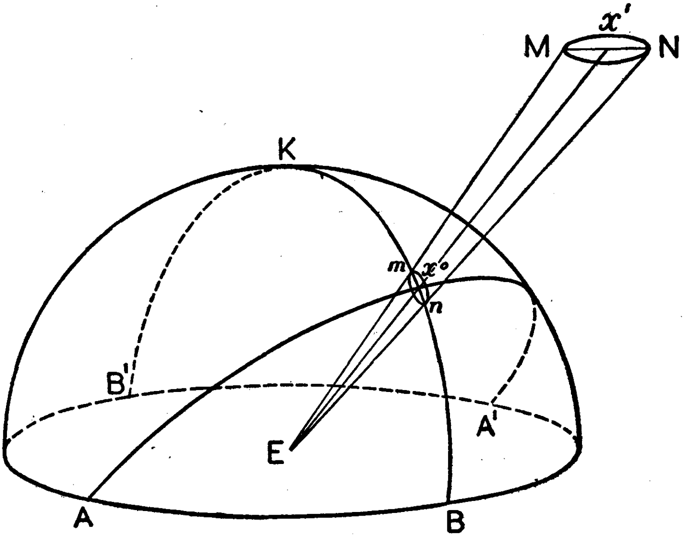

FIG. 60.

If the eccentricity of the Earth's orbit be taken into account, the curve MN will be an ellipse instead of a circle, but its projection *mn* will still be an ellipse.

172. Major and Minor Axes of the Ellipse

We shall now prove the following properties of the small ellipse described during the course of the year by a star whose parallax is P, and celestial latitude b:—

(i) (*a*) *The length of the semi-axis major is* P.

(*b*) *The major axis is parallel to the ecliptic.*

(c) *When the star is displaced along the major axis it has no parallax in latitude.*

(d) *At these times the Sun's longitude differs from the star's by* 90°.

(ii) (a) *The length of the semi-axis minor is P sin b.*

(b) *The minor axis is perpendicular to the ecliptic.*

(c) *When the star is displaced along the minor axis it has no parallax in longitude.*

(d) *At these times the Sun's longitude is either equal to the star's, or differs from it by* 180°.

On the celestial sphere let x_0 denote the heliocentric position of the star, $ABA'B'$ the ecliptic, K its pole, BKx_0B the secondary to the ecliptic through the star. Then, if S is the Sun, the star x_0 is displaced to x, where

$$x_0x = P \sin x_0S.$$

(i) The displacement is greatest when $\sin x_0S$ is greatest, and this happens when

$$\sin x_0S = 1, \quad x_0S = 90°.$$

If, therefore, we take A, A' on the ecliptic so that

$$x_0A = x_0A' = 90°,$$

A, A' are the corresponding positions of the Sun.

Now A, A' are the poles of BKB', and therefore the great circles Ax_0A' is a secondary to BKB'. Hence, if a, a' denote the displaced positions of the star, aa' is perpendicular to KB, and is therefore parallel to the ecliptic. Also :—

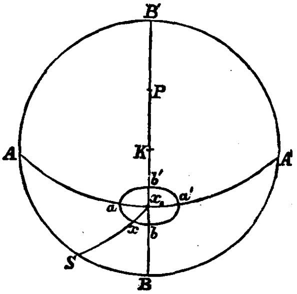

FIG. 61.

$$x_0a = x_0a' = P \sin 90° = P\,;$$

therefore the semi-axis major of the ellipse is P.

Since $AB = A'B = 90°$, the star's longitude (ΥB) differs from the Sun's longitude at A or A' by 90°.

And since the star is displaced parallel to the ecliptic, its latitude, or angular distance from the ecliptic, is unaltered, and therefore the parallax in latitude is zero.

(ii) The parallactic displacement is least when $\sin x_0S$ is least, and this happens when S is at B. For B is the point on the ecliptic nearest to x_0. Also, since

$$\sin x_0B' = \sin (180° - x_0B) = \sin x_0\mathrm{B},$$

it follows that the parallactic displacement is also least when S is at B′.

If, therefore, b, b' be the extremities of the minor axis, the arc bb' is along KB, and is therefore perpendicular to the ecliptic. Also :—

$$x_0b = x_0b' = P \sin x_0B = P \sin b,$$

therefore the semi-axis minor is $P \sin b$.

When the Sun is at B, it has the same longitude as the star; when at B', the longitudes differ by 180°.

And since the star is displaced in a direction perpendicular to the ecliptic, its longitude ♈B is unaltered; therefore the parallax in longitude is zero.

The parallax in latitude is evidently equal to the apparent angular displacement of the star resolved parallel to x_0K, and its maximum value is x_0b, or x_0b'. The parallax in longitude is not equal to the star's angular displacement perpendicular to Kx_0, but to the change of longitude thence resulting, and this is measured by the angle xKx_0. Hence, in Fig. 61,

(i) The maximum parallax in latitude

$$= x_0b = P \sin l.$$

(ii) The maximum parallax in longitude

$$= \angle x_0Ka = x_0Ka'$$
$$= x_0a/\sin Kx_0$$
$$= P/\cos x_0B = P \sec l.$$

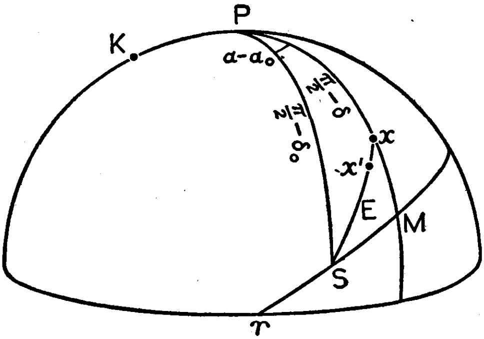

FIG. 62.

173. Annual Parallax in Right Ascension and Declination

In Fig. 62, P is the pole and K is the pole of the ecliptic. x is the star and S is the Sun. a, δ denote the R.A. and Dec. of the star and a_0, δ_0 the R.A. and Dec. of the Sun.

Parallax displaces the star from x to x', where $xx' = P \sin E$, E being the angular distance of x from the Sun.

A displacement $P \sin E$ along xS can be resolved into $P \sin E \sin \theta$ perpendicular to xM and $P \sin E \cos \theta$ along xM, where θ is the angle SxM.

In the triangle PSx, the sides $Px = \frac{\pi}{2} - \delta$, $PS = \frac{\pi}{2} - \delta_0$, and the angle $SPx = a - a_0$ are known. Hence the side $Sx = E$ and the angle $SxP = \pi - \theta$ can be determined. We have :—

$$\frac{\sin E}{\sin (a - a_0)} = \frac{\cos \delta_0}{\sin \theta}.$$

Hence $\quad P \sin E \sin \theta = P \sin (a - a_0) \cos \delta_0$.

This is the component of the displacement perpendicular to xM. The corresponding displacement in right ascension is $P \sin (\alpha - \alpha_0) \cos \delta_0 \sec \delta$. This must be subtracted from the true R.A. to obtain the apparent R.A. as affected by parallax or added to the apparent R.A. to obtain the true R.A.

The component of the displacement along xM is $P \sin E \cos \theta$. By the use of formula (3) of Art. 10, this is equal to

$$P\,\{\cos (\alpha - \alpha_0) \sin \delta \cos \delta_0 - \cos \delta \sin \delta_0\}.$$

This quantity must be subtracted from the true Dec. to obtain the apparent Dec. as affected by parallax, or added to the apparent Dec. to obtain the true Dec.

174. Annual Parallax in Latitude and Longitude

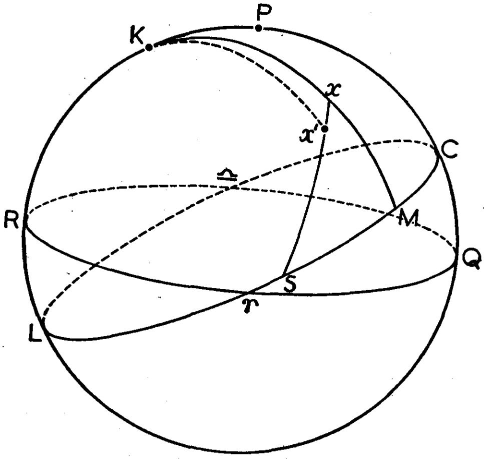

FIG. 63.

Referring to Fig. 63, P is the pole of the equator, R ♈ Q ♎, and K is the pole of the ecliptic L ♈ C ♎. x is the heliocentric position of the star as seen from the Sun, x' is the position as displaced by parallax. M is the foot of the secondary circle from the pole of the ecliptic, K, to the ecliptic. S denotes the position of the Sun.

In the triangle SxM, Sx is E, the angular distance of the star from the Sun. SM is $(l - l_0)$, where l, l_0 are the longitudes of the star and Sun respectively; xM is the latitude, b, of the star. The angle at M is a right-angle.

The parallactic displacement is $xx' = P \sin E$. This can be resolved into components $P \sin E \sin SxM$ perpendicular to xM and $P \sin E \cos SxM$ along xM.

For the compound perpendicular to xM, we have :—

$$P \sin E \sin SxM = P \sin (l - l_0).$$

The corresponding difference of longitude is $P \sin (l - l_0) \sec b$.

The component along xM is $P \sin E \cos SxM$. But, using the formulae for a right-angled triangle, $\cos SxM = \tan b \cot E$.

Hence $P \sin E \cos SxM = P \cos E \tan b = P \sin b \cos (l - l_0)$ since $\cos E = \cos b \cos (l - l_0)$.

The parallax in longitude is therefore $P \sec b \sin (l - l_0)$ and the parallax in latitude is $P \sin b \cos (l - l_0)$.

These quantities must be added to the observed longitude and latitude respectively to obtain the corresponding heliocentric quantities.

EXAMPLES

1. If a, a' be the true and apparent altitudes of a planet affected by parallax, prove the equation $a = a' + P \cos a'$.

2. A planet, whose distance is 16 million miles, has declination + 20°. Find the parallax in R.A., at a place in latitude 50° N, for hour-angles 6h. and 18h.

3. Given that the Moon's horizontal parallax is 55′ 25″, when its semi-diameter s 15′ 6″, find the semi-diameter when its horizontal parallax is 61′ 24″.

4. The observed meridian zenith distance of the Moon's upper limit is 58° 28′ 21″. The horizontal parallax is 60′ 16″, the semi-diameter 16′ 25″, and the latitude of the place of observation is 50° 45′. Find the Moon's declination, given that the constant of refraction is 58.2″.

5. Prove that cosec 8·76″ = 23546 approximately, and thence that the distance of the Sun is nearly 81 million geographical miles, the angle 8·76″ being the Sun's parallax, and a geographical mile subtending 1′ at the Earth's centre.

6. Find the Sun's diameter in miles, taking the Sun's parallax as 8·8″, its angular diameter as 32′, and the Earth's radius as 3,960 miles.

7. A spot at the centre of the Sun's disc is observed to subtend an angle of 5″. What is its absolute diameter?

8. If the annual parallax be 2″, determine the distance of the star, taking the Sun's distance to be 90,000,000 miles. Hence, deduce the distance of a star whose parallax is 0·2″.

9. Find, roughly, the distance of a star whose parallax is 0·5″, given that the Sun's parallax is 9″, and the Earth's radius is 4000 miles.

10. The parallax of 61 *Cygni* is 0·3″, and its proper motion, perpendicular to the line of sight, is 5″ a year; compare its velocity in that direction with that of the Earth in its orbit round the Sun.

11. Account for the following phenomena: (i) all stars in the ecliptic oscillate in a straight line about their mean places in the course of the year; (ii) two very near stars in the ecliptic appear to approach and recede from one another in the course of the year.

CHAPTER VIII

ABERRATION

175. Aberration

In the two preceding chapters, we have discussed the displacements of the apparent position of a body produced by refraction and by geocentric and annual parallax. In the present chapter, we discuss further effects that give rise to small changes in the positions of celestial bodies.

The Aberration of Light is a displacement of the apparent directions of stars, due to the effect of the Earth's motion on the direction of the *relative* velocity with which their light approaches the Earth.

The velocity of light has been measured by laboratory experiments in a variety of ways. Recent determinations by Michelson have given a velocity of 299,796 kilometres or 186,285 miles a second: this may be taken as the most probable value.

The rays of light emanating from a star travel in straight lines through space* with a velocity of about 186,285 miles per second. We *see* the star when the rays reach our eye, and the appearance presented to us depends solely on how the rays are travelling at that instant. If the Earth were at rest, and there were no refraction, we should see the star in its true direction, because the light would be travelling towards our eyes in a straight line from the star. But in every case the direction in which a star is *seen* is the direction of approach of the light-rays from the star at the instant of their reaching the eye.

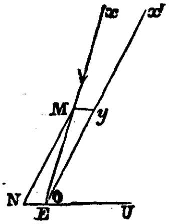

FIG. 64.

Now the velocity of approach is the *relative velocity* of the light with respect to the observer. If the observer is in motion, this relative velocity is partly due to the motion of the light and partly due to the motion of the observer. If the observer happens to be travelling towards or away from the source of light, the only effect of his motion will be to increase or decrease the velocity of approach of the light, without altering its direction, but if he be moving in any other direction, his own motion will alter the direction of the relative velocity of approach, and will therefore alter the direction in which the star is seen.

Suppose the light to be travelling from a distant star x in the direction xO. Let V be the velocity of light, and let it be represented by the length MO. Suppose also that an observer is travelling along the direction NO with velocity u, represented by the straight line NO. Then, if we regard O as a *fixed point*, the light is approaching O with velocity represented by *MO*. Also since the observer is approaching *O* with velocity represented by *NO*, the point *O* is approaching the observer *N* with an equal and opposite velocity represented therefore by *ON*. Hence the whole relative velocity with which the light is travelling towards the observer is the resultant of the velocities represented by *MO* and *ON*. By the Triangle of Velocities this resultant velocity is represented in magnitude and direction by *MN*. Hence *MN* represents the direction of approach of the light towards the observer's eye. Therefore when the observer has reached *O* the star is seen in the direction *Ox′* drawn parallel to *NM*, although its real direction is *Ox*.

---

* Of course the rays are refracted when they reach the Earth's atmosphere, but the effects of refraction can be allowed for separately.

In consequence, the star appears to be displaced from its true position x to the position x' This displacement is called the *aberration* of the star, and its amount is, of course, measured by the angle xOx'. This angle is sometimes called the *angle of aberration* or the *aberration error*.

176. Illustrations of Relative Velocity and Aberration

The following simple illustrations may possibly assist the reader in understanding more thoroughly how aberration is produced.

(1) Suppose a shower of rain-drops to be falling perfectly vertically, with a velocity, say, of 40 feet per second. Then, if a man walk through the shower, say with a velocity of 4 feet per second, the drops will appear to be coming towards him, and therefore to be falling in a direction inclined to the vertical. Here the man is moving towards the drops with a horizontal velocity of 4 feet per second, and therefore the drops appear to be coming towards the man with an equal and opposite horizontal velocity of 4 feet per second.

Their whole relative velocity is the resultant of this horizontal velocity and the vertical velocity of 40 feet per second with which the drops are approaching the ground. By the rule for the composition of velocities, this relative velocity makes an angle $\tan^{-1} \frac{4}{40}$ or $\tan^{-1} \cdot 1$ with the vertical. Hence the man's own motion causes an apparent displacement of the direction of the rain from the vertical through an angle $\tan^{-1} \cdot 1$. This angle corresponds to the angle of aberration in the case of light.

(2) Suppose a ship is sailing due south, and that the wind is blowing from due west with an equal velocity. Then to a person on the ship the wind will appear to be blowing from the south-west, its southerly component being due to the motion of the ship, which is approaching the south. In this case the ship's velocity causes the wind to apparently change from west to south-west, *i.e.* to turn through 45°. We might, therefore, consistently say that the " angle of aberration " of the wind was 45°.

177. Annual and Diurnal Aberration

A point on the Earth's surface is moving through space with a velocity compounded of (i) the orbital velocity of the Earth in the ecliptic about the Sun; (ii) the velocity due to Earth's rotation about the poles.*

---

* There are still other velocities, viz. that due to the Sun's motion in space (Art. 392) These, however, are constant for long periods.

These give rise to two different kinds of aberration, known respectively as *annual* and *diurnal aberration.* Now the Earth's orbital velocity is about $2\pi \times 93{,}000{,}000$ miles per annum, or rather over 18 miles per second, while the velocity due to the Earth's rotation at the equator is roughly $2\pi \times 4000$ miles per day, or 0·3 miles per second. The former velocity is about $\frac{1}{10000}$ of the velocity of light, and therefore the annual aberration is a small though measurable angle. The latter velocity is only $\frac{1}{60}$ as great; hence the diurnal aberration is much smaller and less important. For this reason the term "aberration" always signifies annual aberration, unless the word "diurnal" is also used. We shall now consider the effects of annual aberration, leaving diurnal aberration till the end of this section.

178. To Determine the Correction for Aberration on the Position of a Star

Let Ox be the actual direction of a star x seen from the Earth at O; OU the direction of the Earth's orbital motion at the time of observation. On Ox take OM representing on any scale the velocity of light, and draw MY parallel to OU, and representing on the same scale the velocity of the Earth. Then YO represents the relative velocity of the light in magnitude and direction, so that OYx' is the direction in which the star x is *seen* (Fig. 65).

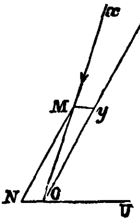

FIG. 65.

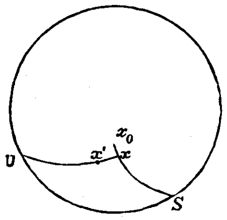

FIG. 66.

For if ON be drawn parallel and equal to YM, the parallelogram of velocities $MNOY$ shows that MO, the actual velocity of the light-rays in space is the resultant of the two velocities YO and NO, or YO and MY, and therefore YO is the required relative velocity.

Since Ox, Ox', and OU all lie in one plane, it follows, by representing their directions on the celestial sphere, that *a star is displaced by aberration along the great circle joining its true place to the point on the celestial sphere towards which the Earth is moving.*

The displacement xOx' is called the star's *aberration error.* Let it be denoted by y, and let

$u = NO =$ velocity of Earth, $V = MO =$ velocity of light.

Then the triangle OMY gives :—

$$\frac{\sin MOY}{\sin MYO} = \frac{MY}{MO} = \frac{u}{V};$$

$$\text{or}\quad \sin y = \frac{u}{V}\sin MYO = \frac{u}{V}\sin UOx'.$$

The aberration error y is, therefore, greatest when $UOx' = 90°$. Let its value, then, be k. Putting $UOx' = 90°$, we have :—

$$\sin k = u/V\,;$$

$$\sin y = \sin k \sin UOx'.$$

The angle UOx' is called the *Earth's Way* of the star, and k is called the *Coefficient of Aberration.* Since y and k are both small, we have, approximately

$$y = k \sin \text{(Earth's way)},$$

$$k \text{ (in circular measure)} = u/V\,;$$

and, therefore, if y'', k'' denote the number of seconds in y, k respectively

$$y'' = k'' \sin \textit{(Earth's Way)},$$

$$k'' = \frac{180 \times 60 \times 60}{\pi}\,\frac{u}{V} = 206{,}265 \times \frac{\textit{velocity of Earth}}{\textit{velocity of light}}.$$

179. General Effect of Aberration on the Celestial Sphere

Neglecting the eccentricity of the Earth's orbit, the direction of motion of the Earth, in the ecliptic plane, is always perpendicular to the radius vector drawn to the Sun. Hence, on the celestial sphere, the point U, towards which the Earth is moving, is on the ecliptic, at an angular distance 90° behind the Sun. This point is sometimes called the *apex of the Earth's Way.*

Let x' denote the observed position of the star. Draw the great circle $x'U$, and produce it to a point x, such that

$$xx' = k \sin x'U.$$

Then x represents the star's true position, corrected for aberration.

Conversely, if we are given the true position x, we can find the apparent position x' by joining xU and taking

$$xx' = k \sin xU,$$

for it is quite sufficiently approximate to use $k \sin xU$ instead of $k \sin x'U$.

We thus have the following laws :—

(i) *Aberration produces displacement in the apparent position of a star towards a point U on the ecliptic, distant* 90° *behind the Sun.*

(ii) *The amount of the displacement varies as the sine of the Earth's Way of the star,* i.e. *the star's angular distance from the point U.*

180. Comparison between Aberration and Annual Parallax

The student will not fail to notice the close analogy between the corrections for aberration and annual parallax.

The point U for the former corresponds to the point S for the latter, in determining the direction and magnitude of the displacement. In fact, *the aberration error of a star is exactly the same as its parallactic correction would be three months earlier* (when the Sun was at U) *if the star's annual parallax were* k.

There is, however, this important difference that *the annual parallax depends on a star's distance, whilst the constant of aberration* k *is the same for all stars.*

For k depends only on the ratio of the Earth's velocity to the velocity of light, and not on the star's distance. The value of k in seconds is about 20·47″; for rough purposes it may be taken as 20·5″.

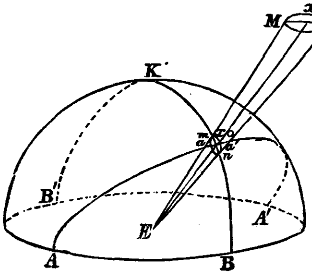

Fig. 67.

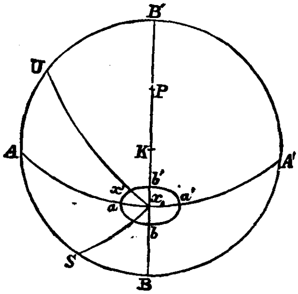

Fig. 68.

181. To show that the Aberration Curve of a Star is an Ellipse

This result, which follows immediately from the analogy between aberration and parallax, may be proved independently as follows:— On Ex (Fig. 67), the true direction of a star Ex, take x to represent the velocity of light, and xM to represent the Earth's velocity. Then MO meets the celestial sphere in m, the star's apparent position.

As the Earth's direction of motion in the ecliptic varies, while its velocity remains constant, M describes a circle about x as centre in a plane parallel to the ecliptic plane. The projection of this circle on the celestial sphere is an ellipse, and this is the curve traced out by a star during the year in consequence of aberration.

PARTICULAR CASES.—*A star in the ecliptic* oscillates to and fro in a *straight line*, or more accurately an arc of a great circle of length $2k$. *A star at the pole of the ecliptic* revolves in a small circle of radius k (*cf.* Art. 171).

182. Major and Minor Axes of the Aberration Ellipse

By writing U for S and k for P in the investigation of Art. 172, we obtain the analogous results relating to the ellipse described by a star in consequence of aberration, namely :—

(i) (*a*) *The length of the semi-axis major is k.*

(*b*) *The major axis of the ellipse is parallel to the ecliptic.*

(*c*) *When the star is displaced along the major axis it has no aberration in latitude.*

(*d*) *At these times the Sun's longitude is either equal to the star's, or differs from it by* 180°.*

(ii) (*a*) *The length of the semi-axis minor is k sin l.*

(*b*) *The minor axis is perpendicular to the ecliptic.*

(*c*) *When the star is displaced along the minor axis, it has no aberration in longitude.*

(*d*) *At these times the Sun's longitude differs from the star's by* 90°.

COROLLARY.—The maximum aberration in longitude $= k \sec l$ (*cf.* Art. 172, ii).

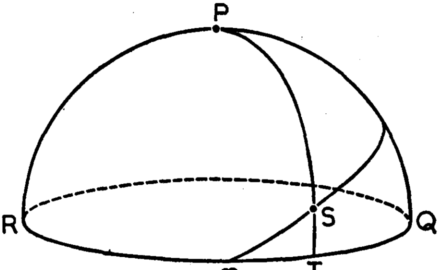

FIG. 69.

183. Aberration in Right Ascension and Declination

The effect of aberration in right ascension and declination can be derived in exactly the same way as in Art. 173 for annual parallax. In Fig. 62, the point S is now a point on the ecliptic whose longitude is 90° smaller than that of the Sun. If α_0, δ_0 are the R.A. and Dec. of this point, we have, as in Art. 173,

$$\text{Aberration in R.A.} = k \sec \delta \sin (\alpha - \alpha_0) \cos \delta_0.$$

$$\text{Aberration in Dec.} = k \{\cos (\alpha - \alpha_0) \sin \delta \cos \delta_0 - \cos \delta \sin \delta_0\}.$$

We have now to express α_0, δ_0 in terms of the Sun's longitude l_0.

In Fig. 69, in the spherical triangle ΥST, ΥT is α_0, ST is δ_0, ΥS is $l_0 - 90°$ and the angle $S\Upsilon T$ is ϵ, the obliquity of the ecliptic. From the formulae for a right-angled triangle, we have

$$\sin \alpha_0 \cos \delta_0 = \sin (l_0 - 90^\circ) \cos \epsilon = - \cos l_0 \cos \epsilon$$

$$\cos \alpha_0 \cos \delta_0 = \cos (l_0 - 90^\circ) = \sin l_0$$

$$\sin \delta_0 = \sin (l_0 - 90^\circ) \sin \epsilon = - \cos l_0 \sin \epsilon.$$

---

* Note that (i, *d*) and (ii, *d*) are the reverse of the corresponding properties in Art. 172.

We obtain, therefore, *aberration in R.A.*

$$= k \sec \delta \sin (\alpha - \alpha_0) \cos \delta_0$$
$$= k \sec \delta (\sin \alpha \cos \alpha_0 - \cos \alpha \sin \alpha_0) \cos \delta_0$$
$$= k \sec \delta (\sin \alpha \sin l_0 + \cos \epsilon \cos \alpha \cos l_0)$$

and *aberration in Dec.*

$$= k \{(\cos \alpha \cos \alpha_0 + \sin \alpha \sin \alpha_0) \sin \delta \cos \delta_0 - \cos \delta \sin \delta_0\}$$
$$= k \{\cos \alpha \sin \delta . \sin l_0 - \sin \alpha \sin \delta \cos \epsilon \cos l_0 + \sin \epsilon \cos \delta \cos l_0\}$$
$$= k \{\sin \delta (\cos \alpha \sin l_0 - \cos \epsilon \sin \alpha \cos l_0) + \sin \epsilon \cos \delta \cos l_0\}.$$

These quantities must be added to the apparent R.A. and Dec. to obtain the true values.

184. Aberration in Latitude and Longitude

The expressions for the aberration in latitude and longitude can be derived in exactly the same way as those for annual parallax in latitude and longitude in Art. 174. In Fig. 63, xx' becomes $k \sin E$, where E is the distance xS, S now being a point whose longitude is 90° less than that of the Sun.

The expressions for aberration in latitude and longitude are therefore obtained from those for annual parallax by substituting the constant of aberration k for P and $l_0 - 90°$ for l_0. We thus obtain :—

$$\text{Aberration in longitude} = k \sec b \sin (l - l_0 + 90°)$$
$$= k \sec b \cos (l - l_0)$$
$$\text{Aberration in latitude} = k \sin b \cos (l - l_0 + 90°)$$
$$= - k \sin b \sin (l - l_0)$$

These quantities must be added to the observed longitude and latitude to obtain the true values.

*185. Effect of Eccentricity of Earth's Orbit

Owing to the elliptic form of the Earth's orbit the Earth's velocity is not quite uniform, and therefore the coefficient of aberration is subject to small variations during the year. The earth's velocity is greatest at perihelion and least at aphelion. The angular velocities at those times are inversely proportional to the squares of the corresponding distances from the Sun, but the actual (linear) velocities are inversely proportional to the distances themselves, and these are in the ratio of $1 - e : 1 + e$, or $1 - \frac{1}{60} : 1 + \frac{1}{60}$. Since the coefficient of aberration is proportional to the Earth's velocity, its greatest and least values are therefore in the ratio of 61 : 59, and are respectively $\frac{61}{60}$ and $\frac{59}{60}$ of its mean value.

Moreover, the direction of the Earth's motion is not always exactly perpendicular to the line joining it to the Sun, hence the "apex of the Earth's way," towards which a star is displaced, may be distant a little more or less than 90° from the Sun at different seasons.

The aberration curve is still an ellipse. The student who has read the more advanced parts of particle dynamics may know that the curve MN, traced out by

M, is in this case the "hodograph" of the Earth's orbital motion. It is also known, in the case of elliptic motion, such as the Earth's, that this hodograph is a circle, whose centre does not, however, quite coincide with x. Hence the aberration curve hk is an ellipse.

186. Discovery of Aberration

Aberration was discovered by Bradley, in 1725, in the course of a series of observations made with a zenith sector on the star *γ Draconis* for the purpose of discovering its annual parallax. The star's latitude was observed to undergo small periodic variations during the course of the year, and these differed from the variations due to annual parallax in the fact that *the displacement in latitude was greatest when the Sun's longitude differed from that of the stars by* 90°; that is, at the time *when the parallax in latitude should be zero* (Art. 172, i, c). It can be seen also from Art. 174 that the parallax in latitude is zero when $l - l_0 = \pm 90°$.

The fact that the phenomenon recurred annually led Bradley to suppose that it was intimately connected with the Earth's motion about the Sun, and he was thus led to adopt the explanation which we have given above. It will be seen that the peculiarity which led Bradley to discard annual parallax as an explanation is quite in harmony with the results of Arts. 182 and 184.

187. To Determine the Constant of Aberration by Observation

The constant k can best be found by observing the declinations of stars with a zenith telescope or with a transit circle.

If δ is the true declination of the star and δ_1 is the observed declination, we can write :—

$$\delta - \delta_1 = kF_1$$

where kF_1 is the expression given in Art. 183. At another season of the year, observations are made again. If δ_2 is the observed declination :—

$$\delta - \delta_2 = kF_2$$

$$\text{whence } \delta_2 - \delta_1 = k(F_1 - F_2) \text{ determines } k.$$

The seasons of the year at which observations are made are chosen to give the largest range in $(F_1 - F_2)$, in order to give the most favourable conditions for an accurate determination of the constant of aberration. These conditions can be seen from the expression in Art. 183 for the aberration in declination. As ϵ is about $23\frac{1}{2}°$, the first two terms do not differ greatly from the value obtained by putting $\cos \epsilon = 1$ (the correct value being ·92). The first two terms then become $\sin \delta \sin (l_0 - \alpha)$. Stars of high declination and values of $(l_0 - \alpha)$ near 6h. and 18h. are therefore required. The conditions are satisfied if the stars are on the meridian at about 6h. and 18h. local mean time. Observations should therefore be made as near 6h. as conditions permit at one season of the year and near 18h. six months later. Corrections must be applied for refraction and for precession.

The constant of aberration is difficult to determine accurately. Systematic errors are liable to enter when observations taken at different seasons of the year and at different times of the night have to be compared.

The star γ *Draconis* was chosen by Bradley because it was nearly in the zenith and its position was therefore but little affected by refraction. The star is very favourable in another respect, for its longitude is very nearly 270°. It therefore lies very nearly in the "solstitial colure," its declination circle passing nearly through the pole of the ecliptic.

At the vernal equinox, the star's longitude is less than the Sun's by 90°, and it is therefore displaced away from the poles of the ecliptic and equator through a distance k'' sin l, its declination being therefore decreased by k'' sin l. At the autumnal equinox its declination is increased by k'' sin l.

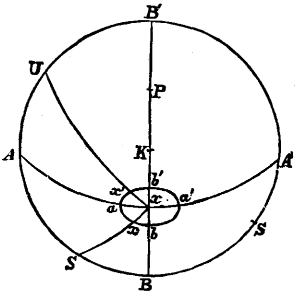

Fig. 70.

Hence the difference of the apparent declinations $= 2k''$ sin l, and this is also the difference of the star's apparent meridian zenith distances. By observing these, k'' may be found, l being of course known. The value of k'' is very approximately 20·47″.

An alternative method of finding the aberration constant is to measure with a spectroscope the exact positions of the spectral lines of selected stars near the ecliptic at different seasons of the year. The motion of the Earth to or from the stars alters the wave length of the light, and causes a shift of the spectral lines to and fro; from this, the ratio of the Earth's speed to that of light can be deduced, whence we obtain the aberration constant and the Sun's distance.

188. Relation between the Coefficient of Aberration, the Sun's Parallax, and the Velocity of Light

We have seen (Art. 178) that :—

$$k'' = \frac{180 \times 60 \times 60}{\pi} \frac{u}{V} \text{ .... (i)}$$

where k'' is the coefficient of aberration in seconds, u the velocity of the Earth, V that of light, both of which we will suppose measured in miles per second.

Now let r represent the radius of the Earth's orbit (supposed circular) in miles. Then in one sidereal year, or 365¼ days, the Earth travels round its orbit through a distance $2\pi r$ miles. Hence the Earth's velocity in miles per second is $u = \dfrac{2\pi r}{365\frac{1}{4} \times 24 \times 60 \times 60}$. Substituting in (i), we have :—

$$k'' = \frac{15}{365\frac{1}{4}} \frac{r}{V}.$$

If, therefore, the coefficient of aberration be determined by observation and V, the velocity of light be measured experimentally, the Sun's distance r can be found. Thus the Sun's parallax can be calculated from the coefficient of aberration and the velocity of light or, conversely, the coefficient of aberration can be calculated from the sun's parallax and the velocity of light.

It can be shown that, when the fact that the Earth's orbit is not circular but is an ellipse of eccentricity e is taken into account, this expression should be changed to

$$k'' = \frac{15}{365\frac{1}{4}} \cdot \frac{r}{\sqrt{1 - e^2}\, V}.$$

Now if ρ is the Earth's radius, the Sun's parallax, P, is given by

$$P'' = \frac{180 \times 60 \times 60}{\pi} \frac{\rho}{r}.$$

$$\text{Therefore } P''k'' = \frac{180 \times (60)^3}{\pi \times 1461} \frac{\rho}{\sqrt{1 - e^2}\, V}.$$

Putting $\rho = 3963$ miles, $V = 186{,}285$ miles per sec, $e = 0{\cdot}01674$ we obtain

$$P''k'' = 180{\cdot}3.$$

189. Planetary Aberration

The direction of any planet is affected by aberration, which is due partly to the motion of the Earth, and partly to that of the planet itself. For, during the time occupied by the light in travelling from a planet to the Earth, the planet itself will have moved from the position which it occupied when the light left it.

We shall, however, show that *the direction in which a planet is seen at any instant was the actual direction of the planet relative to the Earth at the instant previously when the light left the planet.*

Let t be the time required by the light to travel from the planet to the Earth. Let P, Q be the positions of the planet and Earth at any instant; P', Q' their positions after an interval t. The light which leaves the planet when at P reaches the Earth when it has arrived at Q'; the direction of the actual motion of the light is, therefore, along PQ'. But PQ' and QQ' are the spaces passed over by the light and the Earth respectively in the time t (and QQ' is so small an arc that it may be regarded as a straight line). Therefore

$$QQ' : PQ' = \text{velocity of Earth} : \text{velocity of light.}$$

Hence it follows from Art. 178 that the line PQ represents the direction of relative velocity of the light with respect to the Earth. Therefore, when the Earth is at Q' the planet is seen in a direction parallel to PQ, and its apparent direction is exactly what its real direction was at a time t previously.

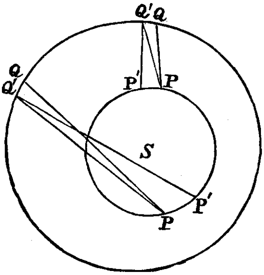

FIG. 71.

The same is true in the case of the Sun or a comet, or any other body, provided that the time taken by the light from the body to reach the Earth is so small that the Earth's motion does not change sensibly in direction in the interval.

The *aberration* of the planet at any instant is the angle between the apparent direction PQ and the actual direction $P'Q'$.

Example.—*Find the effect of aberration on the positions of* (i) *the Sun,* (ii) *Saturn in opposition, taking its distance from the Sun to be* $9\frac{1}{2}$ *times the Earth's.*

(i) The light takes 8m. 20s. to travel from the Sun to the Earth, therefore the Sun's apparent coordinates at any instant are its actual coordinates 8m. 20s. previously. Thus, its apparent decl. and R.A. at noon are its true decl. and R.A. at 11h. 51m. 40s.

Now the Sun describes 360° longitude in $365\frac{1}{4}$ days. Hence in 8m. 20s. $= r/V$ seconds it describes

$$\frac{360 \times 60 \times 60}{365\frac{1}{4} \times 24 \times 60 \times 60} \times \frac{r}{V} = \frac{15}{365\frac{1}{4}}\,\frac{r}{V} = k'' \text{ (by Art. 188).}$$

This is otherwise evident from the fact that the Earth's way of the Sun is 90°; and it is at rest, consequently its aberration $= k$.

(ii) The distance of Saturn from the Earth at opposition is $= 9\frac{1}{2} - 1$, or $8\frac{1}{2}$ times the Sun's distance. Light travels over this distance in

$$8\text{m. } 20\text{s.} \times 8\tfrac{1}{2} = 500 \times 8\tfrac{1}{2}\text{s.} = 1\text{h. } 10\text{m. } 50\text{s.}$$

Therefore, the apparent coordinates are the actual coordinates 1h. 10m. 50s. previously.

Thus the observed decl. and R.A. at midnight (24h. 0m. 0s.) are the true decl. and R.A. at 22h. 49m. 10s.

190. Diurnal Aberration

Diurnal aberration is due to the effect of the Earth's diurnal rotation about the poles on the relative velocity of light.

As the Earth revolves from west to east, the portion of the motion of an observer due to this diurnal rotation is in the direction of the east point E of the horizon. The effect of diurnal aberration can thus be investigated by methods precisely similar to those of Art. 178, E taking the place of U.*

Hence, every star x is displaced by diurnal aberration towards the east point E. And if x' be its displaced position, then

$$\text{the displacement } xx' = A \sin xE,$$

$$\text{where circular measure of } A = \frac{\text{velocity of observer}}{\text{velocity of light}},$$

Taking a for the Earth's radius, V for the velocity of light, let the observer's latitude be ϕ. In a sidereal day (86164·1 mean seconds) the Earth's rotation carries the observer round a small circle, whose distance from the Earth's axis is $a \cos \phi$, and whose circumference is, therefore, $2\pi a \cos \phi$. Hence :—

$$\text{Observer's velocity} = \frac{2\pi a \cos \phi}{86164{\cdot}1} \text{ miles per second};$$

$$\text{and the circular measure of } A = \frac{2\pi a \cos \phi}{86164{\cdot}1 \times V};$$

$$\text{Therefore, } A'' \text{ (number of seconds in } A) = \frac{180 \times 60 \times 60}{\pi} \times \frac{2\pi a \cos \phi}{86164{\cdot}1\ V}$$

$$= \frac{15a \cos \phi}{V} \text{ approximately.}$$

Thus the coefficient of diurnal aberration varies as the cosine of the latitude. If K'' denote the coefficient of diurnal aberration at the equator in seconds, we therefore have

$$K'' = \frac{15a}{V} = \frac{15 \times 3963}{186{,}000} = 0{\cdot}32'',$$

$$A'' = K'' \cos \phi = 0{\cdot}32'' \cos \phi.$$

*191. Effect of Diurnal Aberration on Meridian Observations

The correction for diurnal aberration is greatest when the star is 90° from the east point, *i.e.* is on the meridian. In this case, the displacement is perpendicular to the meridian, and is equal to A''.

The star's meridian altitude is thus unaffected, but its time of transit is somewhat retarded at upper culmination, and (for a circumpolar star) accelerated at lower culmination, since the star appears on the meridian, when it is really A'' west of the meridian.

For a star on the equator, seen from the Earth's equator, the retardation of the time of transit would be $\frac{1}{15} K''$ seconds, $= \frac{1}{50}$ of a second nearly, and it would be difficult to observe such a small interval.

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* The student will find it useful to go through the various steps of Arts. 178–181, considering the diurnal motion.

192. To determine the Coefficient of Diurnal Aberration

Diurnal aberration can be calculated if we know the size of the Earth, the period of its rotation, and the speed of light. All these constants are known to at least 4 significant figures; hence diurnal aberration could be calculated, if required, to 4 figures; two figures, however, suffice in practice, and about their value there can be no shadow of doubt. Hence it is useless to attempt to find this quantity by observation, which would be difficult to make with the necessary accuracy, since differential measures, such as are used in parallax work, are here unavailable.

EXAMPLES

1. Suppose the velocity of light to be the same as the velocity of the Earth round the Sun. Discuss the effect on the Pole Star as seen by an observer at the North Pole throughout the year.

2. Sound travels with a velocity 1100 feet per second. Determine the aberration produced in the apparent direction of sound to a person in a railway train travelling at sixty miles an hour, if the source of sound be exactly in front of one of the windows of the carriage.

3. Show that, in consequence of aberration, the fixed stars whose latitude is l appear to describe ellipses whose eccentricity is $\cos l$.

4. How must a star be situated so as to have no displacement due to (i) aberration, (ii) parallax? Where must a star be so that the effect may be the greatest?

5. On what stars is the effect of aberration or parallax to make them appear to describe (i) circles, (ii) straight lines?

6. Show that the effect of annual parallax on the position of a star may be represented by imagining the star to move in an orbit equal and parallel to the Earth's orbit, and that the effect of aberration may be represented by imagining it to revolve in a circle whose radius is equal to the distance traversed by the Earth while the light is travelling from the star.

7. Supposing the star η *Virginis* to be situated (as it nearly is) at the first point of Libra, find the direction and magnitude of its displacement due to aberration about the 21st day of every month of the year, taking the coefficient of aberration to be $20{\cdot}5''$. When is its aberration greatest?

8. At the solstices show that a star on the equator has no aberration in declination. If its R.A. be 22h., show that its time of transit is retarded at the summer and accelerated at the winter solstice by ·68 of a second.

9. If the coefficient of aberration be $20''$, and an error of 2000 miles a second be made in determining the velocity of light, find, in miles, the consequent error in the value of the Sun's mean distance as computed from these data.

10. Show that when a planet is stationary its position is unaffected by aberration.

11. Taking the Earth's radius as 4000 miles, velocity of light 186,000 miles per second, show that the coefficient of diurnal aberration at the equator is about one-third of a second.

12. Prove that the product of a star's parallax and its distance in light-years is equal to the coefficient of aberration divided by 2π.

13. Show the ratio of the coefficients of diurnal and annual aberration is equal to the Sun's parallax in circular measure multiplied by $366\frac{1}{4}$.

EXAMINATION PAPER

1. Explain what is meant by *equatorial horizontal parallax, annual parallax, annual aberration, diurnal aberration.*

2. Given that the radius of the Earth is 3963 miles, the velocity of light is 186,285 miles a second and the Sun's parallax is 8.79″, find the distance of the Sun and the constant of aberration.

3. Given that the Moon's angular diameter 29′ 28″, its horizontal parallax is 54′ 04″, and the radius of the Earth is 3963 miles, find the Moon's diameter in miles. What is the value of the horizontal parallax when the angular diameter is 33′ 28″?

4. Define *parsec* and *light-year* and find how many light-years are equivalent to one parsec.

5. The parallax of a star is to be determined from observation of the displacement of the star in R.A. when on the meridian. Discuss when the observations should be made in order to obtain the largest parallactic displacements, and therefore the most accurate determination of the parallax.

6. Distinguish between solar and stellar parallax. Towards what point does a star seem to be displaced by heliocentric parallax? Find an expression for the displacement.

7. Explain the *aberration of light,* and investigate the direction and magnitude of the displacement which it produces on the apparent position of a star.

8. Show that owing to aberration a star in the pole of the ecliptic appears to describe a circle, and that a star in ecliptic appears to oscillate to and fro in a straight line during the course of the year.

9. Show how the velocity of light may be determined from the aberration of a star when the Sun's mean distance is known.

10. Investigate the general effects of *diurnal aberration* due to the Earth's rotation about its axis. In what direction are stars displaced by diurnal aberration? Show that the coefficient of diurnal aberration at a place in latitude l is $K \cos l$, where K is the coefficient at the equator.

CHAPTER IX

THE MOON

I.—MOTION AND PHASES OF THE MOON: ITS DISTANCE AND DIMENSIONS

193. Motion of the Moon

The Moon describes among the stars a great circle of the celestial sphere, inclined to the ecliptic at an angle of about 5°. The motion is direct, and the period of a complete "sidereal" revolution is about $27\frac{1}{3}$ days. In this time the Moon's celestial longitude increases by 360°. The Moon therefore has a rapid motion in the sky relative to the stars, in an eastward direction. If the Moon is watched on any night when it happens to be in close proximity to a bright star, its eastward motion relative to the star will be readily apparent to the naked eye in the course of a few hours.

When the Moon has the same longitude as the Sun, it is said to be *New Moon*, and the period between consecutive new Moons is called a *Lunation*. When the Moon has described 360° from new Moon, it will again be at the same point among the stars; but the Sun will have moved forward, so that the Moon will have a little further to go before it catches up the Sun again. Hence the lunation will be rather longer than the period of a sidereal revolution, being about $29\frac{1}{2}$ days.

The *Age of the Moon* is the number of days which have elapsed since the preceding new Moon. Since the Moon separates 360° from the Sun in $29\frac{1}{2}$ days, it will separate at the rate of about 12°, or more accurately $12\frac{1}{5}°$, per day, or 30′ per hour. This enables us to calculate roughly the Moon's angular distance from the Sun, when the age of the Moon is given, and conversely, to determine the Moon's age when its angular distance is given.

Example.—*On September 23rd, the Moon is 20 days old. Find roughly its angular distance from the Sun and its longitude on that day.*

(1) In one day the Moon separates $12\frac{1}{5}°$ from the Sun; therefore, in 20 days it will have separated $20 \times 12\frac{1}{5}$, or 244°, and this, or rather $360° - 244° = 116°$, is the required angular distance from the Sun.

(2) On September 23rd the Sun's longitude is 180°; therefore the Moon's longitude is $180° + 244° = 424° = 360° + 64°$, or 64°.

The method of the above example only gives very rough results; for the Moon's motion is far from uniform, and the variations follow very complex laws. Moreover, the plane of the moon's orbit is not fixed, but its intersections with the ecliptic (called the *Nodes*) have a retrograde motion of 19° per year. However, for rough purposes, it is possible to neglect the small inclination of the Moon's orbit, and to consider the Moon in the ecliptic. If greater accuracy be required, the Moon's decl. and R.A. may be found from the *Nautical Almanac.*

194. The Moon's Motion in Declination

The Moon has a rapid motion in declination. If the Moon moved in the ecliptic, its declination would change from 23½° N. to 23½° S. in the course of its monthly motion round the Earth, just as the declination of the Sun varies between these limits in the course of the annual motion of the Earth round the Sun. But the orbit of the Moon is inclined at about 5° to the ecliptic. The range in the declination of the Moon in the course of a lunation will depend upon the position of the nodes of the Moon's orbit with respect to the first points of Aries and Libra.

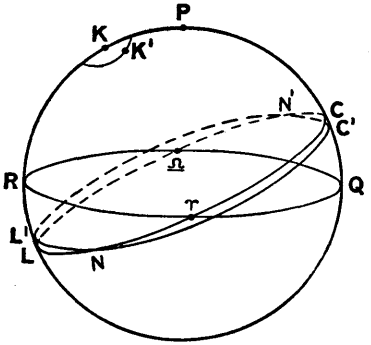

FIG. 72.

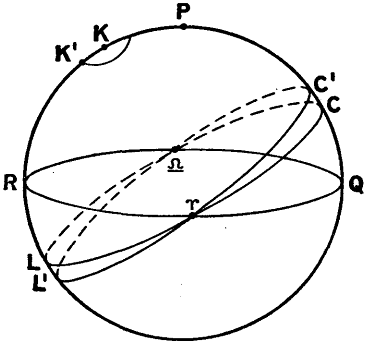

FIG. 73.

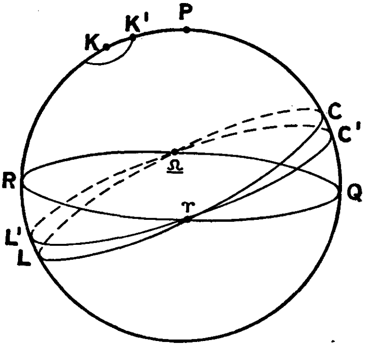

FIG. 74.

The retrograde motion of the nodes of the Moon's orbit, referred to above, causes the pole of the Moon's orbit to describe a small circle, of radius about 5°, around the pole of the ecliptic. In Fig. 72, *P*, *K* are the poles of the equator and the ecliptic. *R*♈*Q*♎ is the equator, *L*♈*C*♎ is the ecliptic. *L'N'C'N* is the great circle described by the Moon, when the pole of its orbit is at *K'*. When the retrograde motion of *N* carries it to ♈, the pole *K'* is on the great circle *PKR*, and between

K and *R* (Fig. 73). The declination of the Moon then has its greatest range in the course of a month: $QC' = QC + CC' = 23\frac{1}{2}^\circ + 5^\circ = 28\frac{1}{2}^\circ$; the declination thus varies between 28½° N. and 28½° S. About 9½ years later, N has moved round to ≏, and the pole of the Moon's orbit is again on the great circle *PKR*, but between *K* and *P*. The range in declination in the course of a month is now only between 18½° N. and 18½° S, (Fig. 74) because $QC' = QC - CC' = 23\frac{1}{2}^\circ - 5^\circ$.

195. Phases of the Moon

The accompanying diagrams will show how the phases of the Moon are accounted for on the hypothesis that the Moon is an opaque body illuminated by the Sun. In the upper figure the central globe represents the Earth, the others represent the Moon in different parts of its orbit, while the Sun is supposed to be at a great distance away to the right of the figure.* The half of the Moon that is turned towards the Sun is illuminated, the other half being dark. The Moon's appearance depends on the relative proportions of the illuminated and darkened portions that are turned towards the Earth. The lower figures, *a*, *b*, *c*, *d*, *e*, *f*, *g*, *h*, represent the appearances of the Moon relative to the ecliptic, as seen from the Earth when in the positions represented by the corresponding letters in the upper figure.

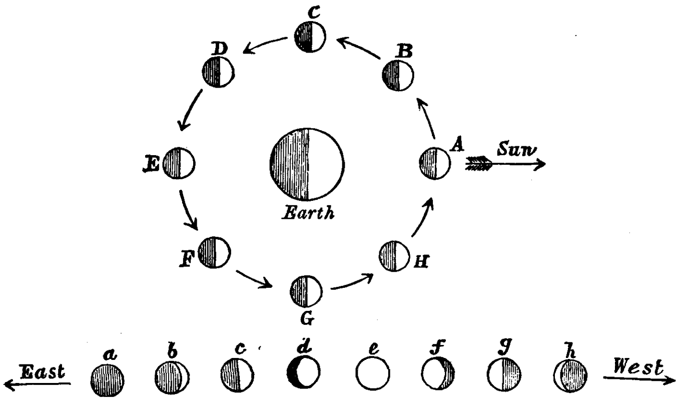

FIG. 75.

At *A*, *a* the Moon is in conjunction, and only the dark part is towards the Earth. This is called **New Moon**.

At *B*, *b* a portion of the bright part is visible as a crescent at the western side of the disc. The Moon's appearance is known as **horned**. The points or extremities of the horns are called the *cusps*.

---

* The Sun's distance is about 390 times the Moon's. If the latter be represented by an inch, the former will be represented by about 11 yards.

At *C*, *c* the Moon's elongation is 90°, and the western half of the disc, or visible portion, is illuminated, the eastern half being dark. The Moon is then said to be **dichotomized.** This is called the *First Quarter*. The Moon's age is about $7\frac{1}{2}$ days.

At *D*, *d* more than half the disc is illuminated. The Moon's appearance is then described as **gibbous.**

At *E*, *e* the Moon is in opposition. The whole of the disc is illuminated. This is called **Full Moon.** The Moon's age is about 15 days.

At *F*, *f* a portion of the disc at the western side is dark. The Moon is again gibbous, but the bright part is turned in the opposite direction to that which it has at *D*, *d*.

At *G*, *g* the Moon's elongation is 270°. The eastern half of the disc is illuminated, and the western half is dark. The Moon is again dichotomized. This is called the *Last Quarter*. The Moon's age is about 22 days.

At *H*, *h* only a small crescent in the eastern portion is still illuminated. The Moon is now again horned, but the horns are in the opposite direction to those in *B*, *b*.

Finally, the Moon comes round to conjunction again at *A*, and the whole of the part towards the Earth is dark.

From new to full Moon, the visible illuminated portion increases, and the Moon is said to be *waxing*. From full to new, the illuminated portion decreases, and the Moon is said to be *waning*.

It will be noticed from a comparison of the figures that the illuminated portion of the visible disc is always that nearest the Sun. Moreover, its area is greater the greater the Moon's elongation.*

196. Relation between Phase and Elongation

Let M (Fig. 76) be the centre of the Moon, MS the direction of the Sun, $E'ME$ that of the Earth. Draw the great circles AMB perpendicular to ME, and CMD perpendicular to MS; the former is the boundary of the part of the Moon turned towards the Earth, and the latter is the boundary of the illuminated portion. Hence the visible bright portion is the lune AMC. The angle of the lune, $\angle AMC$, is equal to $\angle E'MS$. The area of a spherical lune is proportional to its angle. Hence,

$$\frac{\text{area of visible illuminated part}}{\text{area of hemisphere}} = \frac{\angle AMC}{180^\circ} = \frac{\angle E'MS}{180^\circ}$$

$$= \frac{180^\circ - \angle EMS}{180^\circ}.$$

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* The phases of the Moon may be readily illustrated experimentally, by taking an opaque ball, or an orange, and holding it in different directions relative to the light from the Sun or an electric light.

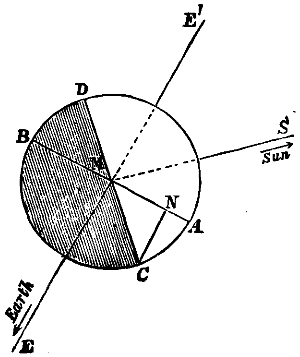


FIG. 76.

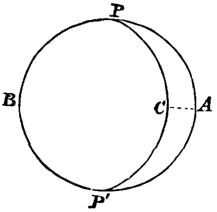


FIG. 77.

But this does not give the *apparent area* of the bright part. For the apparent area of a body is the area of the disc formed by projecting the body on the celestial sphere. If N denote the projection of the point C on the plane AMB (so that CN is perpendicular to BA), the arc AC will be seen in perspective as a line of length AN, and the bright part will be seen as a *plane lune* (Fig. 77), whose boundary PCP' optically forms the half of an ellipse whose major axis is PP', and minor axis $2MN$. It may be shown that

$$\text{area of half-ellipse } PCP' : \text{area of semicircle } PAP' = MN : MA$$

$$\text{and thus area } APCP' : \text{area } APBP' = AN : AB$$

$$= 1 - \cos AMC : 2 = 1 - \cos E'MS : 2.$$

Hence the apparent area of the bright part is proportional to $1 - \cos SME'$.

The angle SME' differs from the Moon's elongation SEM by the small angle ESM (Fig. 78); *i.e.* the angle which the Moon's distance subtends at the Sun. This angle is very small, being always less than 10′. Hence the area of the phase is very approximately proportional to 1 — cos (Moon's elongation).

197. Determination of the Sun's Distance by Aristarchus

From observing the Moon's elongation when dichotomized, Aristarchus (270 B.C. *circ.*) made a computation of the Sun's distance in the following manner. When the Moon is dichotomized, $\angle SME = 90°$, the Moon's elongation

$$\angle SEM = 90° - \angle ESM,$$

and $\cos SEM = EM/ES$. Hence, by observing the angle SEM between the directions from the Earth to the Sun and to the Moon, the ratio of the Sun's distance to the Moon's was computed.

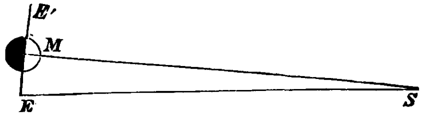


FIG. 78.

This method of estimating the distance of the Sun in terms of that of the Moon is of historical interest, because it was the first serious attempt to obtain information about the distance of the Sun. But it is incapable of giving reliable results, owing to the impossibility of finding the exact instant when the Moon is dichotomized. The Moon's surface is rough, and covered with mountains, and the tops of these catch the light before the lower parts, while throwing a shadow on the portions behind them. Hence the boundary of the bright part is always jagged and is never a straight line, as it would be at the quarters, if the surface of the Moon were perfectly smooth. The angle *SEM* is so nearly a right-angle that a small error in its measurement makes a very large error in the estimation of the Sun's distance. In fact, Aristarchus estimated the Sun's distance as only about 19 times that of the Moon, whereas they are really in the proportion of nearly 400 to 1.

198. Earth-Shine on the Moon.—Phases of the Earth

When the Moon is nearly new the unilluminated portion of its surface is distinctly visible as a disc of a dull-grey colour. This appearance is due to the light reflected from the Earth as "Earth-shine," which illuminates the Moon in just the same way that the moonshine illuminates the Earth at full Moon. From Art. 167, the Earth's superficial area is greater than the Moon's in the proportion of about 40 : 3. Consequently the Earth-shine on the Moon is more than 13 times as bright as the moonshine on the Earth.

The Earth, as seen from the Moon, would appear to pass through phases similar to those of the Moon, as seen from the Earth. The Earth's and Moon's phases are evidently *supplementary*. Thus, when the Moon is new the Earth would appear full, and vice-versa; when the Moon is in the first quarter, the Earth would appear in the last quarter.

Owing, however, to twilight, the boundary of the Earth's illuminated portion would not be so well defined as in the case of the Moon; there would be a gradual shading off from light to darkness, extending over a belt of breadth 18° beyond the bright part. The entire absence of twilight on the Moon is one of the evidences against the existence of a lunar atmosphere similar to that of our Earth. A stronger one is derived from occultations (Art. 152).

199. Appearance of Moon relative to the Horizon

We are now in a position to represent, in a diagram, the Moon's position and appearance relative to the horizon at a given time of day and year when the Moon's age is given.

The ecliptic having been found, as explained in Art. 42, the age of the Moon determines the Moon's elongation, as in Art. 193. Measuring this angle along the ecliptic, we find the Moon's position roughly; for the Moon is never very far from the ecliptic (*cf.* Art. 193). The elongation also determines the phase, and enables us to indicate the appearance of the disc. The bright side or limb is always turned towards the Sun. The cusps, therefore, point in the reverse direction, and the line joining them is perpendicular to the ecliptic.

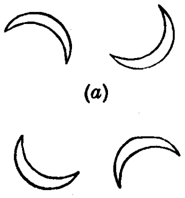
We can also trace the changes in the direction of the Moon's horns relative to the horizon, between its time of rising and setting.

Take, for example, the case when the Moon is a few (say three) days old. The Moon is then a little east of the Sun; therefore the bright limb is at the western side of the disc, and the horns point eastward. Hence, at rising, the horns are pointed downwards, and at setting they are pointed upwards [Fig. 79 (*a*)].

When the Moon is waning, the reverse will be the case [Fig. 79 (*b*)].

200. Summer and Winter Full Moons

From Art. 195, it is apparent that at full Moon the direction from the Earth to the Moon is diametrically opposite to the direction from the Earth to the Sun. If the orbit of the Moon were in the same plane as that of the Sun, *i.e.* in the ecliptic, the Moon would be in the Earth's shadow at full Moon and there would then be an eclipse of the Moon at every full Moon. But as the inclination of the Moon's orbit to the ecliptic is about 5°, there is not normally an eclipse of the Moon at full Moon. It follows that at full Moon the declination of the Moon is nearly equal and opposite to the declination of the Sun, within about 5°. At the winter solstice, the declination of the Sun is 23½° S; the declination of the full Moon at the winter solstice is accordingly 23½° ± 5° N., or between 28½° and 18½° N. At the summer solstice, the declination of the Sun is 23½° N.; the declination of the full Moon is then 23½° ± 5° S., or between 28½° and 18½° S. Since the south zenith distance, when on the meridian, is $(\phi - \delta)$, where ϕ is the latitude, it follows that the meridian zenith distance of the full Moon in northern latitudes is much greater in summer than in winter. Thus, at Greenwich, for instance, whose latitude is approximately 51½° N., the meridian zenith distance of the full Moon at the winter solstice is between 23° and 33°, but at the summer solstice it is between 70° and 80°. For this reason the Moon is said to ride high in the winter and to ride low in the summer.


(*a*)
(*b*)
FIG. 79.

201. Distance and Dimensions of the Moon

The method by which the parallax of the Moon is determined was described in Art. 161. The horizontal parallax depends upon the position of the Moon in its orbit and varies within an appreciable range, because of the considerable eccentricity of the Moon's orbit. As stated in Art. 162, the greatest and least values are in the ratio of about 19 to 17. The mean horizontal parallax corresponds to a mean distance of nearly 240,000 miles. The Moon is thus the Earth's nearest neighbour in space.

The diameter of the Moon is easily inferred from the horizontal parallax, as shown in Art. 167. The diameter of the Moon is about 2,160 miles; its volume is about 1/50th of that of the Earth.

II.—SYNODIC AND SIDEREAL MONTHS—MOUNTAINS ON THE MOON

202. Definitions

In Art. 193 we defined the *lunation* as the period between consecutive new Moons, and showed that it was rather longer than the period of the Moon's revolution relative to the stars. We shall now require the following additional definitions, most of which apply also to the planets.

The *elongation* of the Moon or planet is the difference between its celestial longitude and that of the Sun. If the body were to move in the ecliptic its elongation would be its angular distance from the Sun.

The Moon or planet is said to be in *conjunction* when it has the same longitude as the Sun, so that its elongation is zero. The Moon is in conjunction at *new Moon* (Art. 193). The body is in *opposition* when its elongation is 180°. In both positions it is said to be in *syzygy*. The body is said to be in *quadrature* when its elongation is either 90° or 270°.

The period between consecutive conjunctions is called the *synodic period* of the Moon or planet. The Moon's synodic period is, therefore, the same as a lunation; it is also called a **Synodic Month.** In this period the Moon's elongation increases by 360°, the motion being *direct*.

The period of revolution relative to the stars is called the *sidereal period*; that of the Moon, the **Sidereal Month.**

The average length of the **Calendar Month** in common use is slightly in excess of the synodic month.

203. Relation between the Sidereal and Synodic Months

Let the number of days in a year be Y, in a sidereal month M, and in a synodic month S.

In M days the Moon's longitude increases	360°;
So that in 1 day the Moon's longitude increases	$360°/M$.
Similarly in 1 day the Sun's longitude increases	$360°/Y$,
and the Moon's elongation increases	$360°/S$.

Now, from the definition :—

(Moon's elongation) = (Moon's long.) — (Sun's long.),

and their daily rates of increase must be connected by the same relation ; thus,

$$\frac{360}{S} = \frac{360}{M} - \frac{360}{Y};$$

$$i.e. \quad \frac{1}{S} = \frac{1}{M} - \frac{1}{Y}, \text{ or } \frac{1}{M} = \frac{1}{S} + \frac{1}{Y};$$

$$i.e. \quad \frac{1}{\text{sider. month}} = \frac{1}{\text{synod. month}} + \frac{1}{\text{year}}.$$

Example.—*Find (roughly) the length of the sidereal month, given that the synodic month* (S) = $29\frac{1}{2}$*d., and the year* (Y) = $365\frac{1}{4}$*d.*

Here we have $$\frac{1}{M} = \frac{1}{29\frac{1}{2}} + \frac{1}{365\frac{1}{4}}.$$

To simplify the calculations, we put the relation into the form

$$\frac{29\frac{1}{2} \times 365\frac{1}{4}}{29\frac{1}{2} + 365\frac{1}{4}} = 29\tfrac{1}{2} \times \frac{365\frac{1}{4}}{394\frac{3}{4}} = 29\tfrac{1}{2} \times \left(1 - \frac{29\frac{1}{2}}{394\frac{3}{4}}\right)$$

$$= 29{\cdot}5 - 29{\cdot}5 \times \frac{118}{1579} = 29{\cdot}5 - 2{\cdot}20 = 27{\cdot}3.$$

Hence the sidereal month is very nearly $27\frac{1}{3}$ days.

204. To determine the Moon's Synodic Period

An eclipse of the Sun can only happen at conjunction, and an eclipse of the Moon at opposition, and the middle of the eclipse determines the exact instant of conjunction or opposition, as the case may be. Hence, by observing the exact interval of time between the middle of two eclipses, and counting the number of lunations between them, the length of a single lunation, or synodic period, can be found with great accuracy expressed in mean solar units of time.

The records of ancient eclipses enable us to find a still closer approximation to the mean length of the lunation. From modern observations, the length of a lunation has been found with sufficient accuracy to enable us to tell the exact number of lunations between these ancient eclipses and a recent lunar eclipse (this number being, of course, a *whole number*). By dividing the known interval in days by this number, the mean length of the synodic period during the interval can be accurately found. At the present time the length of a lunation is 29·5305881 days, or 29d. 12h. 44m. 2·7s. nearly.

From this the length of the Moon's sidereal period is calculated, as in Art. 203, and found to be 27d. 7h. 43m. 11·5s. nearly.

205. Heights of Lunar Mountains—First Method

We stated in Art. 197 that the Moon's surface is covered with mountains, and that in consequence the bounding line between the illuminated and dark portions of the disc is always jagged and irregular, while the mountains themselves throw their shadows on the portions of the surface behind them. These circumstances have led to two different ways of measuring the height of the lunar mountains. The first method is as follows :—

If a tower is standing in the middle of a perfectly level plain, it is evident from trigonometry that the length of the shadow, multiplied by the tangent of the Sun's altitude, gives the height of the tower. The same will be true in the case of the shadow cast by a mountain, provided we measure the length of the shadow from a point vertically underneath the summit. Now, in the case of the Moon it is possible, from knowing the Moon's age, to calculate exactly what would be the altitude of the Sun as it would be seen from any point of the lunar surface. The apparent length of the shadows of the mountains can be measured, in angular measure, by means of a micrometer; from this their actual length can be calculated, allowance being, of course, made for the fact that we are not looking vertically down on the shadows, and hence they appear foreshortened. In this way, the height of the mountains can be found.

The principal disadvantage of this method is, that if the surface of the Moon surrounding the mountain should be less flat than it has been estimated, there will be a corresponding error in the height of the mountain. In particular, it would be impossible to apply the method to find the heights of mountains closely crowded together.

206. Heights of Lunar Mountains—Second Method

In treating of the Earth in Art. 90, we showed that one effect of the dip of the horizon is to accelerate the times of rising, and to retard the times of setting of the Sun and stars. We also showed how to calculate the amount of the acceleration if the dip be known. Conversely, if the acceleration in the time of rising be known, the dip of the horizon can be calculated, and from this the height of the observer above the general level of the Earth may be found.

Now precisely the same method may be applied to measure the heights of lunar mountains. When the Moon is waxing the Sun is gradually rising over those parts of the Moon's surface which are turned towards the Earth. The tops of the mountains catch the rays before the lower parts, and, therefore, stand out bright against the dark background of the unilluminated parts below. Similarly, when the Moon is waning, the summits of the mountains remain as bright specks after the lower portions are plunged in shadow. By noticing the exact instant at which the Sun's rays begin or cease to illuminate the summit, this acceleration or retardation, due to dip, may be calculated, and the height of the mountain determined.

If the Moon's surface around the mountain is fairly level, the distance of the mountain from the illuminated portion at the instant of disappearance determined *the distance of the visible horizon* as seen from the mountain. This distance can be calculated from measurements made with a micrometer (proper allowance being made for foreshortening if the mountain is not in the centre of the disc).

Hence the height (h) of the mountain may be calculated by the formula of Art. 87 (i), viz. $h = d^2/2a$, where d is the estimated distance of the horizon, and a the Moon's radius.

III.—THE MOON'S ORBIT AND ROTATION

207. The Moon's Orbit about the Earth

This can be investigated by a method precisely similar to that employed in the case of the Sun (see Art. 128). The Moon's R.A. and decl. may be observed daily by the Transit Circle. The observed decl. must be corrected for refraction and parallax (neither of which affect the R.A., since the observations are made on the meridian). We thus find the positions of the Moon on the celestial sphere relative to the Earth's *centre* for every day at the instant of its transit across the meridian of the observatory.

The Moon's relative distances from the Earth's centre on different days, may be compared by measuring the Moon's angular diameter, with the heliometer. Here, however, another correction for parallax is required. For the observed angular diameters are inversely proportional to the corresponding distances of the Moon from the *observer*, and not from the *centre* of the Earth.

This correction is by no means inconsiderable. Thus, for example, if the Moon be vertically overhead, its distances from the observer and from the Earth's centre will differ by the Earth's radius, *i.e.* by about $\frac{1}{60}$ of the latter distance, and its angular diameter will, therefore, be increased in the proportion of about 60 to 59.

Having thus determined the direction and distance of the Moon's centre, relative to the Earth's centre, for every day in the month, the Moon's orbit may be traced out in just the same way as the Sun's orbit was traced out in Art. 129. It is thus found that the motion obeys approximately the following laws :—

(i) *The Moon's orbit lies in a plane through the Earth's centre, inclined to the plane of the ecliptic at an angle of about* 5° 8′.

(ii) *The orbit is an ellipse, having the Earth's centre in one focus, the eccentricity of the ellipse being about* $\frac{1}{18}$.

(iii) *The radius vector joining the Earth's and Moon's centres traces out equal areas in equal intervals of time.*

The period of revolution is, of course, the sidereal lunar month, as defined in Section II, namely, about 27⅓ days.

The laws which govern the Moon's motion are thus identical with Kepler's laws for the Earth's orbital motion round the Sun (Art. 136).

208. The Eccentricity of the Moon's Orbit

This is found by comparing the Moon's greatest and least distances, which are inversely proportional to its least and greatest (geocentric) angular diameters respectively. The latter are in the ratio of about 17 to 19, and it is inferred that the eccentricity is about $(19-17)/(19+17)$, or $\frac{1}{18}$ (*cf.* Art. 131).

The terms *perigee, apogee, apse line* are used in the same sense as in Art. 130. *Perigee* and *apogee* are the points in the orbit at which the Moon is nearest to and furthest from the Earth respectively. Both are called the *apses* or *apsides,* the line joining them being called the *apse line, apsidal line* or *line of apsides,* according to choice. It is the major axis of the orbit.

As in Art. 133, it follows that the Moon's angular motion in its orbit is swiftest at perigee, and slowest at apogee.

209. Nodes

The points in which the Moon's orbit, or its projection on the celestial sphere, cuts the ecliptic are called the Moon's *Nodes* (*cf.* Art. 193). The line joining them is called the *Nodal Line.* It is the line of intersection of the planes of the Moon's orbit and ecliptic. That node through which the Moon passes in crossing from south to north of the ecliptic is distinguished as the *ascending node,* the other is distinguished as the *descending node.*

210. Perturbations

As the result of observations extending over a large number of lunar months, it is found that the Moon does not describe exactly the same ellipse over and over again, and that, therefore, the laws stated in Art. 207 are only approximate. Even in a single month the departure from simple elliptic motion is quite appreciable, owing chiefly to the disturbance called the Variation (Art. 477). The disturbance known as the Evection (Art. 477) causes the eccentricity to change appreciably from month to month. Further, the motions described in Arts. 211, 212 cause the roughly elliptical orbit slowly to change its position.

The complete investigation of these small changes or *perturbations,* as they are called, belongs to the domain of Gravitational Astronomy.

It will be necessary here to enumerate the chief perturbations, on account of the important part they play in determining the circumstances of eclipses.

211. Retrograde Motion of the Moon's Nodes

The Moon's nodes are not fixed, but have a retrograde motion along the ecliptic of about 19° in a year. This phenomenon closely resembles the retrograde motion of ♈ (Precession, Art. 125), but is far more rapid. Its effect is to carry the line of nodes, with the plane of the Moon's orbit, slowly round the ecliptic, performing a complete revolution in 6793·391 days, or rather over 18·6 years.

One result of this nodal motion is that the angle of inclination of the Moon's orbit to the equator is subject to periodic variations, as already described in Art. 194.

212. Progressive Motion of Apse Line

The line of apsides is not fixed, but has a direct motion in the plane of the Moon's orbit, performing a complete revolution in 3232·575 days, or about nine years. A similar progressive motion of the apse line of the Earth's orbit about the Sun was mentioned in Art. 135. The latter motion is, however, much less rapid, its period being about 108,000 years.

213. Other Perturbations

The inclination of the Moon's orbit to the ecliptic is not quite constant. It is subject to small periodic variations, its greatest and least values being 5° 18½′ and 4° 59½′.

In addition there are variations in the eccentricity of the orbit, in the rates of motion of the nodes, and in the length of the sidereal period. All of these render the *accurate* investigation of the Moon's orbit one of the most complicated problems of Astronomy.

214. The Moon's Rotation

It is a remarkable fact that the Moon always turns the same side of its surface to the Earth. Whether we examine the markings on its surface with the naked eye, or resolve them into mountains and streaks with a telescope, they always appear very nearly the same, although their illumination, of course, varies with the phase.

From this it is evident that the Moon rotates upon its axis in the same "sidereal" period as it takes to describe its orbit about the Earth, *i.e.* once in a sidereal month. It might, at a first glance, appear as if the Moon had no rotation, but such is not the case. To explain this, let us consider the phenomena which would be presented to an observer if situated on the Moon in the centre of the portion turned towards the Earth.

The Earth would always appear directly overhead, *i.e.* in the observer's zenith. But as the Moon describes its orbit about the Earth, the direction of the line joining the Earth and Moon revolves through 360°, relative to the fixed stars, in a sidereal month. Hence the direction of the observer's zenith on the Moon must also revolve through 360° in a sidereal month, and therefore the Moon must rotate on its axis in this period.

The Moon would be said to describe its orbit *without rotation*, if the same points on its surface were to remain always directed towards the same fixed *stars*. Were this the case, *different* parts of the surface would become turned towards the *Earth* as the Earth's direction changed, and *this is not what actually occurs*.

It thus appears that, to an observer on the Moon, the directions of the stars relative to the horizon would appear to revolve through 360° once in a sidereal lunar month. Thus, the sidereal month is the period corresponding to the sideral day of an observer on the Earth. In a similar way, the Sun's direction would appear to revolve through 360° in a synodic month. This, therefore, is the period corresponding to the solar day on the Earth, as is otherwise evident from the fact that the Moon's phases determine the alternations of light and darkness on the Moon's surface, and that they repeat themselves once in every synodic month.

215. Librations of the Moon—Libration in Latitude

If the axis about which the Moon rotates were perpendicular to the plane of the Moon's orbit, we should not be able to see any of the surface beyond the two poles (*i.e.* extremities of the axis of rotation). In reality, however, the Moon's axis, instead of being exactly perpendicular to its orbit, is inclined at an angle of about 6½° to the perpendicular, just as the Earth's axis of rotation makes an angle of about 23° 27′ with a perpendicular to the ecliptic. The consequence is that during the Moon's revolution, the Moon's north and south poles are alternately turned a little towards and a little away from the Earth ; thus, in one part of the orbit we see the Moon's surface to an angular distance of 6° 44′ beyond its north pole, in the opposite part we see 6° 44′ beyond the south pole. This phenomenon is called the Moon's *libration in latitude*. It makes the Moon's poles appear to nod, oscillating to and fro once in every revolution relative to the nodes.

Libration in latitude may be conveniently illustrated by the corresponding phenomenon in the case of the Earth's motion round the Sun, as represented in Fig. 46 (Art. 136). At the summer solstice the whole of the Arctic circle is illuminated by the Sun's rays, and therefore an observer on the Sun (if such could exist) would see the Earth's surface for a distance of 23° 27′ beyond the north pole. Similarly, at the winter solstice an observer on the Sun would see the whole of the Antarctic circle, and a portion of the Earth's surface extending 23° 27′ beyond the south pole.

216. Libration in Longitude

Owing to the elliptical form of the orbit, the Moon's angular velocity about the Earth is not quite uniform, being least at apogee and greatest at perigee. But the Moon rotates about its polar axis with practically uniform angular velocity equal to the *average* angular velocity of the orbital motion (so that the periods of rotation and of orbital motion are equal).

Thus, at *apogee* the angular velocity of rotation is slightly *greater* than that of the orbital motion, and is, therefore, greater than that required to keep the same part of the Moon's surface always turned towards the Earth. In consequence, the Moon will begin to gradually turn round, so as to show a little more of the *eastern* side of its surface.

At *perigee*, the angular velocity of rotation is *less* than that of the orbital motion, and is, therefore, not quite sufficient to keep the same part of the Moon's surface always turned towards the Earth. In consequence we shall begin to see a little further round the *western* side of the Moon's disc.

This phenomenon is called *libration in longitude*. Its maximum amount is 7° 45′; thus, during each revolution of the Moon relative to the apse line, we alternately see 7° 45′ of arc further round the eastern and western sides of the disc than we should otherwise.

217. Diurnal Libration

The phenomenon known as *diurnal libration* is really only an effect of parallax. If the Moon were vertically overhead, and if we were to travel eastwards, we should, of course, begin to see a little further round the eastern side of the Moon's surface. If we were to travel westwards we should begin to see a little further round the western side. Now, the rotation of the Earth carries the observer round from west to east. Hence, when the Moon is rising we see a little further round its western side, and when setting we see a little further round its eastern side, than we should from a point vertically underneath the Moon.

Similarly an observer in the northern hemisphere would always see rather more of the Moon's northern portion, and an observer in the southern hemisphere would see rather more of the southern portion than an observer at the equator.

The greatest amount of the diurnal libration is equal to the Moon's horizontal parallax, and is therefore about 57′. We see 57′ round the Moon's western edge when rising, and 57′ round the eastern edge when setting.

An observer at any given instant sees *not quite half* (49·77 per cent.) the Moon's surface. The visible portion is bounded by a cone through the observer's eye enveloping the Moon, and is less than a hemisphere by a belt of breadth equal to the Moon's angular semi-diameter, *i.e.* about 16′.

218. General Effects of Libration

In consequence of the three librations, about 59 per cent. of the Moon's surface is visible from the Earth at some time or other, instead of rather under 50 (49·77) per cent., as would be the case if there were no libration. At the same time only about 41 per cent. of the surface is always visible from the Earth. The remainder is sometimes visible, sometimes invisible.

To an observer on the surface of the Moon the result of libration in latitude and longitude would be that the Earth, instead of remaining stationary in the sky, would appear to perform small oscillations about its mean position. It would really appear to describe a series of ellipses. The motion of the different parts of the Earth across its disc in the course of the Earth's diurnal revolution would be the only phenomenon resulting from the cause which produces diurnal libration.

219. The Moon and the Calendar: the Epact: the Golden Number: the Sunday Letter

The name "Month" is derived from the Moon, and the majority of the nations of antiquity made their months begin with the New Moon, or more strictly when the Moon's thin crescent could first be seen in the west after sunset. But the months had to be taken as an exact number of days; the usual plan was to make them alternately 29 and 30 days long, so that twelve months made 354 days; actually 12 synodic months make 354·36706 days, or about 11 days less than an average solar year, so that the age of the Moon on January 1st becomes 11 days greater in successive years. This age has received the name of the *Epact,* and is of importance in the calculation of Easter.

The Mohammedan Calendar uses the year of 354 days unmodified, so that each month goes right round the seasons in about 33 years; but some nations of antiquity kept their years in close accord with the solar year by the insertion of a thirteenth lunar month about once in three years. Meton and Euctemon introduced into Greece in B.C. 433 the cycle known as the "Metonic"; it was already known in eastern countries. This rule introduces the thirteenth month seven times in 19 years; this gives 19 × 12 + 7, or 235 lunar months. Their length amounts to 6939·6882 days; now the average length of the Gregorian year is 365·2425 days (see Art. 170); nineteen times this is 6939·6075 days. The difference is only 0·0807 or 1h. 56m.; hence 19 years gives a very close repetition of the dates of New Moon; the error is only about a day after 237 years.

It is said that Meton had the years of this cycle inscribed in golden letters on a temple in Athens, whence the phrase *Golden Number* which is still used to indicate the place of a year in the nineteen-year cycle. The rule for finding the Golden Number is: "It is the remainder when the A.D. date increased by 1 is divided by 19." Hence the year A.D. 0, which is called B.C. 1 in Civil reckoning, had Golden Number 1.

Previously to the change of style in 1582 by Gregory XIII., the rule for finding the Epact was: "The remainder when 11 × (Golden Number — 1) is divided by

30." This would give Epact 0, that is New Moon on January 1st in the year A.D. 0. It is sometimes stated that the New Moon actually occurred then, but it was really 6 days earlier, on December 26th, 2 B.C. The difference arises from the slight error of the Metonic cycle.

In 1582, when the change of style was made, new rules were also made for finding the Epact: (1) The Epact is diminished by 1 in the centennial years 1700, 1800, 1900, etc., that are not leap years (this is called the solar equation). (2) The Epact is increased by 1 in the years 1800, 2100, 2400, etc.; *i.e.* every 300 years, but once in 2,500 years a 400-year interval replaces the 300-year one; thus after 3900, occurs 4300; after 6400, occurs 6800, etc. (this is called the lunar equation). When both equations occur together they compensate each other and there is no change; this occurred in 1800 and will occur again in 2100. The last year that had the Golden Number 1 was 1938; starting with that year (or with 1900, two Metonic cycles earlier) the successive Epacts for 19 years are 29, 10, 21, 2, 13, 24, 5, 16, 27, 8, 19, 0, 11, 22, 3, 14, 25, 6, 17. They then repeat themselves, and the cycle will recur unchanged up to the year 2199; after that they will all be diminished by 1. (For a special reason the number 25 above is to be used as 26 in finding Easter.)

The Gregorian Calendar was adopted by Greece and Russia in the present century, so that its use is now practically universal; in Greece, however, Easter is calculated from the date of the real full Moon, the time being reckoned according to the meridian of Jerusalem, instead of using the Epact.

The Sunday Letter. If we assign the seven letters A to G in regular succession to the days of the year, starting with January 1st, and omitting the leap-day, February 29th, there will be a change each year in the letter for Sunday; and further, Leap years will have two Sunday letters, one for January and February, and the other for the remainder of the year; the following are the Sunday letters from 1940 to 1946: '40 GF, '41 E, '42 D, '43 C, '44 BA, '45 G, '46 F; after 28 years they recur, unless a non-leap centennial year intervenes; in that case they recur after 40 years; thus 1901 was the same as 1861, and so on.

Having calculated the Epact and Sunday letter, the date of Easter is given by a table which will be found in *Encyclopaedia Britannica*, 14th Edition, Vol. 4, p. 572; for Easter to come on March 22nd, its earliest date, we must have Epact 23, Sunday Letter D. It will be seen that there is no 23 in the Epacts up to the year 2199; but in the following century 24 is turned into 23, and Easter is on March 22nd in 2285. The last time it happened was in 1818.

As an example of the use of the Sunday letter, we may take Question 10, p. 66; for a year to have five Sundays in February it must be a leap year, and February 1st must be Sunday. Now February 1st is the 31st day from January 1st, hence its Letter is D; we want a leap year whose first Letter is D; reckoning back from 1928 we find that 1920 was such a year; after that the 28-year cycle will give similar years at 28-year intervals till the end of next century, since 2000 is a leap year; but we have to go back 40 years before 1920, since the non-leap year 1900 is included; thus 1880, 1852, and 1824 were similar years.

It is well to remember that 400 years in the Gregorian calendar are an exact number of weeks, so that the Sunday letters recur after that period. It is easy to see this, since the period is equal to 400 times 52 weeks plus 400 days plus 97 leap-days; and 497 is divisible by 7.

The rule for finding Easter is: The first Sunday after the first full moon (calculated from the Epact) on or after March 21st; if the calculated full moon is on

Sunday, Easter is the following Sunday. The rule was devised to make Easter come at nearly the same time as the Passover, which was a fixed date in the Jewish lunar year.

Widespread endeavours have been made in recent years to alter the reckoning of Easter, *e.g.* to make it the Sunday after the second Saturday in April; there is, however, considerable opposition to such an alteration in certain quarters, and there is little prospect of it being made in the near future.

220. Retardation of Moon's Transit

The eastward motion of the Moon amongst the stars causes a progressive retardation in the time of the Moon's transit across the meridian. The average amount of this retardation can be determined as follows.

Let T_1, T_2 denote the sidereal times of two successive transits of the Moon across a given meridian and α_1, α_2 denote its right ascensions at these transits. Then :—

$$T_1 = \alpha_1\,;\ T_2 = 24h. + \alpha_2$$

If we denote the interval in sidereal hours between the two transits by x, then :—

$$x = T_2 - T_1 = 24h. + (\alpha_2 - \alpha_1)$$

Now if M is the length of the sidereal month (Art. 203) in mean solar days the R.A. of the Moon increases by 24h. in M mean solar days or by $1/M$ hours per mean solar hour.

But $(\alpha_2 - \alpha_1)$ is the increase in the R.A. during x sidereal hours or during $xY/(Y+1)$ mean solar hours, Y being the length of the year in days (365¼ days). Thus :—

$$x = 24\text{h.} + \frac{x}{M}\cdot\frac{Y}{Y+1};$$

$$\text{or}\quad x\left\{\frac{Y+1}{Y} - \frac{1}{M}\right\} = 24\text{h.}\ \frac{Y+1}{Y}.$$

By Art. 203, $\dfrac{1}{M} = \dfrac{1}{S} + \dfrac{1}{Y}$ where S denotes the synodic month.

$$\text{Therefore}\quad x\left(1 - \frac{1}{S}\right) = 24\text{h.}\ \frac{Y+1}{Y}\ \text{in sidereal time}$$

$$= 24\text{h. in mean solar time.}$$

$$\text{or}\quad x = \frac{S}{S-1}.24\text{h.} = 24\text{h.} + \frac{24\text{h.}}{S-1}\ \text{mean solar hours.}$$

Thus the interval between successive transits exceeds 24h. by 24h./$(S-1)$ on the average. Thus the average retardation of successive transits is :—

$$\frac{24}{S-1}\ \text{hours} = \frac{24}{28{\cdot}53}\ \text{hours} = \frac{1440}{28{\cdot}53}\ \text{minutes} = 50{\cdot}5\ \text{minutes}$$

approximately.

221. Retardation of Times of Moon's Rising and Setting. The Harvest Moon

When the time of the Moon's meridian transit is known, the times of rising and setting can be inferred. These depend upon the latitude of the place and the declination of the Moon. If the declination remained constant then, at any given place, the retardation from night to night in the times of rising and setting would be the same as the retardation in the time of meridian transit. But the Moon has a rapid motion in declination (Art. 194), and the change in declination from day to day has a considerable influence on the times of rising and setting.

The Full Moon which occurs nearest the autumnal equinox is called the *Harvest Moon*. In the case of the harvest Moon the daily retardation is less than in the case of any other full Moon, as we shall now show. To simplify our rough explanations we suppose the Moon to be moving in the ecliptic.

When the Moon is in the first point of Aries it is passing from south to north of the equator, and its declination is increasing most rapidly. Now, the arguments of Arts. 107–109 are applicable to the Moon as well as the Sun, and they show that, as the declination increases, there is, in north latitudes, a corresponding increase in the length of time that the Moon is above the horizon. The effect of this increase is to lengthen the interval from the Moon's rising to its transit; this lengthening tends to counterbalance, more or less, the retardation in the time of transit, thus reducing the retardation in the time of *moonrise* to a *minimum*. The retardation in the time of setting is thus a maximum.

Similarly it may be shown that whenever the Moon passes the first point of Libra, the daily retardation of moonrise will be a *maximum*, while that of the time of *setting* will be a *minimum*. These phenomena, therefore, recur once each lunar month.

Now, at harvest time the Sun is near ♎; hence, when the Moon is near ♈ it is full; and the minimum retardation of the Moon's rising, therefore, takes place at full Moon. And since the Moon is then opposite the Sun, it rises at sunset. Both these causes make the phenomenon more conspicuous in itself than at other times, and as the continuance of light is useful to the farmers when gathering in their harvest, the name Harvest Moon has been applied.

At the following full Moon the phenomena are similar but less marked. But as it is now the hunting season, the Moon is called the *Hunter's Moon.*

The phenomenon of the Harvest Moon is most marked when the Moon's orbit is at its maximum inclination to the equator, that is, when the Ascending node of its orbit is at the First Point of Aries (see Art. 211). This happened at the beginning of 1932, and recurs at intervals of 18·6 years from that date.

EXAMPLES

1. If the diameters of the Moon at perigee and apogee are 33′ 10″ and 29′ 26″, respectively, find the eccentricity of the Moon's orbit.

2. Find the approximate altitude of the full Moon at meridian transit at a place in latitude 52° N. on March 21st, June 21st and December 22nd, when the ascending node of the Moon's orbit is at (*a*) the first point of Aries, (*b*) the first point of Libra.

3. If in our latitude, on March 21st, the Moon is in its first quarter, about what time may it be looked for on the meridian, and how long does it remain above the horizon?

4. Show that from a study of the Moon's phases we can infer the Sun to be much more distant than the Moon. Prove that if the synodic period were 30 days, and the Sun only twice as distant as the Moon, the Moon would be dichotomized after only 5 days instead of 7½.

5. Taking the usual values of the Sun's and the Moon's distances, calculate, roughly, the mean value of the angle *ESM* when the Moon is dichotomized.

6. There was an eclipse of the Moon on Jan. 28th, 1888, central at 11.10 in the evening. What was the Moon's age on May 21st of that year?

7. Find approximately the position and appearance of the Moon, relatively to the horizon, in latitude 50° N., in the middle of November at 10 P.M., when it is ten days old.

CHAPTER X

ECLIPSES

I.—GENERAL DESCRIPTION OF ECLIPSES

222. Eclipses

Eclipses are of two kinds, lunar and solar. If at full Moon the centres of the Sun, Earth, and Moon are very nearly in a straight line, the Earth, acting as a screen, will stop the Sun's rays from reaching the Moon, and the Moon will, therefore, be either wholly or partially darkened. This phenomenon is called a *Lunar Eclipse.*

On the other hand, if the three centres are nearly in a straight line when the Moon is new, the Moon, by coming between the Earth and the Sun, will cut off the whole or a portion of the Sun's rays from certain parts of the Earth's surface. In such parts the Earth will be darkened, and the Sun will appear either wholly or partially hidden. This phenomenon is a *Solar Eclipse.*

If the Moon were to move exactly in the ecliptic we should have an eclipse of the Moon at every opposition, and an eclipse of the Sun at every conjunction, for at either epoch the centres of the Earth, Sun, and Moon would be in an exact straight line. In consequence, however, of the Moon's orbit being inclined to the ecliptic at an angle of about 5¼°, the Moon at "syzygy" (conjunction or opposition) is generally so far on the north or south side of the ecliptic that no eclipse takes place. An eclipse only occurs when the Moon at syzygy is very near the ecliptic, and, therefore, not far from the line of nodes (Art. 209).

223. Different Kinds of Lunar Eclipse

Eclipses of the Moon are of two kinds, *total* and *partial*. Let S, E be the centres of the Sun and Earth respectively. Draw the common tangents ABV and $A'B'V$ to the two globes, meeting on SE produced in V, and draw also the other pair of tangents $AB'K'$, $A'BK$ cutting at U, between S and E. If the figure be supposed to revolve about SE, the tangents will generate cones, enveloping the Sun and Earth, and having their vertices at U and V. The space BVB', inside the inner cone, is called the *umbra*; the space between the inner and outer cone is called the *penumbra*.* The character of the lunar eclipses will vary according to the following conditions :—

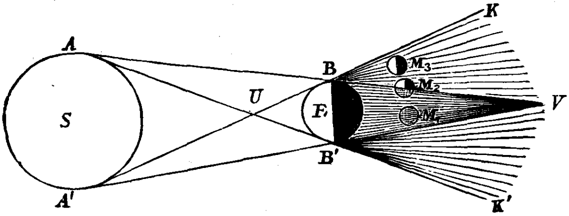


FIG. 80.

(i) If at opposition, the Moon falls entirely within the umbra or inner cone BVB', as at M_1, no portion of the Moon's surface then receives any direct rays from the Sun, and the Moon is therefore plunged in darkness (except for the light which reaches it after refraction by the Earth's atmosphere, as explained in Art. 152). The eclipse is then said to be *total*.

(ii) If the Moon falls partly within and partly without the umbra BVB', as at M_2, the portion within the umbra receives no light from the Sun, and is, therefore, obscured, while the remaining portion receives light from part of the Sun's surface about A, and is, therefore, partially illuminated. The eclipse is then said to be *partial*.

(iii) If the Moon falls entirely within the "penumbra," as at M_3, it receives the Sun's rays from A, but not from A'. There is no true eclipse, but only a diminution of brightness (sometimes called a "*penumbral eclipse*").

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* When the Moon is in the outer part of the penumbra no effect is discernible, but when it approaches close to the umbra there is a very perceptible smokiness on the side next to the umbra.

A lunar eclipse is visible simultaneously from all places on that hemisphere of the Earth over which the Moon is above the horizon at the time of its occurrence.

Near the boundary of the hemisphere there are two strips in the form of lunes, comprising those places respectively at which the Moon sets and rises during the eclipse ; at such places only its beginning or end is seen.

224. Phenomena of a Total Eclipse of the Moon

As the Moon gradually moves towards opposition, the first appearance noticeable is the slight darkening of the Moon's surface as it enters the penumbra. This darkening increases very gradually as the Moon approaches the umbra, or true shadow. At *First Contact* a portion of the Moon enters the umbra, and the eclipse is then seen as a partial eclipse, the dark portion being bounded by the circular arc formed by the boundary of the umbra. As the Moon advances, the dark portion increases till the whole of the Moon is within the umbra, and the eclipse is total. When the Moon begins to emerge at the other side of the umbra, the eclipse again becomes partial, and continues so until *Last Contact*, when the Moon has entirely emerged from the umbra, after which the Moon gradually gets brighter and brighter till it finally leaves the penumbra.

In the case of a partial eclipse, the umbra merely appears to pass over a portion of the Moon's disc, which portion is greatest at the middle of the eclipse.

225. Effects of Refraction on Lunar Eclipses

In Art. 152 it was stated that, owing to atmospheric refraction, the Moon's disc appears of a dull-red colour during the totality of the eclipse. A still more curious phenomenon is noticed when an eclipse occurs at sunset or sunrise. The refraction at the horizon increases the apparent altitudes of the Sun and Moon in the heavens, so that both appear above the horizon when they are just below. Hence a total eclipse of the Moon is sometimes seen when the Sun is shining.

226. Different Kinds of Solar Eclipse

An eclipse of the Sun may be either *total*, *annular*, or *partial*. To explain the difference between the first two kinds of eclipse, let us suppose that the observer is situated exactly in the line of centres of the Sun and new Moon, so that both bodies appear in the same direction. Then, if the Moon's angular diameter is greater than the Sun's, the whole of the Sun will be concealed by the Moon ; the eclipse is then said to be *total*. If, on the other hand, the Sun has the greater angular diameter, the Moon will conceal only the central portion of the Sun's disc, leaving a bright ring visible all round; under such circumstances, the eclipse is said to be *annular*. Lastly, if the observer is not exactly in the line of centres, the Moon may cover up a segment at one side of the Sun's disc; the eclipse is then *partial*.

Now, the Moon's angular diameter varies, according to the distance of the Moon, from 29′ 22″ at apogee to 33′ 28″ at perigee, the corresponding limits for the Sun's diameter being 31′ 28″ at apogee, and 32′ 32″ at perigee. Hence, both total and annular eclipses of the Sun are possible. Thus, when the Sun is in apogee and the Moon in perigee an eclipse must be either total or partial; when the Sun is in perigee and the Moon in apogee, an eclipse must be annular or partial.

227. Circumstances of a Solar Eclipse

Fig. 81 shows the different circumstances under which a solar eclipse is seen from different parts of the Earth.

Draw the common tangents CDQ, $C'D'Q$, CRD', $C'RD$, to the Sun and Moon, orming the enveloping cones DQD' and fRg; these constitute respectively the boundaries of the umbra and penumbra of the Moon's shadow.

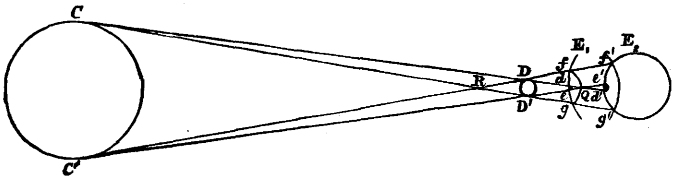

FIG. 81.

First let the umbra DQD' meet the Earth's surface (E_1) before coming to a point at Q, the curve of intersection being de. Also let the penumbra fRg meet the Earth's surface in the curve fg. Then from any place on the Earth within the space de the Sun appears totally eclipsed. At a place elsewhere within the penumbra fg, the Sun appears partially eclipsed, a portion only being obscured by the Moon.

Next let the umbra DQD' come to a point Q before reaching the Earth E_2. Then, if the cone of the umbra be produced to meet the Earth in $d'e'$, an observer anywhere within the space $d'e'$ sees eclipse as an annular eclipse. At any place elsewhere within the penumbra $f'g'$, the eclipse appears partial, as before. At parts of the Earth which fall without the penumbra there is no eclipse. Hence a solar eclipse is only visible over a part of the Earth's surface, and its circumstances are different at different places.

As the Sun and Moon move forward in their relative orbits, and the Earth revolves on its axis, the two cones of the Moon's shadow travel over the Earth, and the eclipse becomes visible from different places in succession. The inner cone traces out on the Earth a very narrow belt, over which the eclipse is seen as a total or annular eclipse, according to circumstances. The outer cone, or penumbra, sweeps out a far broader belt, including that part of the Earth's surface where the eclipse is visible as a partial eclipse.

A total or annular eclipse of the Sun, like a total eclipse of the Moon, always begins and ends as a partial eclipse, the totality or annular condition, as seen from a given station, only lasting for a short period about the middle of the eclipse. The maximum duration of totality is just under eight minutes. This does not occur at the Equator, but about North Latitude 20° when the Sun is near the Apogee.

On 1937, June 8th (Pacific Ocean west of Panama), totality lasted more than 7 minutes, an unusually long duration. The eclipse of 1955, June 20th (Philippine Islands) will also have a totality of more than 7 minutes.

In the case of an annular eclipse, there are two internal, as well as two external, contacts, and the eclipse remains annular during the interval between the internal contacts. This may sometimes be rather more than twelve minutes.

Owing to the limited area of the belt over which a solar eclipse is visible, the chance that any eclipse may be visible at any given place is far smaller than in the case of a lunar eclipse. The chance of an eclipse being *total* at any place is very small indeed. The last eclipses visible as total eclipses in England occurred in 1724 and 1927; the next will take place on June 30th, 1954, in the Shetlands, and August 11th, 1999, in Cornwall. At Greenwich, or any other station, about 43 partial solar eclipses are visible in a century.

At a given station (*i.e.* a point on the Earth's surface, not an extended region) there are on the average slightly under 3 total solar eclipses in 1000 years, and slightly over 4 annular ones.

In the British Isles, including the Shetlands, there is on the average one total eclipse in 50 years, but the distribution is very irregular; there was a blank interval of 203 years between 1724 and 1927, followed by three totalities in just over 72 years.

II.—DETERMINATION OF THE FREQUENCY OF ECLIPSES

228. To Find the Limits of the Moon's Geocentric Position consistent with a Solar or Lunar Eclipse

In Fig. 82, let the plane of the paper represent any plane through the Sun's and Moon's centres; and let ABV and $A'B'V$ represent the common tangents bounding the cone of the Earth's true shadow. Let AUB' be the other common tangent, which goes (nearly) through B'; and let the line SE, joining the centres of the Sun and Earth, meet the common tangents in V and U. Let T, t, t' be those points on ABV and AB' whose distance from E is equal to that of the Moon.

Then, if M_1, M_2 denote the positions of the Moon's centre, when touching the cone BV externally and internally at T, it is evident that a *lunar eclipse* occurs whenever the full Moon is nearer the line of centres than M_1. Hence, if m denote the Moon's angular semi-diameter TEM_1, the Moon's angular distance from EV must be less than VEM_1, or $VET + m$.

Similarly, the lunar eclipse is *total* when the Moon is not further from the line of centres than M_2; for this the Moon's (geocentric) angular distance from the line of centres must be not greater than VEM_2, or $VET - m$.

Let m_1, m_2 be the centres of the Moon at internal and external contact with AB near t. There is evidently a *solar eclipse* visible at some point of the Earth's surface (such as B) as a partial eclipse, if the Moon's angular distance from the Sun is less than SEm_1, or $SEt + m$.

Supposing the Moon's distance to be such that its angular radius is less than that of the Sun, there is an *annular* eclipse whenever the Moon lies wholly within the cone AVA', as at m_2. This requires the Moon's geocentric angular distance from the Sun to be less than SEm_2, or $SEt - m$.

If, however, the Moon is so near that its angular radius is greater than that of the Sun, the angle it subtends is greater than ABA', and therefore there is a *total* eclipse at B whenever the edge of the Moon reaches the internal tangent $A'B$. Taking m_3 to represent the corresponding position of the Moon when touching the other tangent AB' at t' (for the sake of clearness in the figure), we see that, in order that there may be a total eclipse somewhere on the Earth's surface, the geocentric angular distance between the Moon's and Sun's centres must be less than SEm_3 or $SEt' + m$.

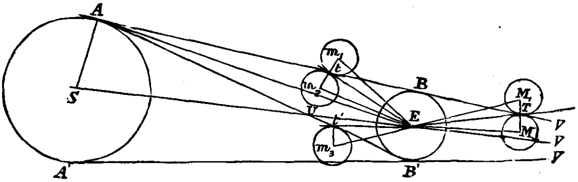

FIG. 82.

Now, as the cone AVA' tapers to a point at V, the breadth of its cross section is greater near m_1, m_2, m_3 than near M_1, M_2, and when the Moon is in syzygy, its angular distance from EV or ES = its latitude. Hence the limits of latitude are greater for a solar than for a lunar eclipse, and therefore the probability of the occurrence of a solar eclipse is greater than the probability of a lunar eclipse. This explains why, on the whole, *solar eclipses are more frequent than lunar.*

Oppolzer's *Canon der Finsternisse* gives 8,000 solar eclipses between — 1207 and A.D. 2161 : 5,200 lunar eclipses between — 1206 and A.D. 2163. Hence there are 20 solar eclipses to 13 lunar ones, or roughly 3 to 2. But at a given station about twice as many lunar ones as solar ones are visible; the reason is that a lunar eclipse is visible over a much larger region on the Earth than a solar one.

We shall now calculate the angles VEM_1, VEM_2, SEm_1, SEm_2, SEm_3. Let p, P denote the horizontal parallaxes of the Moon and

Sun ; m, s their respective angular semi-diameters (Fig. 82). We have :

$$s = \angle SEA,$$
$$p = \angle BTE = \angle BtE = \angle B't'E,$$
$$P = \angle BAE = \angle B'AE,$$
$$\text{and } m = \angle TEM_1 = \angle TEM_2 = \angle tEm_1 = \angle tEm_2 = \angle t'Em_3.$$

For the lunar eclipses we have, from the triangle TEA :—

$$\angle ETB + \angle EAB = 180^\circ - \angle TEA = \angle VET + \angle SEA :$$
$$\text{or} \quad \angle VET = \angle ETB + \angle EAB - \angle SEA = p + P - s ;$$
$$\text{so that} \quad \angle VEM_1 = \angle VET + \angle TEM_1 = \boldsymbol{p + P - s + m} ;$$
$$\text{and} \quad \angle VEM_2 = \angle VET - \angle TEM_2 = \boldsymbol{p + P - s - m} ;$$

For the solar eclipses we have, from the triangle tEA :—

$$\angle EtB - \angle EAB = \angle tEA = \angle SEt - \angle SEA$$
$$\text{or} \quad \angle SEt = \angle EtB - \angle EAB + \angle SEA = p - P + s.$$
$$\text{so that} \quad \angle SEm_1 = \boldsymbol{p - P + s + m},$$
$$\text{and} \quad \angle SEm_2 = \boldsymbol{p - P + s - m}.$$

Lastly from the triangle $t'EA$ we have :—

$$\angle Et'B' - \angle EAB' = \angle AEt' = \angle AES + \angle SEt'.$$
$$\text{Therefore,} \quad \angle SEt' = \angle B't'E - \angle B'AE - \angle AES = p - P - s.$$
$$\text{and} \quad \angle SEm_3 = \boldsymbol{p - P - s + m}.$$

*229. Greatest Latitudes of the Moon at Syzygy

Since S and V are in the ecliptic, it follows that when the Moon is in conjunction or opposition, the plane of the paper in Fig. 82 is perpendicular to the ecliptic. Therefore the angles VEM_1, VEM_2 measure the Moon's latitude at conjunction, and SEm_1, SEm_2, SEm_3 measure its latitude at opposition in the positions represented. The above expressions are, therefore, the greatest possible latitudes at syzygy consistent with eclipses of the kinds named.

Now, taking the mean values we have, roughly :—

$$s = 16' ; \; m = 15' ; \; p = 57' ; \; P = 0' \, 8''.$$

Substituting these values, and collecting the results, we have, roughly, the following limits for the Moon's geocentric latitude, or angular distance from the line of centres :—

(1) *For a lunar eclipse,* $VEM_1 = p + P - s + m = 56'$;

(2) *For a total lunar eclipse,* $VEM_2 = p + P - s - m = 26'$;

(3) *For a solar eclipse,* $SEm_1 = p - P + s + m = 88'$;

(4) *For an annular eclipse,* $SEm_2 = p - P + s - m = 58'$.

Lastly, taking the Sun at apogee, and the Moon at perigee, we have, $m = 17'$ and $s = 16'$ nearly, whence we have, in the most favourable case :—

(4*a*) *For a total solar eclipse,* $SEm_3 = p - P - s + m = 58'$.

230. Ecliptic Limits

From the last results it appears that a lunar eclipse cannot occur unless at the time of opposition the Moon's latitude is less than about 56′, and that a solar eclipse cannot occur unless at conjunction the Moon's latitude is less than about 88′. Now the Moon's latitude depends on its position in its orbit relatively to the line of nodes; hence there will be corresponding limits to the Moon's distance from the node consistent with the occurrence of eclipses. These limits are called the *Ecliptic Limits*.

* The ecliptic limits may be computed as follows :—Let the geocentric direction of the Moon's centre be represented on the celestial sphere by M. Let N represent the node, MH a secondary to the ecliptic. [The *ecliptic limit*, strictly speaking, means the limit of NH measured along the ecliptic, and not that of NM.]

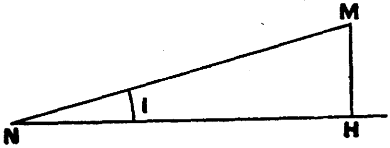

FIG. 83.

Now the limit of latitude MH has been calculated in the last paragraph for the different cases. Let this be denoted by l. Also let I be the inclination of the Moon's orbit to the ecliptic. Then in the sperical triangle NHM, right-angled at H, we have $HM = l$, and $\angle HNM = I$; both of these are known, hence NH can be calculated.

For rough purposes it will be sufficient either to treat the small triangle HNM as a plane triangle or to regard MH as approximately the arc of a small circle, whose pole is N. The first method gives

$$l = NH \tan I; \quad \text{or } NH = l \cot I.$$

Or, adopting the second method, we have

$$l = MH = \angle MNH \times \sin NH = I \sin NH;$$

$$\text{or} \quad \sin NH = l/I,$$

whence the ecliptic limit NH is found.

The exact formula may be given from the solution of right-angled spherical triangles given in Art. 10. Note first that the least distance l between the centres of the Sun and Moon does not occur when the Moon is at M, but at a point K on NM, such that HKN is a right angle: then from the triangle HKN

$$\sin NH = \sin l/\sin I.$$

231. Major and Minor Ecliptic Limits

Owing to the variations in the distances of the Sun and Moon their parallaxes and angular semi-diameters are not quite constant. Hence the exact limits of the Moon's latitude l, as calculated by the method of Art. 230, are subject to small variations. This alone would render the ecliptic limits variable. But there is another cause of variation in the ecliptic limits, arising from the fact that I, the inclination of the Moon's orbit is also variable, its greatest and least values being about 5° 18′ and 4° 59′. However the range in I at the times of eclipses is much smaller, being 5° 15′ to 5° 18′. The mean is 5° 16·8′.

The greatest and least values of the limits for each kind of eclipse are called the *Major* and *Minor Ecliptic Limits.*

For an eclipse of the Moon the major and minor ecliptic limits have been calculated to be about 11° 38′ and 9° 39′ respectively at the present time. For an eclipse of the Sun the limits are 17° 25′ and 15° 23′ respectively. Thus a lunar eclipse *may* take place if the Moon, when full, is within 11° 38′ of a node; and a lunar eclipse *must* take place if the full Moon is within 9° 39′ of a node. Similarly, a solar eclipse *may* take place if the Moon, when new, is within 17° 25′, and a solar eclipse *must* take place if the new Moon is within 15° 23′ of a node.

The mean values of the lunar and solar ecliptic limits are now 10° 29′ and 16° 14′. But the eccentricity of the Earth's orbit is very slowly decreasing; consequently the major limits are smaller and the minor limits larger than they were, say, a thousand years ago. The change is very slight.

232. Synodic Revolution of the Moon's Nodes

An eclipse is thus only possible at a time when the Sun is within a certain angular distance of the Moon's nodes. Hence the period of revolution of the Moon's nodes, relative to the Sun, marks the recurrence of the intervals of time during which eclipses are possible. This period is called the period of a *synodic revolution of the nodes.*

In Art. 211 it was stated that the Moon's nodes have a retrograde motion of about 19° per annum, more exactly 19° 21′. In one year (365d.) the Sun, therefore, separates from a node by $360° + 19° \, 21'$ or 379·35°, hence it separates 360° in $(360 \times 365\frac{1}{4}°)/379{\cdot}35$ days, or about 346·62d. This, then, is the period of a synodic revolution of the node.

In a synodic lunar month ($29\frac{1}{2}$ days), the Sun separates from the line of nodes by an angle $379\frac{1}{3}° \times 29\frac{1}{2} \div 365\frac{1}{4}$, or 30° 40′, a result which will be required in the next paragraph.

233. To Find the Greatest and Least Number of Eclipses possible in a Year

Let the circle in Fig. 84 represent the ecliptic, and let *N*, *n* be the Moon's nodes. Take the arcs *NL*, *NL'*, *nl*, *nl'*, each equal to the lunar ecliptic limit, and *NS*, *NS'*, *ns*, *ns'* each equal to the solar ecliptic limit. Then the *least* value of *SS'* or *ss'* is twice the *minor solar* ecliptic limit, and is 30° 46', and this is greater than 30° 40', the distance traversed by the Sun relative to the nodes between two new Moons. Hence, at least one new Moon *must* occur while the Sun is travelling over the arc *SS'*, and two *may* occur. Therefore *there must be one, and there may be two eclipses of the Sun, while the Sun is in the neighbourhood of a node.*

Again, the *greatest* value of *LL'*, *ll'* is double the *major lunar* ecliptic limit, and is, therefore, 23° 16'. This is considerably less than the space passed over by the Sun relative to the nodes between two full Moons. Hence, there cannot be more than one full Moon while the Sun is in the arc *LL'*, and there may be none. Therefore *there cannot be more than one eclipse of the Moon while the Sun is in the neighbourhood of a node, and there may be none at all.* There must, however, be a penumbral lunar eclipse.

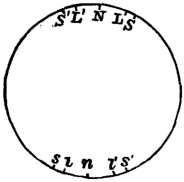

FIG. 84.

234. The Greatest Number of Eclipses possible in a Year

The case most favourable for more than one eclipse is when the Moon is new just after the Sun has come within the solar ecliptic limits, *i.e.* near *S*.

When the Moon is full (about 14¾ days later) the Sun will be near *N*, at a point within the lunar ecliptic limits; there will therefore be an eclipse of the *Moon.*

At the following new Moon the Sun will not have reached *S'*; and there will be a *second* eclipse of the *Sun.*

In six lunations from the first eclipse the Sun will have travelled through just over 180°, and will be within the space *ss'*, near *s*; there will therefore be a *third* eclipse of the *Sun.*

At the next full Moon the Sun will be near *n*, and there will be a *second* eclipse of the *Moon.*

The Sun may just fall within the space *ss'* near *s'* at the next new Moon; there will then be a *fourth* eclipse of the *Sun.*

In twelve lunations from the first eclipse, the Sun will have described about 368°, and will, therefore, be about 8° beyond its first position, and well within the limits *ss'*; there will, therefore, be a *fifth* eclipse of the *Sun.*

About 14¾ days later, at full Moon, the Sun will be well within the lunar ecliptic limits LL', and there will be a *third* eclipse of the *Moon.*

All these eclipses occur in 12½ lunations, *i.e.* 369 days, or a year and four days. We cannot, therefore, have all the eight eclipses in one year, but *there may be as many as seven eclipses in a year, namely, either five solar and two lunar, or four solar and three lunar.* This happened in 1917, and again in 1935.

235. The Least Number of Eclipses possible in a Year

The most unfavourable case is that in which the Moon is full just before the Sun reaches the ecliptic limits at L.

At new Moon the Sun will be near N, and there will be *one solar* eclipse.

At the next full Moon the Sun will have passed L', so that there will be no lunar eclipse. After six lunations the Sun will not have arrived at l.

At the next new Moon the Sun will be within the ecliptic limits, and there will be a *second solar* eclipse.

At the next full Moon the Sun will be again just beyond l', and at 12 lunations from first full Moon, the Sun may again not have quite reached L.

At 12½ lunations there will be a *third solar* eclipse.

The interval between the first and third eclipses will be 12 lunations, or about 354 days. If, therefore, the first eclipse occurs after the 11th day of the year, *i.e.* January 11th, the third will not occur till the following year. Therefore, *the least possible number of eclipses in a year is two: these must both be solar eclipses.*

236. The Saros of the Chaldeans

The period of a synodic revolution of the nodes is (Art. 232) approximately 346·62 days. Hence :—

19 synodic revolutions of the node take 6585·78 days.

Also 223 lunar months = 6585·32 days.

If follows that after 6585⅓ days, or 18 years 11⅓ days,* the Moon's nodes will have performed 19 revolutions relative to the Sun, and the Moon will have performed 223 revolutions almost exactly. Hence the Sun and Moon will occupy almost exactly the same position relative to the nodes at the end of this period as at the beginning, and eclipses will therefore recur after this interval. The period was discovered by observation by the Chaldean astronomers who called it the *Saros*. By a knowledge of it they were usually able to predict eclipses.

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* The date goes forward 12⅓ days when 3 leap-days intervene (this may happen when a year like 1900 occurs in the interval), 11⅓ days when 4 leap-days intervene, and 10⅓ days when 5 leap-days intervene.

A "synodic revolution of the Moon's apsides," or the period in which the Sun performs a complete revolution relative to the Moon's apse line, occupies 411·74 days. Hence sixteen such revolutions occupy 6587·87 days, or about two days longer than the Saros. Therefore the Moon's line of apsides also returns to very nearly the same position relative to the Sun and Moon. Hence, the solar eclipses, as they recur, will be nearly of the same kind (total or annular) in each Saros. The whole number of eclipses in a Saros is about 70. The average of all eclipses from B.C. 1207 to A.D. 2162 shows that there are 20 solar eclipses to 13 lunar.

The Saros enables us to make very fair predictions of future eclipses, if we have the date of the eclipses 18 years earlier. Any eclipse of considerable size is certain to have a successor. Its region of visibility on the earth is shifted westward through about 116°, owing to the fraction of a day in the cycle; three Saroses bring us back to about the same longitude. It is possible to have two successive eclipses visible at the same station if the first occurs early in the morning (or early in the night for lunar eclipses). Thus the successor of that of 1927, June 29th, occurs in the early afternoon of 1945, July 9th, and is a large partial eclipse in England, total in Norway.

The number of times that an eclipse can recur varies between 70 and 85, corresponding to 1,244 and 1,514½ years respectively. The eclipse begins as a small one in the Arctic or Antarctic regions; it increases at each return, finally becoming total or annular; it is a central eclipse for about two-thirds of its whole career; finally it goes off at the opposite pole to that of its first appearance. On the average a new eclipse is born once in 31 years; the death rate is, of course, the same as the birth rate.

Lunar eclipses have a shorter career; they have between 44 and 54 returns, corresponding to 775 and 955½ years respectively.

There is rather a rare type of solar eclipse called annular-total; in these cases the Moon as seen from the beginning and end of the track, where it is in the horizon, is not quite large enough to cover the Sun; but in the middle of the eclipse track the Moon is higher in the sky, and consequently nearer to the observer; it thus becomes large enough to cover the Sun for a short time; such eclipses occur about twice in a Saros. There was one in April, 1912, just total in Portugal; its successor in April, 1930, was just total in California.

The Saros is the most useful of all eclipse cycles; but there are several others, of which two may be mentioned.

(1) A cycle of 6,444 lunations, equal to 190,295·109 days, or 521 years and 3 or 4 days, according to the number of leap-days intervening; this cycle brings back the latitude of the eclipse very closely. An example follows:

— 1199, June 16th, annular in Atlantic.
— 678, June 17th, annular in Sahara.
— 157, June 17th, total in England.
364, June 16th, total in Scotland. Observed as partial at Binchester, England. This is the first British eclipse recorded as observed.
885, June 16th, total in Scotland. Mentioned in *Chronicon Scotorum*.
1406, June 16th, large eclipse in London; total in Belgium.
1927, June 29th, total in Wales, Lancashire, Yorkshire.

It will be seen that in Old Style the date is brought back almost exactly; the jump between 1406 and 1927 is due to the change of Style. One-ninth of this cycle, that is 58 years less about 40 days, is also an eclipse cycle, but a less convenient one.

(2) A cycle of 22,325 lunations, equal to 659,270·38 days, or 1,805 years and a few days. Like the Saros, it restores the diameters and motions of the Sun and Moon almost exactly. As an example, the eclipse of 1927 was preceded by one on 122, June 21st, total about the Shetlands, but in the evening, whereas that of 1927 was in the morning. The name Megalosaros has been suggested for this cycle.

*III.—OCCULTATIONS—PLACES AT WHICH A SOLAR ECLIPSE IS VISIBLE

237. Occultations

When the Moon's disc passes in front of a star or planet, the Moon is said to *occult* it.

An occultation evidently takes place whenever the apparent angular distance of the Moon's centre from the star becomes less than the Moon's angular semi-diameter. As the apparent position of the Moon is affected by parallax, the circumstances of an occultation are different at different places on the Earth's surface.

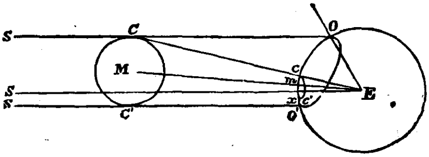

FIG. 85.

Let m denote Moon's angular semi-diameter, p its horizontal parallax. In the figure, let E and M be the centres of the Earth and Moon, and let sC, sC' represent the parallel rays coming from a star, and grazing the Moon's disc. These rays cut the Earth's surface along a curve OO', and it is evident that only to observers at points within this curve is the star hidden by the Moon's disc. Let EC, Es, EM, EC' cut the Earth's surface in c, x, m, c'; the rays EC, EC' cut the Earth's surface in a small circle cc', whose angular radius $mEc = MEC = m$. Let d be the geocentric angular distance sEM between the Moon's centre and the star. Then :—

$$\begin{aligned} \text{Angle } ECO &= \text{angle subtended by Earth's radius } EO \text{ at } C; \\ &= \text{parallax of } C \text{ when viewed from } O; \\ &= p \sin COZ \text{ (Art. 157)}; \\ &= p \sin OEx \text{ (by parallels)}. \\ \text{But } ECO &= CEs = \text{angle subtended by } cx; \end{aligned}$$

So that $\sin OEx = \dfrac{\text{angle } cx}{p}$.

Hence we have the following construction for the curve separating those points on the Earth's surface at which the occultation is visible at a given instant from those at which the star is not occulted. Taking the sublunar point m as pole, describe a circle cc' on the terrestrial globe, with the Moon's angular semi-diameter (m) as radius. Through the substellar point x draw any great circle, cutting this small circle in any point c. Measure along it an arc xO such that $\sin xO$ is always the same multiple $(1/p)$ of xc. The locus of the points O, thus determined, is the curve required.

Half of the circle cc' consists of points under the advancing limb of the Moon; hence, over the portion of the curve OO' corresponding to this half-circle, the occultation is just beginning. At points on the other half of cc' the Moon's limb is receding; hence over the other portion of OO' the star is reappearing from behind the Moon's disc.

Since the greatest and least values of cx in any position are $d + m$ and $d - m$, it is evident that the greatest value of d for which an occultation can take place is when

$$d - m = p; \ d = m + p.$$

238. Occultation of a Planet

If s be a planet, the lines Es, Os, can no longer be regarded as rigorously parallel; but the angle between them, EsO,

$$\begin{aligned} &= \text{angle subtended at } s \text{ by the Earth's radius } EO \\ &= \text{parallactic correction at } O \text{ (Art. 156)} \\ &= P \sin ZOs \text{ (Art. 157)} = P \sin OEx \text{ very nearly.} \end{aligned}$$

As before, $$ECO = p \sin OEx.$$

But $$ECO = EsO + CEs;$$

$$\text{Thus } \ p \sin OEx = P \sin OEx + cx; \ \sin OEx = \frac{cx}{p - P}.$$

With this exception, the construction is the same as for a star.

If the planet be so large that we must take account of its angular diameter, the method of the next paragraph must be used.

239. Eclipse of the Sun

There is a total eclipse of the Sun, provided the Moon's disc completely covers the Sun's; this occurs if the Moon's angular semi-diameter (m) is larger than the Sun's (s), and the apparent angular distance between the Sun's and Moon's centres (as seen from any point at which the eclipse is visible) is less than $m - s$. Hence, if the Moon's angular semi-diameter were reduced to $m - s$, the Sun's centre would then be occulted. Hence the points O, whose locus encloses the places from which the eclipse is visible, can be found as follows :—

With centre m the sublunar point, and angular radius $m - s$, describe a circle. Through the subsolar point x draw any arc of a great circle xc, cutting the circle in c, and take O, on xc produced, such that

$$\sin xO = \frac{xc}{p - P}.$$

For an annular eclipse $m < s$, and the apparent angular distance between the centres is $s - m$; hence the same construction is followed, save that $s - m$ is the angular radius of the small circle first described. For a partial solar eclipse, the angular radius is $s + m$.

When a planet has a sensible disc, the beginning of its occultation may be compared to a partial eclipse of the Sun; and the planet is entirely occulted when the conditions are satisfied corresponding to those for a total eclipse.

Example.—*Supposing the centres of the Earth, Moon, and Sun to be in a straight line and the Moon's and Sun's semi-diameters to be exactly* 17′ *and* 16′, *find the angular radii of the circles on the Earth over which the eclipse is total and partial respectively, taking the relative horizontal parallax as* 57′.

At those points at which the eclipse is total, the apparent angular distance between the centres, as displaced by parallax, must be not greater than 17′—16′, or 1′. Hence, since the centres are in a line with the Earth's centre, the parallactic displacement must be not greater than 1′. Hence, if z be the Sun's zenith distance at the boundary, then $57' \sin z = 1'$; or $\sin z = \frac{1}{57}$, or approximately circular measure of $z = \frac{1}{57}$. So that $z = 1°$ approx. Hence the eclipse is total over a circle of angular radius 1° about the sub-solar point.

Similarly, the eclipse is partial if $57' \sin z < 16' + 17'$, or 33′, or $\sin z < \frac{33}{57}$, or ·58. From a table of natural sines, we find that $\sin^{-1} \cdot 58 = 35\frac{1}{2}°$ roughly; therefore the angular radius is $35\frac{1}{2}°$.

Examples on Eclipses generally.—1. *Find* (*roughly*) *the maximum duration of an eclipse of the Moon, and the maximum duration of totality.*

From Art. 229 we see that a lunar eclipse will continue as long as the Moon's angular distance from the line of centres of the Earth and Sun is less than 58′, and the eclipse will continue total while the angular distance is less than 26′. Hence, the maximum duration of the eclipse is the time taken by the Moon to describe 2 × 58′, or 116′, and the maximum duration of totality is the time taken to describe 2 × 26′, or 52′.

Now the Moon describes 360° (relative to the direction of the Sun) in the synodic month, 29½ days. Therefore, the times taken to describe 116′ and 52 respectively are

$$\frac{29\frac{1}{2} \times 116}{360 \times 60} \text{ and } \frac{29\frac{1}{2} \times 52}{360 \times 60} \text{ days,}$$

i.e. 3h. 48m. and 1h. 42m.,

and these are the maximum durations of the eclipse and of totality. The eclipse of June 26th, 2029, will last 3h. 44m., and be total for 1h. 44m.

2. *Calculate roughly the velocity with which the Moon's shadow travels over the Earth.* (*Sun's distance* = 93,000,000 *miles.*)

The radius of the Moon's orbit being about 240,000 miles, its circumference is about 1,508,000 miles. Relative to the line of centres, the Moon describes the circumference in a synodic month, *i.e.* about 29½ days. Hence its relative velocity is about 1,508,000 ÷ 29½, or 51,000 miles per day, *i.e.* 2,100 miles per hour. If q denote the point where the middle of the shadow reaches the Earth (Fig. 81), and if the Earth's surface at q is perpendicular to Sq, we have :—

$$\begin{aligned} \text{velocity of } q : \text{ vel. of } M &= Sq : SM \\ = 93{,}000{,}000 : 93{,}000{,}000 - 240{,}000 &= 1\cdot0026 \text{ nearly.} \\ \text{Hence the velocity of the shadow at } q &= \text{vel. of } M \text{ very nearly} \\ &= 2{,}100 \text{ miles an hour.} \end{aligned}$$

To find the velocity of the shadow relative to places on the Earth, we must subtract the velocity of the Earth's diurnal motion. This, at the Earth's equator, is about 1,040 miles an hour. Hence, if the Earth's surface and the shadow are moving in the same direction, the relative velocity is about 1,060 miles an hour.

3. *Find the maximum duration of totality of the eclipse of the example on page* 185, *neglecting the obliquity of the ecliptic.*

The angular radius of the shadow being 1°, or about 69½ miles, its diameter is 139 miles. The obliquity of the ecliptic being neglected, the eclipse is central at a point on the equator, and the shadow and the Earth are therefore moving in the same direction with relative velocity 1,060 miles an hour (by Question 2). The greatest duration of totality is the time taken by the shadow to travel over a distance equal to its diameter, *i.e.* 139 miles, and is therefore 139 × 60/1060 minutes, *i.e.* 7·9 minutes (roughly).

EXAMPLES

1. If a total lunar eclipse occur at the summer solstice, and at the middle of the eclipse the Moon is seen in the zenith, find the latitude of the place of observation.

2. If there is a total eclipse of the Moon on March 21st, will the year be favourable for observing the phenomenon of the Harvest Moon?

3. Having given the dimensions and distances of the Sun and Moon, show how to find the diameter of the umbra where it meets the Earth's surface.

CHAPTER XI

THE PLANETS

I.—General Outline of the Solar System

240. Planets

The name *planet*, or "wanderer," was applied by the Greeks to designate all those celestial bodies, except comets and meteors, which changed their position relative to the stars, independently of the diurnal motion; these included the Sun and Moon. At present, however, only those bodies are called *planets* which move in orbits about the Sun. The Sun itself is a *star*, while the Earth is classed among the planets, and the Moon, which follows the Earth in its annual path, and has an orbital motion about the Earth, is described, along with similar bodies which revolve about other planets, as a *satellite* or *secondary*.

241. The Sun, ☉

The *Sun* is distinguished by its immense size and mass. It forms the centre of the solar system, for, in spite of the great distances of some of the furthest planets, the centre of mass of the whole system always lies very near the Sun. The Sun resembles the other fixed stars in being self-luminous, whereas the planets are not self-luminous but are seen only by means of reflected sunlight.

Its diameter is 110 times that of the Earth, or nearly twice as great as the diameter of the Moon's orbit about the Earth. Its mass is about 332,000 times that of the Earth.

From observing the apparent motion of the spots, which are often to be seen on the Sun's disc, it is inferred that the Sun rotates on its axis in the sidereal period of about 25 days.

The planets all revolve round the Sun in the same direction as the Earth. Their motion is, therefore, *direct*.

242. Mercury, ☿

Mercury is the planet nearest the Sun, its distance in terms of the distance of the Earth as unit being 0·39. It is characterised by its small size and small mass, 1/27 that of the Earth, the great eccentricity of its elliptical orbit, amounting to about $\frac{1}{5}$, and the great inclination of the orbit to the ecliptic, namely, about 7°. The sidereal period of revolution round the Sun is about 88 of our days.

Mercury rotates on its axis once in a sidereal period of revolution; consequently it always turns nearly the same face to the Sun, as the Moon does to the Earth (Art. 214).

Owing, however, to the great eccentricity of the orbit, the "libration in longitude" is much greater than that of the Moon, amounting to 47°. Consequently, rather over *one quarter* of the whole surface is turned alternately towards and away from the Sun, *three-eighths* is always illuminated, and three-eighths is always dark.

Mercury resembles the Moon in having apparently no atmosphere. This is shown by its very low albedo, below that of the Moon, and by the absence of an external ring of refracted sunlight when it is entering on the Sun in transit.

243. Venus, ♀

Venus is the next planet, its mean distance from the Sun being 0·72 units. It is the planet that most closely resembles the Earth in size and mass, being slightly smaller and slightly less massive than the Earth. Its orbit is very nearly circular, and is inclined to the ecliptic at an angle of about 3° 23′. Venus revolves about the Sun in a period of 224 days. It has an extensive atmosphere. The rotation period is not known with certainty but is probably somewhere about 30 to 40 days.

244. The Earth, ⊕

The *Earth* comes next, its mean distance being one unit, and its orbit very nearly circular (eccentricity $= \frac{1}{60}$). Its period of revolution in the ecliptic is 365¼ days, and its period of rotation is a sidereal day, or 23h. 56m. mean time. It is the nearest planet to the Sun having a *satellite* (the Moon, ☾), which revolves about it in 27⅓ days.

245. Mars, ♂

Mars is at a mean distance of 1·52 units. It is smaller than the Earth, its diameter being about one half and its mass 1/9th that of the Earth. Its orbit is inclined at less than 2° to the ecliptic, and is an ellipse of eccentricity about $\frac{1}{11}$. It revolves about the Sun in a sidereal period of about 686 days, and rotates on its axis in about 24h. 37m.

Mars has two small *satellites*, which revolve about it in the periods 7½ and 30¼ hours, roughly. The appearance which would be presented by the inner satellite, if observed from Mars, is rather interesting. As it revolves much faster than Mars, it would be seen to rise in the *west* and set in the *east twice* during the night. The outer satellite would appear to revolve slowly in the opposite direction—from east to west. The inner satellite is eclipsed often at opposition, and would appear to transit the Sun's disc often at conjunction.

Mars has an atmosphere, but considerably less dense than that of the Earth. White caps of snow or hoar frost surround each pole, and wax and wane with the seasons. Large dusky areas on the planet were formerly taken for seas: it is now believed that they may be covered with some form of vegetation. The orange-coloured areas are thought to be sandy deserts.

246. The Asteroids

The next planet in order of distance from the Sun is Jupiter. But between Mars and Jupiter a large number of small planets have been discovered, to which the name *Asteroids* or *Minor Planets* has been given. The first of these to be found was discovered in 1801. Since that time a few new asteroids have been discovered almost every year, the total number now known being about 2,000. Most of the discoveries are now made by photography. A very interesting asteroid, named Eros, was discovered by Witt in 1898. Its orbit is inside that of Mars, and at the most favourable oppositions, it approaches the Earth within 15 million miles. Such opportunities are very useful for determining the solar parallax (see Art. 380). Eros was just visible to the naked eye in January 1931.

Ceres is the largest asteroid, with diameter $\frac{1}{5}$ of that of the Moon. Vesta is considerably smaller, but is brighter, and sometimes visible to the naked eye. Among the others Juno, Pallas, and Astraea are the most conspicuous telescopic objects. Many of the smaller asteroids are less than ten miles in diameter, and are probably simply masses of rock flying round and round the Sun.

The periodic time of revolution of the asteroids vary considerably, but their average is about 1,600 days. The orbits are in many cases very oval, the eccentricity of one (Polyhymnia) being over $\frac{1}{3}$, and they are often inclined at considerable angles to the ecliptic, the inclination in the case of Pallas amounting to nearly 35°, while that of Juno is 13°.

A very interesting group of asteroids, known as the Trojans, since they are named after heroes of the siege, are at the same distance from the Sun as Jupiter. Each moves so as always to form approximately an equilateral triangle with the Sun and Jupiter. Five of them are in advance of Jupiter, two behind it.

Two recently discovered asteroids, Ganymede and Hidalgo have large eccentricities. The latter travels out nearly to Saturn's orbit.

247. Jupiter, ♃

Jupiter is at a mean distance of 5·21 units. It revolves round the Sun in a period of twelve years, in an orbit nearly circular and inclined at only $1\frac{1}{3}$° to the ecliptic. It is much the largest and most massive of all the planets. Its diameter is about 11 times and its mass is 317 times the Earth's. Its mean density is only one-quarter of the mean density of the Earth; this is because it possesses a very extensive atmosphere with a depth of many thousands of miles. The period of rotation is somewhat less than 10 hours ; this rapid rotation causes the planet to be perceptibly oblate. Through a telescope the disc is seen to be encircled with a series of *belts* or *streaks* parallel to its equator. On account of their variability, these are supposed to be due to *belts of clouds* in the atmosphere of the planet.

Jupiter is now known to have eleven satellites. The four larger ones are interesting as being the first celestial bodies discovered with the telescope, by Galileo (A.D. 1610). A fairly powerful opera glass will show them.

The orbits of all four are nearly circular, and inclined less than half a degree to the plane of Jupiter's equator, and about 3° to the plane of its orbit. They revolve round Jupiter in periods of 16d. 17h., 7d. 4h., 3d. 13¼h., and 1d. 18½h. The fifth or innermost satellite was discovered in 1892 ; it revolves in a period of nearly 12h., at a mean distance of 70,000 miles from the surface, or 113,000 miles from the centre of Jupiter. The above satellites are frequently *eclipsed* by passing into the shadow cast by Jupiter, or *occulted* when Jupiter comes between them and the Earth.

Six very distant satellites have been detected in recent years. All their orbits are very eccentric and highly inclined to Jupiter's orbit, and the Sun produces large disturbances in their motions. VI. and VII. form a pair with interlocked orbits, their periods of revolution being 250 and 260 days. VIII and IX also form a pair with interlocked orbits, their periods of revolution being about two years ; they are sometimes 20 millions of miles from Jupiter. A curious feature is that VIII and IX go round Jupiter in the opposite direction to that of all the other satellites.

The six outer satellites are all small objects ; the largest of them is about 100 miles in diameter, the smallest probably not more than 15 miles. As seen from Jupiter they would appear as very faint telescopic stars.

248. Saturn, ♄

Saturn is at a mean distance from the Sun of 9·54 units. The periodic time of revolution is 29½ years. The orbit is nearly circular, and inclined to the ecliptic at an angle of 2½°. It is the second largest of the planets, with a diameter about 9 times that of the Earth and a mass 95 times the Earth's. It has the lowest mean density of any of the planets, 7/10ths that of water and one-eighth the mean density of the Earth. This is accounted for by Saturn, relative to its size, having the most extensive atmosphere of any planet. Its period of rotation is rather more than 10h. and it is the most oblate of any of the planets.

Saturn's rings are among the most wonderful objects revealed by the telescope. They appear to be three flat annular discs of extreme thinness, lying in a plane inclined to the ecliptic at an angle of about 28°, and extending to a distance rather greater than the radius of the planet; the middle ring is by far the brightest, while the inner ring is very faint. When the Earth is in the plane of the rings they are seen edgewise, and, owing to their very small thickness, they then become invisible except in the best telescopes.

It is known that the rings consist of a large number of small satellites or meteors. It is certain that they do not consist of a continuous mass of solid or liquid matter. The surface of the planet itself is encircled with belts similar to those on Jupiter.

In addition to the rings, Saturn has nine satellites, all situated outside the rings. The seven nearest move in planes nearly coinciding with that of the rings, while the orbit of the eighth is inclined to it at an angle of 13°. The sixth satellite is by far the largest, having a probable diameter nearly equal to that of Mercury. The eighth has been observed, like our Moon, always to turn the same side towards the planet.[*] The ninth satellite is very minute and very distant. It revolves nearly in the plane of Saturn's orbit, and, like Jupiter's satellites VIII, IX, its motion is retrograde.

The distances of the satellites from Saturn range from 3 to 218 times the planet's semi-diameter, and the corresponding periods range from 22½h. to 550d.

249. Uranus, ♅

Uranus, at mean distance 19·2 units, revolves in an approximately circular orbit, nearly coinciding with the ecliptic, in a period of 84 years. Its diameter is about four times and its mass nearly 15 times the Earth's; the mean density is about the same as that of Jupiter. The period of rotation is about 11h. Uranus was discovered in 1871 by Sir William Herschel, who named it the *Georgium Sidus* in honour of the king.

Uranus is attended by four satellites at least, and these possess the remarkable peculiarity of revolving in a plane nearly perpendicular to the ecliptic and in a retrograde direction. In fact, the plane of their orbits makes an angle of 82° with the ecliptic. Their periods are 2½d., 4d., 8¾d., and 13½d. roughly.

250. Neptune, ♆

The position of *Neptune* was predicted in 1846 almost simultaneously by Adams and Leverrier, from the observed effects of its attraction on the orbital motion of Uranus. It was first discovered by Galle, of

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[*] This is probably true for all satellites, excepting those that are very distant from their primaries.

Berlin, in September 1846, very close to the position which had been computed beforehand. It has a mean distance of 30·1, and it revolves in its orbit in about 165 years. Neptune is practically equal in size to Uranus but is somewhat more massive. Its period of rotation is about 16h.

Neptune has one satellite moving in a retrograde direction in a plane inclined to the ecliptic at about 35°. The plane of its orbit is shifting, which is concluded to be caused by the attraction of Neptune's equatorial protuberance. Hence the orbit plane must make a considerable angle, about 20°, with Neptune's equator. On the other hand, the satellites of Uranus all lie in the same plane, which shows no shift, so that it evidently coincides with Uranus's equatorial plane.

251. Pluto, ♇

A new planet, named Pluto, was discovered photographically at the Lowell Observatory, Flagstaff, Arizona, in 1930. It is a faint object, of the 15th magnitude. Its mean distance from the Sun is 39·5 units, and its orbit, which is transversed in 248 years, has the large inclination to the ecliptic of 17° and an eccentricity of 0·25. Both the inclination and eccentricity are much larger than for any of the other planets. Nothing definite is known about the size and mass of Pluto ; from its faintness, it is probable that it is smaller than the Earth ; its mass is estimated to lie between the masses of the Earth and Venus.

252. Bode's Law

Before the discovery of Neptune, it was noticed that the distances of the planets could be represented approximately by an empirical law, known as Bode's Law. This law is merely a result of observation and does not appear to have any physical significance.

The law is given by the following rule : Write down the series of numbers, 0, 3, 6, 12, 24, 48, 96, 192, (384), (768), in which each number (after the second) is double the preceding. Now add 4 to every term. These numbers represent fairly closely the relative distances of the various planets from the Sun, the distance of the Earth (the third in the series) being taken as 10.

When Bode formulated the law, Uranus, Neptune and Pluto had not been discovered and the minor planets were unknown. The series of numbers given by the law and the mean distances of the planets are as follows :—

(1)	4	7	10	16	28	52	100	196
(2)	☿	♀	⊕	♂	Minor planets	♃	♄	♅
(3)	3·9	7·2	10	15	—	52	95	192

The law is approximately satisfied and the minor planets take the place where the law indicated that a planet was missing. When Uranus was discovered, it was found to be at about the distance given by the law. The law breaks down, however, both for Neptune and Pluto; for these the relative distances given by it are 388 and 772, whereas the true distances are 301 and 396.

253. Tabular View of Solar System

Name of Planet.		Mean Dist. of Planet. / Mean Dist. of Earth.	Periodic Time.		Inclination of Orbit.		Eccentricity of Orbit.	No. of Satellites.
			days =	years	°	′		
Mercury	☿	0·3871	87·97	0·241	7	0	·206	—
Venus	♀	0·7233	224·70	0·615	3	24	·007	—
Earth	⊕	1·0000	365·26	1·000	0	0	·017	1
Mars	♂	1·5237	686·98	1·881	1	51	·093	2
Ceres	⊙	2·7673	1,681·45	4·604	10	37	·076	—
Jupiter	♃	5·2082	4,322·59	11·862	1	18	·048	11
Saturn	♄	9·5388	10,759·20	29·457		29	·056	9 & 3 rings
Uranus	♅	19·1910	30,685·9	84·015	0	46	·047	4
Neptune	♆	30·0707	60,187·6	164·788	1	47	·009	1
Pluto	♇	39·4574	90,471·3	247·697	17	9	·249	—

	Inclin. Equator to Orbit.	Mass. Earth = 1.	Mean Diameter Miles.	Density Water = 1.	Time of Sidereal Rotation.	Oblateness.
☿	Small	0·037	3110	3·5	87·97d.	Small
♀	Unknown	0·816	7705	4·86	Unknown	Small
⊕	23° 27′	1·000	7917	5·52	23h. 56m. 4·1s.	1/296
♂	23° 59′	0·106	4207	3·96	24h. 37m. 22·6s.	1/192
♃	3° 4′	316·9	86740	1·34	9h. 50m. to 9h. 56m.	1/15
♄	26° 50′	94·9	71530	0·71	10h. 14m. to 10h. 38m.	1/10
♅	98° 0′	14·66	31700	1·27	10¾h. about	1/14
♆	29° 0′	17·16	31100	1·58	15⅔h. about	1/40
♇	Unknown	Small (? 0·1)	4000	—	Unknown	—

II.—SYNODIC AND SIDEREAL PERIODS—DESCRIPTION OF MOTION IN ELONGATION OF PLANETS AS SEEN FROM THE EARTH—PHASES

254. Inferior and Superior Planets.—Definitions

In describing the motions of the planets relative to the Earth, it is convenient to divide the planets into two classes; those nearer the Sun than the Earth are called *inferior*, those that are more remote are called *superior*.

The angle of *elongation* is the difference between the geocentric (Art. 137) longitude of the planet and that of the Sun.

It appears from the preceding section that (i) The planets all revolve round the Sun in the same direction ; (ii) The planets which are nearer the Sun travel at a greater speed than those which are more remote.

In order further to simplify the descriptions we shall assume that the planets all revolve uniformly in circles, about the Sun as centre, in the plane of the ecliptic.

255. Changes in Elongation of an Inferior Planet

Let E be the Earth, V an inferior planet moving in the orbit $AUBU'$ about S the Sun. Since SV revolves more rapidly about S than SE, the motion of V relative to E, as it would appear from S, is direct.

SV separates from SE at a rate which is the difference of the rates at which E, V revolve in their orbits. The changes in the positions of the planet relative to the Sun are therefore the same as if E were at rest and V revolved with an angular velocity equal to the excess of the angular velocity of the planet over that of the Earth.

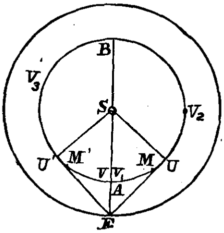

FIG. 86.

Let the line ES meet the orbit of V in A and B. When V is at A or B it has the same longitude as S, and if the planet actually moved in the ecliptic it would be in front of the Sun at A, behind the Sun at B. In reality, owing to the inclination of the orbits, this but rarely happens.

At A, the planet is said to be in *inferior conjunction* with the Sun ; it has the same longitude as the Sun and is nearer than the Sun to the Earth. At B the planet is said to be in *superior conjunction* with the Sun ; it has the same longitude as the Sun but is further away. If we consider the appearances which would be presented on the Sun, the planet is in " heliocentric conjunction " with the Earth at A and in " heliocentric opposition " at B.

After inferior conjunction at A, the planet is seen on the westward side of the Sun, as at V_1. The elongation SEV gradually increases till the planet reaches a point U such that EU is a tangent to the orbit. The planet is then at its greatest elongation, the angle SEU being a maximum.

Subsequently, as at V_2, the elongation diminishes, and the planet approaches the Sun, until superior conjunction occurs, as at B. The planet then separates from the Sun, reappearing on the opposite (eastern) side, as at V_3, attains its maximum elongation at U', and finally comes round again to inferior conjunction at A.

The time between two consecutive conjunctions of the same kind (superior or inferior) is called the *synodic period* of the planet (*cf.* Art. 202), and is the period in which SV separates from SE through 360°.

256. To Find (roughly) the Ratio of the Distance from the Sun of an *Inferior* Planet to that of the Earth

To do this it is only necessary to observe the planet's greatest elongation. For if U, E (Fig. 86) represent the planet and Earth at the instant of greatest elongation, the angle EUS is a right angle, and therefore :—

$$\sin SEU = \frac{SU}{SE};$$

that is : $$\frac{\text{Distance of planet}}{\text{Distance of Earth}} = \text{sine of greatest elongation.}$$

This method is, however, much modified by the fact that the real orbits are not circles, but ellipses.

Examples 1.—*Given that the greatest elongation of Venus is* 45°, *find its distance from the Sun, that of the Earth being* 93,000,000 *miles.*

$$\begin{aligned}\text{Here distance of Venus} &= 93{,}000{,}000 \sin 45^\circ = 93{,}000{,}000 \times \sqrt{\tfrac{1}{2}} \\ &= 93{,}000{,}000 \times \cdot 70711 = 65{,}760{,}000 \text{ miles.}\end{aligned}$$

2.—*Taking the Earth's distance as unity, find the distance of Mercury, having given that Mercury's greatest elongation is* $22\tfrac{1}{2}$°.

$$\begin{aligned}\text{The distance of Mercury} &= 1 \times \sin 22\tfrac{1}{2}^\circ = \sqrt{\{\tfrac{1}{2}(1 - \cos 45^\circ)\}} \\ &= \tfrac{1}{2}\sqrt{(2 - \sqrt{2})} = \cdot 38268.\end{aligned}$$

257. Changes in Elongation of a Superior Planet

Let us now compare the apparent motion of a superior planet J with that of Sun. Since it revolves about the Sun in the same direction as the Earth does, but more slowly, the line SJ will move, relative to SE, in the opposite or retrograde direction. Hence, in considering the changes in the position of the planet relative to the Sun, we may regard SE as a fixed line, and J must then revolve about S in the circle $ARBT$ with a retrograde motion, *i.e.* in the same direction as the hands of a watch.*

At A the planet is in *opposition* with the Sun, and its elongation is 180°. At B it is in *conjunction*, and its elongation is 0°. If, however, we were to refer the directions of the Earth and planet to the Sun, the planet would be in heliocentric conjunction with the Earth at A, and in heliocentric opposition at B.

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* As a simple illustration, both the hour and minute hands of a watch revolve in the same directions, but the minute hand goes faster and leaves the hour hand behind. Hence the hour hand *separates from the minute hand* in the opposite direction to that in which both are moving.

The planet is nearest the Earth at *A*, and since its orbital velocity is constant, its relative angular velocity is then greatest, and the elongation *SEJ* is decreasing at its most rapid rate. As the planet moves round from opposition *A* to conjunction *B*, the elongation *SEJ* decreases continuously from 180° to 0°.

At *R* the elongation is 90°, and the planet is said to be in *quadrature.*

At conjunction, *B*, the elongation is 0° ; and we may also consider it to be 360°. As the planet revolves from *B* to *A*, the elongation (measured round in the direction *BRA*) decreases from 360° to 180°.

At *T* the elongation is 270°, and the planet is again said to be in quadrature.

At *A* the elongation is again 180°, the planet being once more in opposition. After this the elongation decreases from 180° to 0° as before, as the planet's relative position changes from *A* through *R* to *B*.

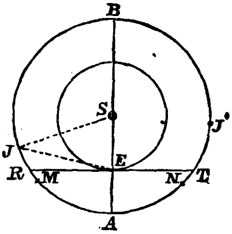

FIG. 87.

The cycle of changes recurs in the *synodic period*, *i.e.* the period between two successive conjunctions or oppositions. We see that the elongation decreases continually from 360° to 0° as the planet revolves from conjunction round to conjunction, and there is no greatest elongation.

258. To Compare (roughly) the Distance of a *Superior* Planet with that of the Earth

Here there is no greatest elongation, and therefore we must resort to another method.

Let the planet's elongation *SEJ* (Fig. 87) be observed at any instant, the interval of time which has elapsed since the planet was in opposition being also observed. Let this interval be t, and let S denote the length of the planet's synodic period. Then, in time S the angle *JSE* increases from 0° to 360° ; therefore, if we assume the change to take place uniformly, the angle *JSE* at time t after conjunction is $= 360° \times t/S$.

Hence, *JSE* is known. Also *JES* has been observed, and *SJE* ($= 180° - JES - JSE$) is therefore also known. Therefore we have, by plane trigonometry,

$$\frac{\textit{Distance of Planet}}{\textit{Distance of Earth}} = \frac{SJ}{SE} = \frac{\sin SEJ}{\sin SJE},$$

which determines the ratio of the distances required.

This method is also applicable to the inferior planets. It is, however, not exact, owing to the fact that the planetary motions are not really uniform (see Art. 265).

259. General Method

It is not necessary to observe the instant of conjunction or opposition. If the synodic period S is known, two observations of the elongation and the elapsed time are sufficient to determine the ratio of the distances. The requisite formulae are more complicated, but they only involve plane trigonometry. We, therefore, leave their investigation as an exercise to the more advanced student.

Example.—*Calculate the distance of Saturn in terms of that of the Earth, having given that* 94 *days after opposition the elongation of Saturn was* 84° 17′, *and that the synodic period is* 376 *days. Given also tan* 5° 43′ = ·1.

Let the Sun, Earth, and Saturn be denoted by S, E, J. In 376 days $\angle JSE$ increases from 0° to 360°. It follows that :—

$$\text{in 94 days after opposition } \angle JSE = 90^\circ;$$

$$\text{also, by hypothesis, } \angle JES = 84^\circ\ 17'.$$

$$\frac{\text{Distance of Saturn}}{\text{Distance of Earth}} = \frac{SJ}{SE} = \tan SEJ = \tan 84^\circ\ 17' = 10.$$

Therefore the distance of Saturn, as calculated from the given data is 10 times that of the Earth.

260. The Synodic Period of an Inferior Planet

This may be found very readily by determining the time between two transits of the planet across the Sun's disc and counting the number of revolutions in the interval.

For a superior planet this is not possible, and we must, instead, find the interval between two epochs at which the planet has the same elongation.

261. Relations between the Synodic and Sidereal Periods

The relation between the synodic and sidereal periods is almost exactly the same as in the case of the Moon, the only difference being that the planets revolve about the Sun and not about the Earth.

The *sidereal period* of a planet is the time of the planet's revolution in its orbit about the Sun relative to the stars. The *synodic period* is the interval between two conjunctions with the Earth relative to the Sun. It is the time in which the planet makes one whole revolution as compared with the line joining the Earth to the Sun.

Let S be the planet's synodic period, P its sidereal period, Y the length of a year, that is, the Earth's sidereal period, all the periods being supposed measured in days. Then, in one day :—

the angle described by the planet about the Sun = $360^\circ/P$,
the angle described by the Earth = $360^\circ/Y$,
and the angle through which their heliocentric directions have separated = $360^\circ/S$.

If the planet be inferior, it revolves more rapidly than the Earth, and $360°/S$ represents the angle gained by the *planet* in one day. Therefore:—

$$\frac{360°}{S}=\frac{360°}{P}-\frac{360°}{Y}; \quad \text{or} \quad \frac{1}{\boldsymbol{S}}=\frac{1}{\boldsymbol{P}}-\frac{1}{\boldsymbol{Y}}$$

If the planet be superior, it revolves more slowly than the Earth, and $360°/S$ is the angle gained by the *Earth* in one day. Therefore :—

$$\frac{360°}{S}=\frac{360°}{Y}-\frac{360°}{P}; \quad \text{or} \quad \frac{1}{\boldsymbol{S}}=\frac{1}{\boldsymbol{Y}}-\frac{1}{\boldsymbol{P}}.$$

From these relations, the sidereal period can be found if the synodic period is known, and *vice versa*.

262. Phases of an Inferior Planet

As the planets derive their light from the Sun, they must, like the Moon, pass through different *phases* depending on the proportion of their illuminated surface which is turned towards the Earth.

An inferior planet V will evidently be new at inferior conjunction A, dichotomized like the Moon at its third quarter at greatest elongation U, full at superior conjunction B, dichotomized like the Moon at first quarter when it again comes to greatest elongation at U'. Thus, like the Moon, it will undergo all the possible different phases in the course of a synodic revolution.

There is, however, one important difference. As the planet revolves from A to B its distance from the Earth increases, and its angular diameter therefore decreases. Thus the planet appears largest when new and smallest when full, and the variations in the planet's brightness due to the difference of phase are, to a great extent, counterbalanced by the changes in the planet's distance. For this reason, Venus alters very little in its brightness (as seen by the naked eye) during the course of its synodical revolution.

The phase is determined by the angle SVE, and this is the angle of elongation of the Earth as it would appear from the planet. The illuminated portion of the visible surface of the planet at V is proportional to $180° - SVE$, and the proportion of the apparent area of the disc which is illuminated varies as $1 + \cos SVE$ or $2\cos^2 \frac{1}{2} SVE$. (*Cf.* Art. 196).

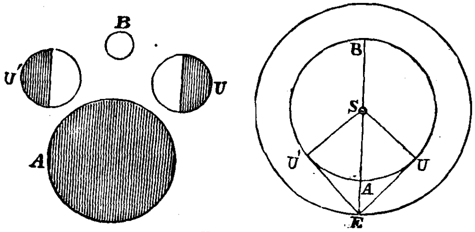

FIG. 88.

The phases of Venus are easily seen through a telescope.

263. Phases of a Superior Planet

For a superior planet J the angle SJE never exceeds a certain value. It is greatest when $SEJ = 90°$, being then the greatest elongation of the Earth as it would appear from the planet. Hence the planet is always nearly full, being only slightly gibbous, and the phase is most marked at quadrature.

The gibbosity of Mars, though small, is readily visible at quadrature, about one-eighth of the planet's disc being obscured. The other superior planets are, however, at a distance from the Sun so much greater than that of the Earth that they always appear very approximately full.

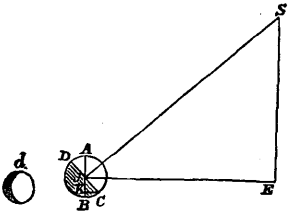

FIG. 89.

A very slight phase can be seen on Jupiter near quadrature with powerful instruments.

III.—KEPLER'S LAWS OF PLANETARY MOTION

264. Kepler's Three Laws

We have already seen that the orbits of most of the planets are nearly circular, their distances from the Sun being nearly constant and their motions being nearly uniform. A far closer approximation to the truth is the hypothesis held for a long time by Tycho Brahe and other astronomers, namely, that each planet revolved in a circle whose centre was at a small distance from the Sun, and described equal angles in equal intervals of time about a point found by drawing a straight line from the Sun's centre to the centre of the circle and producing it for an equal distance beyond the latter point.

The true laws which govern the motion of the planets were discovered by the Danish astronomer Kepler, in connection with his great work on the planet Mars (*De Motibus Stellae Martis*). After nine years' incessant labour the first and second of the following laws were discovered, and shortly afterwards the third.

I. *Every planet moves in an ellipse, with the Sun in one of the foci.*

II. *The straight line drawn from the centre of the Sun to the centre of the planet (the planet's " radius vector ") sweeps out equal areas in equal times.*

III. *The squares of the periodic times of the several planets are proportional to the cubes of their mean distances from the Sun.*

These laws are known as **Kepler's Three Laws.** We have already proved that the first two laws hold in the case of the Earth. The third law is also found to hold good for the Earth as well as the other planets, and this fact alone affords strong evidence that the Earth is a planet.

By the *mean distance* of a planet is meant the arithmetic mean between the planet's greatest and least distances from the Sun. If *p*, *a* (Fig. 90) be the planet's positions at perihelion and aphelion (*i.e.* when nearest and furthest from the Sun respectively), the planet's mean distance $= \frac{1}{2}(Sp + Sa) = \frac{1}{2}\,pa = \frac{1}{2}$ (major axis of ellipse described) (Art. 129).

The periodic times are, of course, the *sidereal periods.* Hence the third law is a relation between the sidereal periods and the major axes of the orbits.

265. Verification of Kepler's First and Second Laws

We shall now roughly sketch the principle of the methods by which Kepler determined the orbit of Mars, and thus proved his First and Second Laws. A verification of the laws in the case of the Earth has already been given, and we have shown (Art. 127) how to determine exactly the position of the Earth at any given time; we may regard this, therefore, as known. We may also suppose the length of the sidereal period of Mars to be known, for the average length of the synodic period may be found, as in Art. 204, and the sidereal period may be deduced by the formulae of Art. 261.

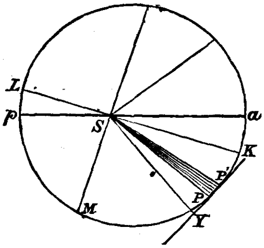

FIG. 90.

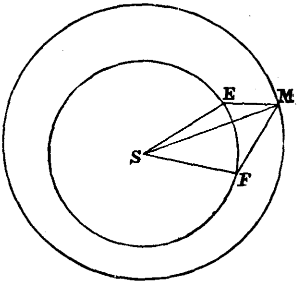

FIG. 91.

Let the direction of the planet be observed when it is at any point *M* in its orbit, the Earth's position being *E*. When the planet has returned again to *M* after a sidereal revolution, the Earth will *not* have returned to the same place in its orbit, but will be in a different position, say *F*. Let now the planet's new direction *FM* be observed.*

---

* For simplicity we suppose Mars to move in the ecliptic plane. The methods require some modification when the inclination of the orbits is taken into account, but the general principle is the same.

From knowing the Earth's motion, we know SE, SF and the angle ESF. From the observations of the two directions of M we know the angles SEM and SFM. These data are sufficient to enable us to solve the quadrilateral $SEMF$.

For join EF. In $\triangle SEF$ we know SE, SF, and $\angle ESF$. Hence we find EF, $\angle SEF$, $\angle SFE$. Hence $\angle FEM$ $(= SEM - SEF)$ and $\angle EFM$ $(= SFM - SFE)$ are known. With these and EF solve $\triangle MEF$ and find EM, MF. Lastly, in $\triangle SEM$ we know SE, EM, and $\angle SEM$, and thus we find SM and $\angle ESM$.

We can thus determine SM and the angle ESM, whence the distance and direction of M from the Sun are found. Similarly, any other position of Mars in its orbit can be found by two observations of the planet's sidereal period separated by the interval of the planet's sidereal revolution.

In this way, by a series of observations of Mars, extending over two sidereal periods, the planet's direction and distance relative to the Sun can be determined daily, and the whole orbit can thus be plotted out.

This method was that actually adopted by Kepler, except that he had not previously determined the Earth's motion, and believed that it could be accurately represented by Tycho Brahe's hypothesis. This approximation was close enough, for the Earth's orbit is very nearly a circle, and that of Mars, which he was deducing, is very much more eccentric.

266. Verification of Kepler's Third Law

Kepler's Third Law can be verified much more easily, especially if we make the approximate assumption that the planets revolve uniformly in circles about the Sun as centre. The sidereal periods of the different planets can be found by observing the *average* length of the synodic period (the actual length of any synodic period is not quite constant, owing to the planet not revolving with exactly uniform velocity) and applying the equations of Art. 261. The distance of the planet may be compared with that of the Earth, either by observing the greatest elongation (Art 256) in the case of an inferior planet, or by the method of Art. 258. It is then easy to verify the relation between the mean distances and periodic times of the several planets.

In the table of Art. 253, the student will have little difficulty in verifying (especially if a table of logarithms be employed) that the square of the ratio of the periodic time of the planet to the year (or periodic time of the Earth) is in every case equal to the cube of the ratio of the planet's mean distance to that of the Earth.* The data being only approximate, however, the law can only be verified as approximately true, although it is in reality accurate.*

---

* In other words, 2 log (period in years) = 3 log (distance in terms of Earth's distance).

Owing to the importance of Kepler's Third Law, we append the following examples as illustrations.

Examples.—1. *Given that the mean distance of Mars is* 1·52 *times that of the Earth, find the sidereal period of Mars.*

Let T be the sidereal period of Mars in days. Then, by Kepler's Third Law,

$$\left\{\frac{T}{365\frac{1}{4}}\right\}^2 = (1{\cdot}52)^3 = 3{\cdot}5118;$$

$$\text{giving } T = 365\tfrac{1}{4} \times \sqrt{(3{\cdot}5118)} = 365\tfrac{1}{4} \times 1{\cdot}874 = 684{\cdot}5.$$

Hence, from the given data, the period of Mars is 1·874 of a year, or 684·5 days.

Had we taken the more accurate value of the relative distance, viz. 1·5237, we should have found for the period the correct value, namely, 687 days.

2. *The synodic period of Jupiter being* 399 *days, find its distance from the Sun, having given that the Earth's mean distance is* 92 *million miles.*

Let T be the sidereal period of Jupiter. Then, by Art. 264,

$$\frac{1}{T} = \frac{1}{365\frac{1}{4}} - \frac{1}{399} = \frac{33\frac{3}{4}}{365\frac{1}{4} \times 399},$$

$$\text{or } T = \frac{399}{33\frac{3}{4}} \times 365\tfrac{1}{4} \text{ days} = \frac{399}{33\frac{3}{4}} \text{ years} = 11{\cdot}82, \text{ or nearly 12 years.}$$

Let a be the distance of Jupiter in millions of miles. Then, by Kepler's Third Law,

$$\left(\frac{a}{92}\right)^3 = \left(\frac{12}{1}\right)^2 = 144.$$

$$\text{So that } a = 92 \times \sqrt[3]{(144)} = 92 \times 5{\cdot}24 = 482;$$

that is, Jupiter's distance is 482 millions of miles.

By taking $T = 11{\cdot}82$ and the Earth's distance as 92·04, we should have found the more accurate value 477·6 for Jupiter's distance in millions of miles.

267. Satellites

The motions of the satellites about any planet are found to obey the same laws as those which Kepler investigated for the orbits of the planets. For example, the Moon's orbit about the Earth is an ellipse, and (except so far as affected by perturbations) satisfies both of Kepler's First and Second Laws. When a number of satellites are revolving round a common primary (*i.e.* planet) as is the case with Jupiter, the squares of their periodic times are found, in every case, to be proportional to the cubes of their mean distances from the planet.†

Example.—*Compare* (*roughly*) *the mean distances of its two satellites from Mars.*

The periodic times are 30¼h. and 7½h. respectively, and these are in the ratio (nearly) of 4 to 1.

---

* See, however, Art. 431.

† Of course the relation does not hold between the periodic times and mean distances of satellites revolving round *different* planets, nor between those of a satellite and those of a planet.

Hence the mean distances are as $4^{\frac{2}{3}} : 1$, or $\sqrt[3]{16} : 1$.

Now, $2\sqrt[3]{16} = \sqrt[3]{128} = 5$ very nearly (since $5^3 = 125$). Hence the mean distances are very nearly in the ratio of 5 to 2.

IV.—MOTIONS RELATIVE TO STARS—STATIONARY POINTS

268. Direct and Retrograde Motion.—(1) An Inferior Planet

We have described (Arts. 258—260) the motion of a planet relative to the Sun. In considering its motion *relative to the stars* we must take account of the Earth's motion.

An *inferior planet* moves more swiftly than the Earth. Hence at inferior conjunction the line AE (Fig. 92) joining them is moving in the direction of the hands of a watch. The planet therefore appears to move *retrograde.* At greatest elongation (U, U') the planet's *own* motion is in the line joining it to the Earth, and hence produces no change in its direction; but the Earth's *direct* motion causes the line EU or EU' to turn about U or U' with a rotation *contrary* to that of the hands of a watch; and therefore the apparent motion is *direct.* Over the whole portion UBU' of the relative orbit both the Earth's motion and the planet's combine to make the planet's apparent motion *direct.* There must, therefore, be two positions M between A and U and N between U' and A, at which the motion is checked and reversed. At these two positions the planet is said to be *stationary.*

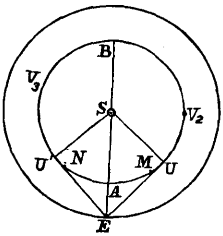

FIG. 92.

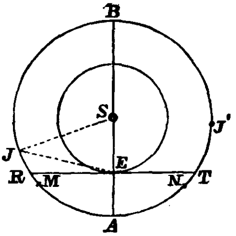

FIG. 93.

269. Direct and Retrograde Motion.—(2) A Superior Planet

A *superior planet* moves *slower* than the Earth; hence at opposition the line EA (Fig. 93) joining them is turning in the direction of the hands of a watch. The planet therefore appears to move *retrograde.* At quadrature (R, T) the Earth is moving along RET; hence its motion produces no change in the planet's direction. Hence the planet's direct motion about the Sun makes its apparent motion also *direct.* In all parts of the arc RBT the orbital velocities of Earth and planet conspire to produce direct motion. Hence the planet is *stationary* at M, between A and R, and at N between T and A.

In both cases the longitude increases from M to N and decreases from N to M; hence it is a maximum at N and a minimum at M.

After a complete synodic revolution the planet's elongation is the same as at the beginning, and the Sun's longitude has been increased; therefore the planet's longitude has also *increased.* Hence the *direct* preponderates over the *retrograde* motion.

270. Alternative Explanation

We may also proceed as follows. Let E, J represent two planets at heliocentric conjunction. Let E_1, E_2, E_3, ..., J_1, J_2, J_3, ..., be their successive positions after a series of equal intervals. To find the apparent motion of J among the stars, as seen from E, take any point E, and let $E1$, $E2$, $E3$, ... (Fig. 95) be parallel respectively to E_1J_1, E_2J_2, E_3J_3, ... Then the points 1, 2, 3, ... represent J's direction as seen from E at a series of equal intervals, starting from opposition.

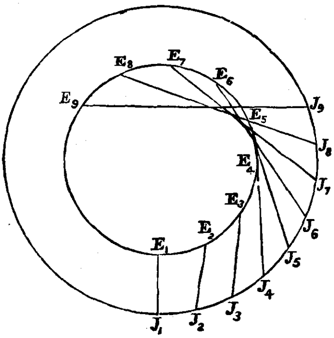

FIG. 94.

Again, if $J1$, $J2$, $J3$ be taken parallel to J_1E_1, J_2E_2, ... (Fig. 96), the points 1, 2 now represent E's direction as seen from J.

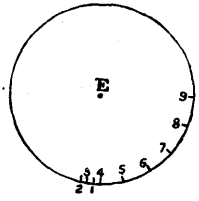

FIG. 95.

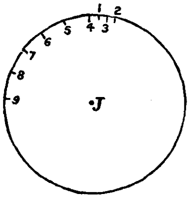

FIG. 96.

We observe from Figs. 95, 96 that the relative motion is retrograde from 1 to 2, and becomes direct near 3. At the instant at which this takes place, either planet must be stationary, relative to the other. Since J_4E_4 is nearly a tangent to E's orbit, E is near its greatest elongation, and J is near quadrature at the positions 4; hence, E appears stationary from J between inferior conjunction and greatest elongation; and J appears stationary between opposition and quadrature.

We notice that $J1$, $J2$, ... are parallel to $E1$, $E2$, but measured in opposite directions, showing that the motion of E relative to J is the same (direct, stationary, or retrograde) as that of J relative to E.

271. Effects of Motion in Latitude

Hitherto we have supposed the planet to move in the ecliptic. When, however, the small inclination of the orbit to the ecliptic is taken into account, it is evident that the planet's latitude is subject to periodic fluctuations.

The points of intersection of the planet's orbit with the ecliptic are (as in the case of the Moon) called the *Nodes*. Whenever the planet is at a node its latitude is zero; and this happens twice in every *sidereal* period of revolution.

272. Loops of Retrogression

A planet is stationary when its longitude is a maximum or minimum, but unless its latitude should happen to be a maximum at the same time, the planet does not remain actually at rest. When the change from direct to retrograde motion, and *vice versa*, is combined with the variations in latitude, the effect is to make the planet describe a zigzag curve, sometimes containing one or two loops, called "*loops of retrogression.*" This is readily verified by observation.

Fig. 97 is an example of the path of Venus in the neighbourhood of its stationary points, the numbers representing its positions at a series of intervals of ten days. Here, the planet is stationary close to the node N, between 4 and 5, and it describes a loop in the neighbourhood of the stationary point near 9, where its motion changes from retrograde to direct.

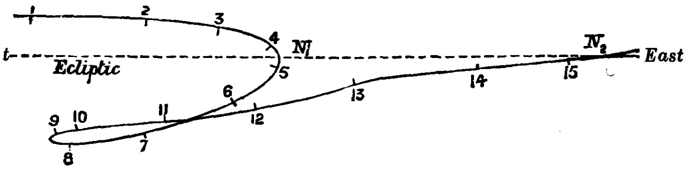

Fig. 97.

The student will find it an instructive exercise to trace out the path of any planet in the neighbourhood of its retrograde motion, using the values of its decl. and R.A., at intervals of a few days, as tabulated in the *Nautical Almanac* or *Whitaker's Almanack*.

273. To find the Condition that two Planets may be Stationary as seen from one another, assuming the Orbits Circular and in one Plane

Let P, Q be the positions of the planets at any instant; P', Q' their positions after a very short interval of time.

Then, if PQ and $P'Q'$ are parallel, the direction of either planet, as seen from the other, is the same at the beginning and end of the interval; that is, P is stationary as seen from Q, and Q is stationary as seen from P.
]

Let u, v represent the orbital velocities of the planets P, Q; a, b the radii SP, SQ respectively.

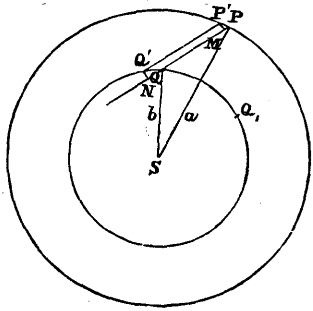

FIG. 98.

Draw $P'M$, $Q'N$ perpendicular to PQ. Then, in the stationary position, we must have $P'M = Q'N$.

But PP', QQ', being the arcs described by the two planets in the same interval, are proportional to the velocities, u, v. Therefore $P'M$, $Q'N$ are proportional to the component velocities of the planets perpendicular to PQ. These component velocities must, therefore, be equal, and we have

$$u \sin P'PM = v \sin Q'QN.$$

Whence, since $P'P$ is perpendicular to SP and $Q'Q$ to SQ,

$$u \cos SPQ = v \cos SQN = -v \cos SQP \text{ ....(i)},$$

and this is the condition that the planets may be stationary relative to one another.

274. To find the Angle between the Radii Vectores in the Stationary Position, and the Period during which a Planet's Motion is Retrograde

By projecting SQ, QP on SP, we have

$$a = b \cos PSQ + PQ \cos SPQ.$$

Similarly $$b = a \cos PSQ + PQ \cos SQP.$$

Thus $$\cos SPQ : \cos SQP = a - b \cos PSQ : b - a \cos PSQ.$$

Whence, by (i), $$u\,(a - b \cos PSQ) + v\,(b - a \cos PSQ) = 0;$$

$$\text{or } \cos PSQ = \frac{au + bv}{av + bu} \text{ .... (ii)}$$

By means of Kepler's Third Law, we can express the ratio of u to v in terms of a and b. For if T_1, T_2 denote the periodic times, then evidently—

$$uT_1 = 2\pi a;\quad vT_2 = 2\pi b;$$

$$\text{and}\quad u : v = aT_2 : bT_1.$$

But $$T_1 : T_2 = a^{\frac{3}{2}} : b^{\frac{3}{2}};$$

$$\text{so that}\quad u : v = \sqrt{b} : \sqrt{a}.$$

Substituting in (ii), we have :—

$$\cos PSQ = \frac{a\sqrt{b} + b\sqrt{a}}{a\sqrt{a} + b\sqrt{b}} = \frac{(ab)^{\frac{1}{2}}\,(a^{\frac{1}{2}} + b^{\frac{1}{2}})}{a^{\frac{3}{2}} + b^{\frac{3}{2}}} = \frac{\sqrt{(ab)}}{a - \sqrt{(ab)} + b}.$$

[From this result it may be easily deduced that

$$\tan \tfrac{1}{2} PSQ = \left(\frac{1 - \cos PSQ}{1 + \cos PSQ}\right)^{\frac{1}{2}} = \frac{\sqrt{b} - \sqrt{a}}{\sqrt{(a + b)}}\Big].$$

In the above investigation PSQ is the angle through which SQ separates from SP between heliocentric conjunction and the stationary point. Hence, since $\angle PSQ$ increases from 0° to 360° in the synodic period S, the time taken from conjunction to the stationary point is :—

$$S \times \frac{\angle PSQ}{360^\circ}.$$

If $\angle PSQ_1 = \angle PSQ$, there is another stationary point before conjunction, when the planets are in the relative positions P, Q. Hence, the interval between the two stationary positions is twice the time taken by the planets to separate through $\angle PSQ$, and is therefore :—

$$2S \times \frac{PSQ}{360^\circ} = S \times \frac{PSQ}{180^\circ}.$$

This represents the interval during which the motion of either planet, as seen from the other, is *retrograde*. During the remainder of the synodic period the motion is direct, and the time of *direct* motion is therefore

$$S - S \times \frac{PSQ}{180^\circ} = S \times \frac{180^\circ - PSQ}{180^\circ}.$$

EXAMPLES

1. The Earth revolves round the Sun in 365·25 days, and Venus in 224·7 days. Find the time between two successive conjunctions of Venus.

2. If Venus and the Sun rise in succession at the same point of the horizon on the 1st of June, determine roughly Venus' elongation.

3. Find the ratio of the apparent areas of the illuminated portions of the disc of Venus when dichotomized and when full, taking Venus' distance from the Sun to be $\frac{9}{11}$ of that of the Earth.

4. Mars rotates on his axis once in 24½ hours, and the periods of the sidereal revolutions of his two satellites are 7½ hours and 30 hours respectively. Find the time between consecutive transits over the meridian of any place on Mars of the two satellites respectively.

5. A small satellite of the Earth is eclipsed at every opposition. Find an expression for the greatest inclination which its orbit can have to the plane of the ecliptic.

6. If the periodic time of Saturn be 30 years, and the mean distance of Neptune 2,760 millions of miles, find (roughly) the mean distance of Saturn and the periodic time of Neptune. (Earth's mean distance is 92 millions of miles.)

7. If the synodic period of revolution of an inferior planet were a year, what would be its sidereal period, and what would be its mean distance from the Sun according to Kepler's Third Law ?

8. Jupiter's solar distance is 5·2 times the Earth's solar distance; find the length of time between two conjunctions of the Earth and Jupiter.

9. Saturn's mean distance from the Sun is nine times the Earth's mean distance. Find how long the motion is retrograde, having given $\cos^{-1}\frac{3}{7} = 65°$.

10. Show that if the planets further from the Sun were to move with greater velocity in their orbits than the nearer ones, there would be no stationary points, the relative motion among the stars being always *direct*. What would be the corresponding phenomenon if the velocities of two planets were *equal*?

EXAMINATION PAPER

1. Explain the apparent motion of a superior planet. Illustrate by figures.

2. Describe the apparent course among the stars of an inferior planet as seen from the Earth, and the changes in appearance which the planet undergoes.

3. Define the *sidereal* and *synodic period* of a superior or inferior planet, and find the relation between them. Calculate the synodic period of a superior planet whose period of revolution is thirty years.

4. How is it that Venus alters so little in apparent magnitude (as seen by the naked eye) in her journey round the Sun? Why does not Jupiter exhibit any notable phases?

5. State Bode's Law connecting the mean distance of the various planets from the Sun.

6. Prove that the time of most rapid approach of an inferior planet to the Earth is when its elongation is greatest, and that the velocity of approach is then that under which it would describe its orbit in the synodic period of the Earth and the planet. Give the corresponding results for a superior planet. (The orbits are to be taken circular and in the same plane.)

7. What is meant by *stationary points* in the apparent motion of a planet? Prove that, if a planet Q is stationary as seen from P, then P will be stationary as seen from Q.

8. State Kepler's Three Laws, and, assuming the orbits of the Earth and Venus to be circular, show how the Third Law might be verified by observations of the greatest elongation and synodic period of Venus.

9. Find the periods during which Venus is an evening star and morning star respectively, being given that the mean distance of Venus from the Sun is ·72 of that of the Earth.

10. Having given that there will be a full Moon on the 5th of June, that Mercury and Venus are both evening stars near their greatest elongations, that Mars changed from an evening to a morning star about the vernal equinox, and that Jupiter was in opposition to the Sun on April 21st, draw a figure of the configuration of these heavenly bodies on May 1st. (All these bodies may be supposed to move in one plane.)

CHAPTER XII

PRECESSION AND NUTATION

I. PRECESSION

275. Motion of the First Point of Aries

In Art. 125 it was mentioned that the first point of Aries is not fixed, but has a slow retrograde movement along the ecliptic, amounting to about 50·26″ in a year. Because of this motion of the first point of Aries the sidereal day, which is defined as the interval between two consecutive transits of the first point of Aries, is slightly shorter than the true period of rotation of the Earth (Art. 28).

276. Precession of the Equinoxes

The phenomenon of precession was discovered by Hipparchus about 125 B.C. From the comparison of the positions of the brighter stars given by observations extending over some years, Hipparchus found that the longitude of each star increased by about 50″ a year but that the latitude remained unaltered.

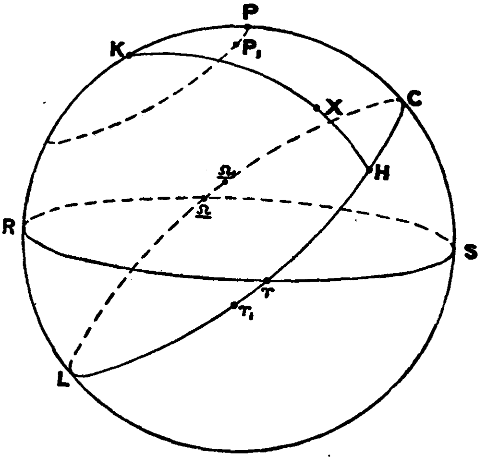

FIG. 99.

In Fig. 99, the great circles R♈S♎, L♈C♎ denote the equator and ecliptic respectively, P, K being their poles. If X is a star and H is the intersection of the ecliptic with the secondary through K and X, the latitude of X is XH; the longitude is ♈H. Hipparchus concluded that, since the latitudes XH do not alter, the ecliptic is a fixed circle on the celestial sphere; but since the longitudes ♈H are increasing by about 50″ a year, the point ♈, the intersection of the equator with the ecliptic, is moving backwards along the ecliptic at this rate.

If ♈$_1$ denotes the position of ♈ at a subsequent epoch, the pole of the equator will have moved from P to a point P_1, the pole of the ecliptic remaining fixed at K. It is found that the inclination of the equator to the ecliptic is not changed by precession. This inclination is the angle C♈S, which is also equal to the angle between the two poles

P and K. The point P_1 must therefore be at the same distance from K as P is; it must accordingly move along a small circle, with K as its pole. The pole of the equator (the celestial pole) completes a circuit of this small circle in the precessional period of 25,800 years.

If we think of the stars as fixed on the celestial sphere (actually, of course, it is the directions to the stars that are fixed), the ecliptic and its pole K being also fixed, whilst the equator, and therefore also ♈, as well as P are moving, it is clear that the right ascensions and declinations of stars are changed by considerable amounts in the course of the precessional cycle. In particular, the Pole Star, which at the present time marks approximately the position of the north celestial pole, will no longer be close to the pole when a few thousand years have elapsed. After about 13,000 years, it will be about 47° from the north pole.

The physical causes of precession are considered in Chap. XVIII.

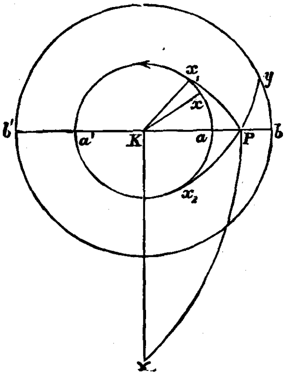

FIG. 100.

277. Various Effects of Precession

Since the R.A. and decl. of a star depend only on the *relative* positions of the star and equator, their variations due to precession are just the same as they would be if the equator and ecliptic were fixed, and the stars had a *direct* motion of rotation, of 50·26″ per annum, about the pole of the ecliptic.

If we make this supposition, the stars will describe circles about K in a period of 25,800 years.

(i) If a star's distance Kx from the pole of the ecliptic is less than the obliquity ϵ, or its latitude (l) greater than $90° - \epsilon$, it will describe a circle $ax_1a'x_2$ (Fig. 100), of radius $90° - l$, not enclosing the pole P, and its greatest and least N.P.D. will be

$$Pa' = \epsilon + (90° - l), \quad Pa = \epsilon - (90° - l).$$

Also the star's R.A. will fluctuate between the values ♈Px_1 and ♈Px_2. Now ♈ is the pole of PK; hence KP♈ is a right angle, and ♈$PK = 270°$; therefore the maximum and minimum R.A. are $270° + KPx_1$, and $270° - KPx_1$.

(ii) If, on the other hand, the star's latitude is $< 90° - \epsilon$, it will describe a circle byb' enclosing the pole P. Its greatest and least N.P.D. will be :—

$$Pb' = (90° - l) + \epsilon. \quad Pb = (90° - l) - \epsilon.$$

The stars R.A. will continually increase from 0° to 360°.

In either case the star's N.P.D. will increase as its longitude increases from 90° (at *a* or *b*) to 270° (at *a′* or *b′*), and will decrease over the other half of the path.

The Pole Star will, after a time, move away from the pole, and its place will be then occupied in succession by other stars whose latitude is very nearly $= 90^\circ - \epsilon = 66^\circ\ 33'$. If l, L be the latitude and longitude of such a star, it will be nearest the pole in an interval of $(90^\circ - L) \div 50{\cdot}26''$ years, and its N.P.D. will then be $(90^\circ - l) \sim \epsilon$.

That precession has shifted the equinoctial points from the constellations Aries and Libra, into Pisces and Virgo, has already been mentioned. Since there are twelve signs of the zodiac, the equinoctial points shift from one constellation into the next in 25,800/12 years, *i.e.* about 2,150 years. This is an average value ; the signs are all equal in breadth but the constellations are not.

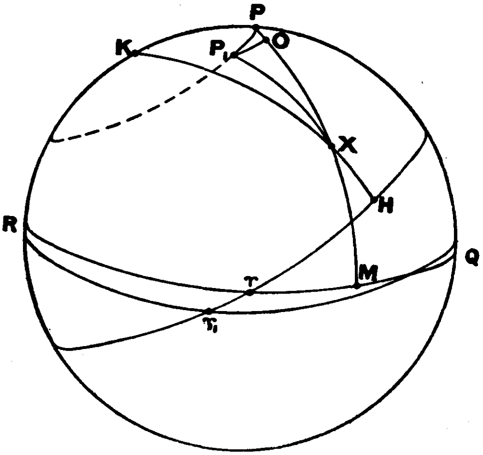

FIG. 101.

278. Precession in Right Ascension and Declination

In the preceding paragraph, the effects of precession on the R.A. and Dec. in the course of long intervals of time have been considered. It is convenient to derive expressions for the changes in R.A. and Dec. produced by precessions in the course of short intervals of time. Several observations of the R.A. and Dec. of a star may be obtained, for instance, in the course of a few years. Before these can be compared with one another, corrections must be applied for the effect of precession.

The expressions for the effect of precession on the right ascension and declination of a star can be derived in the following way.

In Fig. 101, P represents the position of the celestial pole at a certain epoch, P_1 its position one year later when ♈ has moved to $♈_1$, where $♈♈_1 = \Psi$, the precessional motion of the first point of Aries in one year. K is the pole of the ecliptic $L♈C$.

We denote the R.A. and Dec. of the star referred to the equator $R♈Q$ and equinox ♈ by α, δ ; the R.A. and Dec. referred to the equator $R\Upsilon_1 Q$ and equinox Υ_1 by α_1, δ_1. Then $(\alpha_1 - \alpha)$, $(\delta_1 - \delta)$ will be small quantities.

From P_1 drop a perpendicular P_1O to PX. Since PP_1 is small, the triangle P_1PO can be treated as plane. $PX = 90° - \delta$; $OX = P_1X = 90° - \delta_1$. Hence $PO = (\delta_1 - \delta)$. Also the angle $KPX = RM = R\Upsilon + \Upsilon M = 90° + \alpha$. But the angle $KPP_1 = 90°$; hence angle $P_1PO = \alpha$. Also the angle $PKP_1 = \Upsilon\Upsilon_1 = \Psi$ and since $KP = \epsilon$, $PP_1 = \Psi \sin \epsilon$ (by Art. 5). Thus :—

$$PO = \delta_1 - \delta = PP_1 \cos P_1PO$$

$$\text{or} \quad \delta_1 - \delta = \Psi \sin \epsilon \cos \alpha.$$

Also, in the triangle KPX, we have $KX = 90° - b$, where b is the latitude of the star, XH; $KP = \epsilon$; $PX = 90° - \delta$, $\angle KPX = 90° + \alpha$. Hence by formula (1), Art. 10,

$$\sin b = \cos \epsilon \sin \delta - \sin \epsilon \cos \delta \sin \alpha \quad \dots\dots\dots\dots (1)$$

Similarly from the triangle KP_1X, we have :—

$$\sin b = \cos \epsilon \sin \delta_1 - \sin \epsilon \cos \delta_1 \sin \alpha \quad \dots\dots\dots\dots (2)$$

Now we can write, since $(\delta_1 - \delta)$ and $(\alpha_1 - \alpha)$ are small :—

$$\sin \delta_1 = \sin \{\delta + (\delta_1 - \delta)\} = \sin \delta + (\delta_1 - \delta) \cos \delta;$$

$$\cos \delta_1 = \cos \{\delta + (\delta_1 - \delta)\} = \cos \delta - (\delta_1 - \delta) \sin \delta;$$

$$\sin \alpha_1 = \sin \{\alpha + (\alpha_1 - \alpha)\} = \sin \alpha + (\alpha_1 - \alpha) \cos \alpha.$$

By substituting these expressions for $\sin \delta_1$, $\cos \delta$, and $\sin \alpha$, in the right-hand side of (2) and equating to the right hand side of (1), we obtain

$$(\alpha_1 - \alpha) \sin \epsilon \cos \alpha = (\delta_1 - \delta)(\cos \epsilon + \sin \epsilon \tan \delta \sin \alpha).$$

But $\delta_1 - \delta = \Psi \sin \epsilon \cos \alpha$. It follows that :—

$$\alpha_1 - \alpha = \Psi (\cos \epsilon + \sin \epsilon \sin \alpha \tan \delta).$$

The yearly changes in R.A. and Dec. due to the annual precession Ψ are therefore given by :—

R.A. $\quad \Psi \cos \epsilon + \Psi \sin \epsilon . \sin \alpha \tan \delta$

Dec. $\quad \Psi \sin \epsilon . \cos \alpha.$

If Ψ is expressed in seconds of arc, then the effect in R.A. and Dec. will be expressed in seconds of arc. The effect in R.A. is obtained in seconds of time by dividing by 15.

Putting $\Psi = 50{\cdot}265''$, $\epsilon = 23° \ 27'$, we obtain :—

$$\Psi \cos \epsilon = 46{\cdot}11'' = 3^{s}{\cdot}074; \quad \Psi \sin \epsilon = 20{\cdot}02'' = 1^{s}{\cdot}335$$

The annual precessional changes are therefore

in R.A. $\quad + 3^{s}{\cdot}074 + 1^{s}{\cdot}335 \sin \alpha \tan \delta$

in Dec. $\quad + 20{\cdot}02'' \cos \alpha.$

It will be noted that the effect in declination depends on the R.A. alone. The effect in R.A. becomes very great near the Pole, when tan δ becomes large. The formula does not give the precession in R.A. accurately near the Pole, and more exact methods must be used for high declinations. The quantity Ψ is called the *constant of precession.*

II. NUTATION

279. Nutation in Longitude

The precession of the equinoxes is caused by the gravitational attraction of the Sun and Moon on the Earth, which is spheroidal in shape (Art. 98). The manner in which these attractions give rise to precession is discussed in Chapter XVIII. If the Sun and the Moon moved in circular orbits and if the orbit of the Moon were in the same plane as the orbit of the Sun, the motion of the equinoxes would be the uniform motion that we have termed precession. But the distances of both Sun and Moon from the Earth are slightly variable. The orbit of the Moon is, moreover, inclined at an angle of about 5° to the ecliptic, so that the Moon is sometimes above and sometimes below the ecliptic plane; the orbital plane, moreover, does not maintain a constant direction with respect to the stars, its nodes having a retrograde motion of about 19° a year (Art. 211). In consequence, the continued gravitational attraction of the Sun and Moon, which gives rise to the precession of the equinoxes, is variable both in magnitude and direction. The retrograde motion of the equinoxes is, accordingly, not uniform, as already mentioned in Art. 56.

The motion of the equinoxes can be represented as the combination of a uniform motion and a variable motion. The derivation of the expression representing the motion is beyond the scope of this book. The principal terms can be expressed by a formula of the type :—

$$\Psi t + a \sin ☊ + b \sin 2 ☊ + c \sin 2\odot + d \sin 2 ☾ .$$

This expression represents the motion in t years. ☊ denotes the longitude of the Moon's ascending node ; the terms with coefficients a and b are due to the movement of the orbital plane of the Moon. $\odot$ denotes the Sun's longitude ; the term with coefficient c is due to the variation in the distance of the Sun from the Earth. ☾ denotes the Moon's longitude ; the term with coefficient d is due to the variation in the distance of the Moon from the Earth.

The term Ψt corresponds to a uniform motion at of an amount Ψ a year. Ψ is the constant of precession. The terms

$$a \sin ☊ + b \sin 2 ☊ + c \sin 2\odot + d \sin 2 ☾$$

together with a number of smaller terms, which can be derived by theoretical investigation, are termed the *nutation in longitude.* They are usually denoted collectively by $\Delta\Psi$. The most important term is the first term, which has for its period the time of sidereal revolution of the Moon's nodes, *i.e.* about 18 years 220 days.

280. Nutation in Obliquity

The precessional motions of the equinoxes are not accompanied by any change in the obliquity, ϵ,—the inclination of the equator to the ecliptic. But the same causes that give rise to the nutation in longitude produce also small changes in the obliquity, which are usually denoted collectively by $\Delta\epsilon$. The principal terms are of the form

$$\Delta\epsilon = a_1 \cos \Omega + b_1 \cos 2\Omega + c_1 \sin 2\odot + d_1 \sin 2\leftmoon .$$

The combination of the nutation in longitude and the nutation in obliquity can be represented by a motion of the pole of the equator. Instead of describing a small circle around the pole of the ecliptic, as represented in Fig. 101, it moves in a wavy curve about this small circle as a mean position. In Fig. 102, $P_1P_3P_5P_7$ represents a portion of the small circle; $P_1P_2 \ldots P_8$ a portion of the actual path described by the celestial pole.

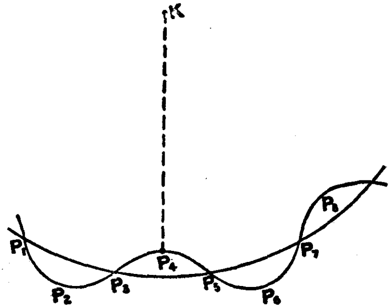

FIG. 102.

The word *nutation* means nodding; it is because of the nodding motion of the pole, illustrated in Fig. 102, that the phenomenon is called nutation.

The movement of the Pole from, say, P_2 to P_4 is necessarily accompanied by a change in the obliquity of the ecliptic, which is measured by the angular distance of the celestial pole, P, from the pole of the ecliptic. The positions P_2, P_6 would correspond to the extreme large values of the obliquity; the positions P_4, P_8 would correspond to the extreme small values.

281. Discovery of Nutation

Nutation was discovered by Bradley soon after his discovery of aberration, while continuing his observations on the star γ *Draconis* and on a small star in the constellation *Camelopardus*, by its effect on the declinations of these stars. The peculiarity which led him to separate nutation from aberration was their difference of period. The period of the former phenomenon is about 19 years, while that of the aberration displacement is only a year. Had the observed variations in declination been due to aberration alone, the declination would always have had the same apparent value at the same time of year, but such was not the case.

Newton had, sixty years previously (1687), proved the existence of nutation from theory, but had supposed that its effects would be inappreciable.

282. Nutation in Right Ascension and Declination

The effect of nutation on the R.A. and Dec. of a star can be derived by considering separately the nutation in longitude and the nutation in obliquity. The effect of nutation in longitude, $\Delta\Psi$, on the R.A. and Dec. can be derived in the same way as for precession in Art. 278 and the formulae there obtained can be used, on replacing Ψ by $\Delta\Psi$. To investigate the effects of the nutation in obliquity, we suppose that ♈ remains fixed, but that the obliquity changes.

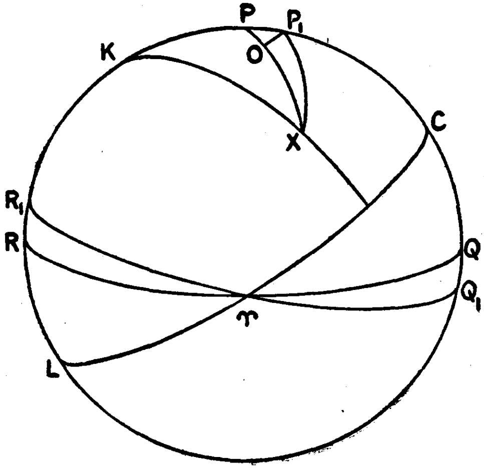

FIG. 103.

In Fig. 103, P is the pole of the equator $R♈Q$ at a certain time; P_1 is the pole of the equator $R_1♈Q_1$ at some subsequent time. The change in the obliquity, $\Delta\epsilon$, is equal to PP_1 or to QQ_1. If X is a star, and from P_1 the perpendicular P_1O is drawn to PX, the small triangle PP_1O can be treated as a plane triangle. Since the angle $♈PQ$ is 90° and $♈PX$ is α, the R.A. of the star, the angle P_1PO is $90° - \alpha$. Also $PX = 90° - \delta$, $P_1X = 90° - \delta_1$ and $P_1X = OX$; therefore $PO = (\delta_1 - \delta)$. But $PO = PP_1 \cos P_1PO$ or:—

$$\delta_1 - \delta = \Delta\epsilon \sin \alpha.$$

In the triangle KPX, the angle $PKX = 90° - l$, the side $KX = 90° - b$, where l, b are the longitude and latitude of X. The angle KPX is $90° + \alpha$ and the side PX is $90° - \delta$. Formula (5) of Art. 10 gives:—

$$\cos \alpha \cos \delta = \cos l \cos b$$

In the same way from triangle KP_1X

$$\cos \alpha_1 \cos \delta_1 = \cos l \cos b.$$

Therefore $$\cos \alpha_1 \cos \delta_1 = \cos \alpha \cos \delta.$$

As $(\alpha_1 - \alpha)$, $(\delta_1 - \delta)$ are small, we can write :—

$$\cos \alpha_1 = \cos \{\alpha + (\alpha_1 - \alpha)\} = \cos \alpha - (\alpha_1 - \alpha) \sin \alpha,$$

$$\cos \delta_1 = \cos \{\delta + (\delta_1 - \delta)\} = \cos \delta - (\delta_1 - \delta) \sin \delta.$$

When $\cos \alpha_1$, $\cos \delta_1$, are multiplied together, the term in $(\alpha_1 - \alpha)(\delta_1 - \delta)$ can be neglected, being the product of two small quantities. We thus obtain :—

$$\cos \alpha \cos \delta - (\alpha_1 - \alpha) \sin \alpha \cos \delta - (\delta_1 - \delta) \cos \alpha \sin \delta = \cos \alpha \cos \delta,$$

$$\text{or} \quad (\alpha_1 - \alpha) = - (\delta_1 - \delta) \cot \alpha \tan \delta.$$

Since $(\delta_1 - \delta) = \Delta\epsilon \sin \alpha$, we obtain :—

$$(\alpha_1 - \alpha) = - \Delta\epsilon \cos \alpha \tan \delta.$$

The combined effect of the nutation in longitude, $\Delta\Psi$, and the nutation in obliquity, $\Delta\epsilon$, is thus :

$$\alpha_1 - \alpha = (\cos \epsilon + \sin \epsilon \sin \alpha \tan \delta) \Delta\Psi - \cos \alpha \tan \delta \, \Delta\epsilon$$

$$\delta_1 - \delta = \sin \epsilon \cos \alpha \, \Delta\Psi + \sin \alpha \, \Delta\epsilon.$$

III. APPARENT AND MEAN PLACES OF A STAR

283. Apparent Place, Mean Place and True Place of a Star

Suppose a number of observations of a star or other celestial body have been obtained on different nights and it is desired to compare them and to combine them to give the most accurate position. How is this to be done? Because of the motion of the Earth round the Sun between the observations, the displacements caused by aberration are different for the various observations; because of the movement of the equator and equinox, due to precession and nutation, between the observations, the observed positions are not directly comparable, even after the displacements due to aberration have been allowed for. In the case of a relatively near body, such as the Moon, a planet, or the Sun, the observed positions will be affected by parallax; in the case of a star, they will be affected by annual parallax, though for the majority of the stars the displacements caused by annual parallax are so small that they can be neglected. In order to compare observations at different times, it is necessary to refer them to a common basis. The way in which this is done will now be described.

The *apparent place* of a star or other celestial body is its position on the celestial sphere, as it would be seen from the centre of the moving Earth, referred to the true equator and true equinox at the instant of observation.

The position of a star, as observed with the meridian circle, after correction for refraction, is the apparent place. The position of a planet, as observed with the meridian circle, must be corrected for refraction and parallax to obtain the apparent place of the planet.

The *true place* of a star is its position on the celestial sphere, as it would be seen from the Sun, referred to the true equator and true equinox at the instant of observation.

The apparent place is equal to the true place plus the corrections due to aberration and annual parallax.

The *mean place* of a star is its position on the celestial sphere, as it would be seen from the Sun, referred to the *mean* equator and equinox at the beginning of a year (so that the effect of nutation is neglected).

If observations of a star, made at different times, are reduced to mean places for the beginning of the same year they become directly comparable with one another.

To reduce from apparent place at one time to the mean place at another time, corrections must be applied for the effects of the following :—

1. Precession.
2. Nutation.
3. Aberration.
4. Annual parallax.
5. The motion of the star itself in the heavens, known as its proper motion.

Of these causes, 3, 4, 5 change the actual direction in which the star is seen ; 1 and 2 do not affect the actual direction of the star but change the co-ordinates by means of which the position of the star is expressed.

It is usual to reduce from the apparent or observed position to the mean position for the beginning of the year of observation ; if observations in different years are to be compared, it is then necessary to reduce from the mean position for the beginning of one year to the mean position for the beginning of another year. These two steps will be considered in turn.

The correction for annual parallax can usually be neglected, being very small except for the few nearest stars. If the star has an appreciable parallax, corrections can be applied as described in Art. 173.

284. Reduction from Apparent Place to Mean Place for the beginning of the Year of Observation, or vice versa

We suppose that (α, δ) are the apparent R.A. and Dec. and that (α_0, δ_0) are the mean R.A. and Dec. for the beginning of the year. Let τ be the fraction of the year that has elapsed at the time of observation ; then $\Psi\tau$ is the motion of the equinox in this interval. $\Delta\Psi$, $\Delta\epsilon$ denote the nutation in longitude and in obliquity in the same interval.

(*a*) Consider first the effects of *precession and nutation.* From Arts. 278, 282, we obtain :—

$$\alpha - \alpha_0 = (\cos \epsilon + \sin \epsilon \sin \alpha \tan \delta)(\Psi\tau + \Delta\Psi) - \cos \alpha \tan \delta \Delta\epsilon,$$

$$\delta - \delta_0 = \sin \epsilon \cos \alpha \, (\Psi\tau + \Delta\Psi) + \sin \alpha \, \Delta\epsilon.$$

Put $\Psi \cos \epsilon = m$; $\Psi \sin \epsilon = n$,

$$A = \tau + \frac{\Delta\Psi}{\Psi};\ B = -\Delta\epsilon;$$

$$a = m^s + n^s \sin \alpha \tan \delta \quad \left(\text{where } m^s = \frac{m''}{15},\ n^s = \frac{n''}{15}\right),$$

$$b = \frac{1}{15} \cos \alpha \tan \delta; \qquad a' = n'' \cos \alpha; \qquad b' = -\sin \alpha.$$

We then obtain :—

$\alpha - \alpha_0 = Aa + Bb$ (expressed in seconds of time).

$\delta - \delta_0 = Aa' + Bb'$ (expressed in seconds of arc)

(*b*) Consider next the effects of *aberration.* From Art. 183, remembering that the expressions there given are to be added to apparent R,A. (α, δ) to obtain true R.A. (α_0, δ_0), we have :—

$$\alpha - \alpha_0 = -k \sec \delta\ (\sin \alpha \sin l_0 + \cos \epsilon \cos \alpha \cos l_0)$$

$$\delta - \delta_0 = -k \{\sin \delta\ (\cos \alpha \sin l_0 - \cos \epsilon \sin \alpha \cos l_0) + \sin \epsilon \cos \delta \cos l_0\}.$$

Put $C = -k \cos \epsilon \cos l_0$ $\qquad D = -k \sin l_0$

$$c = \frac{1}{15} \cos \alpha \sec \delta, \qquad c' = \tan \epsilon \cos \delta - \sin \alpha \sin \delta,$$

$$d = \frac{1}{15} \sin \alpha \sec \delta, \qquad d' = \cos \alpha \sin \delta.$$

We then obtain :—

$\alpha - \alpha_0 = Cc + Dd$ (expressed in seconds of time),

$\delta - \delta_0 = Cc' + Dd'$ (expressed in seconds of arc).

(*c*) Consider finally the *effects of proper-motion.* The motion of a star in space will cause a displacement of the position of the star on the celestial sphere, unless the motion of the star happens to be in the line of sight to or from the observer. The proper-motion of a star is its angular displacement on the celestial sphere. If the annual proper-motion in R.A. is μ seconds of time and on Dec. is μ' seconds of arc, then in the interval τ from the beginning of the year, we have

$$\alpha - \alpha_0 = \mu\tau, \quad \delta - \delta_0 = \mu'\tau.$$

285. The Besselian Day Numbers

Combining together the various terms, we have

$$\alpha - \alpha_0 = Aa + Bb + Cc + Dd + E + \mu\tau \quad \ldots\ldots\ldots(1)$$

$$\delta - \delta_0 = Aa' + Bb' + Cc' + Dd' + \mu'\tau \quad \ldots\ldots\ldots\ldots(2)$$

The term E in the first of these has been added for the sake of completeness. It is a small precessional term caused by the action of the planets, which affect all stars equally.

In these expressions, the quantities a, b, c, d and a', b', c', d' do not depend upon the time of observation, but only upon the R.A. and Dec. of the star, and the quantities m, n. As m, n, α, δ vary slowly, the quantities a, b, c, d and a', b', c', d' for a given star can be considered as constant for several years. They are called *star constants*.

The quantities A, B, C, D and E are independent of the position of the star but vary from day to day. They were first introduced by the astronomer Bessel and are called the *Besselian day numbers*. Their values, which are the same for any star, are tabulated for each day of the year in the *Nautical Almanac*.

To determine the apparent place of a star at a given date from the mean place at the beginning of the year, or *vice versa*, the star constants a, b, c, d and a', b', c', d' are calculated; the Besselian day numbers are taken from the *Nautical Almanac* for the appropriate date. The formulae, (1) and (2), which include also the effect of proper-motion, give the required reduction from mean place to apparent place or *vice versa*.

286. The Independent Day Numbers

If the values for the star constants are substituted in the formulae (1) and (2) in Art. 285, the terms involving precession and nutation become :—

$$\begin{aligned} Aa + Bb + E &= Am + (An \sin \alpha + B \cos \alpha) \tan \delta + E \\ Aa' + Bb' &= An \cos \alpha - B \sin \alpha. \end{aligned}$$

We introduce three new quantities g, G, f defined by :—

$$g \sin G = B; \qquad g \cos G = An; \qquad f = Am + E.$$

We then obtain

$$\begin{aligned} Aa + Bb + E &= f + g \sin (G + \alpha) \tan \delta; \\ Aa' + Bb' &= g \cos (G + \alpha) \end{aligned}$$

The terms involving aberration become :—

$$\begin{aligned} Cc + Dd &= (C \cos \alpha + D \sin \alpha) \sec \delta. \\ Cc' + Dd' &= C \tan \epsilon \cos \delta + (D \cos \alpha - C \sin \alpha) \sin \delta. \end{aligned}$$

We introduce three new quantities h, H, i defined by :—

$$h \sin H = C; \qquad h \cos H = D; \qquad i = C \tan \epsilon.$$

Then

$$\begin{aligned} Cc + Dd &= h \sin (H + \alpha) \sec \delta. \\ Cc' + Dd' &= h \cos (H + \alpha) \sin \delta + i \cos \delta. \end{aligned}$$

We thus obtain :—

$$\alpha - \alpha_0 = f + g \sin (G + \alpha) \tan \delta + h \sin (H + \alpha) \sec \delta + \mu\tau \ldots (1)$$

$$\delta - \delta_0 = g \cos (G + \alpha) + h \cos (H + \alpha) \sin \delta + i \cos \delta + \mu'\tau \ldots (2)$$

The quantities f, g, h, i, G, H do not depend on α or δ; they are the same for all stars at the same time. They are known as the *Independent Day Numbers* and their values are tabulated in the *Nautical Almanac* for each day of the year. The formulae (1), (2) can be used to obtain apparent place from mean place or *vice versa*, by extracting the appropriate values of the independent day numbers from the *Nautical Almanac* and substituting the appropriate values of α and δ. By the use of the independent day numbers, the computation of the star constants is avoided. If a single position of a star has to be reduced from mean to apparent place or *vice versa*, it is generally more convenient to use the independent day numbers; if a number of positions of the same star have to be computed, it is more convenient to use the Besselian day numbers, because the same star constants are required for each reduction.

Examples.—1. *The mean place of Capella for* 1942 *is R.A.* 5*h.* 12*m.* 24·01*s. Dec.* 45° 56′ 28·6″ *N. Find the apparent place on October 27th,* 1942, *using the Besselian day numbers. The proper-motion of Capella is* + 0ˢ·0081 *in R.A. and* —0·422″ *in Dec. per year.*

R.A. 5h. 12m. 24·01s. = 78° 6′ 0″;

$\sin\alpha = +\cdot 9785 \quad \cos\alpha = +\cdot 2062;$

$\sin\delta = +\cdot 7186 \quad \cos\delta = +\cdot 6954 \quad \sec\delta = +1\cdot 4378 \quad \tan\delta = +1\cdot 0331$

Therefore $\sin\alpha\tan\delta = +1\cdot 0109 \qquad \cos\alpha\tan\delta = +\cdot 2130$

$\sin\alpha\sec\delta = +1\cdot 4069 \qquad \cos\alpha\sec\delta = +\cdot 2965$

From the *Nautical Almanac* we find that for 1942

$m = 3^s\cdot 07312 \quad n = 1^s\cdot 33622 = 20\cdot 0432'' \quad \tan\epsilon = +\cdot 4337$

From the formulae for a, b, c, d; a', b', c', d', we find that

$a = +4\cdot 4239 \quad b = +\cdot 0142 \quad c = +\cdot 0198 \quad d = +\cdot 0938$

$a' = +4\cdot 133 \quad b' = -\cdot 9785 \quad c' = -\cdot 4014 \quad d' = +\cdot 1482$

Also from the *Nautical Almanac* for 1942, for October 27th.

$A = +\cdot 6233 \quad B = +7\cdot 88 \quad C = +15\cdot 76 \quad D = +11\cdot 13 \quad E = -\cdot 001$

Also $\tau = \cdot 819$, so that $\mu\tau = +0^s\cdot 007$; $\mu'\tau = -0\cdot 346''$.

We find that :—

$$\begin{aligned}\alpha - \alpha_0 &= Aa + Bb + Cc + Dd + E + \mu\tau\\ &= +2\cdot 757 + 0\cdot 112 + 0\cdot 312 + 0\cdot 043 - 0\cdot 001 + 0\cdot 007 = +4^s\cdot 23\\ \delta - \delta_0 &= Aa' + Bb' + Cc' + Dd' + \mu'\tau\\ &= +2\cdot 58 - 7\cdot 71 - 6\cdot 33 + 1\cdot 65 - 0\cdot 35 = -10\cdot 2''.\end{aligned}$$

Hence apparent place on October 27th is :—

R.A. 5h. 12m. 28·24s.; Dec. 45° 56′ 18·4″

2. *Find the apparent place of Capella for the same date, using the independent day numbers.*

From the *Nautical Almanac* for 1942 we obtain, for October 27th :—

$f = +1^s\cdot 914, \quad g = 14\cdot 77'', \quad h = 19\cdot 30'', \quad i = +6\cdot 84''$

$G = 2\text{h. } 9\cdot 0\text{m}, \quad H = 3\text{h. } 39\cdot 1\text{m}.$

We have :— $G + \alpha = 7\text{h. } 21\cdot 4\text{m.} = 110° 21'$

$\sin(G + \alpha) = +\cdot 9376 \qquad \cos(G + \alpha) = -\cdot 3477$

$H + \alpha = 8\text{h. } 51\cdot 5\text{m.} = 132° 52' 30''$

$\sin(H + \alpha) = +\cdot 7328 \quad \cos(H + \alpha) = -\cdot 6804.$

Whence $g \sin (G + \alpha) \tan \delta = + 14{\cdot}306'' = + 0^s{\cdot}954$

$h \sin (H + \alpha) \sec \delta = + 20{\cdot}334'' = + 1^s{\cdot}356$

and $\alpha - \alpha_0 = f + \frac{1}{15} g \sin (G + \alpha) \tan \delta + \frac{1}{15} h \sin (H + \alpha) \sec \delta + \mu\tau$

$= + 1^s{\cdot}914 + 0^s{\cdot}954 + 1^s{\cdot}356 + 0^s{\cdot}007 = + 4^s{\cdot}23$

Also $g \cos (G + \alpha) = - 5{\cdot}14''$

$h \cos (H + \alpha) \sin \delta = - 9{\cdot}44'' \qquad i \cos \delta = + 4{\cdot}76''$

and $\delta - \delta_0 = g \cos (G + \alpha) + h \cos (H + \alpha) \sin \delta + i \cos \delta + \mu'\tau$

$= - 5{\cdot}14'' - 9{\cdot}44'' + 4{\cdot}76'' - 0{\cdot}35'' = - 10{\cdot}2''.$

which agree with the values for $(\alpha - \alpha_0)$, $(\delta - \delta_0)$ found in Example 1.

287. To Reduce from Mean Place for One Year to the Mean Place for Another Year

The mean place is referred to the mean equator and equinox, *i.e.* the position that the equinox would have if it moved uniformly and was not affected by nutation. To transfer the mean position of a star from one epoch to another epoch, we have therefore to take into account only the effects of precession and the proper-motion of the star. If the precession were constant in value, the effect of precession would be obtained by multiplying the precession in R.A. and Dec. by the interval in time. But the precessional constants, m and n (Art. 284) change slowly with the time and it is necessary to take this change into account.

Neglecting proper-motion and considering only the effect of precession, if α_0 is the R.A. at time $t = 0$ and α is the R.A. at time t, we can write

$$\alpha = \alpha_0 + p \,.\, t + \tfrac{1}{2} s \,.\, t^2.$$

The quantity s is called the *secular variation,* because it measures the rate of change of the precession p. This may be seen as follows: if α_1, α_2 denote the values of α at times t_1, t_2 we have

$$(\alpha_2 - \alpha_1) = p\,(t_2 - t_1) + \tfrac{1}{2} s\,(t_2^2 - t_1^2),$$

$$\text{or} \quad \frac{\alpha_2 - \alpha_1}{t_2 - t_1} = p + \tfrac{1}{2} s\,(t_1 + t_2).$$

The quantity on the right hand side of the equation is the average rate of change of α in the time interval $(t_2 - t_1)$. When t_2 is very near to t_1, we obtain, as the rate of change of α *at the time* t_1, $p + st_1$. It is clear that s measures the rate of change of the precession with the time.

Thus if p is the precession at time $t = o$, the precession at the time $t = \dfrac{t_1}{2}$ is $p + \frac{1}{2} st_1$.

But

$$\begin{aligned} \alpha_1 &= \alpha_0 + p.t_1 + \tfrac{1}{2} st_1^2 \\ &= \alpha_0 + (p + \tfrac{1}{2} st_1)\, t_1 \\ &= \alpha_0 + \left(\text{precession at time } \frac{t_1}{2}\right) \times t_1. \end{aligned}$$

Thus we can obtain the mean place at time $t = t_1$ from the mean place at time $t = 0$ by adding to the latter the precession at the mid-epoch multiplied by the time interval. If the precession at this epoch is not known, we can compute it if we know the precession and secular variation for another epoch.

It is usual to give in star catalogues the *annual precession* for the epoch of the catalogue and the *secular variation per century.* When the annual proper-motion of the star is known, it is combined with the annual precession to give the *annual variation.*

The formula which gives the right ascension at a subsequent epoch is then $a = a_0 + pt + \frac{s}{200} t^2$ with a similar formula for the declination.

Example.—*Compute the mean position of Capella for* 1950, *given that the position for* 1925 *is R.A. 5h.* 11*m.* 8·68*s., Dec.* + 45° 55′ 24·6″. *The annual variations for* 1925 *in R.A. and Dec. are* + $4^s{\cdot}4299$ *and* + 3·818″ *respectively, and the secular variations* + $0^s{\cdot}0152$ *and* — 0·634″.

$$\text{Here } \frac{t^2}{200} = \frac{25 \times 25}{200} = 3{\cdot}125$$

We have :—

(α, δ) 1950	5^h	11^m	$8{\cdot}68^s$	+	45°	55′	24·6″
25 × ann. var.	+	1	50·75	+		1	35·5
$3\frac{1}{8}$ × sec. var.	+		·05	—			2·0
(α, δ) 1925	$5^{h\cdot}$	$12^{m\cdot}$	$59{\cdot}48^s$	+	45°	56′	581″

EXAMPLES

1. Show that, owing to precession, the right ascension of a star at a greater distance than $23\frac{1}{2}$° from the pole of the ecliptic will undergo all possible changes, but that a star at a less distance than $23\frac{1}{2}$° will always have a right ascension greater than twelve hours.

2. Prove that for a short time precession does not alter the declinations of stars whose right ascensions are 6h., or 18h.

3. Exhibit in a diagram the position of the pole star (R.A. = 1h. 37m., decl. = 88° 56′) relative to the poles of the equator and ecliptic, and hence show that owing to precession its R.A. is increasing rapidly, but that its polar distance is decreasing.

4. The present position of a star is R.A. 18h., decl. 40°S. Find its position at the end of half the precessional period.

5. The present position of a star is R.A. 18h. del. 23° 27′S. Find its position at the end of one quarter of the precessional period.

6. Given that the precessional constants, m, n for 1900 are $m = 3^s{\cdot}0723$, $n = 1^s{\cdot}3365 = 20{\cdot}047''$, find the position of a star for which the precessions in both R.A. and decl. are zero.

7. Given that the precessional constants for 1900 are as in Example 6 and for 1950 are $m = 3^s{\cdot}0733$, $n = 1^s{\cdot}3362 = 20{\cdot}043''$, find the mean place for 1950 of a star whose mean place for 1900 is R.A. 0h. decl. 45°N.

8. With the same data as in the preceding question, find the mean place for 1900 of a star whose mean place for 1950 is R.A. 6h. decl. 45°S.

9. Find the mean position of γ Mensae for 1915, given that the mean position for 1950 is R.A. $5^h\ 33^m\ 51{\cdot}20^s$, decl. 76° 22′ 39·9S″. and that the annual and secular variations are $-\,2^s{\cdot}3737$ and $+\,0^s{\cdot}0482$ in R.A.; $+\,2{\cdot}576''$ and $+\,0{\cdot}338''$ in decl.

10. Show that the sidereal day, defined as the period of rotation of the Earth with respect to the First Point of Aries is 0·009 seconds shorter than the period of rotation with respect to the stars.

EXAMINATION PAPER

1. Give a general description of precession. Does precession change the position of (*a*) the equator, (*b*) the ecliptic among the stars?

2. Show that the precessions in R.A. and decl. can be expressed in the form $m + n \sin \alpha \tan \delta$ and $n \cos \alpha$ respectively.

3. Define the apparent place, true place and mean place of a star. Explain how to obtain the mean place at one epoch from the mean place at another epoch.

4. Explain how the Besselian day numbers are used to obtain the apparent place of a star from its mean place at the beginning of the year.

5. The R.A. and decl. of a star are changed by precession, nutation, aberration and annual parallax. The first two of these do not change the direction in which the star is seen, but the direction is changed by the last two. Explain why this is so.

6. Describe the general sequence of changes of the R.A. and decl. of the present Pole Star in the course of the precessional cycle.

CHAPTER XIII

THE OBSERVATORY

288. Determination of R.A. and Dec.

One of the most important problems of practical astronomy is to determine, by observation, the right ascension and declination of a celestial body. We have seen in Chapter II that these co-ordinates not only suffice to fix the position of a star relative to neighbouring stars, but they also enable us to find the direction in which the star may be seen from a given place at a given time of day on a given date (Art. 42). Moreover, it is evident that by determining every day the declination and right ascension of the Sun, the Moon, or a planet, the paths of these bodies relative to the stars can be mapped out on the celestial sphere and their motions investigated.

In Section II of Chapter II we showed that the right ascension and declination of a star can be determined by observations made when the star is on the meridian. We proved the following results :—

The star's R.A. measured in time is equal to the time of transit indicated by a sidereal clock (Art. 29).

The star's north decl. δ can be found from z its meridian zenith distance, and ϕ the latitude of the observatory by the formula

$$\delta = \phi + z$$

where if the decl. is south δ is negative, and if the star transits south of the zenith z is negative (Art. 30).

Lastly, ϕ can be found by observing the altitudes of a circumpolar star at its two culminations, and is therefore known (Art. 31).

Hence the most essential requisites of an observatory must include (i) a clock to measure sidereal time, (ii) a telescope so fitted as to be always pointed in the meridian, provided with graduated circles to measure its inclination to the vertical, and with certain marks to fix the position of a star in its field of view.

289. The Sidereal Clock

The *Sidereal Clock is* a clock regulated to indicate sidereal time. It should be set to mark 0h. 0m. 0s. at the time when the first point of Aries crosses the meridian. It will gain about 4 minutes per day on an ordinary clock, or a whole day in the course of a year.

A clock of high precision is needed as the standard clock of an observatory. The quality of a clock is judged by the uniformity of its rate. If the rate is uniform, the error of the clock at any instant can be accurately interpolated from adjacent determinations of its error, but if the rate is subject to erratic variations the interpolated error will not be the true error. It is customary for precision clocks to be enclosed in airtight cases, so that the pressure within the case remains constant. The rate of the clock is thus freed from the effects of changing atmospheric pressure. It is also usual for the clock to be mounted in a chamber maintained at a uniform temperature, for it is not possible to compensate the pendulum so accurately that changes of rate with changes of temperature are entirely eliminated.

A pendulum clock is merely a device to count up the swings of the pendulum and any uniformly recurring phenomenon that can be counted may be adapted as a clock. Thus a tuning fork may be maintained in vibration by electrical means and the vibrations can be counted. The most accurate type of clock consists of a small quartz bar or ring maintained in vibration at a rapid rate, usually about 100,000 vibrations a second. These vibrations are counted up electrically. The highest precision pendulum clocks are accurate to about 0·01 seconds a day, whilst quartz crystal clocks give an accuracy of about 0·001 seconds a day.

The astronomical clock is provided with means for sending an electrical signal at uniform intervals, usually every second, which can be recorded on a chronograph.

290. The Astronomical Telescope

The *Astronomical Telescope* (Fig. 104) consists essentially of two convex lenses, or systems of lenses, O and O', fixed at opposite ends of a metal tube, and called the *object-glass* and *eye-piece* respectively. The former lens receives the rays of light from the stars or other distant objects, and forms an inverted image (ab) of the objects. The centre O of the round object-glass is called its optical centre and the image is produced as follows :—Let AAA be a pencil of rays from a distant star. By traversing the object-glass these rays are refracted or bent towards the middle ray AO, which alone is unchanged in direction. The rays all converge to a common point or focus at a point a in AO produced, and, if received by the eye after passing a, they would appear to emanate from a luminous point or image of the star at a.

Similarly, the rays BBB, coming from another distant star, will converge to a focus at a point b in BO produced, and will give the effect of an image of the star at b. All these images (a, b) lie in a certain plane FN, called the focal plane of the object-glass, and they form a kind of picture or image of such stars as are in the field of view.

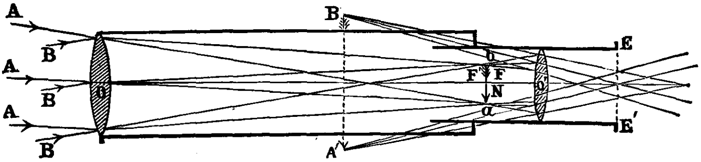

FIG. 104.

The eye-piece O' acts as a kind of magnifying glass, and enlarges the image ab just as if it were a small object placed in the focal plane FN. The figure shows how a second image $A'B'$ is formed by the direction of the pencils of light after refraction through O'. This is the final image seen on looking through the telescope. The eye must be placed in the plane EE, so as to receive the pencils from A', B'.

If, now, a framework of fine wires or spider's threads (Fig. 106) be stretched across the tube in the focal plane FN, these wires, together with the image (ab), will be equally magnified by the eye-piece. They will thus be seen in focus simultaneously with the stars, and the field of view will appear crossed by a series of perfectly distant lines, which will enable us to fix any star's position, and thus determine its exact direction in space. Suppose, for example, that we have two wires crossing one another at the point F', and the telescope is so adjusted that the image of a star coincides with F', then we know that the star lies in the line joining F' to the optical centre O of the object-glass.

The *scale* of the telescope, *i.e.* the linear distance in the focal plane corresponding to a given angular separation on the celestial sphere, is proportional to the focal length of the object-glass, *F*. The *angular magnification*, or the ratio of the apparent angular separation of the images *A′*, *B′* to the angular separation of the stars *A*, *B*, is measured by *F*/*f*, the ratio of the focal length of the object glass, *F*, to the focal length of the eye-piece, *f*. The magnification obtainable with a given telescope can be increased by using an eye-piece of shorter focal-length.

291. The Transit Circle

The *Transit Circle* (Figs. 105, 107) is the instrument used for determining both right ascension and declination. It consists of a telescope *ST*, attached perpendicularly to a rigid axis, *WPPE*, hollow in the interior. The extremities of this axis are made in the form of cylindrical pivots, *E*, *W*, which are capable of revolving freely in two fixed forks, called Y's, from their shape. These Y's rest on piers of solid stone, built on the firmest possible foundations, and they are carefully fixed, so as to keep the axis horizontal and pointing due east and west.

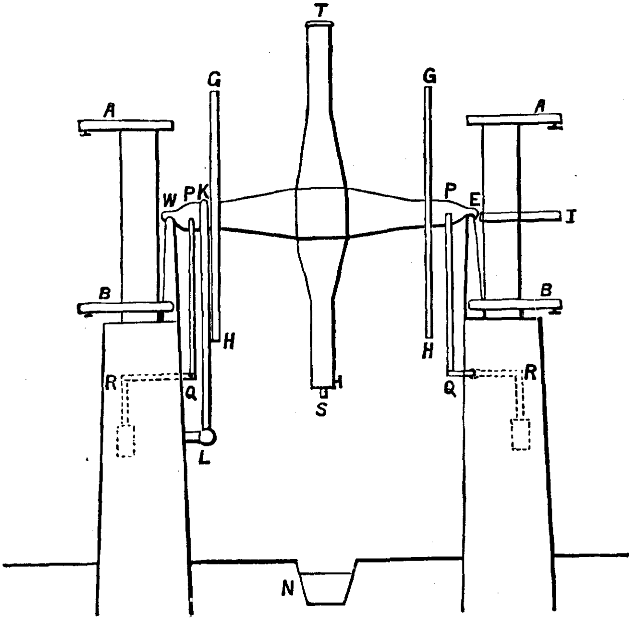

FIG. 105.

In order to reduce the wear of the pivots through friction, some form of counterpoise is employed to reduce the pressure on the bearings. In the device shown in the figure, the axis of the telescope is partially supported at *P*, *P* upon friction rollers (not represented in the figure) attached to a system of levers (Q, Q) and counterpoises (R, R) placed within the piers. These support about four-fifths of the weight of the telescope, leaving sufficient pressure on the Y's to ensure their keeping the axis fixed.

Within the telescope tube, in the focal plane of the object-glass (Art. 290), is fixed a framework of cross wires, presenting the appearance shown in Fig. 106. Five, or sometimes seven, wires appear vertical, and two appear horizontal. Of the latter, one bisects the field of view; the other is movable up and down by means of a screw, whose head is divided by graduation marks which indicate the position of the wire.

The line joining the optical centre of the object-glass to the point of intersection of the middle vertical wire with the fixed horizontal wire is called the *Line of Collimation*. The wires should be so adjusted that the line of collimation is perpendicular to the axis about which the telescope turns. For this purpose the framework carrying the wires can be moved horizontally, by means of a screw, into the right position. If the Y's have been accurately fixed, then, as the telescope turns, the line of collimation will always lie in the plane of the meridian. Hence, when a star transits we shall, on looking through the telescope, see it pass across the middle vertical wire.

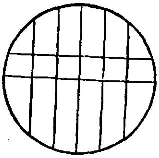

FIG. 106.

Attached to the axis of the telescope, and turning with it, are two *graduated circles*, *GH*, having their circumferences divided into degrees, and further subdivided at (usually) intervals of 5′. By means of these graduations the inclination of the line of collimation to the vertical is read off by aid of several fixed compound microscopes, *A*, *I*, *B*, pointed towards the circle. One of these microscopes (*I*), called the *Pointer* or *Index*, is of low magnifying power, and shows by inspection the number of degrees and subdivisions in the mark of the circle, which is opposite a wire bisecting its field of view. The pointer should read zero when the line of collimation points to the zenith, and the graduations increase as the telescope is turned northwards.

292. Reading Microscopes

In addition to the pointer there are four (sometimes six) other microscopes, called *Reading Microscopes*, arranged symmetrically round each circle, as at *ABCD* (Fig. 107). These serve to determine the number of minutes and seconds in the inclination of the telescope, by means of the following arrangement. Inside the tube of each microscope in the focal plane of its object-glass* is fixed a graduated scale *NL* (Fig. 108) in the form of a strip of metal with fine teeth or notches. This scale, and the image of the telescope circle, formed by the object-glass of the microscope, are simultaneously viewed by the eye-glass, and present the appearance shown in Fig. 108.

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* A compound microscope, like a telescope, consists of an object-glass, which forms an image of an object, and an eye-piece which enlarges this image. A scale of wires fixed in the plane of the image will, therefore, be seen in distinct focus, like the wires in the telescope.

A small hole ○ marks the middle notch, and 5 notches correspond to a division of the telescope circle, hence the number of notches from the hole to the next division of the circle gives the number of minutes to be added to the pointer reading.

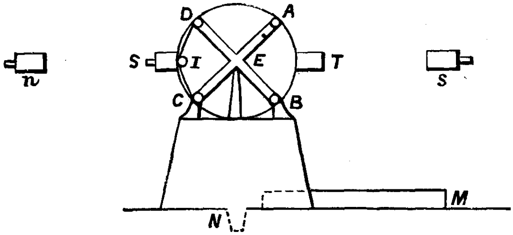

FIG. 107.

To read off the number of seconds, a pair of parallel wires, *SE*, are attached to a framework, and can be moved across the field of view by means of a screw. One whole turn takes the wires from one notch of the metal scale to the next, *i.e.* over a space representing 1′ on the telescope circle; and the head of the screw is divided into 60 parts, each, therefore, representing 1″. The wires are adjusted so that the graduation on the telescope circle appears midway between them, and the reading of the screw-head then gives the number of seconds. With practice, tenths of a second can be estimated.

The four microscopes of one of the circles are all read, and the best result is obtained by taking the mean of the readings.

293. Clamp and Tangent Screw

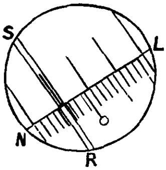

FIG. 108.

When it is required to rotate the telescope of the transit circle very slowly, this is done by means of the bar represented at *LK* in Fig. 105. The telescope axis may be firmly clamped to this bar by means of a *clamp* (not represented in the figure) which grips the rim of one of the circles. When this has been done, the bar *KL*, and with it the telescope, may be slowly turned by means of a horizontal screw at *L*, called the *Tangent Screw*, and provided with a long handle attached to it by a "universal joint." This handle is held by the observer, and he can thus turn the tangent screw without ceasing to watch the stars.

294. Arrangements for Illumination

As most observations are conducted at night, the wires in the telescope and the graduations of the circles must be illuminated. This is done by a lamp placed in front of one of the pivots, the light from which is concentrated by means of a lens in front and a mirror behind. Part of the rays are reflected, by a complicated arrangement of mirrors and prisms, so as to illuminate the parts of the graduated circle viewed by the microscopes. The rest of the light passes down the hollow axis to a ring-shaped mirror, whence it is reflected up to the wires; thus the wires appear as dark lines on a faintly illuminated background. An alternative method is to have a small reflector, with a matt white surface, cemented to the centre of the object-glass, and illuminated by a small electric lamp at the side of the tube. This secures that the illuminating light and the starlight are parallel; this is important, as otherwise an apparent shift in the position of the wires takes place.

There is also another arrangement for illuminating the wires from in front, if desired, so that they appear bright on a dark ground. This is not, however, advisable, as it is found that the bright wires are liable to an appreciable apparent shift, compared with the dark ones.

295. Taking a Transit

If a star is to be observed with the transit circle, its R.A. and decl. must have been *roughly* estimated beforehand; hence, its meridian Z.D. [= (star's decl.) — (observer's lat.)] is known roughly. Before the star is expected to cross the meridian, the telescope is turned by hand until the pointer indicates this roughly determined Z.D.; this adjustment is sufficiently accurate to ensure the star traversing the field of view. The telescope is then clamped. The star is soon seen to enter the field of view, the diurnal rotation of the Earth causing it to travel through the field of view in a horizontal direction. The observer records the instant of its passage across each of the vertical wires by pressing a button on the instrument, which closes an electric circuit and sends a current through to the *Chronograph.* This is an instrument which may take various forms: the two that are most common are the barrel chronograph and the tape chronograph. Whatever form it takes, the signals from the clock are automatically recorded along with the signals sent from the instrument by the observer. The times by the clock at which these signals are sent can be read off at leisure after the observations have been completed.

The barrel chronograph consists of a cylindrical barrel which is made to turn slowly and uniformly about an axle by clockwork. The barrel is covered with paper and a pen, mounted on a carriage, which is traversed slowly along as the barrel turns, describes a spiral trace on the paper. The pen is supported from the armature of a small electromagnet and when a current is sent through this the armature gives a slight flick, causing the pen to make a kick at right angles to the direction of the trace.

In the tape chronograph, a paper tape is fed uniformly through the instrument and the pen records on this tape. The pen may take the form of a fine syphon tube, mounted from the coil of a loud speaker wire, which has little inertia and a rapid response.

296. The Impersonal Micrometer

A further development in recent years is to employ a single movable wire instead of several fixed ones for recording the transit of the star. The single wire is traversed along by a screw which terminates in two large heads, one east, one west of the tube. These heads are turned by the two hands, which are shifted alternately, so that the motion can be made smooth and continuous. The observer keeps the wire central on the star while it is passing the central part of the field. An agate disc is attached to one of the heads, which has metal studs inserted, flush with the surface, at certain points in its rim ; as these studs pass a metal point pressed against the rim by a spring, a current passes to the chronograph. The registration is thus automatic, and it is found that "Personal Equation" (see Art. 301) is almost entirely eliminated. The positions of the contacts relatively to the line of collimation are found by placing the metal point centrally on each stud, and reading the micrometer head.

In some instruments the wire is moved automatically by an electric motor, there being a device for modifying the speed to correspond with the star's declination. In this case, the observer has only to apply the small corrections necessary to keep the wire central on the star.

297. Corrections

After the transit of a star has been observed, certain corrections have to be allowed for in practice before its true R.A. and decl. are obtained. These corrections, which depend on errors of adjustment, observation, etc., may be conveniently classified as follows :—

A.—Corrections required for the Right Ascension :

1. Error and rate of the astronomical clock.
2. Personal equation of the observer.
3. Errors of adjustment of the transit circle, including
    - (*a*) Collimation error.
    - (*b*) Level error.
    - (*c*) Azimuth error.
    - (*d*) Irregularities in the form of the pivots.
    - (*e*) Corrections for the "verticality" and "wire intervals."

B.—Corrections required in finding the Declination:

1. Reading for zenith point, or for the nadir, horizontal or polar point.
2. Errors of imperfect centering of the circles.
3. Errors of graduation.
4. Errors of " runs " in the reading microscopes.
5. Error of horizontality in horizontal wire.
6. Error of curvature in the star's path.

A correction is always regarded as positive when it must be added to the *observed* value of a quantity in order to get the *true* value, negative if it has to be subtracted.

A.—CORRECTIONS REQUIRED FOR THE RIGHT ASCENSION

298. Clock Error and Rate

A good astronomical clock can generally be regulated so as not to gain or lose more than about 0·05s. in a sidereal day. It is necessary to apply a correction to the time indicated, owing to the clock being either fast or slow.

The *Error* of a clock is the amount by which the clock is *slow* when it indicates 0h. 0m. 0s. Thus, the error must be added to the indicated time in order to obtain the correct time. If the clock is fast, its error is negative.

The *Rate* of the clock is the increase of error during 24 hours. It is, therefore, the amount which the clock *loses* in the 24 hours. If the clock *gains*, the rate is negative.

The rate of a clock is said to be *uniform* or *constant* when the clock loses equal amounts in equal intervals of time.

299. Correction for Error and Rate

If the error of a clock and its rate (supposed uniform) are known, the correct time can be readily found from the time shown by the clock. The method will be made clear by the following example :—

Example.—*If the error of an astronomical clock be* 2·52s., *and its rate be* 0·44*s., find to the nearest hundredth of a secomd the correct time of a transit, the observed time by the clock being* 19*h*. 23*m*. 25·44*s*.

Here in 24h. the clock loses 0·44s.

So that in 1h. it loses $\frac{1}{24} \times$ 0·44s. = 0·0183s.

Hence, loss in 19h. = 0·0183s. × 19 = 0·348s.,

and loss in 23m. = 0·007s.

At 0h. 0m. 0s. the clock error is = 2·52s.;

Thus at 19h. 23m. 25·44s., clock is too slow by 2·52s. + 0·355s. = 2·88s.,

and the correct time = 19h. 23m. 25·44s. + 2·88s. = 19h. 23m. 28·32s.

300. Determination of Error and Rate of Clock

The clock error is found by observing the transits of known stars, *i.e.* stars whose R.A. and decl. are known.

If the clock were correct, the time of transit (when corrected for all other errors) would be equal to the star's R.A. (see § 30). If this is not the case, we have evidently

(Clock error) = *(Star's R.A.)* — *(observed time of transit)*.

This determines the clock error at the time of transit.

To find the rate, the transits of the same star or other known stars are observed on consecutive or neighbouring nights.

Let t and $t-x$ be the observed times of transit; then x is the amount the clock has lost in 24 hours, *i.e.* the rate of the clock. Therefore

(Rate of Clock) = *(observed time of 1st transit)*
— *(observed time of 2nd transit)*.

Having found the rate of the clock and its error at the time of transit, the error at 0h. 0m. 0s. may be found by subtracting the loss between 0h. 0m. 0s. and the transit.

Stars used in finding clock error are known as *Clock Stars.*

301. Personal Equation

Personal Equation is the error made by any particular observer in estimating the time of a transit.

Of two observers, one may habitually estimate the transit too soon, another may estimate it too late, but experience shows that the error made by each observer in taking times of transit by the same method is approximately constant.

If all observations are made by the same individual there will be no need to take account of personal equation, because the error made in taking a transit will be compensated by the error made in observing the clock stars to set the clock. If the two operations are performed by different observers, we must allow for the *difference* of their personal equations.

Personal equation may be measured by an apparatus for observing the transit of a fictitious star, *i.e.* a bright point moved by clockwork; in this case the actual time of its transit is known, and can be compared with the observed time. Personal equation is positive if the observer is early, so that the correction must be *added* to the observed time to get the true time.

In the tapping method of recording transits, the personal equations of different observers may differ by half a second or more. By the use of the travelling wire, the personal equations are reduced to a few hundredths of a second; because with this form of micrometer they are substantially eliminated, it is usually termed the *impersonal micrometer.*

302. Errors of Adjustment of the Transit Circle

If the transit circle is in perfect adjustment, the line of collimation of the telescope must always lie in the plane of the meridian. If not, we must correct for the small errors of adjustment. The conditions required for perfect adjustment, together with the corresponding corrections when these conditions are not fulfilled, may be classified as follows :—

(*a*) The line of collimation should be perpendicular to the axis about which the telescope rotates. If not, the corresponding correction is called *Collimation Error.*

(*b*) The axis of rotation must be horizontal. *Level Error.*

(*c*) The axis must point due east and west. *Azimuth Error.*

(*d*) The pivots resting on the Y's must be truly turned, and form parts of the same circular cylinder. Correction for *shape of pivots.*

(*e*) The vertical wires in the transit must be truly vertical (*i.e.* parallel to the meridian) and equidistant. *Verticality* and *Wire Intervals.*

303. Collimation Error

We have seen (Art. 291) that the framework carrying the vertical wires in the transit telescope can be adjusted by a screw, so that collimation error can be corrected. Suppose, for simplicity, that no other error is present. Then the line of collimation will always make a constant small angle with the meridian, and this angle will measure the collimation error.

To correct this error, two telescopes, called Collimators, are pointed towards each other, one due north, the other due south of the instrument (n, s, Fig. 107). Both contain adjustable "collimating marks," formed by cross wires in their focal planes. The transit telescope being first pointed vertically, and two apertures in the side of its tube being uncovered, the observer looks through the telescope s, and sees through the apertures into the telescope n. He then brings the wires in s into coincidence with the images of the wires in n; he then knows (from the optical theory of the telescope) that the lines of collimation of n, s are *parallel.* Suppose (*e.g.*) that they make a small unknown angle x'' W. of S., and E. of N., respectively.

He now looks through the transit telescope into the collimator s. He adjusts the middle vertical wire of the transit to coincide with the image of the cross mark in s, reading the graduated screw by which the adjustment is made. The line of collimation of the transit is now x'' *west* of the meridian. He points the telescope into n, and similarly adjusts the wires: the line of collimation is now x'' *east* of the meridian. He now turns the adjusting screw to a reading *midway* between the two observed readings; the line of collimation is then in the meridian, and collimation error has been removed.

In practice, the screw is set at a standard reading near the line of collimation, and the outstanding error is allowed for.

The collimation error is usually expressed in time. If, on account of collimation error, c seconds must be added to the observed time of transit of an equatorial star to obtain the true time of transit, then $c \sec \delta$ seconds must be added to the observed time of transit of a star of declination δ.

304. Level Error

This is measured by the inclination to the horizon of the axis of rotation of the telescope. It causes the line of collimation to trace out, on the celestial sphere, a great circle inclined to the meridian at an angle equal to the level error.

Level error is found by pointing the telescope (corrected for collimation error) downwards over a trough of mercury (*N*, Figs. 105, 107, 111).

An eye-piece is provided, called a " collimating, or Bohnenberger, eye-piece " (*EF*, Fig. 111), containing a plate of glass *M*, which reflects the light from a lamp straight down the tube. The mercury will form a reflected image of the telescope, which may be treated just as if it were a real telescope or collimator. By the law of reflection, if the middle wire coincides with its image, the line of collimation will be vertical, and (since there is no collimation error) there will be no level error. If not, the wires are moved by the screw until the vertical wire coincides with its image. The observer reads the angle through which the screw has been turned, and thus measures the level error.

In practice, instead of making two images coincide, it is more accurate to make them touch each other alternately on the two sides, and take the mean.

Suppose the level error is b'', the east end of the axis being the lower. It therefore points to a point E_1 on the celestial sphere which is on the prime vertical, but below E by the amount $EE_1 = b$ (Fig. 109).

Then, when the telescope is rotated, its axis will move in the great circle SZ_1N, whose pole is E_1. A star X on this great circle will appear to be on the meridian, though actually it has the easterly hour angle $QPM = QM$. If X' is the point where X crosses the meridian, $XX' = ZZ_1 \cos X'Z = b \cos(\phi - \delta)$ since $QX' = \delta$ and $QZ = PN = \phi$. Also $XX' = QM \cos \delta$. Thus the hour angle $QM = b'' \cos(\phi - \delta) \sec \delta = \frac{1}{15} b \cos(\phi - \delta) \sec \delta$ seconds of time. This must be added to the observed time of transit to obtain the true time.

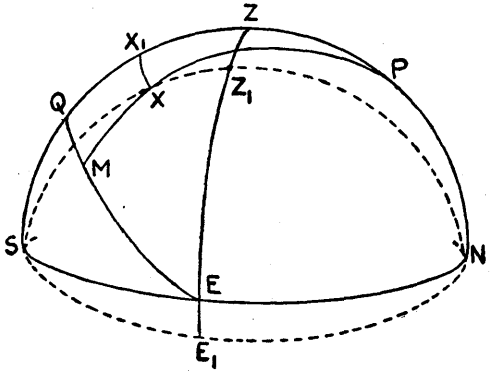

FIG. 109.

305. Azimuth Error

Azimuth error is measured by the small angle which the axis of rotation of the telescope makes with the plane of the prime vertical. It causes the line of collimation of an otherwise correctly adjusted transit circle to describe a great circle through the zenith whose inclination to the meridian is equal to the deviation error.

Suppose the azimuth error is k'', the axis being inclined by the amount k to the north of east, so that it points to E_1 on the horizon (Fig. 110), where $EE_1 = k$.

As the telescope is rotated, the axis moves in a great circle passing through the zenith and a point S_1 on the horizon such that $SS_1 = k$. A star X appears to be on the meridian when its easterly hour-angle is $XPX' = MQ$.

$$XX' = SS_1 \sin XZ = k \sin(\phi - \delta)$$

and the hour-angle $QM = k'' \sin(\phi - \delta) \sec\delta = \frac{1}{15} k \sin(\phi - \delta)\sec\delta$ seconds of time. This must be added to the observed time of transit to obtain the true time.

The azimuth error can be determined by observing the times of upper and lower transit of a circumpolar star. It can easily be shown that, at lower transit, the quarterly $\frac{1}{15} k \sin(\phi + \delta) \sec\delta$ seconds must be subtracted from the observed time of transit to obtain the true time. The observed interval between upper and lower transit will therefore differ from 12h. by the amount $\frac{1}{15}k\,\{\sin(\phi - \delta) + \sin(\phi + \delta)\}\sec\delta = \frac{2}{15} k \sin\phi$ seconds. By observing the difference of the intervals the azimuth error, k, can therefore be found.

In case the double observation of a close polar star is impracticable, a single observation of any known star near the pole combined with another known star will give the error.

306. Combined Effect of Collimation, Level and Azimuth Errors

From the results of Arts. 304, 305, 306, it appears that if the collimation error is c seconds of time, the level error is b'' and the azimuth error is k'', the observed time of transit must be increased by the quantity t to obtain the true time of transit, where

$$\begin{aligned} t &= c \sec\delta + \tfrac{1}{15} b \cos(\phi - \delta)\sec\delta + \tfrac{1}{15}k \sin(\phi - \delta)\sec\delta \\ &= m + n\tan\delta + c\sec\delta \end{aligned}$$

where

$$m = \tfrac{1}{15}(b\cos\phi + k\sin\phi)$$
$$n = \tfrac{1}{15}(b\sin\phi - k\cos\phi).$$

The quantities, m, n do not involve the declination of the star, but only the level and azimuth errors.

It is usual to determine c by means of the collimators, as explained in Art. 383. m and n can then be obtained by observing the times of transit of two stars of known R.A. and decl., one near the equator, for which $\tan\delta$ is small and one near the Pole, for which $\tan\delta$ is large.

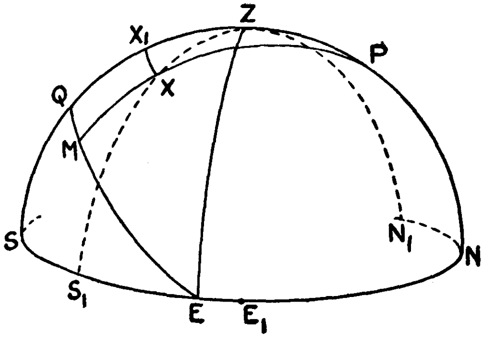

FIG. 110.

307. The Correction for the Shape of the Pivots

This correction is rather complicated, but, in a good instrument, it should be very small. When the pivots are much worn by friction, they should be re-turned.

The errors may be measured by making a small mark on the end of each pivot, and observing, by means of reading microscopes, the motions of the marks as the instrument is slowly turned round. If the pivots are true, the marks should remain fixed, or describe circles.

308. Verticality of the Wires

Verticality of the wires may be tested by observing one of the collimators, whose cross wires are adjusted as in Art. 303. If the cross wires always appear to intersect on the middle wire of the transit when the instrument is turned through any small angle, we know that the middle wire is vertical.

309. Wire Intervals

By *Equatorial Wire Intervals* are meant the intervals of time taken by a star on the equator in passing from one vertical wire of the transit to the next.

If the intervals between successive wires are unequal, the mean of the times of transit over the wires will not in general be the same as the time of transit over the middle wire. We may imagine a straight line so drawn across the field of view that the time of transit across it is exactly equal to the mean of the times of transit over the five or seven wires. This line is called the *Mean of the Wires*.

By carefully determining the equatorial wire intervals, the very small interval between the transits over the mean of the wires and over the middle wire can be found.

For a star not in the equator, the wire intervals are proportional to the secant of the declination.

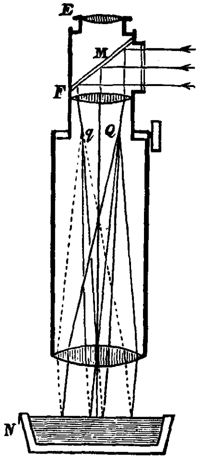

Fig. 111.

B.—CORRECTIONS REQUIRED IN FINDING THE DECLINATION OF A STAR

310. Zenith Point

In Art. 291 we stated that the pointer of the transit circle is usually adjusted to read 0° 0′ when the line of collimation is pointed to the zenith. But it would be very difficult to adjust the microscopes to give a mean reading of exactly 0° 0′ 0″ for the zenith. Hence it is necessary to determine the *zenith point*, or zenith reading, and in calculating the meridian Z.D. of any star, this must be subtracted from the reading for the star.

Let Z and N be the readings when the telescope is pointed to the zenith and nadir, respectively, H and H' the readings for the north and south points of the horizon ; then evidently,

$$Z = H - 90^\circ = N - 180^\circ = H' - 270^\circ.$$

Also, if x is the reading for the meridian transit of any star, then star's meridian Z.D. $= x - Z$, if north of the zenith, or, $= 360^\circ - (x - Z)$, if south of the zenith.

311. To Find the Nadir Point

To find the *Nadir Point* use is made of the *Collimating Eye Piece*. already mentioned in Art. 304, and represented in Fig. 111. It consists of two lenses E, F, between which is a plate of glass, M, inclined at an angle of 45° to the axis. This plate illuminates the wires from above by partially reflecting the light from a lamp on them, at the same time allowing them to be seen through the eye-glass, E.

The telescope is pointed downwards over the trough of mercury, N ; and the rays of light from any one of the wires, Q, will produce by reflection a distinct image of the wire at q in the focal plane. By turning the telescope with the tangent screw, the fixed horizontal wire may be made to coincide with its image ; it will then be vertically over the "optical centre" of the object-glass (Art. 290). The line of collimation will, therefore, point to the nadir, and the nadir reading is given by the pointer and microscopes. Substracting 180°, we have the zenith reading.

312. Determination of Horizontal Point.—Method of Double Observation

Both the horizontal reading and the meridian altitude of a star can be determined by observing the star, both directly and by reflection, in a trough of mercury placed in a suitable position (*M*, Figs. 107, 112).

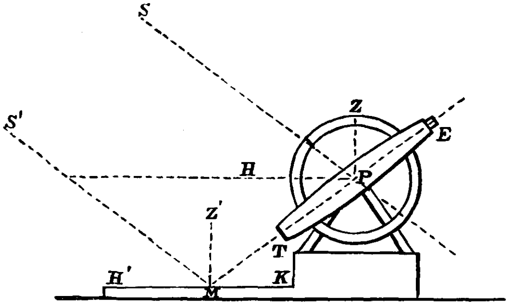

FIG. 112.

Fig. 112 illustrates the method of double observation. Let *PZ* be the direction of the line of collimation corresponding to the zero reading, *PH* the horizontal direction, *PS* and *MTP* the directions of the star viewed directly and its image viewed by reflection. The reading of the circle for the direct observation is the angle *ZPS*, the reading for the reflection is the angle *ZPM*.

Since the angles of reflection and incidence *S'MZ'*, *TMZ'* at the mercury are equal, and *MS'*, *PS* are parallel, we have evidently $\angle SPH = S'MH' = TMK = MPH$; so that star's altitude, $SPH = \frac{1}{2}\,SPM$; $= \frac{1}{2}\,(ZPM - ZPS)$

= half the difference of the two readings.

Also : Horizontal reading, $ZPH = \frac{1}{2}\,(ZPM + ZPS)$;

= half the sum of the two readings.

Subtracting 90° from the north horizontal point, the zenith point is found.

313. Polar Point

In order to find the declination of a star by means of the transit circle, it is necessary to know the reading when the telescope is pointed to the pole. This may be found, just as in Art. 31, by observing the upper and lower transits of a circumpolar star. The mean of the two readings gives the polar point.

The N.P.D. of any star is found by taking the difference of the readings for the star and the polar point. The declination is, of course, the complement of the N.P.D.

We may also find declinations thus :—Since angles are measured from the zenith northwards, it is evident (by drawing a figure or otherwise) that the reading for the point of the equator above the horizon is given by

$$\text{Equatorial point} = (\text{Polar point}) + 270°.$$

Since the decl. is the angular distance from the equator, we have

$$(\text{North Decl.}) = (\text{Reading for star}) - (\text{Equatorial point}).$$

If the star transits north of the zenith, its reading must be increased by 360°.

The latitude of the observatory is given by :—

$$\begin{aligned} \text{Latitude} &= \text{Altitude of pole} \\ &= (\text{North horizontal point}) - (\text{Polar point}). \end{aligned}$$

314. Errors of Graduation

The operation of testing the accuracy of the graduations on the circles of the transit circle is very long and laborious. One of the two graduated circles is so attached to its axis, that it can be turned through any angle relative to the telescope. Then, by reading the microscopes belonging to both circles, every graduation on one circle is compared with every graduation on the other circle, and any errors of graduation are thus detected and measured. The effect of such errors is much reduced by using all the four microscopes, and taking the mean of their readings.

315. Errors due to Imperfect Centering of the Circles

By taking the mean of the microscopic readings, all errors due to imperfect centering are eliminated. In proof, let us suppose that only two microscopes (*A*, *C*, Fig. 107) are used, but that these are opposite to one another. If the circle is truly centered, with its centre on the line *AC*, the two readings will differ by 180°. If, now, the graduated circle is displaced, without being rotated, till its centre is at a distance h from *AC*, then the points of the scale, now under *AC*, will be at distances h from the points formerly under *AC*, both being displaced in the same direction. Hence, since both readings are measured the same way round the circle, one will be increased and the other will be decreased by the same angle. The arithmetic mean of the two readings will, therefore, be unaltered by the displacement of the centre, and will be independent of any small error due to imperfect centering. The same is, of course, true of the mean reading for the other pair of microscopes, *B*, *D*.

A knowledge of the error of centering is not necessary, but if desired it may be ascertained thus: set the circle so that the pointer microscope reads successively 0°, 5°, 10°, 15° ... 350°, 355°, and read two opposite microscopes *A* and *C* in each position; then draw a curve with pointer reading as abscissa, and $A - C$ as ordinate; this will be approximately a sine-curve like either of the curves in Fig. 28;

the error of centering is one-quarter of the vertical distance from the highest to the lowest point of the curve. Graduation error may be allowed for if necessary, but it is generally small compared with error of centering.

316. Error of Runs

In the reading microscopes, one turn of the micrometer screw should move the parallel wires over a space corresponding to exactly 1′ on the graduated circle, so that the wires should be brought from one mark of the circle to the next by exactly five turns of the screw. In practice it will probably be found that rather more or rather less than five turns will be necessary. In this case the readings of the teeth and of the micrometer screw-head will differ slightly from true minutes and seconds of arc on the circle, and a correction will be required. This error is called the *Error of Runs*.

317. Error of Inclination

The angle between the R.A. and declination wires cannot generally be altered once the wires are mounted. It is important to make the R.A. wires exactly vertical, hence the others will not generally be quite horizontal. The error can be determined by observing a number of stars at equal distances east and west of the centre, and determining the mean difference of micrometer readings. The correction is directly proportional to the distance from the centre.

Connected with this is the *Error of Sag*, which occurs if the wire is mounted loosely, and drops in the middle. It is very difficult to correct for this error if present, but it is found that with well-mounted wires its effects are inappreciable. To test for it, bisect a star very near the equator every ten seconds while passing across the field, and plot the readings with time as abscissa ; if there is no sag, they will lie along a straight line, if there is sag, along a curve.

318. Error of Curvature

The path of a non-equatorial star is a small circle, whereas the trace of the declination wire on the sky is a great circle. Now it is generally inconvenient to make declination observations at the centre of the field ; indeed it is impossible if the moving wire method is employed for observing R.A., since this needs both hands. If the declination is observed s seconds before or after the star passes the centre, the correction for curvature is $\cdot 000273\ s^2 \sin 2$ N.P.D. seconds ; but if it is observed at a mark at such a distance from the centre that an equatorial star would take t seconds to traverse it, the correction is $\cdot 000545\ t^2 \cot$ N.P.D. seconds. For north declinations the observed place is north of the meridian place ; for south ones south. For example let $s = 30$ s., N.P.D. $= 45°$; then $s^2 = 900$, $\sin 2$ N.P.D. $= 1$. The correction is $\cdot 0273 \times 9 = 0{\cdot}25''$.

319. Collimation, Level and Azimuth Errors

These errors have no appreciable effect on observations for declination, provided that such errors are small compared with the star's N.P.D. Hence, they may be left out of account, except in observations of stars very near the Pole.

320. General Remarks

We first described the Transit Circle, and the methods of " taking a transit " ; we afterwards described the corrections which must be applied to the results of the observations in finding the right ascension and declination of a star. In practical work, the various errors must be determined before or during the course of the observations. Among these, collimation, level and azimuth errors, and the nadir point should be found daily, as they may be affected by heat or cold, or by shaking the instrument.

Clock error and rate are also determined daily by observing certain "clock stars." These consist of the brighter stars in the equatorial belt of the sky, whose positions have been determined with great accuracy.

321. Observations on the Sun, Moon, and Planets

The positions of the Sun, Moon, and Planets are defined by the co-ordinates of their *centres*. In finding these, the angular diameters must be taken into account.

In observing the Moon or a planet, the fixed horizontal wire is adjusted to touch the illuminated edge of its disc, and the times at which its edge touches the vertical wires are observed. To find the co-ordinates of the centre, a correction is made for the angular semi-diameter of the body, which must be determined independently. It must not be forgotten that the image formed by the telescope is *inverted*.

In observing the Sun, after the first limb is observed in R.A. the instrument is moved to bring the north limb (say) near the horizontal wire, clamped, and the microscopes read by a second observer, while the first observer places the horizontal wire tangent to the limb, and records the micrometer reading; the process is repeated with the south limb, and finally the second limb is observed in R.A. When the Moon is very nearly full, the same method of observation is used. In finding the time of transit, the times of contact of the disc on arriving at and leaving each wire are separately observed; their arithmetic mean for any wire is the time of transit of the centre.

322. The Equatorial Telescope

The transit instrument can move in only one plane—the plane of the meridian, and observations of celestial objects with it are restricted to their passages across the meridian. If a telescope is required to point to any desired point of the sky, it must have two degrees of freedom or, in other words, be provided with two axes about which it can turn.

A theodolite has two degrees of freedom. The telescope is supported by a horizontal axis, about which it can turn in a vertical plane, to enable objects at different altitudes to be observed. The pillar carrying the telescope can turn about a vertical axis, so that objects in different azimuths can be observed. The two motions of which a theodolite is capable are thus motions in altitude and azimuth.

For the observation of celestial objects these motions are not convenient. The diurnal motion of a body across the sky causes both its altitude and azimuth to change continuously. To keep the body in the centre of the field of view of a telescope with an altitude-azimuth or altazimuth mounting, as such a mounting is usually called, the telescope must be continuously moved in altitude and also in azimuth. It is much more convenient to mount the telescope so that its two motions are in hour-angle (or right ascension) and in declination. For most objects, with the exception of the Moon, the declination remains constant or nearly constant during the observations, so that to keep the object in the centre of the field of view it is necessary to move the telescope only about the hour-angle axis. As, moreover, the hour-angle of a celestial body changes at a rate that is practically uniform, the motion about this axis can be provided by appropriate clock-work or other mechanical means.

This form of mounting is called the Equatorial. It is shown schematically in Fig. 113. The framework carrying the telescope turns as a whole about an axis *AB*, which is supported at *A* and *B*, so as to be parallel to the axis of the Earth. *BA* thus points towards the pole and is accordingly called the *polar axis*. Attached perpendicularly to this axis, and turning with it, is a graduated circle, called the *Hour Circle*, which is read by a pointer microscope, *N*.

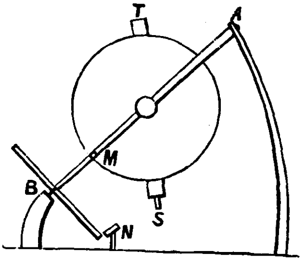

FIG. 113.

The framework *AB* carries a secondary axis, perpendicular to the primary axis, called the *declination axis*, and the telescope *ST* is attached perpendicularly to this axis, about which it is free to turn. The axis of the telescope carries a second graduated circle, called the *Declination Circle*, which is read by the pointer microscope *M*.

The declination circle should read zero when the telescope is pointed in the plane of the equator, and the hour circle should read zero when the telescope is in the plane of the meridian. If now the telescope is pointed towards any celestial body, the readings of the two microscopes will give, respectively, the declination and hour angle of the body.

When it is required to observe the same body continuously with the equatorial, the declination circle is clamped, and the observer must slowly rotate the hour circle by hand, so as to keep the body observed in the field of view.

In large instruments the hour circle can be attached to a clamp which is worked by clockwork in such a manner that the whole framework turns uniformly round the primary axis AB once in a sidereal day. This motion will ensure that the star under observation shall always remain in the centre of the field of view.

The pointer-microscope of the hour circle may be made to revolve with the clamp, and to mark zero when the telescope is pointed towards the first point of Aries; its reading will then give the right ascension of any observed star. But the declination and right ascension cannot be determined with any great degree of accuracy by reading the circles of the equatorial. These circles are provided merely to enable the telescope to be pointed in the appropriate direction for the observation of any particular object whose right ascension and declination are known.

323. Uses of the Equatorial

Amongst these the following may be mentioned :—

(i) "Differential" observations, *i.e.* micrometric observations of the relative distances and positions of two near objects simultaneously visible. Most observations of comets and minor planets are made by measuring their distance and direction from neighbouring stars whose positions are known; sometimes these measures are made on photographs.

(ii) Observations of the appearance, structure and magnitude of the celestial bodies.

(iii) Stellar photography.

(iv) Spectroscopic observations, with the aid of a spectrograph attached to the telescope, which are usually made photographically.

The exposures, in uses (iii) or (iv), may last for several hours, during which time the telescope must follow accurately the diurnal motion of the object. In photographic observations, it is usual to mount a second but smaller telescope alongside AB; the object to be photographed is set on cross-wires in the focal plane of this telescope. The clockwork, or other driving mechanism of the telescope, is provided with a differential motion, under the control of the observer, by means of which the motion of the telescope can be slightly accelerated or retarded. During the exposure, the observer watches the image of the object in the auxiliary or guiding telescope and operates the differential control as necessary so that the image remains accurately bisected by the cross wire. It is thus possible to correct for irregularities in the mechanical drive of the telescope, movements of the image due to atmospheric tremors, and small displacements due to change of atmospheric refraction in the course of the exposure.

324. Micrometers

Any instrument used for measuring the small angular distance between two bodies simultaneously visible in the field of view of a telescope is called a *Micrometer*. Thus the moveable horizontal wire in the transit circle, with its graduated screw, is a micrometer, for if the instrument be so adjusted that the fixed wire crosses one star, while the moveable wire crosses another neighbouring star, the distance between the wires, as read off on the screw head, gives the difference of declination of the stars. The moveable wire in the field of view of the reading microscope is identical in principle with a micrometer.

325. The Screw and Position Micrometer

This instrument (Fig. 114) serves to find both the angular distance between two neighbouring stars and the direction of the line joining them. It contains a framework of wires placed in the focal plane of the telescope. Two of these wires are parallel, and one of them can be separated from the other by turning a screw with a graduated head. A third wire, which we will call the "transverse wire," is fixed in the framework perpendicular to the two former. The whole apparatus, together with the eye piece of the telescope, can be rotated so that the wires may appear in any required direction across the field of view. A graduated circle, called the *Position Circle*, is attached to the eye-piece, and measures the angle through which it has thus been turned. Besides the wires, the framework contains a transverse strip of metal marked with notches, at distances apart corresponding to complete turns of the micrometer screw, an arrangement similar to that employed in the reading microscope (Art. 292).

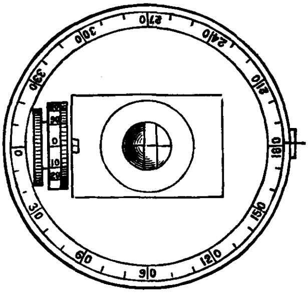

FIG. 114.

In observing two stars, the equatorial and micrometer are so adjusted that one of the stars may appear at the intersection of the two fixed wires, while the other appears at the intersection of the fixed and moveable wires.

The number of notches of the scale, together with the reading of the screw-head, determine the distance between the images of the stars in turns and parts of a turn of the screw-head. To find the angular distance between the stars, we only require to multiply by the known angular distance corresponding to one turn of the screw.

The reading of the position circle determines the direction of the small arc joining the stars. The position-circle should read zero if the stars have the same R.A. Then the reading in any other position will determine their *position angle, i.e.* the angle which the line joining the stars makes with a declination circle through one of the stars.

326. Dollond's Heliometer

Dollond's heliometer is another form of micrometer, depending on the principle that if the object-glass of an astronomical telescope be cut across in two, each half will form an image of the whole field of view, in the same way as if the lens were still complete. In the heliometer one half of the object-glass can be made to slide along the other, the separation of the two halves being read off on an accurately graduated scale.

Suppose that we want to measure the angular diameter of the Sun (S, Fig. 115). When the halves of the object-glass are together, so that their optical centres coincide, one image of the Sun will be formed. When the two halves are separated, two separate images will be formed in the focal plane of the telescope, and will be seen simultaneously. The half-lenses are separated, till the two images *touch*, as *ab* and *bc*.

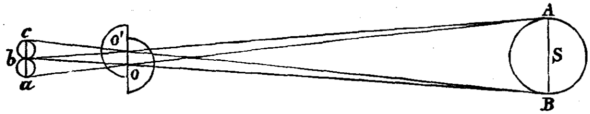

FIG. 115.

Let O, O' be the optical centres of the two halves of the objective. The distance OO' is read off on the scale ; from this reading the Sun's angular diameter may be found.

For at b, the point of contact of the images, the half-lens O forms an image of the lower limb B, and the half-lens O' forms an image of the upper limb A. Hence, BOb and $AO'b$ are straight lines, and ObO' is the angular diameter BbA. But the focal length Ob is known. Hence, if OO' is also known, the angular diameter ObO' can be found.

In practice, the scale reading is taken with the two halves of the object glass in the positions shown in the diagram. The two halves are then interchanged, so that O is brought to O' and *vice versa*, the two images again touching : the scale reading in this position is also taken. The separation of the two halves of the object glass in either position is half the difference of the two scale readings.

In measuring the angular distance between two stars, the heliometer is adjusted so that the image of one star formed by one half-lens O coincides with the image of the other star formed by the other half-lens O'. The principle is the same as before. In order that the two images may be brought into coincidence, the direction of separation of the two halves of the object glass must be parallel to the great circle through the two stars. The entire object glass is arranged so that it can be rotated and a graduated circle is provided for reading off its orientation. To measure the distance between two stars, the orientation and separation of the two half-lenses are adjusted until the images of the stars are brought into coincidence. The positions of the two half-lenses are then interchanged, as previously described. The object glass is then turned through 180°, so that the direction of separation is again parallel to the direction from one star to the other and the operations are repeated.

327. Preliminary Checks when using Micrometer

To find the angular distance corresponding to a revolution of the micrometer screw, the simplest plan is to observe the Sun's diameter, and to compare the reading with its known value. The latter is given in the *Nautical Almanac* for every day. The distance between two known stars in the same field may also be used. The Pleiades are convenient for this purpose.

To test the zero reading of the position circle, the equatorial is pointed to a star near the equator, and fixed, and the micrometer is turned till the diurnal rotation causes the star to run along the transverse wire. The circle should then read 90°.

EXAMPLES

1. Describe the altazimuth type of mounting. Why is it not so well suited for continuous observations as the equatorial, and, in particular, why is it quite unsuitable for stellar photography?

2. How may we most easily set the astronomical clock?

3. Show that the rate of a clock might be found by observations on successive nights with *any* telescope provided with cross wires, and pointed constantly in a fixed direction.

4. Distinguish, with examples, *direct* and *retrograde* angular motion. Is R.A. measured direct or retrograde?

5. Show that in latitude 45° the interval between the time of any star's passing due east and its time of setting is constant.

6. Show that, if a transit circle be not centred truly, the consequent error can be eliminated by taking the mean of the readings of the microscopes.

7. In a double observation made with the transit circle, the readings of the pointer directly and by reflection are $59° \ 35'$ and $125° \ 20'$; the means of the microscope readings are in the two cases $3' \ 42''$ and $1' \ 13''$. The moveable wire reads $+ \ 2''$, and the reflected star runs along the *fixed* horizontal wire. Find the zenith reading.

8. The level error of a transit instrument is $12''$ (east end of axis high); the azimuth error is $25''$ (north of east); the collimation error is $3''$ (east). If the

latitude is 51°, find the correction required to the observed time of transit of a star whose declination is 12° S.

9. In latitude 35°, a star of declination 85°N is observed to transit 22·5s. late; a star of declination 5°S is observed to transit 7·8s. late. The clock is 8·6s. slow and the collimation error is zero. Find the level and azimuth errors.

10. Find the decl. of *a Ophiuchi* from the following observations, made at Greenwich (lat. 51° 28′ 38″ N.) :—Pointer reading 321° 10′, microscope readings, 1′ 2″, 0′ 50″, 0′ 46″, 0′ 58″, the zenith reading being 0° 0′ 16″.

11. Find also the R.A. of *a Ophiuchi.* Given: Time by sidereal clock = 17h. 29m., the numbers of seconds at the transits over the five wires being 37·4s., 50·2s., 1m. 2·9s., 1m. 15·2s., 1m. 27·4s. Clock error = − 10·6s.; personal equation = + 0·4s.

EXAMINATION PAPER

1. Classify the various observations which are taken in astronomical investigations, and state the respective instruments which may be used for those observations.

2. Define the *right ascension* and *declination* of a star, and describe shortly the principles of the methods of finding them.

3. Describe how the time of transit of a star across each of the five or seven wires of a transit instrument is observed, and explain how the time of transit across the meridian is deduced. Define the *equatorial interval* of two wires.

4. Describe the Reading Microscope, and show how the zenith distance of a star may be found by direct observation with the transit circle.

5. Enumerate the errors of a transit instrument, and explain how level error may be measured and corrected.

6. Explain what is meant by *collimation error,* and draw a diagram showing the circle traced out on the celestial sphere by the line of collimation in an instrument which has a small collimation error east of the meridian. Is the correction, to be applied to the times of transit, positive or negative in such a case ?

7. Describe the Equatorial, and explain the adjustments and principal uses of the instrument.

8. What is meant by the *error* and *rate* of a clock, and the *personal equation* of an observer ? How are they usually found ?

9. On 1st March, 1940, the time of transit of *β Librae,* at Greenwich, was observed to be 15h. 12m. 46·86s., and on the 3rd March the observed time was 15h. 12m. 45·44s. The tabular R.A. of the star was 15h. 13m. 47·96s. Find the error and rate of the clock on 3rd March.

CHAPTER XIV

THE DETERMINATION OF POSITION ON THE EARTH

I.—INSTRUMENTS USED IN NAVIGATION

328. Introduction

Among the different uses to which Astronomy has been put, perhaps the most important of all is its application to finding the geographical latitude and longitude of any place on the Earth from observations of celestial bodies. Such observations may be made for either of the following purposes :—

1. The determination of the exact latitude and longitude of an observatory. These must be known accurately before the co-ordinates of a star can be found or observations taken at different observatories can be compared.

2. The construction of maps. The geographical latitude and longitude of a place form a system of co-ordinates which enable us to represent its exact position on a map.

3. The determination of the position of a ship or aircraft. This is the most useful application of all ; on a long sea voyage it is necessary to calculate daily the ship's latitude and longitude correct to within a mile or so. The position of an aircraft does not need to be determined with such accuracy ; an error of five miles is usually immaterial.

Now, owing to the motion and rocking of a ship or aircraft, all the astronomical instruments hitherto described are useless at sea or in the air. The navigator is therefore obliged to have recourse to others which are unaffected by the unsteadiness of the motion. The two instruments best fulfilling this condition are the Sextant and the Chronometer, which we shall now describe.

329. The Sextant

The use of the sextant is to measure the angular distance between two objects by observing them both simultaneously. It consists of a brass framework forming a sector *CDE* graduated along the circular arc or *limb DE* ; the angle *DCE* is usually about 60° or rather more. To the centre *C* of the arc is fixed an arm *BI*, capable of turning about *C*, and which carries the small mirror *B*, called the *index glass*. Another small mirror *A*, called the *horizon-glass*, is fixed to the arm *CD*, making an angle of about 60° with *BD*. Of this mirror half the back is usually silvered, the other half being transparent. Finally, at *T* is fixed a telescope, pointed towards *A* in such a manner as to receive the rays of light from the mirror *B* after reflection at *A* (Figs. 116, 117).

On looking through the telescope T we shall see two sets of images, for objects at H will be seen directly through the unsilvered part of the mirror A, while objects at S will be seen after two reflections at the mirrors B and A. The mirror is so near the object glass of the telescope as to be quite out of focus; hence these two sets of images will not appear separate, but will overlap one another.

The arm BI carries at I an index mark or pointer by which its position can be read off on the graduated scale DE. The pointer should read zero when the mirrors A, B are parallel (as in the position $B'E$, Fig. 116). When this is the case, the two images of any very distant object H will coincide. For when a ray of light is reflected in succession at two parallel mirrors, its final direction is parallel to its initial direction. Hence if $H'CAT$ represents the path of a ray of light from the object H, as reflected in succession at B' and A, the portion AT is parallel to $H'C$, and therefore coincides with the ray HAT, by which the object is seen directly.

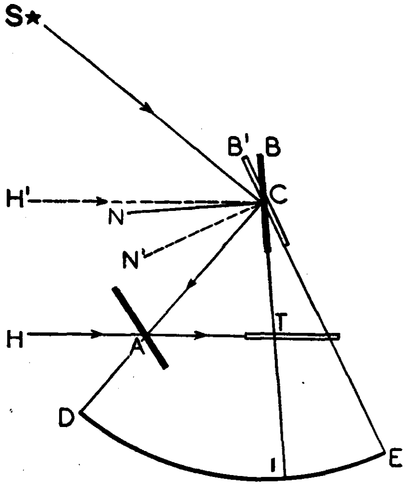

FIG. 116.

Now let it be required to find the angular distance between the two objects H and S. To do this, the mirror B is rotated by means of the arm BI until the image of S (formed by the two reflections) is seen to coincide with H. The angle ECI, through which the mirror B has been turned from its orginal position, is then half the required angular distance between H, S.

For draw CN', CN perpendicular to the two positions B', B of the mirror respectively. Since in reflection at a plane mirror the angles of incidence and reflection are equal.

$$\angle N'CH' = \angle ACN' \text{ and so } \angle ACH' = 2\,\angle ACN';$$

also

$$\angle NCS = \angle ACN \text{ and so } \angle ACS = 2\,\angle ACN.$$

$$\text{Hence } \angle ACS - \angle ACH' = 2\,(\angle ACN - \angle ACN'),$$

$$\text{i.e. } \angle H'CS = 2\,.\,\angle N'CN = 2\,.\,\angle ECI;$$

or the angular distance between the objects is double the angle ECI.

On the scale *ED*, every half-degree is marked as 1°. The reading of the pointer *I* will therefore give *double* the angle *ECI*, and this is the angular distance required.

The coincidence of the two images in the field of view of the sextant will not be affected by any small displacement of the instrument in its own plane. This peculiarity renders the sextant particularly useful on board ship, where it is impossible to hold the instrument perfectly steady.

330. Shades, Clamp and Tangent Screw, Reading Glass, Vernier

For viewing the Sun, the sextant is provided with *shades*. These consist simply of plates of glass blackened for the purpose of reducing the great intensity of the Sun's rays. There are two sets of shades, *G*, *G*, hinged to the frame *CE* in such positions that one set can be inserted between *A* and *C*, to deaden the rays from *S*, while the other set can be turned behind *A* to deaden the rays from *H*. They are called respectively the "index shades" and "horizon shades."

The arm or index bar *BC* is furnished with a *clamp*, by means of which it can be clamped at any desired part of the graduated limb *DE*. When this has been done the arm can be moved slowly by means of a *tangent screw* *K*, and in this way can be adjusted with great precision.

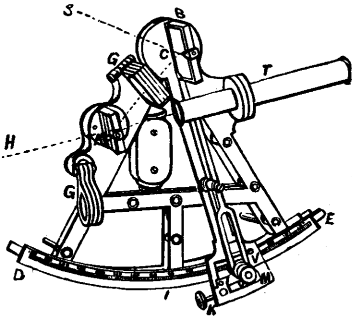

FIG. 117.

The arc *DE* is usually graduated to divisions of 10′,* and is read by means of the lens *M*, called the *reading glass*. But the index bar also carries a scale *V* called a *Vernier* (Art. 331) which, sliding beside the scale on the limb, enables us to read off observations to within 10″.

331. The Vernier

The Vernier is a scale the distance between whose graduations is 10′ — 10″, *i.e.* 9′ 50″, or 10″ less than the distance between the graduations on the limb. These graduations are marked 0″, 10″, 20″, etc., being measured in the same direction as on the limb. For example, let us suppose the zero point on the vernier is between the marks 26° 20′ and 26° 30′ on the limb. We take the reading by the limb as 26° 20′. We then look along the vernier scale until we find that *one of the marks on it exactly coincides with one of the marks on the limb.* Suppose that this is the 25th graduation from the zero point of the vernier, *i.e.* the point marked 4′ 10″. We add this 4′ 10″ to the 26° 20′ read on the limb, and the sum gives the correct reading, namely, 26° 24′ 10″.

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* Of course these divisions are only 5′ apart, but in what follows we shall speak of half-minutes as minutes.

The principle is as follows. Let us denote by P the mark which coincides on the two scales.

Then from zero of vernier scale to P is 25 divisions of *vernier*, *i.e.* an arc of $25 \times (10' - 10'')$.

Also from 26° 20′ of scale on limb to P is 25 divisions of *limb*, *i.e.* an arc of $25 \times 10'$.

From 26° 20′ on limb to 0 of vernier therefore represents an arc of :—

$$25 \times 10' - 25 \times (10' - 10'')\ ;\ i.e.\ 25 \times 10'',\ \text{or}\ 4'\ 10''.$$

Hence the zero mark of the vernier scale is at a distance 26° 20′ + 4′ 10″ from the zero on the limb, and the reading is 26° 24′ 10″.

332. The Errors of the Sextant

These errors need not be described in detail. If the sextant does not read zero when the two mirrors are parallel, it is said to have an *Index Error*, and a constant correction for index error must be added to all readings made with the instrument. There are also errors due to eccentricity or want of coincidence between the centre about which the index bar turns and the centre of the limb, errors of graduation, etc.

Modern sextants are on the same principle as the one figured, but have better optical arrangements and complete graduated circles, and are much smaller. The graduations are read by opposite microscopes, which correct for eccentricity.

333. To Determine the Index Error of the Sextant

In all good sextants the graduated limb is continued backwards for about 5° behind the zero point. This portion of the limb is called the *arc of excess*, and is used for finding the index error, as follows. The Sun or Full Moon is observed; the two images of its disc are brought into contact. Let e be the index-error, r the sextant reading, D the angular diameter of the disc, then we have evidently $D = r + e$. Now let the index bar be moved along the arc of excess until the images again touch, the image which was before uppermost being undermost. If the reading on the arc of excess be $-r'$, we have now—$D = -r' + e$, or $D = r' - e$. Hence :—

$$2e = r' - r.$$

334. To Take Altitudes at Sea by the Sextant

The principal use of the sextant is for finding altitudes. Now the altitude of a star is its distance from the *nearest* point of the celestial horizon. To find this, the sextant is so adjusted that the reflected image of the star appears to lie on the offing or visible horizon*; when the plane of the sextant is slightly turned, the image of the star should just graze the horizon without going below it. The sextant reading then gives the star's angular distance from the nearest point of the "offing." Subtract the dip of the horizon and the correction for refraction, both of which are given in books of mathematical tables. The star's true altitude is thus obtained.

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* Such observations are made in twilight or bright moonlight, the offing being invisible in a dark sky.

335. To Take the Altitude of the Sun or Moon

In observing the Sun's altitude, the "index" shades must be turned into position between the two mirrors, and the instrument adjusted so that the Sun's lower limb appears just to graze the horizon. The reading of the sextant, when corrected for dip and refraction, gives the altitude of the Sun's lower limb. Add the Sun's angular semi-diameter; the altitude of the Sun's centre is obtained.

Both the Sun's altitude and its angular diameter may be obtained by observing the altitudes of the upper and lower limbs. The difference of the two corrected readings gives the Sun's angular diameter, and half the sum of the readings gives the altitude of the Sun's centre.

If this method is used, allowance must be made for the change in the Sun's altitude between the observations. For this purpose, three observations must be made. First take the altitude of the Sun's lower limb, then of the upper limb, and lastly, again of the lower limb. Also note the time of each observation. The difference between the first and third readings determines the Sun's motion in altitude; from this, by a simple proportion, the change in altitude between the first and second observations is found, and thus the altitude of the lower limb at the second observation is known. We can now find the Sun's angular diameter, and the altitude of its centre at the second observation.

Let t_1 = time of 1st observation, when a = alt. of *lower* limb;
t_2 = time of 2nd observation, when b = alt. of *upper* limb;
t_3 = time of 3rd observation, when a' = alt. of *lower limb*;

Then in time $t_3 - t_1$, the alt. of lower limb increases $a' - a$.

$$\text{so that in time } t_2 - t_1 \text{ it increases } (a' - a) \times \frac{t_2 - t_1}{t_3 - t_1}.$$

Hence if a_2 denote the alt. of lower limb at second observation, we have :—

$$a_2 = a + (a' - a)\frac{t_2 - t_1}{t_3 - t_1} = \frac{(t_3 - t_2)\,a + (t_2 - t_1)\,a'}{t_3 - t_1}.$$

This finds a_2, and we then have :—

$$\text{Sun's angular diameter} = b - a_2.$$

$$\text{Alt. of Sun's centre at second observation} = \tfrac{1}{2}(b + a_2).$$

In taking the altitude of the *Moon,* the altitude of the *illuminated* limb must be observed, and the angular semi-diameter, as given in the *Nautical Almanac* must be added or subtracted, according as the lower or upper limb is illuminated.

336. Artificial Horizon for Land Observations

Owing to the absence of a well-defined offing on land, an *artificial horizon* must be used. This is simply a shallow dish of mercury, protected in some manner from the disturbing effect of the wind. The sextant is used to observe the angular distance between a star and its image as reflected in the mercury Half this angular distance is the star's apparent altitude ; correcting this for refraction, the true altitude is obtained (*cf.* Art. 312).

As the limb of the sextant is generally an arc of not more than 70°, the instrument will not measure angular distances of more than 140°, and it can, therefore, only be used with an artificial horizon for altitudes of under 70°. For greater altitudes the zenith sector must be used.

At sea, where altitudes are measured from the offing, this objection does not apply. On account of the motion of the vessel an artificial horizon is useless ; hence, no observations can be taken when the offing is ill-defined, which frequently happens, especially at night. The mariner is, for this reason, chiefly dependent upon observations of the Sun and Moon, and such stars of the first magnitude, or planets, as are visible about dusk.

337. The Air Sextant

There is not usually a well-defined offing available for observations made from an aircraft and even if there were, the correction for dip of the horizon would be large and dependent upon an accurate knowledge of the height of the aircraft. An artificial horizon of the type used for observations on land cannot be used in the air. It therefore becomes necessary to provide an artificial horizon in some other way.

This is achieved by the use of a bubble of air in a spherical level filled with liquid. The bubble takes up the highest position in the level, so that the line through the centre of the bubble and the centre of the spherical level is vertical.

In Fig. 118, *A* and *B* are the two sextant mirrors. *G* is the bubble, *K* is a collimating lens, *M* a mirror and *L* a plane glass inclined to *M* at an angle of 45°. The rays of light from the bubble pass through the lens *K*, and are reflected at the mirror *M*. After passing again through the lens *K*, they emerge as a parallel beam, which is partially reflected by the glass *L* into the horizontal direction. To an eye at *E* the bubble therefore appears to be in the horizontal plane. The observation consists in making the star *S* coincide with the centre of the bubble.

The reading of the sextant after correction for refraction and index error gives the altitude of the star S (no correction for dip required).

Since the mirrors L and M are inclined to one another at an angle of 45°, the vertical ray through G will be turned through twice this angle and therefore brought into the horizontal direction.

The accuracy of the observation is dependent upon the aircraft maintaining a steady speed in a straight line during the observation. Any acceleration of the aircraft will be equivalent to an additional force acting on the liquid of the bubble which, when combined with the force of gravity, will give a spurious direction of gravity. Under these circumstances the bubble G will not remain at the highest point of the spherical level and the direction from G to the centre of the level will then not be vertical; the rays finally reflected from the glass L will not be horizontal and a spurious value of the altitude of the star S will be obtained. A sequence of readings are therefore taken in rapid succession, during the course of which the pilot of the aircraft maintains a course as steady and uniform as possible. By means of an averaging device the mean of the readings is directly obtained.

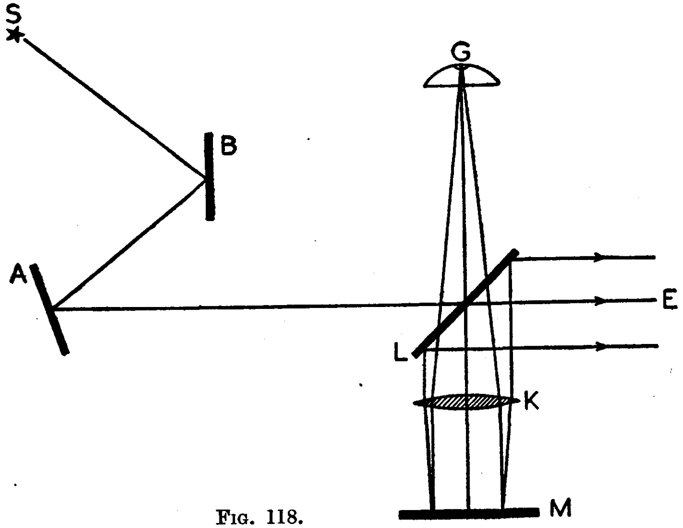

FIG. 118.

338. The Chronometer

The *Chronometer* is the form of timepiece used on board ship, and in all observations in which clocks are unavailable, owing to their want of portability. In principle, the chronometer is simply a large and very accurately constructed watch; its rate of motion being controlled, not by a pendulum, but by a balance-wheel, which oscillates to and fro under the influence of a steel hair-spring. In order that the chronometer may go at a uniform rate, the balance-wheel is constructed in such a manner that its time of oscillation is unaffected by changes of temperature. If the wheel were made of one continuous piece of metal, any increase of temperature would cause the whole to expand, and the couple exerted by the spring would not reverse its motion so readily, so that the time of oscillation would be increased. To obviate this, the rim of the wheel is made in several (generally three) disconnected arcs, each being formed of steel within and of brass without. When the temperature rises, the supporting arms or spokes expand, pushing the arcs outward; but in each arc the outer half of brass expands more than the inner half of steel, and this causes it to curl inwards, bringing the extremity actually nearer the centre than it was before. The arcs carry small screw weights, and by adjusting these nearer to or further from the supports, the compensation can be arranged with great accuracy.*

Another peculiarity of the chronometer consists in the "detached escapement." The action of the main spring, while keeping up the oscillations, must not affect their periodic time, and to secure this condition the escapement is so arranged that the balance-wheel is only acted on during a very small portion of each oscillation.

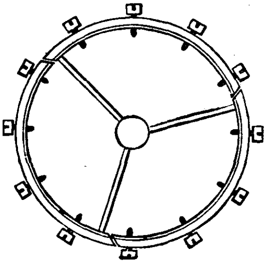
FIG. 119.

The chronometer is usually suspended in gimbals, in such a manner that when the vessel rolls the instrument always swings into a horizontal position; the gimbals also serve to protect it from violent shaking.

The chronometer is not used in aircraft, because of its size and weight and the limitations of space. A high quality watch is employed instead which, though not capable of so high an accuracy of time keeping as the chronometer, is sufficiently accurate for the needs of aircraft, which do not normally remain in the air for more than a few hours at a time and which do not require to know their positions with the same accuracy as ships.

339. Error and Rate of the Chronometer

A chronometer is constructed to keep Greenwich mean solar time. As in the case of the astronomical clock, the amount that a chronometer is slow when it indicates noon is called its *error*, and the amount which it loses in 24 hours is called its *rate*. If the chronometer is fast, the error is negative; if it gains, the rate is negative.

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* The student who has read a little Rigid Dynamics will notice that the compensation must be so arranged that the "moment of inertia" of the balance-wheel is unaffected by the temperature.

The essential qualification of a good chronometer is that its rate must be quite uniform. It is not necessary that the rate shall be *zero*, provided that its amount is known, since a correction can easily be applied to obtain the correct time from the chronometer reading. The error of the chronometer can be controlled and determined by means of radio time signals, giving Greenwich mean time, transmitted from numerous stations at various times during the day.

II.—FINDING THE LATITUDE BY OBSERVATION

340. Classification of Methods

The methods of finding latitude may be conveniently classified as follows:—

A. *Meridian Observations.*

(1) By a single meridian altitude of the Sun or a known star.

(2) By meridian altitudes of two stars, one north and one south of the zenith, taken with the sextant.

(3) By two observations of a circumpolar star.

B. *Observations not made on the Meridian.* ("*Ex-meridian Observations.*")

(4) By a single observed altitude, the local time being known.

  (4A) By "circum-meridian altitudes."

  (4B) By observing the altitude of the Pole Star.

(5) By observations of two altitudes.

(6) By the Prime Vertical instrument.

We now proceed to examine the various methods in detail, but it must be premised that the "ex-meridian" methods cannot be thoroughly explained without spherical trigonometry.

341. Latitude by a Single Meridian Altitude

Let S (Fig. 120) represent the position of the Sun or a star of known declination when southing.

Let the meridian altitude sS be observed, and let it be $= a$; also let z be the meridian Z.D. ZS, so that $z = 90° - a$. Let δ be the known N. decl. QS, and ϕ the required N. latitude QZ. Then in the figure we have:—

$$QZ = QS + SZ;$$

$$\text{or}\quad \phi = \delta + z$$

$$= \delta + 90° - a$$

which determines ϕ.

If the declination be south, δ must be taken negative; if the body transits between the zenith and the north pole, z must be taken negative; and ϕ will be negative if the latitude is south. The first formula will then be applicable in all cases.

In order that the second formula may be universally applicable, a must be the angular distance from the south point of the horizon. If the star transits north of the zenith, as at x, and a_1 denote the altitude nx, the angular distance sx is $a = 180° - a_1$. Therefore :—

$$\phi = \delta + a_1 - 90°.$$

In the case of a circumpolar star x' observed at inferior culmination, the declination $\delta = 90° - Px' = 180° - Qx'$. Hence, $Qx' = 180° - \delta$, and the formula gives

$$\phi = 180° - \delta + z = 180° - \delta - z_1$$
$$i.e.\quad \phi = 90° - \delta + a_1,$$

where z_1 is the north zenith distance and $= -z$.

In numerical calculations the student will find it advisable, in every case, to draw a suitable diagram, and not to rely on mere formulae.

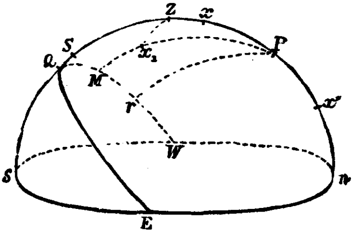

FIG. 120.

342. Finding Latitude at Sea

In finding the latitude at sea, the Sun's meridian altitude is found by means of the sextant in the following manner :—Begin to observe the altitude of either limb about ten minutes before apparent noon, and as the Sun's altitude continues to increase, continue to move the index bar of the sextant with the tangent screw, so that the image of the Sun continues to touch the visible horizon. When the Sun has passed the meridian, and its altitude begins to decrease, the adjustment of the sextant *must not be reversed,* but should be stopped. The reading then gives the greatest altitude of the observed limb.

Owing to the north or south motion of the ship, and to a less extent, to the Sun's motion in declination, the greatest altitude is, in general, slightly greater than the meridian altitude. A correction may be applied, taken from a table. From the sextant reading subtract the corrections for dip and refraction, also add or subtract the Sun's semi-diameter according as the lower or upper limb is observed; thus the Sun's meridian altitude is found.

The Sun's declination is then to be found from the *Nautical Almanac,* which gives the declination at Greenwich midnight, and its hourly rate of variation. To apply the latter correction, the Greenwich time of the observation must be known *roughly* by the ship's chronometer. The declination at the time of the observed transit can then be found. The latitude is given by the formula.

343. Finding Latitude on Land

In finding the latitude on land, by this method, the meridian altitude of a fixed star can be observed with a sextant furnished with an artificial horizon, the declination of the star being found from astronomical tables.

If the Sun be observed, a dark glass cap may be fitted on to the telescope, instead of using the shades. The altitude of the Sun's centre might be found by adjusting the two images to *coincide*, but it is much more easy to adjust the images to *touch*, and thus to find the altitude of the lower or upper limb, preferably the former. Add or subtract the Sun's semi-diameter according to circumstances; thus the meridian altitude of the centre is found.

The meridian Z.D. of a star may also be observed by the zenith telescope (Art. 97). By selecting a star which transits near the zenith, the liability of error in the correction for refraction may be greatly reduced. This is the method employed by a number of observatories engaged in studying the small *Variations of Latitude.* There are two terms in this variation, one with a period of 14 months, the other with an annual period. They are both small fractions of a second of arc.

344. Finding Latitude in a Fixed Observatory

In a fixed observatory, the meridian altitude is found by the transit circle.

The best determinations of the latitude are those resulting from a large series of observations of different stars, extending over a considerable number of years; from such observations on different stars the latitude of the transit circle can be found to within a small fraction of a second, representing a distance of only a few yards.

Example.—*In longitude* 8° 12′ *E.* (*roughly*) *with an artificial horizon, the meridian reading of the sextant for the Sun's lower limb was observed to be* 107° 59′ 48″. *Barometer* 30·7 *inches, the thermometer* 72°. *Find the latitude, given the following data :—*

		°	′	″	
⊙'s (Sun's) decl. at Greenwich noon, Ap. 11th	=	8	19	4	From *Nautical Almanac*
Hourly variation of decl.	=			55·1	
⊙'s semi-diameter	=		15	59	
Mean refraction at altitude 54°	=			41	From Tables.
Correction for barometer	=			+1	
„ *for thermometer*	=			—2	

The calculation is best arranged as follows :—

° ′ ″
(i) Double observed alt. of lower limb ... = 107 59 48
Observed alt. ... ... = 53 59 54
Corrected refraction at this alt.
(which is nearly 54°) ... ... = 40 (—)

True alt. of lower limb ... ... = 53 59 14
Ang. semi-diam. ... ... = 15 59 (+)

Merid. alt. ⊙'s centre ... ... = 54 15 13
Subtract from 90

Merid. Z.D. of ⊙'s centre ... ... = 35 44 47 S. ....(i)

M. S.
(ii) Long. 8° 12′ E. in time ... ... = 32 48
Time of observation is ... ... = 32 48 *before* Greenwich noon.

° ′ ″
⊙'s decl. at Greenwich noon April 11th ... = 8 19 4N. (increasing).
Variation in 30m. before noon ... ... = 27 (—)
" 2m. 48s. (about) ... ... = 3 (—)

⊙'s decl. at time of observation ... ... = 8 18 34 N.
Add ⊙'s merid. Z.D. from (i) ... ... = 35 44 47 S.

Required north latitude... ... ... = 44° 3′ 21″.

345. To Find the Latitude by Sextant Observations of the Meridian Altitudes of Two Stars which Culminate on Opposite Sides of the Zenith

This is really only a modification of the first method. Two stars of known declination are selected which culminate, one south and the other north of the zenith, at very nearly the same altitude. The latitude is calculated independently from observations of the meridian altitudes of either star, and the mean of the two results is taken as the correct latitude. This method possesses the following advantages :—

(1) There is no need to correct the observed altitudes for dip of the horizon ;

(2) The result is unaffected by any constant instrumental errors (index error, etc.) which affect both altitudes equally ;

(3) The correction for refraction is reduced to a minimum, or even entirely eliminated, if the altitudes are almost equal.

For let δ_1, δ_2 be the north declinations of the two stars ;
z_1 (south) and z_2 (north) their true meridian Z.D.'s ;
a_1 and a_2 their observed meridian altitudes ;
u_1 and u_2 the corrections for refraction ;
D the dip of the horizon ;
e the correction for constant instrumental errors.

For true meridian altitudes of the two stars we have :—

$$90^\circ - z_1 = a_1 + e - D - u_1,$$
$$90^\circ - z_2 = a_2 + e - D - u_2.$$

The two observations give, therefore, for the latitude (by Art. 341)

$$\phi = \delta_1 + z_1 = \delta_1 + 90^\circ - a_1 - e + D + u_1,$$
$$\phi = \delta_2 - z_2 = \delta_2 - 90^\circ + a_2 + e - D - u_2.$$

Therefore, taking the mean of the two results,

$$\begin{aligned}\phi &= \tfrac{1}{2}(\delta_1 + \delta_2 + z_1 - z_2)\\ &= \tfrac{1}{2}\{\delta_1 + \delta_2 + (a_2 - a_1) - (u_2 - u_1)\},\end{aligned}$$

a result involving no corrections beyond the difference of refractions, $u_2 - u_1$.

Moreover, if the altitudes a_1 and a_2 are greater than 45°, and their difference $(a_2 - a_1)$ is less than a degree, then $\frac{1}{2}(u_2 - u_1)$ is $<1''$, and therefore the refraction correction may be entirely neglected.

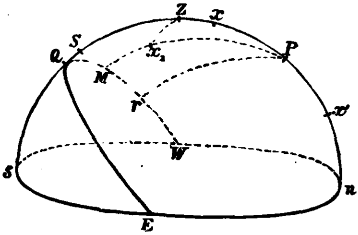

FIG. 121.

346. Latitude by Circumpolars

This method has already been mentioned in Art. 31 but we will here repeat the investigation for convenience.

Let x, x' (Fig. 121) represent the positions of a circumpolar star at its upper and lower transits. Let its meridian altitudes nx and nx' be observed, and let their *corrected* values be a_1 and a_2 respectively.

$$\begin{aligned}\text{Since } Px &= \text{star's N.P.D.} = Px',\\ nP &= \tfrac{1}{2}(nx + nx'),\\ \text{or} \qquad \phi &= \tfrac{1}{2}(a_1 + a_2).\end{aligned}$$

In this formula no knowledge of the star's declination is required, but the observed altitudes require to be corrected for refraction, etc.

The circumpolar method is most useful in determining the latitude of a fixed observatory, because this must be done before the declination of any star can be determined. The transit circle is used to determine the meridian altitudes at the two culminations.

By observing two or more circumpolars the correction for refraction may be found, as in Art. 145, and the observed altitudes may then be corrected for refraction.

As the declinations of a large number of stars are given in astronomical tables, the circumpolar method is never used at sea. It would possess no advantage, and would have the disadvantage of requiring a correction for the change in the ship's place between the two culminations.

Examples.—1. *The observed meridian altitude of β Ceti* (*decl.* 18° 36′ 44·5″ *S.*) *is* 36° 43′ 12″, *and that of α Ursae Minoris* (*decl.* 88° 41′ 53·1″ *N.*) *at its upper culmination is* 36° 9′ 57″, *both altitudes being measured from the "offing," and the dip being unknown. Find the latitude, given*

Refraction at alt. 36° = 1′ 20″; *at alt.* 37° = 1′ 17″.

This is an example of the method of Art. 345. The calculation stands thus:—

β Ceti (south),		*α Ursae Minoris* (north).
36° 43′ 12″	Observed altitudes	36 9′ 57″
— 0 1 18	Refraction corrections	— 0 1 19·5
36 41 54	Corrected altitudes	36 8 37·5
90 0 0		90 0 0
+53 18 6 S.	Zenith distances	—53 51 22·5 N.
—18 36 44·5 S.	Declinations	+88 41 53·1 N.
34 41 21·5 N.	Calculated Latitudes	34 50 30·6 N.

Thus, lat. by star north of zenith = 34° 50′ 30·6″ N.
" " south " = 34 41 21·5 N.

2)69 31 52·1

Mean latitude = 34° 45′ 56″ N.

Here, owing to dip, one of the calculated latitudes is 4′ 34·6″ too great, and the other is 4′ 34·5″ too small, but the mean of the two results is the correct latitude.

2. *The observed altitudes of β Ursae Minoris at lower and upper culmination are* 29° 58′ 15″ and 60° 45′ 3″. *Find approximately the latitude, assuming the coefficient of refraction to be* 57″.

By the "tangent formula," refraction at altitude 30° (approx.)

$= 57'' \tan 60° = 57'' \times \sqrt{3} = 57'' \times 1{\cdot}732 = 1' \, 39''$.

Refraction at alt. 60° $= 57'' \tan 30° = 57'' \times \sqrt{3}/3 = 1' \, 39'' \div 3 = 33''$.

Hence true alt. at lower culmination = 29° 58′ 15″ — 1′ 39″ = 29° 56′ 36″
" " " upper " = 60 45 3 — 33″ = 60 44 30

2) 90 41 6

Required North Latitude = 45° 20′ 33″

We come now to the determination of latitude by *ex-meridian observations.*

347. To find the Latitude by a Single Altitude, the Local Time being known

If the local time be known, a single altitude of the Sun or a known star is sufficient to determine the latitude.

For let S be the observed body, Z the zenith, P the pole.

Then in the spherical triangle PZS, the known local time enables us to find the hour angle ZPS. For, if the Sun be observed:—

its hour angle ZPS = 15 × (apparent local time)

= 15 × (mean local time + equation of time);

and if a star be observed :—

its hour angle $ZPS = 15 \times$ (local sidereal time — star's R.A.)

Also $ZS =$ observed body's Z.D. $= 90° -$ (observed altitude) ;

$PS =$ observed body's N.P.D. $= 90° -$ (known decl.).

Hence, ZS, PS, and the angle ZPS are known. These data completely fix the spherical triangle ZPS, and from them ZP can be found.

By formula (1) of Art. 10,

$$\sin a = \sin \delta \sin \phi + \cos \delta \cos \phi \cos h$$

which determines ϕ when a, δ, h are known. To solve this equation, put

$$\sin \delta = A \cos \theta$$

$$\cos \delta \cos h = A \sin \theta.$$

These two equations determine A and θ. We then have :—

$$A \sin (\phi + \theta) = \sin a$$

which determines $(\phi + \theta)$ and, therefore, ϕ.

348. To find the Latitude by Circum-meridian Altitudes

This is a particular case of the method last described. In attempting to find the latitude by meridian observations, it may happen that passing clouds prevent the body from being observed at the instant of transit. In this case the latitude can be found from the observed altitude when very near the meridian. The hour angle ZPS is then small, and the difference between the observed and meridian altitudes is also small. This difference is called the " Reduction," and is found by approximate methods. Take a number of altitudes of the body before and after passing the meridian.

349. To find the Latitude by a Single Altitude of the Pole Star

The N.P.D. of *Polaris* is only 1° 1′ in 1941, and is diminishing by 18″ per annum. Hence, if its altitude is observed, the latitude may be found by adding to, or subtracting from, this altitude, a small correction, never greater than 1° 1′ for the next three centuries.

Tables for facilitating the reduction were given in the *Nautical Almanac* up to 1916, but have now been discontinued. A table for this purpose is now given in the *Air Almanac.* Treating the triangle as plane, the difference between the star's altitude and the latitude is Star's N.P.D. multiplied by cos Hour-Angle. The altitude is greater than the latitude when the star is within 6h. of its upper culmination.

350. To find the Latitude by Observation of Two Altitudes

By observing the altitudes of two known stars, both the latitude and the local sidereal time can be found.

The same method can be employed to determine the latitude by two observations of the Sun's altitude, separated by a known interval of time.

The necessary calculations are very complicated, involving spherical trigonometry, and they cannot be materially simplified even by the use of tables.

A very useful *geometrical* construction, enabling us, from the two observed altitudes, to indicate the exact position of a ship on a globe without calculation, will be detailed in Section VI.

351. To find the Latitude by the Prime Vertical Instrument

The latitude of a fixed observatory may be found by means of an instrument similar to the transit circle, but whose telescope turns in the plane of the prime vertical instead of the meridian. A star will cross the middle wire of such an instrument when its direction is either due east or west; the times of the two transits are observed. Let S, S' be the positions of a known star at its eastern and western transits, Z the zenith, P the pole. The sidereal interval between the two transits determines the angle SPS', and this is evidently twice the angle ZPS. Hence $\angle ZPS$ is known. Also PS, the star's N.P.D., is known, and PZS is a right angle. The spherical triangle ZPS is completely determined, and the colatitude ZP can be found. Formula (1), of Art. 10 gives

$$\cos h = \tan \delta \cot \phi$$

which determines the latitude ϕ.

The times of the transits are unaffected by refraction, and this fact constitutes the principal advantage of the method.

III.—TO FIND THE LOCAL TIME BY OBSERVATION

352. The Methods Employed

In determining the longitude of a place on the Earth, the first step is to find the local time by observations of the hour angle of a known celestial body. If the time indicated by a chronometer or clock at instant of observation be also noted, we shall find the difference between the true local time and the indicated time. This difference is the error of the clock on local time.

In Art. 54 we described one instrument for observing local time—the sun-dial. This cannot, however, be used except for very rough observations, as the boundary of the shadow cast by the style is not sufficiently well defined to admit of accurate measurements.

For this reason the local time is usually found by one or other of the following methods:—

(1) By meridian observations.
(2) By equal altitudes.
(3) By a single altitude, the latitude being known.
(4) By observation of two altitudes.

353. Local Time by Meridian Observations

In a fixed observatory, the local sidereal time is found by means of the transit circle, as explained in Arts. 20, 300. The transit of a known star is observed; the local sidereal time of transit is equal to the star's R.A., and is therefore known.

Or by observing the transit of the Sun's centre, the time of *apparent* local noon may be found. The equation of time is the difference between mean time and apparent time and is given in the *Nautical Almanac* for 0h. G.M.T. each day. The value of the equation of time for the instant of observation is obtained by interpolation. Hence the local mean time is found.

These methods are not available at sea, as the transit circle cannot be used. It might be thought that we could use a sextant to ascertain the instant when the body's altitude is greatest, but, for a short interval before and after the transit, the altitude remains very nearly constant; it is therefore impossible to tell with any degree of accuracy when it is a maximum.

On the other hand, a slight error in the time of observation does not affect the altitude perceptibly, so that the meridian altitude may be observed with great accuracy, as in Art. 342.

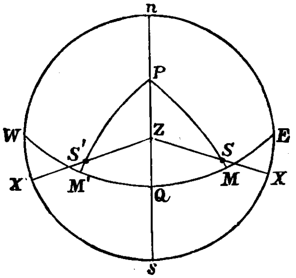

FIG. 122.

354. Method of Equal Altitudes

When it is required to find the local time from observations taken with a sextant, the simplest method is as follows:—Observe the altitude of any celestial body some time before it culminates. After the body has passed the meridian, observe the instant of time when its altitude is again the same as it was at the first observation. Half the sum of the times of the two observations gives the time of transit.

For let S, S' be the two observed positions of the body, Z the zenith, and P the pole. The altitudes of SX, $S'X'$ being equal, the zenith distances are equal; or

$$ZS = ZS'.$$

Also $PS = PS'$, and the spherical triangles ZPS, ZPS' have ZP in common.

Therefore $$\angle SPZ = \angle ZPS'.$$

Now let t_1 and t_2 be the times of the two observations, t the time of transit. Then $t - t_1$ is the time taken to describe the angle SPZ

and $t_2 - t$ is the time taken to describe the angle ZPS'. Since the two angles are equal :—

$$t - t_1 = t_2 - t;$$
$$t = \tfrac{1}{2}(t_1 + t_2).$$

From the time of transit the local time can be found, as in the last article.

355. Observing the Equal Altitudes with a Sextant

In observing the equal altitudes with a sextant the following method is used :—At the first observation clamp the index bar at an altitude slightly greater than that of the body. Continue to observe the body as it rises, till its image is in contact with the horizon, and note the instant of time (t_1) at which this happens. Keep the index bar clamped until the second observation; commence observing the body again just before it has reached the same altitude again, and note the instant of time (t_2) when its image is again in contact with the horizon. The two observed times (t_1, t_2) are the times of equal altitude.

If an artificial horizon be used, we must observe the two instants of time (t_1, t_2) when the two images are in contact.

356. Equation of Equal Altitudes

If the Sun be the observed body, its declination will, in general, change slightly between the two observations; hence PS will not be exactly equal to PS', and the angles SPZ, ZPS' will not be quite equal. For this reason a small correction must be applied, in order to allow for the effect of the change of declination. This correction is called the *Equation of Equal Altitudes*, and may be found from tables which have been calculated for the purpose.

At Sea allowance must also be made for the change of position of the ship between the two observations, and this correction is also effected by means of tables.

357. Advantages of the Method of Equal Altitudes

(1) The results are unaffected by errors of graduation of the sextant, for the actual readings are not required.

(2) The semi-diameter of the observed body need not be known.

(3) The observed altitudes, being equal, are equally affected by refraction, and no refraction correction need be made, other than the small ones depending on change in the barometer and thermometer between the observations.

(4) The dip of the horizon need not be known, provided that it is the same at both observations.

358. Finding Time of Apparent Noon with a Gnomon

With a *Gnomon*, the time of apparent noon can be roughly found in a very simple manner. A rod is fixed vertically in a horizontal plane, and on the latter are drawn several circles, concentric with the base of the rod. Let the times be observed, before and after noon, when the extremity of the shadow cast by the rod just touches one of these circles. At these two instants the Sun's altitudes are, of course, equal, and therefore the time of apparent noon is the arithmetical mean between the observed times. Greater accuracy is secured by mounting a small bead on a needle at the top of the rod, and observing the centre of the shadow of the bead. This is a more definite point than the shadow of the top of the rod.

Example.—*The shadow of a vertical stick at Land's End (long.* 5° 40′ *W.) is observed to have the same length at* 9*h.* 27*m.* A.M. *and* 3*h.* 1*m.* 40*s.* P.M., *Greenwich time. Find the equation of time on the day of observation.*

Greenwich mean time of local apparent noon is :—

$$\tfrac{1}{2}\{9\text{h. } 27\text{m. } 0\text{s.} + 3\text{h. } 1\text{m. } 40\text{s.} - 12\text{h.}\} = 14\text{m. } 20\text{s.}$$

But, by Art. 64, Greenwich mean time of local mean noon = 22m. 40s.

Thus local mean noon is 8m. 20s. after local apparent noon. When the local mean time is 12h. the local apparent time is 12h. 8m. 20s. Thus (Art. 47) the equation of time is + 8m. 20s.

359. Latitude by the Method of Equal Altitudes

The latitude may also be found by the method of equal altitudes, though the calculations require spherical trigonometry. For this purpose, the altitude at either observation must be read off on the sextant, and corrected for refraction, dip, etc. The zenith distance SZ is therefore known The angle SPZ is also known, being half the angle described in the interval $t_2 - t_1$, and PS, being the complement of the declination, is also known. The spherical triangle ZPS is therefore completely determined, and ZP, which is the complement of the latitude, can be found. The formula is the same as that given in Art. 347.

360. Local Time by a Single Altitude, the Latitude being known

This is the converse of the method for finding the latitude (Art. 347). If the altitude of a known body, S, be observed in known latitude, we know ZS, SP, PZ, the declination and the latitude respectively ; hence the hour angle SPZ, and the local time, may be found. The formula is the same as that given in Art. 347.

*361. Local Time by Two Altitudes

The method of Art. 350 determines, not only the latitude, but also the hour angles of the bodies at the two observations, and these determine the local time. The method of equal altitudes is in reality only a particular case.

IV.—DETERMINATION OF THE MERIDIAN LINE

362. The Methods Employed

Before setting up a transit circle or equatorial in a fixed observatory, it is necessary to know with considerable accuracy the direction of the meridian line, *i.e.* the line joining the north and south points of the horizon. At sea, the directions of the cardinal points are determined by a mariner's compass; but here, too, it is of great use, on long voyages, to determine the *variation of the compass*, or the deviation of the magnetic needle from the meridian line. This deviation is different at different parts of the Earth.

There are three ways of finding the meridian line: first, by two observations of a celestial body at equal altitudes; second, by a single observation of the azimuth; third, by one or more observations of the Pole Star.

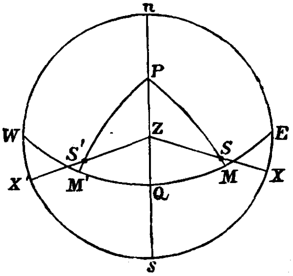

FIG. 123.

363. Meridian Line by Equal Altitudes

When a body has equal altitudes before and after culmination, the corresponding azimuths are equal and opposite.

For if S, S' denote the two positions of the body, the triangles ZPS, ZPS' are equal in all respects; so that

$$\angle PZS = \angle PZS' \quad \text{or} \quad \angle nZS = \angle nZS'.$$

364. Use of Method

At Sea, the Sun's compass bearing may be taken, and the local time noted; from the latter and the latitude, the Sun's azimuth may be derived from Burdwood's or other Azimuth Tables, and hence the variation of the compass may be found.

On Land, we may observe the directions of the shadow cast by a vertical rod on a horizontal plane when it has equal lengths; for this purpose we mark the points at which the end of the shadow just touches a circle concentric with the base of the rod (*cf.* Art. 358). Bisecting the angle between the two directions the north and south points are found.

*365. Meridian Line by a Single Observation

If the direction of the vertical plane through a single celestial body S be observed at any instant, the direction of the meridian line may be found by means of spherical trigonometry.

For if any three parts of the triangle ZPS are known, the triangle is completely determined, and the angle PZS can be found. (See Fig. 123).

The azimuth $sZS = 180° - PZS$, and is then known; hence the meridian line ZS is found.

Now the sides PS, ZS, ZP are the complements of the declination, the altitude, and the latitude; and the hour angle ZPS is known, if the local time be known. Any three of these data are sufficient to determine the angle PZS.

Thus the Sun's direction, *either* at sunrise *or* at sunset, determines the meridian line, if either the local time or latitude is known. This should not be used if accuracy is required; refraction near the horizon is uncertain.

366. Meridian Line by Observations of the Pole Star

The direction of the meridian may be very accurately determined by observations of the star *Polaris*. If the azimuthal readings of this star be observed at the two instants when it is furthest from the meridian, east and west respectively, the reading for the meridian is half their sum. The observations may be made with a theodolite. The azimuth at either observation is a maximum, and it remains very nearly constant for a short interval before and after attaining its maximum. Hence, a slight error in the time of observation will not perceptibly affect the azimuth. The same method is applicable to any star which culminates between the pole and the zenith.

The *Nautical Almanac* gives each year a table in which the azimuth of Polaris is tabulated for north latitudes 10°, 15°, 20°, then at intervals of 2° to 60°, then at intervals of 1° to 70°, for intervals of 10m. in hour angle. By simple interpolation the azimuth at any instant and for any latitude from 10°N. to 70°N. can be obtained. If then the azimuthal reading of Polaris at this instant is observed, the reading for the meridian is readily inferred.

An alternative method is that employed in finding the azimuth error of the transit circle (see Art. 305). If the telescope always moves in the plane of the meridian, the interval from upper to lower culmination, and the interval from lower to upper culmination, will both be exactly twelve sidereal hours. If not, the small amount by which the vertical plane swept out by the telescope is east or west of the meridian, can be found by observing the amounts by which the two intervals are greater and less than 12h.

V.—LONGITUDE BY OBSERVATION

367. Various Methods

In Section III of the present chapter we showed how the local time can be found by observing the celestial bodies. When this has been done, the longitude of the place of observation may be found by comparing the observed local time with the corresponding Greenwich time. For in Art. 64 we showed that if the longitude of a place west of Greenwich be L°, then

$$(\text{Greenwich time}) - (\text{local time}) = \tfrac{1}{15}L\text{h.} = 4L\text{m.};$$

whence, knowing the difference of the two times, L may be found.

The determination of longitude is thus reduced to the determination of Greenwich time at the instant of observation. Before the days of telegraphy, this was not an easy matter. The method generally used involved the transport of a large number of chronometers. These were carefully rated and their errors determined, either at Greenwich or at a place whose longitude was known. They were then transported to the place whose longitude was required. If the rates of the chronometers had remained constant, Greenwich time could be inferred from each chronometer by allowing for its accumulated error. But the rates of even the best chronometers are liable to small erratic changes, so that only by chance would any one chronometer give the Greenwich time correctly. The purpose of transporting a large number of chronometers was to reduce the uncertainty, by using the average of the times given by each. But the method is at best an unsatisfactory one and incapable of giving a longitude with precision.

Longitudes at sea were determined by the *method of lunar distances.* In this method the Moon, by its rapid motion among the stars, takes the place of a chronometer, its position relative to the neighbouring stars determining the Greenwich time. The Moon moves through 360° in $27\frac{1}{3}$ days; hence it travels at the relative rate of about 33′ per hour, or rather over 1″ in every 2s., and this motion is sufficiently rapid to render it available as a timekeeper. The angular distance of the star from the nearest or furthest point of the Moon's disc (according as one or other is illuminated) is measured with a sextant, the exact chronometer time of the observation being noted. The application of the semi-diameter (taken from the *Almanac*) gives the distance from the Moon's centre. The method involves laborious calculations and is not capable of very high precision. It served to provide a control on the rate of the chronometer. The method was used so seldom, however, that since the year 1907 the *Nautical Almanac* has discontinued the publication of the lunar distances of stars. It is now only of historic interest.

The submarine telegraph cable provided an improved method for the determination of longitudes on land, but this has now been entirely superseded by radio telegraphy, both on shore and at sea.

368. Longitude by the Chronograph

When two observatories are in telegraphic communication, the local time may be readily signalled from one to the other by means of the electric current, and the difference between the longitudes thus determined.

This method is employed in connection with the chronographic method of recording transits, the chronographs being connected by the telegraph line, so that a transit is recorded nearly simultaneously at both stations. Let us call the two stations A and B. When the star crosses the meridian at A, the observer presses the button of his chronograph. Let t_1, t_2 be the times of transit at A as thus recorded at A and B respectively. When the same star crosses the meridian at B, the times of transit are again recorded at A and B. Let these recorded times be T_1 and T_2 respectively.

The transmission of the signal from one station to the other is not quite instantaneous, because a small interval of time must always elapse before the current has attained sufficient strength to make the signal at the distant station. Let this interval be x. Then the transit at A will be recorded too late at B by the amount x, and the transit at B will be recorded too late at A by the same amount x.

When this correction is applied, the true times of the two transits, as determined by the chronograph record at A, will be t_1 and $T_1 - x$. Hence, if L denote the difference of longitude in time measured westwards from A to B, the chronograph record at A gives

$$L = T_1 - x - t_1.$$

Again, the true times of the two transits, as determined by the chronograph record at B, will be $t_2 - x$ and T_2. Hence the chronograph record at B gives

$$L = T_2 - (t_2 - x) = T_2 - t_2 + x.$$

By addition, we have :—

$$2L = T_1 - t_1 + T_2 - t_2\,; \quad \text{or} \quad L = \tfrac{1}{2}\,(T_1 - t_1 + T_2 - t_2),$$

a result which does not involve x. Thus we see that, by using both chronograph records, and taking the mean of the separately calculated differences of longitude, the corrections due to the time occupied by the passage of the signals are entirely eliminated. Actually it suffices to find the error of the clock at each station by star-transits, and then compare the clocks by chronograph signals.

*369. Elimination of Personal Equation

In the above investigation we have taken no account of the personal equations of the two observers. But if e is the correction for personal equation of the observer at A, and E is that of the observer at B, the observed times t_1, t_2 must both be increased by e, and T_1, T_2 must both be increased by E. Introducing these corrections, the formula gives

$$L = \tfrac{1}{2}\,(T_1 - t_1 + T_2 - t_2) + (E - e).$$

To eliminate the corrections, let the two observers change places, and repeat the operations, and let the new recorded times of transit be denoted by accented letters. The correction E must now be applied to the times t_1', t_2', and the correction e must be applied to T_1', T_2'. Therefore :—

$$L = \tfrac{1}{2}(T_1' - t_1' + T_2' - t_2') + (e - E).$$

By again taking the mean of the two results we get

$$L = \tfrac{1}{4}\{(T_1 - t_1 + T_2 - t_2) + (T_1' - t_1' + T_2' - t_2')\},$$

a result in which the personal equation is eliminated.

The introduction of the " moving wire " method of taking transits (Art. 296) has practically eliminated personal equation, so that exchange of observers is no longer considered necessary.

370. Longitude by Radio Time Signals

This method is now used almost exclusively both on land and at sea. Radio time signals are sent out daily, at specified Greenwich mean times, from various radio stations. Thus, for instance, time signals are sent out daily from Greenwich Observatory, via the Rugby radio station, at 10h. and 18h. G.M.T. The signals are used to determine the error of the clock or chronometer that provides the standard of time at the place on land whose longitude is required or on the ship at sea. By observing the signals on successive days, a continuous check on the error and rate of the clock or chronometer is obtained, so that the error at any intermediate time can be inferred.

The local time is found as described in Section III. The corresponding Greenwich time at the same instant is inferred by applying the appropriate correction to the time given by the clock or chronometer. Hence the longitude east or west of Greenwich can be inferred.

The radio time signals sent out from the Greenwich or any other Observatory are based on a predicted clock error and may not be transmitted at the exact hour of Greenwich Mean Time ; the signals are sent from the Observatory to the radio station by land line and though allowance is made for the time taken in transmission over the line, this time is subject to small variations. The difference between the time at which the signal is actually transmitted and the time at which it should be transmitted is a small quantity, rarely as great as 0·05s., and can for most purposes be neglected. Where great accuracy is required, however, as in surveying operations, the appropriate corrections must be applied. Corrections to the actual times of transmission of the signals from Greenwich Observatory are published at approximately monthly intervals. For high accuracy, the travel time of the radio signal (with a speed equal to that of light) must also be allowed for.

Examples (1).—*At apparent noon a chronometer indicates 19h. 33m. 25s., Greenwich mean time, and the equation of time is + 2m. 1s. Find the longitude.*

As the equation of time is		+	2m.	1s.
Greenwich mean time — local mean time	=	19h.	35m.	26s.
Mult. by 15, we have long. W. of Greenwich	=	293°	51′	30″
or sub. from 360°, long. E. of Greenwich	=	66°	8′	30″

(2).—*Find the longitude, from the following data :—Sun's computed hour angle* = 75° *E. Time by chronometer* = 11*h.* 7*m.* 31*s. Equation of time* = — 3*m.* 55*s. Correction for error and rate,* — 1*m.* 18*s.*

(i) Here ⊙'s hour angle in time	=	5h. before noon		
So that apparent local time	=	7h.	0m.	0s.
Equation of time	=		3	55 (—)
Thus mean local time	=	7h.	3m.	55s.
(ii) Observed time	=	11h.	7m.	31s.
Correction	=		—1	18
Greenwich time	=	11	6	13
		7	3	55
W. Long. in time	=	4	2	18
				15
Therefore *required long.*	=	60°	34′	30″ W.

(3).—*On June* 29*th, from a ship in the North Atlantic Ocean, the Sun was observed to have equal altitudes when the chronometer indicated* 11*h.* 27*m.* 26*s. and* 6h. 48*m.* 32*s. At noon on June* 25*th, the chronometer was* 3*s. too fast, and it gains* 8*s. a day. The equation of time on June* 29*th at* 3 *p.m. was* — 2*m.* 58*s. Find the ship's longitude.*

The process stands as follows :—		H.	M.	S.
Chronometer time of first observation	=	11	27	26
,, ,, ,, second observation + 12h. ...	=	18	48	32
	2)	30	15	58
		15	7	59
Hence the chronometer time of local apparent noon ...	=	3	7	59
Correction for chronometer error June 25th = — 3s. ,, ,, ,, rate in 4 days = — 32s. ,, ,, ,, ,, ,, 3 hours = — 1s.	=			—36
Thus Greenwich time of local apparent noon	=	3	7	23
Add equation of time (since mean noon occurs first) ...	=		—2	58
Greenwich time of local mean noon	=	3	4	25
				15
Longitude west of Greenwich	=	46°	6′	15″

(4).—*At* 17*h. by a chronometer, the Greenwich mean time was found to be* 16*h.* 59*m.* 57·2*s. It was taken to a place A, and indicated* 4*h., when the local mean time was* 3*h.* 47*m.* 46·9*s.*; *and when it indicated* 11*h., the Greenwich time was* 11*h.* 0*m.* 9·7*s. Find the longitude of A in time and in angle.*

Here, at 17h., the chronometer error by Greenwich time was —2·8s.
„ „ 24+11h. „ „ „ „ „ +9·7s.
Therefore, in 18h. the chronometer lost 12·5s.;

and the loss in 11h. $= \frac{11}{18} \times$ 12·5s. = 7·64s. nearly;

Thus, the Greenwich time, when the chronometer indicated 4h., was
= 4h. — 2·8s. + 7·64s. = 4h. 0m. 4·84s.,
and the local time at the same instant was = 3h. 47m. 46·9s.
Therefore required longitude = 12m. 17·9s. W. = 3° 4′ 29″ W.

(5).—*As a ship starts from Liverpool, its chronometer indicates 0h., and is correct by Greenwich mean time. After* 16 *days, as it reaches Quebec, the chronometer indicates 7h. 0m. 23s., and Quebec time is 2h. 5m. 42s. Nearly seven days afterwards, the ship departs at Quebec 0h., the chronometer then reading 4h. 54m. 39s.; and when it reaches Liverpool, after a voyage of just over fourteen days, it is found to be 17s. slow by Greenwich mean time. Find the longitude of Quebec.*

By Quebec time, the ship stayed in port 7d. —2h. 5m. 42s. = 6d. 21h. 54m. 18s.
By chronometer, the ship stayed in port 7d. 4h. 54m. 39s. —7h. 0m. 23s.
= 6d. 21h. 54m. 16s.

Thus in 7 days in port, chronometer lost ... ... 2s.
But in 37 days altogether, „ „ ... ... 17s.
Therefore, in 30 days at sea, „ „ ... ... 15s.
and in 16 days, from Liverpool to Quebec, it lost ... ... 8s.
But chronometer time on arrival was ... ... 7h. 0m. 23s.
Greenwich time was therefore ... ... 7h. 0m. 31s.
And local time was ... ... 2h. 5m. 42s.

The difference = longitude of Quebec (in time) — 4h. 54m. 49s. or *Longitude of Quebec (in angle) = 73° 42′ 15″W.*

[Actually Quebec does not now keep local time, but time differing by exactly 5 hours from Greenwich time.]

VI.—GEOGRAPHICAL POSITION OF A CELESTIAL BODY AND THE POSITION CIRCLE

371. Geographical Position of a Celestial Body

The observation of a celestial body made by the navigator of a ship or aircraft gives, after correction for refraction and also, except when a bubble sextant is used, for the dip of the horizon gives the altitude of the body at a definite instant of Greenwich mean time.

The data given by this observation enable the point *U* on the Earth's surface, which has the celestial body in its zenith, to be determined. This point is called the *sub-solar point* in the case of the Sun and the *sub-steller point* in the case of a star; it is generally known as the *geographical position of the body.*

As the celestial body is in the zenith at the point *U* on the Earth's surface, the latitude of *U* is equal to the declination of the body. The *Nautical Almanac* tabulates the R.A.'s and decls. of the celestial bodies at certain intervals of G.M.T. The positions of the Sun and planets are given for each day at 0h. G.M.T.; the position of the Moon is given at each hour of G.M.T.; the positions of the stars are given at 10-day intervals. By interpolating the declination for the G.M.T. of the observation, the latitude of *U* is obtained.

The longitude of *U*, measured westwards from Greenwich, is the Greenwich hour-angle of the body observed. If the Sun is the observed body, the Greenwich hour-angle is the apparent solar time (measured from apparent noon), at Greenwich, which may be found by adding the equation of time to the mean time (measured from midnight) and adding or subtracting 12h. If the body is the moon or a planet or a star, it is necessary to find first the Greenwich sidereal time of the observation, as described in Art. 60 or Art. 61. The Greenwich hour-angle is then obtained by subtracting the right ascension of the body, interpolated for the G.M.T. of observation, from the Greenwich sidereal time.

Thus both the longitude and the latitude of the geographical position, *U*, are obtained.

372. The Position Circle

The zenith distance of a celestial body, as seen from any place on the Earth's surface, is equal to the angular distance of the place from the geographical position *U*, being in each case the angle between the direction towards the zenith at the given place and at *U*. Hence, the places at which the celestial body has a given zenith distance all lie on a small circle of the celestial globe, whose pole is at the geographical position *U* and whose angular radius is equal to the zenith distance of the body.

This small circle on the Earth's surface is called a *position circle*.

An observation of the zenith distance of a celestial body therefore gives the information that the observer is on a certain circle on the Earth's surface, provided that the G.M.T. of the observation is known. The position of observation may be assumed to be known approximately, from dead reckoning or otherwise. In the vicinity of the position of observation, the position circle will be practically a straight line; the observation therefore provides the information that the ship or aircraft was, at a certain instant, somewhere on a definite line, called the *position line*.

Though by means of a single observation the position of the ship or aircraft cannot be fixed more definitely, the information that it is on a certain position line may be of great value. If, for instance, it was desired to make a small island out at sea, a convenient course might be followed until a position line was obtained that passed over the island. Then, by changing course and running down the position line, the island would be found.

If two different bodies are observed at the same G.M.T., two position circles can be drawn, which intersect in two points. The observer must be at one or other of these points of intersection; the observer's position being known approximately, one of these points of intersection can be ruled out of consideration, and the observer's position is therefore definitely fixed. The two observations will not normally be made at the same G.M.T.; it will then be necessary to allow for the motion of the observer between the two observations.

373. The Intercept

For the practical application of the position line method of determining position, it is convenient to make use of an *assumed position*. The speed and course of the ship or aircraft being known and allowance being made for drift from tides or wind, a position known as the *dead reckoning position* is obtained. This position or any convenient position in its vicinity can be adopted as the *assumed position*; the final position deduced from the observations will be independent of the particular assumed position that is used.

In Fig. 124 P is the Pole, U the geographical position of the observed body, D is the assumed position of the observer, AJB is the position circle, J the point in which the great circle connecting U and D cuts the position circle.

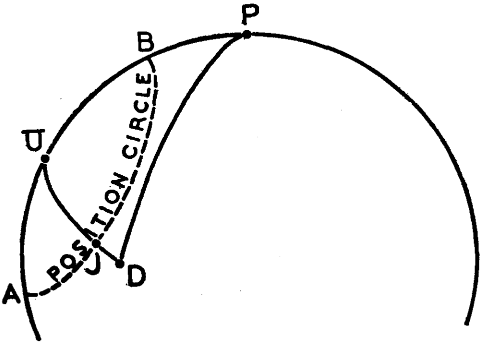

FIG. 124.

In the spherical triangle DPU, DP is $90° - \phi$, where ϕ is the latitude of the assumed position; UP is $90° - \delta$, where δ is the declination of the observed body at the instant of observation. Also the angle DPU is equal to the hour-angle of the celestial body for the instant of observation and for the position D. This angle is $(h - \lambda)$, where h is the Greenwich hour-angle of the observed body and λ is the longitude of the assumed position D, which is positive if D is west of Greenwich and negative, if east of Greenwich: thus the angle DPU is known.

Thus in the triangle DPU, the two sides PD, PU and the included angle DPU are known. By the formulae of spherical trigonometry, the side DU and the angle PDU can therefore be calculated. The formulae are (DU being denoted by D) :—

$$\cos D = \sin \phi \sin \delta + \cos \phi \cos \delta \cos (h - \lambda)$$

gives D, and D being known, the angle $PDU = A$ is given by :—

$$\sin \delta = \sin \phi \cos D + \cos \phi \sin D \cos A.$$

Special volumes of tables are available from which the distance and azimuth of U from D can be obtained for given hour-angle, latitude and declination. When using such tables, it is convenient to adopt for the assumed position one whose longitude and latitude are such that no interpolation in the tables is required for hour-angle or latitude ; the only interpolation then needed is for the declination.

Now JU is equal to the observed zenith distance (*i.e.* 90° minus observed altitude), because the position circle is a small circle with this radius. Hence JU is known ; DU has been calculated. Thus the angular distance DJ is obtained by subtracting the *observed* zenith distance from the *computed* zenith distance. This distance is called *the intercept* and will always be a small quantity, because the assumed position will not be far from the true position, which is somewhere on the position circle. This makes it possible to treat the plotting of the position line as a problem in plane trigonometry. The intercept DJ, in minutes of arc, is equal to the distance of J from D in nautical miles. If DJ is negative, J lies on UD produced.

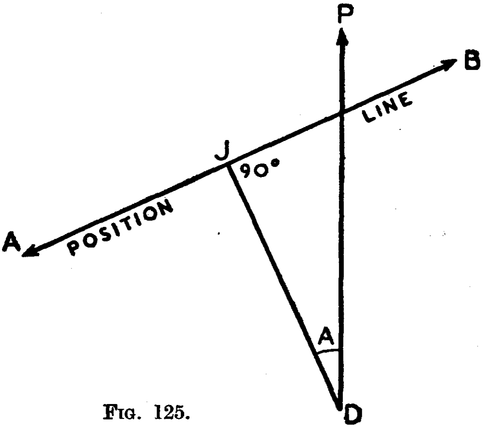

FIG. 125.

In Fig. 125, D represents the assumed position, plotted on a chart. DP is the north direction on the chart. From D draw DJ making with DP the angle A, the azimuth derived above. On DJ mark off the point J, so that DJ, on the scale of the chart, is equal to the intercept in nautical miles (if the intercept is negative, J will lie on the opposite side of D). From J draw the line AJB at right angles to DJ. This is the required position line. It is the only portion of the position circle with which the navigator is concerned. The same line would have been obtained if any other assumed position had been used, provided only that it is sufficiently near the true position for the small arc DJ of a great circle to be treated as a straight line. In practice the errors will not be great if DJ does not exceed 20 nautical miles ; if it proves to be in excess of this, it is advisable to change the assumed position to one that is nearer the position line.

374. Determination of Position by Position Line Method

Two position lines must be drawn to determine the position of a ship or aircraft. Since it must lie on each of the two position lines, the position is given by their point of intersection. This assumes that both observations are made at the same instant of G.M.T. Though it is not possible for one observer to observe two objects simultaneously, it is possible for the first object to be observed again after the second object, so that the mean of the two observations agrees with in line with the second observation.

If the two objects are not observed at the same time, it is necessary to transfer the first position to allow for the motion of the ship or aircraft between the two observations. In Fig. 126, *D* is the assumed position for the first observation, *AB* the position line and *DJ* the intercept. Suppose *JJ′* represents in magnitude and direction the motion of the observer in the interval between the two observations. Then, *at the instant of the second observation*, the observer must lie on a line *A′B′* drawn through *J′* parallel to *AB*. *J′* can be used as the assumed position for the second observation or any convenient position *D′*, near *J′*, can be used. The second position line *PQ* is drawn from the results of the second observation. The intersection *O* of the two lines *PQ, A′B′* is the position at the instant of the second observation.

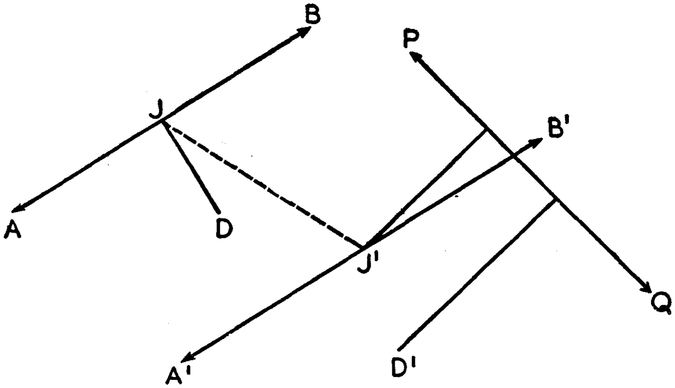

FIG. 126.

An error in observation will result in an error in drawing the position line. For a given error in the position line, the error in the position of the point of intersection *O* will be greater the more acute the angle at which the two position lines cut one another. This angle should be as near to 90° as possible and should not in any case be smaller than 30°. But since the azimuth of the position line differs by 90° from the azimuth of the object observed, this implies that the azimuths of the two objects should differ by at least 30° in order to get a good intersection of the position lines and should preferably be as near to 90° as possible.

375. The Air Almanac

It has been shown in Art. 371 that the geographical position of a body is determined by the Greenwich hour-angle and the declination of the body for the G.M.T. of the observation. In the *Nautical Almanac* (and also in the *Abridged Nautical Almanac*, which is specially adapted for the use of seamen) the tabulated quantities are right ascension and declination; the R.A. is given in time (hours, minutes and seconds) and the declination in arc. The Greenwich hour-angle must be computed and converted into arc.

The *Air Almanac* has been arranged to facilitate the determination of position in the air. Because of the speed with which an aircraft moves, the air navigator requires to draw the position line with the minimum of delay after making the observation. The *Air Almanac* accordingly tabulates the two quantities that are required for obtaining the geographical position: the Greenwich hour-angle (G.H.A.) in arc (*not* in time) and the declination. Also in the *Nautical Almanac* the tabulated quantities for the Sun, Moon and planets are given at infrequent intervals—daily intervals for the Sun and planets, hourly intervals for the Moon—so that the position for the time of observation has to be carefully interpolated. In the *Air Almanac* the tabulated quantities for the Sun and Moon are given at intervals of 10 minutes; those for the planets are given at intervals of one hour (though it is intended shortly to give the positions of the planets also at intervals of 10 minutes). The interpolation for the time of observation is simple and is facilitated by special tables.

For the stars, the *Nautical Almanac* tabulates right ascensions and declinations and gives also the Greenwich sidereal time at 0h. each day (*i.e.* at Greenwich midnight). The Greenwich hour-angle of the star at the instant of observation must be computed. The *Air Almanac* gives data for 50 of the brightest stars and instead of right ascension tabulates what is called the *sidereal hour-angle.* The sidereal hour-angle of a star is the hour-angle measured westwards from the First Point of Aries. It is therefore equal to 360° minus the R.A. of the star, and is expressed in arc. The G.H.A. of a star is equal to the G.H.A. of the First Point of Aries plus the sidereal hour-angle of the star. The sidereal hour-angle of a star is a quantity that varies very slowly through the year, depending only upon the slight change in apparent position of the star. The G.H.A. of the First Point of Aries, on the other hand, is a quantity that varies rapidly. It is necessary to interpolate it for the G.M.T. of the observation. This quantity is tabulated at intervals of 10 minutes of time; a table is provided to facilitate interpolation for the time of observation.

In the *Abridged Nautical Almanac*, used for surface navigation at sea, the data are given to an accuracy of one-tenth of a minute of arc. In the *Air Almanac* the data are given to the nearest minute of arc, which is adequate for the requirements of air navigation.

EXAMPLES

1. At noon on the longest day a circumpolar star is passing over the observer's meridian, and its zenith distance is the same as that of the Sun's centre; at midnight it just grazes the horizon. Find the latitude.

2. On January 2nd, on a ship in the North Atlantic in longitude 48° W., it was observed that the Sun's meridian altitude was 15° 21′ 45″. The Sun's declination at noon at Greenwich on the same day was 22° 54′ 33″, and the hourly variation 13·78″. Find the ship's latitude.

3. Show how to find the latitude by observing the *difference* of the meridian zenith distances of two known stars which cross the meridian on opposite sides of the zenith at nearly equal distances from it. Explain whether the stars chosen should be near to or remote from the zenith. Give also the advantages and disadvantages of this method of finding the latitude, as compared with the method of circumpolars.

4. On a certain day the observed meridian altitude of *a Cassiopeiae* (declination 55° 49′ 11·1″ N.) was 85° 10′ 18″. The eye of the observer was 18 feet above the horizon, and the error for refraction for the altitude of the star is 5″; determine the latitude.

5. A vertical rod is fixed exactly in the centre of a circular fountain basin, and it is observed that on the 25th of July the extremity of the shadow exactly reaches the margin of the water at 10h. 7m. A.M., and at 2h. 25m. P.M. The equation of time on that day is + 6m. What is the error, compared with local time, of the watch by which these observations were taken?

6. A chronometer is set by the standard clock at Greenwich at 6 A.M. It is then taken to Shepton Mallet, and indicates noon when the local time is 11h. 49m. 50s. The chronometer is then brought back to Greenwich, and indicates 9 P.M., when the correct time is 8h. 59m. 55s. Find the longitude of Shepton, supposing the chronometer rate uniform.

7. Amerigo Vespucci is said to have found his longitude in latitude 10° N. in the following manner. At 7.30 P.M. the Moon was 1° E. of Mars, at midnight the Moon was 5½° E. of Mars. The Nuremberg time of conjunction of the Moon and Mars was midnight. Hence he calculated that his longitude was 82½° W. of Nuremberg. Discuss the accuracy of the method, and point out the necessary corrections.

8. A chronometer whose rate is uniform is found at Greenwich to have an error of δ_1 hours when the time which it indicates is t_1. It is then taken to a place A, and when it indicates t_2 it is found that the excess of the observed *local* time of the place A over t_2 is δ_2 hours. It is now brought back to Greenwich, and the chronometer time and error are observed to be t_3 and δ_3 hours respectively. Prove that the longitude of A east of Greenwich is

$$15\,(\delta_2 t_3 + \delta_3 t_1 + \delta_1 t_2 - t_2\delta_3 - t_3\delta_1 - t_1\delta_2)/(t_3 - t_1) \text{ degrees.}$$

9. The sidereal times of transit of a certain star across the meridian of an observatory A, as recorded at A, and by a telegraphic signal at B, are t_1, t_2 respectively. The sidereal times of transit of the same star across the meridian of B, recorded by telegraphic signal at A, and at B, are T_1, T_2 respectively. If the signals take the same time to travel in either direction, show that the difference of the longitudes of B and A in angular measure

$$= \tfrac{15}{2}(T_1 - t_1 + T_2 - t_2).$$

10. The altitudes of two known stars are observed at a given instant of time. Show how to find on a terrestrial globe the places at which the stars are vertically overhead, and give a geometrical construction for the place of observation.

11. In Question 10, find the condition that there should be two, one, or no possible positions of a ship at which the altitudes of the known stars have certain given values.

12. Find the geographical position of the Sun at G.M.T. 8h. 30m. 25s. given that its decl. is 13° 52′ N. and that the equation of time is — 4m. 17s.

13. Find the geographical position of *Antares* (R.A. 16h. 25m. 43s. decl. 26° 18′ S.) at G.M.T. 3h. 15m. 30s. on April 1st, given that the sidereal time at April 1d. 0h. is 12h. 36m. 48s.

14. Two stars, A, B, were observed simultaneously in a position assumed to be 52° 40′ N., 40° 0′ W. The following results were obtained :

	Observed Z.D.	*Computed Z.D.*	*True bearing.*
A	50° 48′	50° 42′	52°
B	35° 31′	35° 34′	135°

Find by graphical methods the observed position.

EXAMINATION PAPER

1. Give a description of the sextant, and explain how to use it for taking altitudes (1) at sea, (2) on land.

2. How does a chronometer differ from an ordinary watch? What are its *error* and *rate*?

3. Prove that a single meridian altitude of a star, whose declination is known, will determine the latitude. Why is a zenith sector sometimes preferred to a transit circle for this purpose?

4. Show how the latitude is determinable by two meridian observations of a circumpolar star. Why is this method not generaliy applicable on board ship?

5. Show how to find the latitude of a place (1) by observing the Sun's altitude at a given time; (2) by the Prime Vertical Instrument.

6. Describe the method of equal altitudes for finding the time of transit of a celestial body. If the times be observed by the ship's chronometer, show how to find the longitude.

7. How is the difference of longitude determined by electric telegraph? Explain how the personal equation and the time of transmission of the signal are eliminated.

8. Knowing the Greenwich time and observing the altitude and direction of the Sun, show how to construct graphically on a globe the position of the ship without any calculation whatever.

9. What is meant by the geographical position of a celestial body? Show how to find the geographical position of (*a*) the Sun, (*b*) a star, for a given instant of Greenwich Mean Time.

10. Describe the arrangement of data in the *Air Almanac*, for (*a*) Sun, Moon and planets; (*b*) the stars. Explain the advantages of this arrangement.

CHAPTER XV

THE DISTANCES OF THE SUN AND STARS

I.—Determination of the Sun's Parallax by Observations of a Superior Planet at Opposition*

376. Introduction

In Chapter VII, Section I, we explained the nature of the correction known as parallax, and showed how to find the distance of a celestial body from the Earth in terms of its parallax. We also described two methods of finding the parallax of the Moon or of a planet in opposition—the first by meridian observations at two stations, one in the northern and the other in the southern hemisphere (Art. 161); the second by micrometric observations made at a single observatory shortly after the time of rising and shortly before the time of setting of the planet or observed body (Art. 163).

In both methods the position of the body is compared with that of neighbouring stars. This is impossible in the case of the Sun, for the intensity of the Sun's rays necessitates the use of darkened glasses in observations of the Sun, and these render all near stars invisible.

Of course the star could theoretically be dispensed with in the method of Art. 161, but only (as there explained) at a great sacrifice of accuracy; and if a star is used which crosses the meridian at night, the temperature of the air has changed considerably, and the corrections for refraction are therefore quite different, besides which other errors are introduced by the change of temperature of the instrument.

In Art. 197 we described a method, due to Aristarchus, in which the ratio of the Sun's to the Moon's distance was determined by observing the Moon's elongation when dichotomized, but this method was rejected, owing to the irregular boundary of the illuminated part of the disc, and the consequent impossibility of observing the instant of dichotomy.

377. Classification of Methods

The principal *practicable* methods of finding the Sun's distance may be conveniently classified as follows:—

A. *Geometrical Methods*

(1) By observations of the parallax of a superior planet at opposition (Section I).

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* The student will find it of great advantage to revise Section I. of Chapter VII before commencing the present Section.

(2) By observations of a transit of the inferior planet Venus (Section II).

B. *Optical Methods* (Section III)

(3) By the eclipses of Jupiter's satellites (Roemer's Method).

(4) By the aberration of light.

C. *Gravitational Methods* (Chapter XVIII, Section IV).

(5) By perturbations of Venus or Mars.

(6) By lunar and solar inequalities.

378. To Find the Sun's Parallax by Observation of the Parallax of Mars

By observing the parallax of Mars when in opposition, the Sun's parallax can readily be found. For the observed parallax determines the distance of Mars from the Earth, and this is the *difference* of the distances of the Sun from the Earth and Mars respectively. The *ratio* of their mean distances may be found, if we assume Kepler's Third Law (Art. 264), by comparing the sidereal period of Mars with the sidereal year, and is therefore known. Hence the distance of either planet from the Sun may readily be found, and the Sun's parallax thus determined.

The parallax of Mars in opposition may be observed by either of the methods described in Chapter VIII, Section I. The method of Art. 161 (by meridian observations at two stations) was employed by E. J. Stone in 1865. The observations were made at Greenwich and at the Cape, and the Sun's parallax was computed as 8·94″. The method of Art. 163 was employed by Gill in 1879, using the heliometer, with the result 8·78″. This method was also used by Spencer Jones and Halm at the favourable opposition of 1924: visual observations with the heliometer gave 8·76″ and photographic observations gave 8·81″. The red colour of Mars is disadvantageous to this method. Atmospheric refraction is greater for blue light than for red light; because Mars is redder than the average star, it is displaced towards the zenith by refraction by a smaller amount than the stars. But the effect of parallax is to displace a body away from the zenith; relative to the stars Mars is effectively displaced away from the zenith by refraction. The measured differential displacement of Mars is interpreted as parallax; as a consequence of the differential refraction, too large a value of the parallax is to be expected by the photographic method. In visual observations with the heliometer, there are systematic personal errors, which are too complicated to discuss here.

Example.—*If the parallax of Mars when in opposition be* 14″, *find the Sun's parallax, assuming the distances of the Sun from the Earth and Mars to be in the ratio of* 10 : 16.

The distance of the Earth from Mars in opposition is the difference of the Sun's distances from the two planets. Hence

$$\text{Distance of Earth from Mars : Distance of Earth from Sun} = 16 - 10 : 10 = 3 : 5.$$

But the parallax of a body is inversely proportional to its distance (Art. 158), so that:—

$$\text{Parallax of Sun : Parallax of Mars} = 3 : 5;$$

$$\text{or } \textit{Sun's parallax} = \frac{3 \times 14''}{5} = 8{\cdot}4''.$$

379. Effect of Eccentricities of Orbits

Owing to the eccentricities of the orbits of the Earth and Mars, their distances from the Sun when in opposition will not in general be equal to their mean distances, and therefore their ratio will differ from that given by Kepler's Third Law. But, by the method of Art. 128, the Earth's distance at any time may be compared with its mean distance, and similarly, since the eccentricity of the orbit of Mars and the position of its apse line are known, it is easy to determine the ratio of Mars' distance at opposition to its mean distance, and thus to compare its distance with that of the Earth.

380. Sun's Parallax by Observations on the Asteroids

The Sun's parallax may also be found by observing the parallax of one of the asteroids when in opposition, the method being identical with that employed in the case of Mars. In this way Gill, by heliometer observations of Victoria, Iris and Sappho obtained the value 8·804″. By far the best asteroid for the purpose is Eros (see Art. 246). At the fairly favourable opposition of Eros in 1901, when its nearest approach to the Earth was 30 million miles, a great number of observatories cooperated in observing and photographing the asteroid and surrounding stars. The value of 8·807″ for the solar parallax was deduced by Hinks. In 1931 there was a still more favourable opposition, when Eros approached to a distance of about 16 million miles from the Earth; a more extensive programme of observations was arranged, in which many observatories used the photographic method. From these observations, a solar parallax of 8·790″ was deduced by Spencer Jones. This is the most reliable determination yet made. It corresponds to a mean distance of the Sun from the Earth of 93,005,000 miles.

Examples.—1. *The parallax of Mars was observed when it was in opposition in perihelion, the Earth's distance from the Sun being then* 1·01 *times its mean distance; the observed value was* 23·48″. *Calculate the Sun's mean parallax, the eccentricity of Mars' orbit being* $\frac{1}{11}$, *and its period* 1·88 *years.*

Let r, r' denote the mean distances of the Earth and Mars from the Sun respectively. By Kepler's Third Law we have

$$\frac{r'^3}{r^3} = \frac{(1{\cdot}88)^2}{1^2}; \quad \frac{r'}{r} = (1{\cdot}88)^{\frac{2}{3}} = 1{\cdot}523.$$

The perihelion distance of Mars from the Sun $= r'\left(1 - \frac{1}{11}\right)$

$$= \left(1 - \tfrac{1}{11}\right) \times 1{\cdot}523r = (1{\cdot}523 - {\cdot}138)r = 1{\cdot}385r.$$

The Earth being distant $1{\cdot}01r$ from the Sun, is ${\cdot}375r$ from Mars, the three bodies being in a straight line at opposition.

Therefore, since r is the Sun's *mean* distance from the Earth, we have

$$\frac{\text{Observed parallax of Mars}}{\text{Mean parallax of Sun}} = \frac{r}{{\cdot}375r} = \frac{1}{{\cdot}375} = \frac{8}{3};$$

so that *Sun's mean parallax* $= 23{\cdot}48'' \times \frac{3}{8} = 8{\cdot}805''$.

2. *Find the Earth's mean distance from the Sun, and its distances at perihelion and aphelion, taking the Sun's parallax as* $8{\cdot}79''$.

If a denote the Earth's equatorial radius, we have, approximately,

$$r = \frac{a}{\sin 8{\cdot}79''} = \frac{a}{\text{circ. meas. of } 8{\cdot}79''} = a \times \frac{206{,}265}{8{\cdot}79}.$$

Taking $a = 3963{\cdot}4$, this gives :—

r (*Earth's mean solar distance*) $= 93{,}005{,}000$ *miles*.

correct to the nearest thousand miles.

Also, *perihelion distance from Sun* $= 93{,}005{,}000 \times \left(1 - \frac{1}{60}\right)$

$$= 93{,}002{,}000 - 1{,}550{,}000 = 91{,}455{,}000 \textit{ miles},$$

and *aphelion distance* $= 93{,}002{,}000 \times \left(1 + \frac{1}{60}\right)$

$$= 93{,}002{,}000 + 1{,}550{,}000 = 94{,}555{,}000 \textit{ miles}.$$

II.—TRANSITS OF INFERIOR PLANETS

381. Introduction

When either of the inferior planets Mercury or Venus is very close to one of its nodes (Art. 272) at the time of inferior conjunction, it passes over the Sun's disc, appearing as a black dot.

Transits of Venus are interesting historically, since it was by them that a fairly reliable estimate of the Sun's distance was first obtained. The ratio of the distances of Venus and the Earth from the Sun can be found by the method of Art. 256 ; they are approximately as 18 to 25. Hence when Venus is in inferior conjunction its distance from the Earth is to that of the Sun as 7 to 25. Their parallaxes are therefore as 25 to 7 (Art. 158) and the relative parallax of Venus (*i.e.* the excess of its parallax over the Sun's) is $\frac{18}{7}$ of the Sun's parallax.

This relative parallax was found by two methods : each method required observers to be stationed over widely scattered regions of the Earth's surface, who should observe the exact instants at which the discs of the Sun and Venus touched each other, with the planet lying wholly upon the Sun. The calculation depends on the fact that these observed times will be slightly different at the different stations.

Halley, who in 1676 was the first to suggest the method of transits of Venus for finding the Sun's distance, noted that it was very difficult to ascertain the longitudes of the stations occupied with the accuracy necessary for finding the absolute times of contact; hence he relied on comparisons of the *observed duration* of transit, reckoned from first to second internal contact, as seen from different stations. Some of these stations were chosen as far north as possible, others as far south as possible, so as to displace Venus to the south and north respectively on the Sun's disc. See Fig. 127 where *a* and *b* are the relative positions of Venus on the Sun's disc as seen at *A* and *B* respectively on the Earth.

Delisle's method assumed that the longitudes of the observing stations were known, so that the Greenwich times of the contacts could be determined. Stations were chosen for which internal contact at ingress happened as early and as late as possible; similarly for internal contact at egress. Separate determinations of the parallax result from the observations at ingress and egress, and these serve to check each other. This method is much the best, when the longitudes are accurately known. In 1874 and 1882 longitudes of many of the stations were not accurately known. Favourable stations for Delisle's method can be found in lower latitudes than those for Halley's method. This is an advantage both from greater accessibility and (usually) better weather conditions.

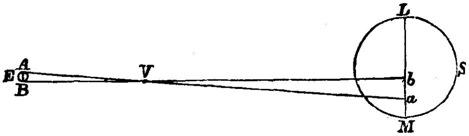

FIG. 127.

382. Recurrence of Transits of Venus

Thirteen revolutions of Venus are very nearly equal to eight of the Earth. Accordingly transits of Venus frequently occur in pairs 8 years apart. After such a pair there is an interval of more than a century before another pair, and these take place at the opposite node. The period of 243 years gives an extremely close approximation to recurrence of circumstances; thus we have December transits in the years 1631, 1639, 1874, 1882, 2117, 2125, etc., the series continuing unbroken for more than a thousand years from the present time; and June transits in 1761, 1769, 2004, 2012, etc., till 2733 when Venus just misses the Sun, crossing it near the centre in 2741.

The first observed transit was that of 1639, seen by Horrocks and Crabtree. The pair of 1761, 1769 gave the first fairly reliable determination of the Sun's distance. Encke deduced the value 95½ million miles, which is about 2½ million miles too great.

The transits of 1874, 1882 were very carefully observed, but discussion of the observations established the fact that the method is not capable of such precision as had been hoped. Its drawbacks are :—

(1) Atmospheric tremors produced by the Sun's heat;

(2) Distortion of the shape of Venus when in contact with the Sun —this is probably due to irradiation, and makes it difficult to determine the time of contact within several seconds;

(3) The difficulty is further increased by the fact that Venus has an atmosphere similar to our own—the portion of Venus outside the Sun is surrounded by a luminous ring, owing to refraction, which increases the difficulty of noting the instant of contact.

When we add that the asteroid Eros (discovered in 1898) is about 3 times a century at a distance from the Earth only half that of Venus, and that its place among the stars can be fixed very accurately by photography, it will be seen that it is unlikely that transits of Venus will ever again be employed to find the Sun's distance. They will doubtless be observed in the future, but the observations will only be used to give better determinations of the elements of Venus's orbit.

383. Transits of Mercury

Transits of Mercury occur much more frequently than those of Venus: there were transits in 1927, 1937 and 1940. The next will occur on November 14th, 1953 and will be visible in Europe. A transit of Mercury occurs either in May or in November. The average interval between them is only $7\frac{2}{3}$ years. They are also free from the objection (3) above, since Mercury has little or no atmosphere. Mercury is, however, much farther from the Earth than Venus is, and its parallax relatively to the Sun is too small to make the method of any use. The parallax of Mercury is $16{\cdot}0''$ in May transits, and $13{\cdot}0''$ in November ones. The parallax relatively to the Sun is $7{\cdot}2''$ and $4{\cdot}1''$ in the two cases, whereas that of Venus is $24{\cdot}5''$.

Examples.—1. *Given that the synodic period of Venus is* 584 *days, and that the difference between the times of ending of a central transit, as seen from opposite ends of that diameter of the Earth that is in the direction of motion, is* 11*m.* 25*s., find the Sun's parallax.*

In 584 days Venus revolves through 360° about the Sun relatively to the Earth; therefore its angular motion per minute is

$$\frac{360 \times 60 \times 60}{584 \times 24 \times 60} = 1{\cdot}541''.$$

Therefore in $11\frac{5}{12}$m. Venus describes an angle $1{\cdot}541'' \times 11\frac{5}{12} = 17{\cdot}60''$. The *Sun's parallax* is half this or $8{\cdot}80''$.

2. *Find the angular rate at which Venus moves across the Sun's disc.*

Let S, E, V denote the Sun, Earth, and Venus respectively (Fig. 139).

From example 1, SV separates from SE with relative angular velocity, about S, of 1·54″ per minute, or 1′32·4″ per hour.

But Venus is nearer the Earth than the Sun in the ratio 28 : 72 (roughly). And we have :—

$$\text{Angular velocity of } EV : \text{ang. vel. of } SV = \frac{1}{EV} : \frac{1}{SV} = 72 : 28 = 18 : 7.$$

Therefore EV separates from ES with angular velocity $= \frac{18}{7} \times 1'32{\cdot}4''$ per hour = 3′57·6″ per hour = 4″ *per minute very nearly.*

III.—OPTICAL METHODS

384. Velocity of Light

We now come to certain methods of finding the Sun's distance which depend on the fact that light is propagated through space with a large but measurable velocity.

The velocity of light has been measured by laboratory experiments in various ways and is known with a high degree of accuracy. The experiments give the velocity of light in air ; the velocity in vacuo can be obtained by multiplying this by the index of refraction of air. The most probable value of the velocity of light in vacuo may be taken as 299,796,000 metres or 186,285 miles a second.

385. Roemer's Method.—The Equation of Light

In chapter XI we stated that Jupiter has four large satellites, which revolve very nearly in the plane of the planet's orbit. Consequently a satellite passes through the shadow cast by Jupiter once in nearly every revolution, and is then eclipsed, as is our Moon in a lunar eclipse.

Since the orbits and periods of the satellites have been accurately observed, it is possible to predict the recurrence of the eclipses, so that when one eclipse has been observed the times at which subsequent eclipses will begin and end can be computed.*

Now, the Danish astronomer Roemer in 1675 observed a remarkable discrepancy between the predicted and the observed times of eclipses. If of two eclipses one happens when Jupiter is near opposition, and the other happens near the planet's superior conjunction, the observed interval between the former and the latter is always *greater* than the computed interval ; similarly the observed interval between an eclipse near superior conjunction and the next eclipse near opposition is always *less* than the computed interval. The eclipses at conjunction are thus always retarded, relatively to those at opposition, by an interval of time which is observed to be about 16m. 40s. As explained by Roemer, this apparent retardation is due to the fact that light travels from Jupiter to the Earth with finite velocity, and therefore takes 16m. 40s. longer to reach the Earth when the planet is furthest away at superior conjunction (B) than when the planet is nearest the Earth at opposition (A).

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* The intervals are not exactly equal, owing to Jupiter's variable velocity.

The relative retardation is the difference between the times taken by the light to travel over the distances AE and BE. But $BE - AE = 2SE$. Therefore the *retardation is twice the time taken by the light to travel from the Sun to the Earth.*

Taking the retardation as 16m. 40s., we see that light takes 8m. 20s. to travel from the Sun to the Earth. This interval is sometimes called the "*equation of light.*"

If we know the equation of light and the velocity of light, we may calculate the Sun's distance. Conversely, if the Sun's distance and the equation of light are known, the velocity of light can be determined.

Knowing the Sun's distance, the Sun's *parallax* can be computed, as in Chapter VII, Section I. The present method differs from those described in Sections I, II, in that it gives the distance instead of the parallax of the Sun. The method is of historic interest only; it is not capable of determining the Sun's distance with great accuracy.

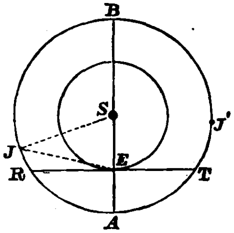

FIG. 128.

Examples.—1. *Find the Sun's distance, having given that the velocity of light is* 186,330 *miles per second, and that eclipses of Jupiter's satellites which occur when the planet is furthest from the Earth, are retarded* 16m. 40s. *relatively to those which occur when the planet is nearest.*

Here the time taken by light to pass over a diameter of the Earth's orbit is 16m. 40s.; therefore light travels from the Sun to the Earth in 8m. 20s., or 500 seconds.

Therefore *the Sun's distance* = 186,330 × 500 miles = 93,165,000 *miles.*

2. *Taking the value of the Sun's distance calculated in the preceding example, the Sun's parallax will be found to be about* 8·78″.

386. The Aberration of Light

It was shown in Art. 188 that there is a relationship between the coefficient of aberration and the parallax of the Sun. If both quantities are expressed in seconds of arc, it was shown that

$$P.k = 180{\cdot}3.$$

Any determination of the coefficient of aberration therefore leads to a value of the parallax of the Sun and thence of the Sun's distance. Methods of determining the constant of aberration were described in Art. 187.

IV.—THE DISTANCES OF THE FIXED STARS

387. Annual Parallax of a Star

The parallax of a star was defined in Art. 168. In Art. 169 it was shown that the reciprocal of the parallax (expressed in seconds of arc) gives the distance of the star in parsecs. It was proved in Art. 171 that any star, owing to parallax, appears to describe an ellipse in the sky in the course of a year. To determine the parallax of a star, it is necessary to measure the displacement of the star by parallax at different times during the year. Because of the great distance of the stars, the parallax is a small quantity and great care and accuracy of measurement is needed to give a reliable result. It is important that the measurements should be made at the times when the parallactic displacements are as large as possible.

It is generally most convenient to measure the component of the parallactic displacement in R.A. It was shown in Art. 173 that this displacement (in arc) is $P \sin (\alpha - \alpha_0) \cos \delta_0$, where α_0, δ_0, are the R.A. and decl. of the Sun and α is the R.A. of the star. The term $\sin (\alpha - \alpha_0)$ can vary between $+1$ and -1, whereas the range of variation of $\cos \delta_0$ is only from $+ \cdot 92$ to $+1$. The most favourable conditions would therefore be secured by making observations at the times when $\alpha - \alpha_0 = \pm 90^\circ$ or $\pm$ 6h. These conditions would be obtained by making observations of the star when it is on the meridian at about 6 p.m. local time and again six months later at about 6 a.m. local time. These ideal conditions are subject to the restriction that the observations of the stars can be made only after sunset; in high latitudes in the summer months, it may not be possible to begin observations for two or three hours after 6 p.m. or to continue them within two or three hours of 6 a.m.

It is necessary to separate the displacement of a star caused by its own proper-motion from the displacement due to parallax. For this purpose observations must be made at a minimum of three epochs, separated by intervals of approximately six months. Comparison between the first and third epochs, when the parallactic displacements are nearly the same, will determine the proper motion of the star and make it possible to disentangle it from the parallactic displacement.

388. Determination of the Annual Parallax of any Star

To determine the Annual Parallax of any star, the following methods have been employed :—

(i) The absolute method, by the transit circle ;

(ii) Bessel's, or the differential method, by the micrometer or heliometer ;

(iii) The photographic method.

The *absolute method* consisted simply in observing with the transit circle the apparent decl. and R.A. of a star at different times in the year. From the small variations in these co-ordinates it is possible to find the star's parallax.

The method was used in 1835 by Henderson at the Cape to prove that the star *α Centauri* had a large parallax, though even here the value obtained was quite rough. It is not capable of giving the degree of refinement now aimed at, and must be regarded as only of historical interest.

389. Bessel's Method

Bessel's method consists in observing with a micrometer (Art. 325) or heliometer (Art. 326) the variations in the angular distance and relative position of two optically near stars during the course of a year.

The stars, being nearly in the same direction, are very nearly equally affected by refraction*, and we may also mention that the same is true of aberration, precession and nutation. These corrections do not therefore sensibly affect the relative angular distance and positions of the stars. On the other hand, the two stars may be at very different distances from the Earth; if so, they are differently displaced by parallax, and their angular distance and position undergo variations depending on their *relative* parallax or *difference* of parallax. Hence, by observing these variations during the year the difference of parallax can be found.

This method does not determine the *actual* parallax of either star. But if one of the observed stars has a large proper motion and the other has a much smaller one or is a much fainter star we can assume that the former is comparatively near the Earth, while the latter is at such a great distance away that its parallax is insensible. Under such circumstances the observed relative parallax is the parallax of the nearer star alone. By making comparisons between the bright star and *several* different faint stars in its neighbourhood, this point may be settled.

If a considerable discrepancy is found in the observed relative parallaxes, one or more of the comparison stars must themselves have appreciable parallaxes, but since the vast majority of stars in any neighbourhood are too distant to have a parallax, we shall be able to find the parallax not only of the star originally observed, but of that with which we had first compared it.

The parallax of a star can *never be negative*; if the relative parallax should be found to be negative, we should infer that the comparison star has the greater parallax, and is therefore nearer the Earth.

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* Refraction is slightly different for stars of different colour; to avoid error from this source, the observations are made near the meridian, and only the right-ascension component of the parallax is measured.

390. The Photographic Method

The *Photographic method* is identical in principle with the last, but instead of observing the relative distances of different stars with a micrometer, portions of the heavens are photographed at different seasons, and the displacements due to parallax are measured at leisure by comparing the positions of any star on the different plates.

All the refined researches on parallax are now photographic, large refractors of great focal length being devoted to this work, in order to obtain great light-gathering power and large scale. Experience has suggested many precautions, such as taking all the photographs near the meridian, so that there is no refraction in R.A., and making the parallax star of the same apparent magnitude as the comparison stars ; this is done by making a small slotted screen rotate in front of its image on the plate. The probable error of a result has been reduced to less than 0·01″.

A star at a distance of 100 parsecs has a parallax of 0·01″, which is not much in excess of the probable error with which the parallax can be measured. The direct measurement of parallaxes is therefore restricted to the comparatively near stars. For stars at much greater distances parallaxes must be determined by indirect methods, which are outside the scope of this book ; for details of these the student is referred to text-books on astrophysics.

391. Parallaxes of certain Fixed Stars

The nearest stars are *Proxima Centauri*, with a parallax of 0·78″, *α Centauri*, with parallax 0·75″, and *Munich* 15040, with 0·55″. Among others, the following may be mentioned : *α Lyrae*, 0·10″, *Sirius*, 0·38″, *Arcturus*, 0·08″, *Polaris*, 0·04″, *α Aquilae*, 0·22″, 61 *Cygni*, 0·30″. Of these, *Proxima Centauri* is of the eleventh magnitude ; *Munich* 15040 is of the tenth magnitude ; a companion to *Sirius* is only of the tenth magnitude. So it is not an *invariable* rule that faint stars are most distant, and have no appreciable parallax. A star may appear faint either because its intrinsic luminosity is low or because it is at a very great distance.

392. Proper Motions.—Binary Stars

Many stars, instead of being fixed in space, are gradually changing their positions. They are then said to have a *proper motion*. This motion may partly belong to the star, but is also partly an apparent motion, due to the fact that the solar system is itself moving through space in the direction of a point R.A. 18h., North decl. about 30°. The displacement can be allowed for approximately if the star's distance is known.

Many of these motions, like that of our own Sun, are apparently *progressive*; *i.e.* the star moves with constant velocity and in the same direction. Others are *orbital*, *i.e.* the star revolves about some other star, or (more accurately) two stars revolve about their common centre of mass. Such a system of stars is called a *Binary Star*. It is usually seen by the naked eye as a single heavenly body, its components being too near to be distinguished. Frequently a system of stars has itself a progressive motion; and sometimes an apparently progressive motion may really be an orbital one, with a period so long that the path has not sensibly diverged from a straight line during the short period for which stellar motions have been watched.

A progressive or orbital motion cannot be confounded with the displacement due to annual parallax, for the former is always in the same direction, and the latter has a period differing from a year, while parallax always produces an *annual* variation.

EXAMPLES

1. Show, by means of a diagram, that the general effect of the Earth's diurnal rotation is to shorten the duration of a transit of Venus, and that this circumstance might be used to find the Sun's parallax.

2. Supposing the equator, ecliptic, and orbit of Venus all to lie in one plane, and that a transit of Venus would last eight hours, at a point on the Earth's equator, if the Earth were without rotation; show that, if the Sun is vertically overhead at the middle of the transit, the duration is diminished by about 9m. 55½s. owing to the Earth's rotation, taking the Sun's parallax to be 8·8″, and the synodic period of Venus to be 584 days.

3. Suppose the velocity of light to be the same as the velocity of the Earth round the Sun. Discuss the effect on the Pole Star as seen by an observer at the North Pole throughout the year.

4. Given that the Sun's parallax is 8·79″ and the radius of the Earth is 3,960 miles, find the distance that the Earth moves while light travels from the Sun to the Earth. (The velocity of light is 186,000 miles a second).

5. The parallax of a minor planet when in opposition is found to be 25·6″. Assuming that the Sun's parallax is 8·79″, find the ratio of the distances of the planet and the Earth from the Sun.

MISCELLANEOUS QUESTIONS

1. Explain the following terms :—*asteroid, libration, lunation, parallax, perihelion, planet's elongation, right ascension, synodical period, syzygies, zenith.*

2. Given that the R.A. of Orion's belt is 80° and its declination 0°, show by a figure its position at different hours of the night about March 21st and September 23rd.

3. Prove that the number of minutes in the dip is equal to the number of nautical miles in the distance of the visible horizon.

4. Show how to determine the latitude of a place by meridianal observations on a circumpolar star, taking into account the refraction error.

5. At what time of the year can the waning moon best be seen?

6. On July 21st at 2 A.M. the Moon is on the meridian. What is the age of the Moon? Indicate the position on the celestial sphere of a star whose declination is 0° and whose R.A. is 30°.

7. Taking the distance of Venus from the Sun to be $\frac{3}{4}$ of that of the Earth, find the ratio of the planet's angular diameters at superior and inferior conjunction and greatest elongation, and draw a series of diagrams showing the changes in the planet's appearance during a synodic period, as seen through a telescope under the same magnifying power.

8. Defining a *lunar day* as the interval between two consecutive transits of the Moon across the meridian, find its mean length in (i) mean solar, and (ii) sidereal units.

9. At what seasons is the aberration of a star least whose R.A. is 90° and whose declination is 60°?

10. Show that the constant of aberration can be determined by observation of Jupiter's satellites, without a knowledge of the radius of the Earth's orbit.

11. How is it possible to calculate separately the aberration—the constant of aberration being supposed unknown—annual parallax, and proper motion of a star, from a long series of observations of the apparent place of a star?

EXAMINATION PAPER

1. Why is the method for finding the Moon's parallax not available in the case of the Sun? Show how the determination of the parallax of Mars leads to the determination of the Sun's parallax.

2. Show how the Sun's parallax can be found by comparing the times of *commencement* or of *termination* of a transit of Venus at two stations not far from the Earth's equator.

3. Show how the Sun's parallax can be found by comparing the *durations* of a transit of Venus at two stations in high N. and S. latitudes. Why is this method not available when the transit is *central*?

4. Describe Bessel's method of determining the annual parallax of a fixed star.

5. How might the Sun's parallax be determined by observations of the eclipses of Jupiter's satellites?

6. At what times of the day and year should observations of a star be made to give the most reliable determination of its parallax? When should observations be made of a star whose R.A. is 6h. and of one whose R.A. is 12h.

7. Show that transits of Venus are more suitable for the determination of the solar parallax than transits of Mercury.

DYNAMICAL ASTRONOMY

CHAPTER XVI

THE ROTATION OF THE EARTH

393. Introductory

In the preceding chapters we have shown how the motions of the celestial bodies can be determined by actual observation, and we have also explained certain resulting phenomena. But no use has yet been made of the principles of dynamics; consequently we have been unable to investigate the *causes* of the various motions. In particular, while we have assumed that the diurnal rotation of the stars is an appearance due to the Earth's rotation, we have not as yet given any definite proof that this is the only possible explanation.

The ancient Greeks accounted for the motions of the solar system by means of the *Theory of Epicycles*, according to which each planet moved as if it were at the end of a system of jointed rods rotating with uniform but different angular velocities. Suppose AB, BC, CD to be three rods jointed together at B, C. Let A be fixed; let AB revolve uniformly about A; let BC revolve with a different angular velocity about B; and let CD revolve with another different angular velocity about C. Then, by properly choosing the lengths and angular velocities of the rods, the motion of D, relative to A, may be made nearly to represent the motion, relative to the Earth, of a planet.

Copernicus (A.D. 1500 *circ.*) was the first astronomer* who explained the motions of the solar system on the theory that the diurnal motion is due to the Earth's rotation, and that the Earth is one of the planets which revolve round the Sun. This theory was adopted by *Kepler* (A.D. 1609 *circ.*) whose laws of planetary motion have already been mentioned (Art. 264). These laws were, however, unexplained until their true cause was found by *Newton* (A.D. 1687) by his discovery of the law of gravitation.

394. Arguments in Favour of the Earth's Rotation

Without appealing to dynamical principles, the probability of the Earth's rotation about its axis (Art. 79) may be inferred from the following considerations:—

(i) If the Earth did not rotate, we should have to imagine the Sun and stars to be revolving about it with enormous velocities. We could not, indeed, assume on that hypothesis quite so great values for the distances of the stars, as those to which we have been led on the received hypothesis. The distance of the Sun is some 24,000 times the radius of the Earth. A speed of 25 million miles per hour would be required to carry it round the Earth in a day. The very small parallaxes of the stars would suffice to prove that their distance was many times as great as the Sun's, and their speed hundreds or thousands of millions of miles an hour or comparable with that of light. It is difficult to conceive of such speed in the case of bodies immensely larger than the Earth.

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* Some of the old Greek philosophers had made similar suggestions; but the evidence was then insufficient to carry conviction.

(ii) The diurnal rotations all take place about the pole, and are all performed in the same period—a sidereal day. This uniformity is a natural consequence of the Earth's rotation, but if the Earth were at rest, it could only be explained by supposing the stars to be rigidly connected in some manner or other. Were such a connection to exist, it would be difficult to explain the proper motions of certain fixed stars, and the independent motions of the Sun, Moon, and planets.

(iii) By observing the motion of the spots on the Sun at different intervals, it is found that the Sun rotates on its axis. Moreover, similar rotations may be observed in the planets; thus, Mars is known to rotate in a period of nearly 24 hours. There is, therefore, nothing unreasonable in supposing that the Earth also rotates once in a sidereal day.

(iv) The phenomenon of *diurnal aberration* affords a proof of the Earth's rotation. Were it not for the difficulty of its observation, this proof alone would be conclusive.

We may mention that diurnal parallax could be equally well accounted for if the celestial bodies revolved round the Earth; not so, however, diurnal aberration.

395. Dynamical Proofs of the Earth's Rotation

The following is a list of the methods by which the Earth's rotation is proved from dynamical considerations :—

(1) The eastward deviation of falling bodies.

(2) Foucault's pendulum experiment.

(3) Foucault's experiments with a gyroscope.

(4) Experiments on the deviation of projectiles.

(5) Observations of ocean currents and trade winds.

(6) Experiments on the differences in the acceleration of gravity in different latitudes, due to the Earth's centrifugal force, as observed by counting the oscillations of a pendulum; combined with

(7) Observations of the figure of the Earth.

396. The Eastward Deviation of Falling Bodies

If the Earth is rotating about its polar axis, those points which are furthest from the Earth's axis move with greater velocity than those which are nearer the axis. Hence the top of a high tower moves with slightly greater velocity than the base. If, then, a stone be dropped from the top of the tower, its eastward horizontal velocity, due to the Earth's rotation, is greater than that of the Earth below, and it falls to the east of the vertical through its point of projection. The same is true when a body is dropped down a mine. This eastward deviation, though small, has been observed, and affords a proof of the Earth's rotation.

Consider, for example, a tower of height h at the equator. If a be the Earth's equatorial radius, the base travels over a distance $2\pi a$ in a sidereal day, owing to the Earth's rotation, while the top of the tower describes $2\pi\ (a + h)$ per sidereal day. Thus, the velocity at the top exceeds that at the bottom by $2\pi h$ per sidereal day.

If h be measured in feet, this difference of velocities is $\pi h/3600$ inches per sidereal second, and is sufficiently great to cause a small but perceptible deviation when a body is let fall from a high tower.

The calculation of the deviation is somewhat difficult, and involves an application of principles which are discussed in the next chapter. If t is the time of flight, v the difference of velocities of the top and the bottom of the tower, the deviation is found to be approximately $\frac{2}{3}vt$, and not vt as might naturally be expected. Owing to the difficulty of measuring this deviation experimentally, the question is chiefly of historic interest.

397. Foucault's Pendulum Experiment

In 1851, Foucault invented an experiment by which the Earth's rotation is very clearly shown. A pendulum is formed of a large metal ball suspended by a fine wire from the roof of a high building, and is set in motion by being drawn on one side and suddenly released; it then oscillates to and fro in a vertical plane. If now the pendulum be sufficiently long and heavy to continue vibrating for a considerable length of time, the plane of oscillation is observed to very gradually change its direction relative to the surrounding objects, by turning slowly round from left to right at a place in the northern hemisphere, or in the reverse direction in the southern. If the experiment is performed in latitude ϕ, the plane of oscillation appears to rotate through $15° \times \sin\phi$ in a sidereal hour, $360° \sin\phi$ in a sidereal day, or 360° in $\operatorname{cosec}\phi$ sidereal days. This apparent rotation is accounted for by the Earth's rotation, as follows:—

(i) Let us first imagine the experiment to be performed at the *north pole* of the Earth. Let the pendulum AB (Fig. 129) be vibrating about A in the arc BB' in the plane of the paper. The only forces acting on the bob are the tension of the string BA and the weight of the bob acting vertically downwards; both are in the plane of the paper. The Earth's rotation about its axis CA produces no forces on the bob. Hence there is nothing whatever to alter the direction of the plane of oscillation; this plane therefore remains fixed in space. But the

Earth is not fixed in space; it turns from west to east, making a complete *direct* revolution in a sidereal day. Hence the plane of the pendulum's oscillation appears, to an observer not conscious of his own motion, as though it rotated once in a sidereal day, in the reverse or *retrograde* direction (east to west). If, however, he were to compare the plane of oscillation not with the Earth but with the stars, whose directions are actually fixed in space, he would see that it always retained the same position relatively to them.

Since, then, the pendulum at the pole of the Earth appears to follow the stars, it evidently appears to rotate in the same direction as the hands of a watch at the north pole, and in the direction opposite to the hands of a watch at the south pole.

(ii) Next suppose the experiment performed at the Earth's *equator*. If the bob be set swinging in the plane of the equator, take this as the plane of the paper (Fig. 130). The direction of the vertical AQC is now rotating about an axis through C perpendicular to the plane of the paper; hence it always remains in that plane. Hence there is nothing whatever to turn the plane of oscillation of the pendulum out of the plane of the Earth's equator. It therefore continues always to pass through the east and west points, and there is *no* apparent rotation of the plane of oscillation.

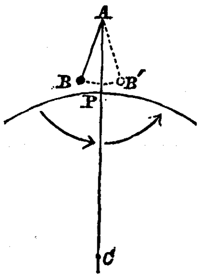

FIG. 129.

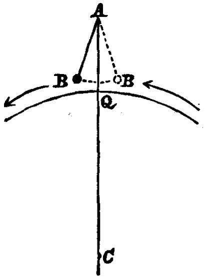

FIG. 130.

If the pendulum does not swing in the plane of the equator, the explanation is much more complicated. As the Earth rotates, the direction of gravity performs a *direct* revolution in a sidereal day. Hence, relative to the point of support, gravity is gradually and continuously turning the bob westwards, in such a way as to keep its mean position always pointed towards the centre of the Earth. When the bob is south of its position of equilibrium, this westward bias tends to turn the plane of oscillation in the clockwise direction, but when the bob is north of the mean position, the westward bias has an equal tendency to turn the plane in the reverse direction. Consequently the two effects counteract one another, and therefore produce no apparent rotation of the plane of oscillation relative to surrounding objects.

(iii) Lastly, consider the case of an observer O in latitude ϕ (Fig. 131). Let n denote the angular velocity with which the Earth is rotating about its pole axis CP. It is a well-known theorem in rigid dynamics that an angular velocity of rotation about any line may be resolved into components about any two other lines, by the parallelogram law, in just the same way as a linear velocity or a force along that line; this theorem is called the Parallelogram of Angular Velocities. Applying it to the angular velocity n about CP, we may resolve it into two components—

$$n \cos PCO \text{ or } n \sin \phi \text{ about } CO,$$

and

$n \sin PCO$ or $n \cos \phi$ about a line CO' perpendicular to CO, and we may consider the effects of the two angular velocities separately.

As in case (i), the component $n \sin \phi$ causes the Earth to turn about CO, without altering the direction in space of the plane of oscillation; this plane, therefore, appears to rotate *relatively* in the reverse or retrograde direction, with angular velocity $n \sin \phi$. As in case (ii), the angular velocity $n \cos \phi$ about CO' produces no apparent rotation of the plane of oscillation relative to the Earth. Hence the plane of oscillation appears to revolve, relative to the Earth, with retrograde angular velocity $n \sin \phi$.

But the angular velocity n

$$= 15° \text{ per sidereal hour}$$
$$= 360° \text{ per sidereal day.}$$

Therefore the plane of oscillation turns through 15° $\sin \phi$ per sidereal hour = 360° $\sin \phi$ per sidereal day, and its period of rotation $= \dfrac{360°}{n \sin \phi}$: that is:—

$$\text{period of rotation} = \operatorname{cosec} \phi \text{ sidereal days.}$$

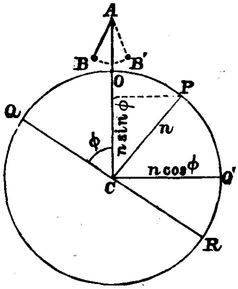

FIG. 131.

398. The Gyroscope or Gyrostat

This is another apparatus used by Foucault to prove the Earth's rotation. It is simply a heavy revolving wheel M (Fig. 132), whose axis of rotation AB is supported by a framework, so that it can turn about its centre of gravity in any manner. Thus, by turning the wheel and the inner frame $ACBD$ about the bearings CD, and then turning the outer frame $DECF$ about the bearings EF, the axis AB (like the telescope in an altazimuth or equatorial) can be pointed in any desired direction. The three axes AB, CD, EF all pass through the centre of gravity of the top; hence its weight is entirely supported, and does not tend to turn it in any way; and the bearings A, B, C, D, E, F are very light, and so constructed that their friction may be as small as possible.

The top may be spun by a string in the usual way, and it continues to spin for a long time.

When a symmetrical body, such as the wheel M, is revolving rapidly about its axis of figure, and is not acted on by any force or couple, it is evident that no change of motion can take place, and therefore the axis of rotation AB must remain fixed in direction. This is the case with the gyroscope, for, from the mode in which the weight of the wheel is supported, there is no force tending to turn it round.

When the experiment is performed it is observed that the axis AB follows the stars in their diurnal motion; if pointed to any star, it always continues to point to that star, its position relative to the Earth changing with that of the star. Hence it is inferred that the directions of the stars are fixed in space, and that the diurnal motion is not due to them, but to the rotation of the Earth.

If while the gyroscope is spinning rapidly any attempt be made to alter the direction of the axis of rotation AB by pushing it in any direction, a very great resistance will be experienced, and the axis will only move with great difficulty. This shows that the small friction at the pivots CD, EF can have but little effect in turning the axis of the top, and therefore the gyroscope spins as if it were practically free, as long as its angular velocity remains considerable.

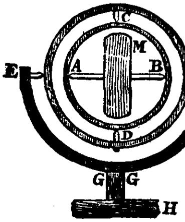

FIG. 132.

399. Further Experiments with the Gyroscope

The following additional experiments with the gyroscope can be also used to prove the Earth's rotation.

Experiment 1.—Let the hoop $CEDF$ be steadily rotated about the line EF. The line AB is no longer free to take up any position, for the pivots C and D obviously force it always to be in a plane through EF and perpendicular to plane $CEDF$. Hence the axis of rotation is no longer able to maintain always the same position, unless that position coincides with EF. The result is that the axis gradually turns about CD till it *does* coincide with EF, the direction of rotation of the wheel being the same as that in which frame is forced to revolve. It will then have no further tendency to change its place. Of course we suppose the hoop turned so quickly that the effect of the slow motion of the Earth is imperceptible.

Experiment 2.—We may now repeat Experiment 1, using the Earth's rotation. Let the framework $CEDF$ be fixed in a horizontal position, the line CD being held pointed due east and west. The axis AB is then free to turn in the plane of the meridian. Now, owing to the Earth's rotation, the framework carrying CD is turning about the Earth's polar axis, and this causes the top to turn *till its axis points to the celestial poles.* The result of experiment agrees with theory, thus affording a further proof of the Earth's rotation about the poles. The gyro-compass, now used on many battle-ships and other large vessels, is essentially a large gyroscope whose rotation is maintained by an electric motor. It points to the astronomical north, not the magnetic north. As the guiding force is more powerful than in the magnetic compass, it is possible to have a single compass at a safe position below the water line, which controls dials in various parts of the ship.

Experiment 3.—Let the framework *CEDF* be clamped in a vertical plane. The axis *AB* can then turn in a horizontal plane, but it cannot point to the pole. It will, however, try to point in a direction differing as little as possible from the direction of the Earth's axis. and will therefore turn till it points due north and south. This has also been verified by actual observation.

Experiments 2 and 3, if performed with a sufficiently perfect gyroscope, would enable us to find the north point, and then to find the celestial pole, and thus determine the latitude without observing any stars. By means of Foucault's pendulum experiment we could also (theoretically) determine the latitude.

400. The Deviation of Projectiles

If we suppose a cannon ball to be fired in any direction, say from the Earth's North Pole, the ball will travel with uniform horizontal velocity in a vertical plane. But, as the Earth rotates from right to left, the object at which the ball was aimed will be carried round to the left of the plane of projection, and therefore the ball will appear to deviate to the right of its mark. At the South Pole the reverse would be the case, because in consequence of the direction of the vertical being reversed, the Earth would revolve from left to right; hence the ball would deviate to the left of its mark. At the equator no such effect would occur.

The deviation, like that in Foucault's pendulum, depends on the Earth's component angular velocity about a vertical axis at the place of observation, and this component, in latitude ϕ, is $n \sin \phi$ (Art. 397, iii). Now the Earth rotates about the poles through 15″ per sidereal second. Hence, if t be the time of flight measured in sidereal seconds, the deviation is given by :—

$$\text{Deviation} = nt \sin \phi = 15'' \,.\, t \sin \phi,$$

and it is necessary to aim at an angle $15'' \,.\, t \sin \phi$ to the *left* of the target in N. lat. ϕ, or $15'' \,.\, t \sin \phi$ to the *right* in S. lat. ϕ. The formula is sufficiently approximate even if t be measured in *solar* seconds. It is necessary to allow for this deviation in gunnery—thus affording another proof of the Earth's rotation.

401. The Trade Winds

The *Trade Winds* are due to a similar cause. The currents of air travelling towards the hotter parts of the Earth at the equator, like the projectiles, undergo a deviation towards the right in the northern hemisphere, and towards the left in the southern. This deviation changes their directions from north and south to north-east and south-east respectively. In a similar manner the Earth's rotation causes a deviation in the ocean currents, making them revolve in a direction opposite to that of the Earth's rotation, which is " counter clockwise " in the N. and " clockwise " in the S. hemisphere. The rotatory motion of the wind in cyclones is also due to the Earth's rotation.

402. Centrifugal Force

If a body of mass m is revolving in a circle of radius r with uniform velocity v under the action of any forces, it is known that the body has an acceleration v^2/r towards the centre of the circle*. Hence the forces must have a resultant mv^2/r acting towards the centre, and they would be *balanced* by a force mv^2/r acting in the reverse direction, *i.e.* *outwards* from the centre. This force is called the *centrifugal force.*

Thus, in consequence of its acceleration, the body appears to exert a centrifugal force outwards. If it be attached to the centre of the circle by a string, the pull in the string is mv^2/r. If m be measured in pounds, r in feet, and v in feet per second, then mv^2/r represents the centrifugal force in *poundals.* Similarly, in the centimetre-gramme-second system of units, mv^2/r is the centrifugal force in *dynes.*

If n represent the body's angular velocity in radians per second, $v = nr$, and the centrifugal force is therefore mn^2r.

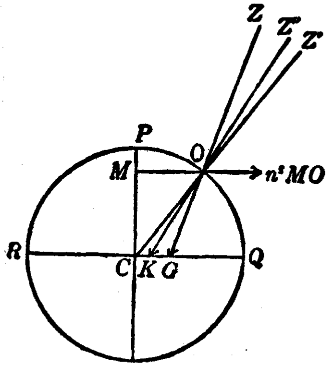

FIG. 133.

403. General Effects of the Earth's Centrifugal Force

If the Earth were at rest the weight of a body would be entirely due to the Earth's attraction. But in consequence of the diurnal rotation the apparent weight is the resultant of the Earth's attraction and the centrifugal force.

Let QOR represent a meridian section of the Earth (Fig. 133). Consider a body of mass m supported at any point O on the Earth's surface. Since the Earth is nearly, but not quite, spherical, the force g_0 of the Earth's attraction on a unit mass is not directed exactly to the Earth's centre, but along a line OK. But, owing to the body's central acceleration along OM, the force which it exerts on the support is not quite equal to the Earth's attraction mg_0, but is compounded of mg_0 acting along OK, and the centrifugal force $m \,.\, n^2 \,.\, MO$ acting along MO.

On KQ take a point G such that

$$KG : KO = n^2 \,.\, MO : g\,;$$

then, by the triangle of forces, OG is the direction of the resultant force exerted by the body on its support, and this force is the apparent weight of the body. Hence, also OG represents the apparent direction of gravity, or the vertical as indicated by a plumb-line. Producing GO, KO to Z, Z'', we see that *the effect of centrifugal force is to displace the vertical from Z'' towards the nearest pole* (P).

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* See any book on Dynamics.

The angle ZGQ measures the (geographical) latitude of the place, and is greater than $Z''KQ$, which would measure the latitude if the Earth were at rest. Hence *the apparent latitude of any place is increased by centrifugal force.*

Again, if the apparent weight be denoted by mg, we have, by the triangle of forces,

$$g : g_0 = GO : KO;$$

now from the figure it is evident that $GO<KO$, and therefore $g<g_0$. Hence *the apparent weight of a body is diminished by centrifugal force.*

404. Effect on the Acceleration of a Falling Body

If a body is falling freely towards the Earth near O, the whole acceleration of its motion in space is due to the Earth's attraction, and is g_0, along OK. But the Earth at O has itself an acceleration n^2OM towards M. Hence the acceleration of the body *relative* to the Earth is the resultant of g_0 along OK, and $n^2 . MO$ along MO, and is therefore g along OG. Hence the body approaches the Earth with acceleration g along OG. *Therefore its relative acceleration is the acceleration due to its apparent weight, that is, to the resultant of the Earth's attraction and centrifugal force.*

405. To Find the Loss of Weight of a Body at the Equator, due to Centrifugal Force

At the equator centrifugal force is directly opposed to gravity; hence, if a denote the Earth's radius CQ,

$$g = g_0 - n^2a.$$

Now we have roughly

$$g_0 = 32{\cdot}18 \text{ feet per second per second,}$$

$$a = 3963 \text{ miles} = 3963 \times 5280 \text{ feet,}$$

and

$$n = 2\pi \text{ radians per sidereal day as that :—}$$

$$n = \frac{2\pi}{86164} \text{ radians per } \textit{mean solar second.}$$

$$\text{Hence } n^2a = \frac{3963 \times 5280 \times 4\pi^2}{86164 \times 86164} = {\cdot}11127,$$

$$\text{and therefore } \frac{n^2a}{g_0} = \frac{{\cdot}11127}{32{\cdot}18} = \frac{1}{289} \text{ nearly.}$$

$$\text{Hence } g = g_0 - \frac{1}{289}g_0,$$

or the effect of the Earth's rotation is to decrease the weight of a body at the equator by about $\frac{1}{289}$ *of the whole.*

For rough calculations it would be sufficient to take $g = 32{\cdot}2$, $a = 3960$ miles, and to neglect the difference between a solar and a sidereal day. This would give $\frac{1}{289}$, as before.

406. To Find approximately the Loss of Weight of a Body and the Deviation of the Vertical due to Centrifugal Force in any given Latitude

Let $\phi = QGO =$ astronomical latitude of O; $D = GOK = ZOZ'' =$ deviation of vertical from direction of Earth's attraction, or increase of latitude due to centrifugal force. We have :—

$$OM = CO \cos COM = a \cos \phi \text{ approximately};$$

where a is the Earth's radius, since the Earth is very nearly spherical, and $\angle COM$ is therefore very nearly equal to the latitude ϕ. Therefore centrifugal force per unit mass at O is :—

$$= n^2 . OM = n^2 . a \cos \phi = \tfrac{1}{289}\, g_0 \cos \phi \text{ (from Art. 405).}$$

Resolving along OG, we have, if g_0 be the Earth's attraction per unit mass at O*,

$$g = g_0 \cos D - n^2 . OM \cos \phi;$$

$$\text{or } g = g_0 - \frac{g_0}{289} \cos^2 \phi \text{ approximately}$$

(since D is small, and $\cos D = 1$ nearly).

Hence, in latitude ϕ, *the Earth's rotation diminishes the weight of a body by approximately* $\frac{1}{289} \cos^2 \phi$ *of itself.*

Resolving perpendicular to OG, we have

$$g_0 \sin D - n^2 OM \sin \phi = 0;$$

$$\text{or} \quad \sin D = \frac{n^2 a \cos \phi \sin \phi}{g_0} = \frac{1}{289} \frac{\sin 2\phi}{2}.$$

Since D is small, this gives approximately :—

$$\text{circular measure of } D = \frac{1}{289} \frac{\sin 2\phi}{2};$$

$$\text{or} \quad D'' \text{ (number of seconds in } D) = \frac{180 \times 60 \times 60}{289 \times 2\pi} \sin 2\phi$$

$$= \frac{206265}{578} \sin 2\phi = 357'' \sin 2\phi.$$

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* Since the Earth is not quite spherical, g_0 is not the same at O as at the equator. The difference may be neglected, however, when multiplied by the small constant $\frac{1}{289}$.

Hence the deviation $D = 5' \, 57''.\sin 2\phi$, *and this is the increase of latitude due to centrifugal force.*

COROLLARY.—The deviation of the vertical due to contrifugal force is greatest in latitude 45° (where $\sin 2\phi = 1$), and is there $5' \, 57''$.

407. Figure of the Earth

In Art. 98 we stated that the form of the Earth has been observed to be an oblate spheroid. Now it has been proved mathematically that a mass of gravitating liquid when rotating takes the form of an oblate spheroid whose least diameter is along its axis of rotation. Thus the Earth's form may be accounted for on the theory that the Earth's surface was formerly in a fluid or molten state, and that it then assumed its present form, owing to its diurnal rotation. We thus have another argument in favour of the Earth's rotation; but it is only fair to say that this theory of the Earth's origin has not been satisfactorily demonstrated. It accounts satisfactorily, however, for the form of the surface of the *ocean*.

This theory may be illustrated by the following general considerations. When a mass of liquid is acted on by no forces beyond the attractions of its particles, it is easy to realise that the whole is in equilibrium in a spherical form, being then perfectly symmetrical.

If, however, the fluid be rotating about the axis PCP', the centrifugal force tends to pull the liquid away from this axis and towards the equatorial plane. The liquid would, therefore, fly right off, but its attraction is always trying to pull it back to the spherical form. Hence, the only effect of centrifugal force (which, for the Earth, is small compared with gravity) is to distort the liquid from its spherical form by pulling it out towards the equator; and it is therefore reasonable to suppose that the fluid will assume a more or less oblate figure, whose equatorial is greater than its solar diameter.

It may also be remarked that the form assumed by the liquid would be such that the effective force of gravity (*i.e.* the resultant of the attraction and centrifugal force) on the surface would everywhere be perpendicular (*i.e.* normal) to the surface.

408. Gravity Observations

If the Earth were a sphere, its attraction g_0 would everywhere tend to its centre, and would be of the same intensity at all points on its surface, while the variations in g, the apparent intensity of gravity, would be entirely due to the Earth's centrifugal force, its value in latitude ϕ being proportional to $1 - \frac{1}{289}\cos^2\phi$ (Art. 406). By comparing the values of g at different places, we should then be able to demonstrate the Earth's centrifugal force, and hence prove its rotation. But, owing to the Earth's ellipticity, its attraction g_0 does not pass through the centre, except at the poles and equator, and its intensity is not everywhere constant. It is, therefore, important to determine experimentally the values of g at different stations. By allowing for centrifugal force, the corresponding values of the Earth's attraction g_0 can be found, and the variations in its intensity at different places afford a measure of the amount by which the Earth differs from a sphere. We thus have a gravitational method of finding the Earth's ellipticity.

But the Earth's ellipticity can also be determined by direct observation, as explained in Chapter IV, Section III. The agreement between the results thus independently obtained furnishes another proof of the Earth's rotation.

In consequence of the Earth's ellipticity it is found (by observation) that the difference in the intensity of gravity between the pole and equator is increased from $\frac{1}{289}$ to $\frac{1}{190}$ of the whole.

409. To Compare the Intensity of Gravity at Different Places

The intensity of gravity may be measured by the force with which a body of unit mass is drawn towards the Earth. This cannot be measured by weighing a body with a *common balance,* because the weights of the body and of the counterpoise, by means of which it is weighed, are equally affected by variations in the intensity of gravity, and two bodies of equal *mass* will, therefore, balance one another when placed in the scale pans, no matter what be the intensity of gravity. In fact, by weighing a body with weights in the ordinary way, we determine only its *mass* and not the absolute force with which it is drawn to the Earth.

We might determine the intensity of gravity by means of a "*spring balance,*" for the elasticity of the spring does not depend on the intensity of gravity, and therefore the extension of the spring gives an absolute measure of the force with which the body is drawn towards the Earth. If the apparatus were to support a mass of one pound, first at the equator and then at the pole, the force on it would be greater at the latter place by about $\frac{1}{190}$, and this spring would there be extended about $\frac{1}{190}$ more. It would be very difficult to construct a spring balance sufficiently sensitive to show such a small relative difference of weight, but it has been done.

Atwood's machine might be used to find g, but this method is not capable of giving very accurate results.

The most accurate method of finding g is by timing the oscillations of a *pendulum* of known length.

[* A theoretical *simple pendulum,* consisting of a mere heavy particle of no dimensions, suspended by a thread without weight, is of course impossible to realise in practice, but the difficulty is overcome by the use of a pendulum called *Captain Kater's Reversible Pendulum.* This pendulum is a bar which can be made to swing about either of two knife-blades fixed at opposite sides of, but unequal distances from, its centre of gravity, and it is so loaded that the periods of oscillation, when suspended from either knife-edge, are equal. It is then known that the pendulum will swing about either knife-edge in just the same manner as if it were a simple pendulum whose whole mass was concentrated at the *other* knife-edge. The distance between the knife-edges is, therefore, to be regarded as the *length of the pendulum.*]

410. Oscillations of a Simple Pendulum

In a simple pendulum, formed of a small heavy particle suspended by a fine light thread of length l, the period of a complete small oscillation to and fro is

$$t = 2\pi \sqrt{\frac{l}{g}},$$

the time of a single swing or "*beat*" being of course half of this. Hence by observing the time of oscillation t and measuring the length l, the intensity of gravity g can be found.

By the "*seconds pendulum*" is meant a pendulum in which one *beat* occupies one second, hence a complete oscillation occupies *two seconds.*

Example.—Having given that the length of the seconds pendulum is 99·39 centimetres, to find g in centimetres per second per second.

$$t = 2\pi\sqrt{l/g} = 2 \text{ seconds, and } l = 99{\cdot}39 \text{ cm.,}$$
$$\text{Therefore } g = \pi^2 l = 99{\cdot}39 \times (3{\cdot}1416)^2 = 981.$$

It is often necessary to compare the lengths of two pendulums whose periods of oscillation are very nearly equal, to find the effect of small changes in the length of a pendulum due to variations in temperature, or, in comparing the intensity of gravity at different places, to find the effect of a small alteration in the value of g on the period of oscillation and on the number of oscillations in a given interval. If the differences are small, the calculations may be much simplified by means of the following methods of approximation.*

411. To Find the Change in the Time of Oscillation of a Pendulum, and in the Number of Oscillations in a given Interval, due to a small Variation in its Length or in the Intensity of Gravity.

If t be the time of a complete oscillation of a pendulum of length l, we have, by Art. 410,

$$t^2 = 4\pi^2\frac{l}{g} \quad \ldots\ldots\ldots\ldots\ldots\ldots \text{(i)}$$

(i) Suppose the length increased to l', and let t' be the new period of oscillation. We have

$$t'^2 = 4\pi^2\frac{l'}{g}.$$

$$\text{Therefore,} \quad \frac{t'^2}{t^2} = \frac{l'}{l},$$

$$\text{therefore also} \quad \frac{t'^2 - t^2}{t^2} \equiv (t' - t)\frac{t' + t}{t^2} = \frac{l' - l}{l}.$$

These formulae are exact. But if l' is very nearly equal to l, t' is very nearly equal to t, and therefore, putting $t + t' = 2t$, we have approximately

$$2\frac{t' - t}{t} = \frac{l' - l}{l},$$

whence, if t, l be known, the change $t' - t$, consequent on the increase of length $l' - l$, may be readily found approximately without the labour of extracting any square roots.

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* The same results can of course be obtained by means of the differential calculus.

(ii) Suppose the intensity of gravity increased to g', the length l being unaltered, and let t' be the new period. Now :—

$$t'^2 = 4\pi^2 \frac{l}{g'},$$

$$\text{Therefore } \frac{t'^2}{t^2} = \frac{g}{g'}$$

$$\text{therefore also } \frac{t'^2 - t^2}{t^2} \equiv (t' - t)\frac{t' + t}{t^2} = \frac{g - g'}{g'}.$$

But, if t, g are very nearly equal to t', g', this gives approximately

$$2\frac{t' - t}{t} = -\frac{g' - g}{g}.$$

(iii) If l and g both vary, becoming l' and g', we have, in like manner

$$\frac{t'^2}{t^2} = \frac{l'g}{lg'}.$$

$$\text{and } \frac{t'^2 - t^2}{t^2} = (t' - t)\frac{t' + t}{t^2} = \frac{l'g - lg'}{lg'} = \frac{l'g - l'g'}{lg'} + \frac{l'g' - lg'}{lg'}$$

$$= \frac{l'}{l}\frac{g - g'}{g'} + \frac{l' - l}{l},$$

or approximately, if the variations are small,

$$2\frac{t' - t}{t} = \frac{l' - l}{l} - \frac{g' - g}{g},$$

showing that the effects of the two variations may be considered separately.

(iv) If n, n' be the number of complete oscillations of the pendulum in a given interval T, and if, in consequence of the change, this number be altered to n', we have

$$nt = n't' = T,$$

$$\text{so that } \frac{n'}{n} = \frac{t}{t'},$$

$$\text{whence } \frac{n' - n}{n} = \frac{t - t'}{t'}.$$

If t' is very nearly equal to t, this gives approximately

$$\frac{n' - n}{n} = -\frac{t' - t}{t} = -\tfrac{1}{2}\frac{l' - l}{l} + \tfrac{1}{2}\frac{g' - g}{g},$$

which determines the number of beats gained by the pendulum in the time T, in consequence of the variations, the original number n being supposed known.

Example.—*Find the number of oscillations gained or lost in an hour by the pendulum of the Example of Art.* 410, *supposing* (*i*) *its length increased to* 1 *metre*; (ii) *the acceleration of gravity increased to* 982; (iii) *both changes made simultaneously.*

(i) The pendulum beats seconds; therefore it performs 3600 half oscillations or 1800 whole oscillations in an hour. Also $l' = 100{\cdot}00$, $l' - l = 0{\cdot}61$, $g' - g = 0$. Hence, if n' be the new number of oscillations in an hour,

$$\frac{n' - 1800}{1800} = -\frac{0{\cdot}61}{2l} = -\frac{0{\cdot}61}{2l'} \text{ (approx.)} = -\frac{0{\cdot}61}{200};$$

$$\text{or } n' - 1800 = -9 \times {\cdot}61 = -5{\cdot}49.$$

Hence the pendulum loses nearly $5\tfrac{1}{2}$ oscillations in an hour, and the number of oscillations is therefore $1794\tfrac{1}{2}$.

$$\text{(ii) Here } \frac{n' - 1800}{1800} = \frac{g' - g}{2g} = \frac{982 - 981}{2 \times 981} = \frac{1}{2 \times 981};$$

$$\text{or } \quad n' - 1800 = \frac{1800}{1962} = {\cdot}9 = 1 \text{ nearly.}$$

Hence the pendulum gains 1 oscillation in an hour, the total number of oscillations being 1801.

(iii) Since from the first cause the pendulum loses $5\tfrac{1}{2}$ oscillations and from the second it gains 1 oscillation, therefore on the whole it loses $5\tfrac{1}{2} - 1$ or $4\tfrac{1}{2}$ oscillations per hour. It therefore performs $1795\tfrac{1}{2}$ oscillations or 3591 *beats* per hour.

412. To Compare the Times of Oscillations of Two Pendulums whose Periods are Very Nearly Equal

If two pendulums of nearly equal periods are simultaneously started swinging in the same direction, the one whose period is a little the shortest will soon begin to swing before the other. After some time it will gain a half oscillation, and the pendulums will then be swinging in opposite directions. After another equal interval, the quicker pendulum will have gained one whole oscillation on the slower, and both will be again swinging together in the same direction. Similarly, every time the quicker pendulum has gained an exact number of complete oscillations on the slower, both will be swinging together in the same direction. Thus, the number of *coincidences*, or the number of times that the two pendulums are together, in any interval, is equal to the number of complete oscillations (to and fro) gained by the quicker pendulum over the slower, *i.e.* the difference between the numbers of complete oscillations performed by the two pendulums.

Thus, if n, n' be the number of oscillations of the slower and faster pendulums in any given interval, then $n' - n$ is the number of oscillations gained by the latter, and is, therefore, the number of "coincidences." If either of the numbers n, n' is known, we can, by counting the coincidences, find the other number.

413. To find g, the Acceleration of Gravity

The simplest plan is to use a Captain Kater's pendulum, the beat of which is very nearly one second. By counting the " coincidences " of the pendulum with the pendulum of a clock regulated to beat seconds during, say, an hour (as shown by the clock) the exact time of oscillation can be found. Moreover, from the number of beats gained or lost, and the observed length of the pendulum, we may calculate the amount by which the length must be increased or decreased in order to make the pendulum beat seconds. The length of the seconds pendulum is thus known, and the value of g can be found.

The reason for using *two pendulums* is that it would be extremely difficult to measure the length of the pendulum of the clock, and it would be equally difficult to find the period of oscillation of a pendulum without comparing it with that of a clock, whose rate can be regulated daily by astronomical observations.

414. To Compare the Value of g at Two Different Stations

The simplest plan is to determine the number of seconds gained or lost in a day by a clock after it has been taken from one station to the other, the length of the pendulum remaining the same. If n, n' be the number of seconds marked by the clock in a day at the two places, we have exactly

$$\frac{n'^2}{n^2} = \frac{g'}{g},$$

or approximately,

$$\frac{n' - n}{n} = \tfrac{1}{2}\,\frac{g' - g}{g},$$

whence the ratio of g' to g may be found.

Here there is no necessity to use a Captain Kater's pendulum, because the length of the pendulum is not required ; hence the ordinary compensating pendulum of the clock answers the purpose. If a *non-compensating* pendulum were used, it would be necessary to make allowance for any change in the length of the pendulum due to variations in temperature.

EXAMPLES

1. A Foucalt's pendulum being set vibrating in latitude 30°, show that after one sidereal day it is again vibrating in the same plane. Find the corresponding interval in latitude 45°.

2. If two *conical pendulums* of equal length revolve in opposite directions, describing cones of equal vertical angle, show that at a place in the northern hemisphere the pendulum which revolves in the same direction as the hands of a watch will have the greater apparent angular velocity, and will gain two complete revolutions on the other in the period in which the plane of Foucault's pendulum turns through 360°. Consider, in the first place, the phenomena at the North Pole. Also describe the corresponding phenomena in the southern hemisphere.

3. If a railway is laid along a meridian, and a train is travelling from the equator towards the pole, investigate whether it will exert an eastward or a westward thrust on the rails, and why.

4. A bullet is fired in N. latitude 45°, with a velocity of 1,600 feet per second, at an elevation 45°. Prove that it must be aimed in a vertical plane 12′ 30″ to the left of the target; and, if this precaution be neglected, calculate how many feet it will deviate to the right.

5. Show that if the Earth were to rotate seventeen times as fast, a body at the equator would have no weight.

6. If the Earth were a homogeneous sphere rotating so fast that bodies at the equator had no weight, show that in any latitude the plumb-line would point to the celestial pole.

7. Would the latitude of Greenwich be increased or decreased by an increase in the speed of the Earth's rotation? If the latitude of a place be 60°, find what would be its latitude if (i) the Earth were reduced to rest, (ii) its angular velocity were doubled.

8. Prove that if the Earth were reduced to rest, a pendulum in latitude 45° would gain one oscillation in every 1156, but if the Earth's angular velocity were doubled, it would lose *three* oscillations in 1156.

9. A clock and a chronometer are taken from London to Gibraltar and it is observed that the clock begins to lose, while the chronometer continues to keep correct time. Why is this?

10. Assuming that a body loses $\frac{1}{183}$ of its weight when taken from the poles to the equator, show that a clock which keeps mean time at the equator would keep sidereal time at the poles, with a rate amounting to only a fraction of a second per day.

11. With the data of the last question, show that the Earth's attractions on a unit mass placed at the equator and at the poles are in the ratio of (nearly) 496 : 497.

12. If a railway train is travelling along the equator from east to west, show that it presses on the rails with a force greater than its apparent weight when at rest. If the train is travelling at forty-five geographical miles per hour, and its mass is 144 tons, find (roughly) in pounds the increase in the downward thrust on the rails.

EXAMINATION PAPER

1. Give reasons for supposing that the diurnal rotation of the heavens is only an appearance caused by a real rotation of the Earth. Name methods by which it has been claimed that this is proved.

2. Describe the *gyroscope experiment*, and the *gyroscope*.

3. Give any theoretical methods of determining latitude without observing a heavenly body.

4. Describe Foucault's experiment for exhibiting the Earth's rotation; and find the time of the complete rotation of the plane of vibration of a simple pendulum freely suspended in latitude 60°.

5. Having given that the Earth's circumference is 40,000 kilometres, find the acceleration of a body at the equator due to the Earth's rotation in centimetres

per second per second, and taking g_0, the acceleration of gravity, to be 981 of these units, deduce the ratio of centrifugal force to gravity at the equator.

6. What is meant by the *vertical* at any point of the Earth's surface? Supposing the Earth to be a uniform sphere revolving round a diameter, calculate the deflection of the vertical from the normal to the surface.

7. State what argument is drawn from the Earth's form to support the hypothesis of its rotation.

8. Why is it that the intensity of gravity is less at the equator than in higher latitudes? Show that the alteration in the apparent weight of a body due to centrifugal force varies nearly as $\cos^2\phi$ where ϕ is the latitude, and state the ratio of centrifugal force to gravity at the equator.

9. If a body is weighed by a spring balance in London and at Quito, a difference of weight is observed. Why is this not observed if an ordinary pair of scales be used?

10. Show that an increase in the intensity of gravity will cause a pendulum to swing more rapidly, and *vice versa*. If the acceleration of gravity be increased by the small fraction $1/r$ of its value, show that a pendulum will gain one complete oscillation in every $2r$.

CHAPTER XVII

THE LAW OF UNIVERSAL GRAVITATION

I.—THE EARTH'S ORBITAL MOTION—KEPLER'S LAWS AND THEIR CONSEQUENCES

415. Evidence in Favour of the Earth's Annual Motion round the Sun

The theory that the Earth is a planet, and revolves round the Sun, was propounded by Copernicus (*circ.* 1530) and received its most convincing proof, over 150 years later from Newton (A.D. 1687), who accounted for the motions of the Earth and planets as a consequence of the law of universal gravitation. This proof is based on dynamical principles; but the following arguments, based on other considerations, afford independent evidence in favour of the theory that the Earth revolves round the Sun rather than the Sun round the Earth.

(i) The Sun's diameter is 110 times that of the Earth's, and it is much easier to believe that the smaller body revolves round the larger, than that the larger body revolves round the smaller.

If the dynamical laws of motion be assumed, and the other planets left out of consideration, the Sun and the Earth would each revolve around their common centre of gravity. The mass of the Earth is only 1/330,000 that of the Sun. Their centre of gravity is less than 300 miles from the centre of the Sun, whose radius is 432,000 miles. The Sun is pulled only very slightly from its position by the attraction of the planets, each of which can be considered to revolve round the Sun.

(ii) The stationary points, and alternately direct and retrograde motions of the planets, are easily accounted for on the theory that the Earth and planets revolve round the Sun (Chap. XI) in orbits very nearly circular, and it would be impossible to give such a simple explanation of these motions on any other theory. It is true that we might suppose, with Tycho Brahé (*circ.* 1600), that the planets revolve round the Sun as a centre, while that body has an orbital motion round the Earth, but this explanation would be more complicated than that which assumes the Sun to be at rest. And it would be hard to explain how such huge bodies as Jupiter and Saturn could be brought to describe such complex paths.

(iii) As seen through a telescope, Venus and Mars are found to be very similar to the Earth in their physical characteristics, and their phases show that, like the Earth and Moon, they are not self-luminous. It is, therefore, only natural to suppose that their property of revolving round the Sun is shared by the Earth. Moreover, the Earth's relative distance from the Sun agrees fairly closely with that given by Bode's law ; hence there is a strong analogy between the Earth and the planets.

(iv) The orbital motion of the Earth is in strict accordance with Kepler's Laws of Planetary Motion. In particular, the relation between the mean distances and periodic times given by Kepler's Third Law (Art. 264) is satisfied in the case of the Earth's orbit.

Moreover, a similar relation is observed to hold between the periodic times of Jupiter's satellites and their mean distances from Jupiter. Hence it is probable that the Earth and planets form, like Jupiter's satellites, one system revolving about a common centre. But it is improbable that the Sun and Moon should both revolve about the Earth, for their distances from it and their periods are *not* connected by this relation.

(v) The changes in the relative positions of two stars during the year in consequence of annual parallax can only be accounted for on the hypothesis either of the Earth's orbital motion, or of a highly improbable rigid connection between all the nearer stars and the Sun, compelling them all to execute an annual orbit of the same size and position.

(vi) The aberration of light affords the most convincing proof of all. In particular, the relation between the coefficient of aberration and the retardation of the eclipses of Jupiter's satellites has been fully verified by actual observations, and affords incontestible evidence that the phenomenon is actually due to the finite velocity of light, And the alternative hypothesis which would account for annual parallax would *not* give rise to aberration, but would produce an entirely different phenomenon. Hence the evidence derived from the aberration of light, unlike the previous evidence, furnishes a conclusive proof, and not merely an argument, in favour of the Earth's orbital motion.

416. Newton's Theoretical Deductions from Kepler's Laws

Kepler's Three Laws of planetary motion (Art. 264) naturally suggest the following questions:—

(1) What makes the planets move in ellipses?

(2) Why does the radius vector from the Sun to any planet trace out equal areas in equal times?

(3) Why are the squares of the periodic times proportional to the cubes of the mean distances from the Sun?

These questions were first answered by Newton about 1687, or nearly sixty years after the death of Kepler. The theoretical interpretation of the Second Law necessarily precedes that of the first; accordingly we now repeat the laws in their new order, together with Newton's interpretations of them.

Kepler's Second Law.—The radius vector joining each planet to the Sun moves in a plane describing equal areas in equal times.

> NEWTON'S DEDUCTION.—*The force under which a planet describes its orbit always acts along the radius vector in the direction of the Sun's centre.*

Kepler's First Law.—The planets move in ellipses, having the Sun in one focus.

> NEWTON'S DEDUCTION.—*The force on any planet varies inversely as the square of its distance from the Sun.*

Kepler's Third Law.—The squares of the periodic times of the several planets are proportional to the cubes of their mean distances from the Sun.

> NEWTON'S DEDUCTION.—*The forces on different planets vary directly as their masses, and inversely as the squares of their distances from the Sun, or, in other words, the accelerations of different planets, due to the Sun's attraction, vary inversely as the squares of their distances from the Sun.*

If, as we have every reason for believing, the planets are material bodies, Newton's laws of motion show that they cannot move as they do unless they are acted on by some force, otherwise they would either be at rest or move uniformly in a straight line. Kepler's Second Law then enables us to determine the direction of this force, his First Law enables us to compare the force at different parts of the same orbit, and his Third Law enables us to compare the forces on different planets.

417. Case of Circular Orbits

We have seen that the orbits of most of the planets are nearly circular, the eccentricities being small, except in the case of Mercury. Before proceeding to the general discussion of the dynamical interpretation of Kepler's Laws, it will be convenient therefore to consider the case where the orbits are supposed circular, having the Sun for centre. Kepler's Second Law shows that under such circumstances the planets will describe their orbits uniformly, and it hence follows that the acceleration of a planet has no component in the direction of motion, but is directed exactly towards the centre of the Sun. The law of force can now be deduced very simply, as follows :—

418. To Compare the Sun's Attractions on Different Planets, assuming that the Orbits are Circular and that the Squares of the Periodic Times are proportional to the Cubes of the Radii

Suppose a planet of mass M is moving with velocity v in a circle of radius r. Let T be the periodic time, P the force to the centre.

Since the normal acceleration in a circular orbit is v^2/r, we have :—

$$P = \frac{Mv^2}{r}.$$

In the period T the planet describes the circumference $2\pi r$, so that

$$vT = 2\pi r.$$

Substituting for v, we have

$$P = \frac{4\pi^2 Mr}{T^2} = \frac{M}{r^2} \times \frac{4\pi^2 r^3}{T^2}.$$

Let M' be the mass of another planet revolving in a circular orbit of radius r', T' its periodic time, P' the force of the Sun's attraction ; then we have in like manner

$$P' = \frac{M'}{r'^2} \times \frac{4\pi^2 r'^3}{T'^2}.$$

By Kepler's Third Law, $\dfrac{r^3}{T^2} = \dfrac{r'^3}{T'^2}$;

Therefore, $P : P' = \dfrac{M}{r^2} : \dfrac{M'}{r'^2}$.

The forces on different planets vary directly as their masses and inversely as the squares of their distances from the Sun.

COROLLARY 1.—Let $P = CM/r^2$; then C is called the *absolute intensity* of the Sun's attraction, and we see that :—

The absolute intensity of the Sun's attraction is the same for all planets.

For $$C = \frac{4\pi^2 r^3}{T^2} = \frac{4\pi^2 r'^3}{T'^2}.$$

The constant C evidently represents the force with which the Sun would attract a unit mass at unit distance, or the acceleration which the Sun would produce at unit distance.

COROLLARY 2.—If another body be revolving in an orbit of radius r' in a period T', under a *different* central force, which produces an acceleration C'/r'^2 at distance r', we have :—

$$C' = \frac{4\pi^2 r'^3}{T'^2} \quad \text{and} \quad C = \frac{4\pi^2 r^3}{T^2};$$

$$\text{or} \quad C'T'^2 : CT^2 = r'^3 : r^3,$$

a formula which enables us to compare the absolute intensities of two *different* centres of force, which attract inversely as the squares of the distances, when the periodic times and distances of two bodies revolving about them are known.

419. To Compare the Velocities and Angular Velocities of Two Planets moving in Circular Orbits

If v, v' are the velocities, n, n' the angular velocities (in radians per unit time), we have :—

$$n = 2\pi/T, \quad n' = 2\pi/T';$$

and $$n : n' = T^{-1} : T'^{-1} = r^{-\frac{3}{2}} : r'^{-\frac{3}{2}}.$$

Also $$v = rn, \quad v' = r'n';$$

so that $$v : v' = r^{-\frac{1}{2}} : r'^{-\frac{1}{2}}.$$

Examples.—1. *If the Earth's period were doubled, find what would be its new distance from the Sun.*

If r, r' be the old and new distances, Kepler's Third Law gives

$$r'^3 : r^3 = 2^2 : 1^2;$$

$$\text{or } r' = r \times \sqrt[3]{4} = 93{,}000{,}000 \times 1{\cdot}587 = 147{,}600{,}000 \text{ miles.}$$

2. *If the Earth's velocity were doubled, its orbit remaining circular, find its new distance.*

Here $$r' : r = v^2 : v'^2 = 1 : 4;$$

$$\text{and } r' = \tfrac{1}{4}r = 23{,}250{,}000 \text{ miles.}$$

3. *If the Earth's angular velocity were doubled, find its new distance.*

The new angular velocity being double the old, the new period would be half the old, and therefore

$$r'^3 : r^3 = (\tfrac{1}{2})^2 : 1^2;$$

$$\text{and } r' = r \times \sqrt[3]{\tfrac{1}{4}} = r/\sqrt[3]{4} = 93{,}000{,}000 \div 1{\cdot}587 = 58{,}600{,}000 \text{ miles.}$$

4. *Find what would be the coefficient of aberration to an observer situated on Venus.*

The coefficient of aberration (in circular measure) is the ratio of the observer's velocity to the velocity of light; hence, if k, k' are the coefficients on the Earth and Venus,

$$\frac{k'}{k} = \frac{v'}{v} = \frac{r'^{-\frac{1}{2}}}{r^{-\frac{1}{2}}} = \sqrt{\frac{r}{r'}} = \sqrt{\frac{100}{72}};$$

$$\text{or } k' = 20{\cdot}47'' \times \sqrt{(1{\cdot}38)} = 20{\cdot}47'' \times 1{\cdot}1785 = 24{\cdot}12''.$$

We shall now prove Newton's deductions from Kepler's Laws, for the general case of elliptic orbits, employing, however, different and simpler proofs than those used by Newton.

420. Areal Velocity.—Definition

If a point P is moving in any path MPK about a centre S, the rate of increase of the area of the sector MSP, bounded by the fixed line SM and the radius vector SP, is called the *areal velocity* of P about the point S.

If the radius vector SP describes equal areas in equal times, in accordance with Kepler's Second Law, the areal velocity of P about S is of course constant, and is then measured by the area of the sector described in a unit of time.

If the rate of description of areas is not constant, we must, in measuring the areal velocity *at* any point, pursue a similar course to that adopted in measuring variable velocity *at* any instant, as follows :—

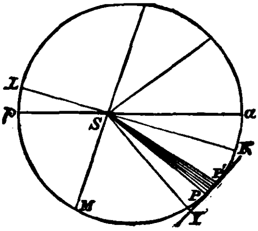

FIG. 134.

If the radius vector describes the sector PSP' in the interval of time t, then the *average areal velocity* over the arc PP' is measured by the ratio

$$\frac{\text{area } PSP'}{\text{time } t}.$$

(Thus the average areal velocity is the areal velocity with which a radius vector, sweeping out equal areas in equal times, would describe the sector PSP' in the same time t.)

The areal velocity *at* a point P is the limiting value of the average areal velocity over the arc PP' when this arc is *infinitesimally small*.

421. Relation between the Areal Velocity and the Actual (linear) Velocity

Let PP' be the small arc described by a body in any small interval of time t. Let v be the actual or linear velocity of the body, h its areal velocity. Since the arc PP' is supposed small, we have

$$PP' = vt, \quad \text{area } PSP' = ht.$$

Draw SY perpendicular on the chord PP' produced. Then

$$\triangle PSP' = \tfrac{1}{2}\,(\text{base}) \times (\text{altitude}) = \tfrac{1}{2}PP' \times SY\,;$$

$$\text{Therefore}\ \ ht = \tfrac{1}{2}vt \times SY, \text{ or } \ h = \tfrac{1}{2}v \,.\, SY.$$

But when the arc PP' is infinitesimally small, PY is the tangent at P, and SY is therefore the perpendicular from S on the tangent at P. If this perpendicular be denoted by o, we have therefore

$$h = \tfrac{1}{2}vp \text{ .... (i)}$$

or (*areal vel. about S*) = ½ (*velocity*) × (*perp. from S on tangent*).

COROLLARY.—For planets moving in circular orbits of radii r, r', we have :—

$$h = \tfrac{1}{2}vr, \text{ and } h' = \tfrac{1}{2}v'r'.$$

But

$$v : v' = r^{-\frac{1}{2}} : r'^{-\frac{1}{2}};$$

$$\text{Therefore } h : h' = r^{\frac{1}{2}} : r'^{\frac{1}{2}};$$

hence the areal velocity of a planet moving in a circular orbit is proportional to the square root of the radius.

422. PROPOSITION I. **If a particle moves in such a manner that its areal velocity about a fixed point is constant, to prove that the resultant force on the particle is always directed towards the fixed point. [Newton's Deduction from Kepler's Second Law.]**

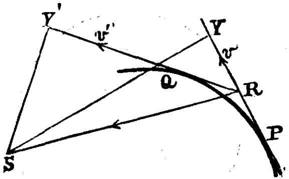

FIG. 135.

Let a body be moving in the curve PQ in such a way that its areal velocity about S remains constant. Let v, v' be the velocities at P, Q, and let PY, QY', the corresponding directions of motion, intersect in R. Drop SY, SY', perpendicular on PY, QY'. Since the areal velocities at P and Q are equal :—

$$v \,.\, SY = v' \,.\, SY'.$$

But

$$SY = SR \sin SRY,$$

$$SY' = SR \sin SRY'.$$

$$\text{Therefore } v \sin SRY = v' \sin SRY'.$$

i.e. Component velocity at P perpendicular to SR = component vel. at Q perp. to SR.

Therefore, as the particle moves from P to Q, its velocity perpendicular to RS is unaltered, and therefore the total change of velocity is parallel to RS.

This is true whether the arc PQ be large or small. But if the arc PQ be taken infinitesimally small, the average rate of change of velocity over PQ measures the acceleration at P, and R coincides with P.

Therefore the direction of the acceleration of the particle at any point of its path always passes through S, and therefore the force acting on the particle also always passes through S.

423. Conversely, if the force on the particle always passes through S, the areal velocity about S remains constant

For in passing from P to Q, the direction of motion is changed from PR to RQ, and the same change of velocity could therefore be produced by a suitable single blow or instantaneous impulse acting at R. And since the force at every point of PQ always passes through S, this equivalent impulse must evidently also pass through S; it must therefore act along RS. Hence the velocity perpendicular to RS is unaltered by the whole impulse, and is the same at P as at Q; therefore

$$v \sin SRY = v' \sin SRY';$$

$$\text{or } v \,.\, SY = v' \,.\, SY';$$

$$\textit{i.e.} \text{ areal vel. at } P = \text{areal vel. at } Q.$$

424. PROPOSITION II. A particle describes an ellipse under a force directed towards the focus; to show that the force varies inversely as the square of the distance. [Newton's Deduction from Kepler's First Law.]

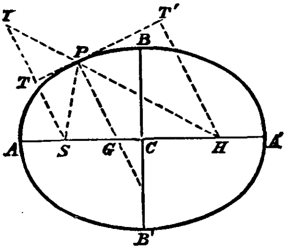

FIG. 136.

If h is the constant areal velocity, we have, by (i) :—

$$v = 2h/p.$$

We will now express the kinetic energy of the particle in terms of r, its distance from the focus. Let its mass be M. In the Appendix (Ellipse 11) it is proved that for the ellipse whose major and minor axes are $2a$, $2b$,

$$\frac{1}{p^2} \equiv \frac{1}{ST^2} = \frac{a}{b^2}\left(\frac{2}{r} - \frac{1}{a}\right).$$

$$\text{whence } v^2 = \frac{4h^2}{p^2} = \frac{4h^2a}{b^2}\left(\frac{2}{r} - \frac{1}{a}\right),$$

and kinetic energy at distance r

$$= \tfrac{1}{2}Mv^2 = \tfrac{1}{2}M\,\frac{4h^2}{p^2} = M\,\frac{2h^2a}{b^2}\left(\frac{2}{r} - \frac{1}{a}\right) \quad \ldots\ldots\ldots\ldots \text{(ii)}$$

If v' is the velocity at distance r', we have, similarly :—

$$\tfrac{1}{2}Mv'^2 = \tfrac{1}{2}M\,\frac{4h^2}{p'^2} = M\,\frac{2h^2a}{b^2}\left(\frac{2}{r'} - \frac{1}{a}\right).$$

and therefore, for the increase of kinetic energy,

$$\tfrac{1}{2}Mv'^2 - \tfrac{1}{2}Mv^2 = \frac{4Mh^2a}{b^2}\left(\frac{1}{r'} - \frac{1}{r}\right) \quad \ldots\ldots\ldots\ldots \text{(iii)}$$

Now the increase of kinetic energy is equal to the work done by the impressed force in bringing the particle from distance r to distance r'. The resolved part of the displacement in the direction of the force is $r - r'$. Hence if P' denote the *average value* of the force between the distances r and r', we have

$$\text{Work done} = P'(r - r') = \tfrac{1}{2}Mv'^2 - \tfrac{1}{2}Mv^2 = \frac{4Mh^2a}{b^2}\left(\frac{1}{r'} - \frac{1}{r}\right)$$

$$= \frac{4Mh^2a}{b^2}\,\frac{r - r'}{rr'} \quad \ldots\ldots\ldots\ldots\ldots\ldots\text{(iv)}$$

$$\text{or} \quad P' = \frac{4Mh^2a}{b^2rr'} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(v)}$$

Put $r' = r$; then the average force P' becomes the actual force P at distance r. Therefore

$$P\ (\text{Force at distance } r) = \frac{4Mh^2a}{b^2r^2} \quad \ldots\ldots\ldots\ldots\ldots\text{(vi)}$$

This is proportional to $1/r^2$. *Therefore the force varies inversely as the square of the distance.*

425. PROPOSITION III. **Having given that the squares of the periodic times of the planets are proportional to the cubes of the semi-axes major of their orbits, to compare the forces acting on different planets. [Newton's Deduction from Kepler's Third Law.]**

Let T be the periodic time of any planet; then, by hypothesis, the ratio $\frac{T^2}{a^3}$ is the same for all planets.

In the last proposition (vi) we showed that the force at distance r is given by

$$P = \frac{4Mh^2a}{b^2r^2}.$$

Let this be put $= MC/r^2$, where C is some constant; then

$$C = \frac{4h^2a}{b^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(vii)}$$

Now in the period T the radius vector sweeps out the area of the ellipse, and this area is πab (Appendix, Ellipse 13). Hence, since the areal velocity is h, we have :—

$$hT = \pi ab.$$

Substituting the value of h from this equation in (vii), we have

$$C = \frac{4\pi^2a^3b^2}{T^2b^2} = \frac{4\pi^2a^3}{T^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(viii)}$$

But a^3/T^2 is the same for all the planets; therefore C is constant for all the planets, and since the force :—

$$P = \frac{MC}{r^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(ix)}$$

it follows that *the forces on different planets are proportional to their masses divided by the squares of their distances from the Sun.*

Or, as in Art. 418, Cor. 1, *the absolute intensity of the Sun's attraction* (C) *is the same for all the planets.*

COROLLARY.—Let accented letters refer to the orbit of another particle revolving round a *different* centre of force of intensity C'. Then, by (viii),

$$T^2C : T'^2C' = a^3 : a'^3.$$

426. Other Consequences of Kepler's Laws

(i) In Art. 133 we showed that, in consequence of Kepler's Second Law being satisfied by the Earth in its annual orbit, the Sun's apparent motion in longitude is inversely proportional to the square of the Earth's distance from it. Since the areal velocity of any planet about the Sun always remains constant, it may be shown in like manner that its angular velocity is inversely proportional to the square of its distance from the Sun.

For, if the planet's radius vector revolves from SP to SP' in the time t, and if the arc PP' is very small, we have

$$\text{area } SPP' = \tfrac{1}{2}SP^2 \times \angle PSP' \text{ (Art. 133),}$$

the angle being measured in radians;

$$\text{Therefore, } \frac{\text{area } SPP'}{t} = \tfrac{1}{2}SP^2 \times \frac{\angle PSP'}{t},$$

i.e. (areal velocity of P) $= \frac{1}{2}SP^2 \times$ (angular velocity of P), provided that the angular velocity is measured in *radians* per unit of time.

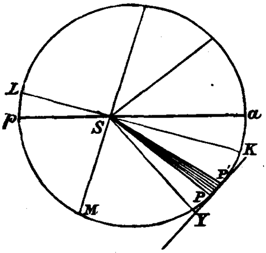

FIG. 137.

If n denote the angular velocity, h the areal velocity, and r the distance SP, we have therefore

$$h = \tfrac{1}{2}r^2n.$$

And since h is constant, n is inversely proportional to r^2.

(ii) If the mass of the planet is M, its momentum is Mv along PY, and the moment of this momentum about S is :—

$$= Mv \times SY = Mvp = 2hM. \text{ (Art. 421).}$$

This is the planet's *angular momentum*, and is constant, since h is constant.

427. Having given, in Magnitude and Direction, the Velocity of a Planet at any point of its Orbit, to construct the Ellipse described under the Sun's Attraction

Let the attraction at distance r be C/r^2 per unit mass, where C is given. Suppose that at the point P of the orbit the planet is moving with velocity v in the direction PT.

We have $v \times ST = 2h$, which determines h. Also, from (vii), $C = 4h^2a/b^2$. Substituting in (ii) we get :—

$$v^2 = C\left(\frac{2}{r} - \frac{a}{1}\right) \ldots\ldots\ldots\ldots\ldots\ldots\ldots (\mathrm{x})$$

Hence, by considering the planet at P, we have :—

$$v^2 = C\left(\frac{2}{SP} - \frac{1}{a}\right) \ldots\ldots\ldots\ldots\ldots\ldots\ldots (\mathrm{x}\,a)$$

Now v, C, and SP are known ; hence the last equation determines the semi-axis major a. If $r = SP$, we have :—

$$2a = \frac{2Cr}{2C - rv^2}.$$

Let H be the other focus of the ellipse. Then it is known (Ellipse 8) that HP, SP make equal angles with PT. Also $SP + HP = 2a$. Hence, we can construct the position of H by making $\angle TPI = \angle TPS$, and producing IP to a point H such that

$$PH = 2a - SP.$$

The ellipse can now be constructed as in Appendix (Ellipse 2).

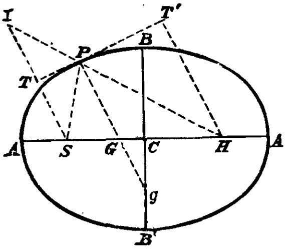

FIG. 138.

COROLLARY 1.—Since $SP + HP = 2a$, equation (x) gives

$$v^2 = \frac{C \,.\, HP}{SP \,.\, a}.$$

COROLLARY 2.—Substituting for h in terms of C, we see from equation (iv) that the work done when the body moves from distance r to distance r' is

$$= MC\left(\frac{1}{r'} - \frac{1}{r}\right).^*$$

Hence the work done by a mass M in falling from distance $2a$ to distance r is

$$= MC\left(\frac{1}{r} - \frac{1}{2a}\right) = \tfrac{1}{2}Mv^2 \ldots\ldots\ldots\ldots\ldots\ldots \text{by (x)}$$

= kinetic energy of the planet when at distance r.

Therefore, if a circle be described about the centre of force S, with radius equal to the major axis $2a$, the velocity at any point of the orbit is that which the planet would acquire by falling freely from the circle to that point under the action of the attracting force.

COROLLARY 3.—If the planet be revolving in a circle, $r = a$, and therefore $v^2 = C/r = C/a$, as in Art. 418.

COROLLARY 4.—If $v^2 = 2C/r$, (x) gives $1/a = 0$; or $a = \infty$. Hence the velocity is that acquired by falling from an infinite distance. In this case, the orbit is not an ellipse, but a *parabola*, a conic section satisfying the "focus and directrix" definition of Appendix (1), but having its eccentricity equal to unity.

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[*] This result is also proved independently in many treatises on dynamics, but a fuller investigation would be out of place here.

If $v^2 > 2C/r$, the velocity is greater than that due to falling from infinity, a comes out negative, and the orbit is a *hyperbola*, a conic section satisfying the focus and directrix definition, but having its eccentricity e greater than unity.

Many comets move in orbits that are sensibly parabolic; however, it can be shown that comets arriving from outside the solar system would have strongly marked hyperbolic orbits, which have never been observed. Hence we infer that all observed comets belong to the solar system, and that the supposed parabolas are really very long ellipses. In a few cases the perturbations produced by the planets change these orbits into hyperbolas that are almost parabolas. Such comets leave our system for ever, unless subsequent perturbations reverse the action.

Example.—*Find how long the Earth would take to fall into the Sun if its velocity were suddenly destroyed.*

If the Earth's velocity were *very nearly*, but not quite destroyed, it would describe a very narrow ellipse, very nearly coinciding with the straight line joining the point of projection to the Sun. The major axis of this ellipse would be very nearly equal to the Earth's initial distance from the Sun, and therefore the Earth would have very nearly gone half round the narrow ellipse when it would collide with the surface of the Sun.

Hence, if r denote the Earth's distance from the Sun, the semi-major axis of the narrow ellipse is $\frac{1}{2}r$, and the periodic time in this ellipse would be $(\frac{1}{2})^{\frac{3}{2}}$ years. The Earth would therefore collide with the Sun in

$$\tfrac{1}{2} \times (\tfrac{1}{2})^{\frac{3}{2}} \text{ years} = \frac{1}{4\sqrt{2}} \text{ years} = \frac{\sqrt{2}}{8} \text{ years}$$

$$= \frac{365}{8} \times 1{\cdot}4142 \text{ days} = 64\tfrac{1}{2} \text{ days nearly.}$$

II.—NEWTON'S LAW OF GRAVITATION—COMPARISON OF THE MASSES OF THE SUN AND PLANETS

428. The Law of Gravitation

In the last section we showed that the Sun attracts any planet of mass M at distance r with a force CM/r^2, where C is a constant. If we assume the truth of Newton's Third Law of Motion (*i.e.* that action and reaction are equal and opposite), the planet must also attract the Sun with an equal and opposite force CM/r^2. Since in the former case the force is proportional to the mass of the *attracted* body, and in the latter to the mass of the *attracting* body, it is reasonable to suppose that the attraction between two bodies is proportional to the mass of each.

Moreover, the motions of the various satellites, such as the Moon, confirm the theory that they revolve in their orbits under the attraction of their respective primary planets. From evidence of this character Newton, after many years of careful investigation, enunciated his *Law of Universal Gravitation*, which he stated thus :—

Every particle in the universe attracts every other particle with a force proportional to the quantities of matter in each, and inversely proportional to the square of the distance between them.

By *quantity of matter* is, of course, meant *mass*, and the word *attracts* implies that the force between two particles acts in the straight line joining them and tends to bring them together.

If M, M' be the masses of two particles, and r the distance between them, the law asserts that either particle is acted on by a force, directed towards the other, of magnitude

$$k\frac{MM'}{r^2},$$

where k has the same value for all bodies in the universe. The constant k is called the *constant of gravitation.*

429. Astronomical Unit of Mass

Taking any fundamental units of length and time, it is possible to choose the unit of mass such that $k = 1$. This unit is called the *astronomical unit of mass.* Hence, if M, M' are expressed in astronomical units, the force between the particles is *equal to* MM'/r^2. It is, however, usually more convenient to keep the unit of mass arbitrary, and to retain the constant k.

430. Remarks on the Law of Gravitation

Newton's Law states that not only do the Sun, the planets and their satellites, and the stars, mutually attract one another, but every pound of matter on one celestial body attracts every other pound of matter, on either the same or another body. But it is well-known that two spheres attract one another in just the same way as if the whole of the mass of either were concentrated at its centre, provided that the spheres are either homogeneous or made up of concentric spherical layers, each of uniform density. Since the Sun and planets are very nearly spherical, and their dimensions are very small compared with their distances, we see that their attractions may be very approximately found by regarding them as mere particles, instead of taking separate account of the individual particles forming them.

Moreover, every planet is attracted by every other planet, as well as by the Sun. But the mass of the Sun, and consequently its attraction, is so much greater than that of any other member of the solar system, that the planetary motions are only very slightly influenced by the mutual attractions. Kepler's Laws, therefore, still hold *approximately*, but the orbits are subject to small and slow changes or perturbations.

The Moon, on the other hand, is far nearer to the Earth than to the Sun; hence the Moon's orbital motion is mainly due to the Earth's attraction. The chief effect of the Sun's attraction on the Earth and Moon is to cause them together to describe the annual orbit; but it also produces perturbations or disturbances in the Moon's relative orbit (Art. 210) with which we are not here concerned.

The fixed stars also attract one another and attract the solar system, which in its turn attracts the stars. The proper motions of stars are probably due to this cause; but when we consider the vast distances of the stars, and remember that the attraction varies inversely as the square of the distance, it is evident that the relative accelerations are mostly too feeble to have produced any sensible changes of motion within historic times, and that countless ages must elapse before such changes can be discerned.

431. Correction of Kepler's Third Law

From the fact that a planet attracts the Sun with a force equal to that with which the Sun attracts the planets, it may be shown that Kepler's Third Law cannot be *strictly* true, as a consequence of the law of gravitation. Not only will the planet move under the Sun's attraction, but the Sun will also move under the planet's attraction. But since the forces on the two bodies are equal, while the mass of the Sun is very great compared with the mass of any planet, it follows that the *acceleration* of the Sun is very small compared with that of the planet, and hence the Sun remains very nearly at rest.

We may, however, obtain a modification of Kepler's Third Law in which the planet's reciprocal attraction is allowed for as follows:— Let S, M be the masses of the Sun and planet; then the attraction between them is

$$k\frac{SM}{r^2}.$$

This attraction, acting on the mass M of the planet, produces an acceleration of the planet towards the Sun equal to

$$k\frac{S}{r^2}.$$

The corresponding attraction on the mass S of the Sun produces an acceleration, in the reverse direction, of

$$k\frac{M}{r^2}.$$

Hence the whole acceleration of the planet *relative* to the Sun is:—

$$k\frac{S+M}{r^2},$$

instead of kS/r^2, as it would be if the Sun were at rest.

Hence the *absolute intensity* of the planet's acceleration towards the Sun is $k(S + M)$, and this depends on the values of both M and S. Let now T be the periodic time, r the planet's mean distance from the Sun, or the semi-axis major of the relative orbit; then, by Art. 418 (for a circular orbit), or Art. 425 (for an elliptical orbit),

$$k(S + M)T^2 = 4\pi^2r^2.$$

If M' be the mass of another planet, we have in like manner for *its* orbit $k(S + M')T'^2 = 4\pi^2r'^3$. Therefore :—

$$T^2(S + M) : T'^2(S + M') = r^3 : r'^3,$$

the correct relation between the periods and mean distances.

It is known that different planets have different masses. Hence, the fact that Kepler's Third Law is approximately true shows that the masses of the planets are small compared with that of the Sun.

432. Motion relative to Centre of Mass

The mutual attractions of the Sun and planet have no influence on the position of the centre of mass (commonly called the "centre of gravity") of the solar system; hence, in considering the relative motions, that point may be treated as fixed. It is known from general dynamical principles that when a system of bodies are under the influence of their mutual reactions or attractions alone, the centre of mass of the whole system is not accelerated. But it may be interesting to prove independently that when two bodies, such as the Sun and a planet, attract one another, they both revolve about their centre of mass.

Let us suppose (to take a simple case) the relative orbit circular and of radius SP $(= r)$, the angular velocity being n. Then, if G be the point about which the planet (P) and Sun (S) revolve, individually, we have

$$n^2 \times GP = \text{accel. of planet} = kS/r^2;$$

$$n^2 \times GS = \text{accel. of Sun} = kM/r^2.$$

Hence $$M \times GP = S \times GS;$$

which relation shows that G is the common centre of mass, as was to be proved.

In the case of three or more bodies, such as the Sun and planets, the centre of mass is still the common centre about which they revolve, but the corresponding investigation is more difficult, owing to the effect of the mutual attractions of the planets in producing perturbations.

It may be mentioned that the mass of the Sun is so large, compared with those of the planets, that, although the further planets are so very distant, the centre of mass of the whole solar system always lies very near the Sun.

433. Verification of the Theory of Gravitation for the Earth and Moon

Before considering the motions of the planets about the Sun, Newton investigated the orbital motion of the Moon about the Earth, to discover whether the Earth's attractive force, which retains the Moon in its orbit, is the same force as that which produces the phenomenon of gravity at the Earth's surface.

If we assume that the force varies inversely as the square of the distance, and that the Moon's distance is 60 times the Earth's radius, the acceleration due to the Earth's gravity at the Moon should be $(\frac{1}{60})^2 g$, where g is the acceleration of gravity on the Earth's surface.

But the acceleration g = 32·2 feet per sec. per sec.;
and accel. at Moon's distance = 32·2/3600 feet per sec. per sec.
= 32·2 feet per min. per min.

From the length of the lunar month and the Moon's distance in miles, Newton calculated what must be the normal acceleration of the Moon in its orbit. At the time of his first investigation (1666) the Earth's radius and the Moon's distance were but imperfectly known, and the Moon's normal acceleration, as thus computed, came out only about 27 feet per minute per minute. Some fifteen years later, the Earth's radius, and consequently the Moon's distance, had been measured with much greater accuracy, and, working with the new values, Newton found that the Moon's normal acceleration to the Earth agreed with that given by his theory.

Taking the lunar sidereal month as 27·3 days, the Earth's radius as 3,960 miles, and the radius of the Moon's orbit as 60 times the Earth's radius, the angular velocity (n) of the Moon, in radians, per minute is $2\pi/27{\cdot}3 \times 24 \times 60$. The Moon's distance in feet (d) is $3960 \times 60 \times 5280$. Hence the Moon's normal acceleration (n^2d) in feet per minute per minute is :—

$$\frac{3960 \times 60 \times 5280 \times 4\pi^2}{(27{\cdot}3)^2 \times 24^2 \times 60^2} = \frac{2 \times 110^2 \times \pi^2}{(27{\cdot}3)^2 \times 10} = 32 \text{ approximately,}$$

thus agreeing with that given by the law of gravitation.

Example.—*Having given that a body at the Earth's equator loses* 1/289 *of its weight in consequence of centrifugal force,* (i) *Calculate the period in which a projectile could revolve in a circular orbit, close to, but without touching the Earth, and* (ii) *Deduce the Moon's distance.*

(i) The centrifugal force on the body would have to be equal to its weight, and would therefore have to be 289 times as great as that at the Earth's equator.

Hence the projectile would have to move $\sqrt{289}$, or 17 times as fast as a point on the Earth's equator, and would therefore have to perform 17 revolutions per day.*

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* *Relative to the Earth* it would perform 16 or 18 revolutions per day, according to whether it was revolving in the same or the opposite direction to the Earth.

Therefore the period of revolution $= \frac{1}{17}$ of a day.

(ii) Assuming the law of gravitation, the periodic times and distances of the projectile and Moon must be connected by Kepler's Third Law. Hence, taking the Moon's sidereal period as $27\frac{1}{3}$ days, we have, if $a =$ Earth's rad., $d =$ Moon's dist.

$$d^3 : a^3 = (27\tfrac{1}{3})^2 : (\tfrac{1}{17})^2;$$

$$\text{or } d^3 = a^3 \times (17 \times 27\tfrac{1}{3})^2 \text{ giving } d = 59{\cdot}99a$$

or distance of Moon $= 60 \times$ Earth's radius almost exactly.

434. Effect of Moon's Attraction.—Moon's Mass

If we take account of the Moon's attraction on the Earth we must introduce a correction analogous to that made in Kepler's Third Law (Art. 431). If M, m are the masses of the Earth and Moon, the whole relative acceleration is $k(M + m)/d^2$, instead of kM/d^2. But, if g_0 is the acceleration of gravity on the Earth's surface, $g_0 = kM/a^2$;

$$\text{or } k = g_0 a^2/M,$$

and, if T is the length of the sidereal month, then, by Art. 431,

$$4\pi^2 d^3 = k(M + m)T^2 = g_0 a^2 \frac{M + m}{M} T^2.$$

$$\text{giving} \quad 1 + \frac{m}{M} = \frac{4\pi^2 d^3}{g_0 a^2 T^2}.$$

This formula might be used (and was used by Airy) to find m/M, the ratio of the Moon's to the Earth's mass, in terms of the observed values of a, d, g_0, T. It is not, however, a very accurate method, owing to the smallness of m/M.

435. To find the Ratio of the Masses of the Sun and Earth

Let S, M, m be the masses of the Sun, Earth, and Moon, d, r the distances of the Moon and Sun from the Earth, T, Y the lengths of the sidereal lunar month and year respectively. Then, if k be the gravitation constant, the Earth's attraction on the Moon is $= kMm/d^2$, and its intensity is kM.

The Sun's attraction on the Earth is $= kSM/r^2$, and its intensity is kS. Therefore, by Art. 425, Corollary,

$$kM \,.\, T^2 = 4\pi^2 d^3, \ kS \,.\, Y^2 = 4\pi^2 r^3;$$

$$\text{giving} \quad S : M = \frac{r^3}{Y^2} : \frac{d^3}{T^2}, \quad \text{or} \quad \frac{S}{M} = \frac{r^3}{d^3}\frac{T^2}{Y^2};$$

whence the ratio of the Sun's to the Earth's mass may be found.

If we take account of the attraction of the smaller body on the larger, the whole acceleration of the Earth, relative to the Sun, is $k(S + M + m)/r^2$ (since the Sun is attracted by the Moon as well as the Earth), and that of the Moon, relative to the Earth, is $k(M + m)/d^2$. Hence the corrected or more exact formula is

$$S + M + m : M + m = \frac{r^3}{Y^2} : \frac{d^3}{T^2}.$$

Since the Moon's mass is about $\frac{1}{81}$ of that of the Earth, the first or approximate formula can only be used if the calculations are not carried beyond two significant figures.

In this manner it is found that the Sun's mass is about 331,100 times that of the Earth.

Examples.—1. *Compare, roughly, the masses of the Earth and Sun, taking the Sun's distance to be* 390 *times the Moon's, and the number of sidereal months in the year to be* 13.

We have $$S : M = \frac{390^3}{13^2} : 1^2;$$

and $$\frac{\text{mass of Sun}}{\text{mass of Earth}} = \frac{390^3}{13^2} = 30^2 \times 390 = 351{,}000$$

To the degree of accuracy possible with these data, the Sun's mass is therefore 350,000 times that of the Earth.

2. *Find the ratio of the masses, taking the Moon's mass as* $\frac{1}{81}$ *of the Earth's, and the number of sidereal months in the year as* $13\frac{1}{3}$.

$$\frac{S + M + m}{M + m} = \frac{390^3}{(13\frac{1}{3})^2} = \frac{390^3 \times 3^2}{40^2} = \frac{5338710}{16} = 333669;$$

and $S = 333668\,(M + m) = 333668\,(1 + \frac{1}{81})\,M = 337{,}787\,M.$

436. To Determine the Mass of a Planet which has one or more Satellites

The method of the last paragraph is obviously applicable to the case of any planet which has a satellite. We require to know the mean distance and the periodic time of the satellite. The former may be easily found by observing the maximum angular distance of the satellite from its primary, the distance of the planet from the Earth at the time of observation having been previously computed. The periodic time of the satellite may also be easily observed.

Let M', m' be the masses of the planet and satellite, d' their distance apart, r' their distance from the Sun, T' the period of revolution of the satellite, Y' the planet's period of revolution round the Sun. Using unaccented letters to represent the corresponding quantities for the Earth and Moon we have, roughly,

$$\frac{4\pi^2}{k} = \frac{M'T'^2}{d'^3} = \frac{SY'^2}{r'^3} = \frac{SY^2}{r^3} = \frac{MT^2}{d^3},$$

or, more accurately,

$$\frac{4\pi^2}{k} = \frac{(M' + m')\,T'^2}{d'^3} = \frac{(S + M' + m')\,Y'^2}{r'^3}$$

$$= \frac{(S + M + m)\,Y^2}{r^3} = \frac{(M + m)\,T^2}{d^3};$$

whence the mass of the planet, or, more correctly, the sum of the masses of the planet and satellite, may be determined in terms of the mass of the Sun, or the sum of the masses of the Earth and Moon. We do not require to know the periodic time and mean distance of the planet from the Sun, since the above expressions enable us to express the required mass, $M' + m'$, in terms of the year and mean distance of the Earth, or in terms of the lunar month and the mean distance of the Moon.

Example.—*Find the mass of Uranus in terms of that of the Sun, having given that its satellite* Titania *revolves in a period of* 8 *days* 17 *hours at a distance from the planet* = ·003 *times the distance of the Earth from the Sun.*

Let M be the mass of Uranus, then we have :—

$$M : S = \frac{d^3}{T^2} : \frac{r^3}{Y^2},$$

and, by Kepler's Third Law, r^3/Y^2 is the same for Uranus as for the Earth. Hence

$$M : S = \frac{(\cdot 003)^3}{(8\text{d. }17\text{h.})^2} : \frac{1^3}{(365\text{d. }6\text{h.})^2}$$

$$\text{or} \quad \frac{M}{S} = \left(\frac{3}{1000}\right)^3 \times \left(\frac{365\text{d. }6\text{h.}}{8\text{d. }17\text{h.}}\right)^2 = \frac{27}{10^9} \times \left(\frac{8766}{209}\right)^2 = \frac{1}{21053} \text{ nearly.}$$

Thus, the mass of Uranus is to that of the Sun in the ratio of 1 to 21,053.

437. The Masses of Mercury and Venus

The masses of Mercury and Venus (which have no satellites) could *theoretically* be found by determining their mean distances from the Sun by direct observation, and comparing them with those calculated from their periodic times by Kepler's Third Law. For, if M' is the mass of such a planet, we have

$$\frac{(S + M')\mathrm{Y}'^2}{r'^3} = \frac{(S + M + m)\mathrm{Y}^2}{r^3}.$$

This enables us to find the sum of the masses of the Sun and planet, and, the Sun's mass being known, the planet's mass could be found.

This method is, however, worthless, because the masses of Mercury and Venus are only about $\frac{1}{7500000}$ and $\frac{1}{408000}$ of that of the Sun, and in order to calculate one significant figure of the fraction M'/S it would be necessary to know all the data correct to about seven significant figures, a degree of accuracy unattainable in practice. For this reason it is necessary to calculate the masses of these planets by means of the perturbations they produce on one another and on the Earth; these perturbations will be discussed in the next chapter.

438. Centre of Mass of the Solar System

When the masses of the various planets have been found in terms of the Sun's mass, the position of the centre of mass of the system can be found for any given configuration, and can thus be shown to lie always very near the Sun.

Examples.—1. *Find the distance of the centre of mass of the Earth and Sun from the centre of the Sun.*

Here the mass of the Sun is 331,100 times the Earth's mass, and the distance between their centres is about 93,000,000 miles. Hence, the centre of mass of the two is at a distance from the Sun's centre of about

$$\frac{93{,}000{,}000}{331{,}100 + 1} = 281 \text{ miles.}$$

2. *Find the centre of mass of Uranus and the Sun, and to show that it lies within the Sun.*

The distance of Uranus from the Sun is 19·2 times the Earth's distance, and its mass is 1/21053 of the Sun's. Hence the C.M. is at a distance from the Sun's centre of

$$\frac{93{,}000{,}000 \times 19{\cdot}2}{21053 + 1} \text{ miles} = 84{,}800 \text{ miles.}$$

The Sun's semi-diameter is 433,200 miles; hence the centre of mass of the Sun and Uranus is at a distance from the Sun's centre of rather less than $\frac{1}{5}$ the radius.

NOTE.—In the case of Jupiter, the mean distance is 5·2 times that of the Earth, and the mass is 1/1050 of that of the Sun; hence the C.M. is at a distance

$$\frac{5{\cdot}2 \times 92{,}000{,}000}{1050 + 1} = 455{,}000 \text{ miles.}$$

This is just greater than the Sun's radius, showing that the centre of mass lies just *without* the Sun's surface.

III.—THE EARTH'S MASS AND DENSITY

439. Mass of the Earth

The so-called "Weight of the Earth" really means the Earth's mass, and the operation called "weighing the Earth," in some of the older text-books, means finding the mass of the Earth. In the last section we explained how to compare the masses of the Sun and certain planets with that of the Earth, and in the next chapter we shall give methods applicable to a planet having no satellites. But before the masses can be expressed in pounds or tons it is necessary to determine the Earth's mass in these units. The methods of doing this all depend on comparing the Earth's attraction with that of a body of known mass and distance; and the only difficulty lies in determining the latter attraction, since the force between two bodies of ordinary dimensions is always extremely small. The following methods have been used. The first two are by far the best.

(1) By the "Cavendish Experiment," or the balance.

(2) By observations of the influence of tides in estuaries.

(3) By the "Mountain" method.

(4) By pendulum experiments in mines.

440. The "Cavendish Experiment"

This experiment owes its name to its having been first used to determine the Earth's mass by Cavendish, about the year 1798. The essential principle of the method consists in comparing the attractions of two heavy balls of known size and weight with the Earth's attraction. Since the attraction of a sphere at any point is proportional directly to the mass of the sphere and inversely to the square of the distance from its centre, it is evident that by comparing the attractions of different spheres—such as the Earth and the experimental ball of metal—we can find the ratio of their masses.

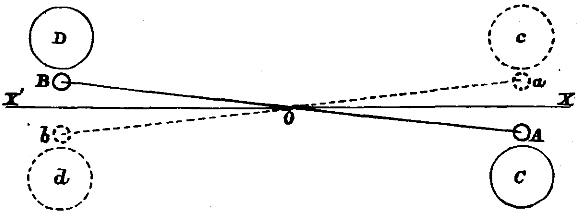

FIG. 139.

The comparison is effected by means of a *torsion balance*. Two equal small balls A, B are fixed to the ends of a light beam suspended from its middle point O by means of a slender vertical thread or "torsion fibre," so as to be capable of twisting about O in a horizontal plane (the plane of the paper in Fig. 139). Two heavy metal balls C, D, are brought near the small balls A, B (as shown in the figure), and their attraction causes the beam to turn about O, say from its original position of rest XX' to the position AB. As the beam turns the fibre twists; this twisting is resisted by the elasticity of the fibre, which produces a couple, proportional to the angle of twist XOA, tending to untwist it again. Let us call this couple $f \times \angle XOA$, where f is a constant depending on the fibre, called its "*torsional rigidity.*"

The beam AB assumes a position of equilibrium when the moments about O of the attractions of the large spheres C, D on the balls A, B, just balance the "untwisting couple" $f \times \angle XOA$. The angle XOA

being measured, and the dimensions of the apparatus being supposed known, the attractions of the spheres can now be determined in terms of the torsional rigidity.

The value of f is found in terms of absolute units of couple by observing the time of a small oscillation of the beam when the balls A, B have been removed. [The beam will then swing backwards and forwards like the balance wheel of a chronometer. The greater the torsional rigidity, the more frequently will it reverse the motion of the beam, and the more frequent will be the oscillations.*]

Hence finally the attractions between the known masses C, D and A, B are found in terms of known units of force, and by comparing these attractions with that of gravity the Earth's mass is found.

In practice, instead of measuring the angle XOA, the masses C, D are subsequently placed on the reverse side of the beam, say with their centres at c, d, and they now deflect the beam in the reverse direction, say to ab. The angle measured is the whole angle aOA, and this angle is *twice* the angle XOA, if the positions CD and cd are symmetrically arranged with respect to the line XOX'.

In the experiments of Cavendish the beam AB was six feet long, and the masses C, D were balls of lead a foot in diameter. Recently, however, C. V. Boys, by the use of a quartz fibre for the suspending thread, has performed the experiment on a much smaller scale, the whole apparatus being only a few inches in size and being highly sensitive. He used cylinders instead of spheres for the attracting bodies, and this introduces extra complications in the calculations. His researches, and those of C. Braun, who used a similar method, agreed in making the Earth's density 5·527 times that of water.

Although the above description shows the general principle of the method, many further precautions are required to ensure accuracy. A full description of these would be out of place here.

441. The Common Balance Method

The *common balance* has also been used to determine the Earth's mass. In this case the differences of weight of a body are observed when a large attracting mass is placed successively above and below the scale-pan containing it.

Example.—*Find the Earth's mass in tons, having given that the attraction of a leaden ball, weighing* 3 *cwt., on a body placed at a distance of* 6 *inches from its centre is* ·0000000432 *of the weight of the body.*

Let M be the mass of the Earth in tons. The mass of the ball in tons is $= \frac{3}{20}$.

The Earth's radius in feet $= 3960 \times 5280 = 20{,}900{,}000$ roughly; and the distance of the body from the ball in feet $= \frac{1}{2}$.

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* The student who has read Rigid Dynamics should work out the formula.

Hence, since the attractions of the Earth and ball are proportional directly to the masses and inversely to the squares of the distances from their centres,

$$\cdot 0000000432 : 1 = \frac{\frac{3}{20}}{(\frac{1}{2})^2} : \frac{M}{(20{,}900{,}000)^2};$$

$$\text{or} \quad M = \frac{(20{,}900{,}000)^2 \times \frac{3}{20}}{\frac{1}{4} \times \cdot 0000000432} = \frac{3 \times 209^2 \times 10^{10}}{5 \times 432 \times 10^{-10}}$$

$$= \frac{3}{5} \times \frac{209^2}{432} \times 10^{20} = \frac{43681 \times 3}{\cdot 2160} \times 10^{20} = 6067 \times 10^{18} \text{ tons.}$$

442. Determination of the Earth's Mass by Observations of the Attraction of Tides in Estuaries

A method which admits of very great accuracy is that in which the mass of the Earth is found by comparing it with that of the water brought by the tide into an estuary. Consider an observatory situated (like Edinburgh Observatory) due south of an arm of the sea, whose general direction is east and west. The direction of its zenith, as shown either by a plummet or by the normal to the surface of a bowl of mercury, is not the same at high tide as at low, because the additional mass of water at high tide produces an attraction which deflects the plummet and the nadir point northward, and hence displaces the zenith towards the south. Hence the latitude of the observatory is less at high tide than at low ; and the difference is a measurable quantity. The great advantage of this method is that the mass which deflects the plumb-line can be measured with great certainty ; for the density of the sea-water is exactly known (and, unlike that of the rocks in the next methods, is uniform throughout) and the shape and height of the layer of water brought in are known from the ordnance maps, and the tide measurements at the port.

443. Mass of Earth by the Pendulum Method

In the *Pendulum Method* the values of g, the acceleration of gravity, are compared by comparing the oscillations of two pendulums at the top and bottom of a deep mine. The difference of the two values is due to the attraction of that portion of the Earth which is above the bottom of the mine ; this exerts a downward pull on the upper pendulum, and an upward pull on the lower one.

If the Earth were homogeneous throughout, the values of g at the top and bottom would be directly proportional to the corresponding distances from the Earth's centre. If this is not observed to be the case, the discrepancy enables us to find the ratio of the density of the Earth to that of the rocks in the neighbourhood of the mine. If the latter density is known, the Earth's density can be found, and knowing its volume, its mass can be computed. But this method is very liable to considerable errors, arising from imperfect knowledge of the density of the rocks overlying the mine.

444. Mass of the Earth by the Mountain Method

The *Mountain Method* compares the attraction of the Earth with that of a mountain. It was used by Maskelyne at Schiehallien in Scotland. This mountain runs due E. and W.; then at a place at its foot on the S. side the attraction of the mountain will pull the plummet of a plumb line towards the N., and at a place on the N. side the mountain will pull the plummet to the S. Hence the Z.D. of a star, as observed by means of zenith sectors, will be different at the two sides, and from this difference the ratio of the Earth's to the mountain's attraction may be found.

In order to deduce the Earth's density it is then necessary to determine accurately the dimensions and density of the mountain. This renders the method very inexact, for it is impossible to find with certainty the density of the rocks throughout every part of the mountain.

445. Determination of Densities. Gravity on the Surface of the Sun and Planets

When the mass and volume of a celestial body have been computed, its average density can, of course, be readily found. By dividing the mass in pounds by the volume in cubic feet, we find the average mass per cubic foot, and since we know that the mass of a cubic foot of water is about 62½ lbs., it is easy to compare the average density with that of water. The determination of densities is particularly interesting, on account of the evidence it furnishes regarding the physical condition of the members of the solar system. The Earth's relative density is about 5·527.

Knowing the ratios of the mass and diameter of the Sun or a planet to that of the Earth, we can compare the intensity of its attraction at a point on its surface with the intensity of gravity on the Earth.

It may be noticed that attraction of a sphere at its surface is proportional to the product of the density and the radius.

For the attraction is proportional to mass ÷ (radius)^2, and the mass is proportional to the density × (radius)^3; the attraction at the surface is therefore proportional to the density × radius.

Examples.—1. *Find the Earth's average density and mass, having given that the attraction of a ball of lead a foot in diameter, on a particle placed close to its surface, is less than the Earth's attraction in the proportion of* 1 : 20,500,000, *and that the density of lead is* 11·4 *times that of water.*

Let D be the average density of the Earth. Then, since the radii of the Earth and the leaden ball are ½ and 20,900,000 feet respectively, and the attractions at their surfaces are proportional to their densities multiplied by their radii, so that :—

$$1 : 20{,}500{,}000 = 11{\cdot}4 \times \tfrac{1}{2} : D \times 20{,}900{,}000\,;$$

and average density of Earth $D = 5{\cdot}7 \times \frac{205}{209} = 5{\cdot}6.$

Hence the average mass of a cubic foot of the material forming the Earth is $5{\cdot}6 \times 62{\cdot}5$ pounds.

But the Earth is a sphere of volume $\frac{4}{3}\pi(20,900,000)^3$ cubic feet.

Hence mass of Earth, with these data, $= \frac{4}{3}\pi \times 209^3 \times 10^{15} \times 5{\cdot}6 \times 62{\cdot}5$ pounds

i.e. Mass of Earth $= 1338 \times 10^{22}$ pounds $= 597 \times 10^{19}$ tons.

2. *Calculate the mean density of the Sun from the following data :—*

Mass of Sun = 330,000 . (*mass of Earth*) ;

Density of Earth = 5·58 ;

Sun's parallax = 8·8″ ; *Sun's angular semi-diameter* = 16′.

The radii of the Sun and Earth being in the ratio of the Sun's angular semi-diameter to its parallax, we have :—

$$\frac{\text{Sun's radius}}{\text{Earth's radius}} = \frac{16'}{8{\cdot}8''} = \frac{960}{8{\cdot}8} = 109{\cdot}1\,;$$

volume of Sun $= (109{\cdot}1)^3$. (vol. of Earth) = 1,298,000 . (vol. of Earth) roughly.

But mass of Sun = 330,000 . (mass of Earth) ; so that :—

$$\frac{\text{density of Sun}}{\text{density of Earth}} = \frac{330}{1298} = \frac{1}{3{\cdot}9} \text{ very nearly};$$

Whence the density of Sun = 1·4.

3. *Find the number of poundals in the weight of a pound at the surface of Jupiter, taking the planet's radius as* 43,200 *miles and density* $1\frac{1}{3}$ *times that of water.*

Taking the Earth's radius as 3960 miles and density as 5·58, we have

(gravity at surface of Jupiter) : (gravity on Earth) $= 1{\cdot}33 \times 43,200 : 5{\cdot}58 \times 3960$.

But at the Earth's surface the weight of a pound = 32·2 poundals ;

Therefore on the surface of Jupiter the weight of a pound :—

$$= 32{\cdot}2 \times \frac{1{\cdot}33 \times 43200}{5{\cdot}58 \times 3960} \text{ poundals} = 83{\cdot}7 \text{ poundals.}$$

EXAMPLES

1. Taking Neptune's distance from the Sun as 30 times the Earth's distance, and the Earth's velocity as 18·6 miles per second, find the orbital velocity of Neptune.

2. If we suppose the Moon to be 61 times as far from the Earth's centre as we are, find how far the Earth's attraction can pull the Moon from rest in a minute.

3. If the Earth possessed a satellite revolving at a distance of only 6,000 miles from the Earth's surface, what would be approximately its periodic time, assuming the Earth to be a sphere of 4,000 miles radius ?

4. Assuming the distance between the Earth's centre and the Moon's to be 240,000 miles, and the period of the Moon's revolution 28 days, find how long the month would be if the distance of the Moon were 80,000 miles.

5. Calculate the mass of the Sun in terms of that of Mars, given that the Earth's mean distance and period are 92×10^6 miles and $365\frac{1}{4}$ days, and the mean distance and period of the outer satellite of Mars are 14,650 miles and 1d. 6h. 18m.

6. Show that the periodic time of an asteroid is $3\frac{1}{2}$ years, having given that its mean distance is 2·305 times that of the Earth.

7. Show that we could find the Sun's mass in terms of the Earth's, from exact observation of the periods and mean distances of the Earth and an asteroid, by the error produced in Kepler's Third Law in consequence of the Earth's mass.

8. Show that an increase of 10 per cent. in the Earth's distance from the Sun would increase the length of the year by 56·14 days.

9. The masses of the Earth and Jupiter are approximately $\frac{1}{300000}$ and $\frac{1}{1000}$ respectively of the Sun's mass, and their distances from the Sun are as 1 : 5. Show that Kepler's Laws would give the periodic time of Jupiter too great by more than 2 days.

10. Prove that the mass of the Sun is 2×10^{27} tons, given that the mean acceleratiom of gravity on the Earth's surface is 9·81 metres per second per second, the mean density of the Earth is 5·53, the Sun's mean distance $1{\cdot}5 \times 10^8$ kilometres, a quadrant of the Earth's circumference 10,000 kilometres, and taking a metre cube of water to be a ton.

11. Having given that the constant of aberration for the Earth is 20·49″, and that the distance of Jupiter from the Sun is 5·2 times the distance of the Earth from the Sun, calculate the constant of aberration for Jupiter.

12. If the mass of Jupiter is $\frac{1}{1050}$ of the mass of the Sun, show that the change in its constant of aberration caused by taking into account the mass of Jupiter is 0·004″ nearly (see Question 11).

13. Find the centre of mass of Jupiter and the Sun. Hence find the centre of mass of Jupiter, the Sun, and Earth, (1) when Jupiter is in conjunction, (2) when in opposition. (Sun's mass = 1,048 times Jupiter's = 332,000 times Earth's. Jupiter's mean distance = 480,000,000 miles; Earth's = 93,000,000 miles.)

14. If the intensity of gravity at the Earth's surface be 32·185 feet per second per second, what will be its value when we ascend in a baloon to a height of 10,000 feet? (Take Earth's radius = 4,000 miles and neglect centrifugal force.) Would the intensity be the same on the top of a mountain 10,000 feet high? If not, why not?

15. Show how by comparing the number of oscillations of a pendulum at the top and bottom of a mountain of known density, the Earth's mass could be found.

16. How would the tides in the Thames affect the determination of meridian altitudes at Greenwich observatory theoretically?

17. If the mean diameter of Jupiter be 86,000 miles, and his mass 315 times that of the Earth, find the average density of Jupiter.

18. If the Sun's diameter be 109 times that of the Earth, his mass 330,000 times greater, and if an article weighing one pound on the Earth were removed to the Sun's surface, find in poundals what its weight would be there.

19. Taking the Moon's mass as $\frac{1}{81}$ that of the Earth, show that the attraction which the Moon exerts upon bodies at its surface is only 1-6th that of gravity at the Earth's surface.

20. If the Earth were suddenly arrested in its course at an eclipse of the Sun, what kind of orbit would the Moon begin to describe?

EXAMINATION PAPER

1. State reasons for supposing that the Earth moves round the Sun, and not the Sun round the Earth.

2. State Kepler's Laws, and give Newton's deductions therefrom.

3. If the Sun attracts the Earth, why does not the Earth fall into the Sun?

4. Show that the angular velocities of two planets are as the cubes of their linear velocities.

5. State Newton's Law of Gravitation, and prove Kepler's Third Law from it for the case of circular orbits, taking the planets small.

6. Explain clearly (and illustrate by figures or otherwise) what is meant by a force varying inversely as the square of the distance.

7. Are Kepler's Laws perfectly correct? Give the reason for your answer. What is the correct form of the Third Law if the masses of the planets are supposed appreciable as compared with the mass of the Sun?

8. How can the mass of Jupiter be found?

9. Show that if a body describes equal areas in equal times about a point, it must be acted on by a force to that point.

10. Find the law of force to the focus under which a body will describe an ellipse; and if C be the acceleration produced by the force at unit distance, T the periodic time, and $2a$ the major axis of the ellipse, find the relation between C, a, T.

CHAPTER XVIII

FURTHER APPLICATIONS OF THE LAW OF GRAVITATION

I.—THE MOON'S MASS—CONCAVITY OF LUNAR ORBIT

446. The Earth's Displacement due to the Moon

In Section II of the last chapter we saw that when two bodies are under their mutual attraction they revolve about their common centre of mass. Thus, instead of the Moon revolving about the Earth in a period of 27⅓ days, both bodies revolve about their centre of mass in this period, although from the Moon's smaller size its motion is more marked.

In this case both the Earth and Moon are under the attraction of a third body—the Sun—which causes them together to describe the annual orbit. But the Sun's distance is so great compared with the distance apart of the Earth and Moon, that its attraction is very nearly the same, both in intensity and direction, on both bodies. To a first approximation, therefore, the resultant attraction of the Sun is the same as if the masses of both the Earth and Moon were collected at their common centre of mass. Hence it is strictly the centre of mass of the Earth and Moon, and not the centre of the Earth, which revolves in an ellipse about the Sun with uniform areal velocity, in accordance with the laws stated in Art. 136. And, owing to the revolution of the Moon, the Earth's centre revolves round this point once in a sidereal month, threading its way alternately in and out of the ellipse described, and being alternately before and behind its mean position.

This displacement of the Earth has been used for finding the Moon's mass in terms of the Earth's, by determining the common centre of mass of the Earth and Moon, as follows.

Let E_1, M_1, G_1 (Fig. 140) be the positions of the centres of the Earth and Moon, and their centre of mass, at the Moon's last quarter, E_2, M_2, G_2 and E_3, M_3, G_3 their positions at new Moon and at first quarter respectively, S the Sun's centre.

Then, at last quarter, E_1 is behind G_1, and the Sun's longitude, as seen from E_1, is less than it would be as seen from G_1 by the angle E_1SG_1. At first quarter, E_3 is in front of G_3, and therefore the Sun's longitude is greater at E_3 than at G_3 by the angle G_3SE_3. If, then, the observed co-ordinates of the Sun be compared with those calculated on the supposition that the Earth moves uniformly (*i.e.* with uniform areal velocity), its longitude will be found to be decreased at last quarter and increased at first quarter.

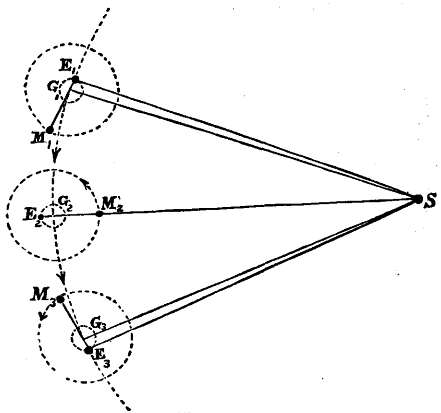

FIG. 140.

From observing these displacements the Moon's mass may be found. For, knowing the angle of displacement E_1SG_1 and the Sun's distance, the length E_1G_1 may be found. Also the Moon's distance E_1M_1 is known. And, since G_1 is the centre of mass of the Earth and Moon,

$$\text{mass of Moon} : \text{mass of Earth} = E_1G_1 : G_1M_1 ;$$

whence the mass of the Moon can be found.

The Sun's displacement at the quarters could be found by meridian observations of the Sun's R.A. with a transit circle. The displacement of the Earth will also give rise to an apparant displacement, having a period of about one month, in the position of any near planet; this could be detected by observations on Mars, when in opposition, similar to those used in finding solar parallax (Art. 378). The most accurate determination of the mass of the Moon was derived in this way from the extensive observations of the asteroid Eros at the time of its unusually close approach to the Earth in 1931 (Art. 380). The mass of the Moon was found to be 1/81·27 of that of the Earth. The Moon's *density*, as thus deduced, is about 3·44 or $\frac{3}{5}$ of that of the Earth.

The fact of the Moon's orbit being inclined to the ecliptic makes the Earth's centre move alternately above and below the ecliptic plane in addition to the oscillation in longitude. This gives the Sun a latitude (as seen from the Earth's centre) of the same sign as that of the Moon, and approximately proportional to the latter: its greatest value is about 0·6″.

Example.—*Compare the masses of the Moon and Earth, having given that the Sun's displacement in longitude at the Moon's quadratures is equal to $\frac{3}{4}$ of the Sun's parallax.*

Since $\angle E_1SG_1 = \frac{3}{4}$ the angle subtended by Earth's radius at S, therefore

$E_1G_1 = \frac{3}{4}$ (Earth's radius).

But

$E_1M_1 = 60$ (Earth's radius);

so that $E_1M_1 = 80 \,.\, E_1G_1$;

and $G_1M_1 = 79 \,.\, E_1G_1$,

and mass of Moon : mass of Earth $= E_1G_1 : G_1M_1 = 1 : 79$.

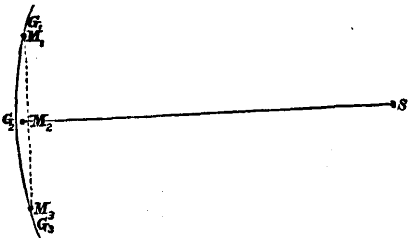

FIG. 141.

447. Application to Determination of Solar Parallax

If the Moon's mass be found by any other method, the above phenomena give us a means of finding the Sun's parallax and distance. For we then know $E_1G_1 : G_1M_1$, and therefore E_1G_1 and the angle E_1SG_1 is found by observation. But the exact ratio of E_1SG_1 to the parallax is known, for it is equal to that of E_1G_1 to the Earth's radius; hence the Sun's parallax and distance can be found. Since the Moon's mass can be found by many different methods, this method is quite as accurate as many that have been used for finding the solar parallax.

448. Concavity of the Moon's Path about the Sun

The Moon, by its monthly orbital motion about the Earth, threads its way alternately inside and outside of the ellipse which the centre of mass of the Earth and Moon describes in its annual orbit about the Sun. Hence the path described by the Moon in the course of the year is a wavy curve, forming a series of about thirteen undulations about the ellipse. It might be thought that these undulations turned alternately their concave and convex side towards the Sun, but the Moon's path is really always concave: that is, it always bends towards the Sun, as shown in Fig. 141, which shows how the path passes to the inside of the ellipse without becoming convex.

To show this it is necessary to prove that the Moon is always being accelerated towards the Sun. Let n, n' be the angular velocities of the Moon about the Earth and the Earth about the Sun respectively. Then, when the Moon is new, as at M_2 (Fig. 142), its acceleration towards G_2, relative to G_2, is $n^2 . M_2G_2$. But G_2 has a normal acceleration n'^2G_2S towards S. Hence the resultant acceleration of the Moon M_2 towards S is $n'^2G_2S - n^2M_2G_2$.

Now, there are about $13\frac{1}{2}$ sidereal months in the year; therefore $n = 13\frac{1}{2}n'$. Also E_2S is nearly 400 times E_2M_2, and therefore G_2S is slightly over 400 times G_2M_2. Therefore roughly

$$n'^2G_2S : n^2M_2G_2 = 400 : 182 \text{; and therefore } n'^2G_2S > n^2G_2M_2.$$

Thus, the resultant acceleration of M_2 is directed *towards*, not *away from* S, even at M_2, where the acceleration, relative to G_2, is directly opposed to that of G_2. Therefore the Moon's path is constantly being bent (or deflected from the tangent at M_2) in the direction of the Sun, and is concave towards the Sun.

II.—THE TIDES

In the last section we investigated the displacements due to the Moon's attraction on the Earth as a whole. We shall now consider the effects arising from the fact that the Moon's attractive force is not quite the same either in magnitude or direction at different parts of the Earth, and shall show how the small differences in the attraction give rise to the tides.

449. The Moon's or Sun's Disturbing Force

Let C, M be the centres of the Earth and Moon; ACA' the Earth's diameter through M; B, B' points on the Earth such that $MC = MB = MB'$. Let M, m denote the masses of the Earth and Moon, a the Earth's radius, d the Moon's distance.

The resultant attraction of the Moon on the Earth as a whole is kMm/CM^2, and the Earth is therefore moving with acceleration km/CM^2 towards the common centre of mass of the Earth and Moon, as shown in Arts. 432, 434.

(i) Now at the sublunar point A the Moon's attraction on unit mass is km/AM^2 and is greater than that at C (since $AM < CM$). Hence the Moon tends to accelerate A more than C and thus to draw a body at A *away from* the Earth, with relative accleration F, where

$$F = km\left(\frac{1}{AM^2} - \frac{1}{CM^2}\right) = km\,\frac{CA\,(CM + AM)}{CM^2 . AM^2}$$

$$= km\,\frac{a(2d-a)}{d^2(d-a)^2} = km\,\frac{2a}{d^3}\,\frac{1 - a/2d}{(1 - a/d)^2}.$$

Since a/d is a small fraction, we have, to a first approximation,

$$F = km\,\frac{2a}{d^3} = 2k\,\frac{m}{d^3}\,CA.$$

(ii) At A' the Moon's attraction per unit mass is $km/A'M^2$, and is less than that at C, since $A'M > CM$. Hence the Moon tends to accelerate C more than A', and thus to draw the Earth *away from* A' with relative acceleration F', where

$$F' = km\left(\frac{1}{CM^2} - \frac{1}{A'M^2}\right) = km\frac{CA'(CM + A'M)}{CM^2 . AM'^2}$$

$$= km\frac{a(2d + a)}{d^2(d + a)^2} = km\frac{2a}{d^3}\frac{1 + a/2d}{(1 + a/d)^2}.$$

To a first approximation, therefore,

$$F' = km\frac{2a}{d^3} = 2k\frac{m}{d^3}CA'.$$

Thus a body *either* at A or A' tends to separate from the Earth, as if acted on by a force *away from* C, of magnitude approximately $= 2kma/d^3$ per unit mass.

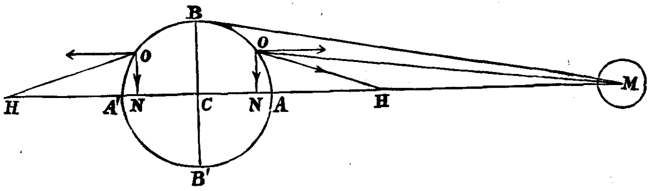

FIG. 142.

(iii) Consider now the effect of the Moon's attraction on a body at B. This produces a force per unit mass of km/BM^2, which may be resolved into components

$$k\frac{m}{BM^2} \times \frac{CM}{BM} \text{ parallel to } CM,$$

and

$$k\frac{m}{BM^2} \times \frac{BC}{BM} \text{ along } BC.$$

Since we have taken $BM = CM$, the first component is equal to km/CM^2; that is, to the force at C. This component therefore tends to make a body at B move with the rest of the Earth, and produces no relative acceleration. Therefore the Moon tends to draw a body at B *towards* the Earth with relative acceleration f, represented by the second component; thus

$$f = km\frac{BC}{BM^3}.$$

The point B is approximately the end of the diameter BCB' perpendicular to AC (since BM, CM, $B'M$ are nearly parallel in the neighbourhood of the Earth).

Hence the relative acceleration at B is approximately perpendicular to CM, and its magnitude

$$f = km\,\frac{a}{d^3} = km\,\frac{BC}{d^3}.$$

Similarly at B' the Moon tends to draw a body *towards* C, with relative acceleration $f = kma/d^3$.

At either of these points, B, B', therefore, a body tends to approach the Earth, as if acted on by a force *towards* the Earth's centre, of magnitude kma/d^3 per unit mass. Generally, the Moon's attraction at any point O tends to accelerate a body, relatively to the Earth, as if it were acted on by a force depending on the difference in magnitude and direction between the Moon's attractions at that point and at the Earth's centre. This apparent force is called the *Moon's disturbing force or tide-generating force.* We see that the disturbing force produces a pull along AA' and a squeeze along BB'.

A similar consequence arises from the attraction of the Sun. The Sun's *actual* attraction on the Earth as a whole keeps the Earth in its annual orbit, but the variations in the attraction at different points give rise to an apparent distribution of force on the Earth which is the *Sun's disturbing force or tide-generating force.*

450. To find approximately the Moon's or Sun's Disturbing Force at any Point

Let O be any point of the Earth. Draw ON perpendicular on CM (Fig. 142). Then the difference of the Moon's attractions at O and N tends to accelerate O towards N, with a relative acceleration $km.NO/d^3$ [by Art. 449 (iii)]. Also, the difference of the attractions at N, C tends to accelerate N away from C with a relative acceleration $2km\,.\,CN/d^3$ [by Art. 449 (i)].

The whole acceleration of O, relative to C, is compounded of these two relative accelerations. Therefore, if X, Y be the components of the disturbing force at O in the directions CN, NO,

$$X = 2km\,.\,\frac{CN}{d^3},\quad Y = -\,km\,.\,\frac{NO}{d^3}.$$

Hence we derive the following geometrical construction:—*On CN produced take a point H such that $NH = 2CN$. Then the line OH represents the disturbing force at O in direction, and its magnitude is*

$$F = km\,.\,\frac{OH}{d^3}.$$

The Sun's tide-raising force may be found exactly in the same way. The force is everywhere directed towards a point on the diameter of the Earth through the Sun, found by a similar construction to the above.

And if r, S denote the Sun's distance and mass, the force is proportional to S/r^3 instead of m/d^3.

In all these investigations we see that the *tide-raising force due to an attracting body is proportional directly to its mass and inversely to the* cube (not the square) *of its distance.*

From this it is easy to compare the tide-raising forces due to different bodies acting at different distances.

Examples.—1. *Compare the tide-raising forces due to the Sun and Moon.*

The masses of the Sun and Moon are respectively 331,000 and $\frac{1}{81}$ times the Earth's mass. Also, the Sun's distance is about 390 times the Moon's. Therefore

Sun's tide-raising force : Moon's tide-raising force

$$= \frac{331,000}{(390)^3} : \frac{1}{81} = 331 : \frac{(39)^3}{3 \times 3^3} = 331 \times 3 : 13^3 = 993 : 2197 = 3 : 7 \text{ nearly.}$$

Thus the Sun's tide-raising force is about three-sevenths of that of the Moon.

2. *Find what would be the change in the Moon's tide-raising force if the Moon's distance were doubled and its mass were increased sixfold.*

If f, f' be the old and new tide-raising forces at corresponding points,

$$f' : f = \frac{6}{2^3} : \frac{1}{1^3} \quad \text{or} \quad f' = \tfrac{3}{4}f.$$

Therefore the tide-raising force would have three-quarters of its present value.

3. *Compare the Moon's tide-raising forces at perigee and apogee.*

The greatest and least distances of the Moon being in the ratio of $1 + \frac{1}{18}$ to $1 - \frac{1}{18}$, or 19 to 17 (Art. 208), the tide-raising power at perigee is greater than at apogee in the ratio of $19^3 : 17^3$ or $6859 : 4913$, or roughly $7 : 5$.

4. *Compare the maximum and minimum values of the Sun's tide-raising force.*

The eccentricity of the Earth's orbit being $\frac{1}{60}$, these are in the ratio of $(1 + \frac{1}{60})^3 : (1 - \frac{1}{60})^3$, or approximately $1 + \frac{3}{60} : 1 - \frac{3}{60}$, or $21 : 19$. As before, the force is greatest at perigee and least at apogee.

451. The Equilibrium Theory of the Tides

Let us imagine the Earth to be a solid sphere covered with an ocean of uniform depth. If we plot out the disturbing forces at different points of the Earth by the construction of Art. 450, we shall find the distribution represented in Fig. 143, the lines representing the forces both in magnitude and direction. Here the disturbing force tends to raise the ocean at the sub-lunar point A and at the opposite point A', and to depress it at the points B, B'. At intermediate points it tends to draw the water away from B and B', towards A and A'.

Hence the surface of the ocean will assume an oval form, as represented by the thick line in Fig. 143, and there will be high water at the sublunar point A and the opposite point A', low water along the circle of the Earth BB', distant 90° from the sublunar point. Thus we have the same tides occurring simultaneously at opposite sides of the Earth.

It may be shown that the oval curve *aba'b'* is an ellipse whose major axis is *aa'*. The surface of the ocean, therefore, assumes the form of the figure produced by revolving this ellipse about its *major* axis. This figure is called a prolate spheroid, and is thus distinguished from an oblate spheroid, which is formed by revolution about the minor axis.

But though this is the form which the ocean would assume if it were at rest, a stricter mathematical investigation shows that the Earth's rotation would cause the surface of the sea to assume a very different form. In fact, if the Earth were covered over with a *sufficiently shallow* ocean of uniform depth, and rotating, we should really have *low tide* very near the sublunar point *A* and its antipodal point *A'*, and high tide at the two points on the Earth's equator distant 90° from the Moon (Fig. 144).

If the Moon were to move in the equator, the equilibrium theory would always give low water at the poles. This phenomenon is uninfluenced by the Earth's rotation, and since the Moon is never more than about 28° from the equator, we see that the Moon's tide-raising force has the general effect of drawing some of the ocean from the poles towards the equator.

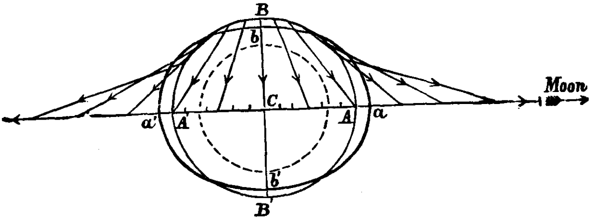

FIG. 143.

452. Canal Theory of the Tides

As an illustration, let us consider what would happen in a circular canal, not extremely deep, supposed to extend round the equator of a revolving globe. Then, in Fig. 144, it is clear that the direction of the disturbing force would, if it acted alone, cause the water in the quadrants *AB* and *AB'* to flow towards *A*; and, in the quadrants *A'B* and *A'B'*, towards *A'*. Hence this force acts in the *same* direction as the Earth's rotation in the quadrants *B'A* and *BA'*, and in the *opposite* direction in *AB* and *A'B'*. Hence, as the water is carried from *A* to *B*, it is constantly being retarded, from *B* to *A'* it is accelerated, from *A'* to *B'* it is retarded, and from *B'* to *A* it is again accelerated, the average velocity being, of course, that of the Earth's rotation. Hence the velocity is *least* at *B* and *B'*, and *greatest* at *A* and *A'*.

Now, it is easy to see that when water moves steadily in a uniform canal it must be shallow where it is swift and deep where it is slow. For, if we consider any portion of the canal, say AB, the quantity that flows in at one end A is equal to the quantity that flows out at the other end B. But it is evident that if the depth of the canal were doubled at any point without altering the velocity of the liquid, twice as much liquid would flow through the canal; consequently, in order that the amount which flows through might be the same as before, we should have to *halve* the velocity of the liquid. This shows that where the canal is deepest the water must be travelling most slowly. Conversely, where the velocity is least the depth must be greatest, and where the velocity is greatest the depth must be least. Hence the depth is *least* at A and A', and *greatest* at B and B', just the opposite to what we should have expected from the equilibrium theory.

In a canal constructed round any parallel of latitude the same would be the case; and hence, if we could imagine a uniform ocean replaced by a series of such parallel canals, low tide would occur at every place when the Moon was in the meridian.

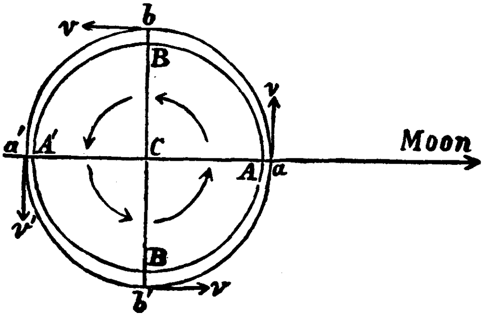

FIG. 144.

This theory (due to Newton), though sounder than Laplace's equilibrium theory, is still not quite mathematically correct. The true explanation of the tides, even in an ocean of uniform depth, is far more complicated, and quite beyond the scope of this book.

453. Lunar Day and Lunar Time

According to either hypothesis, the recurrence of high and low water depends on the Moon's motion relative to the meridian; hence, in investigating this, it is convenient to introduce another kind of time, depending on the Moon's diurnal motion.

The *lunar day* is the interval between two consecutive upper transits of the Moon across the meridian.

In a lunation, or $29\frac{1}{2}$ mean solar days, the Moon performs one direct revolution relative to the Sun, and therefore performs *one* retrograde revolution *less* relative to the meridian. Thus $29\frac{1}{2}$ mean days $= 28\frac{1}{2}$ lunar days; hence for the mean length of a lunar day we have :—

lunar day $= (1 + \frac{2}{57})$ mean solar days $=$ 24h. 50m. 32s. nearly.

The *lunar time* is measured by the Moon's hour angle, converted into hours, minutes, and seconds, at the rate of 15° to the hour.

454. Semi-diurnal, Diurnal and Fortnightly Tides

It has been found convenient to regard the tides produced by the Moon's disturbing force as divided into three parts, whose periods are half a day, a day and a fortnight, the " day " being the lunar day of the last paragraph.

If we adopt the equilibrium theory as a working hypothesis, the lunar tide must be highest when the Moon is nearest to the zenith or nadir. Hence high tide takes place at the Moon's upper and lower transits, when its zenith distance and nadir distance are least respectively. But, for a place in N. lat. (Fig. 145) when the Moon's declination is N., it describes a small circle $Q'R'$, and its least zenith distance ZQ is less than its least nadir distance NE' ; hence the two tides are unequal in height. This phenomenon can be represented by supposing a *diurnal tide*, high only once a lunar day, combined with a *semi-diurnal tide*, high twice in this period.

Again, the Moon's meridian Z.D. and N.D. go through a complete cycle of changes, owing to the change of the Moon's declination, whose period is a month. But after *half* a month, the Moon's declination will have the same value but opposite sign, and hence the diurnal circles $Q'R'$, $Q''R''$, equidistant from the equator QR, are described at intervals of a fortnight. But $NR'' = ZQ'$, $ZQ'' = NR'$; hence the two tides have the same heights. This can be represented by supposing a *fortnightly tide* of the proper height combined with the diurnal and semi-diurnal ones.

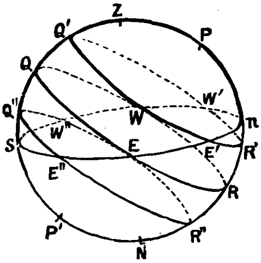

FIG. 145.

In just the same way the smaller tides caused by the Sun may be artificially represented by combining a *diurnal* and *semi-diurnal tide* (the solar day being used) and a *six-monthly tide.*

455. Spring and Neap Tides.—Priming and Lagging

We have hitherto considered chiefly the tides due to the action of the Moon. In reality, however, the tides are due to the combined action of the Sun and Moon, the tide-raising forces due to these bodies being in the proportion of about 3 to 7 (Ex. 1, Art. 450). We shall make the assumption that the height of the tide at any place is the *algebraic sum* of the heights of the tides which would be produced at that place by the Sun and Moon separately.

At new or full Moon the Sun is nearly in the line AA', and the tide-raising powers of the Sun and Moon both act in the same direction, and tend to draw the water from B, B' to A, A' ; hence the whole tide is that due to the *sum* of the separate disturbing forces of the Sun and Moon. The tides are then most marked, the height of high water and depth of low water being at their maximum. Such tides are called *Spring Tides.* We notice that the height of the spring tide $= 1 + \frac{3}{7}$ or $\frac{10}{7}$ of that of the lunar tide alone.

At the Moon's first or last quarter the Sun is in a line BB' perpendicular to AA'. Hence the Sun tends to draw the water away from A, A' to B, B', while the Moon tends to draw the water in the opposite direction. The Moon's action being the greater, preponderates, but the Sun's action diminishes the tides as much as possible. The variations are therefore at their minimum, although high water still occurs at the same time as it would if the Sun were absent. These tides are called *Neap Tides*. The height of the neap tide is the difference of the heights of the lunar and solar tides, and is therefore $\frac{4}{7}$ of that of the lunar tide.

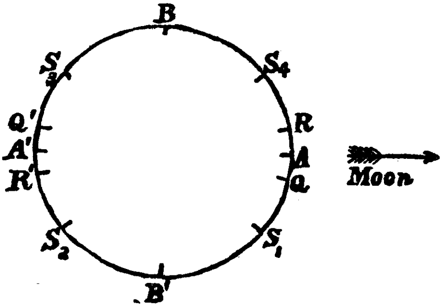

FIG. 146.

Hence spring tides and neap tides are in the ratio of (roughly) 10 to 4.

For any intermediate phase of the Moon, the Sun's action is somewhat different.

Between new Moon and first quarter, the Sun is over a point S_1 behind A. Here the Moon tends to draw the water towards A, A', and the Sun tends to draw the water towards S_1 and the antipodal point S_3. Therefore the combined action tends to draw the water towards two points Q, Q' between A and S_1 and between A and S_3 respectively, whose longitudes are rather less than those of A and A' respectively. The resulting position of high water is therefore displaced to the west, and the high water occurs *earlier* than it would if due to the Moon's influence alone. The tides are then said to *prime.*

Between first quarter and full Moon the Sun is over a point S_2 between B' and A', and the combined action of the Sun and Moon tends to draw the water towards two points R, R', whose longitudes are slightly greater than those of A, A'. The resulting high tides are therefore displaced eastwards, and occur *later* than they would if the Sun were absent. The tides are then said to *lag.*

Between full Moon and last quarter the Sun is over some point S_3 between B and A', but the antipodal point S_1 is between A and B'; hence the tide *primes*.

Between last quarter and new Moon, when the Sun is at a point S_4 between B and A, it is evident in like manner that the tide *lags*.

Hence we have the following:—

Spring Tides occur at the syzygies
(conjunction and opposition).
Neap Tides occur at the quadratures.
From syzygy to quadrature, the tide primes.
From quadrature to syzygy, the tide lags.

The heights of the spring and neap tides vary with the varying distances of the Sun and Moon from the Earth. Spring tides are the highest possible when both the Sun and Moon are in perigee, while neap tides are the most marked when the Moon is in apogee but the Sun is in *perigee* (because the Sun then pulls *against* the Moon with the greatest power, as far as the Sun's action is concerned). Both the spring and neap tides, and also the priming and lagging, are on the whole most marked when the Sun is near perigee, *i.e.* about January.

It may be here stated, without proof, that, taking the Sun's and Moon's tide-raising forces to be in the proportion of 3 to 7, the maximum interval of priming or lagging is found to be about 51 minutes.

456. Establishment of the Port

Both the equilibrium and canal theories completely fail to represent the actual tides on the sea, owing to the irregular distribution of land and water on the Earth, combined with the varying depth of the ocean. These circumstances render the prediction of tides by calculation one of the most complicated problems of practical astronomy, and the computations have to be based largely on previous observations. In consequence of the barriers offered to the passage of tidal waves by large continents, lunar high tide does not occur either when the Moon crosses the meridian, as it would on the equilibrium theory, or when the Moon's hour angle is 90°, as it would on the canal theory. But this continental retardation causes the high tide to occur later than it would on the equilibrium theory, by an interval which is constant for any given place. This interval, reckoned in lunar hours, is called the *Establishment of the Port* for the place considered. Thus the establishment of the port at London Bridge is 1h. 58m., so that lunar high water occurs 1h. 58m. after the Moon's transit, *i.e.* when the Moon's hour angle, reckoned in time, is 1h. 58m.

The same causes affect the solar tide as the lunar, hence the *Sun's* hour angle (or the local apparent time) at the *solar* high tide is also equal to the establishment of the port.

The actual high tide, being due to the Sun and Moon conjointly, is earlier or later than the lunar tide by the amount of priming or lagging.

By adding a correction for this to the establishment of the port, the lunar time of high water may be found for any phase of the Moon; and we notice in particular that at the Moon's four quarters (syzygies and quadratures), the lunar time of high water is equal to the establishment of the port. And, knowing the lunar time of high water, the corresponding mean time can be found, for

(mean solar time) — (lunar time)
= (mean ⊙'s hour angle) — (☾'s hour angle)
= (☾'s R.A.) — (mean ⊙'s R.A.)

[since R.A. and hour angle are measured in opposite directions].

Now the Moon's R.A. is given in the *Nautical Almanac* for every hour of every day in the year. Also the mean Sun's R.A. at midnight is the sidereal time of mean midnight, and can be obtained from the *NauticalAlmanac*. Hence the mean Sun's R.A. [which = (sidereal time) — (mean time)] is easily found for any intermediate time.

Hence the mean time of high water can be readily found. The establishments of different ports, and the times of high water at London Bridge, are given in the Admiralty Tide Tables, etc.

457. Approximate Calculation

If only a *very rough* calculation is required, we may proceed as in Arts. 39, 193. We assume the Moon's R.A. to increase uniformly; we shall then have

(☾'s R.A.) — (⊙'s R.A.) = (☾'s elongation);
or (solar time) = (lunar time) + (☾'s elongation).

Knowing the Moon's age, its elongation may be found, as in Art. 193, and this must be converted into time, at the rate of 1h. to 15°. We then have

(time of high water) = (establishment) + (amount of lag.) + (☾'s elongation in time).

Example.—*Find, roughly, the time of high water at the Moon's first quarter, at London Bridge.*

Here there is no priming or lagging. Hence the lunar time, or ☾'s hour-angle at high water, is equal to the establishment, or 1h. 58m. Also the Moon's elongation is 90°. Hence the Sun's hour angle, in time, = 1h. 58m. + 6h., and high water occurs about 7h. 58m.

458. Tidal Constants

The excess of the establishment of the port at any place, over that at London Bridge, expressed in mean time, is sometimes called the *Tidal Constant* of that place.

If we assume the amount of priming or lagging to be the same at both places, the tidal constant is the difference between the times of high water at London Bridge and the given place. Hence, knowing the tidal constant and the time of high water at London Bridge, the time at any other place can be found.

Tables of tidal constants, and of the heights of the spring and neap tides at different places, are given in *Whitaker's Almanack*.

Example.—*Find the times of high water at Cardiff and Portsmouth on January 25th,* 1892, *the tide intervals from London Bridge being* + 4*h.* 58*m. and* — 2*h.* 17*m.*

From the Almanack we find times of high water at London Bridge are :—

	Jan. 24th	Jan. 25th	
	9h. 15m. aft.	9h. 53m. morn.,	10h. 31m. aft.
Add	4h. 58m.	4h. 58m.	4h. 58m.
Times at Cardiff are (Jan. 25th)	2h. 13m. morn.	2h. 51m. aft.	
Again, subtract from first line		2h. 17m.	2h. 17m.
times at Portsmouth are (Jan. 25th)		7h. 36m. morn.,	8h. 14m. aft.

459. The Masses of the Sun and Moon

These can be compared by observing the relative heights of the solar and lunar tide, the relative distances of the Sun and Moon being known. Or, if the ratio of the masses be supposed known, the distances could be compared by this method. In this manner Newton (A.D. 1687) found the masses of the Moon and Earth to be in the proportion of 1 : 40 nearly. D. Bernoulli (1738) found 1 : 70, and Lubbock (1862) found 1 : 67·3. The two last make the Moon's mass a little too great. Newton makes it double what it ought to be.

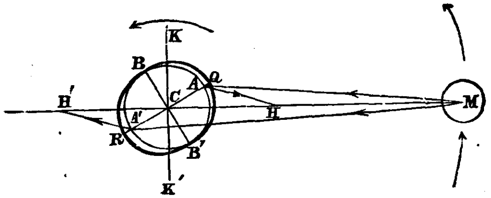

FIG. 147.

460. Effects of Tidal Friction.—Retardation of Earth's Rotation.—Acceleration of Moon's Orbital Motion

All liquids possess a certain kind of friction, known as " viscosity," which tends to resist their motion when they are changing their form, and to convert part of their kinetic energy into heat. Owing to this friction between the Earth and the oceans, the Earth, in its diurnal rotation, tends to carry the tidal wave round slightly in front of the point underneath the Moon, taking the positions of high water forward from the line $H'CM$ to $A'CA$. The Moon, on the contrary, tends to draw the water back from A, A', the disturbing forces AH, $A'H'$ forming a couple, which is resisted only by the Earth's friction. Hence the ocean exerts an equal frictional couple on the Earth, and this couple tends to diminish the angular velocity of the Earth's diurnal rotation, and thus increase its period.

Therefore tidal friction tends to gradually lengthen the day.

But if the Moon exerts a couple on the Earth, tending to retard it, the Earth must exert an equal and opposite couple on the Moon, tending to accelerate it. The portion of the ocean heaped up at A, being nearer the Moon, exerts a greater attraction than that at A', in addition to which the angle CMA is very slightly greater than CMA'. Hence the resultant of the attractions of equal masses of water at A and A' acts on M in a direction slightly in front of MC, and tends to pull the Moon forward. This tends to increase the Moon's areal velocity. (Compare Art. 423). Since the areal velocity of a body revolving in a circle varies as the square root of the radius (Art. 421, Cor.), the Moon's distance must be gradually increased by this means, and hence also its periodic time.*

Therefore tidal friction tends to increase the Moon's distance and to lengthen the month.

The rate of increase of the Moon's distance at the present time is about five feet in a century.

The final effect of tidal friction must be to equalize the lengths of the day and lunar month. The angular velocities of the Earth and Moon both decrease, but the effect of the couple, in producing retardation, is far more considerable on the Earth than on the Moon.

The student who has not read rigid dynamics may illustrate this statement by the comparative ease with which a small top can be spun with the fingers, and the great difficulty of imparting an equal singular velocity to the same body by whirling it round in a circle at the end of a string of considerable length. The top represents the Earth, and the body on the long string the Moon.

In rigid dynamics it is shown that when a system of bodies are revolving under their mutual reactions, their *angular momentum*, or *moment of momentum* about their centre of mass, remains constant. Hence the decrease in the Earth's angular momentum is equal to the increase in that of the Moon. Now the angular momentum of a particle revolving in an orbit is twice the product of its mass into its areal velocity, and this is also approximately true of the Moon. Hence, since the Moon's distance from the common centre of mass is far greater (about sixty times greater) than the distance of any point on the Earth from its axis of rotation, it is evident that the same change in angular momentum produces far more effect on the angular velocity of the Earth than on that of the Moon.

The lengthening of the day by tidal friction amounts to about 0·002 of a second in the course of a century. The lengthening, though slow, is cumulative and eventually the periods of rotation of the Earth and Moon will be equalized. The day and the month will then be of equal length, each being equal to about 47 of our present days. It has been estimated that it will need 50,000 million years to bring this about. The Earth will then always turn the same face to the Moon and the

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* This increase of the distance more than counterbalances the tendency to increase the Moon's actual velocity. For the actual velocity is *inversely* proportional to the square root of the distance (Art. 419), and therefore diminishes as the distance increases. Similarly, the *angular* velocity is decreased.

Moon will continue, as it does at present, to turn the same face towards the Earth. Hence there will be no lunar tides and the retardation due to lunar tidal friction will no longer exist.

The solar tides will, however, still continue to exist, provided that water surfaces still exist on the Earth; indeed, even apart from this condition, considerable bodily tides in the solid crust are known to occur. Further, the lunar tidal force will only amount to one-third of its present value, owing to its increased distance from the Earth; the solar tides will, therefore, be the stronger and will further lengthen the day. The lunar tidal wave will then go round the Earth in the reverse direction, and the Moon as a result will again approach the Earth.

461. Variations in the Length of the Day

In the preceding section it has been shown that the day is slowly lengthening as a consequence of tidal friction. But in addition to this lengthening, astronomical observations have established the occurrence of irregular variations in the length of the day. The day provides the fundamental unit of time and the positions of the Sun, Moon and planets, given in the *Nautical Almanac* are computed on the assumption that the unit of time is invariable. A change in the unit of time will cause apparent displacements of Sun, Moon and planets which, when measured in arc, will be proportional to their rates of angular motion. It is from the similarity in the apparent displacements of these bodies that the changes in the length of the day can be inferred. The changes, which occur rather suddenly, can be as great as 0·004 seconds in the length of the day, which may either be increased or decreased. Tidal friction can never shorten the day; it can only lengthen it. Hence some other cause must be operative, which must produce a change of moment of inertia of the Earth. As the angular momentum remains constant, a decrease in the moment of inertia will result in an increase in the angular velocity or, in other words, in a decrease in the length of the day. An increase in the moment of inertia will result in an increase in the length of the day. A slight expansion or contraction of the Earth as a whole may occur; a change of 6 inches in the radius of the Earth would be sufficient to produce a change in the length of the day of the order of magnitude that is observed. Such a change in the radius of the Earth could not be detected by direct observation. Alternatively, it may be that there is some readjustment from time to time of the layers of equal density within the Earth.

462. The Moon's Form and Rotation

The theory of tidal friction affords a simple explanation of how it is that the Moon always turns the same face to the Earth. Remembering that the Earth's mass is 81 times the Moon's, but that its radius is about four times as great, the Earth's tide-raising force at a point on the Moon would be about 81/4, or over twenty times as great as the Moon's on the Earth. Early in the Moon's history, whilst it was still hot and before it had solidified, huge tides were produced on the Moon by the attraction of the Earth. The Moon was then much nearer to the Earth than it is now and the lengths of the day and month were shorter. The huge tides on the Moon, by their friction, rapidly lengthened the Moon's period of rotation. The period lengthened until it became equal to the period of revolution of the Moon about the Earth. The Moon thus always turned the same face to the Earth, and it has since continued to do so.

If the Moon was then not quite solid, the Earth's tide-raising force, which had then become constant, must have drawn it out into the form demanded by the equilibrium theory, namely, to a first approximation, a prolate spheroid, with its longest diameter pointed towards the Earth.

It may easily be seen, from the expressions in Art. 449 that the tide-raising force of a body is slightly greater at the point just under it than at the opposite point (when we do not only consider *approximate* values). Hence the Moon is not quite spheroidal, but is more drawn out on the side towards the Earth than on the remote side. Its form is, therefore, that of an *egg*, the small end being towards the Earth. This result of theory cannot, of course, be confirmed by direct observation, the remote side being invisible; but Hansen, by the theory of perturbations, has shown that the Moon's centre of mass is further from the Earth than its centre of figure, thus furnishing independent evidence in favour of the theory.

463. Application to Solar System

Since the Sun's tide-raising force on different planets varies inversely as the cube of their distance, the solar tides are far greater on the nearer planets than on those more remote. It is, therefore, quite natural to suppose that the effects of tidal friction may have produced such a great retardation in the rotations of Mercury, and possibly also of Venus, that one or both of these bodies already turn the same face towards the Sun. The remoter planets, must necessarily take a much longer time to undergo the necessary retardation, and it would be very unnatural to expect Neptune, for example, always to turn the same face to the Sun. Observation has shown that Mercury always turns the same face towards the Sun, so that the length of the day (as defined by the period of rotation) and the year on Mercury are the same. The period of rotation of Venus is not known with certainty. It is known to be appreciably longer than the period of rotation of the Earth but not to be so long as the period of revolution of Venus around the Sun. It appears probable that the rotation period of Venus is of the order of 30 days. Tidal friction has therefore slowed down the rotation of Venus but not sufficiently to cause it always to turn the same face to the Sun.

III.—PRECESSION AND NUTATION

464. Precession

In Arts. 125, 276 we stated that the plane of the Earth's equator is not fixed in space, but that its intersections with the ecliptic have a slow retrograde motion. This phenomenon, which is known as Precession, is due to the fact that the Earth is not quite spherical, and that, in consequence of its spheroidal form, the Sun's and Moon's attractions exert a disturbing couple on it.

465. The Sun's and Moon's Disturbing Couples on the Earth

Let the plane of the paper in Fig. 148 contain the Earth's polar axis *PP'*, and the Moon's centre *M*, say at the time when the Moon's south declination is greatest.

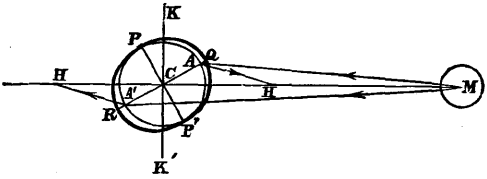

FIG. 148.

Inside the Earth inscribe a sphere *PAP'A'*, touching its surface at the poles. Then we may (for the sake of illustration) regard the protuberant portion of the Earth outside this sphere as a kind of tide firmly fixed to the Earth, and the arguments of the last section (Art. 460) show that the variations in the Moon's attraction at different points give rise to a distribution of disturbing force identical with the tide-raising force, tending to draw this protuberant part with its longest diameter *QR* pointing towards the Moon. The Moon's attraction on the matter inside the inscribed sphere passes exactly through the Earth's centre *C*, and produces no such couple ; but the disturbing forces at *A*, *A'*, which are represented by *AH*, *A'H'*, form a couple on the protuberant parts, *AQ*, *A'R*, tending to turn the diameter *A'A* towards *CM*. The same is true of the disturbing forces at any other pair of opposite points of the Earth in the quadrants *HCK*, *H'CK'*. Of course there are couples in the two other quadrants tending in the reverse direction, but they have less matter to act on, and are therefore insufficient to balance the former couples.

When the Moon is at the opposite point of its orbit, *i.e.* at its greatest N. declination, it is again in the line *CH'*, and again tends to draw the

Earth's equatorial plane towards the line HH'. For any intermediate position of the Moon the couple is smaller, and it vanishes when the Moon is *on* the equator ; still, on the whole, *the Moon's disturbing force always tends to draw the plane of the Earth's equator towards the plane of the Moon's orbit.*

Similarly, *the Sun's disturbing force always tends to draw the plane of the Earth's equator towards the ecliptic.*

Since the Moon's nodes are rotating (Art. 211), the plane of the Moon's orbit is not fixed ; but it is inclined to the ecliptic at a small angle (5°), while the plane of the equator is inclined to the ecliptic at a much larger angle ($23\frac{1}{2}$°). The *average* effect of the Moon's disturbing couple is thus to pull the Earth's equator towards the plane of the ecliptic. This tendency is increased by the Sun's disturbing couple ; and the two are proportional to the Sun's and Moon's tide-producing forces, *i.e.* as 3 : 7 roughly. For this reason, the resulting phenomenon is sometimes called *luni-solar* precession.

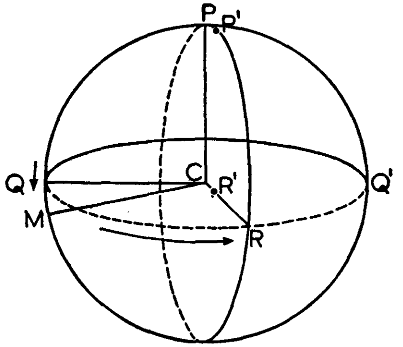

FIG. 149.

466. Effect of the Couple on the Earth's Axis

If the Earth were without rotation, the tendency of this couple would be to bring the plane of the equator into coincidence with the ecliptic, with the result that the equator would oscillate from side to side of the ecliptic, like a pendulum under gravity. But the rapid diurnal motion of the Earth entirely alters the phenomena.

Let CR be a semi-diameter of the Earth, perpendicular to CP and CM (the direction of the Moon). The precessional couple would, alone, produce a slow rotation in the direction PQM ; *i.e.* about CR. If now the Earth's rotation be represented in magnitude and direction by CP, measured along the Earth's axis, this additional rotation must be represented by a very short length CR', measured along CR.

Take PP', equal and parallel to CR' ; then, since PP' is very small, CP' is of almost exactly the same length as CP. But angular velocities, and momenta *about* lines which represent them in magnitude, are compounded by the same law as forces, velocities, etc. [*cf.* Art. 397 (iii)] *along* the same lines of corresponding magnitudes.

Hence, the resultant axis of rotation is shifted from CP to CP', *in a direction perpendicular to the plane of the acting couple.*

A full explanation of what follows would be impossible without a close acquaintance with rigid dynamics. But it is evident that a body flattened at the poles will spin more readily about the line CP than about any other line drawn in its substance. Hence it is easy to understand that the polar axis CP is *itself* deflected towards CP', and thus moves perpendicular to the acting couple.

This motion can be illustrated by that of a rapidly spinning top, or of a gyroscope, the phenomena of which can readily be investigated by experiment.

467. Precession of a Spinning Top

Experiment 1.—Let a top be set spinning rapidly about its extremity, in the opposite direction to the hands of a watch, as seen from above, the top being supported at a point on its axis below its centre of gravity. The weight of the top, acting vertically through the centre of gravity, tends to upset the top by pulling its axis out of vertical. But if the top is spinning sufficiently rapidly, we know that it will not fall, the only effect of gravity being to make it " reel." *i.e.* to cause its axis of rotation to describe a cone about the vertical through the point of support, revolving slowly in the counter-clockwise direction. This slow revolution may be called the *precession* of the top, and the experiment shows that when a top is acted on by a couple (such as that due to its weight) tending to pull its axis away from the vertical, it precesses in the *same* direction in which it is spinning.

Experiment 2.—Now suppose the top *suspended* from its upper extremity, being thus supported above its centre of gravity. The couple due to the weight and the reaction of the support, now tends to draw the axis of the top *towards* the vertical. In this case the axis of the top will be found to slowly describe a cone in the *opposite* direction ; that is, the top now precesses in the *opposite* direction to that in which it is spinning.

Experiment 3.—Suppose the top supported as in Experiment 1. If we give the top a push away from the vertical, its axis will not move in this direction, but its precessional motion will increase. If we give a push in the direction of precession, its axis will approach the vertical. If we push the axis in the direction of the vertical, it will not move towards the vertical, but its rate of precessional motion will be increased, *i.e.* the top will acquire an additional increased precessional motion. If we push it in the direction opposite to that of precession, the axis will begin to move away from the vertical. In every case the axis of the top moves in a direction perpendicular to the direction of the force acting on it, and therefore a couple acting on a very rapidly spinning top produces displacement of the axis in a plane *perpendicular* to the plane of the couple.*

[If we push the top by pressing the side of a pencil against its axis, it thus always moves *in the direction in which the axis would roll along* the side of the pencil. Of course the displacement of the axis is *not due* to rolling, as may easily be

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* These experiments may easily be performed by the reader with any good-sized top.

shown by repeating the same experiment with a gyroscope, this time pushing one of the hoops carrying the top instead of touching the top itself; here no such rolling is possible.]

468. Precession of the Earth's Axis

On the celestial sphere, let P, K be the poles of the equator and ecliptic respectively. The Sun's disturbing couple and the mean couple due to the Moon tend to pull the Earth's equator towards the ecliptic, or to pull the polar axis P towards the axis of the ecliptic K. Hence the Earth behaves like a top suspended from above its centre of gravity, and the polar axis slowly describes a cone about the axis of the ecliptic, revolving in the opposite direction to that of the Earth's rotation, *i.e.* in the retrograde direction.[*] The pole P therefore slowly describes a small circle, PP' about K, the pole of the ecliptic, with angular radius PK, equal to the obliquity of the ecliptic, *i.e.* 23° 27′. As the pole revolves from P to P' it carries the equator from ♈Q♎ to ♈′Q♎′, thus carrying the equinoctial points ♈ and ♎ slowly backwards along the ecliptic. The average angle ♈♈′, or PKP'[†], described in a year, is 50·2″, and P therefore performs a complete revolution about K in 25,800 years.

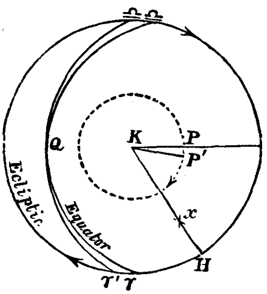

FIG. 150.

The position of the *ecliptic* is not affected by precession. Hence the *celestial latitude* xH *of any star* x *remains constant, and its celestial longitude* ♈H *increases by the amount of precession* ♈♈′, *that is, at the rate of* 50·2″ *per year.*

A star's declination and right ascension are, however, continually changing. This change is, of course, due to the motion of the equator, and not of the star. Thus, as P moves to P', the N.P.D. of the star x decreases from Px to $P'x$, and its R.A. changes from ♈ Px to ♈′$P'x$. (The circles ♈P, ♈′P', xP, xP', are not represented, in order not to complicate the figure unnecessarily.) The declinations of some stars are increasing, of others decreasing. From R.A. 6h. to 18h. north declinations are diminishing, south ones increasing: from R.A. 18h. to 6h. the reverse happens. Expressions for the changes of a star's right ascension and declination caused by precession were derived in Chapter XII, Art. 278.

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[*] See also Fig. 149. If K be pole of ecliptic (CK nearly perpendicular to CM) it is evident that as P travels towards P' it moves in the retrograde direction about K.

[†] P♈ and K♈ are each 90°; therefore ♈ is pole of arc KP; and ∠ ♈KP is a right angle. Similarly, ♈′ KP' is a right angle;

$$\text{Therefore } \angle PKP' = \angle ♈ K ♈' = \text{arc } ♈\ ♈',$$

since ♈♈′ ♎ is a *great circle*, whose pole is K.

469. Nutation of the Earth's Axis

In treating of precession, we have supposed the Earth's poles to describe small circles uniformly about the poles of the ecliptic. This they would do if the Sun's and Moon's disturbing couples on the Earth were always constant in magnitude, and always tended to pull the Earth's poles directly towards the poles of the ecliptic. But the couples, so far from being constant, are subject to periodic variations, in consequence of which the Earth's poles really describe a wavy curve (shown in Fig. 151), threading alternately in and out of the small circle which would be described under precession alone if the couple were constant. This phenomenon is called *Nutation*, because it causes the Earth's poles to *nod* to and from the pole of the ecliptic.

Nutation is really compounded of several independent periodic motions of the Earth's axis; the most important of these is known as *Lunar Nutation*, and has for its period the time of a sidereal revolution of the Moon's nodes, *i.e.* about 18 years 220 days. The effect of lunar nutation may be represented by imagining the pole P to revolve in a small ellipse about its mean position p as centre, in the above period, in the retrograde direction, while p revolves about K, the pole of the ecliptic, with the uniform angular velocity of precession of 50·2″ per annum. The major and minor axes of the little ellipse are along and perpendicular to Kp respectively, their semi-lengths being $pa = 9''$ and $pb = 6{\cdot}8''$ respectively. The angle $pKb = bp/\sin Kp = 6{\cdot}8''$ cosec 23° 27′ = 17·1″ nearly.

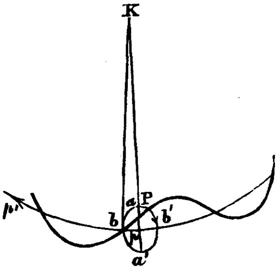

FIG. 151.

470. General Effects of Lunar Nutation

In consequence of lunar nutation, the obliquity of the ecliptic is subject to periodic variations. For this obliquity is equal to the arc KP, and as P revolves about its mean position from one end to the other of the major axis of the little ellipse, the arc KP becomes alternately greater and less than its mean value Kp, by 9″. Thus the greatest and least values of the obliquity of the ecliptic differ by 18″, and the obliquity fluctuates between limits differing by this amount once in 18·6 years. But as, in addition, the obliquity is diminishing 0·47″ each year owing to planetary action, the limits are not fixed.

Again, when the pole is at an extremity of the minor axis b, it has regreded further than its mean position p by the angle pKb, which we have seen is about 17·1″. Hence, also, the first point of Aries has regreded 17·1″ further than it would have gone had its motion been uniform. Similarly, at b' it has regreded 17·1″ less than it would have done if moving uniformly. Hence the first point of Aries oscillates to and fro about its mean position through an arc of 34·2″ in the period of 18⅔ years. while its mean position moves through an angle 18⅔ × 50·2″, or about 15′ 37″.

The angular distance between the true and mean positions of the first point of Aries is called the *Equation of the Equinoxes.* It is, of course, equal to the angle pKP.

Nutation does not affect the position of the ecliptic; hence the *latitudes* of stars are unaltered by it. Their apparent *longitudes* are, however, increased by the equation of the equinoxes. Both this cause and the varying obliquity of the ecliptic produce variations in a star's R.A. and decl; expressions for these were derived in Chapter XII, Art. 282.

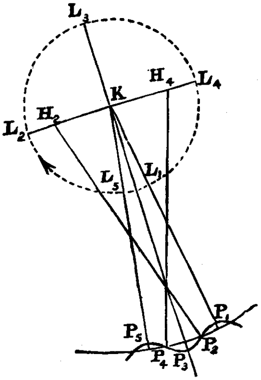

FIG. 152.

471. Physical Cause of Nutation

If the Moon were to move exactly in the ecliptic, the average couples exerted by the Moon as well as the Sun would both tend to pull the Earth's pole directly towards K, the pole of the ecliptic. But the Moon's orbit is inclined to the ecliptic at an angle of 5°; hence, if L be its pole, $KL = 5°$, and the Moon's average disturbing couple tends to pull the pole P towards L instead of K. When we consider the Sun's action also, the resultant of the two couples tends to pull the pole towards a point H which is intermediate between K and L, but nearer to L (because the Moon's disturbing couple is about 2⅓ times the Sun's). Hence the pole P moves off in a direction perpendicular to HP, and not to KP. In consequence of the rotation of the Moon's nodes, L, and therefore also H, revolves in a small circle about P in the period of 18⅔ years (see Fig. 152).

Let L_1, L_2, L_3, L_4, L_5 be the positions of L, and P_1, P_2, P_3, P_4, P_5 the positions of P, when the angle PKL is 0°, 90°, 180°, 270°, 360° respectively, H_2, H_4 the positions of H corresponding to L_2, L_4. Then at P_1 and P_3 the couple is directed towards K, and therefore P is then moving perpendicular to KP. At P_2 the couple is directed towards H_2, and the pole P_2 moves perpendicularly to H_2P_2, thus passing from the inside to the outside of the small circle described by its mean position. Similarly, at P_4 the pole, by moving perpendicularly to H_4P_4, passes back from the outside to the inside of the small circle which it would describe if the couple were always directed towards K.

Thus the wavy form of the curve described by P is accounted for. And since the whole space P_1KP_5 or L_1KL_5, traversed in a revolution of L, is very small, the period of oscillation is almost exactly that of revolution of the Moon's nodes. It is exactly the same as their period of revolution relatively to the equinox, which is 5 days in excess of their sidereal period.

Again, the Moon's couple depends on the angular distance PL, and is greater the greater this distance (as may easily be seen by Art. 465). Hence the resultant couple, and therefore also the precessional motion, is least at P_1 and greatest at P_3. This accounts for the variable rate of motion of P, which gives rise to the equation of the equinoxes.

472. Solar and Fortnightly Nutations

The variations in the intensity of the Sun's and Moon's disturbing couples during their orbital revolutions give rise to two other kinds of nutation. Let us first consider the variations in the Sun's disturbing couple, which produce Solar Nutation. It appears from Art. 465, that the couple vanishes when the Sun is on the equator, and that it is greater the greater the Sun's declination. Also it is readily evident from Fig. 148 that the couple in general acts in a plane through the Sun and the Earth's poles, tending to turn the poles more nearly perpendicular to the direction of the Sun. This shows that the couple is not really directed towards the pole of the ecliptic (though this is its average direction for the year) except at the solstices (Fig. 153).

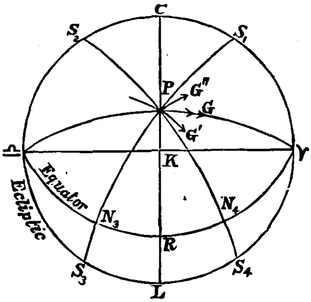

FIG. 153.

Now at the vernal equinox, when the Sun is at ♈, the couple vanishes, and therefore the Earth's tendency to precession, due to the Sun, vanishes. Between the vernal equinox and the summer solstice, when the Sun is at S_1, the couple is along S_1P away from S_1, and this tends to make the pole precess along PG' perpendicularly to S_1P. At the summer solstice the couple along CP is a maximum, and tends to produce precession along PG perpendicular to KP. At S_2 the couple along S_2P tends to make the pole precess in the direction PG''. At the autumnal equinox, ♎, the couple, and therefore the velocity of solar precession, vanishes. At S_3 the Sun's declination is negative, and the couple tends to draw P *towards* S_3; hence the Earth again tends to precess along PG'. At the winter solstice the direction of precession is again along PG, and the precessional velocity again a maximum. Finally, at S_4 the direction of precession is again along PG''.

Hence the variations in the Sun's declination cause the pole to thread its way in and out of the circle it would describe under uniform precession once every six months, and to cause the velocity of revolution about K to fluctuate in the same period. This gives rise to the nutation known as *Solar Nutation*, whose period is half a tropical year. In the case of the Moon the corresponding phenomenon is known as *Fortnightly Nutation*, and its period is half a month; the explanation is exactly the same.

The variations in the obliquity of the ecliptic due to these two causes are small, because, owing to the comparatively small period in which they recur, the pole has not time to oscillate to and from K to any great extent. Moreover, the couple, and therefore the rate of motion of P, decreases as the inclination of PG' to PG increases. When the Sun is at ♈ or ♎ the displacement, if it existed, would be along PK, in the most advantageous direction for producing nutation, but at this instant the couple vanishes.

The solar nutation accounts for the term $c \sin 2\odot$ in Art. 279, and the fortnightly nutation accounts for the term $d \sin 2$ ☾. The corresponding variations in the obliquity are given in Art. 280.

The solar nutation only displaces the pole about 1·2″ to or from K, and the displacement due to fortnightly nutation is 0·1″ at most. The effects on the equation of the equinoxes are more apparent. Under the Sun's action alone, the pole would come to rest *twice a year*, viz. at the equinoxes, and under the Moon's action its rate of motion would vanish *twice a month*, viz. when the Moon crossed the equator. At all other times the couples tend to produce *retrograde—never direct*—motion of the pole about K. Hence the precessional motion can never vanish unless the Sun and Moon should happen to cross the equator *simultaneously*.

IV.—LUNAR AND PLANETARY PERTURBATIONS

473. Universality of Gravitation

In consequence of the universality of gravitation, every body in the solar system has its motion more or less disturbed by the attraction of every other body. Kepler's Laws (with the modification of the Third Law given in Art. 431) would be strictly true only if each planet were attracted solely by the Sun, and each satellite described its relative orbit solely under the attraction of its primary. Hence the fact that these laws very nearly agree with the results of observation shows that the mutual attractions of the planets are small compared with that which the Sun exerts on each of them, and that, in the orbital motion of a satellite, by far the greater part of the relative acceleration is due to the attraction of the primary.

474. Lunar Perturbations

We have seen, in Section I, that the Moon's motion consists of two component parts, a monthly orbital motion relative to the Earth—or, more strictly, relative to the centre of mass of the Earth and Moon—and the annual orbital motion of this centre of mass in an ellipse about the Sun. If the acceleration of the Sun's attraction were the same in magnitude and direction at the Moon as at the Earth, it would be exactly the acceleration required to produce the latter motion, and the relative orbit of the Moon about the Earth would be determined by the Earth's attraction alone. This is very nearly the case, owing to the great distance of the Sun. But the small differences of the accelerations caused by the Sun's attraction on the Earth and Moon tend to modify the relative motion of these two bodies, by giving rise to *perturbations* (Art. 210). The relative accelerations thus produced may be represented by a distribution of *disturbing force* due to the Sun, just in the same way that the relative accelerations of the oceans, which cause the tides, are determined by distributions of disturbing force due to the Sun and Moon. And since the Sun's distance is nearly 400 times the Moon's, the expressions for the disturbing force, corresponding to those investigated in Art. 450, are sufficiently approximate to account for the more important lunar perturbations.

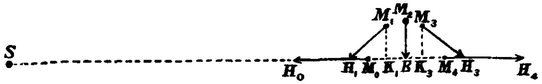

FIG. 154.

Let S, E, M denote the centres of the Sun, Earth, and Moon. Drop MK perpendicular on ES, and on EK produced take $KH = 2EK$. Then, if S denote the mass and r the distance of the Sun, the Sun's disturbing force produces at M a relative acceleration along MH of magnitude $kS \,.\, MH/r^3$, its components being $k \,.\, S \,.\, MK/r^3$ along MK and $2k \,.\, S \,.\, EK/r^3$ parallel to EK.

This force tends to accelerate the Moon *towards* the Earth at quadrature (M_2), and *away* from the Earth at conjunction and opposition (M_0, M_4). At any other position it accelerates the Moon towards a point (H_1) in the line ES, and thus makes the Moon tend to approach the Sun, if its elongation (M_1ES) is less than 90°; but it accelerates the Moon towards a point (H_3) away from the Sun if its angle of elongation from the Sun be obtuse.

475. The Rotation of the Moon's Nodes

Let CL represent the ecliptic, $N_1M_1N_1'$ the great circle which the Moon would appear to describe on the celestial sphere if there were no disturbing force acting upon it, and let H, between N_1 and N_1', on the ecliptic, represent either the Sun's position on the celestial sphere or that of the point antipodal to it. Then the reasoning of the last paragraph shows that the disturbing force acts in the plane HEM_1, and therefore has a component at M_1 directed along the tangent to the great circle M_1H.

Now let us suppose that the Moon is revolving under the Earth's attraction alone, but that on arriving at M_1 it is acted on by a sudden impulse or blow directed towards H. Clearly the effect of such an impulse is to bend the direction of motion inward, from M_1N_1' to M_1N_2', and the Moon will then begin to describe a great circle M_1N_2', which, if produced both ways, will intercept the ecliptic at points N_2, N_2' *behind* N_1, N_1'. The inclination of the orbit to the ecliptic will also be diminished slightly if M_1 is within 90° of N_1; for the exterior angle $M_1N_1H > M_1N_2H$, *since the sides of the triangle* $M_1N_1N_2$ *are each less than* 90°. But when the Moon comes to M_2, let another impulse act towards H. This will deflect the direction of motion from M_2N_2' to M_2N_3', and the Moon will now begin to describe the great circle $N_3M_2N_3'$, whose nodes N_3, N_3' are still further behind their initial positions. The inclination of the orbit to the ecliptic will, however, be increased this time.

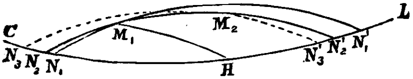

FIG. 155.

It is easy to see that the same general effect takes place when the Moon is acted on by a *continuous force*, always tending *towards* the ecliptic, instead of a series of impulses. Such a force continuously deflects the Moon's direction of motion, and draws the Moon down so that it returns to the ecliptic more quickly than it would otherwise. Hence the Moon, after leaving one node, arrives at the next before it has quite described 180°, and the result is an apparent *retrograde* (*never direct*) motion of the nodes, combined with periodic, but small, fluctuations in the inclination of the orbit.

The retrograde motion of the Moon's nodes is, in some respects, analogous to the precession of the equinoxes, and, although the analogy is somewhat imperfect, the former phenomenon gives an illustration of the way in which the latter is produced. If the Earth had a string of satellites, like Saturn's rings, closely packed together in a circle in the plane of the equator, the Sun's disturbing force, ever accelerating them towards the ecliptic, would, as in the case of the Moon, cause a retrograde motion of the points of intersection of all their paths with the ecliptic, and this would give the appearance of a kind of retrograde *precession* of the plane of the rings. If the particles, instead of being separate, were united into a solid ring, the general phenomena would be the same. And it is not unnatural to expect that what occurs in a simple ring should also occur, to a greater or less degree, in the case of other bodies that are somewhat flattened out perpendicularly to their axis of rotation, such as the Earth, thus accounting for the precession of the equinoxes. (Of course this is only an *illustration*, not a rigorous proof; in fact, if the Earth were *quite* spherical it would behave very differently.)

*476. Perturbations due to Average Value of Radial Disturbing Force

Let d be the Moon's distance. Then, when the Moon is in conjunction or opposition, the Sun's disturbing force acts *away* from the Earth, and is of magnitude $2kSd/r^3$ (Fig. 154). When the Moon is in quadrature the disturbing force acts *towards* the Earth, but is only half as great. Hence, on the *average*, the disturbing force tends to pull the Moon *away* from the Earth.

In consequence, the Moon's average centrifugal force must be rather less than it would be at the same distance from the Earth if there were no disturbing force, and the effect of this is to *make the month a little longer* than it would be otherwise for the same distance of the Moon.

Moreover, the disturbing force increases as the Moon's distance increases, but the Earth's attraction diminishes, being proportional to the inverse square of the distance; this has the effect of making the whole *average* acceleration along the radius vector *decrease more rapidly as the distance increases* than it would according to the law of inverse squares. The result of this cause is *the progressive motion of the apse line.* It is difficult to explain this in a simple manner, but the following arguments may give some idea of how the effect takes place. At apogee the Moon's average acceleration is less, and at perigee it is greater than if it followed the law of inverse squares and had the same mean value. Hence, when the Moon's distance is greatest, as at apogee, the Earth does not pull the Moon back so quickly, and it takes longer to come back to its least distance, so that it does not reach perigee till it has revolved through *a little more than* 180°. Similarly, at perigee the greater average acceleration to the Earth does not allow the Moon to fly out again quite so quickly, and it does not reach apogee till it has described *rather more than* 180°. Hence, in each case, the line of apsides moves forward on the whole.

*477. Variation, Evection, Annual Equation, Parallactic Inequality

When the Moon is nearer than the Earth to the Sun (M_1, Fig. 154), the Moon is more attracted than the Earth, and therefore the disturbing force is towards the Sun (Art. 474). Its effect is, therefore, to accelerate the Moon from last quarter to conjunction, and to retard it from conjunction to first quarter. When the Moon is more distant than the Earth from the Sun (M_3, Fig. 154), it is less attracted than the Earth, and therefore the disturbing force is away from the Sun. Thus the Moon is accelerated from first quarter to full Moon, and retarded from full Moon to last quarter. Hence we see that the Moon's motion in each case must be swiftest at conjunction and opposition, and slowest at the quadratures. This phenomenon is known as the *Variation.*

The force *towards the Earth* is greatest at the quadratures, and least at the conjunction and opposition, since at the former the Sun pulls the Moon towards, and at the latter away from the Earth. Either cause tends to make the orbit more curved at the quadratures and less curved at the syzygies. For, if v is the velocity, R the radius of curvature, then v^2/R = normal acceleration. Hence R is greatest, and the orbit therefore least curved, when v is greatest, and the normal acceleration is least. The effect of this cause would be to distort the orbit, if it were a circle, into a slightly oval curve, which would be most flattened, and therefore narrowest (compare arguments of Arts. 98, 99), at the points towards and opposite the Sun; most rounded, and therefore broadest, at the points distant 90° from the Sun.

Of course the Moon's undisturbed orbit is not really circular, but elliptic, and far more elliptic than the oval into which a circular orbit would be thus distorted. However, the elliptic and variational inequalities act independently of each other. The effect of the latter on the Moon's longitude vanishes at all four quarters; it makes the longitude exceed its undisturbed value by 35′ at the first and fifth octants, and fall short of that value by 35′ at the third and seventh octants. The octants are arcs of 45° round the orbit, the first being midway between new Moon and first quarter.

The largest of the lunar inequalities is known as the evection, and depends on the changing position of the apse line with regard to the Sun in successive lunations. It has two effects :—

(i) the apse line, while advancing on the whole, has an oscillatory movement, and retrogrades about the times when perigee occurs at first and third quarters.

(ii) The eccentricity also oscillates, reaching its greatest value 0·066 when perigee occurs at new or full Moon, and its least value 0·044 when it occurs at first or third quarter.

These various results can be deduced, at least approximately, by elementary methods. But it will suffice here to confine ourselves to the backward motion of the apse when perigee occurs at first quarter. The disturbing force at the quarters is directed towards the Earth, and varies as M_2E (Fig. 154). As in Art. 476 it can be shown that an additional inward force at perigee makes the apse advance, while a similar force at apogee makes it retrograde. But since M_2E is greater at apogee than at perigee, and also since the Moon remains longer in the outer half of its orbit, the action at apogee preponderates and the apse on the whole retrogrades at these times. The tangential disturbing forces vanish at the quarters (see Fig. 154), but it can be shown that their action before and after the quarters assists the retrogradation of the apse.

The two evection effects may be combined into a single term in the longitude amounting to 77′, with a period of 31·812 days, and a term in the radius vector with the same period. The period of the variation is half a lunation or 14·765 days. It is worthy of note that if a total eclipse of the Sun occurs at perigee, the elliptic inequality, variation and evection all combine to bring the Moon nearer to the Earth, and so to increase the duration of totality, which may amount to 7m. 40s. in the most favourable case. The duration exceeds 7m. in June 1937 and June 1955 (see Art. 227).

The Sun's disturbing force is greatest when the Sun is nearest, and least when the Sun is furthest. These fluctuations, between perihelion and aphelion, give rise to another perturbation, called the *annual equation,* whose most noticeable effect consists in the consequent variations in the length of the month (Art. 476).

If, instead of resorting to a first approximation, we employ more accurate expressions for the Sun's disturbing force on the Moon, it is evident that this force is greater when the moon is near conjunction than at the corresponding position near opposition; just as the disturbing force which produces the tides is really greater under the Moon than at the opposite point. Hence the Moon is more disturbed from last quarter through new Moon to first quarter than from first quarter through full Moon to last quarter. Hence the time of first quarter is slightly retarded, and that of last quarter accelerated. This is called the Moon's *Parallactic Inequality*. Its amount is proportional to kSd^2/r^4, instead of kSd/r^3 (like the other perturbations). For many reasons this perturbation is of considerable use in determinations of the Sun's mass and distance.*

478. Planetary Perturbations

The Sun's mass is so great, compared with the masses of the planets, that the orbital motion of one planet about the Sun is but slightly affected by the attraction of any other planet. The mutual attractions of the planets, and their actions on the Sun, give rise to small *planetary perturbations*, which cause each planet to diverge slowly from its elliptical orbit, besides accelerating or retarding its motion.

Since the orbital motions of the planets are all usually referred to the Sun as their common centre or "origin," and not to the centre of mass of the solar system, the perturbations of one planet, due to a second, depend, not on the actual acceleration produced by the latter, but on the differences of the accelerations which it produces on the former planet and on the Sun.

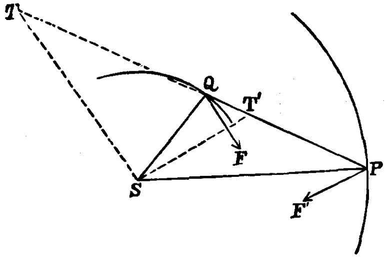

FIG. 156.

479. Geometrical Construction for the Disturbing Force

The approximate expressions, investigated in Art. 474, for the Sun's disturbing force on the Moon, are inapplicable to the disturbing force of one planet on another, because the distance of the *disturbing* body from the centre of motion is no longer very large, compared with that of the *disturbed* body. We must, therefore, adopt the following construction (Fig. 156):—

Let P, Q be two planets, of masses M, M'; S the Sun. Then the planet P produces an acceleration kM/PQ^2 on Q along QP, and an acceleration kM/PS^2 on S along SP. To find the acceleration of Q, relative to S, due to this cause, take a point T on PQ such that $PT : PS = PS^2 : PQ^2$. Then the accelerations of S, Q, due to P, are $kM . SP/SP^3$ and $kM . TP/SP^3$ respectively. Hence, by the triangle of accelerations, the acceleration of Q, relative to S, is represented in magnitude and direction by $kM . TS/SP^3$. Therefore the disturbing force per unit mass on Q, due to P, is parallel to TS, and of magnitude $kM \, . \, TS/SP^3$.

---

* The resulting displacement of the Moon exceeds 2′, a considerable quantity. It was from a study of this inequality that Hansen announced that Encke's value of the Sun's distance (95 million miles) was too great.

Similarly, if we take a point T' on QP such that $QT' : QS = QS^2 : QP^2$, the disturbing force per unit mass on P, due to Q, is parallel to $T'S$, and is of magnitude $kM' \, . \, T'S/SQ^3$.

The disturbing force on Q, due to P, and that on P, due to Q, are not equal and opposite, because they depend on the planets' attractions on S, as well as on their mutual attractions.

When $PQ = PS$, the points Q, T evidently coincide, and the disturbing force on Q is along the radius vector QS. When $PQ < PS$, $PT > PQ$, so that the disturbing force on Q tends to pull Q about S (as in Fig. 156) *towards* P, and when $PQ > PS$, the disturbing force tends to push Q about S *away* from P.

Similarly, when $QP = QS$, the disturbing force on P is along PS. When $QP < QS$ it tends to pull P about S *towards* Q, and when $QP > QS$, it tends to push P about S *away* from Q.

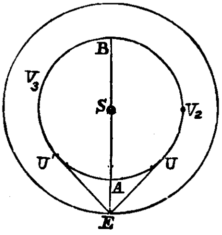

FIG. 157.

*480. Periodic Perturbations on an Interior Planet

Let us consider, in the first place, the perturbations produced by one planet E on another planet V, whose orbit is nearer the Sun; as, for example, the perturbations produced by the Earth on Venus, by Jupiter or Mars on the Earth, or by Neptune on Uranus.

Let A, B be the positions of the planet, relative to E, when in heliocentric conjunction and opposition respectively; U, U' points on the relative orbit such that $EU = EU' = ES$. (These points are near, but not quite coincident with the positions of greatest elongation). Then, if we only consider the component relative acceleration of V perpendicular to the radius vector VS, this vanishes when the planet is at U or U', as shown in the last paragraph.

The tangential acceleration also vanishes at A and B. Over the arc $U'AU$ the relative acceleration is *towards* E, therefore the planet's orbital velocity is accelerated from U' to A; similarly it is retarded from A to U.

Again, at a point V_2 on the arc UBU', the relative acceleration is away from the Earth, and this accelerates the planet's orbital velocity between U and B, and retards it between B and U'.

It follows that V is moving most swiftly at A and B, and most slowly at U and U'. Hence, if we neglect the eccentricity of the orbit, we see that the planet, after passing A, will shoot ahead of the position it would occupy if moving uniformly; thus the disturbing force displaces the planet *forwards* during its path from A to near U. Somewhere near U, when the planet is moving with its least velocity, it begins to lag behind the position it would occupy if moving uniformly; thus from near U to B the disturbing force displaces the planet *backwards*. Similarly it may be seen that from B to near U' the planet is displaced *forwards*, and from near U' to A it is displaced *backwards*.

The principal effect of the component of the disturbing force along the radius vector, is to cause rotation of the planet's apsides, as in the case of the Moon. The direction of their rotation depends on the direction of the force, and is not always direct.

Owing to the inclination of the planes of the orbits of E, V, the attraction of E, in general gives rise to a small component perpendicular to the plane of V's orbit, which is *always directed towards* the plane of E's orbit. This component produces *rotation of the line of nodes*, or line of intersection of the planes of the two orbits. This rotation is *always in the retrograde direction*, and is to be explained in exactly the same way as the rotation of the Moon's nodes.

It is thus a remarkable fact that since all the bodies in the solar system (except the satellites of Uranus, Neptune, outer ones of Jupiter and Saturn, and many comets) rotate in the direct direction, all the planes of rotation and revolution, and all their lines of intersection (*i.e.* the lines of nodes, and the lines of equinoxes) in the whole solar system, with the above exceptions) have a *retrograde* motion.

*481. Periodic Perturbations of an Exterior Planet

The accelerations and retardations produced by a planet E on one J, whose orbit is more remote from the Sun, during the course of a synodic period, may be investigated in a similar manner to the corresponding perturbations of an interior planet, assuming the orbits to be nearly circular.

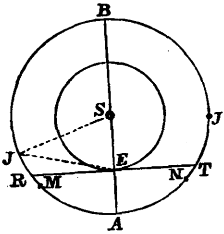

FIG. 158.

If SJ is less than $2SE$ there are two points M, N on the relative orbit at which $EM = EN = ES$. At these points the disturbing force is purely radial, and it appears, as before, that the planet J is accelerated from heliocentric conjunction A to M, and from heliocentric opposition B to N; retarded from N to A, and from M to B.

If $SJ > 2SE$, then $ES < EA$; hence the attraction of E is greater on the Sun than on J, and the disturbing force therefore always accelerates the planet J towards B. Thus the planet's orbital velocity increases from A to B, and decreases from B to A, and it is greatest at B and least at A. Therefore from B to A the planet is displaced in advance of its mean position, and from A to B falls behind its mean position.

The effects of the radial and orthogonal components of the disturbing force in altering the period and causing rotation of the apse line, and regression of the nodes, can be investigated in the same way for a superior as for an inferior planet.

*482. Inequalities of Long Period

If the orbits of the planets were circular (except for the effects of perturbations), and in the same plane, their mutual perturbations would be strictly periodic, and would recur once in every synodic period. Owing, however, to the inclinations and eccentricities of the orbits, this is not the case. The mutual attractions of the planets produce small changes in the eccentricities and inclinations, and even in their periodic times, which depend on the positions of conjunction and opposition relative to the lines of nodes and apses. Neglecting the motion of these latter lines, the perturbations would only be strictly periodic if the periodic times of two planets were commensurable, the period of recurrence being the least common multiple of the periods of the two planets. But when the periodic times of two planets are *nearly but not quite* in the proportion of two small whole numbers, *inequalities of long period* are produced, whose effects may, in the course of time, become considerable.

Thus, for example, the periodic times of Jupiter and Saturn are *very nearly but not quite* in the proportion of 2 to 5. If the proportionality were exact, then 5 revolutions of Jupiter would take the same time as 2 revolutions of Saturn; and, since Jupiter would thus gain three revolutions on Saturn, the interval would contain 3 synodic periods. Thus, after 3 synodic periods had elapsed from conjunction, another conjunction would occur at exactly the same place in the two orbits, and the perturbations would be strictly periodical.

But, in reality, the proportionality of periods is not exact; the positions of every third conjunction are very slowly revolving in the direct direction. They perform a complete revolution in 2,640 years. But there are three points on the orbits at which conjunctions occur, and these are distant very nearly 120° from one another. When the positions of conjunction have revolved through 120°, they will again occur at the same points on the orbits, and the perturbations will be of the same kind as initially. The time required is one-third the above period, or 880 years. The last maximum displacement of the planets occurred about 1790, when Jupiter was 19·8′ in advance of the undisturbed position, Saturn 48·4′ behind it. The displacements will have zero value about 2010.

Again, the periodic times of Venus and the Earth are nearly in the proportion of 8 to 13; consequently 5 conjunctions of Venus occur in almost exactly 8 years, thus giving rise to perturbations having a period of 8 years. But the proportion is not exact, and, consequently, there are other mutual perturbations having a very long period.

One of the most important *secular* perturbations is the alternate increase and decrease in the eccentricity of the Earth's orbit. At the present time the orbit is becoming gradually more nearly circular. In about 24,000 years the eccentricity will be a minimum, and will then once more begin to increase.

483. Gravitational Methods of Finding the Sun's Distance

The Earth's perturbations on Mars and Venus furnish a method of finding the Sun's distance. For the magnitude of these perturbations depends on the ratio of the Earth's mass, or rather the sum of the masses of the Earth and Moon (since both are instrumental in producing the perturbations), to the Sun's mass. Hence, if S, M, m denote the masses of the Sun, Earth, and Moon, it is possible, from observations of these perturbations, to find the ratio of $(M+m):S$.

But, if r, d be the distances of the Sun and Moon from the Earth, T and Y the length of the sidereal lunar month and year, we have, by Kepler's corrected Third Law,

$$(M+m)\,T^2 : (S+M+m)\,Y^2 = d^3 : r^3\,;$$

whence the ratio of r to d is known. If, now, the Moon's distance d be determined by observation in any of the ways described in Chapter IX, or by the gravitational method of Art. 433, the Sun's distance r may be immediately found. This method is not capable, however, of giving the Sun's distance with an accuracy comparable to what is attainable by the geometrical methods. It was used by Levenier in 1782. From observations of certain perturbations of Venus he found a value of 8·86″ for the Sun's parallax, while the rotation of the apse line of Mars gave the value 8·87″.

The lunar perturbations also furnish data for determining the Sun's distance, the principal of these being the parallactic inequality of the Moon (Art. 478). This is best determined from observations of the times of occultations of stars by the Moon, which determine the position of the Moon with a higher accuracy than meridian observations. The coefficient of the parallactic inequality is about 126″ or some 14 times greater than the solar parallax; the advantage of this method of determining the solar parallax is that the uncertainty of the results is only 1/14th of the uncertainty of the determination of the parallactic inequality. But this determination depends essentially upon the comparison of the positions of the Moon in different parts of its orbit. Between new Moon and full Moon, a star, when occulted by the Moon, disappears at the dark limb and reappears at the bright limb; between full moon and new Moon it disappears at the bright limb and reappears at the dark limb. Observations at the bright limb are apt to be unreliable, except for very bright stars, as the star may be lost in the Moon's glare; there is also a possibility of systematic error in observations of reappearance, because of the possibility that the star may not be seen until a moment or two after its reappearance. If observations at the dark limb only are used, the angular semidiameter of the Moon at mean distance must be known with great accuracy; any error in this quantity will have a systematic effect on the determination of the parallactic inequality. On the other hand, observations of both disappearance and reappearance of the same star provide a determination of the Moon's semidiameter, but as observations at the bright limb are involved, there is a risk of systematic error. It is therefore difficult to ensure that the determination of the Sun's parallax by this method is free from systematic errors. By this method Cowell and Brown obtained a value of 8·78″; Battersmann obtained 8·79″ and Spencer Jones obtained 8·796″.

484. Determination of Masses

The mass of any planet which is not furnished with a satellite can be determined in terms of the Sun's mass by means of the perturbations it produces on the orbits of other planets. The amount of these perturbations is always proportional to the disturbing force, and this again is proportional to the *mass* of the disturbing planet. In this manner the mass of Venus has been found to be about 1/408,000 of the Sun's mass, and that of Mercury about 1/10,500,000.

485. The Discovery of Neptune

The narrative of the discovery of Neptune is one of the most striking and remarkable in the annals of theoretical astronomy, and forms a fitting conclusion to this chapter.

In 1795, or about 14 years after its discovery, the planet Uranus was observed to deviate slightly from its predicted position, the observed longitude becoming slightly greater than that given by theory. The discrepancy increased till 1822, when Uranus appeared to undergo a retardation, and again to approach its predicted position. About 1830 the observed and computed longitudes of the planet were equal, but the retardation still continued, and by 1845 Uranus had fallen behind its computed position by nearly 2′.

As early as 1821, Alexis Bouvard pointed out that these discrepancies indicated the existence of a planet exterior to Uranus, but the matter remained in abeyance until 1844, when J. C. Adams, in Cambridge, and Le Verrier, in Paris, independently and almost simultaneously, undertook the problem of determining the position, orbit, and mass of an unknown planet which would give rise to the observed perturbations. Adams was undoubtedly the first by a few months in performing the computations, but the actual search for the planet at the observatory of Cambridge was delayed from pressure of other work. Meanwhile Le Verrier sent the results of his calculations to Galle, of Berlin, who, within a few hours of receiving them, turned his telescope towards the place predicted for the planet, and found it within about 52′ of that place. Subsequent examination of star catalogues showed that the planet had been previously observed on several occasions, but had always been mistaken for a fixed star.

It will be seen from Art. 481 that the acceleration of Uranus up to 1822, and its subsequent retardation, are at once accounted for by supposing an exterior planet to be in heliocentric conjunction with Uranus about the year 1822. But Adams and Le Verrier sought for far more accurate details concerning the planet. At the same time the data afforded by the observed perturbations of Uranus were insufficient to determine all the unknown elements of the new planet's orbit, and therefore the problem admitted of any number of possible solutions. In other words, any number of different planets could have produced the observed perturbations.

To render the problem less indeterminate, however, both astronomers assumed that the disturbing body moved nearly in the plane of the ecliptic and in a nearly circular orbit, and that its distance from the Sun followed Bode's Law.

The latter assumption led to considerable errors, including an erroneous estimation of the planet's period by Kepler's Third Law. For when Neptune was observed, its distance was found to be only

30·04 times the Earth's distance, instead of 38·8 times, as it would have been according to Bode's Law. Nevertheless, the actual planet was subsequently found to account fairly well for all the observed perturbations of Uranus.

The discovery of Neptune affords most powerful evidence of the truth of the Law of Gravitation, and so indeed does the theory of perturbations generally. The fact that the planetary motions are observed to agree closely with theory, that computations of astronomical constants (such as the Sun's and Moon's distances), based upon gravitational methods, agree so closely with those obtained by other methods, when possible errors of observation are taken into account, affords an indisputable proof that the resultant acceleration of any body in the solar system can always be resolved into components directed to the various other bodies, each component being proportional directly to the mass and inversely to the square of the distance of the corresponding body. Such a truth cannot be regarded as a fortuitous coincidence; it can only be explained by supposing every body in the universe to attract every other body in accordance with Newton's Law of Universal Gravitation.

From the analysis of the small remaining residuals in the position of Uranus by Pickering and Lowell, the approximate position of a hypothetical planet beyond Neptune, which was assumed to produce them, was predicted. Search was made at the Lowell Observatory and a new planet, Pluto, was found there on January 21st, 1930. It is of the 15th magnitude and its distance from the Sun is 39·5 astronomical units. The mass of Pluto is not accurately known but is certainly much smaller than the mass predicted by Lowell; the perturbations of Uranus produced by a planet with a mass of the order of that of the Earth would be smaller than the uncertainties of the determination of position of Uranus. It therefore seems that the discovery of Pluto was the result of a fortunate chance coincidence between the position predicted by Lowell and the position in which Pluto happened to be at the time.

EXAMPLES

1. If the Sun's parallax be 8·80″, and the Sun's displacement at first quarter of Moon 6·52″, calculate the mass of the Moon, the Earth's radius being taken as 3,963 miles.

2. Supposing the Moon's distance to be 60 of the Earth's radii, and the Sun's distance to be 400 times that of the Moon, while his mass is 25,600,000 times the Moon's mass, compare the effects of the Sun and Moon in creating a tide at the equator, in the event of a total eclipse occurring at the equinox.

3. If the Earth and Moon were only half their present distance from the Sun, what difference would this make to the tides? Calculate roughly what the

proportion between the Sun's tide-raising power and the Moon's would then be, assuming that the Moon's distance from the Earth remained the same as at present.

4. Taking the Moon's mass as $\frac{1}{80}$ of the Earth's, and its distance as 60 times the Earth's radius, show that the Moon's tide-raising force increases the intensity of gravity by 1/17,280,000 when the Moon is on the horizon, and that it decreases the intensity of gravity by 1/8,640,000 when the Moon is in the zenith.

5. Compare the heights of the solar tides on the Earth and on Mercury, taking the density of Mercury to be two-thirds of that of the Earth, its diameter ·38 of the Earth's diameter, and its solar distance ·38 of the Earth's solar distance.

6. Explain how the pushing forward of the Moon by the tidal wave enlarges the Moon's orbit.

EXAMINATION PAPER

1. Show that the Moon's orbit is everywhere concave to the Sun.

2. Show that the tide-raising force of a heavenly body is nearly proportional to its (mass) $\div$ (distance)^3.

3. How is it that we have tides on opposite sides of the Earth at once?

4. Explain the production of the tides on the equilibrium theory.

5. Define the terms *spring tide, neap tide, priming* and *lagging, establishment of the port, lunar time.*

6. What is meant by the expression "Luni-solar Precession"? Describe the action of the Sun and of the Moon in causing the Precession.

7. Give a general description of Precession. Does precession change the position of (*a*) the equator, (*b*) the ecliptic among the stars?

8. Describe *nutation.* What is the cause of Lunar Nutation? What is meant by the *equation of the equinoxes*?

9. Give a brief account of the discovery of Neptune.

10. Explain how the retrograde motion of the Moon's nodes is caused by the Sun's attraction on the Earth and Moon.

APPENDIX

I.—PROPERTIES OF THE ELLIPSE

For the benefit of those readers who have not studied Conic Sections, we subjoin a list of those properties of the ellipse which are of astronomical importance. The proofs are given in books on Conic Sections.

(1) Definition.—A **conic section** is a curve such that the distance of every point on it from a certain fixed point is proportional to its perpendicular distance from a certain fixed straight line.

The fixed point is called the **focus**, the fixed line is called the **directrix**, and the constant ratio of distances is called the **eccentricity**.

If this constant ratio or eccentricity is less than unity, the curve is called an **ellipse**. In this case the curve assumes the form of a closed oval, as shown in the figure.

If S is the focus, and if from A, P, L, P', A', etc., any points on the curve, perpendiculars AX, PM, etc., be drawn on the directrix, and if the eccentricity be e, the definition requires that

$$e = \frac{SA}{AX} = \frac{SP}{PM} = \frac{SL}{LK}$$

$$= \frac{SP'}{P'M'} = \frac{SA'}{A'X} = \text{etc.},$$

and that e is less than unity.

The other conic sections, the *parabola* and *hyperbola*, are defined by the same property, save that in the former $e = 1$, and in the latter $e > 1$; but they are of little astronomical importance, except as representing the paths described by non-periodic comets.

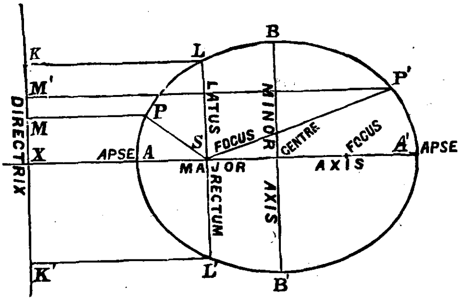

Fig. 159.

(2) An ellipse has **two foci** (each focus having a corresponding directrix), and the sum of the distances of any point from the two foci is constant.

Thus in Fig. 160, S, H are the two foci, and the sum $SP + PH$ is the same for all positions of P on the curve.

From this property an ellipse may easily be drawn. For, let two small pins be fixed at S and H, and let a loop of string SPH be passed over them and round a pencil-point P; then, if the pencil be moved so as to keep the string tight, its point P will trace out an ellipse. For $SP + PH + HS =$ constant, and so $SP + PH =$ constant.

(3) For all positions of P on the ellipse, SP is inversely proportional to $1 + e \cos ASP$, so that

$$SP\,(1 + e \cos ASP) = l = \text{constant},$$

e being the eccentricity and SA the line through S perpendicular to the directrix.

(4) The line joining the two foci is perpendicular to the directrices.

The portion of this line (AA'), bounded by the curve, is called the **major axis** or **axis major.** Its middle point C is called the **centre** and the curve is symmetrical about this point.

The line BCB', drawn through the centre perpendicular to ACA' and terminated by the curve, is called the **minor axis** or **axis minor.** The lengths of the major and minor axes are usually denoted by $2a$ and $2b$ respectively.

(5) The extremities A, A' of the major axis are called the **apses** or **apsides.** Since, by (2), $SP + HP$ is constant, therefore, taking P at A or A'.

$$\begin{aligned} SP + HP &= SA + HA = SA' + HA' \\ &= \tfrac{1}{2}(SA + HA + SA' + HA') \text{ evidently} \\ &= AA' = 2a. \end{aligned}$$

Taking P at B, $SB + HB = 2a$; and so

$$SB = HB = a = CA.$$

(6) The eccentricity $e = CS/CA$; so that $CS = e \,.\, CA$, and :—

$$b^2 = CB^2 = SB^2 - CS^2 = a^2 - a^2e^2 = a^2(1 - e^2);$$
$$e^2 = (a^2 - b^2)/a^2.$$

Hence also :—

$$SA = CA - CS = a(1 - e) \text{ and } SA' = CA' + CS = a(1 \times e).$$

(7) The **latus rectum** is the chord LSL' drawn through the focus perpendicular to the major axis AA'. Its length is $2l$, where $l = a(1 - e^2)$. Also l is the constant of (3), for when P coincides with L, $ASP = 90°$; so that $\cos ASL = 0$, and $SL = l$. [Fig. 159].

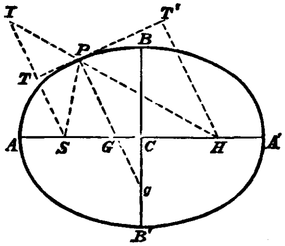

FIG. 170.

(8) The **tangent** $T'PT$ and **normal** PGg, at P, bisect respectively the exterior and interior angles (SPI, SPH) formed by the lines SP, HP.

(9) If the normal meets the major and minor axes in G, g,

$$PG : Pg = CB^2 : CA^2 \quad (\equiv b^2 : a^2).$$

(10) If ST, drawn perpendicular on the tangent at P, meets HP produced in I, then evidently $SP = IP$; and

$$HI = SP + HP = 2a \quad \text{[by (2)]}.$$

If HT' is the other focal perpendicular on the tangent, it is known that rectangle $ST.HT' = \text{constant} = b^2$.

(11) Relation between the focal radius *SP* and the focal perpendicular on the tangent *ST*

Let $SP = r$, $ST = p$. Thus :—

$$\cos TIP = \cos TSP = p/r.$$

By Trigonometry, $SH^2 = IS^2 + IH^2 - 2 \,.\, IS \,.\, IH \cos SIH$; giving :—

$$4a^2e^2 = 4p^2 + 4a^2 - 8pa \times p/r;$$

or
$$\frac{a^2(1 - e^2)}{p^2} = \frac{2a}{r} - 1,$$

or by (6)
$$\frac{b^2}{p^2} = \frac{2a}{r} - 1 = \frac{2a - r}{r} = \frac{HP}{SP}.$$

This may also be proved from the similarity of the triangles SPT, HPT', which gives $ST \,.\, HT' = SP : HP$; whence :—

$$ST^2 : ST \,.\, HT' = SP : HP \text{ and } ST \,.\, HT' = b^2 \text{ (10)};$$
$$\text{and } p^2 : b^2 = r : 2a - r.$$

(12) If a circular cone (*i.e.* either a right or oblique cone on a circular base) is cut in two by a plane not intersecting its base, the curve of section is an *ellipse*. More generally, the form of a circle represented in perspective, or the oval shadow cast by a spherical globe or a circular disc on any plane, are *ellipses*. A circle is a particular form of ellipse for the case where $b = a$ and $e = 0$.

(13) The *area of the ellipse* is πab.

II.—TABLE OF ASTRONOMICAL CONSTANTS

(Approximate values, calculated, when variable, for the Spring Equinox, A.D. 1900).

THE CELESTIAL SHPERE

Latitude of London (Greenwich Observatory),	51° 28′ 38″.
,, Cambridge Observatory,	52° 12′ 51″.
Obliquity of Ecliptic,	23° 27′ 8″.

OPTICAL CONSTANTS

Coefficient of Astronomical Refraction,	57″.
Horizontal Refraction,	33′.
Coefficient of Aberration,	20·471″.
Velocity of Light in miles per second,	186,285.
,, ,, ,, metres ,,	299,796,000.
Equation of Light,	8m. 18·6s.

TIME CONSTANTS

Sidereal Day in mean solar units = 1 — 1/366¼ days = 23h. 56m. 4·1s.
Mean Solar Day in sidereal units = 1 + 1/365¼ days = 24h. 3m. 56·5s.

Year, Tropical, in mean time,	365d. 5h. 48m. 45·98s.
,, Sidereal, ,,	365d. 6h. 9m. 8·97s.
,, Anomalistic, ,,	365d. 6h. 13m. 48·09s.
,, Civil, if the number of the year is not divisible by 4, or, if it be divisible by 100, but not by 400,	365 days.
In other cases,	366 ,,
Month, Sidereal,	27·32166d. = 27d. 7h. 43m. 11·4s.
,, Synodic,	29·53059d. = 29d. 12h. 44m. 30s.
Metonic Cycle,	235 Synodic Months = 6939·69d. = 19 tropical years (all but 2 hours).
Period of Rotation of Moon's Nodes (Sidereal),	6793·391d. = 18·60 yr.
,, ,, ,, ,, (Synodic),	346·644d. = 346d. 14½h.
,, ,, ,, Apsides (Sidereal),	3232·575d. = 8·85 yr.
,, ,, ,, ,, (Synodic),	411·74d.
Saros,	223 Synodic Months = 6585·321d. = 18·0300yr. = 18 yr. 10 or 11 days. = 19 Synodic periods of Moon's Nodes (very nearly). = 16 ,, ,, ,, Apsides (nearly).
Equation of Time, Maximum due to Eccentricity,	7·6m.
,, ,, ,, ,, Obliquity,	9·9m.

THE EARTH

Equatorial Radius,		3,963·35 miles.
Polar ,,		3,950·01 ,,
Mean ,,		3,958·90 ,,
Equatorial Circumference,		24,902·4 ,,
	360 × 60 =	21,600 geographical miles.
	$4 \times 10^7 =$	40,076,494 metres.
Ellipticity or Compression,		1 ÷ 297.
Eccentricity,		·0820.
Density (Water = 1),		5·527.
Mass,		$5,880 \times 10^{18}$ tons.
Mean Acceleration of Gravity in ft. per sec. per sec.,		32·18 (at Paris).
Ratio of Centrifugal Force to Gravity at Equator,		1 ÷ 289.
Eccentricity of its Orbit,		1 ÷ 60.
Annual Progressive Motion of Apse Line,		11·25″.
,, Retrograde Motion of Equinoxes (Precession),		50·26″.
Period of Precession,		25,695 years.
,, Nutation,		18·6 ,,
Greatest change in Obliquity due to Nutation,		9·23″.
Maximum Equation of Equinoxes,		17·1″.

THE SUN

Mean Parallax,	8·790″.
,, Angular Semi-diameter,	16′ 1″.
,, Distance in miles,	93,005,000.
Diameter in miles,	866,600.
,, in Earth's diameters,	109.
Density in terms of Earth's,	¼.
,, (taking water as 1),	1·4.
Mass in terms of Earth's,	331,100.
Period of Axial Rotation at Equator,	25d. 5h. 37m.

THE MOON

Mean Parallax,	57′ 2·63″.
,, Angular Semi-diameter,	15′ 34″.
,, Distance in miles,	238,862.
,, ,, in Earth's radii,	60·27.
,, ,, in terms of Sun's distance,	1/389.
Diameter in miles,	2,160.
,, in terms of Earth's,	3/11.
Density in terms of Earth's,	·61.
,, (taking water as 1),	3·4.
Mass, in terms of Earth's,	1/81½.
Eccentricity of Orbit,	1/18.
Inclination of Orbit to Ecliptic,	5° 8′ 40″.
Ecliptic Limits, Lunar,	11° 38′ and 9° 39′.
,, ,, Solar,	17° 25′ and 15° 23′.
Tide-raising force of Moon in term of Sun's,	7/3.

ANSWERS

Note.—Where only rough values of the astronomical data are given in the questions, the answers can only be regarded as rough approximations, not as highly accurate results. It is impossibie to calculate results correctly to a greater number of significant figures than are given in the data employed, and any extra figures so calculated will necessarily be incorrect. As the use of working examples is to learn astronomy rather than arithmetic, it is advisable to supply from memory the rough values of such astronomical constants as are not given in the questions. These values will thus be remembered more easily than if the more accurate values were taken from the tables on pages 375, 376, though reference to the latter should be made until the student is familiar with them.

Examples (p. 36).

1. Only their *relative* positions are stated; these do not completely fix them.

2. 6 P.M., 6 A.M.; on the meridian. **8.** On September 19th.

9. (i) Early in July; (ii) middle of June—the Sun passes it about June 26th.

10. 304° = 20h. 16m.; at 8h. 17m. P.M.

11. Near the S. horizon about 10 P.M. early in October.

12. 38° 27′, 51° 33′, 28° 5′, or if Sun transits N. of zenith 8° 27′, 81° 33′, 58° 5′.

Examination Papers (p. 37).

7. 30°. **8.** 61° 58′ 37″, 15° 4′ 21″. **9.** 6h. 43m. 16s. (roughly).

10. The figure should make *Capella* slightly W. of N., altitude about 15°; *a Lyræ* a little S.E. of zenith, altitude about 75°; *a Scorpii* slightly W. of S., altitude about 12°; *a Ursæ Majoris* N.W., altitude about 60°.

Examples (p. 65).

1. Retrograde. **3.** + 3·9m. **6.** 347 centuries exactly.

7. Star's hour angle = 4h. 43m. 31s., N.P.D. = 53°.

8. October 28th, 15h. 39m. 27·32s.

10. 12h. 27m. 13·26s. at Louisville = 18h. 9m. 13·26s. at Greenwich.

Examination Paper (p. 65).

4. — 10m.; morning 20m. longer. **5.** See § 58.

8. (i) 7h. 13m. 5s.; (ii) 7h. 12m. 48s. **9.** June 26th.

10. 1824, 1852, 1880, 1920, 1948.

Examples (p. 80).

2. 4,267 ft.

3. a°N., L° — 90°W. and a°S., L° + 90° W., if L° = W. longitude given place.

5. 13m. **6.** 39·8 miles. **7.** 3960. **8.** 6084 ft. **10.** 49′ 6″ per hour.

Miscellaneous Questions (p. 81).

2. N.P.D. = 85°, hour angle = 30° W.

3. Because declination circle has not been defined.

5. 22h. 40m., 9h. 20m., 14h. 0m., 19h. 36m. **10.** 52″.

Examination Paper (p. 82).

1. 24,840 miles, 3,953 miles.
2. 3·285 ft., 6,084 ft., 1·69 ft. per second. 3. 50·7 ft.
5. 3,437,700 fathoms, 6,366,200 metres (roughly), $1{,}851{\cdot}\dot{8}5\dot{1}$ metres.

Examples (p. 106).

5. 45°. 7. Star, 6h. 15m. 26·35s.; Sun, 0h. 13m. 51·90s.
10. 3481 : 3721, or 29 : 31 nearly.

Examination Paper (p. 107).

3. See §§ 114, 134.
8. 0h. 36m. 21·26s. (Note that the clock has a losing rate of 3m. 22·05s. on sidereal time; it gives solar time approximately.)

Examples (p. 119).

3. 3,963 miles.
4. From 50° 9′ 47″ to 49° 59′ 55″ (refraction at altitude 5° is 9′ 47″ by tables).
7. 84° 33′; 377 miles or 327 nautical miles.

Examination Paper (p. 120).

4. 6′. 7. 44° 58′ 54″. 10. 1h. 12m.

Examples (p. 137).

2. — 34·9″ at 6h.; + 34·9″ at 18h., additive to R.A. 3. 16′ 44″.
4. 7° 9′ 51″ S. 6. 432,000 miles. 7. 2,250 miles.
8. 9,282,000 and 92,820,000 million miles respectively.
9. 37·8 million million miles = $37{\cdot}8\times10^{12}$ miles. 10. $5 : 0{\cdot}6\pi$ or 2·65 : 1.

Examples (p. 150).

1. It will always appear half-way between its actual direction and a point on the ecliptic 90° behind Sun. Path is roughly a small circle of angular radius 45°.
2. 4° 35′.
4. (i) On ecliptic 90° from Sun. (ii) In same or opposite direction to Sun. Effects greatest along great circles distant 90° from these points.
5. (i) At either pole of ecliptic. (ii) In ecliptic.
7. Jan. 21st, 10·25″ Eastwards; Feb., 17·75″ E.; Mar., 20·50″ E; April, 17·75″ E.; May, 10·25″ E.; June, 0″; July, 10·25″ Westwards: Aug., 17·75 W.; Sept., 20·50″ W.; Oct., 17·75″ W.; Nov., 10·25″ W.; Dec., 0″.
9. 973,800 miles.

Examination Paper (p. 151).

2. 93 million miles; 20·51″. 3. 4,320 miles; 61′ 24″.

Examples (p. 171).

1. ·0597. 2. (a) 43·1°; 19·7°; 66·6°. (b) 32·9°; 9·4°; 56·3°.
3. At 6 p.m.; about same length as Midsummer Sun, *i.e.* 16½h.
4. See § 197. 5. 8′ 48″. 6. 10d. 4¼h. at noon.
7. Gibbous, bright limb turned slightly below direction of W. Hour angle = 30°, Dec. = 0°.

Examples (p. 186).

1. 23½° S. 2. Favourable if moon passes from N. to S. at ecliptic on March 21st.

Examples (p. 207).

1. 291·96 days, or, if conjunctions are of the same kind, 583·92 days.
2. 40°. 3. 19 : 6, or nearly 3 : 1. 4. 10·9h., 134h.
5. $p+P-s$ with notation of § 228. 6. 888 million miles, 164 yrs.
7. 6 months; $\sqrt[3]{\frac{1}{4}}$ or ·63 of Earth's mean distance. 8. 398 days.
9. $\frac{3}{8}$ of a year = 137 days.
10. Stationary at heliocentric conjunction only, never retrograde.

Examination Paper (p. 208).

3. $1\frac{1}{29}$ years = 378 days.
4. See §§ 262, 263. The alterations in Venus's brightness are really not inconsiderable (see Ex. 3, p. 207).
6. Most rapid approach at quadrature; velocity that with which the *Earth* would describe its orbit in synodic period.
9. 287 days.
10. Draw the circular orbits about ☉, radii 4, 7, 10, 16, 52 (§ 252). The *heliocentric* longitudes (measured from ☉♈) are roughly as follows: ☿ 153°, ♀ 175°, ⊕ 220°, ♂ 20°, ♃ 211°. The ☾ should be drawn close to ⊕ at an elongation ☉ ⊕ ☾ = 90° at first quarter.

Examples (p. 222).

4. R.A. 6h., Dec. 6° 54′ N. 5. R.A. 0h., Dec. 0°.
6. R.A. 6h., Dec. 66° 29′ S. or R.A. 18h., Dec. 66° 29′ N.
7. R.A. 0h. 2m. 33·590s., Dec. 45° 16′ 42·35″ N.
8. R.A. 5h. 58m. 33·177s., Dec. 45° S.
9. R.A. 5h. 35m. 14·58s.; Dec. 76° 24′ 7·9″ S.

Examples (p. 245).

5. Interval = 12 sidereal hours. 7. 2° 29′ 58·5″. 8. + 1·35s.
9. Level error + 9·76″; azimuth error + 19·42″. 10. 12° 39′ 16″ N.
11. 17h. 29m. 52·42s.

Examination Paper (p. 246).

6. Positive. 9. 1m. 2·52s., + 0·71s.

Examples (p. 278).

1. 37° 49′. 2. 51° 44′ 26·09″.
4. 50° 54′ 58·6″ or 60° 43′ 23·6″ according as star transits N. or S. of zenith.
5. — 10m., *i.e.* 10m. fast. 6. 2° 32′ W.
12. Long. 3h. 33m. 52s. E.; lat. 13° 52′ N.
13. Long. 0h. 32m. 53s. E.; lat. 26° 18′ S.
14. 52° 33·3′ N.; 40° 4·0′ W.

Examples (p. 291).

3. It will always appear half-way between its actual direction and a point on the ecliptic 90° behind the Sun. Path is roughly a small circle of angular radius 45°.

4. 9240 miles. **5.** 1·344.

Miscellaneous Questions (p. 291).

5. In the autumn. **6.** 17d. 5h.; star is on equator, hour angle 60° E.

7. $1 : \sqrt{7} : 7$. **8.** 24h. 50m. 30s. mean units = 24h. 54m. 35s. sidereal units.

9. At the equinoxes.

Examination Paper (p. 292).

6. (a) About March 21st at 6 p.m. and Sept. 23rd at 6 a.m. (b) About June 21st at 6 p.m. and Dec. 22nd at 6 a.m., or as near these times as the beginning and end of astronomical twilight permit.

Examples (p. 308).

1. $12 \sqrt{2}$ sidereal hours = 16h. 58m. 5s. sidereal time.

2. Pendulum revolving in direction of hands of watch will have less velocity in S. hemisphere.

7. Increased (i) 59° 54′ 51″; (ii) 60° 15′ 27″. **12.** 109 lb.

Examination Paper (p. 309).

3. By observing deviation of a projectile (§ 400), or by § 397 or § 399.

4. $16\sqrt{3} = 27{\cdot}7157$ sidereal hours = 1d. 3h. 33m. mean time.

5. 3·368 cm. per sec. per sec.; $\frac{1}{239}$. **9.** See § 409.

Examples (p. 334).

1. 3·40 miles per sec. **2.** 15½ ft., or, if $g = 32{\cdot}2$, 15·576 ft.

3. 5h. 35m. **4.** 5·39 days. **5.** 2,959,000. **11.** 8·98″.

13. The distances from the centre of the Sun are 457,579 miles, 457,579 + 278 miles, and 457,579 — 281 miles; but these results can only be considered as approximate.

14. 32·155 greater, owing to attraction of mountain.

17. ·253 of Earth's density; 1·415, taking water = 1.

18. 894 poundals.

20. From the first the path would be concave to the Sun (see § 448); at the beginning the strong perturbations produced by the Earth would prevent the orbit from being an ellipse; but after a time it would approximate to an ellipse, somewhat more eccentric than the Earth's present orbit, with its aphelion near the place where it left the Earth.

Examples (p. 371).

1. $\frac{1}{80{\cdot}34}$. **2.** 2 : 5.

3. 24 : 7, by Ex. 1, § 450 Cor., or 16 : 5, using result of last example.

5. Tide on Mercury is higher in proportion 3 : ·2888, or 135 : 13, or 10 : 1 nearly.

Examination Paper (p. 372).

7. (*a*) Yes; (*b*) No.

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