THE DYNAMICS OF
MECHANICAL FLIGHT

Lectures delivered at the Imperial College of Science and Technology, March, 1910 and 1911

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INTRODUCTION

The lectures were delivered in the Imperial College of Science and Technology in March 1910 and 1911, under the title "The Dynamics of Mechanical Flight," and they are given here in the form in which they were delivered.

The subject was then beginning to take hold of the public imagination, consequent on Blériot's feat of crossing the Channel on July 25, 1909, and the great strides made in the interval since in Mechanical Flight.

The possibility of Human Flight has been an obsession of the imagination of Man from the earliest times recorded, for which an extensive article should be consulted in the Denkschrift der I.L.A. (der Internationalen Luftschiffahrt Austellung) Frankfurt, 1910, Band I, p. 118, *Flugprobleme in Mythus Sage und Dichtung*.

In the Greek Mythology, Demeter rides in a car drawn by flying dragons. and Homer describes the flight of Hera in her chariot, Iliad V., 750; and then there is the legend of Icarus and his father Dædalus "who taught his son the office of a fowl, and yet for all his wings the fool was drowned"; and another legend of Archytas of Tarentum, "aerias tentasse domos" with his invention of a flying mechanical bird.

Æschylus in his Prometheus has described the arrival in a flying chariot of the chorus of the Ocean Nymphs, followed by their father Oceanus on a four-legged bird, anxious to return in a single flight from the Scythian desert and the Caucasus to beyond the Pillars of Hercules and over the Atlantic.

The fabulous Life of Alexander the Great, by the Pseudo-Callisthenes, was a favourite book of the Middle Ages. Alexander is described here to have placed a yoke on the neck of two strong eagles that had been kept fasting for three days. When Alexander took his seat on the yoke, the eagles flew up with him in the air, wherever he pointed his spear, because the head of it carried a large lump of liver. This flying machine of Alexander is illustrated in manuscripts with four or eight eagles or griffins, shown in the vignette on p. 7, and representations in sculpture are to be seen in St. Mark's, Venice, and the cathedral of Basle.

The Tartar and Chinese legend of the Bronze Horse is reproduced in the Squires Tale of Chaucer—

> "Him who left half told
> The story of Cambuscan bold,
> And of the wondrous horse of brass,
> On which the Tartar King did ride" ·

known to us more recently in the operatic version of the *Cheval de bronze* of Scribe and Auber; parodied also by Cervantes in Don Quixote II. in the description of Clavileño.

Chaucer goes into detail of the Magic Steed, but he did not realise the difficulty of the mechanical problem in his jaunty description—

> "This same stede shall bere you ever-more
> With-outen harm, til ye be ther yow leste,
> Though that ye slepen on his bak or reste,
> And turn ayeyn, with wrything of a pin.
>
> .
>
> But whan yow list to ryden any-where,
> Ye moten trille a pin, stant in his ere—
> Bid him descend, and trille another pin.
>
> . . .
>
> Trille this pin, and he wol vanishe anon.
>
> . .
>
> He that it wroughte coude ful many a gin."

So Hecate, too, in Macbeth, reaching for her stick, a broomstick, and saying, "I am for the air"; like Abaris on his arrow, putting a girdle round about the Earth between meals.

But Dr. Johnson's "Rasselas," Chapter VI., "A Dissertation on the Art of Flying," anticipates very accurately the gliding experiments of Lilienthal and Pilcher, now resumed by the Wright brothers. The description is so curiously apposite that an extract may well find a place here:—

Among the artists that had been allured into the Happy Valley, to labour for the accommodation and pleasure of its inhabitants, was a man eminent for his knowledge of the mechanic powers, who had contrived many engines, both for use and recreation.

This artist was sometimes visited by Rasselas, who was pleased with every kind of knowledge, imagining that the time would come when all his acquisitions would be of use to him in the open world.

He came one day in his usual manner, and found the master busy in building a sailing chariot. He saw that the design was practicable for a level surface, and with expressions of great esteem solicited its completion.

"Sir," said the master "you have seen but a small part of what the mechanic arts can perform. I have long been of opinion that instead of the tardy conveyance of ships and chariots, man might use the swifter migration of wings, that the fields of air are open to knowledge, and that only ignorance and idleness need crawl on the ground."

"The labour of rising from the ground will be great" said the artist "as we see in the heavier domestic fowls; but as we mount higher the earth's attraction and the body's gravity will be gradually diminished, till we arrive at a region where man will float in the air without any tendency to fall; no care will then be necessary but to move forward, which the gentlest impulse will effect."

"Nothing," replied the artist, "will ever be attempted if all possible objections must first be overcome. If you will favour my project I will try the first flight at my own hazard. I have considered the structure of all volant animals and find the folding continuity of the bat's wing most easily accommodated to the human form.

"Upon this model I will begin my task to-morrow, and in a year expect to tower into the air beyond the malice and pursuit of man."

The Prince visited the work from time to time, observed its progress, and remarked many ingenious contrivances to facilitate motion and unite levity with strength.

The artist was every day more certain that he should leave vultures and eagles behind him, and the contagion seized upon the Prince. In a year the wings were finished, and on a morning appointed the maker appeared furnished for flight, on a little promontory; he waved his pinions awhile to gather air, then leaped from his stand, and in an instant dropped into the lake.

His wings, which were of no use in the air, sustained him in the water, and the Prince drew him to land, half dead with terror and vexation.

These extracts from Rasselas will show that Dr. Johnson had realised with more accuracy than Cyrano de Bergerac the difficulty of the problem to be solved.

Herr von Lilienthal's experiments, followed by Prof. Fitzgerald and Mr. Pilcher, have borne out to some extent his prophecy, which we hope will not come true in the gliding experiments of the Wright brothers, now in progress.

But viewed by the cold, calculating eye of mechanical science, the usual poetical description is seen to be hopelessly absurd and impossible; especially since Mr. Maxim proved to demonstration with his machine the enormous power required, out of all proportion to the size, for man to emulate the bird.

How came the obsession to take so firm a hold on the human imagination as to the possibility of flight by a man like a bird? Here is a subject worthy of careful historical examination.

The pterodactyl of palæontology goes to prove that flight was practised by an animal much larger than now is capable, judging also by his degenerate descendant archæopteryx.

A mere decrease of gravity, due to the Earth not having shrunk so much, would not account for it, as the air density would have diminished in proportion.

Flight would be no easier if gravity was halved, as the density of the air would be halved at the same time.

We are led to conjecture that the atmosphere was formerly much denser than now, and that a large portion has leaked away and escaped into space, just as free hydrogen will now escape; and that the supply has not been renewed by the comets, as imagined by Newton in the "Principia," Lib. III. :—

"Porro suspicor Spiritum illum, qui Aeris nostri pars minima est, sed subtilissima et optima, et ad rerum omnium vitam requiritur, ex Cometis præcipue venire."

The cooling of the Earth's surface and the deposition of the moisture in the air would not account for a diminution of atmospheric density, as aqueous vapour is only about half as dense as air; but the carbonic acid may have been more abundant, before it was absorbed by a dense vegetation.

A writer, Mr. Harle, in the American journal "Cosmos," on Atmospheric Pressure in past Geological Ages, has come to the conclusion that air density must formerly have been very much greater, judging from the pterodactyl with wing span over thirty feet, equal to a Bleriot machine, and fossil dragon-flies of the carboniferous era, over three feet between the tip of the wings.

The winged figures of Assyrian art may prove an inherited reminiscence of the time of the pterodactyl, in an atmosphere of perhaps two-fold density.

The figures have come down to us through the Greek Mythology to the representation by the modern artist of the winged angel, described by Milton in "Paradise Lost," and painted in the cathedral altar-piece.

But while the poet and artist invites us to consider his human figure, Isis, Nike, Michael, as life size with wings proportioned to suit the æsthetic sense, mechanical science of similitude requires our imagination to reduce the scale to the size of the present dragon-fly, if flight is to be maintained on unassisted one-man power, in the tenuity of our present atmosphere.

Pope, however, has grasped the principle of Mechanical Similitude in his "Rape of the Lock," where he arms his Gnome and Ariel Sprite with a bodkin spear:

> "Know, then, unnumbered Gnomes around thee fly,
> The light Militia of the lower sky."

A thought echoed by Tennyson's "Airy navies of the blue."

In all the recent rapid progress there is one difficulty still unsolved; a name has not been discovered for the flying machine of universal acceptance; and here the poverty comes out of the English language in the formation of a new compound word.

The German calls it a dragon, as in the car of Demeter; and Demoiselle (dragon-fly) is the name given by Santos Dumont to his mount.

The lady of the "Rape of the Lock," reminded by—

> "Think what an Equipage thou hast in Air,
> And view with scorn two Pages and a Chair."

might say then, "Bring the dragon-fly round to the door," or dragon instead of the brougham.

Santos Dumont takes the dragon-fly as his mechanical model; but the two inches of length of the insect becomes twenty-four feet in the Demoiselle, a linear scale of 144 to 1, and the speed must be raised twelve-fold, to over sixty miles an hour, from five or six in the dragon-fly.

Size and weight mount up, then, 144 times 144 times 144 times, three million-fold, and a few grains in the fly become nearly half a ton in the Demoiselle. And the power required is another twelve times more, so we have to multiply by twelve seven times over, thirty-six million times, to go from the horse power of the fly to the dragon.

This is the simple calculation which the mathematician has brought to the attention of the mechanical enthusiast, and so the mathematician, as Clifford or Herbert Spencer, is accused of proving that Human Flight was an impossibility.

And so it was, as a retrospect of the last few years will show. It was impossible till the motor was available, strong and light enough to lift itself and three- or four-fold weight, including a man pilot. For the development of the motor we have to thank the Associated Motor Car industry, but for which we should still be waiting.

Such a mechanical genius as Maxim, working about fifteen years ago, and with unlimited resources, was not able to achieve a flight, and so we may be sure no one else could. A few poor aeronautical enthusiasts might still be working, and for a hundred years more, without arriving at a result presented to them now ready made.

All such experimenters, however, were doing valuable work in preparing the way for success, Langley for instance, subsidised by the American Government, so that the moment the motor was available, man could fly.

But not with one man power, as in the artist's picture. The pilot likes to ride on 50 horse-power at least, and the power is mounting up to 100 horse-power and more, before he will secure the speed to become the real mechanical enlargement of the model dragon-fly.

Roger Bacon has described the direction he would like to follow in his inventions, impossible to carry out in his own day from lack of mechanical skill.

Beginning with the Motor Car, he would follow on with the Flying Machine, and then the Diving Dress, and the Suspension Bridge perhaps. But Bacon is as cautious as Herodotus in distinguishing between what he has heard only, and seen himself, as we read in these extracts—

De secretis operibus artis et naturæ, Caput IV.

De instrumentis artificiosis mirabilibus.

— Currus etiam possent fieri ut sine animali moveantur cum impetu inæstimabili, ut existimantur currus falcati fuisse quibus antiquitus pugnabatur.

Possunt etiam fieri instrumenta volandi, et homo sedens in medio instrumenti revolvens aliquod ingenium, per quod alæ artificialiter compositæ aerem verberent, ad modum avis volantis.

Possunt etiam fieri instrumenta ambulandi in mari et in fluviis ad fundum sine periculo corporali. Nam Alexander magnus his usus est, ut secreta maris videret, secundum quod Ethicus narrat astronomus.

Hæc autem facta sunt antiquitus, et nostris temporibus.

Et certum est, præter instrumentum volandi quod non vidi nec hominem qui vidisset cognovi, sed sapientem qui hoc artificium excogitavit explicité cognosco.

(British Museum M.S.)

LECTURE I

GENERAL PRINCIPLES OF FLIGHT, LIGHT AND DRIFT

A SIMPLE preliminary calculation will show the requirements indispensable for a flight in the air of an aeroplane machine.

Take an aeroplane, AA', rectangular, moving horizontally at a slope a in still air with velocity Q f/s (feet per second) (Figs. 1, 2, 4).

This is the velocity as observed by a spectator standing on the ground.

But the pilot on his seat. looking ahead, is unconscious of his own motion, and feels the air blowing past with velocity Q f/s; and the dynamical problem is the same from each point of view: of the pilot who thinks himself stationary, as a Wright glider, and the air blowing past, or of the spectator standing in still air, and watching the pilot flying past with velocity Q f/s.

But when the spectator is standing in a wind blowing over the ground with velocity W f/s, he will notice a difference in the speed of the machine, $Q \mp W$ f/s, according as it is flying up the wind or down.

The pilot, however, will not be aware of a difference any more than a bird in a gale, as his machine cleaves the air with the same speed as before.

Suppose a bird then, or an aviator is to make a flight in a wind; he will start for preference against the wind, so as to increase his relative velocity, and so rise quicker. It is easier then, to rise against the wind than with the wind behind; and to alight also, provided the pilot does not descend into a region of relative calm.

The aeroplane AA' is now supposed up in the air, with the wind blowing past it at Q f/s; and a pressure difference arises in consequence, which gives a resultant thrust, T lb, on the under side of the plane.

The vertical component, $T \cos \alpha$, is called the Lift, as it is required to lift the weight, W lb; and $T \sin \alpha$, the horizontal

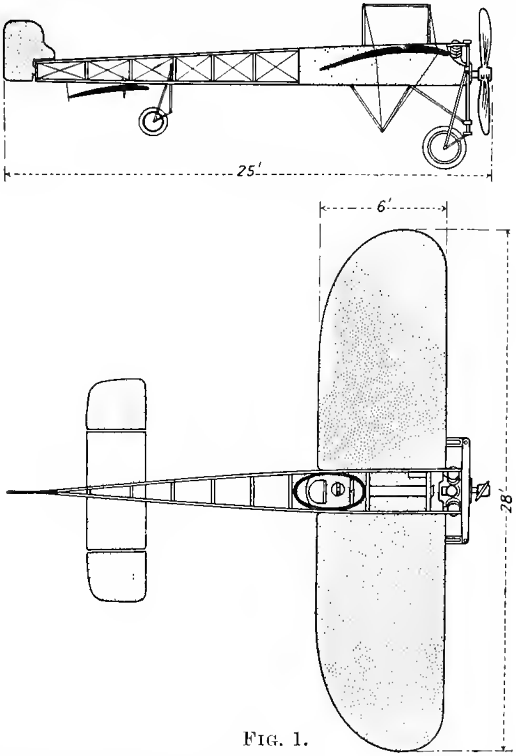


FIG. 1.

component, is called the Drift, and this drift has to be overcome by the thrust of the screw, working at

(1) $$T \sin \alpha \frac{Q}{550}, \text{ effective horse power (E.H.P.)}$$

reckoning 1 H.P. at 550 ft-lb/sec, that is, 33,000 ft-lb/min on Watt's estimate. The ratio then of

(2) $$\frac{\text{E.H.P.}}{\text{lift}} = \frac{T \sin \alpha \frac{Q}{500}}{T \cos \alpha} = \frac{Q \tan \alpha}{550} = \frac{Q}{550\, n}, \text{ or } \frac{\text{lift}}{\text{E.H.P.}} = \frac{500\, n}{Q}$$

if the slope of the aeroplane is reckoned at 1 in n; but it is prudent to double the E.H.P. to arrive at an estimate of the I.H.P. (indicated horse power), reckoning engine and propeller efficiency at 50 per cent.

When the speed is given by S in m/h (miles an hour) the factor 550 is replaced by 375, which is 33,000 ÷ 88, as 1 mile an hour is 88 feet per minute.

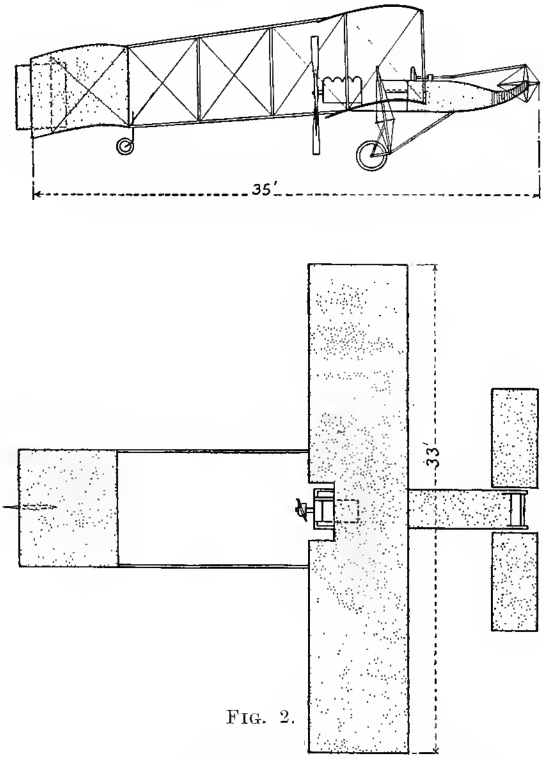


FIG. 2.

If the weight to be lifted is m times the weight of the motor, the motor must weigh

(3) $$\frac{550\, n}{Q\, m} \text{ lb/E.H.P.}$$

to be halved with 50 per cent. efficiency to

(4) $$\frac{550\, n}{2\, Q\, m} \text{ lb/I.H.P.}$$

so that at $S = 40$ m/h, say Q = 60 f/s, and with $n = 5$, $m = 2$, as in Maxim's machine, the motor must weigh

$$\frac{550 \times 5}{2 \times 60 \times 2} = \frac{2{,}750}{240}, \text{ say } 11, \text{ lb/I.H.P.},$$

the figure given by Maxim; and Herbert Spencer and Clifford could tell the aeronaut with certainty that his ideal must await the motor which could be brought down to this weight, 11 lb/I.H.P., and lower still, before his machine could fly.

The petrol motor had not come to any perfection in the day of Maxim's experiment, 1895, and so he was obliged to carry a boiler up with his engine. This implied a great increase in size all round, so that Maxim's machine (Fig. 3) weighed close on four tons, quite unmanageable in the light of modern experience.

An aviator is akin to the imperfect orator, as his chief difficulty arises when he wants to get down; and the shock of landing of four tons rather abruptly would be terrifying, with a boiler close by full of steam at 275 lb/inch^2.

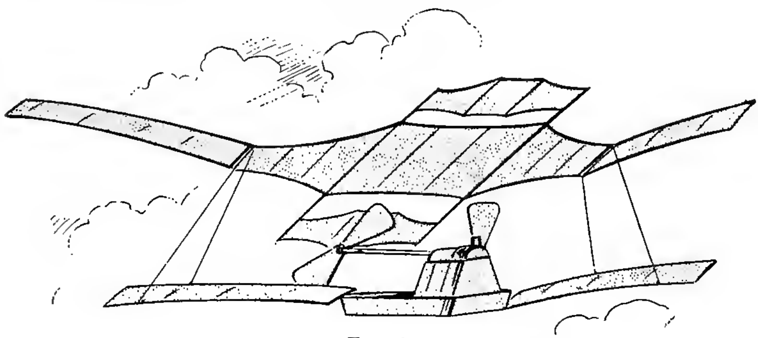


FIG. 3.

A modern locomotive will weigh complete about 80 tons, and will indicate 1,000 H.P. at 50 to 60 m/h; the

$$\text{H.P./ton is then } 12{\cdot}5, \text{ and the lb/H.P.} = \frac{2{,}270}{12{\cdot}5} = 180,$$

so that no locomotive can be expected to fly, still less a marine engine, with its additional weight of condenser.

This simple arithmetical calculation, of Herbert Spencer and Clifford, it can hardly be called mathematical, is enough to show why mankind was compelled to await the advent of the light petrol motor; and man might still be waiting, but for the previous development of the motor car.

A list is given here of the chief dimensions of the early pioneering machines; the list can be added to, and brought up to date with each successive pattern and development.

	MAXIM. (*Nature*, Aug. 1, 1895.)	WRIGHT. (Details missing.)	FARMAN. (*Flight*, Oct. 10, 1909.)
Speed S (m/h) ...	40	—	45
Q (f/s)	60	—	66
Wing Area A (ft^2)	4000	—	410 (biplane)
Slope 1 in n = ...	4 to 5	—	—
Lift or Weight, lb,	10,000—8,000	—	1,212 + pilot
Drift, lb, ...	2,000	—	—
$\frac{\text{I.H.P.}}{\text{Lift}}$ = ...	$\frac{2\times 66}{554\times 8}=0{\cdot}048$	—	—
I.H.P.	384	—	—
Weight of Motor ...	4,000	—	—
in lb/H.P. ...	11	—	—

VOISIN. (*Flight*, Aug. 14, 1909.)	ANTOINETTE. (*Flight*, Oct. 20, 1909.)	BLERIOT. (*Flight*, July 31, 1909.)	SANTOS-DUMONT DEMOISELLE. (*Flight*, Oct. 2 and 9, 1909.)
—	—	—	—
—	—	—	—
450 (biplane)	365	200	115
—	—	—	—
1,150	1,040 + pilot	700	400
230	—	140	80
—	—	—	—
55	—	34	20
400	—	200	110
8	—	6	5·5

But in the machine of the present day the H.P. of the motor has risen from 75 to 150, and the weight to over 500 kg, or half a ton.

The diagrams, Fig. 1 of a monoplane, and Fig. 2 of a biplane, are based on figures in the aeronautical journal *Flight*, as well as the numerical data above.

The year 1909 may be taken as the epoch when the dreams of fancy were realised which have inspired the artist and poet of romance as far back in history as we can trace, through Tennyson, Chaucer, up to Homer and his unknown predecessor, Egyptian, Babylonian, and Chaldæan.

But now anxiety is past for the motor, we can calculate dimension and design for a given weight or lift at a speed assigned; and the first mathematical requirement is a determination of the thrust T lb, over an area A ft^2, driven through the air with velocity Q f/s, at a slope α or 1 in n.

An exact mathematical treatment brings in the Schwarz-Christoffel theory of conformal representation (mapping), and its first application by Helmoltz and Kirchhoff to the stream lines of liquid through an orifice or past a barrier, and of the discontinuity arising at an edge.

This theory will represent the substance of these Lectures; but we may lead up to it through the ancient assumption of Newton, which treats the medium as devoid of rigidity; in the language of the "Principia," p. 219, second edition;

"In Mediis quæ rigore omni vacant resistantiæ corporum sunt in duplicata ratione velocitatum."

The medium of air is taken then to behave like a cloud of particle dust, as assumed also in the modern kinetic theory of a gas.

Taking C ft^3 of the medium to weigh 1 lb, we say that the specific volume (S.V.) is C ft^3/lb.

Thus, for air in a room, 12 to 13 ft^3 weigh 1 lb; an average S.V. of 12·5 ft^3/ lb, or 28,000 ft^3/ ton.

A room 30 feet cube will then contain nearly a ton of air.

The density of air would thus be 0·08 lb/ft^3, but, as usual in all calculation, we prefer to work with the integers of the reciprocal.

If the plane is A ft^2 in area, and sloped at an angle α, the vertical aspect in front elevation is $A \sin \alpha$ ft; and the air received on the plane is

(5) $$QA \sin \alpha \text{ ft}^3/\text{sec, or } \frac{QA}{C} \sin \alpha, \quad \text{lb/sec.}$$

Taking the air particles as inelastic, they slide along the plane after impact without interference; their momentum perpendicular to the plane is reduced to zero; so that in t seconds, the thrust T lb will generate or destroy Tt sec/lb of momentum in reducing $\frac{QA}{C} t \sin \alpha$ lb from velocity $Q \sin \alpha$ to 0; and then

(6) $$Tt = \frac{QA}{C} t \sin \alpha \frac{Q \sin \alpha}{g}, \; T = \frac{A}{C}\frac{Q^2}{g} \sin^2 \alpha, \quad \text{lb.}$$

The gravitation measure of force and momentum is employed. as in all engineering calculation.

The theory is exact when the direction of the leaving stream is compelled to be parallel to the plane, as would be the case for a horizontal jet of vertical cross-section $B = A \sin \alpha$ ft, if it was received on a plane barrier of unlimited extent, and then

(7) $$T = \frac{B}{C}\frac{Q^2}{g} \sin \alpha, \quad \text{lb.}$$

The theory can be employed in all calculations of hydraulic machinery, as a turbine or Pelton wheel (and a screw propeller), where the water is compelled to follow a definite path.

Later on an examination will be made of the action of a screw propeller, and the various theories that have been advanced; and the theory which seems to hold the ground with the naval architect is the one given by Rankine in the "Transactions of the Institution of Naval Architects, 1865," in which it appears that Rankine's formula is a direct result of the Newtonian theory and assumption just stated.

In air the particles may be supposed elastic, endowed with a coefficient of restitution e; and this makes

(8) $$T = (1 + e)\frac{A}{C}\frac{Q^2}{g}\sin^2 a, \qquad \text{lb},$$

or

(9) $$\frac{T}{A} = (1 + e)\frac{Q^2}{Cg}\sin^2 a, \qquad \text{lb/ft.}^2$$

On this assumption for an aeroplane, the lift or weight which can be lifted is

(10) $$W = T\cos a = (1 + e)\frac{A}{C}\frac{Q^2}{g}\sin^2 a\cos a, \qquad \text{lb},$$

at a rate

(11) $$\frac{W}{A} = (1 + e)\frac{Q^2}{Cg}\sin^2 a\cos a, \qquad \text{lb/ft}^2,$$

and to overcome the drift $T\sin a$, the

(12) $$\text{H.P./ft}^2 = \frac{T}{A}\sin a\,\frac{Q}{550} = (1 + e)\frac{Q^3}{550\,Cg}\sin^3 a.$$

In a numerical calculation it is convenient to take, in round numbers,

(13) $$g = 32 \text{ f/s}^2, \text{ with } C = 12{\cdot}5 \text{ ft}^3/\text{lb},$$

for air, as this makes

(14) $$Cg = 400,$$

so that we can write

(15) $$\frac{T}{A} = (1 + e)\left(\frac{Q\sin a}{20}\right)^2, \qquad \text{lb/ft.}^2$$

(16) $$\text{H.P./ft}^2 = (1 + e)\left(\frac{Q\sin a}{20}\right)^2 \cdot \frac{Q\sin a}{550} = \frac{20}{550}(1 + e)\left(\frac{Q\sin a}{20}\right)^3$$

With normal impact $a = 90°$, and even with $e = 0$, this gives

(17) $$\frac{T}{A} = \left(\frac{Q}{20}\right)^2 = 25\left(\frac{Q}{100}\right)^2, \qquad \text{lb/ft}^2;$$

a wind of 20 f/s giving a pressure of 1 lb/ft²; or if the velocity

Q f/s is the equivalent of S m/h,

(18) $$\frac{S}{60} = \frac{Q}{88}, \; Q = \frac{22}{15} S,$$

(19) $$\frac{T}{A} = 25 \frac{484}{225} \left(\frac{S}{100}\right)^2, \text{ say } 54 \left(\frac{S}{100}\right)^2, \quad \text{lb/ft.}^2$$

Normal atmosphere pressure is reckoned at 15 lb/inch^2, which is 2,160 lb/ft^2, nearly 1 ton/ft.^{2}

But even at a speed of $S = 100$ m/h, the normal pressure in (19) is 54 lb/ft^2, which is only $\frac{1}{40}$th of an atmosphere, or $\frac{3}{4}$ inch in the mercury barometer, so that air compression may be ignored, and the density taken as uniform, until we come to the high velocity usual in the screw propeller.

These numbers, even with $\epsilon = 0$, are greatly in excess of the result of recent experiment and theory; although at the Forth Bridge, in a gale of 78 m/h, a pressure of 70 lb/ft^2 was recorded; but this abnormal result may be attributed to gustiness.

To drive the plane normally would then require

(20) $$\text{H.P./ft}^2 = \left(\frac{Q}{20}\right)^2 \frac{Q}{550} = \frac{1}{220}\left(\frac{Q}{10}\right)^3 = \frac{1{,}000}{220}\left(\frac{Q}{100}\right)^3,$$
$$\text{say } 4{\cdot}5 \left(\frac{Q}{100}\right)^3, \text{ or } 14 \left(\frac{S}{100}\right)^3.$$

On the other hand, with the thrust varying as the square of sin α, the formula gives numbers much too low at a small slope of the plane.

Apply the formulas to Maxim's machine, where the area $A = 4{,}000$ ft^2, at a slope of 1 in 5, so that

sin α or tan $\alpha = \frac{1}{5}$, and cos $\alpha = 0{\cdot}98$ may be replaced by unity.

At a speed of $Q = 100$ f/s the lift would be 4,000 lb, only half the weight to be lifted, so that the wing area should be double, 8,000 ft^2; and the E.H.P. would be $\frac{4{,}000}{5} \times \frac{100}{550}$, say 150, requiring nearly the full I.H.P. of 360, with a propulsive efficiency of 40 per cent.; this would prove that the machine cannot leave the ground at this speed.

But the modern mathematical theory of Kirchhoff, to which we return later, confirmed by Langley's experiments, proves that the lift is given more nearly by

$$(21) \qquad A\left(\frac{Q}{20}\right)^2 \tfrac{1}{4}\pi \sin \alpha, \quad \text{instead of } A\left(\frac{Q}{20}\right)^2 \sin^2 \alpha,$$

so that for a small angle α the lift varies more nearly as $\sin \alpha$ than $(\sin \alpha)^2$, much greater than given by the ancient theory.

Applying this again to Maxim's machine at a speed $Q = 100$ f/s, the lift

$$(22) \qquad T = 4{,}000\left(\frac{100}{20}\right)^2 \times 0{\cdot}7854 \times 0{\cdot}2 = 16{,}000 \text{ lb},$$

double the weight to be lifted; and if the speed is reduced to S=45 m/h, $Q = 66$ f/s, the lift is halved, and the machine could rise at this speed, with the engines working up to full 360 H.P.

But at Baldwyn's Park, the run of about a quarter of a mile was not long enough to get up this speed.

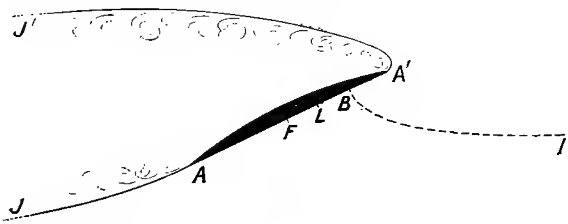


FIG. 4.

For normal incidence on Kirchhoff's theory the factor

$$(23) \qquad \frac{\frac{1}{4}\pi}{1 + \frac{1}{4}\pi} = 0{\cdot}44, \ \pi = \frac{22}{7},$$

is required to reduce Newton's coefficient; this makes the formula for wind pressure

$$(24) \qquad 0{\cdot}44 \times 54 \left(\frac{S}{100}\right)^2 = 24 \left(\frac{S}{100}\right)^2 \qquad \text{lb/ft}^2$$

much lower than $32 \left(\frac{S}{100}\right)^2$, as given by Stanton and Eiffel.

The wide divergence of the Newton formula is explained on the Kirchhoff theory by the shape of the stream lines, AJ and $A'J'$, as they leave an edge, A and A', of the barrier AA'. These are not straight and in the line AA', as in Newton's theory, but curved backward, as in Fig. 4, forming a skin or bounding surface between the wind and the dead air behind the barrier.

(Experiment with a knife-blade held in the jet of water from a tap.)

We proceed now to the object advertised in these Lectures, the determination of the *Stream Lines past a Plane Barrier, and of the Discontinuity arising at an Edge* (Report 19, Advisory Committee for Aeronautics, Wyman, Fetter Lane, E.C., 1910) as it is determined by recent mathematical research; and a digression must be made on the history of the subject inaugurated by Helmholtz and Kirchhoff, about 1868.

Although Helmholtz was the pioneer of the theory, in 1868, we begin with the result arrived at by Kirchhoff, about a year after, in 1869, as it gives a solution for an aeroplane AA' (Fig. 4.)

The stream of air, coming from infinity at I with relative velocity Q, is split along a curved stream line IB, and divides at B on AA' into the two stream lines BAJ and $BA'J'$, where AJ and $A'J'$ are curved stream lines proceeding from the edge A and A'.

Along AJ and $A'J'$ the velocity must be constant and equal to Q, so as to ensure that the pressure in the dead air behind AA' is the constant atmospheric pressure; and the analytical difficulty is to secure this condition, a difficulty which Kirchhoff was the first to overcome.

Kirchhoff's result states that the motion of the medium, treated as homogeneous, is given by

$$\text{(25)} \qquad \operatorname{ch} \Omega = \cos \alpha + \frac{N}{\sqrt{(w_B - w)}},$$

where α denotes the angle the stream at IJJ' makes with the plane AA', and N is a constant.

In the notation of this subject

$$\text{(26)} \qquad z = x + yi, \quad w = \phi + \psi i,$$

ϕ denoting the velocity function at the points (x, y) such that the velocity q is given by the downward gradient$-\frac{d\phi}{ds}$ of the function ϕ, and ψ is the conjugate stream function, constant along a stream line.

Also, if the velocity q at (x, y) makes an angle θ with Ox,

$$\text{(27)} \qquad \frac{d\phi}{dx} = \frac{d\psi}{dy} = - q \cos \theta, \frac{d\phi}{dy} = -\frac{d\psi}{dx} = - q \sin \theta;$$

and then

$$\text{(28)} \qquad \frac{dw}{dz} = \frac{d\phi}{dx} + i \frac{d\psi}{dx} = - q \cos \theta + i q \sin \theta.$$

The function ζ and Ω is now introduced, defined by

$$\text{(29)} \qquad e^{\Omega} = \zeta = - Q \frac{dz}{dw} = \frac{Q}{q} (\cos \theta + i \sin \theta) = \frac{Q}{q} e^{\theta i}.$$

$$\text{(30)} \qquad \Omega = \log \frac{Q}{q} + \theta i.$$

$$\text{(31)} \qquad \operatorname{ch} \Omega = \operatorname{ch} \log \frac{Q}{q} \cos \theta + i \operatorname{sh} \log \frac{Q}{q} \sin \theta,$$

to be employed in equation (25).

Along the dividing stream line we take

(32) $$\psi = 0, \quad w_B - w = \phi_B - \phi,$$

and from I to B

(33) $$\infty > \phi > \phi_B, \ \phi_B - \phi \text{ is negative, } \sqrt{(\phi_B - \phi)} \text{ imaginary,}$$

so that

(34) $$\operatorname{ch} \log \frac{Q}{q} \cos\theta = \cos\alpha, \quad \operatorname{sh} \log \frac{Q}{q} \sin\theta = \frac{N}{\sqrt{(\phi - \phi_B)}}.$$

But beyond B

(35) $$\phi_B > \phi > -\infty, \quad \sqrt{(\phi_B - \phi)} \text{ and } \sqrt{(w_B - w)} \text{ is real}$$

$$\operatorname{sh} \log \frac{Q}{q} \sin\theta = 0,$$

so that either

(36) $$\sin\theta = 0, \quad \theta = 0 \text{ along } BA, \quad \theta = \pi \text{ along } BA';$$

or else

(37) $$\operatorname{sh} \log \frac{Q}{q} = 0, \quad \frac{Q}{q} = 1, \ q = Q,$$

as required along the stream line AJ and $A'J'$; and then

(38) $$\operatorname{ch} \log \frac{Q}{q} = 1, \cos\theta - \cos\alpha = \frac{N}{\sqrt{(\phi_B - \phi)}}.$$

Along the stream line AJ

(39) $$\frac{d\phi}{ds} = -Q, \quad \phi_A - \phi = Qs,$$

if the arc $AP = s$ is measured from A, and

(40) $$Qs = \phi_A - \phi_B + \phi_B - \phi = \phi_A - \phi_B + \frac{N^2}{(\cos\theta - \cos\alpha)}$$

$$= N^2\left[\frac{1}{(\cos\theta - \cos\alpha)^2} - \frac{1}{(1 - \cos\alpha)^2}\right]$$

the intrinsic equation of the stream $A'J'$ with $0 < \theta < \alpha$; with a similar expression for $A'J'$.

For normal incidence (Fig. 5) $\alpha = 90°$, $\cos\alpha = 0$, and the curve AJ is the evolute of a catenary, given by the intrinsic equation

$$Qs = N^2 \tan^2\theta,\ s = c\tan^2\theta.$$

The theory requires the existence of the counter current BA' (*nappe dorsale* in French), passing over the attacking front edge A' of the plane AA'; this *nappe dorsale* is usually omitted in the diagram of popular explanation as insensible, but the existence is revealed in a photograph of a current, either of air or water.

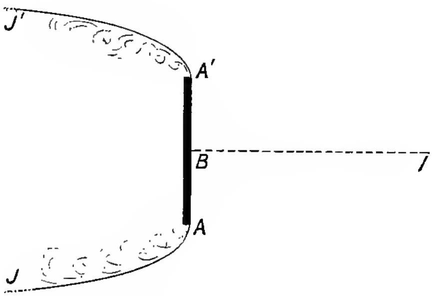


FIG. 5.

It will be proved in the sequel that

(41) $$\frac{BA'}{AA'} = \frac{\frac{1}{2}(1 - \cos\alpha) - \frac{1}{4}\sin\alpha\sin 2\alpha + \frac{1}{4}\alpha\sin\alpha}{1 + \frac{1}{4}\pi\sin\alpha}$$

and at a small angle α, in radians, we may replace

(42) α by $\sin\alpha + \frac{1}{6}\sin^3\alpha$, and $\cos\alpha$ by $1 - \frac{1}{2}\sin^2\alpha - \frac{1}{8}\sin^3\alpha$,

giving the approximation

(43) $$\frac{BA'}{AA'} = \frac{17}{48}\sin^4\alpha.$$

Thus at a slope of 1 in 5,

44 $$\sin\alpha = 0{\cdot}2,\ \frac{BA'}{AA'} = \frac{1}{2{,}000} = 0{\cdot}0005$$

and even at 2 in 5,

(45) $$\sin\alpha = 0{\cdot}4, \quad \frac{BA'}{AA'} = 0{\cdot}008;$$

so that the counter current would still be hardly perceptible. But with normal incidence (Fig. 5),

(46) $$\alpha = \tfrac{1}{2}\pi, \ \sin\alpha = 1, \quad \frac{BA'}{AA} = \tfrac{1}{2},$$

and the current divides equally.

Anticipating other results of Kirchhoff's theory, to be proved hereafter, we find

(47) $$\frac{T}{A} = \frac{Q^2}{Cg}\,\frac{\frac{1}{4}\pi\sin\alpha}{1+\frac{1}{4}\pi\sin\alpha}, \quad \text{lb/ft}^2, \quad \text{versus}\ \frac{T'}{A} = \frac{Q^2}{Cg}\sin^2\alpha,$$

of the ancient theory; also that the centre of pressure L is in front of F the centre of AA', between F and A', where

(48) $$\frac{FL}{AA'} = \frac{\frac{3}{16}\cos\alpha}{1+\frac{1}{4}\pi\sin\alpha}, \quad \text{versus}\ FL = 0.$$

The look of these formulas (47) and (48) suggests a geometrical representation, as in Fig. 6, associated with the ellipse, whose focal polar equation is

(49) $$r = \frac{b^2}{a + c\cos\theta} = \frac{l}{1 + e\cos\theta},$$

(50) $$\cos\theta = \sin\alpha, \text{ and eccentricity } e = \tfrac{1}{4}\pi = 0{\cdot}7854, \text{ or } \tfrac{11}{14}.$$

Take FX as the unit to represent $\frac{Q^2}{Cg}$ geometrically, the average pressure in lb/ft² on Newton's theory for normal incidence, $\alpha = \frac{1}{2}\pi$; and make

(51) $$\frac{FL_o}{FX} = \tfrac{1}{4}\pi, \quad \frac{FA}{FX} = \frac{\frac{1}{4}\pi}{1+\frac{1}{4}\pi} = 0{\cdot}44.$$

Draw the ellipse APL_o, with focus F, vertex A, semi-latus rectum FL_o, and eccentricity $\frac{1}{4}\pi$. Draw the semicircle FQX on the diameter FX, cutting FP in Q. Then

$$(52)\qquad \frac{\dfrac{T}{A}}{\dfrac{Q^2}{Cg}} = \frac{\frac{1}{4}\pi \sin \alpha}{1+\frac{1}{4}\pi \sin \alpha} = \frac{LP}{FX}, \quad \frac{\dfrac{T'}{A}}{\dfrac{Q^2}{Cg}} = \sin^2 \alpha = \frac{LP'}{LX},$$

if QP' is drawn parallel to FL; so that the curves L_oPA $L_oP'X$ represent T and T' graphically.

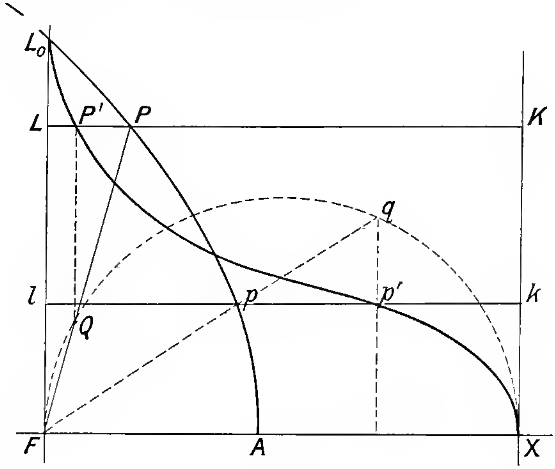


FIG. 6.

Also

$$(53)\qquad \frac{FL}{FL_o} = \frac{FP}{FL_o}\cos \alpha = \frac{\cos \alpha}{1+\frac{1}{4}\pi \sin \alpha}$$

giving L the centre of pressure when FL_o is $\frac{3}{16}$ the breadth of the plane AA'; and L lies inside the middle $\frac{3}{8}$ of AA'.

This can be verified experimentally with a plane plate AA' in a current of water; pivoted like a balanced rudder about an axis through L, and then measuring α.

If L coincides with F, the plane sets itself at right angles to the current with $\alpha = \frac{1}{2}\pi$; as L moves away from F, α diminishes to zero when $FL = \frac{3}{16} AA'$; as L is placed still further away from F, the plane AA' still remains in the line of the stream.

The vane of a weathercock could thus be pivoted so as to point at any assigned angle with the wind; pivoted at the centre F the vane would set itself across the wind.

The movement of L towards the leading edge A' shows why a flat plate if free tends to set itself broadside to the relative stream as a position of stable equilibrium, seen realised in a falling card or leaf; and it explains the instability of the axial motion of an elongated body like a ship.

The stability of the course of a ship is secured only by a constant attention to the helm; but in a flying machine, by increase of Aspect Ratio, making the spread of the wings much greater than the axial depth, the vertical rudder requires little attention, but the vertical stability of the course requires incessant control by the horizontal rudder.

The curve L_0P for moderate value of α is seen running for some distance outside the curve L_0P'; and this shows how on Kirchhoff's treatment the lift may be obtained with a wing area much smaller than was credited on the ancient theory, although a smaller pressure is assigned for normal impact.

For instance, with an average $n = 6$, and at 45 m/h,

$$(54) \qquad \frac{\text{lb. of lift}}{\text{thrust H.P.}} = \frac{375 \times 6}{45} = 50,$$

and on the ancient theory,

$$(55) \qquad \frac{\text{wing area in ft}^2}{\text{Lift in lb}} = \frac{A}{T' \cos \alpha} = \frac{1}{(1+e)\dfrac{Q^2}{Cg} \sin \alpha \cos \alpha}$$

$$\text{practically } \frac{n^2}{\left(1 + e\right)\left(\dfrac{Q}{20}\right)^2},$$

for a small angle of 1 in n; and with $n = 6$, $Q = 66$,

$$(56) \qquad \frac{\text{Area}}{\text{Lift}} = \frac{36}{\left(1 + e\right)\left(\dfrac{66}{20}\right)^2} = \frac{3 \cdot 3}{1 + e},$$

varying from 3·3 to 1·65 with e from 0 to 1.

This is very much greater than is required in an actual machine; Blériot takes $\frac{400}{700}$ ft²/lb, the Demoiselle $\frac{115}{400}$ ft²/lb.

But an inspection of the figure shows that at small value of a the lift given on the Kirchhoff theory is much greater, requiring smaller wing area for given lift,

(57) $$\frac{\text{Wing area}}{\text{lift}} = \frac{1}{\left(\frac{Q}{20}\right)^2} \frac{1 + \frac{1}{4}\pi \sin a}{\frac{1}{4}\pi \sin a \cos a},$$

and this can be replaced by

(58) $$\frac{1}{\left(\frac{Q}{20}\right)^2} \frac{n + \frac{1}{4}\pi}{\frac{1}{4}\pi}, \text{ or practically } \frac{n+1}{20\left(\frac{Q}{100}\right)^2} \text{ or } \frac{n+1}{42\left(\frac{S}{100}\right)^2}.$$

This, for $n = 6$, $S = 75$, works out to

$$\frac{7}{42\left(\frac{45}{100}\right)^2}, \text{ or a little less than } 1;$$

still much greater, nearly double, than is required in practice for wing area per lb of lift.

The discrepancy is attributed to the gain in efficiency due to camber of the wing; and a practical formula in general use is

(59) $$\frac{\text{Lift}}{\text{Area}} = \frac{1}{n}\left(\frac{S}{8}\right)^2, \quad \text{lb/ft}^2,$$

where n is the slope of the chord of the camber.

We have shown that the counter current is insensible at the leading edge A' of attack; and as 1 in n is understood as the slope of the chord of the camber, the slope at the rear edge A is about 2 in n, which reduces the ft²/lb of wing area to a close agreement with practice.

But, as stated in the first clause of Report 19, there is no exact analytical theory at present for the calculation of a stream past a cambered wing, unless of two planes bent at an angle, and here the complication becomes almost intractable for practical use.

A liberal estimate of petrol and lubricating oil consumed is at 1 lb/H.P. hour, or one gallon per 10 H.P. hour: thus in a flight of 100 miles in two hours and a half at 40 m/h with a 50 H.P. Gnome engine, the quantity required would be

$$50 \times 2\tfrac{1}{2} = 125 \text{ lb, bulking } \frac{12{\cdot}5}{\frac{2}{3}} = 18{\cdot}75, \text{ say 20 gallons,}$$

with a petrol of S.G. $\frac{2}{3}$.

With given lift, n varies as Q^2, making the

$$\frac{\text{H.P.}}{\text{lb}} = \frac{Q}{550\,n} \propto \frac{1}{Q},$$

and the hours of a journey varying inversely as Q, the H.P. hours vary inversely as Q^2; so that about half the petrol is required to be carried if the speed over the journey is increased from 40 to 60.

On the ancient theory, n^2 varies as Q^2, making the H.P./lb constant, and the H.P. hours of a journey, and the petrol to be carried, inversely as the speed, or two-thirds for an increase of speed from 40 to 60 m/h.

This calculation ignores friction and head resistance, but it indicates in a general way the advantage and economy of high speed in flight.

Stability too is improved by speed, as well as economy in fuel.

LECTURE II

CALCULATION OF THRUST AND CENTRE OF PRESSURE OF AN AEROPLANE

THE first lecture was confined to generalities, and mathematical detail required for a complete study of the subject, was avoided as much as possible.

But now we begin with a description of the Schwarz-Christoffel method of Conformal Representation, or Mapping, which shows us how to re-invent the original Helmholz-Kirchhoff solutions, and to extend them with ease and certainty to a large number of similar problems of greater generality, for which Report 19 may be consulted, on the *Stream Line past a Plane Barrier*, 1910.

The notation required has been given already in Lecture I., and we now proceed to state the theorem of Conformal Representation in the essential form as required for the subsequent application.

A point P, whose position is given by the vector $z = x + yi$, is to travel round a closed polygon or curve; and it is required to represent z as a function of some variable u, such that

(i.) u is real and diminishes from $+\infty$ to $-\infty$ as P performs a circuit of the polygon,

(ii.) points inside the polygon are to correspond to $u = p + qi$, but to $u = p - qi$ for points outside.

The relation will be given by

$$(1) \qquad \frac{dz}{du} = M(u-a)^{-\frac{\alpha}{\pi}},$$

and the product of factors of the same form; and here $u = a$ at a corner of the polygon, where the direction, or course, changes suddenly through α the *exterior* angle of the polygon.

The representation is called Conformal, because a small square on the z diagram corresponds to a small square on the u diagram, and the relative bearing of adjacent points is preserved for the two maps z and u, as they may be called; and there is thus no distortion although there is rotation and change of scale.

Conformal representation is then the same problem as mapping, and the first application on a large scale was the construction of a map on a given system of projection.

Thus for instance the relation

(2) $$z = \tan \tfrac{1}{2} u$$

connects the z stereographic projection of a hemisphere with the u Mercator chart. (Fig. 7.)

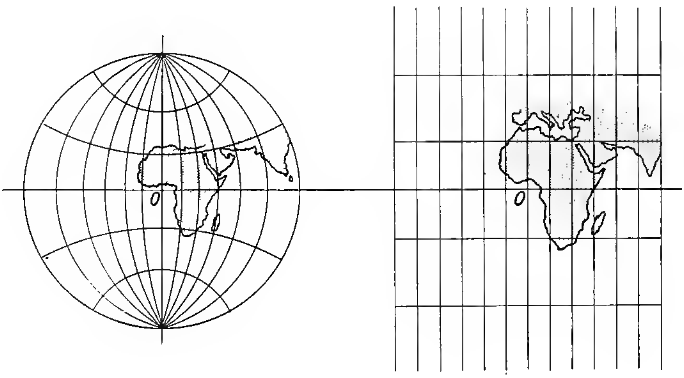


FIG. 7.

Suppose P travels round the polygon so that the inside is to the right hand (" starboard " a sailor would say), as in going round the clock; the angle α is positive when the change of course is to the right or starboard, due to porting the helm.

If the helm is put to starboard the angle α would be negative.

Considering a single factor of (1), the relation

(3) $$\frac{dz}{du} = M\left(u - a\right)^{-\frac{\alpha}{\pi}},$$

where M may be complex, and we put

(4) $$M = Ne^{\theta i} = N(\cos\theta + i\sin\theta);$$

and this leads to

(5) $$\frac{dx}{du} = N\left(u - a\right)^{-\frac{a}{\pi}} \cos\theta, \quad \frac{dy}{du} = N\left(u - a\right)^{-\frac{a}{\pi}} \sin\theta;$$

so that

(6) $$\frac{dy}{dx} = \tan\theta, \text{ so long as } u > a,$$

and P describes a straight line at an angle θ with Ox. (Fig. 8.)

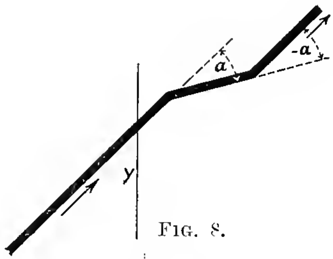


FIG. 8.

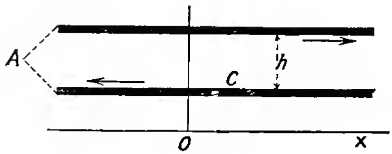


FIG. 9.

But as u passes through a to the other side, where $a > u$, and $u - a$ negative,

(7) $$\frac{dz}{du} = Ne^{i\theta}(-1)^{-\frac{a}{\pi}}(a - u)^{-\frac{a}{\pi}}$$

$$= Ne^{i\theta} e^{-\iota a}(a - u)^{-\frac{a}{\pi}} \text{ with } -1 = e^{i\pi}$$

$$= Ne^{(\theta - a)}(a - u)^{-\frac{a}{\pi}}$$

$$= N\left[\cos(\theta - a) + i\sin(\theta - a)\right](a - u)^{-\frac{a}{\pi}}.$$

(8) $$\frac{dx}{du} = N\cos(\theta - a)(a - u)^{-\frac{a}{\pi}}, \quad \frac{dy}{du} = N\sin(\theta - a)(a - u)^{-\frac{a}{\pi}},$$

(9) $$\frac{dy}{dx} = \tan(\theta - a),$$

so that P describes a new straight line, at an angle $\theta - a$ with Ox.

In the case of a single angle A, an integration gives

$$(10) \qquad z - z_A = M \frac{(u - a)^{1 - \frac{a}{\pi}}}{1 - \frac{a}{\pi}}.$$

The case of $a = \pi$ requires special consideration ; here the interior angle of the polygon is zero, and the point A is at infinity ; the two sides of the polygon are parallel, shown meeting at A in the diagram in the conventional way of Euclid I. 27 (Fig 9) ; and

$$(11) \qquad \frac{dz}{du} = \frac{M}{u - a}, \quad z - z_c = M \log \frac{u - a}{c - a}.$$

If the two sides are parallel to Ox, M is real, and equal to B suppose ; and if h is the distance between them, we shall find $h = \pi B$.

The point P goes off to infinity at A along the lower line to the left, and returns from infinity towards the right along the upper line, as u diminishes through a ; and from (11)

$$(12) \qquad x - x_c + i(y - y_c) = B \log \frac{u - a}{c - a}$$

$$x - x_c + i(y - y_c - h) = B \log \frac{a - u}{c - a}$$

$$(13) \qquad \frac{u - a}{c - a} = e^{\frac{x - x_c}{B}} \left(\cos \frac{y - y_c}{B} + i \sin \frac{y - y_c}{B} \right)$$

$$= e^{\frac{x - x_c}{B}} \left(- \cos \frac{y - y_c - h}{B} - i \sin \frac{y - y_c - h}{B} \right),$$

which requires

$$(14) \qquad \cos \frac{h}{B} = -1, \ \sin \frac{h}{B} = 0, \ \frac{h}{B} = \pi,$$

so that the conformal relation for the space between two parallel lines in distance h apart is

$$(15) \qquad z - z_c = \frac{h}{\pi} \log \frac{u - a}{c - a}.$$

The point P now describes the lower line parallel to Ox from right to left, and goes off to infinity on the left hand at A, where $u = a$; it turns the corner here through an angle $\alpha = \pi$ to starboard and comes back along another parallel line at a distance h above, so as to have the interior of the space between the two parallel lines on the right hand.

The same argument as in (3) applies to each of the corners, A, A_1, A_2, . . . of the polygon, where $u = a, a_1, a_2, \ldots$, and to each of the factors in (1) so that

$$(16) \qquad \frac{dz}{du} = M (u - a)^{-\frac{\alpha}{\pi}} (u - a_1)^{-\frac{\alpha 1}{\pi}} u - a_2)^{-\frac{\alpha 2}{\pi}} \ldots,$$

and no break occurs in the direction of motion except at the factor corresponding to the corner traversed of the polygon, so that all the factors may have their variation ignored at this point.

The integration in (16) now requires special consideration, and is not always tractable, as we see in the sequel.

In the applications to streaming motion we do not consider the $z = x + yi$ polygon, as in the Electrical Applications given in J. J. Thomson's Researches; but we draw the polygon for

$$w = \phi + \psi\, i, \text{ and } \Omega = \log \frac{Q}{q} + \theta\, i,$$

and determine the conformal representation of each in terms of the same variable u by means of the appropriate expression for $\frac{dw}{du}$ and $\frac{d\Omega}{du}$ formed as in (1).

Herein is the great improvement on the original method of Helmholtz and Kirchhoff, introduced by J. T. Michell, *Phil. Trans.*, 1890, and improved by A. E. H. Love, *Proc. Cam. Phil. Soc.*, 1891.

Then if we have found $w = f(u)$, $\Omega = F(u)$ by an integration,

$$(17) \qquad -Q \frac{dz}{du} = \zeta \frac{dw}{du} = e^{F(u)} f'(u),$$

and another integration will give z as a function of u; but this integration need not be carried out, except when required for the determination of a geometrical length on the diagram.

The fluid is bounded by one or more stream lines, over which ψ is constant, and represented by parallel lines in the w diagram, and the angle α in the w polygon is either $+\pi$ or $-\pi$, so that

(18) $\frac{dw}{du}$ = product of factors of the form $(u-a)^{-1}$ or $(u-b)^{+1}$

which can be resolved into a quotient and partial fractions of the form

$$\frac{M}{u-a}.$$

Now, considering the Ω polygon, where

(19) $$\Omega = \log \frac{Q}{q} + \theta i,$$

the Ω polygon is composed either of parallel lines of constant θ and variable $\log \frac{Q}{q}$, corresponding to a boundary or barrier of z, or else a line at right angles of constant velocity $q=Q$, making $\log \frac{Q}{q} = 0$, while θ is variable, as over the free surface of a jet.

Here is the advantage of $\Omega = \log \zeta$ over $\zeta = \frac{Q}{q} e^{\theta i}$, as constant $q = Q$ would give an arc of a circle on the ζ diagram, and θ would be constant along a radius; and so the procedure of Helmholtz and Kirchhoff in employing ζ is not the simplest, and was much improved by Planck's idea of using $\log \zeta = \Omega$.

Begin with the application to Kirchhoff's problem, where a plane barrier AA' like an aeroplane is placed at an angle α across an infinite current of fluid, moving when undisturbed with velocity Q.

In the disturbed motion a wake is formed in rear of AA', which may be supposed still or turbulent, but at constant atmospheric pressure; and the wake is bounded by the two free surfaces AJ, $A'J'$, extending to infinity at J and J', over which the pressure is constant and atmospheric.

The fluid is bounded by the single stream line $JABA'J'$, over which we take $\psi = 0$ (Fig. 10).

At the branch point B, where the stream divides, the velocity is zero, and $\frac{dw}{du} = 0$ when $u = b$.

The w diagram consists of the single straight line $\psi = 0$, but doubled back on itself at $u = b$, so that coming from infinity at j along the under side of the line with the area to the right, a turn to port must be made on arriving at b by starboarding the helm and the turn must be made through two right angles to return along the upper side of the line, making $\alpha = -\pi$.

As $u = b$ at the only corner of the w diagram,

$$(20) \qquad \frac{dw}{du} = M(u-b)^{-\frac{\alpha}{\pi}} = M(u-b), \quad w - w_B = \tfrac{1}{2}M(u-b)^2$$

and it is convenient to take $\frac{1}{2}M = -1$,

$$(21) \qquad w_B - w = (u-b)^2,$$

as before in (25), Lecture I.

The dotted line bi in the prolongation corresponds to the part of the stream line $\psi = 0$ along the curved dividing line BI in the stream, but this does not form part of the boundary, and along it, $w_B - w$ is negative,

$$(22) \qquad u = b + i\sqrt{(w - w_B)},$$

which is imaginary.

On the Ω diagram, as u diminishes from j to a, $q = Q$, $\log\frac{Q}{q} = 0$, and θ diminishes from α to 0, so that the line ja is described.

Passing from a to b, $\theta = 0$, and ab is described at right angles, and extending to infinity, since $q = 0$ at b, $\log\frac{Q}{q} = \infty$.

As u passes through b, θ changes suddenly from 0 to π, so that the Ω diagram continues in a straight line $b\,a'$ at a height π above ab, and arrives at a' on the line $aj\,a'$ where $\log\frac{Q}{q} = 0$.

Beyond a', from a' to j, $\log \dfrac{Q}{q} = 0$, and θ diminishes from π to α, so that j' rejoins j, and the circuit is complete.

Then in accordance with the fundamental theorem (1) of conformal representation,

$\dfrac{d\Omega}{du}$ is composed of the factors $(u-b)^{-1}$, $(u-a)^{-\frac{1}{2}}$, $(u-a')^{-\frac{1}{2}}$;

(23) $$\frac{d\,\Omega}{du} = \frac{N}{u-b}\,\frac{1}{\sqrt{(u-a,\,u-a')}}$$

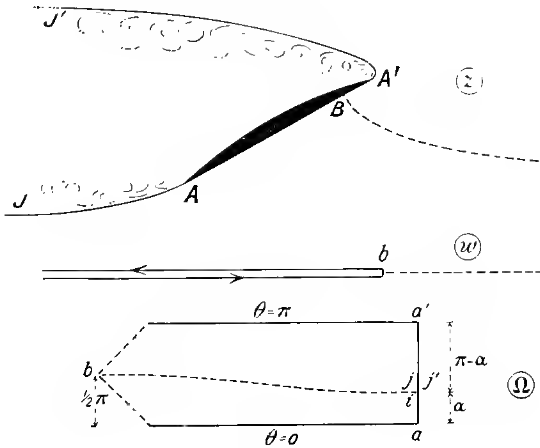


FIG. 10.

In the neighbourhood of $u = b$, when the chief variation is due to $u-b$, we may replace $u-a$, $u-a'$ by $b-a$, $b-a'$, and put

(24) $$\frac{d\,\Omega}{du} = \frac{N}{u-b}\,\frac{1}{\sqrt{(b-a\,.\,b-a')}},$$

and since Ω increases by π as u diminishes through b, equation (14) shows that

(25) $$\frac{N}{\sqrt{(b-a\,.\,b-a')}} = 1,$$

(26) $$\frac{d\,\Omega}{du} = \frac{1}{u-b}\sqrt{\frac{b-a\,.\,b-a'}{u-a\,.\,u-a'}}.$$

D 2

Integrating

(27) $$\Omega = 2\,\mathrm{ch}^{-1}\sqrt{\frac{a-b\,.\,u-a'}{a-a'\,.\,u-b}} = 2\,\mathrm{sh}^{-1}\sqrt{\frac{a'-b\,.\,u-a}{a-a'\,.\,u-b}}$$

$$= 2\log\frac{\sqrt{(a-b\,.\,u-a')}+\sqrt{(a'-b\,.\,u-a)}}{\sqrt{(a-a'\,.\,u-b)}}$$

$$= \log\frac{\sqrt{(a-b\,.\,u-a')}+\sqrt{(a'-b\,.\,u-a)}}{\sqrt{(a-b\,.\,u-a')}-\sqrt{(a'-b\,.\,u-a)}}$$

by theorems, of the Integral Calculus, which ought to be familiar to the student of this subject.

At $u = j$, $\Omega = \alpha i$,

(28) $$\mathrm{ch}\,\tfrac{1}{2}\alpha i = \cos\tfrac{1}{2}\alpha = \sqrt{\frac{a-b\,.\,j-a'}{a-a'\,.\,j-b}}$$

$$\mathrm{sh}\,\tfrac{1}{2}\alpha i = i\sin\tfrac{1}{2}\alpha = i\sqrt{\frac{b-a'\,.\,j-a}{a-a'\,.\,j-b}}$$

and it is convenient to take $j = \infty$, making

(29) $$\cos\tfrac{1}{2}\alpha = \sqrt{\frac{a-b}{a-a'}}, \quad \sin\tfrac{1}{2}\alpha = \sqrt{\frac{b-a'}{a-a'}},$$

(30) $$\mathrm{ch}\,\tfrac{1}{2}\Omega = \cos\tfrac{1}{2}\alpha\sqrt{\frac{u-a'}{u-b}}, \quad \mathrm{sh}\,\tfrac{1}{2}\Omega = \sin\tfrac{1}{2}\alpha\sqrt{\frac{a-u}{u-b}},$$

(31) $$\mathrm{ch}\,\Omega = \mathrm{ch}^2\tfrac{1}{2}\Omega + \mathrm{sh}^2\tfrac{1}{2}\Omega$$

$$= \cos\alpha + \sin^2\alpha\,\frac{a-a'}{u-b},$$

as stated before in (25) I.; also

(32) $$-Q\frac{dz}{dw} = \zeta = \frac{Q}{q}e^{\theta i} = e^{\Omega}$$

$$= \left[\cos\tfrac{1}{2}\alpha\sqrt{\frac{u-a'}{u-b}} + \sin\tfrac{1}{2}\alpha\sqrt{\frac{a-u}{u-b}}\right]^2$$

Over the plane AA', $a > u > a'$, $\theta = 0$, $dz = dx$,

(33) $$Q\,dx = -\zeta\,dw$$

$$= -\left[\cos\tfrac{1}{2}\alpha\sqrt{\frac{u-a'}{u-b}} + \sin\tfrac{1}{2}\alpha\sqrt{\frac{a-u}{u-b}}\right]^2 2\,(b-u)\,du$$

$$= 2[\cos\tfrac{1}{2}\alpha\sqrt{(u-a')} + \sin\tfrac{1}{2}\alpha\sqrt{(a-u)}]^2\,du,$$

and integrating with respect to u, from a' to a,

$$(34) \qquad Q.AA' = \int_{a'}^{a} [2\cos^2 \tfrac{1}{2}\alpha\,(u - a') + 2\sin^2 \tfrac{1}{2}\alpha\,(a - u) + 2\sin\alpha\sqrt{(a - u.\,u - a')}]\,du$$

$$= \cos^2 \tfrac{1}{2}\alpha\,(a - a')^2 + \sin^2 \tfrac{1}{2}\alpha\,(a - a')^2 + \sin\alpha\,\tfrac{1}{4}\pi\,(a - a')^2,$$

since the last integral represents graphically $\sin\alpha$ times the area of a circle on the diameter $a—a'$; and

$$(35) \qquad AA' = (1 + \tfrac{1}{4}\pi\sin\alpha)\frac{(a - a')^2}{Q}.$$

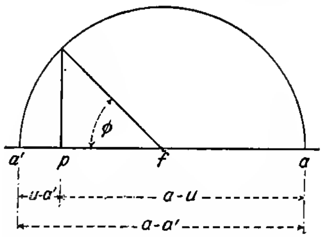


FIG. 11.

To express any length such as $A'P = x$, as a function of u, it is convenient to introduce a variable angle ϕ (not to be confused with the velocity function ϕ), as shown in the diagram (Fig. 11);

$$(36) \qquad a - u = (a - a')\cos^2 \tfrac{1}{2}\phi,\ u - a' = (a - a')\sin^2 \tfrac{1}{2}\phi,$$
$$du = (a - a')\,\tfrac{1}{2}\sin\phi\,d\phi.$$

$$(37) \qquad Qdx = (a - a')^2 \sin^2 \tfrac{1}{2}(\alpha + \phi)\sin\phi\,d\phi$$

$$(38) \qquad \frac{dx}{AA'} = \frac{\sin^2 \frac{1}{2}(\alpha + \phi)\sin\phi\,d\phi}{1 + \frac{1}{4}\pi\sin\alpha}$$

$$= \frac{\frac{1}{2}\sin\phi - \frac{1}{2}\cos\alpha\cos\phi\sin\phi + \frac{1}{2}\sin\alpha\sin^2\phi}{1 + \frac{1}{4}\pi\sin\alpha}d\phi$$

$$(39) \qquad \frac{PA'}{AA'} = \frac{\frac{1}{2}(1 - \cos\phi) - \frac{1}{4}\cos\alpha\sin^2\phi + \frac{1}{4}\sin\alpha(\phi - \sin\phi\cos\phi}{1 + \frac{1}{4}\pi\sin\alpha}$$

and $\dfrac{BA'}{AA'}$ is obtained as in (41), Lecture I., by putting $\phi = \alpha$.

As the velocity diminishes from Q on the skin of the jet at atmospheric pressure to q something less in the interior, the dynamic head diminishes by $\frac{Q^2 - q^2}{2\,g}$, and the static pressure head increases by the same amount; so that the gauge pressure in the interior, the excess over atmospheric pressure, becomes

(40) $$\frac{Q^2 - q^2}{2\,Cg} = \frac{Q^2}{2\,Cg}\left(1 - \frac{q^2}{Q^2}\right).$$

Along AA', $\theta = 0$ (Fig. 12)

(41) $$\zeta = \frac{Q}{q} = \left[\cos\tfrac{1}{2}\alpha\sqrt{\frac{u - a'}{u - b}} + \sin\tfrac{1}{2}\alpha\sqrt{\frac{a - u}{u - b}}\right]^2,$$

$$\frac{q}{Q} = \left[\cos\tfrac{1}{2}\alpha\sqrt{\frac{u - a'}{u - b}} - \sin\tfrac{1}{2}\alpha\sqrt{\frac{a - u}{u - b}}\right]^2,$$

(42) $$Q\,dx = -\frac{Q}{q}dw = \frac{Q}{q}2\,(u - b)\,du,$$

and the thrust dT in the length dx is given by

(43) $$dT = \frac{Q^2}{2\,Cg}\left(1 - \frac{q^2}{Q^2}\right)dx = \frac{Q^2}{2\,Cg}\left(1 - \frac{q^2}{Q^2}\right)\frac{Q}{q}\,2\,(u - b)\,\frac{du}{Q}$$

$$= \frac{Q}{Cg}\left(\frac{Q}{q} - \frac{q}{Q}\right)(u - b)\,du$$

$$= \frac{Q}{Cg}.\quad 2\sin\alpha\sqrt{(a - u\,.\,u - a')}\,du$$

$$= \frac{Q}{2\,Cg}\sin\alpha\,(a - a')^2\sin^2\phi\,d\phi,$$

and integrating for the thrust $T\,(PA')$ over PA',

(44) $$T(PA') = \frac{Q}{4\,Cg}\sin\alpha\,(a - a')^2\,(\phi - \sin\phi\cos\phi)$$

(45) $$T(AA') = \frac{Q}{4\,Cg}\sin\alpha\,(a - a')^2\,\pi$$

(46) $$\frac{T(AA')}{AA'} = \frac{Q^2}{Cg}\,\frac{\frac{1}{4}\pi\sin\alpha}{1 + \frac{1}{4}\pi\sin\alpha},$$ lb/ft^2 of average pressure,

(47) $$\frac{T(PA')}{T(AA')} = \frac{\phi - \sin\phi\cos\phi}{\pi}$$

(48) $$\frac{dT}{T(AA')} = \frac{1 - \cos 2\,\phi}{\pi}\,d\phi.$$

Taking moments round A' to determine L,

(49) $$LA' \,.\, T(AA') = \int x\,dT,$$

(50) $$\frac{LA'}{AA'} = \int \frac{x}{AA'} \cdot \frac{dT}{T(AA')}$$

$$= \frac{1}{1 + \frac{1}{4}\pi \sin\alpha} \int_o^\pi \left[\tfrac{1}{2}(1 - \cos\phi) - \tfrac{1}{4}\cos\alpha\sin^2\phi + \tfrac{1}{4}\sin\alpha\,(\phi - \sin\phi\cos\phi)\right] \frac{1 - \cos 2\phi}{\pi}\, d\phi$$

$$= \frac{1}{1 + \frac{1}{4}\pi\sin\alpha} \int (1 - \cos\phi - \tfrac{1}{2}\cos\alpha\sin^2\phi + \tfrac{1}{2}\phi\sin\alpha - \tfrac{1}{2}\sin\alpha\sin\phi\cos\phi - \cos 2\phi + \cos\phi\cos 2\phi + \tfrac{1}{2}\cos\alpha\sin^2\phi\cos 2\phi - \tfrac{1}{2}\sin\alpha\,\phi\cos 2\phi + \tfrac{1}{2}\sin\alpha\sin\phi\cos\phi\cos 2\phi)\,\frac{d\phi}{2\pi}.$$

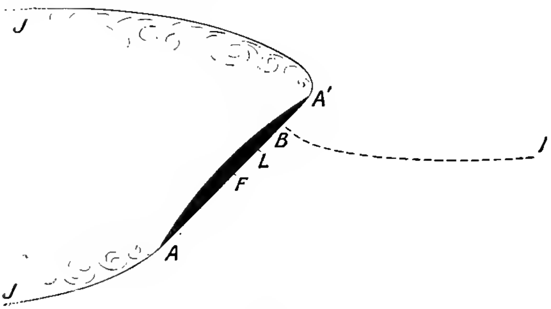


FIG. 12.

Omitting the terms which vanish by inspection,

(51) $$\frac{LA'}{AA'}(1 + \tfrac{1}{4}\pi\sin\alpha) = \int_o^\pi (1 - \tfrac{1}{2}\cos\alpha\sin^2\phi + \tfrac{1}{2}\phi\sin\alpha + \tfrac{1}{2}\cos\alpha\sin^2\phi\cos 2\phi)\,\frac{d\phi}{2\pi}$$

$$= (\pi - \tfrac{1}{4}\pi\cos\alpha + \tfrac{1}{4}\pi^2\sin\alpha - \tfrac{1}{8}\pi\cos\alpha) \div 2\pi$$

$$= \tfrac{1}{2} + \tfrac{1}{8}\pi\sin\alpha - \tfrac{3}{16}\cos\alpha$$

(52) $$\frac{LA'}{AA'} = \tfrac{1}{2} - \frac{\frac{3}{16}\cos\alpha}{1 + \frac{1}{4}\pi\sin\alpha}$$

(53) $$\frac{FL}{AA'} = \frac{\frac{3}{16}\cos\alpha}{1 + \frac{1}{4}\pi\sin\alpha}$$

This last operation for the determination of L is due to Lord Rayleigh; although not so very formidable after all, it must have appeared so to the inventor working out these integrations for the first time with no knowledge of the result.

The three integrations, of thrust T, length x, and centre of pressure L, are typical of what is required in the other extensions of this Kirchhoff problem, which follow in Report 19.

A change of Ω into $n\Omega$ will give the fluid motion past the barrier AA' when it is bent at B to an angle $\frac{\pi}{n}$; and then $A\,B\,A'$ may replace a cambered wing, touching at A and A'.

But now the integrations for $A'B$, BA become intractable, and so also for the thrust T, because

$$(54) \qquad \zeta^n = \left[\cos \tfrac{1}{2}\, a \sqrt{\frac{u-a'}{u-b}} + \sin \tfrac{1}{2}\, a \sqrt{\frac{a-u}{u-b}}\right]^2,$$

$$(55) \qquad Q\frac{dz}{du} = 2\,(u-b)\,\zeta$$

$$= 2\,(u-b)^{1-\frac{1}{n}}\left[\cos \tfrac{1}{2}\, a\ \sqrt{(u-a')} + \sin \tfrac{1}{2}\, a\ \sqrt{(a-u)}\right]^{\frac{2}{n}}.$$

The lift of an aeroplane AA' arises from the gauge pressure on the under face, due to the defect in velocity q, and the opening out of the stream lines in Fig. 12.

The pressure in rear of the plane in its wake is taken as atmospheric, up to and along the skin of the bounding stream AJ and $A'J'$, the air behind being at rest relatively to AA', or in a state of vortical turbulence at average atmospheric pressure.

Any thrust on the plane AA' must be due to an excess of pressure over the atmosphere, *gauge* pressure so called, not *absolute*, on the under attacking face; this gauge pressure arises from a diminution of velocity of q below Q, and diminution of dynamic head, from $\frac{Q^2}{2g}$ to $\frac{q^2}{2g}$, and an equivalent rise of static pressure head, due to the broadening of the interval between the stream lines close to AA', especially in the neighbourhood of B, the branch point.

This shows in a general way why L, the $C.P.$ (centre of pressure) is away from F, the $C.F.$ (centre of figure) and more towards B, so that the tendency of the fluid reaction is to turn the plane more broadside to the stream.

A popular figure of the stream lines past a cambered wing, as here in Fig. 13, showing no such broadening, would imply at once to our eye an absence of all thrust and lift; the figure should be more like Fig. 14.

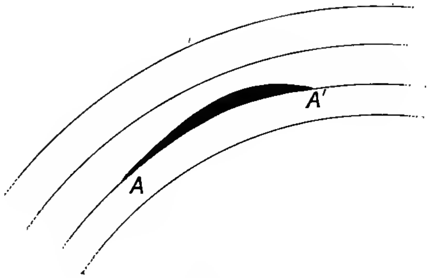


FIG. 13.

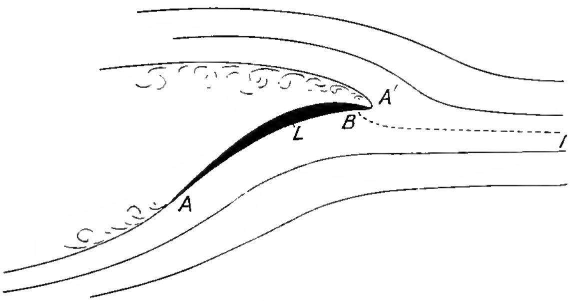


FIG. 14.

In the electrical law of flow where there is no wake, as here in Colonel Hippisley's diagram (Fig. 15), the broadening of the stream is shown near a branch point; but as the stream lines close in behind symmetrically, there is no resultant thrust on the cylinder by the stream tending to move it, but a couple only tending to turn the cylinder broadside to the current, arising from excess of pressure near the two opposite branch points.

The magnitude of the couple on an elliptic cylinder in found to be $\frac{Q^2}{2g}$ sin 2α times the weight of the column of liquid with a circular base on a diameter joining the foci.

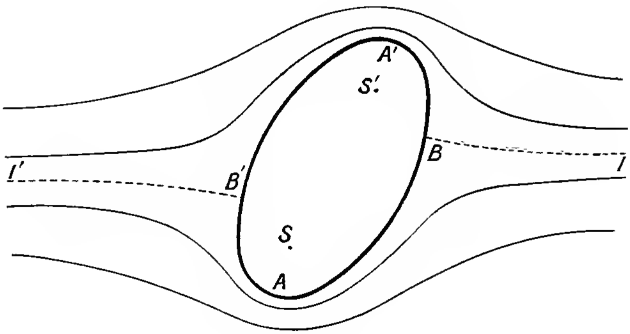


FIG. 15.

The couple is shown experimentally by letting a card drop in the air from the horizontal or vertical position, or by projecting it with a spin, observing how the card turns gradually into a vertical plane.

So, too, the stability is assured of the forward course of the flying machine by making the spread of the wings three or four times the depth from front to back. It is supposed, too, that the stability can be improved by giving the wings a dihedral angle, as at the top of the stroke of the wings of a bird.

The existence of a couple on a flat or elongated body moving in a medium, air or water, tending to set it broadside to the motion is discussed in Thomson and Tait's "Natural Philosophy," and in the Hydrodynamical Treatises of Basset and Lamb, in a treatment rather abstruse.

But an explanation can be given in the manner called "elementary," depending on the principle of momentum.

Take a body like an elliptic cylinder, as in Colonel Hippisley's diagram (Fig. 16), and supposing it moving with velocity Q, and velocity components U and V, being held so as to prevent rotation about the axis of the cylinder.

If there is no surrounding medium, the components of momentum, $W\frac{U}{g}$ and $W\frac{V}{g}$ sec-lb of the body weighing W lb will remain unaltered while the centre moves from O to O', so that the vector OH of momentum will move to $O'H'$ in the same straight line OO', and there is no change of momentum and no force or constraint is required.

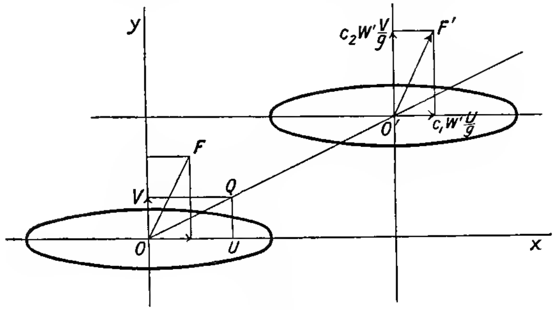


FIG. 16.

But if the body is surrounded by a medium, of which it displaces W' lb, the velocity components U and V will give rise to momentum-components in the medium, $c_1W'\frac{U}{g}$, $c_2W'\frac{V}{g}$, where c_1 and c_2 are different constant numbers depending on the shape of the body; and as the broadside motion will give the greater momentum, $c_2>c_1$.

The body must now be held to prevent rotation; because at O' the momentum of the medium has changed from OF to $O'F'$, not in the same straight line OO'; and the change of momentum is the impulse couple $N_1 = OF\,.\,OD$, acting on the medium, against the clock, OD being the perpendicular on $O'F'$.

The medium reacts on the body with an equal and opposite impulse couple N_1, tending to set it broadside; and if t seconds is the time from O to O', the impulse couple given by its components against the clock is

$$(56)\qquad N_1 = c_2W'\frac{V}{g}.\ Ut - c_1W'\frac{U}{g}.\ Vt$$

$$= (c_2 - c_1)\,W'\frac{UV}{g}\,t, \quad \text{ft-lb-sec,}$$

the accumulated effect in a time t of an incessant couple

$$(57) \qquad N = (c_2 - c_1) W' \frac{UV}{g}, \text{ ft-lb,}$$

and this is the expression found in Hydrodynamics.

For an elliptic cylinder, as in the diagram (Fig. 17), it is found by theory that

$$(58) \qquad c_1 = \frac{b}{a}, \quad c_2 = \frac{a}{b};$$

and for a length l ft in a medium of density w lb/ft^3, $W' = \pi wabl$,

$$(59) \qquad N = \left(\frac{a}{b} - \frac{b}{a}\right) \pi w\, ab\, l \frac{UV}{g}$$

$$= \pi w (a^2 - b^2) l \frac{UV}{g} = W'' \frac{UV}{g}, \quad \text{ft-lb,}$$

where W'' is the weight of medium displaced by a cylinder of cross section the circle on the diameter SS' joining the foci S and S'.

This reduces for a card of breadth $2a$, with $b = 0$, to

$$(60) \qquad N = \pi w a^2 l \frac{UV}{g} = \frac{Q^2}{Cg} A a \pi \sin \alpha \cos \alpha,$$

assuming the electrical law of flow round the edge A and A'.

But this would be qualified by the factor

$$\frac{\frac{3}{32}}{(1 + \frac{1}{4}\pi \sin \alpha)^2}$$

in Kirchoff's discontinuous motion, where the couple, in accordance with (53), would be

$$(61) \qquad \frac{Q^2}{Cg} A a \pi \sin \alpha \cos \alpha \frac{\frac{3}{32}}{(1 + \frac{1}{4}\pi \sin \alpha)^2}.$$

The calculation is not simple of c_1 and c_2 for other figures, such as a solid of revolution; but it is required in the discussion of the stability of an air ship or submarine boat, in its effect on steering and loss of metacentric height.

In artillery the theory is useful for determining on gyroscopic principles the spin requisite for stability of an elongated shot fired from a rifled gun.

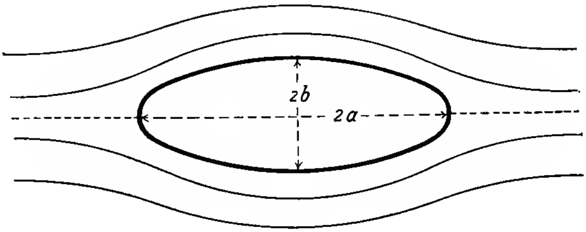


FIG. 17.

When the external shape is assimilated to a prolate spheroid, generated by the revolution of an ellipse of axes $2a$, $2b$ (Fig. 17), and when the length in diameters $2a/2b$ is denoted by x, we find by hydrodynamical theory, on the electrical law of flow,

$$(62) \qquad c_1 = \frac{A}{1 - A}, \quad c_2 = \frac{B}{1 - B},$$

where A and B are determined from the equations,

$$(63) \quad A + 2B = 1,\ x^2A + 2B = \frac{x \operatorname{sh}^{-1} \sqrt{(x^2 - 1)}}{\sqrt{(x^2 - 1)}}$$

$$= \frac{x}{\sqrt{(x^2 - 1)}} \log_e [x + \sqrt{(x^2 - 1)}]$$

changing for an oblate spheroid, $x < 1$, to

$$\frac{x \sin^{-1} \sqrt{(1 - x^2)}}{\sqrt{(1 - x^2)}},$$

For a sphere, $x = 1$, $A = B = \frac{1}{3}$, $c_1 = c_2 = \frac{1}{2}$, and the effective inertia of a sphere is increased by half the weight of the medium displaced.

But for broadside motion of a cylinder, $x = \infty$, $A = 0$, $B = \frac{1}{2}$, $c_2 = 1$; so that the inertia is increased by the weight of the medium displaced.

Thus a spherical balloon or cylindrical air ship of weight W lb, displacing W' lb of air, will start to rise with acceleration

$$(64) \qquad \frac{W' - W}{\frac{1}{2} W' + W} g, \text{ or } \frac{W' - W}{W' + W} g.$$

A spherical bubble of air in water, where W is insensible compared with W', will start up with acceleration $2g$.

A flexible sheet in tension t, separating the current of density w from a dead wake of different density w', as with water past air, and flapping like a flag or shivering as a sail, would swing in waves of length λ advancing with velocity U, such that in accordance with wave motion theory

$$(65) \qquad \frac{2\pi t}{\lambda} = w (U - Q)^2 + w' U^2$$

$$U = \frac{w}{w + w'} Q + \sqrt{\left[\frac{2\pi t}{(w + w')\lambda} - \frac{ww'}{(w + w')^2} U^2\right]}$$

so that the waves break up into vortex motion of

$$(66) \qquad \frac{2\pi t}{\lambda} < \frac{ww'}{w + w'} U^2.$$

The practical question arises as to the advantage of a short length of fringe or flap at the edge A of the aeroplane, so as to retard the formation of a vortex, such as seen at the lee of the chimney top. Some such arrangement can be seen in the new flying machine.

In the absence of these wings the equipotential lines of the electric field would bend round the barrier symmetrically (as in Fig. 18), and IB would be the branch of a hyperbola with foci at A and A', continued on the other branch $B' I'$ at zero potential.

This electric field would represent the analogous streaming motion, realised by coloured filaments in Hele Shaw's experiments, in a viscous liquid, where the motion is slow and no perceptible eddy is formed at A and A'.

The liquid is then said to stream on the electrical law of flow; and the more general case where AA' opens out into a confocal ellipse is shown in Colonel Hippisley's diagram here (Fig. 15), which can be interpreted electrically as representing the disturbance in a uniform field by the presence of an elliptic cylinder to earth.

A magnetic interpretation can also be given to his diagram as a generalisation of Maxwell's Figure XV., of a magnetised cylinder in a uniform magnetic field.

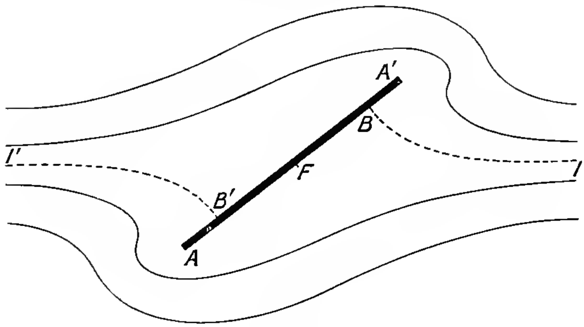


Fig. 18.

It is gratifying and of great assistance, too, when the mathematical analysis of a mechanical question can be made to serve in another interpretation; and so the student of electricity should be interested in following up the electrical analogue of this hydrodynamical problem.

We must suppose the uniform horizontal current to represent a uniform vertical electrical field, and this field to be disturbed by the uninsulated plane strip AA'; while the skin stream lines AJ, $A'J'$ must be replaced by wings of flexible gold or tin foil, as in the electrometer, but stretching away to infinity.

The wake of the plane AA', which to the pilot seems stationary behind him, is being dragged along through the air by the flying machine.

Any machine following which flies into this wake or backwash will seem as if it is entering still air; the lift is lost and the machine will drop. This may happen if the pilot is following at a different level, above or below, so that in passing another machine it is prudent for the pilot to take a course to one side.

LECTURE III

HELMHOLTZ-KIRCHHOF THEORY OF A DISCONTINUOUS STREAM LINE

THE normal conditions of the aeroplane of a machine flying high up in the air are represented on the Kirchhoff theory by the diagram 4 and 10, representing the state of flow relative to the pilot; the analytical conditions have been discussed in Lecture I. and II., and will be found in § 6 of Report 19.

But there is an advantage in extending the theory to a more general case, and beginning as in Report 19, § 2, with a plane barrier oblique to a stream of finite breadth (Fig. 19), as the extension is not essentially more complicated, and it throws light on the simple case of an infinite stream.

No change is required in the Ω diagram (Fig. 19), except that i, j, j', on aa' are now distinct; but now in the w diagram the outside stream lines come from infinity into view, and the diagram has the advantage of being closed, as in Euclid I. 27, so that the difficulties at infinity are kept under observation.

In the w diagram there are three stream lines in the boundary, which we denote by $\psi = m_1, m_2, m_3$; and $\psi = m_1$ is the outer stream line from I to J; $\psi = m_2$ the inner skin from J to A, from A along AA' to the branch point B and then to A', and again along $A'J'$; $\psi = m_3$ is the other outer stream line from J' back to I.

There is also the curved branch IB of the $\psi = m_2$ stream line along which the current divides; but as IB lies in the interior of the fluid, and the velocity q and its direction θ both vary along IB, the corresponding part ib in the w and Ω diagram must be excluded from the boundary; ib is straight in the w diagram, but curved in the Ω diagram, from b at midway height $\frac{1}{2}\pi$ of a and a', to i at height a.

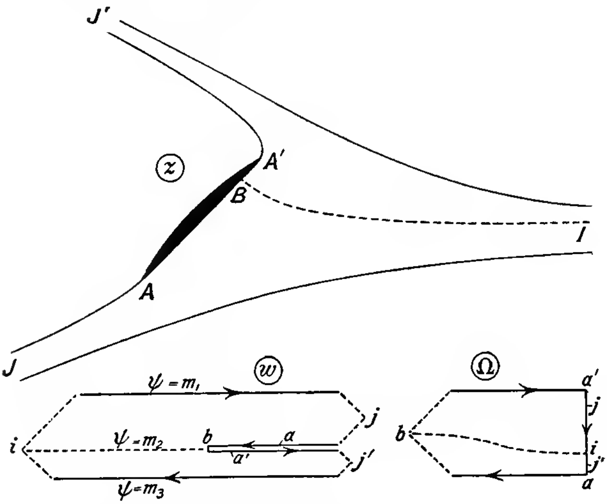


FIG. 19.

The branch point B where the velocity is zero requires $\frac{d\phi}{du} = 0$ at $u = b$, so that, by II. (1),

(1) $$\frac{dw}{du} = M \frac{u - b}{u - j \,.\, u - j' \,.\, u - i}$$

$$= \frac{m_1 - m_2}{\pi} \frac{1}{u - j}$$

$$+ \frac{m_2 - m_3}{u - j'} + \frac{m_3 - m_1}{\pi} \cdot \frac{1}{u - i}$$

$$= \frac{m_1 - m_2}{\pi}\left(\frac{1}{u - j} - \frac{1}{u - i}\right) + \frac{m_2 - m_3}{\pi}\left(\frac{1}{u - j'} - \frac{1}{u - i}\right)$$

when resolved into partial fractions, with the factor appropriate for making w change suddenly by $m_1 - m_2$, $m_2 - m_3$, $m_3 - m_1$, as u passes through j, j', i, in accordance with II. (15).

Integrating,

(2) $w + \text{a constant} = \frac{m_1 - m_2}{\pi} \log \frac{u - j}{u - i} + \frac{m_2 - m_3}{\pi} \log \frac{u - j'}{u - i};$

and with origin at B,

(3) $$w - w_B = \frac{m_1 - m_2}{\pi} \log \frac{u - j}{u - i} \cdot \frac{b - i}{b - j}$$
$$+ \frac{m_2 - m_3}{\pi} \log \frac{m - j'}{m - i} \cdot \frac{b - i'}{b - j'},$$

and taking $i = \infty$ simplifies this to

(4) $$w - w_B = \frac{m_1 - m_2}{\pi} \log \frac{u - j}{b - j} + \frac{m_2 - m_3}{\pi} \log \frac{u - j'}{b - j'}.$$

The z diagram is now as in Fig. 19, and if c_1, c_2 denotes the breadth of the stream at J and J', where the velocity has become uniform and equal to the skin velocity Q,

(5) $$Qc_1 = m_1 - m_2,\ Qc_2 = m_2 - m_3;$$

and putting $c_1 + c_2 = c$, the breadth of the impinging stream at I,

(6) $$Qc = m_1 - m_3;$$

and the curved part BI of the dividing stream line $\psi = m_2$ separates the stream at I into two parts of breadth c_1 and c_2.

Then we can write

(7) $$\pi \frac{w - w_B}{Q} = c_1 \log \frac{u - j}{u - i} \cdot \frac{b - i}{b - j} + c_2 \log \frac{u - j'}{u - i} \cdot \frac{b - i}{b - j'},$$

and at B, where

(8) $$u = b,\ \frac{dw}{du} = 0,$$
$$c_1 \left(\frac{1}{b - j} - \frac{1}{b - i}\right) + c_2 \left(\frac{1}{u - j'} - \frac{1}{u - i}\right) = 0;$$

and with the sequence

$$i\ (\infty) > j > a > b > a' > j' > i\ (-\infty),$$

(9) $$\frac{c_1}{c_2} = \frac{\dfrac{1}{b - j'} + \dfrac{1}{i - b}}{\dfrac{1}{j - b} - \dfrac{1}{i - b}} = \frac{i - j' \,.\, j - b}{i - j \,.\, b - j'}$$

With the value of Ω as before in II. (27), we must write in the region $i > u > a$,

(10) $$\Omega = \log \frac{Q}{q} + \theta i$$

$$= 2i \cos^{-1} \sqrt{\frac{a - b \,.\, u - a'}{a - a' \,.\, u - b}} = 2i \sin^{-1} \sqrt{\frac{b - a' \,.\, u - a}{a - a' \,.\, u - b}}$$

so that $\log \dfrac{Q}{q} = 0$, $q = Q$, as required over the skin of IJ and JA; and

(11) $$\cos^2 \tfrac{1}{2}\theta = \frac{a - b \,.\, u - a'}{a - a' \,.\, u - b}, \quad \sin^2 \tfrac{1}{2}\theta = \frac{b - a' \,.\, u - a}{a - a' \,.\, u - b};$$

and ψ being constant,

(12) $$\frac{dw}{du} = \frac{d\phi}{du} = \frac{d\phi}{ds}\frac{ds}{du} = -Q\frac{ds}{du}, \; w = \text{constant} - Qs;$$

and thence the intrinsic equation is derived of IJ and JA, as given in § 2, Report 19.

The linear scale of the diagram is calculated as explained in § 3 by an integration, expressed in terms of the arbitrary constants a, a', b, j, j', i, in a transcendental form, from which the length AA', AB, BA' is calculated, and the angle α, β, β', at which the stream from I is received by the plane and leaves it again at J, J'.

The inverse process to determine the constants a, a', b, j, j', i from the dimensions of the diagram would be intractable analytically, and would require the trial and error process.

The thrust T is obtained immediately in this case, and without integration, by the principle of momentum; resolving perpendicular to the plane AA,'

(13) $$T = (m_1 - m_3)\frac{Q}{Cg}\sin\alpha - (m_1 - m_2)\frac{Q}{Cg}\sin\beta - (m_2 - m_3)\frac{Q}{Cg}\sin\beta'$$
$$= \frac{Q^2}{Cg}(c\sin\alpha - c_1\sin\beta - c_2\sin\beta').$$

Resolving parallel to the plane

(14) $$0 = (m_1 - m_3)\frac{Q}{Cg}\cos\alpha - (m_1 - m_2)\frac{Q}{Cg}\cos\beta - (m_2 - m_3)\frac{Q}{Cg}\cos\beta.$$

(15) $$0 = c\cos\alpha - c_1\cos\beta - c_2\cos\beta',$$

which is found to verify.

But the determination of L the $C.P.$ is intractable for a stream of finite breadth.

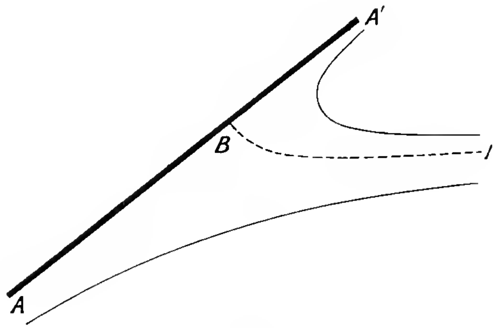


FIG. 20.

The extreme case of an infinite barrier AA' receiving the impact of a jet of finite width is obtained, as shown in Fig. 20, by making $j = a$, $j' = a'$.

Duplication in AA' with the motion reversed will give the direct impact of two unequal jets with equal velocity, as in Fig. 21.

A simple experimental illustration can be given of these problems with a blade held in a jet of water from a tap, or with impinging gas jets.

To return to an aeroplane high up in the air, in a relative current of infinite breadth, we must take i, j, j' equal, and the integration of $\frac{dw}{du}$ leads to an algebraical relation for w, not logarithmic;

(16) $$\frac{dw}{du} = M\frac{u-b}{(u-i)^3}, w - w_B = \frac{\frac{1}{2}M}{i-b}\left(\frac{u-b}{u-i}\right)^2;$$

and writing $\frac{1}{2}M = (i-b)^3$,

(17) $$w - w_B = (u-b)^2\left(\frac{b-i}{u-i}\right)^2,$$

and then, as in (21) II., by taking $i = \infty$,

(18) $$w - w_B = (u-b)^2.$$

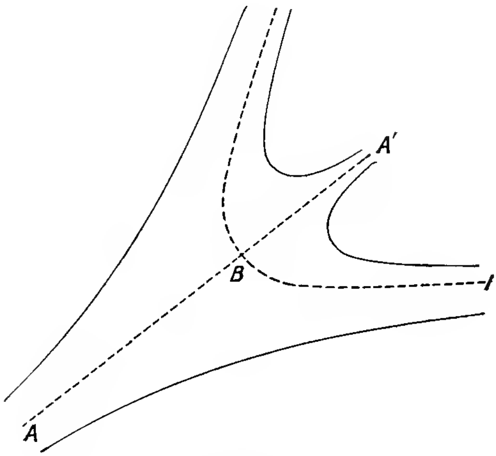


FIG. 21.

But it is quite as simple, and more instructive, to consider a general case, especially as we are then at liberty to alter the sequence of the arbitrary constants, and so make the same analysis serve for different hydrodynamical problems, hitherto considered as distinct.

Previous writers on this subject strive to give special values of these arbitrary constants, such as $Q = 1$, and $a = +1$ at A, $a' = -1$ at A'; but, as we shall see, at a loss of flexibility in the analysis.

The only special value we have adopted so far, is $i = \infty$ occasionally, but better only after the general case has been examined, as the passage to $i = \infty$ is sometimes treacherous.

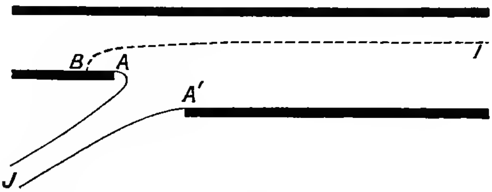


FIG. 22.

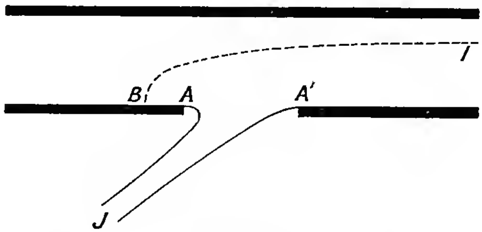


FIG. 23.

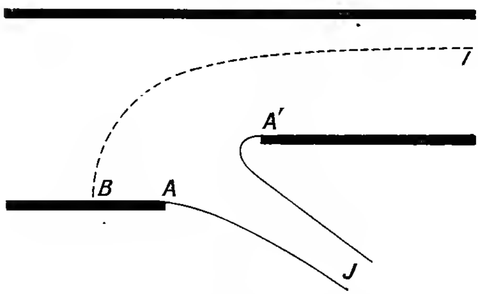


FIG. 24.

To illustrate the flexibility of the analysis, we examine the interpretation of (4), (10) due to a rearrangement of the constants, in the order

(19) $$i(\infty) > a > j > a' > b > j' > i(-\infty).$$

There is no alteration in the w and Ω diagram and relation, but the z diagram changes as shown in Figs. 22, 23, 24, where the barrier AA' is replaced by a slit or leak made in a plane boundary.

If the stream past the slit is very broad, we make $j' = i$, and $m_3 = \infty$, as in § 14, Report 19.

These extensions give an insight into the subject from the point of view of generality, without additional difficulty, and make the initial problem of Kirchhoff appear much easier.

In continuation of these extensions, bring the blade in the experiment close to the tap to imitate the motion when the efflux from a channel is received on the plate (Fig. 25).

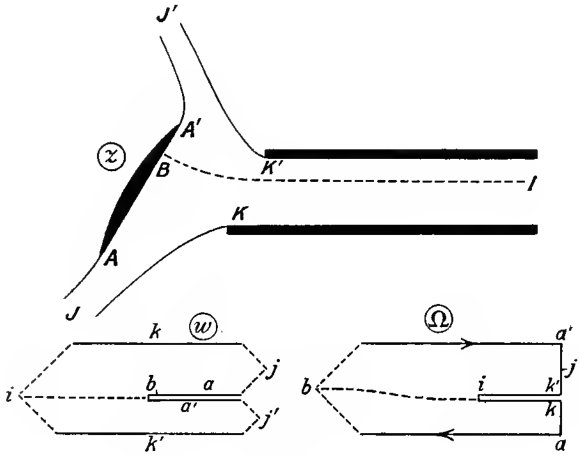


FIG. 25.

Then if K, K' are the ends of the channel in the z diagram, the Ω diagram is modified by an inset kik'; so that with four right angles at $u = a, a', k, k'$, and a zero angle at b, and a turn to port through two right angles at i,

$$(20) \qquad \frac{d\Omega}{du} = N \frac{u - i}{u - b}, \frac{1}{\sqrt{(u - a \,.\, u - a' \,.\, u - k \,.\, u - k')}};$$

and with

$$(21) \qquad (u - b) \frac{d\Omega}{du} = 1, \quad \text{at } u = b,$$

$$N = \frac{\sqrt{(b - a \,.\, b - a' \,.\, b - k \,.\, b - k')}}{b - i};$$

so that, writing

(22) $$u - a \,.\, u - a' \,.\, u - k \,.\, u - k' = U,$$
$$b - a \,.\, b - a' \,.\, b - k \,.\, b - k' = B,$$

(23) $$\frac{d\Omega}{du} = \frac{u - i}{u - b \,.\, b - i} \frac{\sqrt{B}}{\sqrt{U}} = \frac{1}{b - i} \frac{\sqrt{B}}{\sqrt{U}} + \frac{1}{u - b} \frac{\sqrt{B}}{\sqrt{U}}$$

introducing the elliptic integral of the first and third kind (*I. E.I.* and *III. E.I.*).

But there is no alteration in the w diagram and relation of Figs. 10 and 19; and as we proceed to problems of increasing generality, the diagrams may change one at a time, of w and Ω.

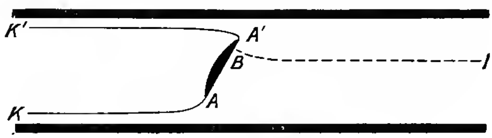


FIG. 26.

The barrier AA' may be placed inside the entrance KK' of the channel, as in Fig. 26, representing a rudder boxed in.

By making $j = k$, $j' = k'$, the barrier becomes of unlimited length, with a barrier AA' across it aslant.

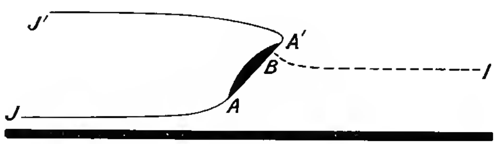


FIG. 27.

Further, by taking $j' = k' = i$, the upper barrier IK' is at infinite height, as in Fig. 27, and the analysis will serve for an aeroplane flying horizontally near the ground, as in making a start in flight.

The analysis is seen to be too complicated for practical application, but § 11 and § 59 of Report 19 show how a solution can be obtained of quasi-algebraical character, when the barrier or rudder AA' is set at an angle $\frac{1}{2}\pi/n$, an aliquot part n of a right angle, the *III. E.I.* now becoming pseudo-elliptic; and for the identification of results it is convenient to take $k' = \infty$,

$$\frac{\sqrt{B}}{\sqrt{U}} = \sqrt{\frac{b-a \,.\, b-a' \,.\, b-k}{u-a \,.\, u-a' \,.\, u-k}} \tag{24}$$

A biplane machine, with two decks, AA', KK', would have the z diagram, with w and Ω as in Fig. 28, and then, with $\psi = m, m'$ over AA', KK'

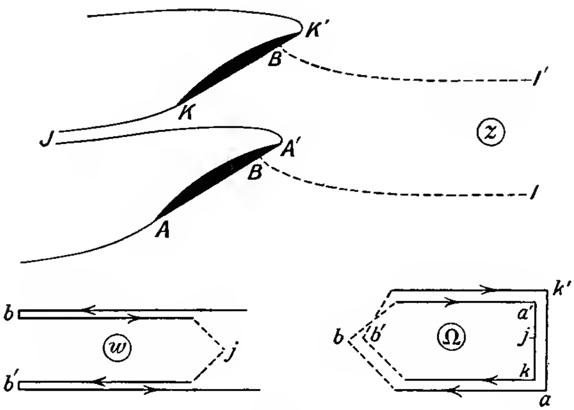


FIG. 28.

$$\frac{dw}{du} = M\frac{u-b \,.\, u-b'}{u-j} = \frac{m-m'}{\pi}\left(\frac{u+j-b-b'}{j-b \,.\, j-b'} + \frac{1}{u-j}\right) \tag{25}$$

$$\pi\frac{w-w_A}{m-m'} \tag{26}$$

$$= \tfrac{1}{2}\frac{(u+k-2b)(u+k-2b')-(a+k-2b)(a+k-2b')}{(j-b)(j-b')}$$

$$+ \log\frac{u-j}{a-j}$$

$$\frac{d\Omega}{du} = \frac{N}{u-b \,.\, u-b'} \cdot \frac{1}{\sqrt{(u-a \,.\, u-a' \,.\, u-k \,.\, u-k')}} \tag{27}$$

$$= \frac{1}{u-b}\sqrt{\frac{B}{U}} - \frac{1}{u-b'}\sqrt{\frac{B'}{U}}$$

introducing the *III. E.I.* again.

Experiment with a fork, two-pronged, held under a tap.

In the short time at disposal in these Lectures it is not possible to follow the elliptic integral interest much further; but these aeronautical applications are calculated to give a rapid insight for those who wish to study the subject.

Thus in § 37, Report 19, an application is made to the vortex in a polygon, an eddy whirlwind, such as Chavez would have to negotiate in a corner of the precipices, when crossing the Simplon, 1909.

The analysis is the same as that required for the electric field of a prismatic condenser, of which the cross section is bounded by concentric similar polygons, with a dielectric interspace; and it is the mean equipotential surface which corresponds to a free surface of the vortex in a polygon (Fig. 29).

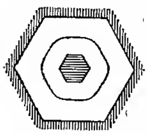


FIG. 29.

A single vortex travelling parallel to the ground, or a pair of such vortices duplicated by reflexion in a plane, will have a curious vortex sheet surface, such that any point on the surface will trace out an *Elastica*, the curve made by bending a straight spring; much as a point on the rim of a wheel will trace out a cycloid over the road, while a particle of the tire describes a circle relatively to the carriage.

The trail of straw and paper of a windy day, left on the pavement near a wall, will imitate the general appearance of the curve for a vertical vortex, such as the pilot may be required to negotiate high up in the air.

But it is the motion of a horizontal vortex near the ground which will increase the difficulty of alighting.

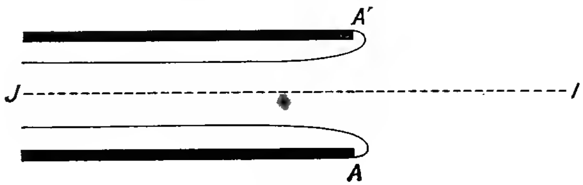


FIG. 30.

The hydrodynamical application of conformal representation originated with Helmholtz in 1868, when he applied it to the efflux of a stream between two parallel walls, extending into the fluid, as in Fig. 30, of a Borda mouthpiece.

But as the general solution is analytically of equal simplicity for a stream issuing from a channel between two parallel walls $I'A'$, IB, blocked partially by a lip BA, as in Fig. 31, we resume the consideration of conformal representation with this application, and treat the original 1868 problem of Helmholtz as a particular case.

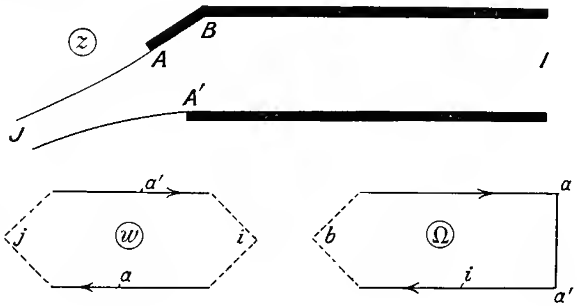


FIG. 31.

The fluid is bounded by two stream lines, $\psi = 0$ and $\psi = m$, so that the w diagram consists of two parallel straight lines, joined up at i and j, as in Euclid I., 27.

The Ω diagram is open rectangle of semi-infinite length, with right angles at $u = a, a'$, and stretching to infinity at $u = b$.

Then

$$(28)\qquad \frac{dw}{du} = \frac{M}{u - i \,.\, u - j} = \frac{m}{\pi}\frac{1}{u - j} - \frac{m}{\pi}\cdot\frac{1}{u - i}.$$

$$(29)\qquad \pi\frac{w - w_B}{m} = \log\frac{u - j}{u - i}\cdot\frac{b - i}{b - j},$$

reducing, when we take $i = \infty$, to

$$(30)\qquad \pi\frac{w - w_B}{m} = \log\frac{u - j}{b - j};$$

and a further simplification can be made by taking $j = 0$,

$$(31)\qquad e^{\pi\frac{w - w_B}{m}} = \frac{u}{b},\quad e^{\pi\frac{w - w_A}{m}} = \frac{u}{a};$$

with the sequence

$$(32) \qquad i\,(\infty) > b > a > 0 > a' > i'\,(-\infty).$$

For the Ω diagram

$$(33) \qquad \frac{d\Omega}{du} = \frac{N}{u-b} \cdot \frac{1}{\sqrt{(u-a\,.\,u-a')}},$$

with

$$(34) \qquad (u-b)\frac{d\Omega}{du} = \frac{\frac{\pi}{2n}}{\pi} = \frac{1}{2n}, \text{ when } u = b,$$

if the exterior angle at B is $\frac{1}{2}\pi/n$, one nth of a right angle;

$$(35) \qquad \frac{d\Omega}{du} = \frac{1}{2n} \cdot \frac{1}{u-b} \cdot \sqrt{\frac{b-a\,.\,b-a'}{u-a\,.\,u-a'}},$$

and thence, by the previous integration in II. (27),

$$(36) \qquad \zeta^n = e^{n\Omega} = \frac{\sqrt{(b-a\,.\,u-a')} + \sqrt{(b-a'\,.\,u-a)}}{\sqrt{(a-a'\,.\,u-b)}}.$$

Along $I'A'$, $-\infty < u < a'$, $\theta = 0$,

$$(37) \qquad \zeta^n = \left(\frac{Q}{q}\right)^n = \sqrt{\frac{b-a'\,.\,a-u}{a-a'\,.\,b-u}} + \sqrt{\frac{b-a\,.\,a'-u}{a-a'\,.\,b-u}}.$$

Along $A'J$, $a' < u < j$, $q = Q$,

$$(38) \qquad \zeta^n = e^{n\delta i} = \sqrt{\frac{b-a'\,.\,a-u}{a-a'\,.\,b-u}} + i\sqrt{\frac{b-a\,.\,u-a'}{a-a'\,.\,u-b}},$$

and so also along JA, $j < u < a$, $q = Q$;
and measuring the arc s from A',

$$(39) \qquad Qs = \phi - \phi_{A'} = w - w_{A'} = \frac{m}{\pi}\log\frac{u}{a'};$$

and if c is the breadth of the jet at J,

$$(40) \qquad Qc = m, \quad u = a'e^{\pi\frac{s}{c}},$$

which combined with

$$(41) \qquad \cos^2 n\theta = \frac{b - a' \,.\, a - u}{a - a' \,.\, b - n}, \qquad \sin^2 n\theta = \frac{b - a \,.\, u - a'}{a - a' \,.\, b - n}$$

will give the intrinsic equation of $A'J$ or JA.

If $\theta = \beta$ at J, where $u = j = 0$,

$$(42) \qquad \cos^2 n\beta = \frac{b - a' \,.\, a}{a - a' \,.\, b} = \frac{1 - \frac{a'}{b}}{1 - \frac{a'}{a}},$$

$$\sin^2 n\beta = \frac{b - a \,.\, - a'}{a - a' \,.\, b} = \frac{1 - \frac{a}{b}}{1 - \frac{a}{a'}};$$

and if q_I denotes the velocity at I, where $u = \infty$, and d the distance between the two planes IA, $I'A'$,

$$(43) \quad q_I d = m = Qc, \quad \left(\frac{d}{c}\right)^n = \left(\frac{Q}{q_I}\right)^n = \sqrt{\frac{b - a'}{a - a'}} + \sqrt{\frac{b - a}{a - a'}}.$$

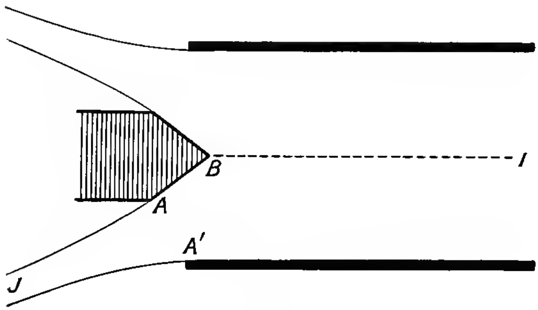


FIG. 32.

The figure can be duplicated about IB, and so give the motion when the channel is blocked by a wedge-shaped pier (Fig. 32).

When A' is carried along on $I'A'$ up to J at an infinite distance, making $a' = j = 0$, the wall $I'A'$ is of infinite length, and the motion may be duplicated again about $I'J$, and so represent the flow of water through the piers of a bridge, wedge-shaped as at Westminster.

Duplicating the original figure once about $I'J'$ will give the efflux from a channel with a conveying mouthpiece, as in Fig. 33.

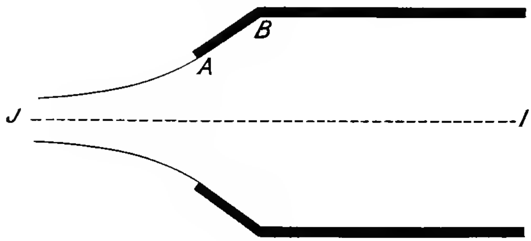


FIG. 33.

Make $b = \infty$, and the flow is obtained through the gap between two walls converging at an angle π/n; and now along the skin of the jet AJ, of breadth Qc at J,

$$(44) \qquad w_A - w = Qs = m\frac{s}{C} = \frac{m}{\pi}\log\frac{a}{u},$$

$$(45) \qquad \sin^2 n\theta = \frac{u}{a} = e^{-\pi\frac{s}{c}},$$

the intrinsic equation of the jet AP.

In Helmholtz's original problem, where the liquid is drawn off between two parallel walls like a Borda mouthpiece (Fig. 30),

$$(46) \qquad n = \tfrac{1}{2}, \quad \sin^2\tfrac{1}{2}\theta = e^{-\pi\frac{s}{c}}, \frac{ds}{d\theta} = -\frac{c}{\pi}\cot\tfrac{1}{2}\theta,$$

and if d is the outside distance between the walls,

$$(47) \quad \tfrac{1}{2}d - c = \int_0^\pi - \sin\theta\, ds = \frac{c}{\pi}\int_0^\pi 2\cos^2\tfrac{1}{2}\theta\, d\theta = c,\ d = 4c,$$

so that the coefficient of contraction is $\tfrac{1}{2}$.

In Helmholtz's next problem, where a slit of breadth d is made in a wall, through which the liquid escapes (Fig. 34),

$$(48) \qquad n = 1, \quad \sin\theta = e^{-\frac{1}{2}\pi\frac{s}{c}} = -\frac{dy}{ds},$$

$$(49) \qquad PM = \int_s^\infty e^{-\frac{1}{2}\pi\frac{s}{c}} ds = \frac{2c}{\pi}e^{-\pi\frac{s}{c}} = \frac{2c}{\pi}\sin\theta, \quad PT = \frac{2c}{\pi},$$

so that the curve AP is the tractrix; and

$$\tfrac{1}{2}d - c = AO = \frac{2c}{\pi}, \tag{50}$$

$$\frac{2c}{d} = \frac{\pi}{2+\pi} = 0{\cdot}611,$$

the coefficient of contraction.

Report 19 may be consulted for application of the same analysis, where the z figure changes with a rearrangement of the order of the arbitrary constants.

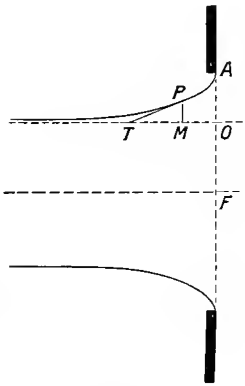


FIG. 34.

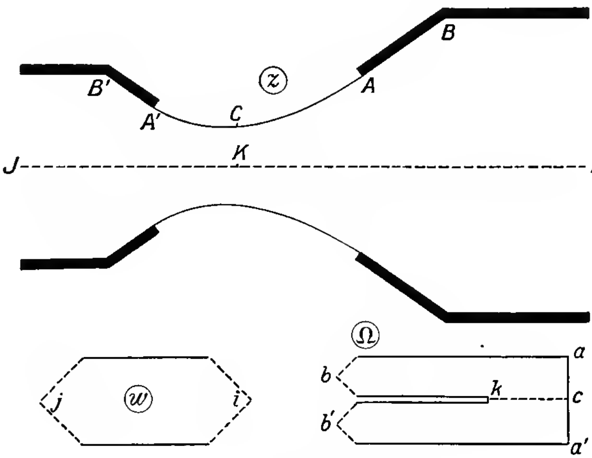


FIG. 35.

As a general exercise on conformal representation, consider the application to the z diagram in Fig. 35, representing uniplane injector flow, with the associated w and Ω diagrams, where the z figure may be supposed duplicated in the median line IJ, and here

$$i(\infty) > b > a > c > a' > b' > j > k > i(-\infty).$$

Next examine the change in the z diagram due to a rearrangement of the sequence of the constants

$$a,\ a',\ b,\ b',\ k.$$

Digression on the Integral Calculus.

Leaving the elliptic integral interest as leading too far, and passing over the analytical expression of w, as given merely by the algebraical and logarithmic function, since $\frac{dw}{du}$ can be resolved into a quotient and partial functions integrable immediately, we concentrate our attention on the determination of Ω from the typical form

(51) $$\frac{d\Omega}{du} = \frac{1}{u-b}\sqrt{\frac{b-a \,.\, b-a'}{u-a \,.\, u-a'}},$$

making

(52) $$(u-b)\frac{d\Omega}{du} = 1, \text{ when } u = b.$$

Returning through Paris in May, 1910, I took the opportunity of attending a lecture at the Sorbonne by M. Marchis, the new professor there of Aeronautics; and I found he devoted a whole lecture to the consideration of a single integral, which with Greek letters we can write

(53) $$\phi = \int \frac{\beta\, d\theta}{\alpha + \gamma\cos\theta},$$

employing a normalising factor β, which is either

$$\sqrt{(\alpha^2 - \gamma^2)} \text{ or } \sqrt{(\gamma^2 - \alpha^2)}.$$

But as M. Marchis was allowed, not six, but some twenty or thirty lectures, he could go thoroughly into this detail, essential in the theoretical study of the aeroplane.

Marchis's integral is identified with the integration required in (27) II. for Ω by putting

(54) $$a - u = (a - a')\cos^2 \tfrac{1}{2}\theta,\ u - a' = (a - a')\sin^2 \tfrac{1}{2}\theta,$$

$$\sqrt{(a - u \,.\, u - a')} = \tfrac{1}{2}(a - a')\sin\theta,$$

$$du = \tfrac{1}{2}(a - a')\sin\theta d\theta,$$

$$\frac{du}{\sqrt{(a - u \,.\, u - a')}} = d\theta,$$

$$a + a' - 2u = (a - a') \cos \theta,$$
$$b - u = b - \tfrac{1}{2}(a + a') + \tfrac{1}{2}(a - a') \cos \theta = \alpha + \gamma \cos \theta,$$
$$\alpha = b - \tfrac{1}{2}(a + a'), \gamma = \tfrac{1}{2}(a - a'),$$
$$\alpha + \gamma = b' - a', \alpha - \gamma = b - a,$$
$$\beta = \sqrt{(b - a \,.\, b - a')},$$
$$\phi = \int \frac{\sqrt{(b - a \,.\, b - a')}\, \theta u}{(b - u)\sqrt{(a - u \,.\, u - a')}} = \Omega i.$$

Or in the hyperbolic form,

(55) $$\beta = \sqrt{(\gamma^2 - a^2)}, \phi = \int \frac{\sqrt{(a - b \,.\, b - a')}}{(b - u)\sqrt{(a - u \,.\, u - a')}} = \Omega.$$

In a complete treatment of integration the hyperbolic function ranks equally with the circular function, direct and inverse; especially in this subject of conformal representation and its hydrodynamical application, where the change from circular to hyperbolic form, and back again, is taking place continually.

Thus, if

(56) $$u - a.\, u - a' \text{ is positive}$$

(57) $$\infty > u > a, \text{ or } a' > u > -\infty,$$

(58) $$u - a = (a - a') \operatorname{sh}^2 \tfrac{1}{2} v, \text{ or } -(a - a') \operatorname{ch}^2 \tfrac{1}{2} v,$$
$$u - a' = (a - a') \operatorname{ch}^2 \tfrac{1}{2} v, \text{ or } -(a - a') \operatorname{sh}^2 \tfrac{1}{2} v,$$
$$\sqrt{(u - a \,.\, u - a')} = \tfrac{1}{2}(a - a') \operatorname{sh} v,$$
$$du = \tfrac{1}{2}(a - a') \operatorname{sh} v \, dv,$$
$$\frac{du}{\sqrt{(u - a \,.\, u - a')}} = dv,$$
$$2u - a - a' = \pm (a - a') \operatorname{ch} v,$$
$$b - u = b - \tfrac{1}{2}(a + a') \mp \tfrac{1}{2}(a - a') \operatorname{ch} v,$$
$$= \alpha + \gamma \operatorname{ch} v$$
$$\alpha = b - \tfrac{1}{2}(a + a'), \gamma = \mp \tfrac{1}{2}(a - a'),$$
$$\alpha - \gamma = b - a', \text{ or } b - a,$$
$$\alpha + \gamma = b - a, \text{ or } b - a',$$

(59) $$\beta^2 = \alpha^2 - \gamma^2 = b - a \,.\, b - a',$$

(60) $$\int \frac{\sqrt{(b - a \,.\, b - a')}}{(b - u)\sqrt{(u - a \,.\, u - a')}} = \int \frac{\beta\, dv}{\alpha + \gamma \operatorname{ch} v}.$$

The contrast is brought out in the geometrical interpretation by means of the ellipse and hyperbola in Figs. 36, 37.

Changing back from the Greek to italic letters, to agree with the ordinary notation, we take

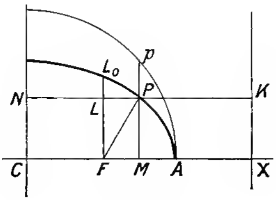


FIG. 36.

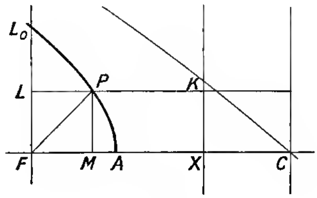


FIG. 37.

$$CA = a,\ CF = c,\ CX = \frac{a^2}{c}$$

$$b^2 = a^2 - c^2$$

$$FP = r = \frac{b^2}{a + c\cos\theta}$$

$$\phi = \int \frac{b\,d\theta}{a + c\cos\theta}$$

$$\phi = \cos^{-1}\frac{c + a\cos\theta}{a + c\cos\theta}$$

$$= \sin^{-1}\frac{b\sin\theta}{a + \cos\theta}$$

With $c = a\cos\alpha$

$$\tan^2\tfrac{1}{2}\phi = \frac{1 - \cos\phi}{1 + \cos\phi}$$

$$= \frac{a - c}{a + c}\cdot\frac{1 - \cos\theta}{1 + \cos\theta}$$

$$= \tan^2\tfrac{1}{2}\alpha\tan^2\tfrac{1}{2}\theta$$

$$r = FP = \frac{c}{a}\cdot PK = \frac{c}{a}(CX - CM)$$

$$= \frac{c}{a}\left(\frac{a^2}{c} - a\cos\phi\right) = a - c\cos\phi$$

$$= \frac{b^2}{a + c\cos\theta}$$

$$CA = a,\ CF = c,\ CX = \frac{a^2}{c}$$

$$b^2 = c^2 - a^2$$

$$FP = \frac{b^2}{a + c\cos\theta}$$

$$\phi = \int \frac{b\,d\theta}{a + c\cos\theta}$$

$$\phi = \mathrm{ch}^{-1}\frac{c + a\cos\theta}{a + c\cos\theta}$$

$$= \mathrm{sh}^{-1}\frac{b\sin\theta}{a + c\cos\theta}$$

With $c = a\sec\alpha$

$$\mathrm{th}^2\tfrac{1}{2}\phi = \frac{\mathrm{ch}\phi - 1}{\mathrm{ch}\phi + 1}$$

$$= \frac{c - a}{c + a}\cdot\frac{1 - \cos\theta}{1 + \cos\theta}$$

$$= \tan^2\tfrac{1}{2}\alpha\tan^2\tfrac{1}{2}\theta$$

$$r = FP = \frac{c}{a}PK = \frac{c}{a}(CM - CX)$$

$$= \frac{c}{a}\left(a\,\mathrm{ch}\,\phi - \frac{a^2}{c}\right) = c\,\mathrm{ch}\,\phi - a$$

$$= \frac{b^2}{a + c\cos\theta}$$

$$CM = x = a \sin \phi$$
$$MP = y = b \cos \phi$$

Circular sector $ACp = \frac{1}{2} a^2 \phi$

Elliptic sector $ACP = \frac{1}{2} ab \phi$

Elliptic sector $AFP = \frac{1}{2} ab\phi - \frac{1}{2} a \sin \phi$

Mean anomaly $nt = \phi - e \sin \phi$, in the elliptic planetary orbit, θ the true anomaly, ϕ the excentric anomaly.

$$CM = x = a \operatorname{ch} \phi$$
$$MP = y = b \operatorname{sh} \phi.$$

Hyperbolic sector $ACP = \frac{1}{2} ab \phi$

Hyperbolic sector $AFP = \frac{1}{2} ac \operatorname{sh} \phi - \frac{1}{2} ab \phi$

Mean anomaly $nt = e \operatorname{sh} \phi - \phi$ in the hyperbolic orbit;

Also, for the integral (60),

$$\frac{d\theta}{d\phi} = \frac{a + c \cos \theta}{b} = \frac{b}{c \operatorname{ch} \phi - a}$$

$$\theta = \int \frac{b\, d\phi}{c \operatorname{ch} \phi - a} = 2 \tan^{-1} \frac{\operatorname{th} \frac{1}{2} \phi}{\tan \frac{1}{2} a}$$

$(a = c \cos \alpha,\ b = c \sin a)$

$$= \cos^{-1} \frac{c - a \operatorname{ch} \phi}{c \operatorname{ch} \phi - a} = \sin^{-1} \frac{b \operatorname{sh} \phi}{c \operatorname{ch} \phi - a}$$

With x as the variable we may take as the typical irrational integrals which occur

(61) $$\int \frac{dx}{y} \text{ and } \int \frac{dx}{(x - p)\, y},$$

$$y = \sqrt{(ax^2 + 2bx + c)},$$

in which the integrations are said to be taken round the conic

(62) $$y^2 = ax^2 + 2bx + c.$$

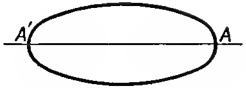
The most general irrational integral of this nature

(63) $$\int \phi\, (x, y)\, dx \text{ being reducible to } \int Pdx + \int Q \frac{dx}{y},$$

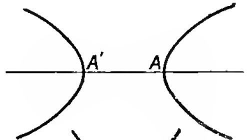
where P and Q are rational fractions, the first integral being required for w and the second for Ω in the previous applications.

There are three regions to consider, according to the shape of the curve of (62).

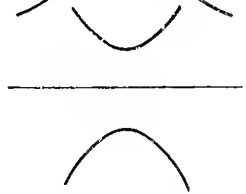
F 2

I. Ellipse (Fig. 38) 

; a negative, $b^2 - ac$ positive.

II. Hyperbola (Fig. 39) 

; a positive, $b^2 - ac$ positive.

III. Conjugate hyperbola (Fig. 40) 
; a positive, $b^2 - ac$ negative.

With a and $b^2 - ac$ both negative, y^2 in (62) would be negative for all real values of x, and the conic imaginary.

	I.	II.	III.
$\int \frac{dx}{y} =$	$\frac{1}{\sqrt{(-a)}} \sin^{-1} y \sqrt{\frac{-a}{b^2-ac}}$	$\frac{1}{\sqrt{a}} \operatorname{sh}^{-1} y \sqrt{\frac{a}{b^2-ac}}$	$\frac{1}{\sqrt{a}} \operatorname{ch}^{-1} y \sqrt{\frac{a}{ac-b^2}}$
$\int \frac{1}{x-p} \frac{dx}{y} =$	$\frac{1}{\sqrt{(-ap^2-2bp-c)}} \sin^{-1} \frac{y}{x-p} \sqrt{\frac{-ap^2-2bp-c}{b^2-ac}}$	$\frac{1}{\sqrt{(ap^2+2bp+c)}} \operatorname{sh}^{-1} \frac{y}{x-p} \sqrt{\frac{ap^2+2bp+c}{b^2-ac}}$	$\frac{1}{\sqrt{(ap^2+2bp+c)}} \operatorname{ch}^{-1} \frac{y}{x-p} \sqrt{\frac{ap^2+2bp+c}{ac-b^2}}$
$\int \frac{M}{x-p} \frac{dx}{y} =$	$\sin^{-1} \frac{y}{x-p} \sqrt{\frac{-ap^2-2bp-c}{b^2-ac}}$	$\operatorname{sh}^{-1} \frac{y}{x-p} \sqrt{\frac{ap^2+2bp+c}{b^2-ac}}$	$\operatorname{ch}^{-1} \frac{y}{x-p} \sqrt{\frac{ap^2+2bp+c}{ac-b^2}}$

where it will be noticed that the second integral is circular or hyperbolic according as y^2 is negative or positive for $x = p$, and the multiplier M is either

$$\sqrt{(-ap^2 - 2bp - c)} \text{ or } \sqrt{(ap^2 + 2bp + c)}.$$

In I. and II. y^2 can be resolved into factors, $x - a$, $x - a'$ suppose; and replacing a by -1 or $+1$, the result can be simplified and the integral corrected for one limit.

I. $a > x > a'$.

$$\int_n^a \frac{dx}{\sqrt{(a - x \,.\, x - a')}} = 2 \sin^{-1}\sqrt{\frac{a - x}{a - a'}} = 2 \cos^{-1}\sqrt{\frac{x - a'}{a - a'}},$$

$$\int_{a'} \frac{dx}{\sqrt{(a - x \,.\, x - a')}} = 2 \cos^{-1}\sqrt{\frac{a - x}{a - a'}} = 2 \sin^{-1}\sqrt{\frac{x - a'}{a - a'}},$$

$$\int_{a'}^a \frac{dx}{\sqrt{(a - x \,.\, x - a')}} = \pi.$$

II. $\infty > x > a$.

$$\int_a^n \frac{dx}{\sqrt{(x - a \,.\, x - a')}} = 2 \,\mathrm{sh}^{-1}\sqrt{\frac{x - a}{a - a'}} = 2 \,\mathrm{ch}^{-1}\sqrt{\frac{x - a'}{a - a'}},$$

$a' > x > -\infty$.

$$\int_n^{a'} \frac{dx}{\sqrt{(a - x \,.\, a' - x)}} = 2 \,\mathrm{ch}^{-1}\sqrt{\frac{a - x}{a - a'}} = 2 \,\mathrm{sh}^{-1}\sqrt{\frac{a' - x}{a - a'}}.$$

Treating the second integral in the same way there are three columns to fill in, where the integral, in the notation above for Ω, is corrected for one limit or the other, and the factors made positive, so that the integral I should be real and positive.

But all these forms are reducible to an integral of the first form $\int \frac{dx}{y}$, by taking $\frac{1}{b - u} = U$ as the variable; thus

$$(64) \qquad I = \int \frac{\sqrt{(a - b \,.\, a' - b)}\, du}{(b - u)\sqrt{(a - u \,.\, a' - u)}}$$

$$= \int \frac{\dfrac{du}{(b - u)^2}}{\sqrt{\left(\dfrac{1}{b - u} + \dfrac{1}{a - b} \,.\, \dfrac{1}{b - u} + \dfrac{1}{a' - b}\right)}},$$

$$= \int \frac{dU}{\sqrt{(U + A \,.\, U + A')}},$$

or an allied form.

I.

$$\infty > b > a > a'$$
$$M = \sqrt{(b - a \,.\, b - a')}$$

$$\infty > u > b$$

$$I = \int_u^\infty \frac{M\,du}{(u - b)\sqrt{(u - a \,.\, u - a')}}$$

(hyperbolic)

$$= 2\,\mathrm{ch}^{-1}\sqrt{\frac{b - a' \,.\, u - a}{a - a' \,.\, u - b}} - 2\,\mathrm{ch}^{-1}\sqrt{\frac{b - a'}{a - a'}}$$

$$= 2\,\mathrm{sh}^{-1}\sqrt{\frac{b - a \,.\, u - a'}{a - a' \,.\, u - b}} - 2\,\mathrm{sh}^{-1}\sqrt{\frac{b - a}{a - a'}}$$

$$b > u > a$$

$$I = \int_a^u \frac{M\,du}{(b - u)\sqrt{(u - a \,.\, u - a')}}$$

(hyperbolic)

$$= 2\,\mathrm{sh}^{-1}\sqrt{\frac{b - a' \,.\, u - a}{a - a' \,.\, b - u}}$$

$$= 2\,\mathrm{ch}^{-1}\sqrt{\frac{b - a \,.\, u - a'}{a - a \,.\, b - u'}}$$

$$(u = b,\ I = \infty)$$

II.

$$a > b > a'$$
$$M = \sqrt{(a - b \,.\, b - a')}$$

$$u > a$$

$$I = \int_a^u \frac{M\,du}{(u - b)\sqrt{(u - a \,.\, u - a')}}$$

(circular)

$$= 2\cos^{-1}\sqrt{\frac{a - b \,.\, u - a'}{a - a' \,.\, u - b}}$$

$$= 2\sin^{-1}\sqrt{\frac{b - a' \,.\, u - a}{a - a' \,.\, u - b}}$$

$$a > u > b$$

$$I = \int_u^a \frac{M\,du}{(u - b)\sqrt{(a - u \,.\, u - a')}}$$

(hyberbolic)

$$= 2\,\mathrm{ch}^{-1}\sqrt{\frac{a - b \,.\, u - a'}{a - a' \quad u - b}}$$

$$= 2\,\mathrm{sh}^{-1}\sqrt{\frac{b - a' \,.\, a - u}{a - a' \,.\, n - b}}$$

$$(u = b,\ I = \infty)$$

III.

$$a > a' > b > -\infty$$
$$M = \sqrt{(a - b \,.\, a' - b)}$$

$$u > a$$

$$I = \int_a^u \frac{M\,du}{(u - b)\sqrt{(u - a \,.\, u - a')}}$$

(hyperbolic)

$$= 2\,\mathrm{sh}^{-1}\sqrt{\frac{a' - b \,.\, u - a}{a - a' \,.\, u - b}}$$

$$= 2\,\mathrm{ch}^{-1}\sqrt{\frac{a - b \,.\, u = a'}{a - a' \,.\, u - b}}$$

$$a > u > a'$$

$$I = \int_u^a \frac{M\,du}{(u - b)\sqrt{(a - u \,.\, u - a')}}$$

(circular)

$$= 2\sin^{-1}\sqrt{\frac{a' - b \,.\, a - u}{a - a' \,.\, u - b}}$$

$$= 2\cos^{-1}\sqrt{\frac{a - b \,.\, u - a'}{a - a' \,.\, u - b}}$$

$$(u = a',\ I = \pi)$$

$$I = \int_{a'}^u \frac{M\,du}{(u - b)\sqrt{(a - u \,.\, u - a')}}$$

$$= 2\sin^{-1}\sqrt{\frac{a - b \,.\, u - a'}{a - a' \,.\, u - b}}$$

$$= 2\cos^{-1}\sqrt{\frac{a' - b \,.\, a - u}{a - a' \,.\, u - b}}$$

$$(u = a,\ I = \pi)$$

$$a > u > a'$$

$$I = \int_u^a \frac{M\,du}{(b-u)\sqrt{(a-u\,.\,u-a')}}$$

(circular)

$$= 2\sin^{-1}\sqrt{\frac{b-a'\,.\,a-u}{a-a'\,.\,b-u}}$$

$$= 2\cos^{-1}\sqrt{\frac{b-a\,.\,u-a'}{a-a'\,.\,b-u}}$$

$$(u = a',\ I = \pi)$$

$$I = \int_{a'}^u \frac{M\,du}{(b-u)\sqrt{(a-u\,.\,u-a')}}$$

$$= 2\sin^{-1}\sqrt{\frac{b-a\,.\,u-a'}{a-a'\,.\,b-u}}$$

$$= 2\cos^{-1}\sqrt{\frac{b-a'\,.\,a-u}{a-a'\,.\,b-u}}$$

$$(u = a,\ I = \pi)$$

$$a' > u > -\infty$$

$$I = \int_u^{a'} \frac{M\,du}{(b-u)\sqrt{(a-u\,.\,a'-u)}}$$

(hyperbolic)

$$= 2\,\mathrm{sh}^{-1}\sqrt{\frac{b-a\,.\,a'-u}{a-a'\,.\,b-u}}$$

$$= 2\,\mathrm{ch}^{-1}\sqrt{\frac{b-a'\,.\,a-u}{a-a'\,.\,b-u}}$$

$$b > u > a'$$

$$I = \int_{a'}^u \frac{M\,du}{(b-u)\sqrt{(a-u\,.\,u-a')}}$$

(hyperbolic)

$$= 2\,\mathrm{ch}^{-1}\sqrt{\frac{b-a'\,.\,a-u}{a-a'\,.\,b-u}}$$

$$= 2\,\mathrm{sh}^{-1}\sqrt{\frac{a-b\,.\,u-a'}{a-a'\,.\,b-u}}$$

$$(u = b,\ I = \infty)$$

$$a' > u > -\infty$$

$$I = \int^{a'} \frac{M\,du}{(b-u)\sqrt{(a-u\,.\,a'-u)}}$$

(circular)

$$= 2\sin^{-1}\sqrt{\frac{a-b\,.\,a'-u}{a-a'\,.\,b-u}}$$

$$= 2\cos^{-1}\sqrt{\frac{b-a'\,.\,a-u}{a-a'\,.\,b-u}}$$

$$a' > u > b$$

$$I = \int_u^{a'} \frac{M\,du}{(u-b)\sqrt{(a-u\,.\,a'-u)}}$$

(hyperbolic)

$$= 2\,\mathrm{sh}^{-1}\sqrt{\frac{a-b\,.\,a'-u}{a-a'\quad u-b}}$$

$$= 2\,\mathrm{ch}^{-1}\sqrt{\frac{a'-b\,.\,a-u}{a-a'\,.\,u-b}}$$

$$(u = b,\ I = \infty)$$

$$b > u > -\infty$$

$$I = \int_\infty^u \frac{M\,du}{(b-u)\sqrt{(a-u\,.\,a'-u)}}$$

(hyperbolic)

$$= 2\,\mathrm{sh}^{-1}\sqrt{\frac{a'-b\,.\,a-u}{a-a'\,.\,b-u}}$$

$$-\ 2\,\mathrm{sh}^{-1}\sqrt{\frac{a'-b}{a-a'}}$$

$$= 2\,\mathrm{ch}^{-1}\sqrt{\frac{a-b\,.\,a'-u}{a-a'\,.\,b-u}}$$

$$-\ 2\,\mathrm{ch}^{-1}\sqrt{\frac{a-b}{a-a'}}$$

$$(u = b,\ I = \infty)$$

The art of Integration of a function is to express the result as a new function of the constituents of the function, leaving them unaltered if possible ; thus

$$\int \sec\theta\, d\theta = \operatorname{ch}^{-1}(\sec\theta) \tag{65}$$

is preferable to log (sec θ + tan θ) or log tan ($\frac{1}{4}\pi + \frac{1}{2}\theta$).

Professor Perry has shown the utility of the idea of *lead* and *lag* ; thus

$$\frac{d}{dt}\frac{\cos}{\sin}(nt+\epsilon) = n\frac{\cos}{\sin}(nt+\epsilon+\tfrac{1}{2}\pi), \tag{66}$$

$$\int\frac{\cos}{\sin}(nt+\epsilon)\,dt = \frac{1}{n}\frac{\cos}{\sin}(nt+\epsilon-\tfrac{1}{2}\pi),$$

a differentiation giving a *lead* of $\frac{1}{2}\pi$ to the phase angle $nt+\epsilon$, and a *lag* of $\frac{1}{2}\pi$ being required in the integration.

So also for the damped vibration,

$$\frac{d}{dt}e^{-mt}\frac{\cos}{\sin}(nt+\epsilon) = \sqrt{(m^2+n^2)}\,e^{-mt}\frac{\cos}{\sin}\left(nt+\epsilon+\pi-\tan^{-1}\frac{n}{m}\right), \tag{67}$$

$$\int e^{-mt}\frac{\cos}{\sin}(nt+\epsilon)\,dt = \frac{e^{-mt}\dfrac{\cos}{\sin}\left(nt+\epsilon-\pi+\tan^{-1}\dfrac{n}{m}\right)}{\sqrt{(m^2+n^2)}}.$$

LECTURE IV

GYROSCOPIC ACTION, AND GENERAL DYNAMICAL PRINCIPLES

A FEW gyroscopic experiments were shown at the end of the last lecture, made with simple apparatus of bicycle wheels, spun by hand, either as an ordinary top (Fig. 41), or like a gyroscope, suspended by a stalk from a vertical axle (a bicycle hub) fastened in a bracket bolted to the under side of a beam, the beam resting on two step ladders for support (Fig. 42).

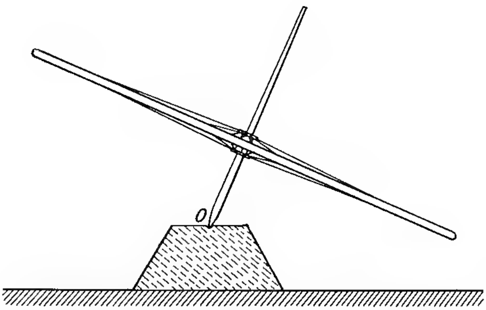


FIG. 41.

On this scale, visible to a large audience, spin sufficient can be given by hand, and no string is required; so also with the Maxwell top (Fig. 43) twirled by a finger and thumb.

The bicycle wheel can also be held by the stalk and brandished, to give the muscular sensation of its gyroscopic action when spinning, and so realise the influence on the steering of a flying machine of the rotation of the motor and screw.

A bicycle wheel on its ball bearings makes an excellent pendulum for experimental illustration. After being tested for balance, and friction in producing a gradual decay of the revolution, the wheel may be put out of balance by an iron bar between the spokes, and then it can swing as a pendulum through an arc small or large, and also make a complete revolution. The laws of pendulum oscillation, depending on the elliptic function, may thus be tested through an angle of oscillation however large.

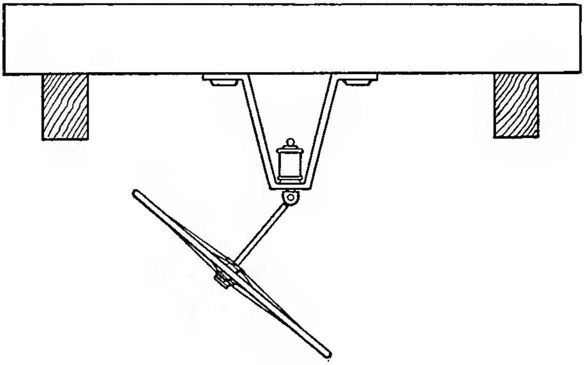


FIG. 42.

To give the exact theory, with as few symbols and formulas as possible, of the steady motion of the gyroscope or top with the axle at a constant angle θ with the vertical, a knowledge is presumed of the general dynamical principles of velocity and momentum, linear and angular, and their vector representation; and use must be made of the dynamical lemma—

"The vector velocity of the momentum, linear or angular, is equal to the vector of the impressed force or couple."

This is Newton's second law of motion, extended to the case of angular momentum; and Maxwell's "Matter and Motion" may be consulted for an elementary demonstration of the essential principle.

The vector representation of a couple, as in Statics, by its axis is thus required also, the couple due here to the preponderance, first moment, or leverage of gravity; the notion, too, and determination of moment of inertia ($M.I.$) or second moment.

In the discussion of the gyroscope in Fig. 42 the *M.I.* is required about the supporting pin O, and this is determined experimentally by letting the wheel swing in one plane like a pendulum, and measuring the length, l feet, of the thread of the simple equivalent pendulum which beats the same time; denoting the *M.I.* by A, then

$$(1) \qquad l = \frac{A}{Wh}, \; A = Whl \text{ (lb-ft}^2\text{)},$$

where Wh, the preponderance in ft-lb, is measured by hooking up the axle to the horizontal position by a spring balance, and taking the leverage as the product of the scale reading in lb, and the distance in feet of the hook from the pin O; having determined W in lb previously, by hanging the wheel from the spring balance, h is then the distance of its $C.G$ from O.

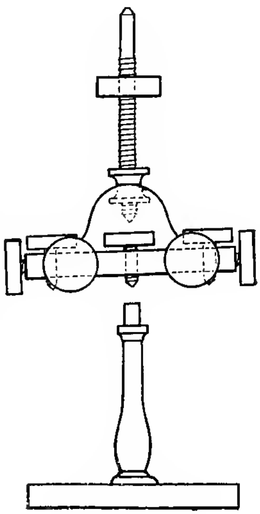


FIG. 43.

So too we require C, the *M.I.* of the wheel about its axle; and C can be determined experimentally, as shown in Fig. 44, with one of the bicycle wheels, by supporting the rim on a knife edge, and noting the length, l' ft, of the equivalent pendulum, and h', the radius of the inside of the rim where the wheel is supported; and then

$$(2) \qquad l' = h' + \frac{C}{Wh'}, \quad C = Wh'(l' - h'), \quad \text{lb-ft}^2.$$

We call W, Wh, A, and C the physical constants of the wheel as a gyroscope or top, and if no spin is given, the wheel can make plane oscillations, like a single pendulum of length l ft, beating time, T seconds in small oscillation, where

$$(3) \qquad l = \frac{A}{Wh}, \quad \text{ft}; \qquad T = \pi\sqrt{\frac{l}{g}} = \pi\sqrt{\frac{A}{gWh}}, \quad \text{secs.}$$

Or, if precessing with angular velocity n (radians/second), in the same period $2\,T$, as a conical pendulum near the vertical,

$$(4) \qquad n = \frac{\pi}{T} = \sqrt{\frac{g}{l}} = \sqrt{\frac{gWh}{A}}, \quad l = \frac{g}{n^2}, \quad A\,\frac{n^2}{g} = Wh \text{ (ft-lb)}.$$

Take this wheel, weighing W lb, and spin it with angular velocity R (rad/sec), so that it makes $\frac{R}{2\pi}$ revolutions per second.

The particles in a concentric ring, of radius r ft, have a velocity rR; and if the density of the material is denoted by w, in lb/ft^3, the energy of a particle, of volume dV ft^3, is

$$(5) \quad wdV\,\frac{r^2R^2}{2g}, \text{ and of the whole body is } \int wdV\,\frac{r^2R^2}{2g} = C\,\frac{R^2}{2g}, \text{ ft-lb,}$$

where $C = \int r^2wdV$ is the $M.I.$ about the axle, in lb-ft^2.

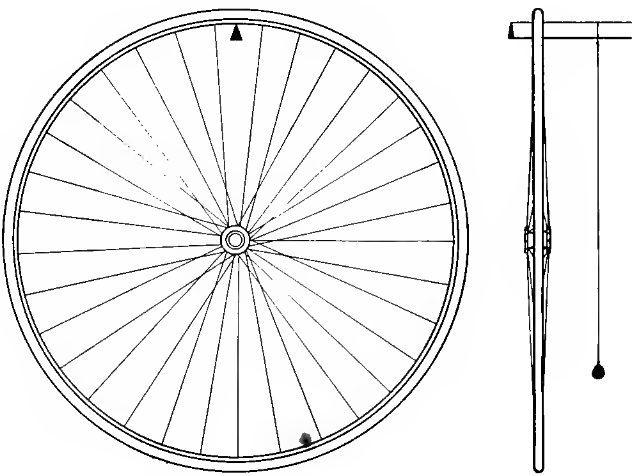


FIG. 44.

When the wheel is moved bodily with velocity v feet/second (f/s), the energy of translation is

$$W\,\frac{v^2}{2g} \text{ (ft-lb), against } C\,\frac{R^2}{2g} \text{ for the energy of rotation.}$$

And just as $W\frac{v}{g}$ is the linear momentum, in sec-lb, so $C\frac{R}{g}$ is the angular momentum (*A.M.*) about the axle in sec-ft-lb.

It is usual to take out W as a factor of C, dividing C into the factors W and k^2, $C = Wk^2$ lb-ft², so that k is a length in feet called the *radius of gyration* (Euler), or *swing radius* (Clifford).

Familiar illustrations of *M.I.* are experienced in the muscular sensation of slamming a door, or brandishing a stick.

Now spin this wheel with angular velocity R, rad/sec, giving *A.M.* $C\frac{R}{g}$, sec-ft-lb, and give the appropriate precession μ so that the axle will move at a constant angle θ with the vertical downward.

As R or μ is increased, the axle can rise higher, pass the horizontal position, and when the beam in Fig. 42 stops the wheel we turn to this other arrangement in Fig. 41, where the wheel spins like a top, with the point in a cup O, and the axle OC' at a constant angle θ with the upward vertical.

Selecting the gyroscope of Fig. 42 for discussion, as the motion is more under control, we draw the associated geometrical representations in Fig. 45, in which OC' is taken to scale to represent the vector of $C\frac{R}{g}$, the *A.M.* about the axle.

Notice that we associate a vector of *A.M.*, or of angular velocity, with a screw; and we select the right-handed screw, so that the vector OC' represents the direction of advance on the screw along the axle due to the rotation R.

This involves spinning the wheel by a push with the left hand in Fig. 42, but with a pull in Fig. 41.

The precession μ is then represented by a vector drawn vertically upward from O, having a component $\mu\cos\theta$ along $C'O$ in Fig. 45, and $\mu\sin\theta$ along OA at right angles; these are the components of angular velocity of the axle or stalk, but the wheel has an independent rotation round the stalk.

The component $\mu\cos\theta$ does not affect the wheel, only the stalk or axle, of which we ignore the inertia; but the wheel rotates freely on the ball bearings of the axle with relative angular velocity

$$R + \mu\cos\theta.$$

But it is the other component $\mu \sin \theta$ which carries the wheel round with the precession μ, and gives it the $A.M.$ about OA,

$$A \frac{\mu \sin \theta}{g}.$$

The vector OK of resultant $A.M.$ has the components OC', representing $C \frac{R}{g}$, and $C'K$, representing $A \frac{\mu \sin \theta}{g}$, working with the gravitation units of the engineer, and keeping g carefully in its right place.

The gravity couple is in the vertical plane COC', and its moment is

(6) $$Wh \sin \theta, \text{ or } A \frac{n^2}{g} \sin \theta, \quad \text{ft-lb},$$

represented by a right-handed screw vector draw O towards us; and this is to be equated to the vector velocity of K.

Now the horizontal component of OK is

(7) $$CK = OC' \sin \theta + C'K \cos \theta,$$

and the vector velocity of K is $\mu . CK$, so that

(8) $$Wh \sin \theta = \mu \left(C \frac{R}{g} \sin \theta + A \frac{\mu \sin \theta}{g} \cos \theta \right)$$

or dividing out $\sin \theta$ and g,

(9) $$An^2 = CR\mu + A\mu^2 \cos \theta,$$

the fundamental relation for steady motion, with the axle at a constant angle θ with the downward vertical.

But if, as in Figs. 41, 46, the angle θ is measured from the upward vertical, the sign of $\cos \theta$ must be changed in (9), and

(10) $$An^2 = CR\mu - A\mu^2 \cos \theta.$$

Writing (10)

(11) $$\cos \theta = \frac{CR}{A\mu} - \frac{n^2}{\mu^2} = \left(\frac{CR}{2An}\right)^2 - \left(\frac{CR}{2An} - \frac{n}{\mu}\right)^2 < \left(\frac{CR}{2An}\right)^2$$

we notice that the top cannot reach the upright position if $CR < 2An$, and the motion is then called *weak*.

In a *strong* motion, $CR > 2\,An$, and $\cos\theta$ can reach 1, after which the factor $\sin\theta = 0$, $\theta = 0$ must be taken, which was discarded from (8).

In OC' make $OL = l = \frac{A}{Wh}$, the length of the equivalent pendulum for plane oscillation; draw LD at right angles to OL cutting OD at right angles to OK in D; and draw DL' horizontal to cut the vertical through O in L'. Then

$$(12)\qquad \frac{OL'}{OL} = \frac{\sin ODL'}{\sin ODL} = \frac{\sin KOC}{\sin KOC'}$$
$$= \frac{KC}{KC'} = \frac{\mu\,.\,KC}{\mu\,.\,KC'} = \frac{An^2\sin\theta}{A\mu^2\sin\theta} = \frac{n^2}{\mu^2},$$

and
$$OL = \frac{g}{n^2}\,,\ \text{so that}\ OL' = \frac{g}{\mu^2}\,,$$

and OL' is the height of a conical pendulum with precession μ, keeping in the vertical plane COC'.

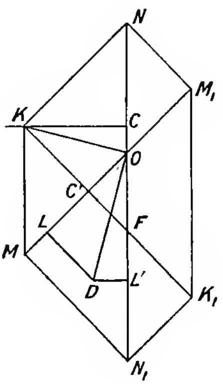


FIG. 45.

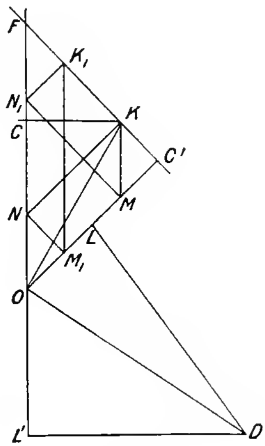


FIG. 46.

We can write

$$(13)\qquad \mu\,.\,CK = A\,\frac{n^2}{g}\sin\,\theta, \qquad \mu\,.\,KN = A\,\frac{n^2}{g},$$
$$C'K = A\,\frac{\mu}{g}\sin\,\theta, \qquad KM = A\,\frac{\mu}{g}.$$

Multiplying, and dividing out μ,

$$(14) \qquad CK \, . \, C'K = \left(A \frac{n}{g} \sin \theta\right)^2, \quad KM \, . \, KN = \left(A \frac{n}{g}\right)^2,$$

so that K lies on a hyperbola, with OC, OC' for asymptotes.

To draw the figure to a geometrical scale, to a length a as unit, we take

$$(15) \qquad \frac{OC'}{a} = \frac{C \frac{R}{g}}{A \frac{n}{g}}$$

$$\frac{C'K}{a} = \frac{A \frac{\mu}{g} \sin \theta}{A \frac{n}{g}} = \frac{\mu}{n} \sin \theta, \quad \frac{KM}{a} = \frac{\mu}{n},$$

$$\mu \frac{C'K}{a} = \frac{A \frac{n^2}{g} \sin \theta}{A \frac{n}{g}} = n, \quad \frac{KN}{a} = \frac{n}{\mu},$$

$$(16) \qquad KM \, . \, KN = a^2$$

But if this condition is not satisfied, the axle requires to be held at the angle θ by means of a revolving gimbal frame COC', providing a couple, N ft-lb, in addition to the gravity couple $Wh \sin \theta$, and

$$(17) \qquad Wh \sin \theta + N = \mu \, . \, \frac{CK}{a} \, . \, A \frac{n}{g}, \quad \text{with } \mu = \frac{n}{\sin \theta} \, . \, \frac{C'K}{a},$$

$$(18) \quad N = A \frac{n^2}{g} \frac{CK \, . \, C'K}{a^2 \sin \theta} - Wh \sin \theta = \left(\frac{KM \, . \, KN}{a^2} - 1\right) Wh \sin \theta.$$

Thus $N = 0$ again, if $KM \, . \, KN = a^2$;
but if $KM \, . \, KN > a^2$, K lies on the concave side of the hyperbola, and N is positive, holding the axle down; and if released the axle would rise.

An increase of the free precession μ would increase $C'K$ and move K outside the hyperbola to the concave side, and the axle would rise, or, stated in Kelvin's words,

"hurry the precession, and the top (or bicycle) rises."

Arrest the precession by clamping the vertical spindle, and the axle falls and swings like a pendulum, having lost the directive gyroscopic effect; but the spin of the wheel exercises a twisting, which is felt as very strong when we try to hold the vertical spindle by hand.

But the wheel in Fig. 42 can be supported by the finger at any angle, and no difference can be felt whether the wheel is at rest or spinning; not until the axle is allowed to fall.

Hammer the spinning top of Fig. 41, 42 with a stick, not too heavy, and the wheel flinches very slightly.

It will be noticed that KC' cuts the hyperbola again, in K_1 on Fig. 46; and if it cuts the vertical in F, $FK_1 = KC'$, and MN_1, M_1N are parallel to KK_1; the precession

$$\mu_1 = \frac{n}{\sin\theta}\frac{C'K_1}{a} = n\frac{K_1M_1}{a}, \tag{19}$$

and is in the opposite direction in Fig. 45 and much larger than μ; so that this motion is more violent, and hurrying the precession will have the opposite effect, of making the axle fall, as we can show experimentally.

Many attempts are being made still to utilise the gyroscope, for steering a constant course automatically, but any such action must be carried out through a light relay, as in the torpedo; if the gyroscope is called on to do any work, it ceases to direct.

Dismount the axle in Fig. 42 and hold it in the hands; or else take the large wheel of Fig. 41, a 52-inch, and brandish it by the stalk, noticing the difference of muscular sensation according as the wheel is spun or not.

With no spin, the wheel moves in the plane of the applied couple, so that the rotation is about the axis of the couple.

But now spin the wheel, with angular velocity R, and $A.M.$ $C\frac{R}{g}$, represented by the vector OC' on the right-handed system.

If the wheel is swung upward when holding the stalk out horizontal, an $A.M.$ is communicated about an axis drawn to the right, and the axle OC' swerves to the right, unless prevented by the muscular action of a couple represented by an axis drawn upward; vice versa when the wheel is swung downward.

Swung to the $\left\{\begin{matrix}\text{right}\\ \text{left}\end{matrix}\right\}$, *A.M.* is communicated about an axis drawn $\left\{\begin{matrix}\text{downward}\\ \text{upward}\end{matrix}\right\}$, and the axle swerves $\left\{\begin{matrix}\text{downward}\\ \text{upward}\end{matrix}\right\}$; and it requires to be held $\left\{\begin{matrix}\text{up}\\ \text{down}\end{matrix}\right\}$ by a couple whose vector axis is drawn to the $\left\{\begin{matrix}\text{right}\\ \text{left}\end{matrix}\right\}$.

The gyroscopic effect on the flying machine of the motor and screws has attracted attention.

The biplane of Wright, Cody and Farman has two screws actuated in opposite direction by chains from the motor, one chain crossed, and here the gyroscopic effect is minimised.

But the monoplanes of Bleriot and Santos Dumont have a single tractor screw in front, and when this is carried on a Gnome motor with cylinders revolving, the effect on the steering must be considerable, due to gyroscopic action.

Putting the helm to port, as a sailor would say, sets up a couple tending to turn the head to starboard, and its axis is downward; so that with a right-handed screw the vector of *A.M.* receives a downward velocity, and the head of the machine descends. To ascend, the helm must be put to starboard.

But if a turn is to be made to starboard, the axis of the couple must point to the right, and the helm of a horizontal rudder must be pushed down vertically, but held up if the head is to turn to port.

With a left-handed screw on a Gnome motor, the gyroscopic influence is reversed; a turn to the right will cause the head to rise and tail to drop. This sensation was dreaded by the pilot, and so he avoided the turn to the right as much as possible.

The couple of reaction of the screw on the frame is a vector along an axis drawn to the rear, tending to make the machine bear heavier on the left wing, as if sailing on the port tack.

Some such bias may help to improve the stability by giving a permanent bias to one side, in preference to the uncertainty of list, as of a crank tender ship.

The muscular sensation when the axle is brandished of a revolving wheel will illustrate the reaction on the bearings of the gyroscopic influence of the revolving machinery due to rolling and pitching of a steamer or flying machine.

A paddle steamer has the main shaft across the ship, and so is affected gyroscopically by the rolling, not pitching. (Worthington, *Dynamics of Rotation.*)

But rolling does not affect the direction of a screw shaft, and pitching causes an extra reaction couple on the bearings acting across the ship.

Suppose the steamer pitches through D° in T seconds; the angular displacement θ being given in radians by

$$\theta = \frac{\pi D}{360} \sin \pi \frac{T}{t}; \tag{20}$$

the angular velocity is given by

$$\frac{d\theta}{dt} = \mu \cos \pi \frac{t}{T}, \quad \mu = \frac{\pi^2 D}{360 T}, \tag{21}$$

and if $C \dfrac{R}{g}$ denotes the angular momentum in lb-ft-sec of the revolving turbines, a couple must be supplied, with vertical axis, of maximum value

$$C \frac{R\mu}{g} = C \frac{\pi^2 D}{360} \frac{R}{gT}$$
$$= Wk^2 \frac{\pi^2 D}{360} \cdot \frac{2\pi N}{60 gT}, \quad \text{ft-tons}, \tag{22}$$

at N revs/minute, with $C = Wk^2$, ton-ft²; and with bearings l ft apart, each carrying $\frac{1}{2} W$,

$$\frac{\text{Force on a bearing across the ship}}{\text{Dead weight on the bearing}} \tag{23}$$

$$= \frac{C \dfrac{R\mu}{g}}{\frac{1}{2} WL} = \frac{4\pi^3 Dk^2 N}{360 \times 60 \times gTL} = \frac{Dk^2 N}{5{,}600 TL}.$$

Working this out for $D = 3$, $N = 150$, $l = 30$, $T = 10$, $k = 3$, the fraction is about $\frac{1}{400}$.

When the wheel is spun rapidly, so that OC' is very much larger than $C'K$ in Fig. 46, K_1 is close to F, and μ_1 is large, giving a violent motion.

But K is close to C', so that we may take

$$(24) \qquad \frac{CK}{a} = \frac{OC'}{a}\sin\theta = \frac{CR}{An}\sin\theta$$

$$(25) \qquad \frac{Wh\sin\theta}{\frac{An}{g}} = n\sin\theta = \mu\frac{CK}{a} = \mu\frac{Ch}{An}\sin\theta,$$

$$\mu = \frac{An^2}{CR},$$

making μ small, and independent of θ; as in the fundamental equation (9), when μ^2 is neglected.

This relation for μ is true accurately when the axle of the top in steady motion is horizontal, and $\cos\theta = 0$; and the approximation for any other angle is useful in popular elementary explanation of gyroscopic motion, such as given in Perry's *Spinning Tops;* although the exact theory, as usual, is after all the simplest.

The approximation in (25) amounts to assuming that the vector OK of resultant *A.M.* is undistinguishable from the axle OC', so that the velocity of C' may be made equal to the couple vector, and it is employed freely in Worthington's *Dynamics of Rotation.*

This is the case with the Earth, where the variation of latitude is insensible, and the approximation was employed by Poinsot in his treatment of Precession and Nutation (*Connaissance des temps,* 1858).

But the explanation still more popular will lead to an erroneous result, which strives to dispense with the idea of angular momentum, and works with angular velocity instead, as it would make

$$(26) \qquad \mu = \frac{n^2}{R}, \text{ instead of } \frac{An^2}{CR}.$$

It is the angular velocity which is visible to the eye, and its vector as giving a line of instantaneous rest, shown by a coloured card on the axle of the Maxwell top; but for complete dynamical treatment the *A.M.* is of greater importance.

The most general motion of the axle would lead too far, where it makes nutations, and describes a path, either undulating or looped or cusped; realised easily with the apparatus of Fig. 42. A condensed treatment will be found in *Notes on Dynamics*, p. 200, and here are two cases which lead to an algebraical solution of simple character.

I. Hold the axle up horizontal, and, with no spin of the wheel, project the axle horizontally; the motion is similar to a spherical pendulum.

II. Spin the wheel and hold the axle up above the horizontal, so that when let fall it starts from a cusp and reaches the horizontal, and rises again to a cusp, and so continues in a succession.

But the general case may prove of very complicated character.

The lecture concludes with a digression on the simple principles of Linear Dynamics; the transcription should go on to a single sheet of paper, but it is all the dynamical theory required for a large number of familiar problems, such as those given in *Notes on Dynamics* (Wyman).

Digression on Linear Dynamics.

A wheeled carriage, electric tram, motor car, or motibus (Fig. 47), W tons, acquires velocity v f/s in t secs through s ft from rest, propelled by a constant force F tons.

In a field of gravity, g f/s², v is acquired in falling freely $\frac{v}{g}$ seconds, through $\frac{v^2}{2g}$ ft.

The car moves as if disturbed by a horizontal field, diluted to

$$\frac{F}{W} g\,;$$

$$(27) \qquad Ft \text{ (or } \int F dt) = W\frac{v}{g} \quad \text{(sec-tons of momentum)}$$

$$(28) \qquad Fs \text{ (or } \int F ds) = W\frac{v^2}{2g} \quad \text{(ft-tons of energy)}$$

$$(29) \qquad \frac{s}{t} = \frac{1}{2}v \quad \text{(the average velocity in f/s)}$$

and Newton's Second Law of Motion is translated by equation (27).

If break resistance B tons brings the car to rest in t' secs and s' ft,

$$(30) \qquad Ft = Bt',\; Fs = Bs'.$$

Curves are drawn for $W\frac{v^2}{2g}$, v, and t; continued in a straight line for the middle part of a run at full speed v; completed where the car comes to a stop.

By giving the energy line a slope of F in W, it will represent the level of apparent gravity to a passenger, perpendicular to the plumb line.

A sudden change in the plumb line will represent the jerk, as at stopping and starting.

A passenger walking out at the front feels the floor sloping down and leans back; the jerk of gravity restored tends to throw him on his back. Vice versa for leaving at the rear, also at starting, and the change from uniform velocity, as felt by the straphanger; although this is smoothed down in practice, as in actual running the transition corners are smoothed down, and not so noticeable.

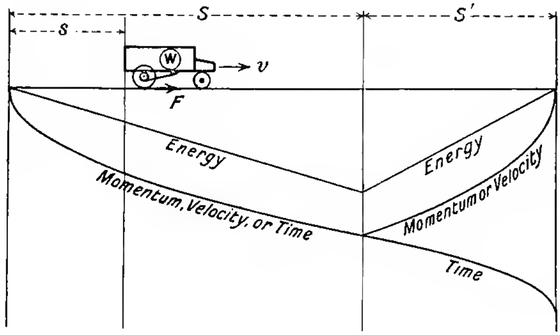


FIG. 47.

Dead resistance may be allowed for by supposing the road slightly up hill at the angle of repose; and no essential alteration is required, except in taking a little off F.

What, for example, is the time gained by cutting out a station on the tube railway? It is the halt and half the time of the stop and start.

And the time curve is the graph of a Bradshaw *Time Table*.

No mention has been made of the acceleration of the car; the idea is difficult and it is proverbial the engine driver cannot grasp it. But the passenger feels it as a tangible sensation, especially on leaving the car at one end or the other.

Acceleration appears, however, in g; and we postulate the theorems that a falling body will acquire

$$v \text{ f/s in } \frac{v}{g} \text{ seconds, falling through } \frac{v^2}{2g} \text{ ft.}$$

And the proper place for g is below v and v^2; it must not go astray under W.

The gravitation measure of force is used, suitable for dynamical questions in the field of gravity in which we live, and employed universally by the engineer.

A second lecture, of one minute, would carry on with the Statics of the alteration of trim of the floor on the springs, due to passengers entering or leaving, and as the carriage is accelerated or retarded.

The Law of the Spring is assumed as an experimental fact, based on Hooke's vague statement of the law—

Ut tensio sic vis.

A third lecture could be devoted to the simple pendulum, and the length required to beat time with the oscillation of a carriage body on the springs, vertical, pitching and rolling.

Thus the vertical oscillation should synchronize with a pendulum of length equal to the set of the springs, the vertical distance the carriage body sinks down on superposition; the law of the spring being supposed to hold.

This is verified with a spring balance and a weight, a 32-lb shot, provided the scale can be graduated uniformly.

But the logical and simple statement of the formula for the beat of the pendulum is

(31) $$T = \sqrt{\frac{l}{L}}, \quad \text{not } \pi\sqrt{\frac{l}{g}},$$

if L denotes the pendulum length which beats the second; as it is L which is determined experimentally, and g is derived from it by the relation

(32) $$g = \pi^2 L.$$

The rolling and pitching oscillation of the carriage body on the springs would introduce the idea of angular inertia, with which this lecture began; this is shown in its simplest form in the carriage wheels, in adding to the linear inertia of the carriage. The measurement of moment of inertia, or second moment, would run into a fourth lecture, provided we had the unlimited time at the disposal of Marchis in his Sorbonne lectures.

LECTURE V

THE SCREW PROPELLER

There is no exact theory, it must be conceded, of universal acceptance for the screw propeller, and reliance is placed chiefly on an empirical factor based on experience and model experiment, employed in a formula which satisfies the condition of mechanical similitude, so as to predict from a small scale experiment the performance to be expected of the full size machine.

A rational theory can be given of a hydraulic machine or turbine, when the water is compelled to follow a definite path; but where the fluid, air or water, is free to take its own course, as in the screw propeller, no exact treatment is possible until the stream lines have been determined.

Where the screw works in open water or air, the stream line is free to take a line of least action, and the shape is influenced to a great extent by the hull and fixtures in the neighbourhood, and the relative position of the propeller, effects which cannot be considered in a single formula.

Numerous theories will be found in the Abstracts of the *Report of the Aeronautical Committee* due to various experimenters, and one initial difficulty is to reconcile the conflicting notation employed by each writer; it is time this notation was standardised.

But the formulas are found to be in general agreement in making the thrust T proportional to

1. The density of the fluid, w or $\frac{1}{C}$, lb/ft³;
2. The disc area, S ft.²;
3. The square of the blade tip velocity, U f/s;
4. The slip s; or more accurately to the product $s\ (1-s)$.

With an empirical factor f the formula for the thrust may then be written

$$(1) \qquad T = fwS\,\frac{U^2}{2g}\,s\,(1-s), \qquad \text{lb},$$

and this is the weight of a cylindrical column of the fluid, of cross section S, and height

$$(2) \qquad f\,\frac{U^2}{2g}\,s\,(1-s), \text{ or } fs\,(1-s) \text{ of } H = \frac{U^2}{2g}.$$

The formula agrees then in making $T=0$ when $s=0$ and there is no slip, and the screw, of uniform pitch, advances in the fluid as if in a solid nut; and also when $s=1$, and there is no advance, and the screw cuts a hole in the fluid and swirls the fluid round.

But with a slip s between 0 and 1, the reaction of the fluid is against the rear of a blade, and a thrust is obtained.

A negative value of s would imply that the screw was being turned by the stream through it, as a windmill or turbine.

Working on the sails of a windmill or ship the wind strikes the rear of a sail and urges it forward, as in Fig. 48.

If OW represents W, the true wind over the water, and OV the velocity V of the ship through the water, then VW represents the apparent wind Q, as felt sweeping across the deck and filling the sail; and the direction of VW is given by the vane on a mast, or smoke from a chimney.

On the Newton theory the thrust on sail area A ft² is given by

$$(3) \qquad T = A\,\frac{(Q \sin \alpha)^2}{gC} = A\left(\frac{Q \sin \alpha}{20}\right)^2, \qquad \text{lb},$$

as we have taken previously when Q is given in f/s; but when the speed is given by K in knots of 100 ft/minute,

(4) $$T = A\left(\frac{K \sin \alpha}{12}\right)^2, \quad \text{lb},$$

because 60 knots is 100 f/s, 12 knots = 20 f/s.

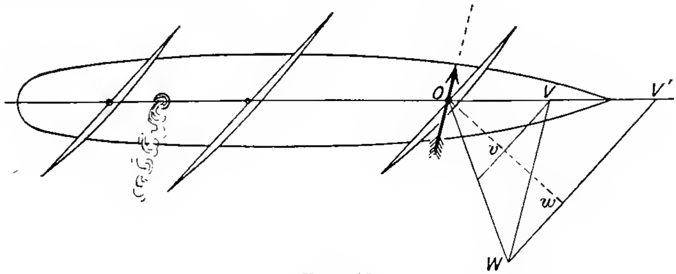


FIG. 48.

On the diagram of velocity of Fig. 48, $VWV' = \alpha$,

(5) $$vw = Ow - Ov,\ Q \sin \alpha = W \sin \beta - V \sin \theta,$$

and the propulsive force in the line of the keel is

(6) $$P = T \sin \theta = A \sin \theta \left(\frac{W \sin \beta - V \sin \theta}{12}\right)^2,$$

if W and V are measured in knots.

One H.P. of 33,000 ft-lb / min. is 330 knot-pounds, so that

(7) $$\text{the sail H.P.} = \frac{AV \sin \theta}{330}\left(\frac{W \sin \beta - V \sin \theta}{12}\right)^2.$$

Working this out for a ship of the size of the *Preussen*, spreading $A = 40{,}000$ ft^2 of canvas, with $\theta = 30°$, $\alpha = 60°$, $Q = 12$ (18) knots, $V = 10$ (15) knots, the sail H.P. is then about 450 (1,534).

An ice boat would run like a windmill unloaded; and at full speed OV', the vane on the mast would point parallel to the sail.

For the screw propeller, as far as theory can go at present, we begin with Rankine's treatment in the *Transactions of the Institution of Naval Architects*, 1865, following his notation as closely as possible.

A screw surface, of a true screw, is swept out by a straight line intersecting an axis at right angles; and the line advances along the axis and turns at the same time in a constant ratio; and the advance for a complete revolution is called the pitch, and denoted by p, and measured in feet.

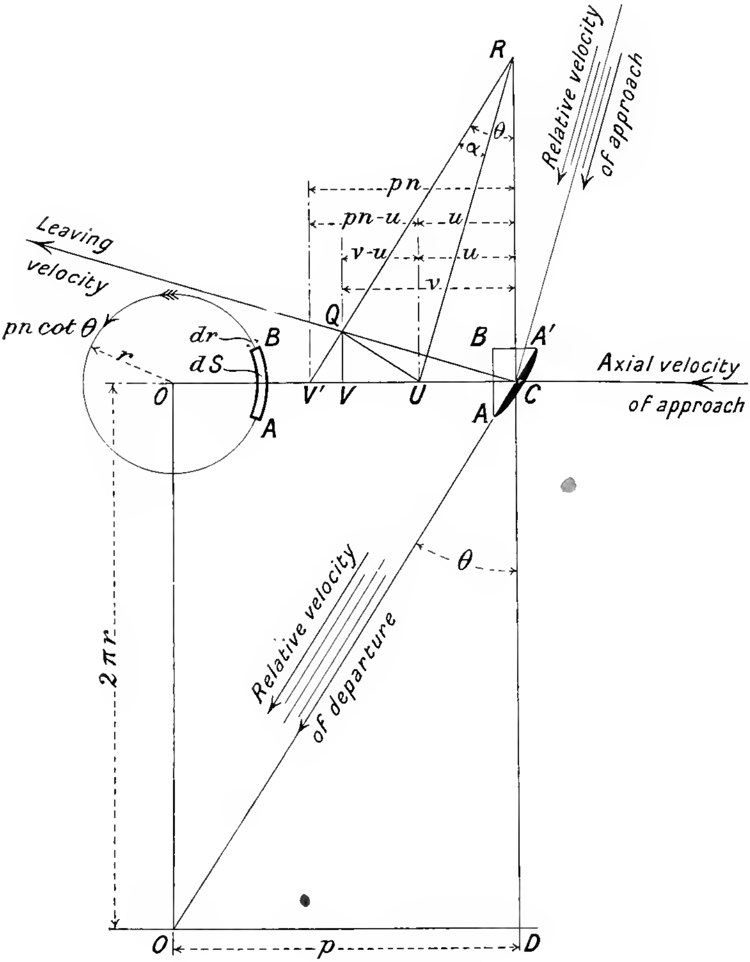

R
Relative velocity
of approach
θ
α
pn
pn-u
u
v-u
u
v
Leaving
velocity
pn cot θ
dr
dS
r
B
A
O
Q
V′
V
U
B
A′
A
C
Axial velocity
of approach
θ
Relative velocity
of departure
2πr
O
p
D

FIG. 49.

Engaging is a fixed nut, the pitch p is the axial advance for a complete revolution of the screw.

A co-axial cylinder will intersect the screw surface in a uniform helix, shown by a straight edge of a flat piece of paper, after the paper is wrapped on a cylinder; also by a thread or tape wound on the cylinder.

When the paper is flattened out again, the helix appears as a straight line OC, and rolling the cylinder on the paper makes the helix roll on the straight line.

Draw OD parallel to the axis of length p; then DC at right angles to OD is $2\pi r$ the circumference of the cylinder, for a radius r feet.

The section of the screw blade on the helix of circumference DC, limited by two planes perpendicular to the axis, is shown in plan by AA', and by AB in end elevation; and so for any other circumference, DC_0 or DC_1, by $OA_0C_0A_0'$, or $OA_1C_1A_1'$ (Fig. 49).

Consider the motion of the fluid relatively to the screw, with the fluid approaching with the axial velocity u f/s, and the screw making n revs/sec.

Draw CR in DC produced, representing to scale $2\pi rn$, the velocity of the point C on the screw; and draw CU representing the velocity u to the same scale; then draw RV' parallel to CO; denote the angle OCD by θ, so that $\tan\theta = \frac{p}{2\pi r}$; and then CV' will represent the velocity pn, which Rankine denotes by v'; and he puts $v' = p'n$, so that Rankine's p' is our p, and there is no need for an accent.

Then RU represents the relative velocity of approach of the fluid with respect to the blade at C, and URV' is called the angle of attack, and denoted by α, so as to correspond with Kirchhoff's diagram (Fig. 4).

Rankine assumes that the fluid is deflected by the blade element AA', so as to stream past it in the direction RV' with relative velocity RQ, where UQ is drawn perpendicular to RV'; the component QU being used up in producing the thrust on AA'.

With respect to the frame which holds the screw, the fluid streams away with velocity represented by CQ, having the axial component CV, denoted by v, and a transverse component VQ.

The velocity $np—u$, represented by UV', is called the *slip velocity*; and its ratio to pn is called the *slip* or *slip ratio* of the screw, and denoted by s; so that

(8) $$s = \frac{pn - u}{pn} = 1 - \frac{u}{pn}, \quad u = pn\,(1 - s),$$

and then

(9) $$v - u = UV = UQ \cos \theta = UV' \cos^2 \theta = (pn - u) \cos^2 \theta,$$

or in Rankine's notation

(10) $$v - u = (v' - u) \cos^2 \theta, \quad v = np\,(1 - s \cos^2 \theta);$$

a result written down by Rankine as if it was obvious, and required no explanation.

Relatively to a blade element AA' of the screw, the fluid approaches on a spiral of angle $\theta - a$, and pitch

(11) $$2\pi\, r \tan\,(\theta - a) = \frac{u}{n},$$

and the fluid is leaving the screw on a spiral of pitch p with respect to the blade; but with respect to the frame, the spiral has an angle CQV, and a pitch

(12) $$2\pi\, r \tan CQV = 2\pi\, r \tan \theta \,\frac{CV}{VV'} = p\,\frac{v}{pn - v} = p\,\frac{1 - s \cos^2 \theta}{s \cos^2 \theta}.$$

Draw AA'' parallel to UR; then if AA'' does not cut the development A_2A_2' of the preceding blade of the screw, and if the fluid is supposed to behave like a dust cloud of non-interfering particles, the fluid crossing $A''A_2'$ will pass through the screw undisturbed by a blade; and it will form a wake like the helical strand of a rope, stationary with respect to the surrounding fluid and of pitch $\frac{u}{n}$; but moving past the frame with velocity u and making n revs/sec.

There is a similar strand proceeding from the rear of a blade, from which all the dust particles must be supposed swept out.

But the particles which strike the blade element, of length AA', and breadth dr, are measured, in lb/sec, by

(13) $$w\ u\ .\ A''A'\ dr = \frac{pn - u}{C} AB\ dr,$$

where $A''A'$ is the parallel to CR, and

(14) $w = \frac{1}{C}$ is the density in lb/ft³, $C = \frac{1}{w}$ is the S.V. in ft³/lb ;

since

(15) $$\frac{A'A''}{AB} = \frac{UV'}{CU} = \frac{pn - u}{u}.$$

If, however, AA'' cuts the preceding blade, so that the blades interfere and screen each other in succession, then no fluid particles escape being received and deflected; so that the fluid acted upon is measured by

(16) $$\frac{u}{C} 2\pi\ r\ dr = \frac{u}{C} dS, \quad \text{lb/sec},$$

and this is contained by the cylindrical sheet of thickness dr and mean radius r, and cross section $dS = 2\pi r dr$.

On these assumptions a blade leaves a vacuum in its rear; but the current of fluid which encounters the face of a blade element AA' is driven off in a stream, parallel to the blade in the motion relative to the screw, but with respect to the frame in the direction CQ.

If unchecked by the surrounding fluid, the stream would continue in a straight line, by what is called sometimes the centrifugal tendency, really the First Law of Motion; and the wake current would grow in diameter in a conical or trumpet shape.

With a screw working in water such a state of centrifugal motion would be impossible unless the water was shattered into drops; but the liquid particles are compelled by the surrounding water to describe spiral lines, of angle θ', and pitch

$$p\left(\frac{1 - s\cos^2\theta}{s\cos^2\theta}\right);$$

and the centrifugal effect must be balanced by a pressure gradient, in a radial direction.

With no such cavitation of the water past the screw, the principle of the equation of continuity in hydrodynamics must be introduced ; but this principle is ignored when the fluid is treated as above by Rankine, as if it was a dust cloud of non-interfering particles, as in the *Kinetic Theory of a Gas.*

So far the theorems are geometrical; a dynamical principle employed by Rankine asserts that the forward thrust P of the screw, in lb, is given by the axial backward momentum in sec-lb, communicated per second.

The increase of axial velocity is from (9)

$$(17) \qquad v - u = (pn - u) \cos^2 \theta,$$

so that in the first case of blades not interfering,

$$(18) \qquad dP_1 = \frac{pn - u}{C} AB \, dr \, \frac{v - u}{g} = \frac{(pn - u)^2}{Cg} \cos^2 \theta \, AB \, dr.$$

Denoting by f_1 the fraction of the disc area occupied by the blade area projected on the plane of the disc area perpendicular to the axis of the screw, so that $AB = 2\pi f_1 r$,

$$(19) \qquad \begin{aligned} dP_1 &= \frac{(pn - u)^2}{Cg} (1 - \sin^2 \theta) \, 2\pi f_1 r dr \\ &= \frac{(pn - u)^2}{Cg} f_1 \left(2\pi \, rdr - \frac{2\pi p^2 \, rdr}{p^2 + 4\pi^2 r^2}\right) \\ &= \frac{(pn - u)^2}{Cg} f_1 \left[dS - \frac{p^2}{4\pi} d \log (p^2 + 4\pi^2 r^2) \right], \end{aligned}$$

where dS denotes the ring element of the disc area S and integrating, with $p^2 + 4\pi^2 r^2 = OC^2$,

$$(20) \qquad P_1 = \frac{(pn - u)^2}{Cg} f_1 \left(S - \frac{p^2}{4\pi} \log \frac{OC_0^2}{OC_1^2}\right) = \frac{(pn - u)^2}{Cg} f_1 SM,$$

suppose, where

$$(21) \qquad \begin{aligned} M &= 1 - \frac{p^2}{4\pi S} \log \frac{OC_0^2}{OC_1^2} \\ &= 1 - \frac{1}{4\pi^2} \cdot \frac{\log \frac{OC_0^2}{OD^2} - \log \frac{OC_1^2}{OD^2}}{\frac{OC_0^2}{OD^2} - \frac{OC_1^2}{OD^2}} \end{aligned}$$

If D denotes the extreme diameter of the screw, D_1 of the boss, and U the tip velocity due to the screw revolving n times a second, $U = \pi Dn$, f/s; and then

$$(22) \qquad P_1 = \frac{p^2 n^2}{Cg} s^2 f_1 SM$$
$$= \frac{p^2}{Cg} \frac{U^2}{\pi^2 D^2} s^2 f_1 \tfrac{1}{4}\pi (D^2 - D_1^2) M$$
$$= w \frac{p^2}{2\pi} \left(1 - \frac{D_1^2}{D^2}\right) \frac{U^2}{2g} s^2 f_1 M, \quad \text{lb.}$$

But in the second case when the blades begin to interfere,

$$(23) \qquad dP_2 = \frac{u}{C} dS \frac{v-u}{g} = \frac{u(pn-u)}{Cg} \cos^2\theta \, . \, 2\pi r dr$$
$$= \frac{u(pn-u)}{Cg} \left(dS - \frac{2\pi p^2 r dr}{p^2 + 4\pi^2 r^2}\right)$$

$$(24) \qquad P_2 = \frac{u(pn-u)}{Cg} SM$$
$$= \frac{p^2 n^2}{Cg} s(1-s)SM$$
$$= w \frac{p^2}{2\pi} \left(1 - \frac{D_1^2}{D^2}\right) \frac{U^2}{2g} s(1-s)M;$$

this is the formula of Coriolis, agreeing in shape with the empirical formula (1).

For a screw working under water without cavitation, where the axial velocity increases from u to v in passing through the screw, the principle of continuity requires the cross section of the stream to diminish, inversely as the axial velocity.

A radial current motion must then exist inside the screw stream; and in the formula of Coriolis the stream wake leaving the screw would be of reduced diameter.

Rankine, from observation of the screw propeller, prefers to take the screw wake as of full screw diameter, make $wvdS$, lb/sec, the quantity of water acted on by the element AA', and so giving an element of thrust

(25) $$dP_3 = wvdS\,\frac{v-u}{g} = w\,\frac{v(pn-u)}{g}\cos^2\theta dS$$

(26) $$dP_3 - dP_2 = w\,\frac{(v-u)(pn-u)}{g}\cos^2\theta dS$$
$$= w\,\frac{(pn-u)^2}{g}\sin^2\theta\cos^2\theta dS$$
$$= w\frac{p^2n^2}{g}\,s^2\sin^2\theta\cos^2\theta dS.$$

Rankine puts

(27) $$\frac{2\pi r}{p} = \cot\theta = q,\ r = \frac{pq}{2\pi},$$
$$dS = 2\pi rdr = \frac{p^2}{2\pi}\,qdq,\quad \sin^2\theta = \frac{1}{1+q^2},\quad \cos^2\theta = \frac{q^2}{1+q^2},$$

(28) $$dP_3 - dP_2 = w\,\frac{p^2n^2}{g}\,s^2\,\frac{p^2}{2\pi}\,\frac{q^3\,dq}{(1+q^2)^2}$$
$$= w\,\frac{p^4}{4\pi}\,\frac{n^2}{g}\,s^2\left[\frac{2qdq}{1+q^2} - \frac{2qdq}{(1+q^2)^2}\right]$$

(29) $$P_3 - P_2 = w\,\frac{p^4}{2\pi^3D^2}\,\frac{U^2}{2g}\,s^2\left(\log\frac{1+q_0^2}{1+q_1^2} - \frac{1}{1+q_0^2} + \frac{1}{1+q_1^2}\right),$$

giving a slight increase in P_3 over the P_2 of Coriolis, depending on s^2 the square of the slip ratio.

In a numerical application the calculation is usually close enough when the area is ignored of the boss of the screw, so that

(30) $$r_1 = 0,\ \theta_1 = \tfrac{1}{2}\pi,\ q_1 = 0,\ S = \pi r_0^2 = \frac{p^2}{4\pi}\cot^2\theta_0,$$

(31) $$M = 1 - \frac{\log\sec^2\theta_0}{\cot^2\theta_0}$$

(32) $$P_3 - P_2 = w\,\frac{p^4}{2\pi^3D^2}\,\frac{U^2}{2g}\,s^2(\log\operatorname{cosec}{}^2\theta_0 + \cos^2\theta_0).$$

A knowledge of the tip velocity U is required in a calculation of the strength of a screw blade, especially with the high velocity required with a wooden propeller of a flying machine.

On a diameter D feet, the ratio of centrifugal force ($C.F.$) to gravity g is given by

(33) $$\frac{C.F.}{\text{gravity}} = \frac{U^2}{\frac{1}{2}Dg},$$

will give an idea of the stress in the material.

If d is the diameter of the circle described by the $C.G.$ of a blade of weight W, the centrifugal pull at the root of the blade

(34) $$W\frac{v^2}{\frac{1}{2}dg} = W\frac{\left(\frac{d}{D}U\right)^2}{\frac{1}{2}dg} = W\frac{4d}{D^2}\frac{U^2}{2g}.$$

For instance, in a screw 7 ft in diameter, 22 ft in circumference, making 1,200 revs per minute, or 20 revs per second, the tip velocity $U = 22 \times 20 = 440$ f/s, which is 300 miles an hour, and at the tip

(35) $$\frac{C.F.}{g} = \frac{(440)^2}{7 \times 16} = 1{,}700 \text{ about.}$$

If the blade weighs 20 lbs and its $C.G.$ describes a circle of 3·5 ft diameter, the pull at the root of the blade is

(36) $$20 \times \frac{2}{7} \times \frac{(440)^2}{64} = 17{,}280 \text{ lbs, nearly 8 tons.}$$

The tension length in a ring of metal on the circumference at this speed U f/s is

(37) $$\frac{U^2}{g} = \frac{(440)^2}{32} = 6{,}050 \text{ ft};$$

and for steel of density 500 lb/ft^3, this would imply a tension of about 9 tons/inch^2.

For a smooth screw, the axial thrust dP implies a circumferential thrust $\tan\theta\, dP$ and so a turning couple, in ft-lb,

(38) $$dL = r\tan\theta\, dP = \frac{p}{2\pi}dP$$

(39) $$\frac{dL}{dP} = \frac{p}{2\pi}, \quad \frac{L}{P} = \frac{p}{2\pi}, \quad 2\pi L = pP;$$

and this is evident from the principle of Virtual Velocity or Work; because $2\pi L$ is the work in ft-lb done by the couple L ft-lb in a complete revolution of 2π radians, and this work with the smooth screw is equal to pP, the work done by the thrust P lb pushing through the pitch p feet.

The shaft horse power (S.H.P.) at n rev/sec is

(40) $$\text{S.H.P.} = \frac{2\pi L n}{550} = \frac{Ppn}{550},$$

while the thrust horse power

(41) $$\text{T.H.P.} = \frac{Pu}{550},$$

and the propeller efficiency e of the screw is measured by

(42) $$e = \frac{\text{T.H.P.}}{\text{S.H.P.}} = \frac{u}{pn} = 1 - s,$$

or

(43) $$\text{efficiency} + \text{slip} = 1.$$

Thus the

(44) $$\text{S.H.P.} = \frac{P_1 pn}{550} = w\frac{pn(pn - u)^2}{550g} f_1 SM,$$

$$\text{or} = \frac{P_2 pn}{550} = w\frac{upn(pn - u)}{550g} SM.$$

If the screw is run as a windmill or turbine, in air or water, so as to take up energy out of the current, np is less than u, and the angle of attack changes to the other side of the blade AA'.

For a given u and by variation of n or np, the S.H.P. transmitted is a maximum when $np = \frac{1}{3}u$ in the first case, but $np = \frac{1}{2}u$ in the second case where the blades interfere.

The old-fashioned windmill with four narrow sails (Fig. 50) will belong to the first class, and will be doing most work when run at one-third the unloaded speed; and then its

(45) $$\text{S.H.P.} = \frac{4}{27}\frac{wu^3}{550g} f_1 SM.$$

But the new Canadian form of windmill (Fig. 51), like a smoke jack, with numerous vanes which screen each other, should run at half the unloaded speed, so as to develop maximum power, and then the

$$\text{(46)} \qquad \text{S.H.P.} = \frac{1}{4}\frac{wu^3}{550g}SM.$$

These theories should be capable of experimental verification on a large scale.

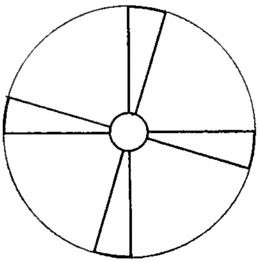


FIG. 50.

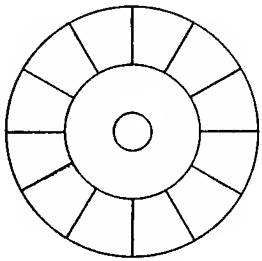


FIG. 51.

The theory of the windmill and air propeller should be the same; but in the treatment above, no notice has been taken of the fluid after it has passed the screw.

The equation of continuity of fluid motion is ignored in the turbulent motion of the wake, and the fluid is assumed to behave as a cloud of dust, so that two currents can pass through each other without interference.

But with a screw under water, where the backward stream leaves the screw with axial velocity v, the stream line must make a spiral of pitch p with respect to the revolving screw, but with pitch $\frac{v}{n} - p$ with respect to the frame.

Also with v greater than u, an inward radial velocity is required to prevent cavitation; the current acted on by the screw is of greater area ahead, and the stream lines converge towards the axis in passing through the screw (Fig. 52).

But no theory so far has been able to assign the amount of this convergence for a screw working in unlimited water.

Let us examine an ideal case, in which the screw is working in a cylindrical tunnel which prevents radial motion in the water, and then continuity and absence of cavitation requires $u = v$, for the water running full bore.

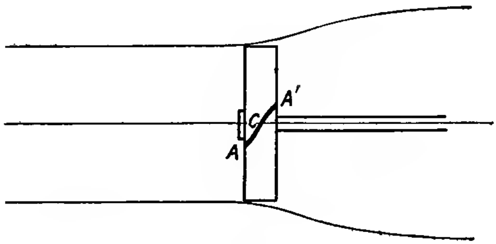


FIG. 52.

There is now no generation of momentum sternward, and the preceding treatment would imply a zero thrust.

The thrust is due, however, to a change of pressure, and the screw acts as a turbine pump.

The water feeds the screw with a uniform stream velocity u, and leaves with the same axial velocity u, but with a transverse velocity

(47) $$(pn - u) \cot \theta = (pn - u) \frac{2\pi r}{p};$$

that is, the angular velocity round the axis is

$$2\pi\left(n - \frac{u}{p}\right), \text{ or the water is making } n - \frac{u}{p}, \text{ revs/sec.}$$

The screw now spins a rope of the wake, at the rate of angular momentum, in lb-ft²/sec,

(48) $$L = w S k^2 u \frac{2\pi\left(n - \frac{u}{p}\right)}{g};$$

or, ignoring the boss, and putting $k^2 = \frac{1}{8} D^2 = \frac{S}{2\pi}$,

(49) $$L = w \frac{u\left(n - \frac{u}{p}\right) S^2}{g}$$

giving a thrust

(50) $$P_4 = \frac{2\pi L}{p} = w \frac{\frac{u}{p}\left(n - \frac{u}{p}\right)}{g} 2\pi S^2;$$

or, expressed in terms of the tip velocity U, and slip s,

(51) $$P_4 = w \frac{U^2}{g}\left(\frac{S}{\pi D}\right)^2 2\pi (s - s^2) = w S \frac{U^2}{2g}(s - s^2),$$

agreeing with formula (1) with $f = 1$, and assuming that the screw wake receives a uniform swirl.

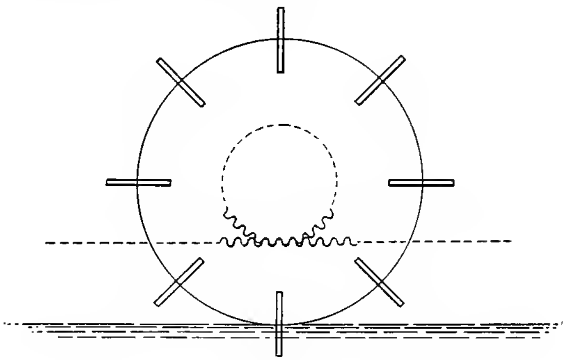


FIG. 53.

In a paddle steamer p should represent the circumference of the pitch circle through the centre of the floats, S the area of a float, and

(52) $$T = fwSu \frac{np - u}{g} = fwS \frac{U^2}{g} s (1 - s).$$

The vessel advances as if a circumference of length $\frac{u}{n}$ engaged in a horizontal rack, to give it a velocity u (Fig. 53).

In sea water, of specific volume 35 ft³/ton or $\frac{35}{2,240} = \frac{1}{64}$ ft³/lb, and with $g = 32$, $Cg = \frac{1}{2}, \frac{w}{g} = 2$.

And if the thrust P_4 is given in tons,

$$Cg = 35 \times 32 = 1,120.$$

Thus for one of the four screws of the Mauretania, 16 ft in diameter, 18 ft pitch, making 150 revs/min with 7 per cent. slip,

$$n = \frac{150}{60} = \frac{5}{2}, \; pn = 45 \text{ f/s, about 27 knots;}$$
$$u = pn\,(1 - s) = 41 \cdot 8 \text{ f/s, say 25 knots;}$$
$$s = 0{\cdot}07,\; 1 - s = 0{\cdot}93,\; S = 64\,\pi^2 \quad \text{ft}^2;$$

we find P_4 is about 90 tons, so that at 42,000 tons displacement, the resistance of the water is equivalent to an incline of 1 in 42,000 ÷ 360, say 120, a resistance of about 20 lb/ton, reckoned as on a railway; and the S.H.P. works out to about 66,000.

These numbers have been manœuvred so as to agree with practical results by the choice of a very small slip, 7 per cent.; an estimate of 15 per cent. would be more likely, but this would double the value of P_4, so that we should require to take $f = \frac{1}{2}$ to bring the former into agreement, and now the speed would have sunk to $45\left(1 - \frac{15}{100}\right) = 38{\cdot}25$ f/s, or 23 knots.

A provisional estimate of f may be made by taking it as the fraction of disc area made by the projected blade area.

Tested on an air propeller, two-bladed, with diameter $D = 15$ ft, and making 450 revs/min with a slip of 36 per cent., and giving a thrust of 1,000 lbs, we should find that we should have to take $f = \frac{1}{18}$, implying that the blades were sections of 10° in the disc area projection.

The thrust P_4 in (51) produces an increase of pressure

(53) $$\frac{P_4}{S} = w\frac{U^2}{2g}s\,(1 - s) \text{ lb/ft}^2,$$

$$\text{or a head of} \frac{U^2}{2g}s\,(1 - s) \text{ ft;}$$

and this is the height to which water could be driven, using the screw enclosed in a tube as a centrifugal pump.

A numerical comparison can be made with a four-stage centrifugal pump, described in the *Engineer*, June 17, 1910, which pumps 110 gallons/minute to a head of 225 feet, at 1,440 revs/min, and B.H.P. 10.

The loss of energy, measured in ft-lb/sec, is

$$(54) \qquad 2\pi Ln - P_4 u ;$$

and of this, half is thrown away in the rotating energy of the wake; the other half is lost by shock on the blade.

This last energy can be recovered if the leading edge A' of the blade is given a zero angle of attack, so that the screw has a gaining pitch, from $\frac{u}{n}$ at A' through its mean value p to some final pitch p', and the blade AA' is cambered.

If the camber is parabolic, so that cot θ increases uniformly in the axial direction, the mean effective pitch p is the harmonic mean of the initial pitch $\frac{u}{n}$ and final pitch p',

$$(55) \qquad \frac{1}{n} = \frac{1}{2}\left(\frac{n}{u} + \frac{1}{p}\right).$$

The system of gaining pitch is always adopted with a turbine, intended to run at a given speed n in a given current u; the guide blades may be taken as the equivalent of another screw fixed in front.

Two screws on the same shaft line were employed in one of the earliest screw steamers, so as to recover the rotational energy of the wake; the system has been brought forward lately by Colonel Rota, of the Italian Navy; the system seems applicable to the Gnome motor on a flying machine, when the shaft is made to carry a screw as well as the cylinders, and is allowed to revolve in the opposite direction.

In this way the revolutions of each screw are halved, while the relative motion of the axle and cylinder remains the same as is desirable in practice.

So long as the slip s is small, the energy recovered from the wake would not be worth the extra weight and complication of a second screw.

But with the two screws the slip s may be made as large as 50 per cent., $s = \frac{1}{2}$, and then each screw is pulling hardest for its weight, so that size can be reduced and weight economised.

With uniform pitch there would be no economy of efficiency, as the energy recovered from the wake is lost again in shock at the second screw; but with appropriate gaining pitch the theoretical efficiency can be made perfect.

Large slip is preferred at sea for driving against a head sea, and diminution of racing.

Racing of the screw is due chiefly to variation of axial flow; the variation of the longitudinal velocity of the water in wave motion has more influence than the accompanying vertical component.

With a fine pitch and small slip this velocity variation causes a rapid change in s and L, not so rapid when s is large.

LECTURE VI

PNEUMATICAL PRINCIPLES OF AN AIR SHIP

The flying machine as a practical success is only some two or three years old; but it looks as if it will displace the air ship balloon, with a large gas bag to give the ascensional force.

The air ship lighter than air is, however, still on its trial, and so we proceed to discuss the pneumatical theory involved; for the detailed calculation a reference must be made to Chapter VIII. of my *Hydrostatics*.

The first practical balloon to make an ascent with a man dates from 1783, the hot air balloon of Montgolfier (Fig. 54).

The legend goes that as Madame Montgolfier's silk dress was airing before a fire, it became inflated and rose to the ceiling.

Montgolfier followed up the idea, on a small scale at first, on the impression that the hot air was some new kind of gas; and finally, as a paper manufacturer, he was able to make a fire balloon large enough to take up the first two real aeronauts, Pilâtre de Rozier and the Marquis d'Arlandes, in November, 1783, from the Château de la Muette in Paris, and so realise finally the dream of the poet and artist of antiquity.

The principle is seen in the ordinary toy hot air balloon; the air in the balloon is rarefied by heat to an extent such as to make the total weight of the balloon, car, and passengers, and of the hot air it contains, equal to or less than the weight of the external cold air displaced.

The experimental laws of pneumatics required in the theory are embodied in the gas equation

$$\frac{pv}{\theta} = R = \frac{p_2 v_2}{\theta_2}, \tag{1}$$

connecting the pressure p, lb/ft^2, specific volume v, ft^3/lb, and absolute temperature θ, which we take Centigrade, with p_2, v_2, θ_2, in another state of the same given quantity of a gas.

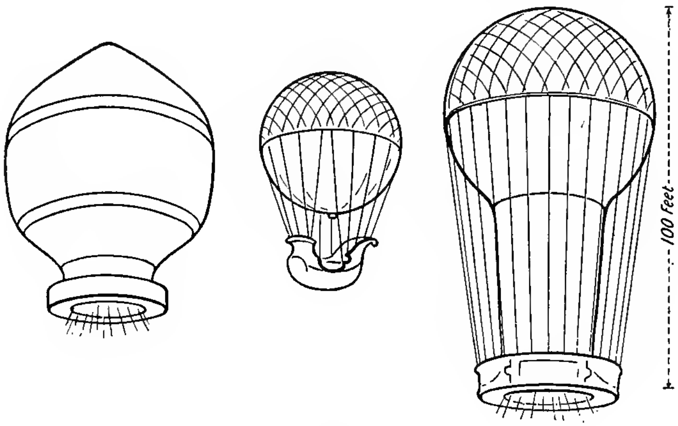


FIG. 54. FIG. 55. FIG. 56.

This equation expresses Boyle's law when the temperature θ is constant, and the law of Charles, when θ varies, and either p or v, one at a time.

Denote by W, lb, the weight of the balloon, car, and aeronauts, corrected for buoyancy of the air, as if weighed in a vacuum, and denote by W' lb the weight of atmospheric air they displace, so that $W - W'$ lb is the apparent weight when weighed in air; denote also by V, ft^3, the capacity of the gas bag of the balloon, so that $M = V\rho$ lb denotes the weight of atmospheric air which fills the balloon, at a density ρ, lb/ft.^{3}

When the air inside is raised in temperature from θ to θ' degrees, absolute Centigrade, part of the air will flow out of the balloon, leaving the rest at the same pressure, p, but at density

(2) $\rho \frac{\theta}{\theta'}$, and therefore of weight $V\rho \frac{\theta}{\theta'} = M \frac{\theta}{\theta'}$, lb.

The balloon will be floating in equilibrium when the weight of the balloon and hot air it contains is equal to the weight of surrounding cold air displaced; that is when

(3)
$$W + M \frac{\theta}{\theta'} = W' + M,$$

$$\frac{\theta}{\theta'} = \frac{M - W + W'}{M}, \quad \frac{\theta' - \theta}{\theta} = \frac{W - W'}{M - W + W'},$$

determining $\theta' - \theta$, the increase of temperature required to rise.

The balloon is now in unstable equilibrium, like a bubble of air compressed to the density of the surrounding water; and it will begin to rise, as it cannot descend.

The balloon will continue to rise and the hot air to escape till another stratum is reached, at height z ft, suppose, where the density is ρ_z and absolute temperature θ_z; and then the pressure p_z is given by the gas equation (1)

(4)
$$p_z = p \frac{\rho_z}{\rho} \frac{\theta_2}{\theta} = \frac{\theta_z}{v_2} \frac{pv}{\theta}.$$

Reference must be made to *Hydrostatics*, Chapter VIII., for the further theory of Montgolfier's hot air balloon; but the principle was soon abandoned in favour of the hydrogen balloon, invented by the chemist Charles a few months later, having an advantage that the lift can be obtained with a gas bag much smaller (Fig. 55).

This is of great importance for military use, where the balloon is tethered, and size must be kept down on account of the wind; also in the large air ship, intending to take up numerous passengers, and to keep up in the air and make a journey as long as possible.

For economy the ordinary balloon of sport is filled with coal gas from the neighbouring works; and on the aeronauts rule that 1,000 ft^3 of this gas will lift about 40 lb, while

$$1{,}000 \text{ ft}^3 \text{ of air, at } 12\cdot5 \text{ ft}^3/\text{lb, weighs } 1{,}000 \div 12\cdot5 = 80 \text{ lb};$$

this makes the S.V. of this gas double that of air, or 25 ft^3/lb.

Generally 1,000 ft^3 of a gas of S.V. n fold of the air will lift P lb, given by

$$(5) \qquad \rho = 80 - \frac{80}{n}\, n = \frac{80}{80 - \rho},$$

represented by a hyperbolic graph.

To rarefy the air in a Montgolfier fire balloon to double S.V. would require the temperature to be raised from 0 to 273° C., from 32 to 520° F.; but this heat would disintegrate a fabric like paper or silk; and if replaced by asbestos the weight of the skin becomes excessive.

With pure hydrogen we may take n as large as 14, so that 1 ton of hydrogen can lift itself and 13 tons more in the air; and as 1 ton of air bulks 2,240 × 12·5 = 28,000 ft^3, 1 ton of hydrogen bulks 28,000 × 14 = 392,000 ft^3; and 2 tons of hydrogen, or 784,000 ft^3 could lift 26 tons; this is the volume of a cylinder 500 ft long and about 50 ft diameter, something like the Admiralty air ship.

Suppose it is required to weigh a ton of hydrogen in the scales and at atmospheric pressure, not compressed to 100 atmospheres in a steel cylinder, 13 ton weights would be required, and placed in the same scale.

But compressed in a cylinder to 100, or generally to x atmospheres, to $\frac{x}{n}$ of air density, the ton of hydrogen would displace $\frac{x}{n}$ of a ton of air, and would weigh an empty cylinder in the other scale and $1 - \frac{n}{x}$ tons.

Thus with $n = 14$, $x = 100$, $1 - \frac{n}{x} = 0\cdot86$, at a volume 3,920 ft^3, say a cylinder 8 ft in diameter and nearly 80 ft long; or 560 cylinders of a usual size, 9 ft long and 1 ft diameter inside.

And the price, too, is an important consideration, nearly £1,000 for a full-size air ship; although it is claimed that hydrogen can be produced at 1*d.* per metre^3 or 35 ft^3 per penny, so that 800,000 ft^3 would cost about £100.

Suppose the gas employed for inflating the balloon is n times lighter than the air, of density $\frac{\rho}{n}$ lb/ft,^{3} and S.V. nv ft^3/lb, $\left(\frac{w}{n}\right.$ and nC in the previous $\left.\text{notation}\right)$; and let U ft^3 of this gas, weighing $P = \frac{U}{nv}$ lb, be allowed to flow into the balloon.

The balloon will be on the point of rising when

$$(6)\quad W + P = W' + nP, \quad P = \frac{W - W'}{n - 1}, \quad U = (W - W')\frac{nv}{n - 1},$$

$$(7)\qquad U = \frac{W - W'}{A}, \text{ where } A = \frac{1}{v} - \frac{1}{nv} = \rho - \frac{\rho}{n},$$

and A is the lift of the gas in lb/ft^3, 1,000 A in lb/1,000 ft.^{3}

The balloon, like the bubble compressed in water, is in unstable equilibrium, and will begin to rise; and to carry the balloon clear of neighbouring obstacles rapidly, it is advisable that the volume U or weight P of gas should be increased, to give an ascensional lift, which at starting will be a force

$$(8)\qquad (n - 1)P - (W - W'), \qquad \text{lb.}$$

As the balloon rises, the gas contained in it will expand until the envelope is completely inflated, and the gas will now occupy V ft^3; this will take place where the density of the air is $\frac{U}{V}\rho$, and $\frac{U}{V}\frac{\rho}{n}$ the density of the gas, the temperature of the gas being supposed unaltered.

The ascensional lift force will now be

$$(9)\qquad (n - 1)P - \left(W - W'\frac{U}{V}\right), \quad \text{lb}$$

$$\text{or } W'\left(1 - \frac{U}{V}\right) \quad \text{lb less than at starting,}$$

The balloon will still continue rising, but now it is very important that the neck of the balloon, should be left open, to allow gas to escape as the balloon rises into the more rarefied air, and so equalise the pressure of the interior gas and surrounding air; otherwise the pressure of the gas, if imprisoned, might burst the balloon, as Charles found when he started experimenting with small hydrogen balloons, sealed up.

At the height z ft, where the density of the air is ρ_z, the ascensional force, in lb, will be

$$(10) \qquad V\rho_z\left(1 - \frac{1}{n} - W - W'\frac{\rho_z}{\rho}\right)$$

$$= [(n-1)\,Q + W']\frac{\rho_z}{\rho} - W$$

$$= (VA + W')\,\frac{\rho_z}{\rho} - W,$$

on putting

$$(11) \qquad Q = V\frac{\rho}{n} = \frac{M}{n}, \quad M - Q = (n - 1)\,Q = VA,$$

where Q denotes the weight of gas, and M of air, of volume V which would fill the balloon on the ground.

The lift is zero and the balloon comes to rest where

$$(12) \qquad \frac{\rho_z}{\rho} = \frac{p_z}{p}\cdot\frac{\theta}{\theta_z} = \frac{W}{VA + W'},$$

and in an isothermal atmosphere,

$$(13) \qquad \frac{\rho_z}{\rho} = \frac{p_z}{p} = e^{-\frac{z}{k}},\ z = k\log\frac{\rho}{\rho_z} = \mu k \log_{10}\frac{VA + W'}{W},$$

where k denotes the height of the homogeneous atmosphere, say 28,000 ft, with an atmospheric pressure on the ground of one ton/ft^2; and $\mu = 2{\cdot}3$, the modulus of the natural logarithm.

Here again the equilibrium is unstable; as if the balloon rises a little more, it loses gas through the neck as well as by diffusion; and if it descends the balloon is bulged in and loses buoyancy displacement, and the pilot recognises this at once by the crackling and pucker of the skin.

A free balloon is either rising or falling, and it must be steered in a vertical plane, either by throwing out ballast or letting off gas by the top valve; but it can be kept at a moderate average height by a rope trailing on the ground, or, in the case of the American Wellman air ship, by the buoy trailing on the sea. But here the violence of this equilibrator among the waves was the cause of failure.

The calculation is given in *Hydrostatics*, § 241, Chapter VIII., of the effect of throwing out ballast.

Pilâtre de Rozier took up the mad scheme of combining the two systems, Montgolfière and Charlière, of Montgolfier and Charles, into one balloon, his Charlo-Montgolfière, as shown in Fig. 56, a spherical gas bag of hydrogen, on the top of a vertical cylinder to carry heated air.

His idea was to preserve the hydrogen sealed up, and to use the hot air as a regulator for rising and descending without carrying ballast, and so keep the air an indefinite time.

The inevitable came to pass very soon, in about half an hour from the start from Boulogne, June, 1785. The gas bag burst as soon as a moderate height was attained, about 5,000 ft, accelerated by the hot air below; the fire blew up the hydrogen and the machine was shattered. A commemoration obelisk is to be seen at Wimille, near Boulogne.

In balloon calculations it is convenient to suppose the total weight, W lb, to be distributed over the skin of the envelope, at a superficial density $m = W/S$, lb/ft^2, S denoting the surface in ft^2.

Then, neglecting W' as insensible,

$$VA - mS \text{ is the lift at the ground,}$$

$$VA\frac{\rho_z}{d} - mS,$$

at a height z, where the density has fallen from ρ to ρ_z.

The balloon will rise to this height z, if

$$(14) \qquad VA\frac{\rho_z}{\rho} = mS, \quad VA = qmS, \quad q = \frac{\rho}{\rho_z}.$$

For a spherical balloon of diameter d ft,

$$(15) \qquad V = \frac{1}{6}\pi d^3,\ S = \pi d^2,\ d = 6\frac{mq}{A},$$

$$(16) \qquad VA = qm\pi\left(\frac{6}{\pi}V\right)^{\frac{2}{3}},\quad V = 36\,\pi\frac{q^3m^3}{A^3},$$

called by the French the equation of the "three cubes"; but it is simpler to work to the diameter d in (15).

The calculation is given on p. 337, *Hydrostatics*, required by the Jesuit Francis Lana, 1670, for his idea of a copper cylinder, exhausted of air, to serve as a balloon; the exhaustion can be carried out by boiling a small quantity of air inside until the steam has driven out all the air; then when sealed up a vacuum would be formed when the steam was condensed.

Thus with copper, 0·01 inch thick, we can take $m = \frac{1}{2}$ lb/ft², and with $A = 0{\cdot}08$ lb/ft³, $d = 37{\cdot}5\,q$; so that on the ground, with $q = 1$, $d = 37{\cdot}5$ ft, and the copper sphere is just about to rise; and taking $d = 100$ ft,

$$q = \frac{Ad}{6m} = \frac{8}{3},\quad \frac{\rho_z}{\rho} = \frac{3}{8},$$

about 5 miles high in an isothermal atmosphere.

The idea was very creditable, but it did not reckon with the collapsing pressure of the atmosphere; and at that date, 1670, hydrogen was unknown, and even the name *gas* had not been invented; and it was only the accidental observation made by Montgolfier mentioned above which showed, 100 years later, that the balloon problem was feasible, with a light envelope, where the gauge pressure in the interior was very low and the stress in the envelope correspondingly small.

The gauge pressure (excess over atmospheric pressure outside) at a height y ft above the lower end of the neck or appendix tube will be

$$(17) \qquad \left(\rho - \frac{\rho}{n}\right) z = Ay, \quad \text{lb/ft}^2,$$

at the ground, changing to $\frac{\rho_z}{\rho} Ay$ at a height z in the air; this is neglecting the variation of density of the air and gas in a height so small comparatively as y ft.

For the cylindrical gas bag on an air ship of diameter d ft, and p diameters long (Fig. 57), we may put

(18) $$V = \tfrac{1}{4}\pi d^3 p,\ S = \pi d^2 p,$$

(19) $$\tfrac{1}{4}\pi d^3 pA = qm^3 S = qm\pi d^2 p,\ d = \frac{4\,mq}{A}.$$

(20) $$V = 16\,\pi\,p\,\frac{m^2 q^3}{A^3}.$$

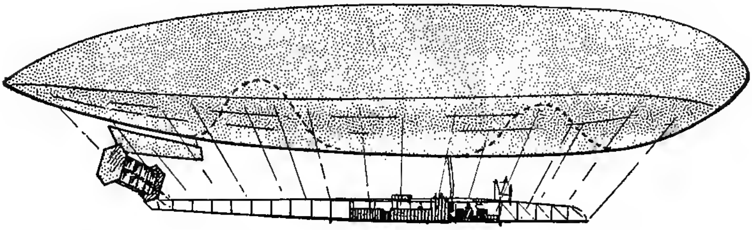


FIG. 57.

Thus for an air ship 500 ft long by 50 ft diameter, $d = 50$, $p = 10$, weighing 25 tons,

(21) $$m = \frac{25 \times 2{,}240}{\pi \times 500 \times 50},$$

and filled with hydrogen of $n = 14$,

(22) $$A = \left(1 - \frac{1}{14}\right)\frac{1}{12{\cdot}5} = \frac{52}{700} = 0{\cdot}075 \text{ lb/ft}^3,$$

(23) $$q = \frac{Ad}{4m} = \frac{52}{700} \times \frac{50 \times \pi \times 500 \times 50}{4 \times 25 \times 2{,}240} = 1{\cdot}3,$$

(24) $$z = 2{\cdot}3\,k \log_{10} q = 7{,}000 \text{ ft},$$

the height ascended in an isothermal atmosphere, with $k = 27{,}000$ ft.

In the non-rigid air ship the cylindrical envelope forms one continuous gas bag, and this must be kept constantly inflated.

If this should become partially deflated, like an ordinary spherical balloon, the indentation would affect the stability as it runs from one end of the belly to the other; and in the middle it would cause the gas bag to lose longitudinal stiffness, and allow it to double up like a bolster, as in Fig. 58.

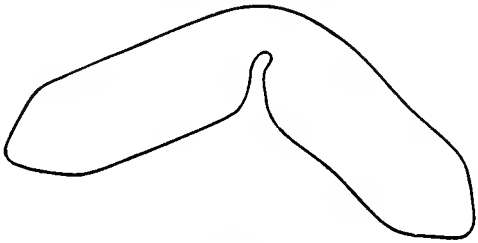


FIG. 58.

A ballonet is required, and this is an interior bag which can be inflated by an air-pump to fill up the space left vacant by the escape of the hydrogen through the appendix in the ascent, and to keep the balloon distended during the descent, and on the ground, in spite of gas leakage.

One ballonet at each end inside is useful, shown by the dotted lines in Fig. 57, to serve like a trimming tank of a steamer, for keeping the air ship on an even keel.

In the air ship of rigid type the hydrogen is carried in a number of separate gas bags enclosed in a lattice frame cage, and covered with a smooth envelope to diminish frictional drag, so that this type is equivalent to a series of spherical balloons harnessed together in a horizontal line.

The gondolas are suspended below, as in Fig. 57, carrying the engines and crew.

Consider the power required to drive this air ship at a given speed, Q f/s, or S m/h, against the resistance of skin friction of the air.

It is estimated as the result of experiment that frictional drag is about $\frac{1}{400}$ of pressure due to normal impact, taken above at $\left(\frac{Q}{20}\right)^2$ lb/ft²: so the friction is taken at $\left(\frac{Q}{400}\right)^2$ lb/ft².

Caution is required here not to take double or half this value; as experimenters have an awkward habit of tabulating frictional drag sometimes for both sides of the surface in their experiments.

Here $S = \pi \times 50 \times 500 = 78{,}540$, say 80,000 ft²;

$$(25) \qquad F = S\left(\frac{Q}{400}\right)^2 = \frac{1}{2} Q^2 \quad \text{lb.}$$

$$(26) \qquad \text{T.H.P.} = \frac{FQ}{330} = \frac{Q^3}{660}.$$

Thus, if a speed of 45 m/h is required,

$$Q = 66, \qquad \text{T.H.P.} = \frac{66^3}{660} = 453{\cdot}6.$$

With screws working at 25 per cent. slip, their efficiency is 0·75, 75 %, so that S.H.P. = 575, requiring probably 700 I.H.P. or more; or in a tabular form,

S	30	36	40	45
Q	44	53	58	66
S.H.P.	170	390	390	575

so that 360 I. H. P. should be expected to give a speed somewhere between 30 and 36 m/h.

FINIS.

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