A TEXTBOOK

ON

SHEET-METAL PATTERN DRAFTING

INTERNATIONAL CORRESPONDENCE SCHOOLS
SCRANTON, PA.

V 1.

ARITHMETIC
INSTRUMENTAL DRAWING
GEOMETRICAL DRAWING

WITH PRACTICAL QUESTIONS AND EXAMPLES
ON ARITHMETIC AND THEIR ANSWERS

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PREFACE.

The Bound Volumes of the Sheet-Metal Pattern Drafting Course contain twenty-two Instruction Papers. The subject of sheet-metal patterncutting is here, for the first time, treated in an exhaustive and comprehensive manner. Beginning with the elementary processes of Arithmetic, the subjects of Geometry and Mensuration are unfolded to the student in a gradual manner, while the application of these subjects to the practical needs of the trade worker is made so obvious that the workman cannot fail to derive the utmost benefit therefrom. These Volumes, therefore, will make a desirable addition to any sheet-metal worker's library, and will serve as an admirable work of reference. The Instruction Papers in pamphlet form may be conveniently kept in the tool-box at the shop; the numerous tables, rules, and constructions will then be within easy reach when needed.

The method of numbering the pages, cuts, articles, etc. is such that each paper and part is complete in itself; hence, in order to make the indexes in the Bound Volumes intelligible, it was necessary to give each paper and part a number. This number is placed at the top of each page, on the headline, opposite the page number; and to distinguish it from the page numbers it is preceded by the printer's section mark (§). Consequently, a reference such as page 9, § 3, would be readily found by looking along the inside of the headlines until § 3 is found, and then through § 3 until page 9 is found.

The Examination Questions and the Keys are given the same section numbers as the Instruction Papers to which they belong, and are grouped together at the end of the first volume. Examination Questions and Keys are provided only for the first twelve sections. Reduced copies of the Drawing Plates that accompany the later sections of the Course are inserted as folders throughout the text of the respective sections to which they refer.

The first volume contains twelve sections on arithmetic and mensuration in which the fundamental operations needed by the pattern draftsman in his calculations are fully, clearly, and concisely set forth; a section on instrumental drawing in which instruments and their uses, together with the preliminary operations of the drawing board, are minutely explained; and a section on geometrical drawing in which the principles of geometry are illustrated in their practical applications to the wants of the pattern-cutter.

The second volume contains eight sections, or Instruction Papers, which comprise the technical, or advanced, portion of the Course. In Practical Projection the subject of working drawings is very carefully explained and the student is taught both how to make and how to read mechanical drawings. Copious illustrations are given and no pains have been spared in order to make this subject clearly understood. In Development of Surfaces the forms drawn by the student in the preceding section are developed—that is, patterns, or lay-outs, for their surfaces are defined—and the student is here made familiar with the general principles that govern all sheet-metal pattern work. Practical Pattern Problems is here complete in three sections, much of the tabular matter which appeared in the first edition of the first section having been incorporated into the earlier mathematical portion of the Course. In these sections the practical applications of the draftsman's art to the sheet-metal worker's trade are shown in an extended list of representative problems. These problems are grouped in such a manner that workmen engaged in the different branches of the trade may find their particular problems under separate headings, thus greatly facilitating the work of referring thereto. In Architectural Proportion the cornice maker and the architectural sheet-metal worker will find the classic orders carefully explained, and may easily ascertain by reference to the pages of this section, just what proportions to give any particular design they may be called on to construct. Development of Moldings contains explicit instructions for every conceivable form of cornice pattern. Miters of every variety are explained and the student is shown the shortest possible method of arriving at the exact lay-out. The section on Skylights contains accurate developments for skylight patterns in every form of bar. The usual constructions are given, and full-sized drawings may be made directly from the illustrations when required.

All these papers have been prepared with especial reference to their practical utility; they are complete and concise, and have been suited to the needs of the sheet-metal worker—they contain nothing that is not of real value to him. No unnecessary operations are shown, and in every case the shortest and most practical methods have been adopted. These papers have been prepared by our own experienced mathematicians and sheet-metal experts; they contain such tables of weights, measures, dimensions, etc. as are needed by the mechanic when he is called on to estimate the costs and capacities of various articles. The entire Course is directly suited to the needs of the practical patterncutter.

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ARITHMETIC.

(PART 1.)

INTRODUCTION.

PRELIMINARY DEFINITIONS.

NUMBER.

1. The idea of number is first derived from counting distinct objects, as when a child counts his fingers.

An **abstract number** is the simple direct answer to the question "How many?" For example, the answer to the question "How many dots are there in Fig. 1?" is the abstract number *twelve.*

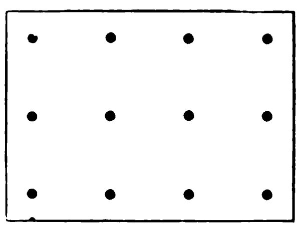


FIG. 1.

2. By counting we can compare the magnitudes of any two groups of distinct similar objects; for example, we can compare the size of a crowd of men in New York with the size of a crowd in Chicago by counting the number of men in each crowd. But it is often necessary to compare two similar magnitudes, each of which does not consist of distinct objects; for example, we may desire to compare the length of one iron bar with the length of another bar. This comparison would be effected by measuring the length of each bar in feet or inches.

3. To **measure** a quantity is to find how often it contains a known quantity of the same kind. A known quantity that is used to measure quantities of the same kind is called a **unit quantity** or a **unit.** Thus, a **foot** is a unit for measuring length. In counting a group of distinct objects, if the objects are counted one by one, each distinct object is regarded as a unit; if the objects are counted in pairs, each pair of objects is regarded as a unit; if the objects are counted in dozens, a dozen objects is regarded as a unit, and so on. Thus, if we count a lot of toothpicks one by one, each toothpick is a unit; if we say that a locomotive has four pairs of wheels, a pair of wheels is the unit; if we say that there are six dozen eggs in a box, a dozen eggs is the unit, and so on.

4. The abstract number *four* contains the abstract number *one* four times; and a length of four inches contains one inch four times; thus the abstract number *one* measures the abstract number *four* just as one inch measures a length of four inches. Now, one inch is called a unit of length, because it measures length; and in the same way, the abstract number *one* is called an **abstract unit,** or **unity,** because it measures abstract numbers.

5. **Like units** are units that measure quantities of the same kind, and **unlike units** are those that measure quantities of different kinds. For example, a pound and an ounce are like units, for they are both used to measure weight. A **linear** foot is used to measure length, and a **square** foot is used to measure area; therefore, a linear foot and a square foot are unlike units.

In reference to units of length, the word *linear* is usually omitted; thus, a linear foot is usually called simply a foot.

6. The magnitude of a quantity is expressed by an abstract number and a unit of the same kind as the quantity itself. The abstract number is called the **numerical measure**; thus, *five inches* expresses a certain length of which *five* is the numerical measure.

7. The complete expression of a magnitude is called a **concrete number**; thus, *two yards* is a concrete number. A concrete number consists of a concrete unit that tells the kind of the quantity and an abstract number that tells how often the concrete unit is contained in the magnitude measured; for example, the concrete number *ten dollars* means *ten times one dollar*.

8. **Like numbers** are those that express quantities of the same kind and **unlike numbers** are those that express quantities of different kinds. Thus, *four feet* and *three inches* are like numbers because they both express lengths; but *four feet* and *three quarts* are unlike numbers.

9. **Direct and Indirect Measurement.**—Every practical application of science involves measurement of some kind. In some cases, we can tell, by direct comparison, how often the unit is contained in the quantity to be measured; this is called **direct measurement.** For example, to measure a line by a two-foot rule is a case of direct measurement. Here the length of the line is determined by a direct comparison with the rule.

In most cases, the quantity to be measured is not directly compared with the unit. For example, the area of a surface is not found by laying the unit area (a square inch, or a square foot) upon the surface and finding how often the unit area must be applied to the given surface in order to cover the whole of it. On the contrary, to find the area of a surface, we measure certain lengths, and then obtain the measure of the area by performing certain arithmetical calculations. Measuring an area is therefore a case of *indirect measurement*.

The direct method is seldom applicable except for measuring lengths or weights; the magnitudes of other quantities usually have to be determined by calculation.

In this Arithmetic we shall teach so much of the art of calculation as is necessary to enable a practical man to make the ordinary calculations that are required in his work.

NOTATION AND NUMERATION.

10. The first numbers are named *one*, *two*, *three*, *four*, *five*, *six*, *seven*, *eight*, *nine*, *ten*, *eleven*, *twelve*. The first nine of these numbers are represented by the marks, or symbols: 1, 2, 3, 4, 5, 6, 7, 8, 9. Since there is no limit to numbers, it would be inconvenient to have an independent name and an independent symbol for each number. We shall therefore explain how the inconvenience of having a multitude of disconnected names and symbols for numbers is avoided.

11. A systematic method of expressing numbers by names is called a **system of numeration.**

A systematic method of representing numbers by symbols is called a **system of notation.**

12. Man having ten fingers, the number *ten* came to be the basis of the common system of numeration. In this system, the first ten numbers have independent names, which are given in Art. **10.** The names of other numbers are formed from these ten words with a few additional independent words.

The number after ten receives the name *eleven*, which is derived from the Gothic words, *ain* (one) and *lif* (ten); thus, eleven means one and ten. The number after eleven has the name *twelve*, which is derived from the Gothic words, *twa* (two) and *lif* (ten); so that twelve means two and ten. The names of the succeeding seven numbers end in the syllable *teen*. *Thirteen* is an abbreviation for three and ten; *fourteen* is an abbreviation for four and ten; and so on to *nineteen*, which means nine and ten.

The number after nineteen is ten and ten, or two tens; this number is called *twenty*, which is a contraction for two tens. The next number is called *twenty-one*, which is a contraction for two tens and one. In like manner, three tens is called *thirty;* and so on to nine tens, which is called *ninety*. A number consisting of ten tens is called a *hundred;* a number consisting of ten hundreds is called a *thousand;* and a number consisting of ten hundred thousand is called a *million.* A *billion* is a thousand millions, and a *trillion* is a thousand billions; but the names billion and trillion, as well as the names of larger numbers, are seldom required.

13. A very clear idea of this system of naming numbers can be obtained by counting a lot of wooden toothpicks. Tie the toothpicks in bundles of ten, until less than ten are left, and suppose there are four single toothpicks left. Then tie up the bundles in larger bundles, each of which contains ten of the smaller bundles, until the number of smaller bundles remaining is less than ten, and suppose there are three small bundles left. Suppose also that there are six of these larger bundles, each of which evidently contains a hundred toothpicks. The whole number of toothpicks in this lot is six hundreds, three tens, and four. This number is usually expressed as "six hundred thirty-four." The number of bundles of ten is counted in exactly the same way as the number of single toothpicks in a lot containing less than ten. In fact, each bundle of ten is regarded as representing a new unit. In the same way, each bundle of a hundred is regarded as representing another new unit.

14. A group of ten units is called a **unit of the second order**; a group of a hundred units is called a **unit of the third order**; a group of a thousand units is called a **unit of the fourth order**; and so on. It is customary also to speak of a unit as a **simple unit,** or a **unit of the first order**; thus, the number *five* means five simple units, or five units of the first order.

15. The first three orders constitute the first class of units, or the **first period**; the second three orders constitute the second class of units, or the **second period**; and so on. The annexed table exhibits the relation between the orders and periods.

Fourth Period. Billions.			*Third Period. Millions.*			*Second Period. Thousands.*			*First Period. Units.*		
Twelfth Order. (Hundreds of Billions.)	Eleventh Order. (Tens of Billions.)	Tenth Order. (Billions.)	Ninth Order. (Hundreds of Millions.)	Eighth Order. (Tens of Millions.)	Seventh Order. (Millions.)	Sixth Order. (Hundreds of Thousands.)	Fifth Order. (Tens of Thousands.)	Fourth Order. (Thousands.)	Third Order. (Hundreds.)	Second Order. (Tens.)	First Order. (Units.)

16. The nine symbols 1, 2, 3, 4, 5, 6, 7, 8, and 9 are called **digits,** from the Latin word *digitus*, which means a finger. If columns are ruled for the several orders of units, as in the table of Art. **15,** any number can be represented by means of the nine digits; thus, in the annexed scheme, Fig. 2, we have written, on successive lines, the three numbers: *three hundred twenty-four; nine hundred eight;* and *seven hundred fifty.* The first of these numbers can be written 324, omitting the names of the orders and the ruled columns, if it is understood that the first digit at the right shall represent units, the second digit tens, and so on. In writing the second of these numbers, we cannot

Hundreds	*Tens*	*Units*
3	2	4
9		8
7	5	

FIG. 2.

omit the names of the orders and the ruled columns, unless we have some way of indicating that there is no digit in the tens column.

For this purpose, there has been introduced the additional symbol 0, which is called *naught*, *cipher*, or *zero*. The second number in the diagram, Fig. 2, can now be written 908, if it is agreed that the digit occupying the first place at the right shall represent units, and the digit (if any) occupying the second place from the right shall represent tens, and so on. The third number in the diagram is written as 750; where the symbol 0 in the first place at the right indicates that there are no units, the digit 5 in the second place from the right means fifty, and the digit 7 in the third place from the right means seven hundreds.

17. The method of representing numbers explained in Art. **16** is called the **Arabic system of notation.** This system, as we have seen, employs ten symbols, namely: 1, 2, 3, 4, 5, 6, 7, 8, 9, and 0. Each of these symbols is called a **figure** and each of the first nine is also called a *digit* (Art. **16**).

18. In order to represent all numbers by means of the ten figures, it has been universally agreed that the number represented by a digit shall depend on the position of the digit. That is, it has been agreed that each digit shall have a local, or relative, value as well as its simple value.

The **simple value** of a digit is the value it has when standing alone. Thus, the digit 5 standing alone means 5 simple units, and this is the simple value of the digit 5.

The **local value** of a digit is the value it represents by virtue of its position in relation to other figures in the same number. Thus, in the number 5304, the digit 3 represents three hundreds, and this is the local value of the digit 3 in this number.

19. *The local value of a digit is increased tenfold by moving the digit one place to the left.*

Thus, in the number 45, the digit 5 represents 5 units;

while in the number 450—where the 5 has been moved one place to the left by putting a cipher in the units place—the 5 represents 5 tens, or fifty; that is, the local value of the digit 5 has been increased tenfold by moving it one place to the left. Likewise, the 4 in 45 represents 4 tens, or forty, while in 450, the 4 has been moved from the tens place into the hundreds place, and the 4 represents 4 hundreds, or ten times forty; that is, the local value of the digit 4 has been increased tenfold by moving it one place to the left.

20. *The local value of a digit is decreased tenfold by moving the digit one place to the right.*

Thus, in the number 560, the digit 5 represents 5 hundreds; and in the number 56, where the 5 has been moved one place to the right, the 5 represents 5 tens, or fifty. In like manner, the 6, which represented 6 tens in 560, being moved one place to the left is decreased tenfold and represents 6 units in 56.

21. The following is the rule for writing in figures a number that has been given in words.

Rule.—*Beginning at the highest order, write the number of units of each order in succession from left to right, taking care to insert ciphers for any orders of units that may be lacking.*

22. To facilitate the reading of a number that is written in figures, it is usual to separate the number, by commas, into periods (Art. **15**) of three figures, or orders, each, beginning at the right or units order. The first period (Art. **15**) is called the **units period**; the second period from the right is called the **thousands period**; and so on. The following rule may be laid down for reading numbers that are written in figures.

Rule.—*After separating the number into periods, begin with the left-hand period, read it as if the figures of that period stood alone, and add the name of the period; then read the next period to the right in the same way; and so on.*

23. The following examples show how a number written in figures is expressed in words; and the same examples show also how a number written in words is expressed in figures. Thus, 345 is read "three hundred forty-five"; 1268 is read "one thousand two hundred sixty-eight"; or, "twelve hundred sixty-eight"; 4007 is read "four thousand seven"; 4070 is read "four thousand seventy"; 56285 is read "fifty-six thousand two hundred eighty-five"; 789563 is read "seven hundred eighty-nine thousand five hundred sixty-three." The number 4007 might also be read "forty hundred seven"; but this method of reading this number is not so short, and therefore not so good, as that given above. On the other hand, it is shorter, and therefore better, to read 1268 as "twelve hundred sixty-eight" than to read it as "one thousand two hundred sixty-eight." The simplest and shortest way of reading a number is always the best.

DECIMALS.

24. A unit may be divided into ten equal parts; each of these parts may be again divided into ten equal parts; and so on. Thus, a dollar is divided into ten dimes, a dime into ten cents, and a cent into ten mills. A dime is the tenth part of a dollar, a cent is the hundredth part of a dollar, and a mill is the thousandth part of a dollar.

25. The mark $ written to the left of the figures denotes dollars. When parts of a dollar are to be written, a point is placed after the dollars, the dimes are written to the right of this point, the cents are written to the right of the dimes, and the mills (if any) are written to the right of the cents. Thus, $24.75 is twenty-four dollars, seven dimes, and five cents, or twenty-four dollars and seventy-five cents; $0.06 is six cents; and $0.006 is six mills.

26. Any other unit can be divided into tenths, hundredths, thousandths, etc., and these parts are written in the same way as the parts of a dollar. Parts so written are called **decimals.** The point to the right of the units figure is called the **decimal point.** *The use of the decimal point is to mark the figure that occupies the units place.* Thus, in the number 4350.678, the decimal point tells us that 0 stands in the units place. The first figure to the left of the units place is 5 tens, the first figure to the right of units place is 6 tenths; the second figure to the left of units place is 3 hundreds, the second figure to the right of units place is 7 hundredths.

27. In using decimals, we must be careful to *mark the units place by placing the decimal point immediately to the right of the figure in the units place.* Then, we continue the notation to the right of the units place precisely as we did to the left of it. The resemblance between the names of the places on the two sides of the units place is shown by the following table, in which the differences are marked by *italics.*

hundreds of thousands	tens of thousands	thousands	hundreds	tens	units	ten*ths*	hundred*ths*	thousand*ths*	ten-thousand*ths*	hundred-thousand*ths*
5	4	3	2	1	0	.1	2	3	4	5

28. Since the notation is the same on both sides of the units place, the statements of Arts. **19** and **20** are applicable on both sides of the decimal point.

29. In this system of notation, each place is said to be **higher** than any place to the right of it, and **lower** than any place to the left of it. Thus, in the number 43.127, the highest place is tens, and the lowest place is thousandths.

30. The following is the usual rule for reading decimals:

Rule.—*Read the decimal as if it were a whole number, and add the fractional name of the lowest place. The word "and" should be pronounced at the decimal point and omitted in all other places.*

Thus, 123.046 is read "one hundred twenty-three and forty-six thousandths"; 78.6543 is read "seventy-eight and six thousand five hundred forty-three ten-thousandths."

It is by no means necessary to adhere rigidly to this method of reading decimals; and in special lines of work, other methods are found convenient. For example, in many kinds of shop work, measuring instruments are used that measure directly to the thousandth part of an inch, and, by means of a special device called a **vernier**, these instruments can be used to measure to the ten-thousandth part of an inch. In using such instruments, it is convenient to read the measurements in the following manner: 1.0567 inch is read "one inch fifty-six thousandths and seven-tenths of a thousandth of an inch."

31. *Annexing or taking away a cipher at the right of a decimal does not change the value of the decimal.*

Thus, .5 is five tenths, .50 is five tenths and no hundredths, and, therefore, .5 and .50 have the same value.

Ciphers at the right of a decimal are usually omitted; sometimes, however, they are retained for the sake of uniformity in reading, or for the sake of preserving the uniformity of a column of figures. For example, instead of $26.70 it would be sufficient to write $26.7; but the cipher is written at the end for uniformity. Again, in using a measuring instrument graduated to thousandths of an inch, the measurement .07 inch would be taken as .070 inch and read "seventy thousandths of an inch."

32. A number that consists solely of complete units is called an **integer,** or a **whole number.** In an integral number, the decimal point is not written, but is understood after the figure in units place. In an integer, the decimal point may be placed after the figure in units place; for example, we may write 1642. if we so desire.

GEOMETRICAL DEFINITIONS.

33. Of the calculations which the mechanic has to make, a very large number deal with geometrical magnitudes. It is convenient, therefore, to give here a few **geometrical definitions.**

34. The word *straight* is derived from the word *stretched*, and a thread stretched by pulling on both ends of it is said to be straight. A **straight line** is the shortest line that can be drawn between two points. Hence, the length of the straight line joining two points is called the **distance** between the two points. A piece of wood or metal whose edge has been made straight for the purpose of ruling straight lines is called a **ruler** or a **straightedge.**

35. The boundaries of a body are called its **surfaces.** A **plane** or **flat surface** is a surface such that if a straightedge is laid upon it in any direction, every point of the straightedge will touch the surface. Such plane or flat surfaces are usually called **planes.** The surface of a small body of still water is a plane.

36. Any combination of points and lines is called a **figure.** A figure that lies altogether in one plane is called a **plane figure.**

In referring to a figure, a point is designated by a letter placed conveniently near the point; thus, in Fig. 3, the left end of the line is referred to as the point A, and the right end of the line is referred to as the point B. When we refer to the entire line in Fig. 3, we speak of it as "the line $A B$."

FIG. 3.

37. A straight line can be **produced** to any extent in either direction by means of a ruler; thus, in Fig. 4, the straight line $A B$ is produced to the points C and D.

FIG. 4.

38. When two lines are cut by a third line, the part of the third line included between the points where it cuts the first two lines is said to be **intercepted** between these two lines, and this part of the third line is called the **intercept** made on the third line by the first two lines. Thus, in Fig. 5, *P Q* is the intercept made on the line *L M* by the two lines *A B* and *C D*. The points *P* and *Q* are called the **points of intersection**; that is, the point *P* is the point of intersection of the lines *A B* and *L M* and *Q* is the point of intersection of the lines *C D* and *L M*.

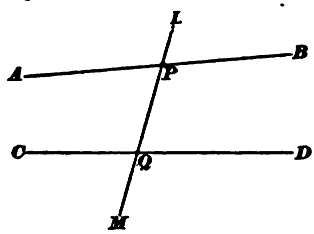


FIG. 5.

39. Two geometrical figures are **equal** if one can be placed on the other so that the two figures coincide throughout their whole extent.

40. To **bisect** any magnitude is to divide it into two equal parts; thus, the straight line *A B*, Fig. 6, is bisected in the point *C* if *A C* is equal to *C B*. When a given magnitude is bisected, each of the parts into which it is divided is one-half of the given magnitude.

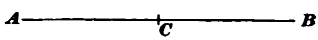


FIG. 6.

41. Dimensions.—It is frequently necessary to mark the measurements of the several parts of a figure upon the drawing; these measurements are usually called **dimensions.**

In marking dimensions, the symbols (′) and (″) are commonly used to denote feet and inches, respectively; thus, 6 feet is denoted by 6′, and 6 inches is denoted by 6″. Thus, in Fig. 7, the length of the line *A B* is 2′ 3″ and the length of the line *B C* is 1′ 6″.

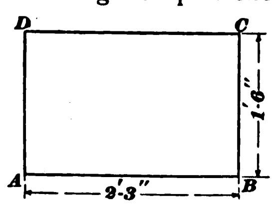


FIG. 7.

ARITHMETIC.

(PART 2.)

ADDITION AND SUBTRACTION.

ADDITION.

1. If we have one bundle of 3 toothpicks, another bundle of 4 toothpicks, and a third bundle of 8 toothpicks, we can easily find by counting that the number of toothpicks in the three bundles taken together is 15. Therefore, we say that 15 is equal to the three numbers 3, 4, and 8 taken together; or that 15 is the *sum* of 3, 4, and 8.

In the same way, if we had bundles containing 984, 345, and 786 toothpicks, respectively, we could find the number of toothpicks in these bundles taken together by counting, but in this case the process of counting would be very tedious. We shall now explain how the number of toothpicks in these bundles taken together can be found without the labor of an actual count.

2. Addition means putting together. In arithmetic, **addition** is the process of finding a number that is equal to two or more numbers taken together.

The number found by adding two or more numbers together is called their **sum,** or **amount.**

3. The sign of addition is + ; it is read *plus*, and means that the numbers between which it is placed are to be added

ADDITION TABLE.

1 + 1 = 2	2 + 1 = 3	3 + 1 = 4	4 + 1 = 5
1 + 2 = 3	2 + 2 = 4	3 + 2 = 5	4 + 2 = 6
1 + 3 = 4	2 + 3 = 5	3 + 3 = 6	4 + 3 = 7
1 + 4 = 5	2 + 4 = 6	3 + 4 = 7	4 + 4 = 8
1 + 5 = 6	2 + 5 = 7	3 + 5 = 8	4 + 5 = 9
1 + 6 = 7	2 + 6 = 8	3 + 6 = 9	4 + 6 = 10
1 + 7 = 8	2 + 7 = 9	3 + 7 = 10	4 + 7 = 11
1 + 8 = 9	2 + 8 = 10	3 + 8 = 11	4 + 8 = 12
1 + 9 = 10	2 + 9 = 11	3 + 9 = 12	4 + 9 = 13
1 + 10 = 11	2 + 10 = 12	3 + 10 = 13	4 + 10 = 14
1 + 11 = 12	2 + 11 = 13	3 + 11 = 14	4 + 11 = 15
1 + 12 = 13	2 + 12 = 14	3 + 12 = 15	4 + 12 = 16
5 + 1 = 6	6 + 1 = 7	7 + 1 = 8	8 + 1 = 9
5 + 2 = 7	6 + 2 = 8	7 + 2 = 9	8 + 2 = 10
5 + 3 = 8	6 + 3 = 9	7 + 3 = 10	8 + 3 = 11
5 + 4 = 9	6 + 4 = 10	7 + 4 = 11	8 + 4 = 12
5 + 5 = 10	6 + 5 = 11	7 + 5 = 12	8 + 5 = 13
5 + 6 = 11	6 + 6 = 12	7 + 6 = 13	8 + 6 = 14
5 + 7 = 12	6 + 7 = 13	7 + 7 = 14	8 + 7 = 15
5 + 8 = 13	6 + 8 = 14	7 + 8 = 15	8 + 8 = 16
5 + 9 = 14	6 + 9 = 15	7 + 9 = 16	8 + 9 = 17
5 + 10 = 15	6 + 10 = 16	7 + 10 = 17	8 + 10 = 18
5 + 11 = 16	6 + 11 = 17	7 + 11 = 18	8 + 11 = 19
5 + 12 = 17	6 + 12 = 18	7 + 12 = 19	8 + 12 = 20
9 + 1 = 10	10 + 1 = 11	11 + 1 = 12	12 + 1 = 13
9 + 2 = 11	10 + 2 = 12	11 + 2 = 13	12 + 2 = 14
9 + 3 = 12	10 + 3 = 13	11 + 3 = 14	12 + 3 = 15
9 + 4 = 13	10 + 4 = 14	11 + 4 = 15	12 + 4 = 16
9 + 5 = 14	10 + 5 = 15	11 + 5 = 16	12 + 5 = 17
9 + 6 = 15	10 + 6 = 16	11 + 6 = 17	12 + 6 = 18
9 + 7 = 16	10 + 7 = 17	11 + 7 = 18	12 + 7 = 19
9 + 8 = 17	10 + 8 = 18	11 + 8 = 19	12 + 8 = 20
9 + 9 = 18	10 + 9 = 19	11 + 9 = 20	12 + 9 = 21
9 + 10 = 19	10 + 10 = 20	11 + 10 = 21	12 + 10 = 22
9 + 11 = 20	10 + 11 = 21	11 + 11 = 22	12 + 11 = 23
9 + 12 = 21	10 + 12 = 22	11 + 12 = 23	12 + 12 = 24

together. Thus, $5 + 6$ is read *five plus six*, and means that 5 and 6 are to be added together.

4. Like numbers (Art. **8**, Part 1) can be added together, but unlike numbers cannot be added together. Thus, 4 dollars can be added to 2 dollars and the sum is 6 dollars; but, 3 dollars cannot be added to 5 inches.

5. The sign of equality is $=$, and is read *is equal to*. Thus, $5 + 6 = 11$ is read *five plus six is equal to eleven*. A statement of equality between two quantities is called an **equation**; thus, $5 + 4 = 9$ is an equation.

The part of the equation to the left of the sign of equality is called the **first member** of the equation, and the part to the right of the sign of equality is called the **second member.** Thus, in the equation $3 + 4 = 7$, $3 + 4$ is the first member, and 7 is the second member. Either member of an equation may be read first; thus, the equation $5 + 7 = 12$ may be read *five plus seven is equal to twelve*, or it may be read *twelve is equal to five plus seven.*

6. The accompanying table gives the sum of every pair of numbers from 1 to 12, inclusive.

7. The student should commit the addition table to memory. He should also practice taking a pair of numbers less than 12 and writing down their sum, until he can do so without hesitation or delay. Let the student write down, as rapidly as possible, the second member of each of the following equations, and then compare what he has written with the addition table, to see if his results are correct..

$3 + 9 =$,	$8 + 7 =$,	$7 + 9 =$,	$7 + 11 =$,
$8 + 11 =$,	$9 + 12 =$,	$6 + 8 =$,	$6 + 4 =$,
$12 + 5 =$,	$4 + 5 =$,	$3 + 12 =$,	$7 + 5 =$,
$11 + 12 =$,	$12 + 6 =$,	$8 + 4 =$,	$9 + 10 =$.

Any student who makes mistakes in these exercises or who cannot write down the second member quickly, should spend a few minutes every day practicing such examples. He can easily make the examples for himself, by writing down pairs of numbers at random. If he continues this practice regularly every day, he will very soon become both accurate and quick. While he is devoting a few minutes every day to exercises of this kind, he can also proceed with the study of arithmetic.

8. The method of finding the sum of a single digit and a number greater than twelve is shown in the following examples:

EXAMPLE 1.—Find the sum of 18 and 7.

SOLUTION.—

$$\begin{array}{r} 18 \\ 7 \\ \hline 25 \end{array} \text{ Ans.}$$

EXPLANATION.—From the addition table, $7 + 8 = 15 = 1$ ten and 5 units $= 10 + 5$. We put down the 5 in the units column and carry 1 to the tens column; then in the tens column we have $1 + 1 = 2$. Thus, the sum of 18 and 7 is 2 tens and 5 units, or 25.

EXAMPLE 2.—Find the sum of 58 and 9.

SOLUTION.—

$$\begin{array}{r} 58 \\ 9 \\ \hline 67 \end{array} \text{ Ans.}$$

EXPLANATION.—From the addition table, we have $9 + 8 = 17$. We write the 7 in the units column and carry 1 ten to the tens column. Then, in the tens column we have $1 + 5 = 6$. Thus, the sum of 58 and 9 is 6 tens and 7 units, or 67.

From these examples, we derive the following rule :

9. Rule.—*To find the sum of a single digit and a number greater than twelve, add the single digit to the units figure of the number greater than twelve, and write the units figure of this sum in the units column. If the sum of the single digit and the units figure of the number greater than twelve is greater than nine, increase the tens figure of the number greater than twelve by unity.*

10. The rule of Art. **9** enables us to find the sum of several numbers each consisting of a single digit.

EXAMPLE.—Find the sum of 7, 4, 6, 8, 7, and 9.

SOLUTION.—

```
 7
 4
 6
 8
 7
 9
——
41  Ans.
```

EXPLANATION.—Beginning at the foot of the column, we have $9 + 7 = 16$, $16 + 8 = 24$, $24 + 6 = 30$, $30 + 4 = 34$, $34 + 7 = 41$. In adding this column, the student should name the following numbers, and no others, 9, 16, 24, 30, 34, 41.

EXAMPLES FOR PRACTICE.

Write down the second member of each of the following equations :

1. $8 + 9 + 7 + 5 + 6 =$ Ans. 35.
2. $7 + 5 + 3 + 4 + 8 =$ Ans. 27.
3. $6 + 7 + 9 + 2 + 9 =$ Ans. 33.

11. The most convenient method of adding several numbers is laid down in the following rule :

Rule.—I. *Write the numbers in columns, taking care to place units under units, tens under tens, tenths under tenths, etc.*

II. *Add the digits in the right-hand column, put down the units of the sum in that column, and carry the tens of the sum to the next column to the left, adding them to the sum of the digits in that column, and so on.*

III. *If 0 occurs in any of the numbers that are being added, it may be passed over without notice; for, evidently adding 0 (nothing) does not affect the result.*

EXAMPLE.—What is the sum of 136, 209, 841, 72, and 367?

SOLUTION.—

```
 136
 209
 841
  72
 367
————
1625  Ans.
```

EXPLANATION.—After placing the numbers in the proper order, begin at the bottom of the right-hand column and add, mentally pronouncing the *different sums.* Do not say 7 and 2 are 9, and 1 are 10 and 9 are 19, and 6 are 25. But say 7, 9, 10, 19, **25**. Pronounce the units figure 5 of the sum emphatically, and write it down while pronouncing it. In the sum of this first column there are 2 tens. Carry the 2 to the next column; the addition of the second column then is: 2, 8, 15, 19, **22**. Carry 2 to the third column, which adds: 2, 5, 13, 15, **16**; the 1 to carry from this column makes 1 in the fourth column. Hence, the required sum is 1,625.

The student will notice that in adding the second column, no attention was paid to the 0 in the number 209.

12. Numbers containing decimals are added exactly like integers. By I of the rule in Art. **11,** the units must be written in the same vertical column. Hence, *the decimal points must also stand in the same vertical column.*

EXAMPLE 1.—Find the sum of 242.1, 26.142, .004, and 7.12.

SOLUTION.—

242.1	
26.142	
.004	
7.12	
275.366	Ans.

Sometimes the decimal point is omitted and replaced by a vertical line; all that is needed is some mark to indicate the units column.

EXAMPLE 2.—Add $76.52, $134.27, $267.85, and $32.20.

SOLUTION.—

$ 76	52	
134	27	
267	85	
32	20	
$510	84	Ans.

EXAMPLE 3.—The drawing in Fig. 1 shows an iron bar with ½-inch holes drilled through it. From the dimensions marked on the drawing,

find (*a*) the length over all (that is, the length from end to end) of the bar, and (*b*) the distance between the two holes marked *A* and *B*, respectively.

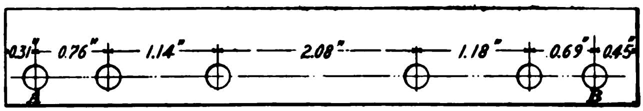


FIG. 1.

SOLUTION.—

(*a*)	(*b*)
0.31	0.76
0.76	1.14
1.14	2.08
2.08	1.18
1.18	0.69
0.69	5.85
0.45	
6.61	

Ans. { (*a*) 6.61 in. (*b*) 5.85 in.

13. Checking Calculations.—In practical affairs, it is of the highest importance that **mistakes** or *blunders* should not be made in any calculation. But it is well known that even the most careful person will make a mistake occasionally; and no one should ever make a calculation without adopting some precaution to detect any mistake he may have made. Any operation that is performed for the purpose of discovering whether a mistake has been made in a calculation is called a **check** upon that calculation.

It is usual to make a distinction between the two words *mistake* and *error*. A mistake, as explained above, is something that can be avoided; and if one has been made, it can be discovered and corrected. On the other hand, an **error** is an unavoidable imperfection in our measurements or their results.

Suppose that a workman makes an iron bar according to the dimensions marked on Fig. 1, Art. **12.** Every practical man knows that in the finished iron bar, one or more of the dimensions will differ slightly from what it ought to be. The amount of this difference will depend on the perfection of the measuring instruments used and the skill of the workman. These unavoidable differences between our measured or calculated dimensions and the actual dimensions we are trying to obtain are called errors.

14. An addition should always be checked by adding the columns down as well as up. Besides this check, the conditions of the problem frequently suggest other convenient checks. A case of this kind is given in the following example.

EXAMPLE.—Check the answers of example 3, Art. **12.**

SOLUTION.—If these answers are correct, the answer (*a*) should be equal to the sum of 0.31 inch, the answer (*b*), and 0.45 inch; as shown in the margin, the sum of 0.31 inch, the answer (*b*), and 0.45 inch is 6.61 inches, which is equal to the answer (*a*). It is probable, therefore, that both answers are correct.

```
0.3 1
5.8 5
0.4 5
-----
6.6 1
```

NOTE.—A check may occasionally fail to disclose a mistake that has been made. For example, if there were a mistake of the same amount in each of the answers (*a*) and (*b*), the check here used would not disclose these mistakes. Suppose that we had obtained for (*a*) the wrong answer 6.51 inches, and for (*b*) the wrong answer 5.75 inches, the answer being in each case 0.1 inch too small. Then applying the check; we would have, as shown in the margin, the sum of 0.31 inch, the wrong answer for (*b*), and 0.45 inch equal to 6.51 inches, which is the wrong answer for (*a*). Thus, in this case, the check gives no indication of the mistakes. A double mistake of this kind rarely occurs.

```
0.3 1
5.7 5
0.4 5
-----
6.5 1
```

EXAMPLES FOR PRACTICE.

Find the sum of:

1. 34.05, 1.76, .32, 25.1, and 3.52. Ans. 64.75.
2. 106.52, 73.954, 24.681, and 35.007. Ans. 240.162.
3. 1.625, 2.75, and .0625. Ans. 4.4375.
4. 35.25, 7.845, .276, and 4. Ans. 47.371.

SUBTRACTION.

15. Subtraction means taking away. In arithmetic, **subtraction** is the process of taking one number from another. Suppose there are 9 toothpicks in a bundle, and 2 toothpicks are removed; then there will remain but 7 toothpicks in the bundle. This operation, which we can perform with our hands, illustrates the arithmetical operation of subtracting 2 from 9.

16. The sign of subtraction is –; it is read *minus* and means that the number after it is to be subtracted from the number before it. Thus, 12 – 7 is read *twelve minus seven* and means that 7 is to be subtracted from 12.

17. The number to be subtracted is called the **subtrahend,** the number from which the subtrahend is taken is called the **minuend,** and the result is called the **remainder,** or **difference.** In the example 9 – 2 = 7, 9 is the minuend, 2 is the subtrahend, and 7 is the remainder.

18. *The sum of the subtrahend and the remainder is equal to the minuend;* in other words, *the remainder is a number that, when added to the subtrahend, produces the minuend.* This is evident, for when a part of any number is taken away, the part that remains added to the part taken away must make up the whole number. For example, 9 – 2 = 7; here 9 is the minuend, 2 is the subtrahend, and 7 is the remainder. In this example, the sum of the subtrahend 2 and the remainder 7 is equal to the minuend 9. In every example in subtraction, then, we have the following equations:

$$(minuend) - (subtrahend) = (remainder),$$

and $$(minuend) = (subtrahend) + (remainder).$$

19. *If the same number is added to both minuend and subtrahend, the remainder is unchanged.*

For example, 8 – 6 = 2. If we add any number, say 10, to both 8 and 6, the difference of the sums will also be 2; thus, 10 + 8 = 18, and 10 + 6 = 16, and the difference between these two sums is 18 – 16 = 2. Hence, *the addition of any number to the minuend is balanced by the addition of the same number to the subtrahend.*

20. The following is the most convenient rule for subtraction:

Rule.—I. *Place the subtrahend under the minuend, so that units shall be under units, tens under tens, etc. If the number of decimal places in the two is unequal, imagine the vacant places to be filled with ciphers.*

II. *Beginning at the right, subtract, if possible, each subtrahend figure from the minuend figure that stands over it; but if the minuend figure is the smaller, add ten to it before subtracting, and after subtracting, add one to the next subtrahend figure.*

The reason for the last part of section II of the rule will be explained in connection with the following example.

EXAMPLE.—From 96.51 subtract 7.35.

SOLUTION.—

minuend =	96.51
subtrahend =	7.35
remainder =	89.16 Ans.

EXPLANATION.—Beginning at the right, we cannot take 5 from 1; therefore, we add 10 to the 1 and say $11 - 5 = 6 =$ first figure of the remainder. By adding 10 to the 1 in the minuend, we increased the minuend by 10 hundredths, or 1 tenth; to balance this, the subtrahend must also be increased by 1 tenth (Art. **19**), and, therefore, 1 must be added to the 3 in the subtrahend. The second figure of the subtrahend is thus increased to 4, and we say $5 - 4 = 1 =$ second figure of the remainder. The third figure, 7, of the subtrahend cannot be taken from the third figure, 6, of the minuend; therefore, 10 is added to the 6 in the minuend, and we say $16 - 7 = 9 =$ the third figure of the remainder. Having increased the minuend by 10 units, or 1 ten, we must also increase the subtrahend by 1 ten; adding 1 to the tens of the subtrahend, we get $0 + 1 = 1$; and we say $9 - 1 = 8 =$ fourth figure of the remainder.

21. Check.—*To check an example in subtraction, add the remainder to the subtrahend; if no mistake has been made, their sum will be equal to the minuend* (see Art. **18**).

EXAMPLE.—Check the example of Art. **20**.

SOLUTION.—

subtrahend =	7.35
remainder =	89.16
subtrahend + remainder =	96.51 Ans.

That is, the sum of the subtrahend and the remainder is equal to the minuend, and the answer is probably correct.

22. The solution of a practical problem often involves both addition and subtraction.

EXAMPLE.—The annexed drawing, Fig. 2, represents a metal plate strengthened by ribs. The upper part of the figure shows the top of the plate, and the lower part of the figure shows the edge of the plate. Find the distance between the inner faces of the ribs; that is, find the distance from *A* to *B*.

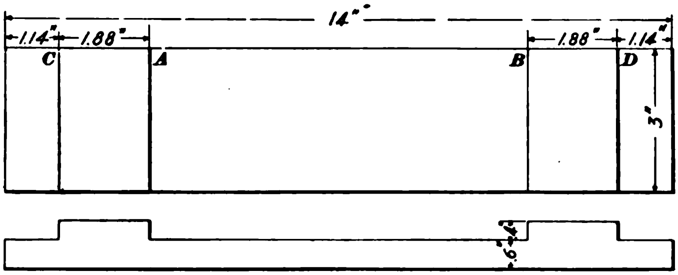


FIG. 2.

SOLUTION.—From the figure, it is evident that the length from *A* to *B* is found by subtracting the sum of 1.14", 1.88", 1.88", and 1.14" from 14".

1.14	
1.88	14.
1.88	6.04
1.14	7.96
6.04	Ans. 7.96".

EXAMPLES FOR PRACTICE.

Solve the following examples:

1. In Fig. 2, find the distance from *C* to *D*, and check your answer. Ans. 11.72".

2. Subtract 104.08 from 208.04. Ans. 103.96.

3. Subtract 3.785 from 44. Ans. 40.215.

4. In the morning a man has $1.15. During the day, he receives $15.00, and pays out the following amounts: $1.35, $2.10, $0.76, $0.14, $1.29, and $6.53. How much money has he at night? Ans. $3.98.

ARITHMETIC.

(PART 3.)

GEOMETRICAL APPLICATIONS OF ADDITION AND SUBTRACTION.

CIRCLES AND ANGLES.

1. A **circle** is a plane figure bounded by a curved line every point of which is equally distant from a point within called the **center.** The line that bounds a circle is called its **circumference.** Any straight line drawn from the center to the circumference is called a **radius** (plural *radii*). Any straight line drawn through the center and terminated at both ends by the circumference is called a **diameter**; evidently, *the length of a diameter is twice the length of a radius.* In Fig. 1, $ABCDEF$ is a circle whose center is O; the straight line EA is a diameter, and the straight lines OA, OE, and OF are radii.

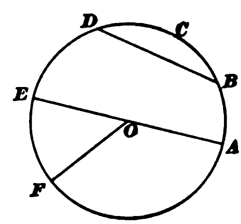


FIG. 1.

2. Any diameter divides the circle into two equal parts called **semicircles.** Thus, in Fig. 1, the figure bounded by the diameter EA and the curved line $ABCDE$ is a semicircle; and the figure bounded by the diameter EA and the curved line AFE is also a semicircle.

3. Any portion of the circumference of a circle is called a **circular arc**; thus, the curved line *B C D*, Fig. 1, is an arc of the circle. Any diameter of a circle divides the circumference into two equal parts, each of which is called a **semi-circumference.** Thus, the arcs *A B C D E* and *A F E*, Fig. 1, are semi-circumferences.

4. A straight line joining two points on the circumference of a circle is called a **chord of the circle.** Thus *B D*, Fig. 1, is a chord of the circle, and *B D* is also said to be the **chord of the arc** *B C D*. Any chord that is not a diameter divides the circumference into two unequal arcs; thus, in Fig. 1, the chord *B D* divides the circumference into the two arcs *B C D* and *B A F E D*; of these, the arc *B C D* is less than a semi-circumference, and the arc *B A F E D* is greater than a semi-circumference. When an arc and its chord are spoken of, it is always the arc less than a semi-circumference that is meant, unless the contrary is stated. An arc less than a semi-circumference is usually referred to by naming the letters at its extremities; thus, the arc *B C D*, Fig. 1, is called the arc *B D*.

5. The portion of a circle included between an arc and its chord is called a **segment of the circle.** Thus, the segment *B C D*, Fig. 1, is the space bounded by the arc *B C D* and its chord *D B*.

A segment whose chord is a diameter is a semicircle (Art. **2**).

6. The portion of a circle included between an arc and the two radii drawn to its extremities is called a **sector of the circle**. Thus, the sector *E O F*, Fig. 1, is the space bounded by the arc *E F* and the two radii *O E* and *O F*.

7. The whole circumference of a circle is divided into 360 equal parts, and each of these equal parts is called a **degree of arc**; a degree of arc is divided into 60 equal parts called **minutes of arc**; and a minute of arc is divided into 60 equal parts called **seconds of arc.** Degrees, minutes, and seconds of arc are used as units for measuring circular arcs.

Since the circumference of every circle contains 360 degrees of arc, the length of a degree of arc differs in different circles. A degree of the earth's equator is a little more than 69 miles long; and a degree of the circumference of a circle whose diameter is 360 inches is 3.1416 inches long.

8. *A semi-circumference contains 180 degrees of arc;* thus, in Fig. 1, each of the semi-circumferences *A B C D E* and *E F A* contains 180 degrees.

9. If a circumference is divided into four equal parts, each of the parts is called a **quadrant.**

A quadrant contains 90 degrees and is equal to one-half of a semi-circumference.

10. An **angle** is the opening between two straight lines that meet at a point; the lines are called the **sides** of the angle, and the point where the sides meet is called the **vertex** of the angle. Thus, in Fig. 2 (*a*), the straight lines *O A* and *O B* form an angle at the point *O*; the lines *O A* and *O B* are the sides of this angle, and the

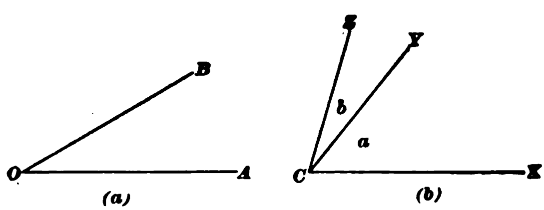


FIG. 2.

point *O* is its vertex. An angle is usually referred to by naming a letter on each of its sides and the letter at the vertex, the letter at the vertex being placed between the other two; thus, the angle in Fig. 2 (*a*) would be referred to as the angle *A O B* or the angle *B O A*.

In Fig. 2 (*b*), the lines *C X* and *C Y* form an angle at the point *C*; the lines *C X* and *C Z* form an angle at *C*; and the lines *C Y* and *C Z* form an angle at *C*. These three angles are referred to as the angle *X C Y*, the angle *X C Z*, and the angle YCZ. When there is but one angle with its vertex at the same point, the angle may be referred to by naming the letter at its vertex; thus, the angle in Fig. 2 (*a*) may be called the angle O. An angle may also be designated by a letter placed within it; thus, the two angles in Fig. 2 (*b*) may be referred to as the angles a and b.

11. We can imagine the angle AOB, Fig. 2 (*a*), to be formed by a straight line turning about the point O, as on a pivot, from the position OA to the position OB. The size of the angle does not depend on the length of the lines OA and OB, but on the amount of turning required to bring a line from the position OA to the position OB. Thus, in Fig. 3, the angle DEF is equal to the angle PQR, though the sides of the angle DEF are shorter than the sides of the angle PQR.

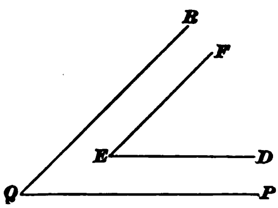


FIG. 3.

12. If a straight line is drawn from a point in another straight line so as to form two equal angles, each of these angles is called a **right angle,** and the lines are said to be **perpendicular** to each other. In Fig. 4, let AOC be a straight line, and let the angles AOB and BOC be equal. Then each of the angles AOB and BOC is a right angle; OB is perpendicular to AC, and AC is perpendicular to OB.

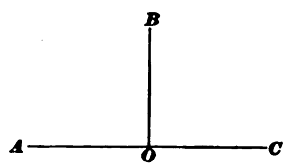


FIG. 4.

13. Any angle that is not a right angle is called an **oblique** angle. An angle less than a right angle is called an **acute** angle, and an angle greater than a right angle is called an **obtuse** angle. In Fig. 5, BOC is an acute angle and COA an obtuse angle.

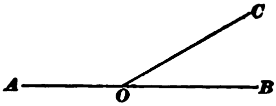


FIG. 5.

14. A right angle is divided into 90 equal parts, called **degrees of angle**; a degree of angle is divided into 60 equal parts, called **minutes of angle**; and a minute of angle is divided into 60 equal parts, called **seconds of angle.** Degrees, minutes, and seconds of angle are used as units for measuring angles.

15. If AB, Fig. 6, is an arc of a circle whose center is O, it can be proved that *the number of degrees of angle in the angle AOB is equal to the number of degrees of arc in the arc AB.* In other words, if a degree of angle is taken as the unit for measuring angles, and a degree of arc as the unit for measuring arcs, then *the angle AOB and the arc AB have the same numerical measure* (Art. **6**, Part 1).

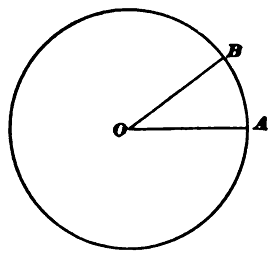


FIG. 6.

For the sake of brevity, this statement is usually expressed in the following abbreviated form.

16. *An angle whose vertex is at the center of a circle is measured by the arc of the circumference included between the sides of the angle.* For example, if the arc AB, Fig. 6, contains 36 degrees and 25 minutes of arc, the angle AOB contains 36 degrees and 25 minutes of angle. It is for this reason that the same names, *degree*, *minute*, and *second* are used for the angular units and for the units of arc. Degrees, minutes, and seconds, both of angle and of arc, are indicated by the marks (°), (′), (″); thus, 32° 15′ 10″ is read *thirty-two degrees fifteen minutes and ten seconds.*

In referring to an angle containing 25 degrees of angle, it is usual to say simply that the angle contains 25 degrees; and in referring to an arc containing 25 degrees of arc, it is usual to say simply that the arc contains 25 degrees. The connection in which the words degree, minute, and second occur generally makes it sufficiently clear whether they are angular units or units of arc.

17. An angle is said to be **inscribed** in a circle when its vertex lies upon the circumference and its sides are chords. The arc included between the sides of an inscribed angle is called its **intercepted arc.** Thus, the angle $A\,B\,C$, Fig. 7, is an inscribed angle whose intercepted arc is the arc $A\,D\,C$.

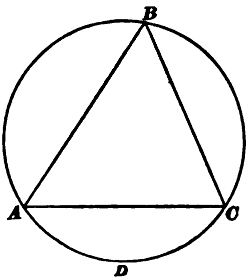


FIG. 7.

18. An angle is said to be inscribed in a segment of a circle when its vertex lies upon the arc of the segment and its sides pass through the extremities of the chord of the segment; thus, the angle $A\,B\,C$, Fig. 7, is inscribed in the segment $A\,B\,C$, since its vertex B lies on the arc $A\,B\,C$, and its sides $B\,A$ and $B\,C$ pass through the extremities of the chord $A\,C$.

19. *An inscribed angle is measured by one-half its intercepted arc.* Thus, in Fig. 7, the inscribed angle $A\,B\,C$ is measured by one-half the intercepted arc $A\,D\,C$.

EXAMPLE.—The angle $A\,B\,C$, Fig. 7, contains 57°; find the number of degrees in the arc $A\,D\,C$.

SOLUTION.—Since the angle $A\,B\,C$ is measured by one-half the arc $A\,D\,C$, the arc $A\,D\,C$ must contain twice as many degrees as the angle $A\,B\,C$. Hence, the arc $A\,D\,C = 57° + 57° = 114°$. Ans.

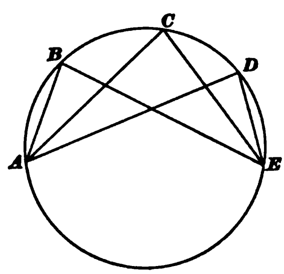


FIG. 8.

20. *All angles inscribed in the same segment are equal.* Thus, in Fig. 8, the inscribed angles $A\,B\,E$, $A\,C\,E$, and $A\,D\,E$ are all equal; for each of them is measured by one-half of the arc $A\,E$.

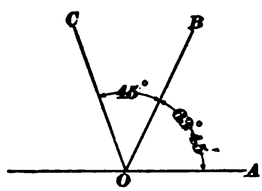


FIG. 9.

21. When dimensions are put on a figure, the magnitudes of angles are marked in the manner shown in Fig. 9, which shows that the angle $A\,O\,B$ contains 63° 45′ and the angle $B\,O\,C$ contains 45°.

EXAMPLE.—In Fig. 10, find the number of degrees in the angle ADC.

SOLUTION.—The angle $A\ D\ B$ = angle $A\ C\ B$, since they are inscribed in the same segment $A\ D\ C\ B$ (Art. **20**); and angle $B\ D\ C$ = angle $B\ A\ C$, since they are both inscribed in the segment $BADC$ (Art. **20**). Therefore, the sum of the angles $A\ D\ B$ and $B\ D\ C$ is equal to the sum of the angles $A\ C\ B$ and $B\ A\ C$; that is, angle $A\ D\ C$ = angle $A\ C\ B$ + angle $B\ A\ C = 32° + 49° = 81°$. Ans.

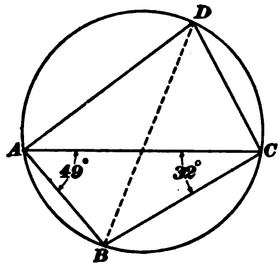


FIG. 10.

22. *An inscribed angle is equal to one-half of the central angle standing on the same intercepted arc.* Thus, in Fig. 11, each of the angles $A\ B\ E$, $A\ C\ E$, and $A\ D\ E$ is equal to one-half of the central angle $A\ O\ E$, which stands on the same intercepted arc $A\ E$. The truth of this statement is evident; for each of the inscribed angles is measured by one-half of the arc $A\ E$, and the central angle $A\ O\ E$ is measured by the whole arc $A\ E$.

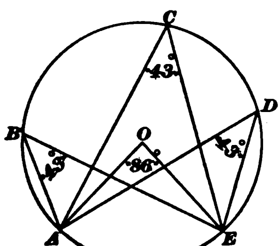


FIG. 11.

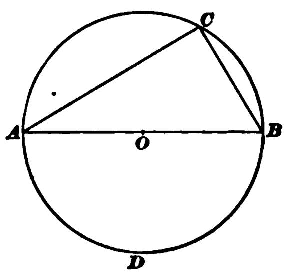


FIG. 12.

23. *Any angle inscribed in a semicircle is a right angle.* Thus, the angle $A\ C\ B$, Fig. 12, is a right angle, for it is measured by one-half of the semi-circumference $A\ D\ B$, and one-half a semi-circumference is a quadrant (Art. **9**).

24. *The shortest line that can be drawn from a point to a line is the perpendicular from that point to the line.* Thus, in Fig. 13, $P\ Q$ is perpendicular to the line $A\ B$, and evidently $P\ Q$ is shorter than any other line PR drawn from P

to the line $A\,B$. Hence, the length of the perpendicular $P\,Q$ is called the **distance** of the point P from the line $A\,B$.

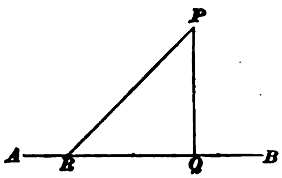


FIG. 13.

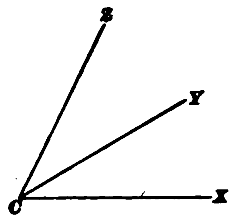


FIG. 14.

25. Two angles that have a common vertex and a common side between them are called **adjacent angles.** Thus, in Fig. 14, the angles $X\,C\,Y$ and $Y\,C\,Z$ are adjacent angles, because they have the common vertex C and the common side $C\,Y$ lying between them. The angles $X\,C\,Z$ and $Y\,C\,Z$, Fig. 14, have the common vertex C and the common side $C\,Z$, but they are not adjacent angles; because, the common side $C\,Z$ does not lie between them and separate one angle from the other.

26. When two straight lines cut, or **intersect,** they form four angles; thus, in Fig. 15, the two straight lines $A\,B$ and $C\,D$ which intersect in the point O form the four angles $B\,O\,D$, $D\,O\,A$, $A\,O\,C$, and $C\,O\,B$. The opposite angles formed by two intersecting lines are called **vertical** angles. Thus, in Fig. 15, $B\,O\,D$ and $A\,O\,C$ are vertical angles; also, $D\,O\,A$ and $C\,O\,B$ are vertical angles.

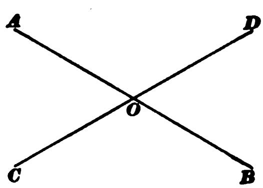


FIG. 15.

27. Rule two intersecting straight lines on a sheet of paper, and put letters on these lines in the manner shown in Fig. 15. Cut out the angle $B\,O\,D$ by cutting along the lines $O\,B$ and $O\,D$, and fit the angle $B\,O\,D$ into the angle $A\,O\,C$. The lines $O\,B$ and $O\,D$ may be longer or shorter than the lines $O\,A$ and $O\,C$, but it will be found that the corner, or angle, $B\,O\,D$ fits exactly into the opening $A\ O\ C$. Thus, the angle $B\ O\ D$ can be made to coincide exactly with the angle $A\ O\ C$; and, therefore, by Art. **39,** Part 1, these two angles are equal. Similarly, it can be shown that the angles $D\ O\ A$ and $C\ O\ B$ are equal.

Hence, we have the following principle:

28. *The vertical angles formed by two intersecting lines are equal.* Thus, in Fig. 15, $B\ O\ D$ is equal to $A\ O\ C$, and $D\ O\ A$ is equal to $C\ O\ B$.

29. *If a straight line is drawn from a point in another straight line, the sum of the two adjacent angles so formed is two right angles, or 180°.*

In Fig. 16, the line $O\ C$ is drawn from the point O in the line $A\ B$, thus forming the two adjacent angles BOC and $C\ O\ A$. With O as center, a circumference is drawn cutting the sides of the angles in the points A, B, and C. The angles $B\ O\ C$ and $C\ O\ A$ are measured, respectively, by the arcs $B\ C$ and $C\ A$ (Art. **16**). Therefore, the sum of the angles $B\ O\ C$ and $C\ O\ A$ is measured by the sum of the arcs $B\ C$ and $C\ A$.

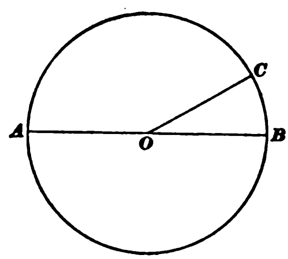


FIG. 16.

The sum of the arcs $B\ C$ and $C\ A$ is equal to a semi-circumference, or 180° of arc (Art. **8**). Hence, the sum of the angles $B\ O\ C$ and $C\ O\ A$ is equal to 180° of angle, or two right angles.

30. *The sum of all the adjacent angles formed on the same side of a straight line is equal to two right angles, or 180°.*

In Fig. 17, the angles $A\ O\ B$, $B\ O\ C$, $C\ O\ D$, and $D\ O\ E$ are all the adjacent angles on one side of the straight line $A\ E$. The angles $A\ O\ B$, $B\ O\ C$, $C\ O\ D$, and $D\ O\ E$ are measured, respectively, by the arcs $A\ B$, $B\ C$, $C\ D$, and $D\ E$. Therefore, the sum of the four angles is measured by the semi-circumference $A\ B\ C\ D\ E$. But the semi-circumference contains 180° of arc (Art **8**). Therefore, the sum of the four angles $A\ O\ B$, $B\ O\ C$, $C\ O\ D$, and $D\ O\ E$, which lie on the same side of the line $A E$, is equal to 180° of angle, or two right angles.

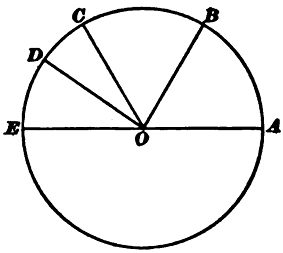


FIG. 17.

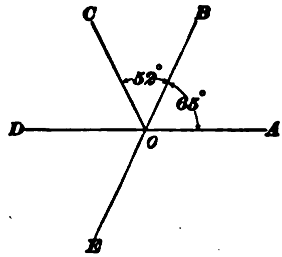


FIG. 18.

EXAMPLE.—In Fig. 18, $A D$ and $B E$ are straight lines intersecting in the point O. From the dimensions marked on the figure, find the number of degrees in each of the angles $C O D$, $D O E$, and $E O A$.

SOLUTION.—By the principle of this article, the sum of the three angles $A O B$, $B O C$, and $C O D$ is 180°; hence, the angle $C O D$ is found by subtracting the sum of the angles $A O B$ and $B O C$ from 180°. The sum of the angles $A O B$ and $B O C$ is $65° + 52° = 117°$.

Therefore,

$$C O D = 180° - 117° = 63°.$$

By Art. **28**, the vertical angles $D O E$ and $A O B$ are equal; therefore, the angle $D O E = 65°$.

Again, the angle $B O D$ is equal to the sum of the angles $B O C$ and $C O D$.

Therefore,

$$B O D = 52° + 63° = 115°.$$

By Art. **28**, the vertical angles $E O A$ and $B O D$ are equal; hence,

$$E O A = B O D = 115°.$$

Thus, we have

$$\left.\begin{array}{l}\text{Angle } C O D = 63°. \\ \text{Angle } D O E = 65°. \\ \text{Angle } E O A = 115°.\end{array}\right\} \text{Ans.}$$

31. *The sum of all the adjacent angles formed in the same plane about the same point is equal to four right angles, or 360°.*

In Fig. 19, the three adjacent angles $A O B$, $B O C$, and $C O A$ are all the adjacent angles formed in one plane about the point O. The sum of the three angles is measured by the sum of the three arcs $A B$, $B C$, and $C A$ (Art. **16**). The sum of the three arcs is equal to the whole circumference, or 360° of arc. Therefore, the sum of the three adjacent angles *A O B*, *B O C*, and *C O A* is equal to 360° of angle, or four right angles.

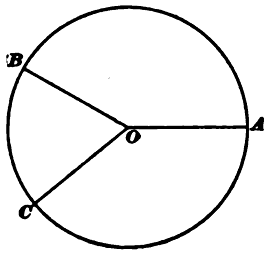


FIG. 19.

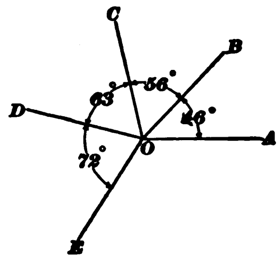


FIG. 20.

EXAMPLE.—In Fig. 20, find the number of degrees in the angle *E O A*.

SOLUTION.—By the principle of this article, the sum of the five angles in Fig. 20 is 360°. Therefore, the angle *E O A* is found by subtracting the sum of the four angles *A O B*, *B O C*, *C O D*, and *D O E* from 360°. The sum of these four angles is 46° + 56° + 63° + 72° = 237°.

Therefore, the angle *E O A* = 360° − 237° = 123°. Ans.

EXAMPLES FOR PRACTICE.

Solve the following examples :

1. In Fig. 21, find the number of degrees in the angle *E O A*.
   Ans. 156°.

2. In Fig. 21, find the number of degrees in the arc *A D*. Ans. 113°.

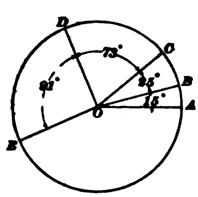


FIG 21.

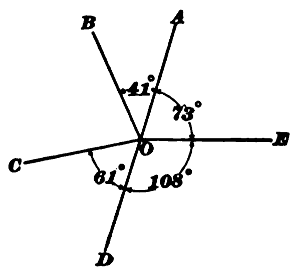


FIG. 22.

3. In Fig. 22, find the number of degrees in the angle *B O C*.
   Ans. 77°.

PARALLELS.

32. Two straight lines in the same plane that never meet, no matter how far they may be produced in both directions, are said to be **parallel**; thus the lines *A B* and *C D*, Fig. 23, are parallel.

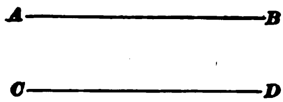


Fig. 23.

33. In Fig. 24, the straight line *X Y* cuts the two parallel lines *P Q* and *R S*, thus forming eight angles. These eight angles are designated by the letters *A*, *a*, *B*, *b*, *C*, *c*, *D*, and *d*, as shown in the figure. The following relations among these angles are very important.

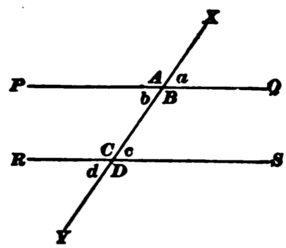


Fig. 24.

34. I. *The angles designated by capital letters in Fig. 24 are equal;* that is, the angles *A*, *B*, *C*, and *D* are all equal.

II. *The angles designated by small letters in Fig. 24 are equal;* that is, the angles *a*, *b*, *c*, and *d* are all equal.

III. *Any angle designated by a capital letter and an angle designated by a small letter in Fig. 24 are together equal to two right angles, or 180°.* That is, the sum of any two angles, as *A* and *b*, of which one is designated by a capital letter and the other by a small letter, is 180°.

Example.—In Fig. 25, find the number of degrees in each of the angles formed by the straight line *L M* cutting the two parallel lines *E F* and *G H*.

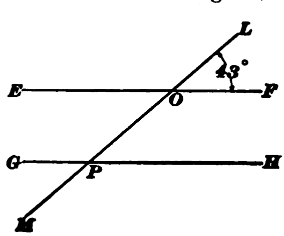


Fig. 25.

Solution.—From the figure, the angle *F O L* = 43°. By Art. **29**, *L O E* = 180° − 43° = 137°. By the principle of this article, we have

$$\left.\begin{array}{l} EOP = OPH = GPM = FOL = 43°. \\ FOP = OPG = HPM = LOE = 137°. \end{array}\right\}$$

Ans.

EXAMPLES FOR PRACTICE.

Solve the following examples:

1. In Fig. 26, the lines KL and MN are parallel and are cut by the straight lines PR and ST. Find the number of degrees in the angle QVM. Ans. 104°.

2. In Fig. 26, find the number of degrees in the angle QVN. Ans. 76°.

3. Find the number of degrees in the sum of the two angles LQV and QVN in Fig. 26. Ans. 180°.

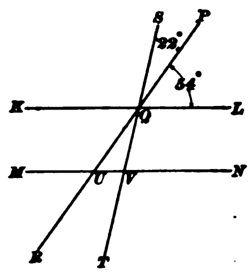


FIG. 26.

35. *If a straight line is perpendicular to one of two parallels, it is perpendicular to the other also.*

Let AB and CD, Fig. 27, be two parallel lines, and let LM be perpendicular to AB. Then it is easy to prove that LM is perpendicular to CD also. For, by Art. **34**, the sum of the angles BLM and LMD is two right angles; and since BLM is a right angle, LMD, also, must be a right angle; therefore, LM is perpendicular to CD (Art. **12**).

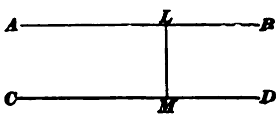


FIG. 27.

36. Let AB and CD, Fig. 28, be two parallel lines. Let LM and PQ be two lines each perpendicular to both of the parallels AB and CD; and let EF be any other line cutting the parallels. Then, *the lengths of LM and PQ are equal and each of them is shorter than EF.*

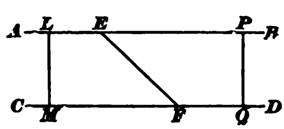


FIG. 28.

37. Hence, the length intercepted by two parallels on any line perpendicular to them is called the **distance** between the two parallels. Thus, the distance between the two parallels AB and CD, Fig. 28, is equal to either of the intercepts LM or PQ.

38. A line is said to be **parallel to a plane** if the line cannot meet the plane, no matter how far both of them are extended.

Two planes are said to be parallel if they cannot meet, no matter how far they may be extended.

39. A **horizontal line** is a line that is parallel to the surface of still water ; and a **vertical line** is a line that has the direction of the plumb line.

40. *When a vertical and a horizontal line meet, they are perpendicular to each other.*

41. In a drawing, it is usual to speak of lines drawn from left to right, or from right to left, as horizontal. In a diagram in a book, a line parallel to the printed lines is called a horizontal line, and a line perpendicular to the horizontal line is said to be vertical.

POLYGONS.

42. A **polygon** is a portion of a plane bounded by straight lines; the boundary lines are called the **sides of the polygon** ; the angles formed by the sides are called the **angles of the polygon**; the vertexes of the angles of the polygon are called the **vertexes of the polygon.** Thus, $ABCD$, Fig. 29, is a polygon bounded by four lines; the sides of this polygon are the four lines AB, BC, CD, and DA; its angles are the four angles ABC, BCD, CDA, and DAB; and its vertexes are the four points A, B, C, and D.

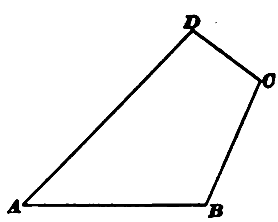


FIG. 29.

43. The number of the vertexes of a polygon is the same as the number of its sides; thus, the polygon $ABCD$, Fig. 29, has four sides and four vertexes.

44. Two straight lines cannot enclose space; and, therefore, the least number of sides that a polygon can have is three.

45. A polygon of three sides is called a **triangle**; a polygon of four sides, a **quadrilateral**; a polygon of five sides, a **pentagon**; a polygon of six sides, a **hexagon**; a polygon of seven sides, a **heptagon**; a polygon of eight sides, an **octagon.**

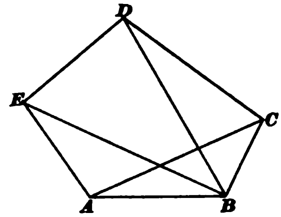


FIG. 30.

46. A diagonal of a polygon is a straight line joining two vertexes that are not adjacent. Thus, *A C*, *B D*, and *B E*, Fig. 30, are diagonals of the pentagon *A B C D E*.

EXAMPLES FOR PRACTICE.

Solve the following examples :

1. The perimeter of a heptagon is 29.5 inches. The lengths, in inches, of six of its sides are 3.25, 4.125, 3.375, 2.75, 4.25, and 3.625; find the length of the seventh side. Ans. 8.125 in.

2. The lengths, in feet, of the sides of a pentagon are 40.5, 30.65, 51.25, 32.75, and 36.85; find the perimeter. Ans. 192 ft.

TRIANGLES.

47. *The sum of the three angles of a triangle is equal to two right angles, or 180°.* Thus, the sum of the three angles *A*, *B*, and *C*, Fig. 31, is $59° + 73° + 48° = 180°$.

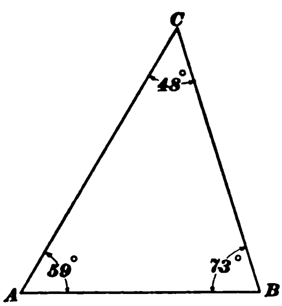


FIG. 31.

48. From Art. **47,** it is evident that if one angle of a triangle is obtuse, the other two angles must be acute. For, if one angle were obtuse and either of the other angles were not acute, the sum of the three angles would be greater than two right angles, which is impossible. For a similar reason, if one angle of a triangle is a right angle, the other two angles must be acute.

49. A triangle that has an obtuse angle is called an **obtuse** triangle. Thus, $A B C$, Fig. 32, is an obtuse triangle, having an obtuse angle at B.

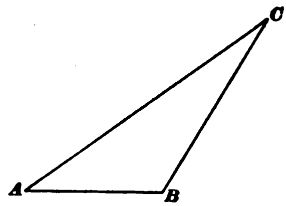


FIG. 32.

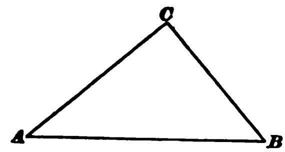


FIG. 33.

50. A triangle that has a right angle is called a **right-angled** triangle, or a **right** triangle; the side opposite to the right angle is called the **hypotenuse,** and the sides containing the right angle are called the **legs.** Thus $A B C$, Fig. 33, is a right triangle; C is the right angle; $A B$ is the hypotenuse; $A C$ and $C B$ are the legs.

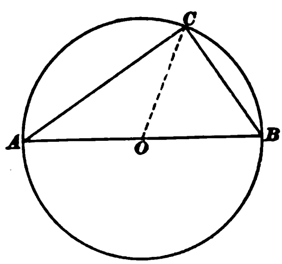


FIG. 34

51. *The circle described on the hypotenuse of a right triangle, as diameter, passes through the vertex of the right angle.* Thus, in Fig. 34, the circle on $A B$, as diameter, passes through C, the vertex of the right angle.

52. *The line joining the middle point of the hypotenuse of a right triangle to the vertex of the right angle is equal to one-half the hypotenuse.* Thus, in Fig. 34, each of the lines $O A$, $O B$, and $O C$ is a radius of the circle; therefore, $O A = O B = O C$. That is, the line $O C$ is equal to one-half of the hypotenuse.

53. A triangle that has three acute angles is called an **acute** triangle. Thus, $A B C$, Fig. 35, is an acute triangle.

54. A triangle that has no two of its sides equal is called a **scalene** triangle. Thus, *A B C*, Fig. 36, is a scalene triangle.

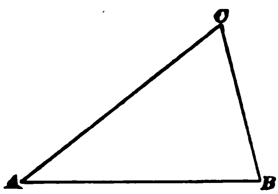


FIG. 35.

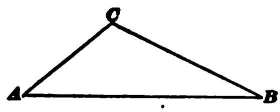


FIG. 36.

55. *In a scalene triangle, the longest side is opposite to the greatest angle, and the shortest side is opposite to the smallest angle.* Thus, in Fig. 36, the longest side *A B* is opposite to the greatest angle *A C B*, and the shortest side *A C* is opposite to the smallest angle *A B C*.

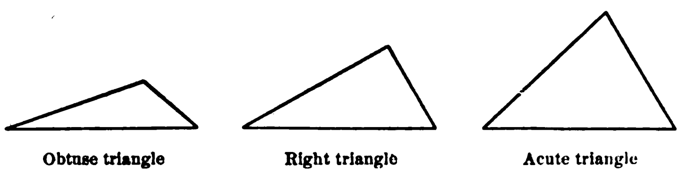


FIG. 37.

Fig. 37 shows a scalene obtuse triangle, a scalene right triangle, and a scalene acute triangle. The student should notice that in each of these triangles the longest side is opposite to the greatest angle.

I. *In an obtuse triangle, the longest side is opposite to the obtuse angle.*

II. *In a right triangle, the longest side is the hypotenuse, which is opposite to the right angle.*

III. *In an acute triangle, the longest side is opposite to the greatest of the three acute angles.*

56. A triangle that has two of its sides equal is called an **isosceles** triangle, and the two equal sides are called its legs; thus, *A B C*, Fig. 38, is an isosceles triangle whose legs are *A B* and *A C*.

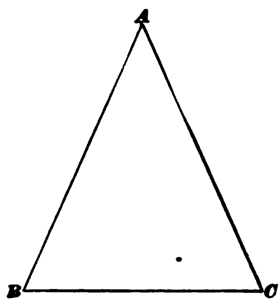


FIG. 38.

57. *The angles opposite the legs of an isosceles triangle are equal.* Thus, the angles *A B C* and *A C B*, Fig. 38, which are respectively opposite to the legs *A C* and *A B*, are equal.

EXAMPLE.—In Fig. 39, *A C* and *A B* are the legs of the isosceles triangle *A B C*; find the number of degrees in the angle *C*.

SOLUTION.—By the principle of this article, the angles *A* and *B* are equal; therefore, the sum of the angles *A* and *B* is $41° + 41° = 82°$. Hence, by Art. **47**, the angle *C* is equal to $180° - 82° = 98°$. Ans.

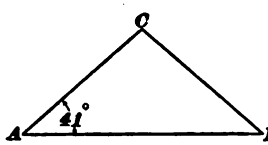


FIG. 39.

58. A triangle that has its three sides equal is called an **equilateral** triangle. Thus, the triangle *A B C*, Fig. 40, is an equilateral triangle.

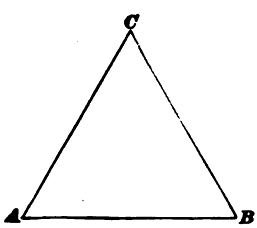


FIG. 40.

59. An equilateral triangle is a particular kind of isosceles triangle. Thus, the triangle *A B C*, Fig. 40, may be regarded as an isosceles triangle whose legs are *A B* and *A C*, as an isosceles triangle whose legs are *B A* and *B C*, or as an isosceles triangle whose legs are *C A* and *C B*.

Regarding the triangle *A B C*, Fig. 40, as an isosceles triangle whose legs are *A C* and *A B*, we see that the angles *A B C* and *A C B* are equal, because these are the angles opposite to the legs (Art. **57**). Regarding *A B C* as an isosceles triangle whose legs are *B A* and *B C*, we see that the angles *A C B* and *B A C* are equal (Art. **57**). Thus, each of the angles ABC and BAC is equal to the angle ACB; and, therefore, the three angles ABC, BAC, and ACB are equal. That is, the three angles of an equilateral triangle are equal.

60. *Each of the three angles of an equilateral triangle contains 60°.* This is evidently true for the three angles must be equal and their sum must be 180°, and we have $60° + 60° + 60° = 180°$.

61. The side on which a triangle is supposed to stand is called its **base.** The angle opposite to the base is called the **vertical angle** of the triangle. The vertex of the vertical angle is called the **vertex** of the triangle.

In a scalene triangle, any side may be regarded as the base. But an isosceles triangle ABC, whose legs are AB and AC, is always supposed to stand on the side BC; the side BC is, therefore, the base of the isosceles triangle, and the point A, in which the legs intersect, is the vertex.

62. Fig. 41 (*a*) and (*b*) shows isosceles triangles. None of the three isosceles triangles in Fig. 41 (*a*) are equilateral. The triangle in Fig. 41 (*b*) is equilateral.

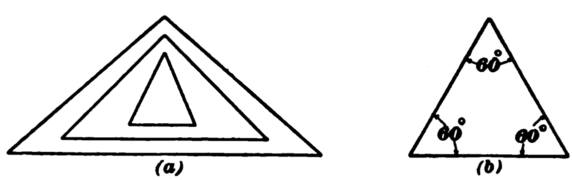


FIG. 41.

In each of the triangles in Fig. 41 (*a*) and (*b*), the student should notice that the angles opposite to the legs are equal.

63. *If two sides of a triangle are unequal, the longer side has the greater angle opposite to it.* Thus, in the triangle ABC, Fig. 42, the side AC is greater than the side BC; and the angle ABC, which is opposite the longer side AC, is greater than the angle BAC, which is opposite the short side BC.

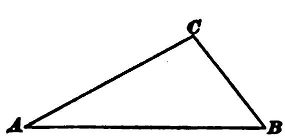


FIG. 42.

64. The perpendicular drawn from the vertex of a triangle to the base, produced if necessary, is called the **altitude** of the triangle. Thus, in Fig. 43 (*a*), *A D* is the altitude of the triangle *A B C*; and in Fig. 43 (*b*), *N P* is the altitude of the triangle *L M N*.

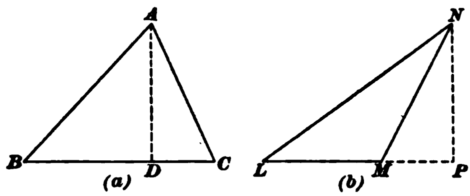


FIG. 43.

When one of the angles at the base of a triangle is obtuse, the perpendicular from the vertex to the base falls outside the triangle, as shown in Fig. 43 (*b*).

65. *The perpendicular drawn from the vertex of an isosceles triangle to the base* (1) *bisects the base;* (2) *bisects the vertical angle*; *and* (3) *bisects the triangle.*

Let *A D*, Fig. 44, be the perpendicular from the vertex *A* to the base *B C* of the isosceles triangle *A B C*. Fold the paper along the line *A D*; then the point *C* will fall on the point *B*, and the part of the figure to the right of *A D* will coincide with the part of the figure to the left of *A D*. Thus, the line *C D* can be made to coincide with *B D*; the angle *C A D* can be made to coincide with the angle *B A D*; and the triangle *C A D* can be made to coincide with the triangle *B A D*. Therefore, by Art. **39,** Part 1, the lines *B D* and *C D* are equal, the angles *B A D* and *C A D* are equal, and the triangles *B A D* and *C A D* are equal. This proves the three principles stated in italics at the beginning of this article.

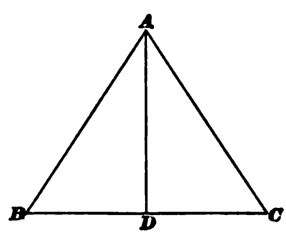


FIG. 44.

66. A line perpendicular to another line at its middle point is called the **perpendicular bisector** of the line. Thus, *P Q*, Fig. 45, is the perpendicular bisector of the line *L M*, since *P Q* is perpendicular to *L M*, and *L O* is equal to *O M*.

67. *Every point on the perpendicular bisector of a line is equidistant from the ends of the line.*

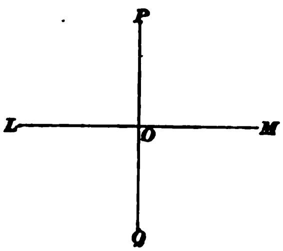


FIG. 45.

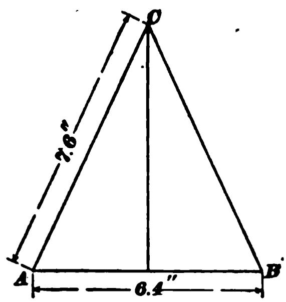


FIG. 46.

EXAMPLE.— C is a point on the perpendicular bisector of $A B$, Fig. 46. Given $A B = 6.4''$ and $A C = 7.6''$; find the perimeter of the triangle $A B C$.

SOLUTION.—By the principle of this article, C is equally distant from A and B; that is, $B C = A C$, or 7.6". Hence, the perimeter of the triangle is $6.4'' + 7.6'' + 7.6'' = 21.6''$. Ans.

EXAMPLES FOR PRACTICE.

Solve the following examples:

1. Find the number of degrees in each of the angles B and C of the triangle $A B C$, Fig. 47.

Ans. $\begin{cases} B = 36°. \\ C = 108°. \end{cases}$

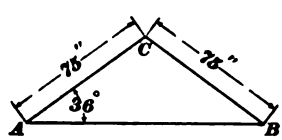


FIG. 47.

2. How many degrees are in each of the acute angles of an isosceles right triangle? Ans. 45°.

3. One of the acute angles of a right triangle contains 30°. How many degrees are in the other acute angle? Ans. 60°.

QUADRILATERALS.

68. The diagonal $A C$, Fig. 48, divides the quadrilateral $A B C D$ into two triangles $A B C$, and $A C D$. The angle $D A B$ is made up of the two angles $C A B$ and $D A C$; and the angle BCD is made up of the two angles BCA and ACD. Hence, the four angles of the quadrilateral are together equal to the sum of the six angles of the two triangles ABC and ACD. But the sum of the six angles of these two triangles is equal to $180° + 180° = 360°$. Hence, the sum of the four angles of the quadrilateral is 360°.

Thus we have the following principle:

69. *The sum of the angles of a quadrilateral is equal to four right angles, or 360°.*

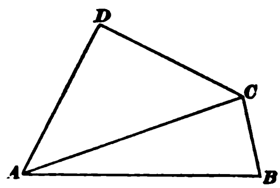


FIG. 48.

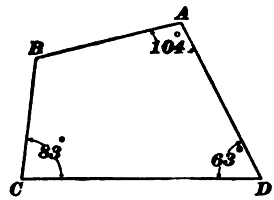


FIG. 49.

EXAMPLE.—In Fig. 49, find the number of degrees in the angle B.

SOLUTION.—The sum of the three angles C, D, and A is $83° + 63° + 104° = 250°$. Therefore, the angle B is equal to $360° - 250° = 110°$. Ans.

70. A polygon is said to be **inscribed** in a circle when each of the vertexes of the polygon lies on the circumference of the circle.

71. *When a quadrilateral is inscribed in a circle, the sum of each pair of opposite angles of the quadrilateral is equal to 180°.* Thus the angle ABC, Fig. 50, is measured by one-half the arc ADC; and the angle CDA is measured by one-half of the arc CBA (Art. **19**). Therefore, the sum of the two angles ABC and CDA is measured by one-half the sum of the two arcs ABC and CDA; that is, the sum of the angles ABC and CDA is measured by one-half of the circumference Therefore, the sum of the opposite angles

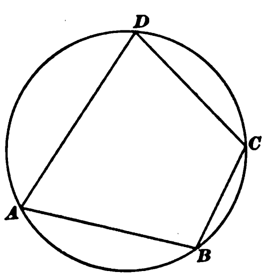


FIG. 50.

A B C and *C D A* is 180°. Similarly, the sum of the opposite angles *A* and *C* is 180°.

EXAMPLE.—In Fig. 51, find the number of degrees in each of the angles *C* and *D*.

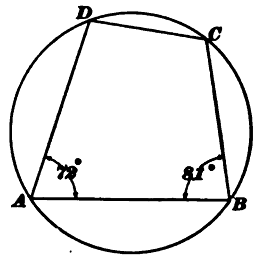


FIG. 51.

SOLUTION.—The sum of the opposite angles *A* and *C* is 180°, and the sum of the opposite angles *B* and *D* is 180°.

Therefore,

angle $C = 180° - 72° = 108°$, and angle $D = 180° - 81° = 99°$. } Ans.

72. A quadrilateral that has no two of its sides parallel is called a **trapezium**; thus, the quadrilateral *A B C D*, Fig. 52, is a trapezium.

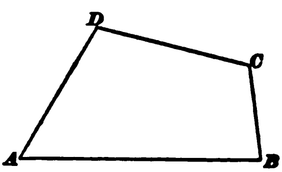


FIG. 52.

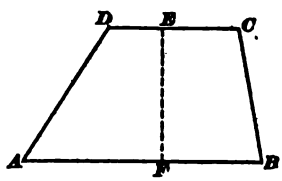


FIG. 53.

73. A quadrilateral that has a pair of parallel sides is called a **trapezoid**; the parallel sides are called its bases; and the perpendicular distance between the bases is called its **altitude.** Thus, *A B C D*, Fig. 53, is a trapezoid, of which *A B* and *D C* are the bases and *E F* the altitude.

EXAMPLE.—In the trapezoid *A B C D*, Fig. 54, find the number of degrees in each of the angles *C* and *D*.

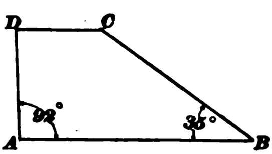


FIG. 54.

SOLUTION.—By Art. **34,** the sum of the angles *B* and *C* is 180°, and the sum of the angles *A* and *D* is 180°.

Therefore,

angle $C = 180° - 35° = 145°$, and angle $D = 180° - 92° = 88°$. } Ans.

74. A **parallelogram** is a quadrilateral whose opposite sides are parallel. Thus, the quadrilateral *A B C D*, Fig. 55, is a parallelogram because the side AB is parallel to the opposite side DC, and the side AD is parallel to the opposite side BC.

75. *The opposite sides and the opposite angles of a parallelogram are equal.* Thus, in Fig. 55, $AB = DC$ and $AD = BC$; also, angle A = angle C and angle B = angle D.

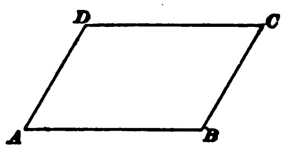


FIG. 55.

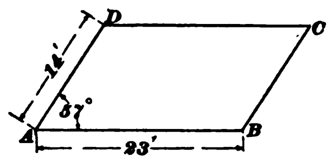


FIG. 56.

EXAMPLE.—Find the perimeter of the parallelogram $ABCD$, Fig. 56, and the number of degrees in each of the angles B, C, and D.

SOLUTION.—We have $BC = AD = 14'$, and $CD = AB = 23'$. Therefore, the perimeter =

$$AB + BC + CD + DA = 23' + 14' + 23' + 14' = 74'.$$

$$\text{Angle } C = \text{angle } A = 57°.$$

Also, angle $B = 180° - 57°$ = angle $D = 123°$ (Art. **34**).
Thus,

$$\left.\begin{array}{r} \text{Perimeter} = 74'. \\ \text{Angle } B = \text{angle } D = 123°. \\ \text{Angle } C = 57°. \end{array}\right\} \text{Ans.}$$

76. The side on which a parallelogram is supposed to stand and the opposite side are called its **lower base** and its **upper base,** respectively. The perpendicular distance between its bases is the **altitude** of the parallelogram. Fig. 57 shows a parallelogram with the length of its base and altitude marked.

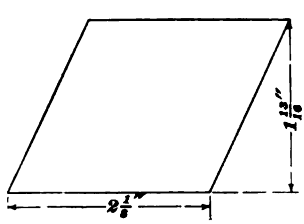


FIG. 57.

77. *The diagonals of a parallelogram bisect each other.* Thus, in the parallelogram $ABCD$, Fig. 58, the diagonals AC and BD intersect in the point O; and we have $AO = OC$, and $BO = OD$.

78. *Either diagonal of a parallelogram divides the parallelogram into two equal triangles.* If the triangle DBC, Fig. 59, is cut out and placed on the triangle BDA, the line DC being placed on the line BA and the line CB on the line AD, then the triangle DBC will coincide exactly

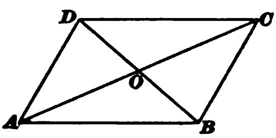


FIG. 58.

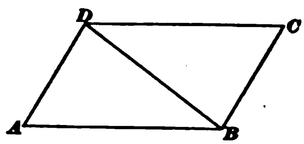


FIG. 59.

with the triangle BDA. Hence, by Art. **39,** Part 1, the triangles DBC and BDA are equal; and, therefore, the parallelogram is divided by the diagonal BD into two equal triangles. Similarly, a diagonal AC would divide the parallelogram into the two equal triangles ABC and ADC.

79. A *rectangle*, Fig. 60, is a parallelogram whose angles are all right angles. When one side of a rectangle is taken as a base, then either of the adjacent sides is the altitude of the rectangle. Thus, in Fig. 60, if AB is taken as the base of the rectangle, either AD or BC is the altitude.

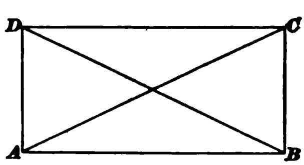


FIG. 60.

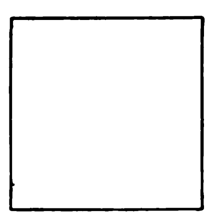


FIG. 61.

80. *The diagonals of a rectangle are equal.* Thus, in the rectangle $ABCD$, Fig. 60, $AC = BD$.

81. A **square,** Fig. 61, is a rectangle whose sides are all equal.

EXAMPLES FOR PRACTICE.

Solve the following examples:

1. In the trapezoid $ABCD$, Fig. 62, find the number of degrees in each of the angles B, C, and D.

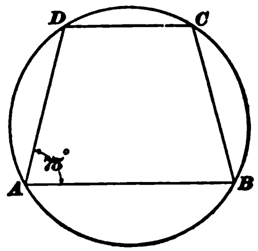


FIG. 62.

Ans. $\begin{cases} \text{Angle } B = 75°. \\ \text{Angle } C = 105°. \\ \text{Angle } D = 105°. \end{cases}$

2. In the parallelogram $ABCD$, Fig. 63, find the number of degrees in each of the angles ABC, BCA, CAB, and DCA.

Ans. $\begin{cases} \text{Angle } ABC = 59°. \\ \text{Angle } BCA = 49°. \\ \text{Angle } CAB = 72°. \\ \text{Angle } DCA = 72°. \end{cases}$

3. By using the principle of Art. **20**, find the number of degrees in the angle AEC, Fig. 64. Ans. 55°.

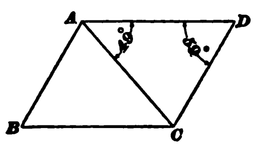


FIG. 63.

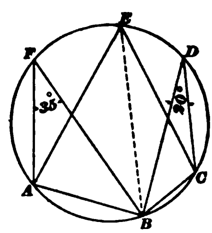


FIG. 64.

4. In Fig. 64, find the number of degrees in the angle ABC.

Ans. 125°

ARITHMETIC.

(PART 4.)

MULTIPLICATION.

1. If we have 5 bundles of toothpicks, each bundle containing 29 toothpicks, we can find the whole number of toothpicks in the 5 bundles by actually counting the toothpicks; but, we can arrive at the same result more easily by writing down the number 29 five times, and adding; thus, we find that the whole number of toothpicks is 145. Here we have a special kind of example of addition in which the numbers to be added are all equal; in other words, we have to find the sum of a certain number of repetitions of the same number. We have seen that this sum can be found either by counting or by addition; we will now explain a method of obtaining this sum that is shorter and easier than addition.

$$\begin{array}{r} 29 \\ 29 \\ 29 \\ 29 \\ 29 \\ \hline 145 \end{array}$$

2. Multiplication is an easy method for finding the sum of a certain number of repetitions of a given number. The number repeated is called the **multiplicand**; the number that shows how often the given number is repeated is called the **multiplier**; and the sum is called the **product**. For example, the sum of seven repetitions of the number 23 is 161; here, 23 is the multiplicand, 7 is the multiplier, and 161 is the product.

$$\begin{array}{r} 23 \\ 23 \\ 23 \\ 23 \\ 23 \\ 23 \\ 23 \\ \hline 161 \end{array}$$

3. The sign of multiplication is ×; it is read *times* or *multiplied by*, according as the multiplier or the multiplicand is written first. For example, 4 × 3 feet = 12 feet is read *four times three feet is equal to twelve feet;* but 3 feet × 4 = 12 feet is read *three feet multiplied by four is equal to twelve feet.*

4. The multiplier is always an abstract number (Art. **1**, Part 1), and the product is always a quantity of the same kind as the multiplicand. For the multiplier shows how many times the multiplicand is repeated, and cannot, therefore, be a concrete number.

5. The multiplier and the multiplicand are called the **factors** of the product.

6. In the arrangement of dots shown in Fig. 1, there are seven vertical columns and five horizontal rows. The whole number of dots may be obtained by counting the number of dots in one vertical column and multiplying by the number of columns; in this way, the whole number of dots is 5 repeated 7 times. But the number of dots can be found equally well by counting the number of dots in one horizontal row and multiplying by the number of rows; in this way, the whole number of dots is 7 repeated 5 times. But the whole number of dots must be the same, no matter what way they are counted; and, therefore, 5 repeated 7 times must be the same as 7 repeated 5 times, or 5 × 7 = 7 × 5. It is evident that the result would hold true for any number of rows and columns. Hence, we are led to the general principle that the product of two factors is the same in whatever order the factors are taken; thus, 2 × 3 = 3 × 2.

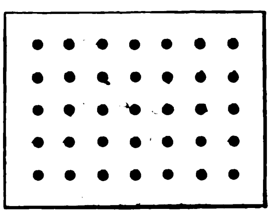


FIG. 1.

7. The following table gives the product of every pair of numbers from 1 to 12 inclusive.

MULTIPLICATION TABLE.

1 × 1 = 1	2 × 1 = 2	3 × 1 = 3	4 × 1 = 4
1 × 2 = 2	2 × 2 = 4	3 × 2 = 6	4 × 2 = 8
1 × 3 = 3	2 × 3 = 6	3 × 3 = 9	4 × 3 = 12
1 × 4 = 4	2 × 4 = 8	3 × 4 = 12	4 × 4 = 16
1 × 5 = 5	2 × 5 = 10	3 × 5 = 15	4 × 5 = 20
1 × 6 = 6	2 × 6 = 12	3 × 6 = 18	4 × 6 = 24
1 × 7 = 7	2 × 7 = 14	3 × 7 = 21	4 × 7 = 28
1 × 8 = 8	2 × 8 = 16	3 × 8 = 24	4 × 8 = 32
1 × 9 = 9	2 × 9 = 18	3 × 9 = 27	4 × 9 = 36
1 × 10 = 10	2 × 10 = 20	3 × 10 = 30	4 × 10 = 40
1 × 11 = 11	2 × 11 = 22	3 × 11 = 33	4 × 11 = 44
1 × 12 = 12	2 × 12 = 24	3 × 12 = 36	4 × 12 = 48
5 × 1 = 5	6 × 1 = 6	7 × 1 = 7	8 × 1 = 8
5 × 2 = 10	6 × 2 = 12	7 × 2 = 14	8 × 2 = 16
5 × 3 = 15	6 × 3 = 18	7 × 3 = 21	8 × 3 = 24
5 × 4 = 20	6 × 4 = 24	7 × 4 = 28	8 × 4 = 32
5 × 5 = 25	6 × 5 = 30	7 × 5 = 35	8 × 5 = 40
5 × 6 = 30	6 × 6 = 36	7 × 6 = 42	8 × 6 = 48
5 × 7 = 35	6 × 7 = 42	7 × 7 = 49	8 × 7 = 56
5 × 8 = 40	6 × 8 = 48	7 × 8 = 56	8 × 8 = 64
5 × 9 = 45	6 × 9 = 54	7 × 9 = 63	8 × 9 = 72
5 × 10 = 50	6 × 10 = 60	7 × 10 = 70	8 × 10 = 80
5 × 11 = 55	6 × 11 = 66	7 × 11 = 77	8 × 11 = 88
5 × 12 = 60	6 × 12 = 72	7 × 12 = 84	8 × 12 = 96
9 × 1 = 9	10 × 1 = 10	11 × 1 = 11	12 × 1 = 12
9 × 2 = 18	10 × 2 = 20	11 × 2 = 22	12 × 2 = 24
9 × 3 = 27	10 × 3 = 30	11 × 3 = 33	12 × 3 = 36
9 × 4 = 36	10 × 4 = 40	11 × 4 = 44	12 × 4 = 48
9 × 5 = 45	10 × 5 = 50	11 × 5 = 55	12 × 5 = 60
9 × 6 = 54	10 × 6 = 60	11 × 6 = 66	12 × 6 = 72
9 × 7 = 63	10 × 7 = 70	11 × 7 = 77	12 × 7 = 84
9 × 8 = 72	10 × 8 = 80	11 × 8 = 88	12 × 8 = 96
9 × 9 = 81	10 × 9 = 90	11 × 9 = 99	12 × 9 = 108
9 × 10 = 90	10 × 10 = 100	11 × 10 = 110	12 × 10 = 120
9 × 11 = 99	10 × 11 = 110	11 × 11 = 121	12 × 11 = 132
9 × 12 = 108	10 × 12 = 120	11 × 12 = 132	12 × 12 = 144

8. The student must commit the multiplication table to memory, and in order to be able to write down quickly the product of any pair of numbers less than 10, he should practice such examples as the following:

Write down the second member of the following equations:

$5 \times 7 =$, $7 \times 8 =$, $9 \times 9 =$, $6 \times 5 =$,
$8 \times 4 =$, $6 \times 7 =$, $8 \times 5 =$, $4 \times 7 =$,
$9 \times 2 =$, $9 \times 8 =$, $7 \times 9 =$, $5 \times 9 =$,
$6 \times 4 =$, $3 \times 8 =$, $9 \times 4 =$, $3 \times 7 =$.

The student can make more of these examples for himself by taking at random any pair of numbers less than 10 and writing down their product.

9. We have

$$2 \times 0 = 0 + 0 = 0,$$
$$3 \times 0 = 0 + 0 + 0 = 0,$$
$$4 \times 0 = 0 + 0 + 0 + 0 = 0,$$

and so on. Therefore, *the product of zero and any number is zero.*

MULTIPLICATION OF INTEGERS.

10. To multiply an integer by a single digit:

EXAMPLE.—Multiply 423 by 5.

SOLUTION.—

$$\begin{array}{rr} \textit{multiplicand} = & 423 \\ \textit{multiplier} = & 5 \\ \hline \textit{product} = & 2115 \end{array} \quad \text{Ans.}$$

EXPLANATION.—It is usual to write the multiplier under the units figure of the multiplicand. From the definition of multiplication in Art. **2**, it appears that multiplying 423 by 5 is the same as adding together five numbers each equal to 423; thus,

$$\left.\begin{array}{r} 423 \\ 5 \\ \hline 2115 \end{array}\right\} \text{ is a contraction for } \left\{\begin{array}{r} 423 \\ 423 \\ 423 \\ 423 \\ 423 \\ \hline 2115 \end{array}\right.$$

In multiplication, we begin at the right. From the multiplication table, we have $5 \times 3 = 15$; hence, 5×3 units $=$ 15 units $=$ 1 ten $+$ 5 units. Write the 5 units in the units place in the product, and reserve the 1 ten to be added to the product of the tens. Again, 5×2 tens $=$ 10 tens; adding the 1 ten reserved, we have 5×2 tens $+$ 1 ten $=$ 10 tens $+$ 1 ten $=$ 11 tens $=$ 1 hundred $+$ 1 ten. Write the 1 ten in the tens place in the product, and reserve the 1 hundred to be added to the product of the hundreds. Then, 5×4 hundreds $=$ 20 hundreds; adding the 1 hundred reserved, we get 5×4 hundreds $+$ 1 hundred $=$ 20 hundreds $+$ 1 hundred $=$ 21 hundreds $=$ 2 thousands $+$ 1 hundred. The 1 hundred is written in the hundreds place in the product; and, as there are no more figures in the multiplicand, the two thousands are written in the thousands place. Therefore, the product is 2,115.

11. *When an integer is multiplied by 10, 100, 1,000, etc., the product is found by simply annexing to the multiplicand as many ciphers as there are ciphers in the multiplier.*

Thus,

$$10 \times 231 = 2{,}310,$$
$$100 \times 231 = 23{,}100,$$
$$1{,}000 \times 231 = 231{,}000,$$
$$10{,}000 \times 231 = 2{,}310{,}000,$$

also,

$$10 \times 4{,}030 = 40{,}300,$$
$$100 \times 4{,}030 = 403{,}000,$$

and so on.

12. *When the multiplier consists of a digit followed by ciphers, the product is found by multiplying the multiplicand by the digit and annexing to the product as many ciphers as there are ciphers in the multiplier.*

For example, let it be required to multiply 3,231 by 30. By Art. **10** we have

$$3 \times 3{,}231 = 9{,}693.$$

And, therefore, $30 \times 3{,}231 = 96{,}930.$

In like manner,

$$300 \times 3{,}231 = 969{,}300,$$
$$3{,}000 \times 3{,}231 = 9{,}693{,}000,$$

and so on.

EXAMPLES FOR PRACTICE.

Solve the following examples:

1. 6 × 3,842. Ans. 23,052.
2. 40 × 749. Ans. 29,960.
3. 300 × 827. Ans. 248,100.
4. 90 × 905. Ans. 81,450.

13. To multiply an integer by a number containing more than one digit.

EXAMPLE.—Multiply 476 by 234.

SOLUTION.—

multiplicand =	4 7 6	
multiplier =	2 3 4	
4 × *multiplicand* =	1 9 0 4	= *1st partial product.*
30 × *multiplicand* =	1 4 2 8 0	= *2d partial product.*
200 × *multiplicand* =	9 5 2 0 0	= *3d partial product.*
234 × *multiplicand* =	1 1 1 3 8 4	= *complete product.*

Therefore, 234 × 476 = 111,384. Ans.

EXPLANATION.—This result may be obtained by writing down 234 numbers each equal to 476 and adding them together. This tedious operation may be divided into three steps: first, we may take the sum of 4 numbers each equal to 476—this sum is the first partial product 1,904; second, the sum of 30 numbers each equal to 476—this sum is the second partial product 14,280; and third, the sum of 200 numbers each equal to 476—this sum is the third partial product 95,200. Finally, adding these sums, or partial products, we get the complete product 111,384. The ciphers at the right of the second and third partial products merely serve to fix the position of the other figures, and may be omitted. It is usual to omit these ciphers and to write the solution in the following form:

```
      4 7 6
      2 3 4
    -------
    1 9 0 4
  1 4 2 8
  9 5 2
-----------
1 1 1 3 8 4   Ans.
```

14. From this example, we derive the following rule for multiplying whole numbers:

Rule.—I. *Write the multiplier under the multiplicand, so that units are under units, tens under tens, etc.*

II. *Beginning at the right, multiply the multiplicand by each successive digit of the multiplier, and place the right-hand figure of each partial product directly under the digit used as multiplier.*

III. *The sum of these partial products is the complete product.*

15. In Art. **6,** it was proved that the product of two factors is the same, no matter which of the factors is taken as the multiplier. This suggests the following method of checking the work of an example in multiplication:

Check.—*To check an example in multiplication, multiply the multiplier by the multiplicand; if this gives the same product as that found by multiplying the multiplicand by the multiplier, the product is probably correct.*

EXAMPLE.—Multiply 653 by 478, and check your answer.

$$
\begin{array}{rr}
\text{SOLUTION.— } \textit{multiplicand} = & 653 \\
\textit{multiplier} = & 478 \\
\hline
& 5224 \\
& 4571 \\
& 2612 \\
\hline
\textit{product} = & 312134 \text{ Ans.}
\end{array}
$$

CHECK.—Using the multiplicand as multiplier, we get

$$
\begin{array}{rr}
\textit{multiplier} = & 478 \\
\textit{multiplicand} = & 653 \\
\hline
& 1434 \\
& 2390 \\
& 2868 \\
\hline
\textit{product} = & 312134 \text{ Ans.}
\end{array}
$$

Since the same product was obtained in both cases, it is probably correct.

16. When the multiplier contains a cipher, we may proceed as in the following examples:

```
     (a)                    (b)
     3124                  20352
     2304                  13002
    -----                  -----
    12496                  40704
    0000                  00000
   9372                  00000
  6248                  61056
  -------              20352
  7197696             ---------
                      264616704
```

In multiplying by a cipher, it is usual to write only the first cipher of the partial product, and then to multiply by the next figure of the multiplier, placing the partial product alongside the cipher. So the solutions of examples (*a*) and (*b*) are usually written thus:

```
            (a)                              (b)
multiplicand =    3124    multiplicand =          20352
  multiplier =    2304      multiplier =          13002
                 -----                            -----
                 12496                            40704
                93720                         6105600
               6248                          20352
               -------                       ---------
     product = 7197696         product =     264616704
```

17. If the multiplier, or multiplicand, or both, ends in one or more ciphers, place the right-hand digit of the multiplier under the right-hand digit of the multiplicand; then, disregarding the ciphers at the right, multiply according to the rule and write as many ciphers at the right of the product as there are ciphers at the right of the multiplicand and multiplier.

EXAMPLE.—Multiply 239,000 by 150.

```
SOLUTION.—  multiplicand =  239000
              multiplier =    150
                            ------
                            1195
                            239
                            --------
                 product =  35850000  Ans.
```

EXAMPLES FOR PRACTICE.

Multiply:

1. 3,842 by 26. Ans. 99,892.
2. 2,170 by 550. Ans. 1,193,500.
3. 2,340 by 30,200. Ans. 70,668,000.
4. 3,257 by 246. Ans. 801,222.

MULTIPLICATION OF DECIMALS.

18. *When a number is multiplied by 10, 100, 1,000, etc., the product is found by simply moving the decimal point in the multiplicand as many places to the* **right** *as there are ciphers in the multiplier, annexing ciphers to the multiplicand if necessary.*

Thus,

$$10 \times 3.1416 = 31.416$$
$$100 \times 3.1416 = 314.16$$
$$1,000 \times 3.1416 = 3,141.6$$
$$10,000 \times 3.1416 = 31,416$$
$$100,000 \times 3.1416 = 314,160$$

When the multiplicand is an integer, the decimal point is understood after the figure in the units place, though it is not written. Thus, $100 \times 52 = 100 \times 52. = 5,200$, which agrees with the result obtained by the rule given in Art. **11**.

19. *When a number is multiplied by .1, .01, .001, etc., the product is found by simply moving the decimal point in the multiplicand as many places to the* **left** *as there are decimal places in the multiplier, prefixing ciphers if necessary.*

Thus,

$.1 \times 56.2 =$ one-tenth of $56.2 = 5.62$.
$.01 \times 56.2 =$ one-hundredth of $56.2 = .562$.
$.001 \times 56.2 =$ one-thousandth of $56.2 = .0562$.

20. The following is the rule for finding the product of any two numbers containing decimals:

Rule.—I. *Disregarding the decimal points, find the product of the two numbers as if they were integers* (Art. **14**).

II. *In the product, point off as many decimal places as there are decimal places in both multiplicand and multiplier, prefixing ciphers to the product if necessary.*

EXAMPLE 1.—Multiply 2.305 by 4.35.

SOLUTION.—

multiplicand =	2.3 0 5
multiplier =	4.3 5
	1 1 5 2 5
	6 9 1 5
	9 2 2 0
product =	1 0.0 2 6 7 5 Ans.

EXPLANATION.—Disregarding the decimal points, we multiply 2,305 by 435. Since there are 3 decimal places in the multiplicand and 2 decimal places in the multiplier, we point off 3 + 2, or 5, decimal places in the product.

EXAMPLE 2.—Multiply 4.325 by 2.8.

SOLUTION.—

multiplicand =	4.3 2 5
multiplier =	2.8
	3 4 6 0 0
	8 6 5 0
product =	1 2.1 1 0 0 = 1 2.1 1 Ans.

EXPLANATION.—In this example, there are 3 decimal places in the multiplicand and 1 decimal place in the multiplier; therefore, there must be 3 + 1, or 4, decimal places in the product. As soon as the decimal point is inserted in the product, the two ciphers at the extreme right should be dropped.

EXAMPLE 3.—Multiply .251 by .13.

SOLUTION.—

multiplicand =	.2 5 1
multiplier =	.1 3
	7 5 3
	2 5 1
product =	.0 3 2 6 3 Ans.

EXPLANATION.—In this example, there are 3 decimal places in the multiplicand and 2 decimal places in the multiplier; therefore, the number of decimal places in the product is 3 + 2, or 5. But there are only four figures in the product of 251 and 13; hence, we must prefix one cipher to this product in order to have 5 decimal places.

EXAMPLES FOR PRACTICE.

Multiply:

1. .00049 by 4.14. Ans. .0020286.
2. 4.32 by 1.7. Ans. 7.344.
3. .714 by 2.5. Ans. 1.785.
4. .004 by .125. Ans. .0005.

IMPORTANT APPLICATIONS.

21. The **area** of a figure means the extent of the surface included within the bounding lines of the figure. The **unit of area** is a square whose side is equal to the unit of length. If an inch is used as the unit of length, the unit of area is a square inch; if a foot is taken as the unit of length, the unit of area is a square foot.

22. Let *A B C D*, Fig. 2, be a rectangle, and suppose that the sides *A B* and *A D* measure 6 inches and 3 inches, respectively. Divide *A B* into 6 equal parts, and through the points of division, draw lines parallel to *A D*. Divide *A D* into 3 equal parts, and through the points of division,

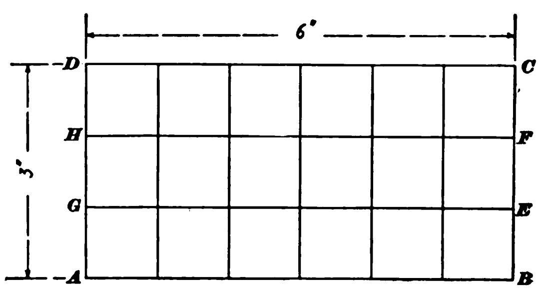


FIG. 2.

draw lines parallel to *A B*. These two sets of parallel lines divide the rectangle *A B C D* into a number of squares, each of which is a square inch. In the row from *A* to *E*, there are 6 squares; in the row from *G* to *F*, there are 6 squares; and in the row from *H* to *C*, there are 6 squares. Thus, the rectangle *A B C D* contains 3 × 6 squares, each of which is a square inch; and therefore the area of the rectangle *A B C D* is 18 square inches. Thus, the number of square inches in the surface of the rectangle *A B C D* is found by multiplying the number of inches in *A B* by the number of inches in *A D*.

The area of any rectangle can be found in this manner; hence, we have the following general statement:

23. Area of a Rectangle.—*The area of a rectangle is equal to the product of its base and altitude.*

This is called a *general* statement, because it is true for all rectangles. This general statement may be expressed thus:

$$(area\ of\ rectangle) = (base) \times (altitude).$$

The base and altitude must be measured by the same unit of length; and, then, the area is expressed in square units of the same name as the unit of length used in measuring the sides. Thus, if the base and altitude are measured in inches, the area is in square inches; if the base and altitude are measured in feet, the area is in square feet.

The student must remember that one concrete number cannot be multiplied by another concrete number; for a multiplier must be an abstract number. The *product of two lines means the product of their numerical measures when measured by the same linear unit.* We cannot multiply the line *A B* by the line *A D*; but we can multiply the *number* of inches in *A B* by the *number* of inches in *A D*.

EXAMPLE 1.—Find the area of a rectangle whose length is 3.5 inches and whose altitude is 2.6 inches.

SOLUTION.—The number of square inches in the area of the rectangle is $3.5 \times 2.6 = 9.1$. Therefore, the area is 9.1 sq. in. Ans.

EXAMPLE 2.—Find the area of a square each of whose sides is 4.5 inches.

SOLUTION.—By Art. **81**, Part 3, a square is a rectangle whose sides are equal. Hence, the area of the square is equal to the product of its base and altitude; that is, the area of the square is 4.5×4.5, or 20.25 sq. in. Ans.

24. The student must observe that *two square inches* does not mean the same as *two inches square.* Fig. 3 (*a*) shows a rectangle 2 inches by 1 inch, and its area is 2 square inches; Fig. 3 (*b*) shows a square 2 inches by 2 inches; that is, a figure 2 inches square, and its area is 4 square inches.

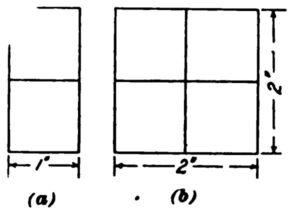


FIG. 3.

25. Area of a Parallelogram. *The area of a parallelogram is equal to the product of its base and altitude.*

Let $A B C D$, Fig. 4, be any parallelogram. Draw $A F$ and $B E$ perpendicular to the two opposite sides $A B$ and $D C$. Then, if the triangle $B E C$ is cut out, it can be placed upon and made to coincide exactly with the triangle $A F D$. Therefore, the rectangle $A B E F$ is equal to the parallelogram $A B C D$. But the area of the rectangle $A B E F$ is equal to the product of $A B$ and $A F$; and, consequently, the area of the parallelogram is equal to the product of its base $A B$ and its altitude $A F$.

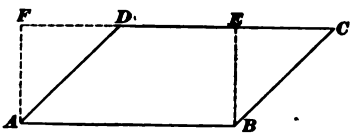


FIG. 4.

EXAMPLE. — Find the area of the parallelogram shown in Fig. 5.

SOLUTION.—The number of square inches in the area of this parallelogram is $2.4 \times 1.3 = 3.12$. Thus, the area is 3.12 sq. in. Ans.

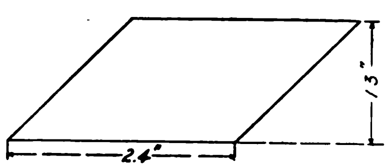


FIG. 5.

26. A **solid** is a figure that has length, breadth, and thickness. The boundaries of a solid are surfaces.

27. A **cube,** Fig. 6, is a solid bounded by six equal square surfaces.

28. A solid bounded by six rectangular surfaces is called a **rectangular solid,** Fig. 7. A cube is a particular kind of a rectangular solid in which the rectangular faces are equal squares.

29. The space included between the bounding surfaces of a solid is called the **cubic contents, capacity, or volume** of the solid.

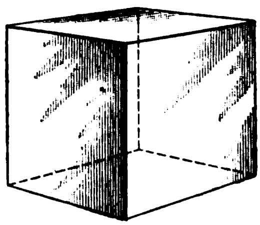


FIG. 6.

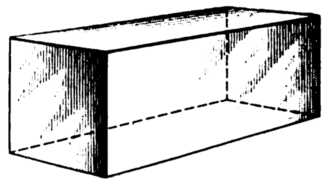


FIG. 7.

30. The **unit of volume** is a cube, each of whose edges is equal to the unit of length. If the unit of length is an inch, a cubic inch is the unit of volume; if a foot is used as a unit of length, the unit of volume is a cubic foot.

31. The **continued product** of several numbers is the result obtained by multiplying the first by the second, and then multiplying this product by the third number, and so on. Thus, the continued product of 2, 3, and 4 is $2 \times 3 \times 4$.

32. Volume of a Rectangular Solid.—*The volume of a rectangular solid is equal to the continued product of its length, breadth, and thickness.*

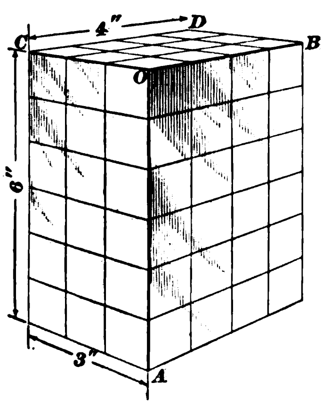


FIG. 8.

In the solid shown in Fig. 8, suppose the length $O A$ is 6 inches, the breadth $O B$ is 4 inches, and the thickness $O C$ is 3 inches. If $O C$ is divided into three equal parts, the solid can be cut into three equal slices each 1 inch thick, as shown in Fig. 9. Each of these slices can be divided again into 6×4 cubic inches, as is evident from Fig. 9. Therefore, the three slices contain $6 \times 4 \times 3$ cubic inches. Thus, the volume of the solid shown in Fig. 8 is $6 \times 4 \times 3$ cubic inches; that is, the volume is equal to the continued product of the length, breadth, and thickness.

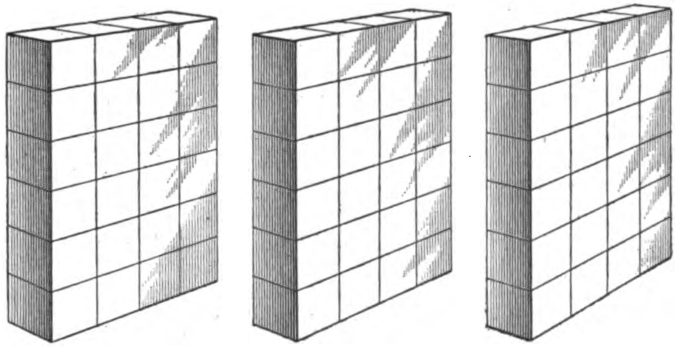


FIG. 9.

EXAMPLE 1.—Find the volume of a rectangular solid whose length, breadth, and thickness are 8.4 inches, 5.5 inches, and 3.5 inches, respectively.

SOLUTION.—The number of cubic inches in the volume of this solid is $8.4 \times 5.5 \times 3.5 = 161.7$. Therefore, the volume is 161.7 cu. in. Ans.

EXAMPLE 2.—Find the volume of a cube whose edge is 3.5 inches long.

SOLUTION.—The volume of the cube is equal to the product of its length, breadth, and thickness; that is, the volume in cubic inches is $3.5 \times 3.5 \times 3.5 = 42.875$. Therefore, the volume is 42.875 cu. in. Ans.

EXAMPLES FOR PRACTICE.

1. Find the area of a square whose side contains 6.5 inches. Ans. 42.25 sq. in.

2. Find the volume of a cube whose edge contains 2.5 inches. Ans. 15.625 cu. in.

3. Find the area of a parallelogram whose length and altitude are 13.5 and 2.4 inches, respectively. Ans. 32.4 sq. in.

4. Find the volume of a rectangular solid whose length, breadth, and thickness are 12.5, 8.5, and 4 feet, respectively. Ans. 425 cu. ft.

PARENTHESES.

33. The **parenthesis** ( ) indicates that the numbers enclosed within it are to be subjected to the same operations; it is called **a symbol of aggregation** because the numbers enclosed within it are to be aggregated or collected into one number. Thus, $(2+3)\times 4$ means that the sum of 2 and 3 is to be multiplied by 4; hence, we have $(2+3)\times 4 = 5\times 4 = 20$. The student must be careful to distinguish between $(2+3)\times 4$ and $2+3\times 4$; for, as we have seen, $(2+3)\times 4 = 5\times 4 = 20$; but, on the other hand, $2+3\times 4 = 2+12 = 14$. In like manner, $(35-3)\times 7$ means that the difference between 35 and 3 is to be multiplied by 7. Hence, $(35-3)\times 7 = 32\times 7 = 224$; but $35-3\times 7 = 35-21 = 14$. The sign of multiplication is usually omitted between two factors one of which is enclosed within a parenthesis. Thus, $3\times(2+5)$ is written $3(2+5)$.

The operations indicated by the signs within the parenthesis should be performed first.

Thus,
$$2(3+5) = 2\times 8 = 16,$$
$$3(2+3-1) = 3\times 4 = 12,$$
$$5(2+1)+4(6+4) = 5\times 3+4\times 10 = 15+40 = 55.$$

34. Any combination of signs that represents a number is called an **expression.** Thus, $2\times 9+6-5$ is an expression. The parts of an expression that are separated by the signs + and − are called the **terms** of the expression. The expression $3\times 5+8\times 7-5$ contains the three terms 3×5, 8×7, and 5.

All operations within any term of an expression are to be performed before that term is combined with any other term. For example, in the expression $5+7\times 2-3\times 6$, 7 is to be multiplied by 2 and the product added to 5 and then from this sum the product 3×6 is to be subtracted, giving the result 1.

EXAMPLE 1.—Find the value of $3+4\times 5+9\times 8$.

SOLUTION.—We have

$$3+4\times 5+9\times 8 = 3+20+72 = 95. \quad \text{Ans.}$$

EXAMPLE 2.—Find the value of $(2 \times 5 + 1)(3 + 5 + 2 \times 11)$.

SOLUTION.—Performing the operations within the parenthesis first, we have

$$\begin{aligned}(2 \times 5 + 1)(3 + 5 + 2 \times 11) &= (10 + 1)(3 + 5 + 22) \\ &= 11 \times 30 \\ &= 330. \quad \text{Ans.}\end{aligned}$$

EXAMPLE 3.—Find the value of $(3 \times 7 - 2 \times 5)(7 \times 2 + 1 - 3 \times 4)$.

SOLUTION.—Performing the operations within the parenthesis first, we get

$$\begin{aligned}(3 \times 7 - 2 \times 5)(7 \times 2 + 1 - 3 \times 4) &= (21 - 10)(14 + 1 - 12) \\ &= (11)(3) = 33. \quad \text{Ans.}\end{aligned}$$

35. *To multiply the sum of two given numbers by a third number is the same as to multiply each of the given numbers by the third number and then take the sum of the products.*

Thus, $5(3 + 7) = 5 \times 3 + 5 \times 7$.

To multiply the difference of two given numbers by a third number is the same as to multiply each of the given numbers by the third number and then take the difference of the products.

Thus, $7(9 - 3) = 7 \times 9 - 7 \times 3$.

36. *Any number that is a factor of every term of an expression is a factor of the whole expression.*

Thus, the number 2 is a factor in every term of the expression $2 \times 3 + 2 \times 5$, and it has been explained in Art. **33** that this expression can be written in the form $2(3 + 5)$, where 2 appears as a factor of the whole expression. This principle is very useful in shortening calculations, as will be seen from the following example:

EXAMPLE.—Find the value of

$$3.1416 \times 2 + 3.1416 \times 1.375 + 3.1416 \times 2.625.$$

SOLUTION.—Since 3.1416 is a factor in every term of this expression, it is a factor of the whole expression; therefore,

$$\begin{aligned}&3.1416 \times 2 + 3.1416 \times 1.375 + 3.1416 \times 2.625 \\ &= 3.1416\,(2 + 1.375 + 2.625) \\ &= 3.1416 \times 6 \\ &= 18.8496. \quad \text{Ans.}\end{aligned}$$

NOTE.—If 3.1416 had not been taken as a factor of the whole expression, the solution would have been as follows:

$$
\begin{aligned}
3.1416 \times 2 &= 6.2832 \\
3.1416 \times 1.375 &= 4.3197 \\
3.1416 \times 2.625 &= 8.2467 \\
\hline
& 18.8496
\end{aligned}
$$

This method requires three separate multiplications, whereas the former solution involves one multiplication only.

EXAMPLES FOR PRACTICE.

Use the principle of Art. **36** to find the value of each of the following expressions:

1. $.5236 \times 3.45 + .5236 \times 7.175 + .5236 \times 9.375$. Ans. 10.472.
2. $3.1416 \times 21.25 + 3.1416 \times 15.78 + 3.1416 \times 7.97$. Ans. 141.372.

ARITHMETIC.

(PART 5.)

DIVISION.

1. Division is the operation by which we determine how many times one number is contained in another number of the same kind, and by which we determine the amount of each part when a given number is divided into a certain number of equal parts. Division is indicated by the sign $\div$, by the colon :, or by a line between the two numbers; thus, each of the expressions $14 \div 2$, $14 : 2$, $\frac{14}{2}$, and 14/2 is read *fourteen divided by two.*

2. We know that 7 dollars is contained in 28 dollars, 4 times. With the multiplication sign this is expressed $\$7 \times 4 = \$28$; with the sign of division it is written as $\$28 \div \$7 = 4$. In this case, division determines how many times the number $7 is contained in the like number $28. If $28 is distributed equally among 7 men, each man receives $4; this is expressed in multiplication as $\$4 \times 7 = \$28$; in division it is written as $\$28 \div 7 = \$4$. Here, division determines the amount of each part when $28 is divided into 7 equal parts.

3. The number divided is called the **dividend,** the number by which it is divided is called the **divisor,** and the result is called the **quotient.**

4. If 31 cents is divided among 7 children, each child receives 4 cents, and there are 3 cents left over. Thus, it is not possible to divide 31 cents exactly into 7 equal parts;

but 31 cents can be divided into 7 parts each containing 4 cents, with 3 cents left over. This is expressed by the equation

$$7 \times 4 \text{ cents} + 3 \text{ cents} = 31 \text{ cents}.$$

Again, 7 cents is contained in 31 cents 4 times, and there are 3 cents left over; that is, 7 cents is not contained an exact number of times in 31 cents, but 7 cents can be taken from 31 cents 4 times and there will be 3 cents left. This gives the equation

$$4 \times 7 \text{ cents} + 3 \text{ cents} = 31 \text{ cents}.$$

In such cases, the quantity left over is called the **remainder.**

The remainder is always a quantity of the same kind as the dividend.

5. From the preceding definitions, it follows that

$$(\textit{dividend}) = (\textit{divisor}) \times (\textit{quotient}) + (\textit{remainder}).$$

When the division is exact, there is no remainder; therefore, for exact division, we have

$$(\textit{dividend}) = (\textit{divisor}) \times (\textit{quotient}).$$

Hence, *when a product and one of its factors are given, the other factor can be found by division.*

6. When the divisor does not exceed 12, the work of division may be arranged in the following form, which is called **short division.**

EXAMPLE.—Divide 3,942 by 9.

SOLUTION.—

$$\begin{array}{r|l} 9 & \underline{3\,9\,4\,2} \\ & \;\,4\,3\,8 \text{ Ans.} \end{array}$$

Use only these words: 9 in 39, 4 and 3 over; in 34, 3 and 7 over; in 72, 8 and no remainder.

EXPLANATION.—From the multiplication table, we know that 9 is contained in 39, 4 times and 3 over; therefore, 9 is contained in 39 hundreds, 4 hundred times and 3 hundreds over. Thus, the hundreds figure of the quotient is 4, which is written under the 9 in the dividend; and the 3 hundreds left over are *carried* and combined with the next figure of the dividend. These 3 hundreds, added to the 4 tens in the dividend, make 34 tens. Now, 34 ÷ 9 gives the quotient 3 and the remainder 7; hence, 34 tens ÷ 9 gives the quotient 3 tens and 7 tens over. Therefore, the tens figure of the quotient is 3, which is written under the 4 in the dividend; and the 7 tens left over must be carried and combined with the next figure of the dividend. These 7 tens, added to 2 units, make 72 units, or, simply, 72. And 72 ÷ 9 gives the quotient 8 and no remainder; therefore, the units figure of the quotient is 8. The complete quotient is 438.

7. Check.—*To check an example in division, multiply the quotient by the divisor, and to this product add the remainder, if any. The sum thus obtained should be equal to the dividend.* (Art. **5.**)

EXAMPLE.—Divide 41,128 by 7 and check the result.

SOLUTION.— 7 | 4 1 1 2 8
5 8 7 5 and 3 over. Ans.

CHECK.—

$$\begin{array}{rr} \textit{quotient} = & 5\,8\,7\,5 \\ \textit{divisor} = & 7 \\ \hline \textit{divisor} \times \textit{quotient} = & 4\,1\,1\,2\,5 \\ \textit{remainder} = & 3 \\ \hline (\textit{divisor}) \times (\textit{quotient}) + (\textit{remainder}) = & 4\,1\,1\,2\,8 \end{array}$$

This is equal to the dividend. The answer, therefore, is probably correct.

8. *When a number is divided by 10, 100, 1,000, etc., the quotient is found by simply moving the decimal point as many places to the* **left** *as there are ciphers in the divisor.*

Thus,

$$785.4 \div 10{,}000 = .07854,$$
$$785.4 \div 1{,}000 = .7854,$$
$$785.4 \div 100 = 7.854,$$
$$785.4 \div 10 = 78.54.$$

When the dividend is an integer, the decimal point is understood after the figure in the units place. Hence,

$$52{,}360 \div 1{,}000 = 52{,}360. \div 1{,}000 = 52.36.$$

9. If the product of two numbers is equal to unity, each of the numbers is said to be the **reciprocal** of the other. For example, the product of 100 and .01 is $100 \times .01 = 1$; therefore, .01 is the reciprocal of 100, and 100 is the reciprocal of .01.

10. *Multiplying by any number is the same as dividing by its reciprocal; and dividing by any number is the same as multiplying by its reciprocal.*

For example, multiplying 785.4 by .01 gives $785.4 \times .01 = 7.854$ (Art. **19,** Part 4), and dividing 785.4 by 100 gives $785.4 \div 100 = 7.854$ (Art. **8**); thus, multiplying by .01 gives the same result as dividing by its reciprocal. The product of 1,000 and .001 is $1{,}000 \times .001 = 1$; thus 1,000 is the reciprocal of .001, and .001 is the reciprocal of 1,000. Now, dividing 785.4 by 1,000 gives $785.4 \div 1{,}000 = .7854$ (Art. **8**), and multiplying 785.4 by .001 gives $785.4 \times .001 = .7854$ (Art. **19,** Part 4); thus, dividing by 1,000 is the same as multiplying by its reciprocal.

11. *When a number is divided by .1, .01, .001, etc., the quotient is found by simply moving the decimal point as many places to the* **right** *as there are figures in the divisor.*

By Art. **10,** we have

$$52.36 \div .1 = 52.36 \times 10,$$
$$52.36 \div .01 = 52.36 \times 100,$$
$$52.36 \div .001 = 52.36 \times 1{,}000,$$

and so on.

By Art. **18,** Part 4, we have

$$52.36 \times 10 = 523.6,$$
$$52.36 \times 100 = 5236.,$$
$$52.36 \times 1{,}000 = 52{,}360.,$$

and so on.

Therefore,

$$52.36 \div .1 = 523.6,$$
$$52.36 \div .01 = 5{,}236.,$$
$$52.36 \div .001 = 52{,}360.,$$

and so on.

And we observe that in each case the quotient is found by moving the decimal point as many places to the right as there are figures in the divisor.

12. When there is no remainder, the divisor and quotient are two factors whose product is the dividend. In multiplication of decimals, we multiply as if the factors were integers, and point off as many decimal places in the product as there are decimal places in both factors (Art. **20**, Part 4). Hence, we divide as if the dividend and divisor were integers, and point off as many decimal places in the quotient as the decimal places in the dividend exceed the decimal places in the divisor. Thus,

$$63.4 \times .07 = 4.438,$$

whence $$4.438 \div .07 = 63.4.$$

In this example, there are three decimal places in the dividend, and two in the divisor; therefore, we point off 3 − 2, or 1, decimal place in the quotient.

Again, $$34{,}125 \times .0008 = 27.3,$$

whence $$27.3 \div .0008 = 34{,}125.$$

In this case, the number of decimal places in the divisor exceeds the number in the dividend; but we can make the number of decimal places in the dividend equal to the number in the divisor by annexing three ciphers to the dividend, which ciphers do not change its value (Art. **31**, Part 1). Then, we divide as if the dividend and divisor were integers, and point off no decimal places in the quotient, since the dividend and divisor have the same number of decimal places; thus,

$$27.3 \div .0008 = 27.3000 \div .0008 = 34{,}125.$$

Sometimes ciphers have to be added after the decimal point in the dividend, in order that the dividend may contain the divisor a certain number of times without a remainder; thus, $\$27 \div 4 = \$27.00 \div 4 = \$6.75$.

13. Hence, we have the following rule for division of decimals:

Rule.—I. *If the dividend has fewer decimal places than the divisor, annex ciphers to the dividend until the number of decimal places in the dividend is equal to the number in the divisor.*

II. *Divide as if the dividend and divisor were integers, annexing ciphers to the dividend when necessary to continue the division.*

III. *Point off as many decimal places in the quotient as the number of decimal places in the dividend, including annexed ciphers, exceeds the number in the divisor.*

EXAMPLE 1.—Divide 4.272 by .8.

SOLUTION.— .8 | 4.2 7 2
5.3 4 Ans.

Disregarding the decimal points, we have 4,272 ÷ 8 = 534. In the quotient we must point off 3 − 1, or 2, decimal places (see III of rule).

CHECK.—

$$\begin{aligned} \textit{quotient} &= 5.34 \\ \textit{divisor} &= .8 \\ \textit{quotient} \times \textit{divisor} &= 4.272 \end{aligned}$$

This is equal to the dividend. The answer, therefore, is correct.

EXAMPLE 2.—Divide 26 inches into 7 equal parts, assuming that measurements are to be made correct to the thousandth part of an inch.

SOLUTION.— 7 | 2 6.0 0 0
3.7 1 4 and 2 thousandths over.

Annex three ciphers to the dividend and continue the division until three decimal places are obtained in the quotient. By annexing more ciphers to the dividend, more decimal places could be obtained in the quotient, and the division would never terminate; by annexing six ciphers to the dividend, we would get

7 | 2 6.0 0 0 0 0 0
3.7 1 4 2 8 5 and 5 millionths over.

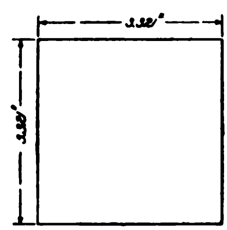
If we continue the division farther, the figures 7, 1, 4, 2, 8, and 5 will repeat again and again in the same order. However, as the measurements are to be made correct to the thousandth part of an inch, the answer 3.714 and 2 thousandths over is all that is needed, and the carrying on of the division to six decimal places was needless work.

CHECK.—	*quotient* =	3.7 1 4
	divisor =	7
		2 5.9 9 8
	remainder =	.0 0 2
	(*quotient*) × (*divisor*) + (*remainder*) =	2 6.0 0 0

This is equal to the dividend.

14. It frequently happens that arithmetical calculations give results containing more figures than are necessary. In any calculation it is a waste of time to carry the calculation to a greater degree of accuracy than is attainable in the actual mechanical measurement of the quantities involved.

15. Suppose that the side of the square in the figure is measured with sufficient care to insure that the measured length of the side is correct to the thousandth part of an inch. This does not mean that the side of the square is exactly 3.321 inches, but that it is nearer to 3.321 inches than to 3.322 inches or to 3.320 inches. Suppose that the actual length of the side is 3.3212 inches; now, 3.322 inches − 3.3212 inches = .0008 inch, and 3.3212 inches − 3.321 inches = .0002 inch, and since .0002 is less than .0008, the supposed actual length, 3.3212 inches is nearer to 3.321 inches than to 3.322 inches; and the length of the side would be taken as 3.321 inches, since the measurements are being made correct only to the thousandth part of an inch. Any length greater than 3.3215 inches is nearer to 3.322 inches than to 3.321 inches; and any length less than 3.3205 inches is nearer to 3.320 inches than to 3.321 inches. Therefore the actual length of the side of this square must lie between 3.3215 inches and 3.3205 inches.



If we could measure it correctly to the ten-thousandth part of an inch, the measure might be 3.3212 inches as supposed above, or 3.3214 inches, or 3.3211 inches, or 3.3209 inches, or any length between 3.3215 inches and 3.3205 inches. The area of the square, by Art. **23**, Part 4, is 3.321 × 3.321 or 11.029041 square inches.

But since it is not known that the length of the side is exactly 3.321 inches, it cannot be asserted that the area is exactly 11.029041 square inches. All that is known is that it is not greater than $(3.3215)^2$, or 11.03236225, nor less than $(3.3205)^2$, or 11.02572025. Now, 11.03236225 is nearer to 11.03 than it is to 11.04 or to 11.02; and 11.02572025 is nearer to 11.03 than to 11.04 or to 11.02. Hence, the area is 11.03 square inches, correct to the hundredth part of a square inch; and with the given degree of accuracy in the measurement of the side, the area cannot be determined to any greater degree of accuracy than the hundredth part of a square inch.

In this example, the side of the square was given correct to four figures, and the calculated area, 11.03 square inches. is also correct to four figures.

16. A number is said to be **correct to four figures** when there are four figures from the first digit at the left of the number to the last accurately determined figure at the right of the number, counting both the first digit and the last accurately determined figure. For example, in Art. **15** the number 3.321, which represents the length of the side of the square, and the number 11.03, which represents its area, are both correct to four figures.

Suppose that the distance between two places is greater than 612,550 feet and less than 612,650 feet; then, correct to four figures, the distance between these two places is 612,600 feet. Again, suppose that a number is known to lie between 3,399.5 and 3,400.5; then, correct to four figures, this number is 3,400. Suppose that a decimal lies between .0020065 and .0020075; then, correct to four figures, the number is .002007.

In counting the number of figures to which a number is correct, we must count from the first digit at the left to the last accurately determined figure at the right, both inclusive. Thus, a number lying between 76.1285 and 76.1275 is 76.128, correct to five figures; a number lying between 713.005 and 712.995 is 713.00, or 713, correct to five figures; a number lying between .000265015 and .000265005 is .00026501, correct to five figures.

17. The degree of accuracy attainable in mechanical measurements varies greatly. With good instruments and careful manipulation, it is sometimes possible to obtain the numerical measure of a magnitude correct to five figures; but it is very seldom possible to attain so great accuracy as this. Therefore, in ordinary calculations it is sufficient to obtain results correct to five figures; and in many cases it is sufficient to obtain four, or even three, figures correct.

18. When a calculated number is to be made correct to a given number of figures, the following rule is universally followed:

Rule.—I. *When some of the superfluous figures are in the integral part of the number, these figures are replaced by ciphers, and the decimal part of the number is dropped.*

II. *When the superfluous figures are all in the decimal part of the number, these figures are dropped.*

III. *If the first of the superfluous figures, counting from the left, is 5 or greater than 5, the preceding figure is increased by unity.*

EXAMPLES.

Calculated number.	*Number correct to five figures.*
6,897,543	6,897,500 (see I).
16,543,691	16,544,000 (see I and III).
219,996	220,000 (see I and III).
53.18912	53.189 (see II).
3.1415926	3.1416 (see II and III).
2.869976	2.87 (see II and III).

19. We will now apply the principle of Art. **18** to some examples in division.

EXAMPLE 1.—Divide 1.36 by 11, and find the quotient correct to five figures.

SOLUTION.— 11 | 1.3 6 0 0 0 0
.1 2 3 6 3 6 and .0 0 0 0 0 4 over.

Since the answer is to be correct to five figures, the sixth figure of the quotient must be obtained in order to determine whether the fifth figure is to be increased by unity or not. Since the sixth figure is 6, the fifth figure must be increased by unity when five figures only are retained (see III of rule, Art. **18**). Therefore, correct to five figures, the quotient is .12364. Ans.

EXAMPLE 2.—Divide .002561 by .08 correct to four figures.

SOLUTION.— .0 8 | .0 0 2 5 6 1 0 0
.0 3 2 0 1 2 and .0 0 0 0 0 0 0 4 over.

Counting from 3, the first digit at the left, the fifth figure is 2; since 2 is less than 5, the preceding figure is not to be increased when four figures only are retained (see III of rule, Art. **18**). Hence, correct to four figures, the quotient is .03201. Ans.

EXAMPLE 3.—Divide 90.997 by .7 correct to five figures.

SOLUTION.— .7 | 9 0.9 9 7 0
1 2 9.9 9 5 and .0 0 0 5 over.

Here the sixth figure of the quotient is 5, and, consequently, the fifth figure must be increased by unity when five figures only are retained. Therefore, correct to five figures, the quotient is 130.00, or 130. Ans.

EXAMPLES FOR PRACTICE.

Find the quotient in each of the following examples:

1.	$7.35 ÷ 3.	Ans. $2.45.
2.	71.34 ÷ 6.	Ans. 11.89.
3.	12.348 ÷ .9.	Ans. 13.72.
4.	1.0044 ÷ .01.	Ans. 100.44.

Find the quotient in each of the following examples correct to three figures.

5.	.17 ÷ 3.	Ans. .0567.
6.	1.37 ÷ 8.	Ans. .171.

20. When the divisor, disregarding the decimal point, exceeds 12, the work of division is arranged in the form illustrated in the following examples. This form is called **long division.** The process of long division is exactly like that of short division, except that in long division the work is written in full.

EXAMPLE 1.—Divide 79.326 by 3.9.

```
SOLUTION.— dividend = 7 9.3 2 6 | 3.9     = divisor
                      7 8       | 2 0.3 4 = quotient.  Ans.
                      ---
                        1 3 2
                        1 1 7
                        -----
                          1 5 6
                          1 5 6
                          -----
```

EXPLANATION.—Place the divisor at the right of the dividend and the quotient under the divisor. Pay no attention to the decimal point until the division is completed (Art. **13**). Since 39 is contained in 79, 2 times, the first figure of the quotient is 2; place 2 in the quotient, multiply the divisor 39 by 2, and subtract the product 78 from 79. The remainder is 1; beside this remainder write the next figure of the dividend, making 13. Now, 39 is not contained in 13; therefore, put 0 in the quotient and bring down the next figure of the dividend, making 132. The divisor 39 is contained in 132, 3 times; put 3 in the quotient, and subtract 39×3, or 117, from 132. The remainder is 15; beside this remainder bring down the next figure of the dividend, making 156. The divisor 39 is contained in 156, 4 times without a remainder. Hence, the quotient is 20.34.

CHECK.—Whenever it is possible, the student should make a rough estimate of the answer before beginning any calculation. In this way, he will readily detect very large mistakes, such as making the answer 100 times too large or 100 times too small by misplacing the decimal point. In this case, $\frac{79.326}{3.9}$ is nearly equal to $\frac{80}{4}$ or 20; therefore, we know that the answer cannot differ much from 20; this would enable us to detect at once a misplacement of the decimal point. Further, applying the usual check for division, we have

$$\begin{array}{rr} \textit{quotient} = & 2\,0.3\,4 \\ \textit{divisor} = & 3.9 \\ \hline & 1\,8\,3\,0\,6 \\ & 6\,1\,0\,2 \\ \hline (\textit{quotient}) \times (\textit{divisor}) = & 7\,9.3\,2\,6 \end{array}$$

This is equal to the dividend.

EXAMPLE 2.—Divide 32.4 by .25.

SOLUTION.— *dividend* = 3 2.4 0 0 | .2 5 = *divisor*
2 5 | 1 2 9.6 = *quotient*. Ans.
7 4
5 0
2 4 0
2 2 5
1 5 0
1 5 0

EXPLANATION.—Annex one cipher to the dividend to make the number of decimal places in the dividend equal to the number in the divisor (Art. **13**). The dividend, then, is 32.40; then, we neglect the decimal point and divide 3,240 by 25. The divisor 25 is contained in 32, 1 time and 7 over; the divisor 25 is contained in 74, 2 times and 24 over; the divisor 25 is contained in 240, 9 times and 15 over. All the figures in the dividend are now exhausted; annex another cipher to the dividend and continue the division. The divisor 25 is contained in 150, 6 times without a remainder. The number of decimal places in the dividend, including the two annexed ciphers, is 3, and the number of decimal places in the divisor is 2. Therefore, in the quotient there must be 3 – 2, or 1, decimal place, which makes the quotient 129.6.

EXAMPLE 3.—Divide .0768 by 2.4.

SOLUTION.— *dividend* = .0 7 6 8 | 2.4 = *divisor*
7 2 | .0 3 2 = *quotient*. Ans.
4 8
4 8

EXPLANATION.—The divisor 24 is contained in 76, 3 times and 4 over. Beside this remainder bring down the next figure of the dividend, making 48. Then, $48 \div 24 = 2$. Since there are 4 decimal places in the dividend and 1 decimal place in the divisor, in the quotient there must be $4 - 1$, or 3, decimal places. In order to point off 3 decimal places in the quotient, we must prefix a cipher; thus, the quotient is .032.

EXAMPLE 4.—Divide 199 by 1.5 correct to four figures.

```
SOLUTION.— dividend = 1 9 9.0 0 0 | 1.5      = divisor
                      1 5         | 1 3 2.6 6 = quotient
                      ---
                        4 9
                        4 5
                        ---
                          4 0
                          3 0
                          ---
                          1 0 0
                            9 0
                            ---
                            1 0 0
                              9 0
                              ---
                              1 0
```

The fifth figure of the quotient is greater than 5; therefore, the fourth figure must be increased by unity when four figures only are retained (see III of rule, Art. **18**). Hence, correct to four figures, the quotient is 132.7. Ans.

21. Those that have little practice in long division usually find difficulty in determining the successive figures of the quotient. No definite rule can be laid down for determining at a glance each successive figure of the quotient; but some useful hints are given in connection with the following examples.

EXAMPLE 1.—Divide 22,878 by 93.

```
SOLUTION.—dividend = 2 2 8 7 8 | 9 3   = divisor
                     1 8 6     | 2 4 6 = quotient.  Ans.
                     -----
                       4 2 7
                       3 7 2
                       -----
                         5 5 8
                         5 5 8
                         -----
```

EXPLANATION.—The divisor 93 is not contained in 22, and, therefore, to find the first figure of the quotient, we must determine how often 93 is contained in 228. Now, 9 is contained in 22, 2 times; hence, we conclude that probably 93 is contained in 228, 2 times, and we try 2 for the first figure of the quotient. When the product 186 ($= 93 \times 2$) is subtracted from 228, the remainder is 42, and bringing down the next figure, 7, we get 427. To find the second figure of the quotient, we must find how often 93 is contained in 427. Since 9 is contained in 42, 4 times, it is probable that 93 is contained in 427, 4 times; therefore, we take 4 as the second figure of the quotient. When the product 372 ($= 93 \times 4$) is subtracted from 427, the remainder is 55, and bringing down the last figure of the dividend, we get 558. Since 9 is contained in 55, 6 times, it is probable that 93 is contained in 558, 6 times; therefore, we take 6 as the third figure of the quotient and we find that 93 is contained in 558, 6 times exactly.

EXAMPLE 2.—Divide 117,552 by 496.

SOLUTION.—*dividend* = 1 1 7 5 5 2 | 4 9 6 = *divisor*

```
        1 1 7 5 5 2 | 4 9 6 = divisor
          9 9 2     | 2 8 7 = quotient.   Ans.
          -----
          1 8 3 5
          1 4 8 8
          -------
            3 4 7 2
            3 4 7 2
            -------
```

EXPLANATION.—The first figure of the quotient is found by dividing 1,175 by 496. Now, 496 is nearly equal to 500, hence, we shall get a good suggestion for the first figure of the quotient by dividing 1,175 by 500, or by dividing 11 by 5. This suggests 2 as the first figure of the quotient. The second figure of the quotient is found by dividing 1,835 by 496. Since 5 is contained in 18, 3 times, we try 3 as the second figure of the quotient. The third figure of the quotient is found by dividing 3,472 by 496. Since 34 contains 5 very nearly 7 times, we try 7 as the third figure.

22. When the second figure of the divisor is less than 5, the successive figures of the quotient can be obtained by using the first digit of the divisor in the manner shown in example 1, Art. **21.** When the second figure of the divisor exceeds 5, the successive figures of the quotient are suggested by increasing the first digit of the divisor by unity, and using this number in the manner shown in example 2, Art. **21.**

EXAMPLES FOR PRACTICE.

Perform the following divisions:

1. .015477 ÷ .3685. Ans. .042.
2. 61.875 ÷ 18.75. Ans. 3.3.
3. 167.22 ÷ 4.5. Ans. 37.16.
4. 2.565 ÷ 3.8. Ans. .675.

In each of the following examples, find the quotient correct to three figures:

5. 101.7 ÷ 2.36. Ans. 43.1.
6. .982 ÷ 76.4. Ans. .0129.

ARITHMETIC.

(PART 6.)

INVOLUTION AND EVOLUTION.

POWERS AND EXPONENTS.

1. If a product consists of equal factors, it is called a **power** of one of those equal factors, and one of the equal factors is called a **root** of the product. The power and the root are named according to the number of equal factors in the product. Thus, 3×3, or 9, is the *second power*, or *square*, of 3; $3 \times 3 \times 3$, or 27, is the *third power*, or *cube*, of 3; $3 \times 3 \times 3 \times 3$, or 81, is the *fourth power* of 3. Also, 3 is the *second root*, or *square root*, of 9; 3 is the *third root*, or *cube root*, of 27; 3 is the *fourth root* of 81.

2. For the sake of brevity,

$$3 \times 3 \text{ is written } 3^2,$$
$$3 \times 3 \times 3 \text{ is written } 3^3,$$
$$3 \times 3 \times 3 \times 3 \text{ is written } 3^4,$$

and so on.

A number written above and to the right of another number, to show how often the latter number is used as a factor, is called an **exponent.** Thus, in 3^{12}, the number ^{12} is the exponent, and shows that 3 is to be used as a factor twelve times; so that 3^{12} is a contraction for

$$3 \times 3 \times 3 \times 3 \times 3 \times 3 \times 3 \times 3 \times 3 \times 3 \times 3 \times 3.$$

In an expression like 3^6, the exponent ^{6} shows how often 3 is used as a factor. Hence, if the exponent of a number is unity, the number is used once as a factor; thus, $3^1 = 3$, $4^1 = 4$, $5^1 = 5$.

3. If the side of a square contains 5 inches, the area of the square contains 5×5, or 5^2, square inches (Art. **23**, Part 4). If the edge of a cube contains 5 inches, the volume of the cube contains $5 \times 5 \times 5$, or 5^3, cubic inches (Art. **32**, Part 4). It is for this reason that 5^2 and 5^3 are called the square and cube of 5, respectively.

INVOLUTION.

4. The process of raising a number to any required power is called **involution.**

EXAMPLE.—What is the fifth power of 2?

SOLUTION.—The fifth power of 2 is

$$2^5 = 2 \times 2 \times 2 \times 2 \times 2 = 32. \quad \text{Ans.}$$

5. *Involution is merely a particular case of continued multiplication in which all the factors are equal.*

EXAMPLE 1.—Find the cube of 1.3.

SOLUTION.—The cube of 1.3 is

$$1.3^3 = 1.3 \times 1.3 \times 1.3 = 2.197. \quad \text{Ans.}$$

EXAMPLE 2.—Find the value of $1.6^2 \times 1.5^2$.

SOLUTION.—We have $1.6^2 = 2.56$ and $1.5^2 = 2.25$.
Therefore,

$$1.6^2 \times 1.5^2 = 2.56 \times 2.25 = 5.76. \quad \text{Ans.}$$

6. Power of a Product.—*A power of the product of two or more factors is equal to the product of the same powers of the factors.*

For example,

$$21^2 = (3 \times 7)^2 = 3^2 \times 7^2,$$
$$(290)^2 = (29 \times 10)^2 = 29^2 \times 10^2,$$
and
$$(1.47)^2 = (147 \times .01)^2 = 147^2 \times .01^2.$$

EVOLUTION.

7. Finding a root of a number is commonly called **extracting the root,** and the process of extracting a root is called **evolution.** Evolution is the inverse of involution; thus, by involution, the square of 13 is $13^2 = 169$, and by evolution the square root of 169 is 13. The symbol $\sqrt{\ }$ is called the **radical sign;** it indicates that a root is to be extracted. The particular root to be extracted is indicated by a number placed before the radical sign; this number is called the **index** of the root and is usually printed in small type in the manner shown in the following examples:

Thus, $\sqrt[2]{9}$ means the *square root* of 9.
$\sqrt[3]{27}$ means the *cube root* of 27.
$\sqrt[4]{81}$ means the *fourth root* of 81.

In the case of the square root, the index is usually omitted; thus, $\sqrt{4}$ is written for $\sqrt[2]{4}$.

8. A straight line drawn over two or more numbers is called a **vinculum.** A vinculum has the same effect as a parenthesis, and indicates that the numbers covered by it are to be subjected to the same operations. Thus, $\sqrt{9 + 16}$ means that 16 is to be added to 9 and then the square root of the sum is to be extracted. In like manner, $\sqrt{4 \times 49}$ means that 49 is to be multiplied by 4 and then the square root of the product is to be extracted.

9. Root of a Product.—*The root of the product of two or more factors is equal to the product of the roots of those factors.*

For example,

$$\sqrt{9 \times 16} = \sqrt{9} \times \sqrt{16}.$$

It is easy to see that this equation is correct. For, $\sqrt{9 \times 16} = \sqrt{144}$, and $12^2 = 144$; therefore, $\sqrt{9 \times 16} = \sqrt{144} = 12$; also, $\sqrt{9} \times \sqrt{16} = 3 \times 4 = 12$. Hence, $\sqrt{9 \times 16} = \sqrt{9} \times \sqrt{16}$.

10. The extraction of a square root by the arithmetical method is an operation that has to be performed very frequently by practical men; therefore, the arithmetical method of extracting the square root of a number is explained in the succeeding articles. It is usual in textbooks on arithmetic to give a somewhat similar method of extracting cube root, but the process is so long and so laborious that it is very rarely used. Therefore, the arithmetical method of extracting cube root is not given in this arithmetic. In practice, cube roots can be obtained easily from a table of cubes, as explained in Arts. **26** to **35.**

SQUARE ROOT.

11. A number that is the square of another number is called a **perfect square**; for example, 1.21 is a perfect square, for it is the square of 1.1.

12. Of the integers that can be expressed by a single digit, the smallest is 1 and the largest is 9; of the integers that can be expressed by two figures, the smallest is 10 and the largest is 99; of the integers that can be expressed by three figures, the smallest is 100 and the largest is 999; and so on.

By multiplication, we find that

$$1^2 = 1, \qquad 9^2 = 81,$$
$$10^2 = 100, \qquad 99^2 = 9,801,$$
$$100^2 = 10,000, \qquad 999^2 = 998,001,$$

and so on.

Therefore, the square of any number expressed by

1 digit contains either 1 or 2 digits,
2 figures contains either 3 or 4 figures,
3 figures contains either 5 or 6 figures,
4 figures contains either 7 or 8 figures,

and so on.

Hence, we have the following principle:

13. *The number of figures in the square root of an integer that is a perfect square can be determined by separating the integer into periods of two figures each, beginning at the right. The number of these periods is equal to the number of figures in the root. If the number of figures in the given integer is odd, the period at the left consists of a single digit only.*

EXAMPLE 1.—The integer 31,505,769 is a perfect square; determine the number of figures in its square root.

SOLUTION.—Beginning at the right we separate the given integer into periods of two figures each; thus, 31'50'57'69. Since the number of these periods is 4, the number of figures in the root is 4. Ans.

EXAMPLE 2.—The integer 841 is a perfect square; find the number of figures in its square root.

SOLUTION.—Beginning at the right, we separate the given integer into periods of two figures each; thus, 8'41. Since there are 2 of these periods, the number of figures in the square root is 2. Ans.

14. *If the integral part of a number is separated into periods of two figures each, beginning at the decimal point, the number of these periods is equal to the number of figures in the integral part of its square root. The period at the left may consist of a single digit only.*

EXAMPLE 1.—How many figures are in the integral part of the square root of 12,795.84?

SOLUTION.—Beginning at the decimal point, separate the integral part into periods of two figures each; thus, 1'27'95.84. Since there are 3 of these periods, the number of figures in the integral part of the square root is 3. Ans.

EXAMPLE 2.—Determine the number of figures in the integral part of the square root of 311.

SOLUTION.—Beginning at the decimal point, separate the number into periods of two figures each; thus, 3'11. Since there are 2 periods, the number of figures in the integral part of the root is 2. Ans.

15. The number of decimal places in the product of two factors is equal to the number of decimal places in both factors (Art. **20,** Part 4). Therefore, the square of any number contains twice as many decimal places as the number itself. For example, 2.5 contains one decimal place, and its square is 6.25, which has two decimal places. Hence, we have the following principle:

16. *The number of decimal places in the square root of a number that is a perfect square can be found by separating the decimal part of the square number into periods of two figures each, beginning at the decimal point. The number of these periods is equal to the number of decimal places in the square root.*

EXAMPLE 1.—The number 7.890481 is a perfect square. How many decimal places are in its square root?

SOLUTION.—Beginning at the decimal point, separate the decimal part of the given number into periods of two figures each; thus. 7.89'04'81. Since there are 3 of these periods, the number of decimal places in the square root is 3. Ans.

EXAMPLE 2.—The number .000225 is a perfect square. How many decimal places are in its square root?

SOLUTION.—Beginning at the decimal point, separate the number into periods of two figures each; thus, .00'02'25. Since there are 3 of these periods, the number of decimal places in the square root is 3. Ans.

17. The following is the rule for extracting the square root of a number:

Rule.—I. *Counting left and right from the decimal point, separate the number into periods of two figures each.*

II. *Find the greatest number whose square is contained in the first, or left-hand, period. Write this number as the first figure of the required root; subtract its square from the first period, and bring down the second period to form the first* **partial dividend.**

III. *Double the part of the root already found, and annex a cipher to it for the* **trial divisor.**

IV. *Divide the trial divisor into the first partial dividend, and take the quotient as the second figure of the root.*

Annex the quotient to the root, and add it to the trial divisor to form the **complete divisor.**

V. *Multiply the complete divisor by the last obtained figure of the root, and subtract the product from the partial dividend. Should the product exceed the partial dividend, then we have taken too large a number for the second figure of the root, and we must try a smaller number.*

VI. *Bring down the third period of the number to form the second partial dividend and repeat operations.*

VII. *If at any stage the trial divisor is not contained in the partial dividend, annex a cipher to the root and also to the trial divisor, and bring down another period.*

VIII. *As soon as all the periods in the integral part of the given number are exhausted, insert the decimal point in the root, and bring down the periods of the decimal part of the given number.*

EXAMPLE 1.—Extract the square root of 154.5049.

SOLUTION.—

trial divisor	*complete divisor*		*root*
		1'5 4.5 0'4 9 (	1 2.4 3 Ans.
		1	
20	22	5 4	
		4 4	
240	244	1 0 5 0	
		9 7 6	
2480	2483	7 4 4 9	
		7 4 4 9	

EXPLANATION.—Counting left and right from the decimal point, we separate the given number into periods of two figures each; thus, 1'54.50'49 (see I of rule). Since there are 2 periods in the integral part of the given number, the number of figures in the integral part of its square root is 2 (Art. **14**). There are 2 periods in the decimal part of the given number; hence, if the given number is a perfect square, there are 2 figures in the decimal part of its square root (Art. **16**). The first period at the left is 1, and its square root is 1. Subtracting 1 ($= 1^2$) from the first period, and bringing down the second period, 54, we get the first partial dividend 54 (see II of rule). The double of the part of the root already found is 2 (= 2 × 1); and annexing a cipher to this 2, we get 20 as the first trial divisor (see III of rule). The trial divisor 20 is contained in the partial dividend 54, 2 times, which suggests 2 as the next figure of the root (see IV of rule). Adding the 2 to the trial divisor 20, we get 22 as the complete divisor (see IV of rule). When 22 is multiplied by 2 and the product 44 subtracted from the partial dividend, the remainder is 10 (see V of rule). Since the periods of the integral part of the given number have all been brought down, we insert the decimal point after the 2 in the root before bringing down the first period of the decimal part (see VIII of rule). Bringing down the next period, 50, we get the second partial dividend 1,050 (see VI of rule). The double of the part of the root already found is 24 (= 2 × 12); and annexing a cipher to this 24, we get the second trial divisor 240 (see III of rule). The trial divisor 240 is contained in the partial dividend 1,050, 4 times, which suggests 4 as the next figure of the root (see IV of rule). Adding 4 to the trial divisor 240, we get the complete divisor 244 (see IV of rule). When the complete divisor 244 is multiplied by 4 and the product 976 subtracted from the partial dividend 1,050, the remainder is 74 (see V of rule). Bringing down the fourth period 49, we get the third partial dividend 7,449 (see VI of rule). The double of the part of the root already found is 248 (= 2 × 124), and annexing a cipher to 248, we get the trial divisor 2,480 (see III of rule). The trial divisor 2,480 is contained in the partial dividend 7,449, 3 times, and this suggests 3 as the fourth figure of the root (see IV of rule). Adding 3 to the trial divisor 2,480, we get 2,483 as the complete divisor. When the product 7,449 (= 2,483 × 3) is subtracted from the partial dividend 7,449, there is no remainder. All the periods of the given number are exhausted, and there is no remainder. Therefore, the square root of 154.5049 is 12.43.

CHECK.—Squaring 12.43, we get

$$12.43^2 = 154.5049.$$

EXAMPLE 2.—Extract the square root of 258.2449.

SOLUTION.—

trial divisor	*complete divisor*	*root*
		2'5 8.2 4'4 9 (1 6.0 7 Ans.
		1
2 0	2 6	1 5 8
		1 5 6
3 2 0 0	3 2 0 7	2 2 4 4 9
		2 2 4 4 9

EXPLANATION.—Counting left and right from the decimal point, we separate the given number into periods of two figures each; thus 2'58.24'49 (see I of rule). This shows that there are 2 figures in the integral part of the root. The greatest number whose square is contained in the left-hand period 2 is 1, and, therefore, 1 is the first figure of the root (see II of rule). Subtracting 1 ($= 1^2$) from the left-hand period 2, and bringing down the next period, 58, we get the first partial dividend 158 (see II of rule). The double of 1, the part of the root already found, is 2 ($= 2 \times 1$), and annexing a cipher to this 2, we get the first trial divisor 20 (see III of rule). The trial divisor 20 is contained in the partial dividend 158, 7 times, which suggests 7 as the second figure of the root. Adding 7 to the trial divisor 20, we get the complete divisor 27. But $27 \times 7 = 169$, which is greater than the partial dividend 158, and, therefore, 7 is too large a number for the second figure of the root (see V of rule). Taking 6 instead of 7 as the second figure of the root, we get 26 as the complete divisor. Subtracting the product 156 ($= 26 \times 6$) from the partial dividend 158, we get the remainder 2 (see V of rule). Since all the integral periods have been brought down, we insert the decimal point after the 6 in the root (see VIII of rule). Bringing down the next period, 24, we get 224 as the second partial dividend (see VI of rule). The second trial divisor 320 is not contained in the second partial dividend 224; therefore, we put 0 in the root, annex a cipher to the trial divisor, and bring down the next period (see VII of rule). The third partial dividend is 22,449, and the third trial divisor is 3,200.

Since 3,200 is contained in 22,449, 7 times, the next figure of the root is 7. Adding 7 to 3,200, we get the complete divisor 3,207. When the product 22,449 (= 3,207 × 7) is subtracted from the partial dividend 22,449, there is no remainder. Therefore, 16.07 is the required square root.

CHECK.—By multiplication, we get

$$16.07^2 = 258.2449.$$

EXAMPLE 3.—Find the square root of .000576.

SOLUTION.—

trial divisor	*complete divisor*	*root*
		.0 0'0 5'7 6 (.0 2 4 Ans.
		4
40	44	176
		176

EXPLANATION.—Since the first period is 00, the first figure of the root must be 0. The second period is 05, or 5. The greatest number whose square is contained in 5 is 2; therefore, 2 is the second figure of the root. Subtracting 4 ($= 2^2$) from the 5, and bringing down the next period, 76, we get the first partial dividend 176. The double of 2, the part of the root already found is 4 (= 2 × 2), and annexing a cipher to this 4, we get the first trial divisor 40. The trial divisor 40 is contained in the partial dividend 176, 4 times; hence, the next figure of the root is 4. Adding 4 to 40, we get the complete divisor 44. When the product 176 (= 44 × 4) is subtracted from the partial dividend 176, there is no remainder. Therefore, .024 is the required square root.

CHECK.—By multiplication, we get

$$.024^2 = .000576.$$

18. The square root of a number that is not a perfect square can be obtained correct to any required number of figures.

EXAMPLE 1.—Find the square root of 3 correct to four figures.

SOLUTION.—

trial divisor	*complete divisor*	*root*
		3.0 0′0 0′0 0 (1.7 3 2 Ans.
		1
2 0	2 7	2 0 0
		1 8 9
3 4 0	3 4 3	1 1 0 0
		1 0 2 9
3 4 6 0	3 4 6 2	7 1 0 0
		6 9 2 4
3 4 6 4 0		1 7 6 0 0

EXPLANATION.—Since we require four figures in the root, we must annex ciphers enough after the decimal point to give 4 periods when the number is separated into periods of 2 figures each; thus, 3.00′00′00. Then, we proceed as in the preceding examples. After the fourth figure, 2, of the root is obtained, the remainder is 176. Annexing two ciphers to this remainder, we get the next partial dividend 17,600; and the next trial divisor 34,640 is not contained in the partial dividend 17,600; therefore, the fifth figure of the root is 0. Hence, correct to four figures, the square root of 3 is 1.732.

NOTE.—The exact square root of 3 cannot be found; in other words, 3 is not a perfect square. Although we cannot find a number whose square is exactly equal to 3, yet by continuing the process of extracting the square root far enough, we can find a number whose square differs from 3 by as small an amount as we please. Thus, $(1.732)^2 = 2.999824$, which differs from 3 by less than .0002; and, correct to four figures, we have $(1.732)^2 = 3$.

EXAMPLE 2.—Find the square root of 1.16964 correct to four figures.

SOLUTION.—

trial divisor	*complete divisor*	*root*
		1.1 6′9 6′4 0 (1.0 8 1 Ans.
		1
2 0 0	2 0 8	1 6 9 6
		1 6 6 4
2 1 6 0	2 1 6 1	3 2 4 0
		2 1 6 1
2 1 6 2 0		1 0 7 9 0 0

Explanation.—To get four figures of the root, the given number must be divided into 4 periods; therefore, we annex a cipher to the given number in order to complete the third period in the decimal part. After the fourth figure, 1, of the root is obtained, the remainder is 1,079; forming the next partial dividend 107,900 and the next trial divisor 21,620, we see that the fifth figure of the root is less than 5. Therefore, correct to four figures, the root is 1.081.

EXAMPLES FOR PRACTICE.

Find the square root of each of the following numbers correct to four figures:

1. 186,624. Ans. 432.
2. 198.137. Ans. 14.08.
3. .003025. Ans. .055.

USE OF TABLE OF SQUARES AND CUBES.

19. The accompanying table gives the square and cube of every number from 1 to 99, inclusive. The squares are divided into periods of two figures each in accordance with Art. **13.** The cubes are divided into periods of three figures each in accordance with a principle of Art. **29.**

20. In any number, the figures beginning with the first digit at the left and ending with the last digit at the right are called the **significant figures** of the number. Thus, the number 405,800 has the four significant figures 4, 0, 5, 8; and the number .000090067 has the five significant figures 9, 0, 0, 6, and 7.

The part of a number consisting of its significant figures is called the **significant part** of the number. Thus, in the number 28,070, the significant part is 2807; in the number .00812, the significant part is 812; and in the number 170.3, the significant part is 1703.

In speaking of the significant figures or of the significant part of a number, we consider the figures, in their proper order, from the first digit at the left to the last digit at the right, but we pay no attention to the position of the decimal

SQUARES AND CUBES.

No.	Square.	Cube.	No.	Square.	Cube.	No.	Square.	Cube.
1	1	1	34	11'56	39'304	67	44'89	300'763
2	4	8	35	12'25	42'875	68	46'24	314'432
3	9	27	36	12'96	46'656	69	47'61	328'509
4	16	64	37	13'69	50'653	70	49'00	343'000
5	25	125	38	14'44	54'872	71	50'41	357'911
6	36	216	39	15'21	59'319	72	51'84	373'248
7	49	343	40	16'00	64'000	73	53'29	389'017
8	64	512	41	16'81	68'921	74	54'76	405'224
9	81	729	42	17'64	74'088	75	56'25	421'875
10	1'00	1'000	43	18'49	79'507	76	57'76	438'976
11	1'21	1'331	44	19'36	85'184	77	59'29	456'533
12	1'44	1'728	45	20'25	91'125	78	60'84	474'552
13	1'69	2'197	46	21'16	97'336	79	62'41	493'039
14	1'96	2'744	47	22'09	103'823	80	64'00	512'000
15	2'25	3'375	48	23'04	110'592	81	65'61	531'441
16	2'56	4'096	49	24'01	117'649	82	67'24	551'368
17	2'89	4'913	50	25'00	125'000	83	68'89	571'787
18	3'24	5'832	51	26'01	132'651	84	70'56	592'704
19	3'61	6'859	52	27'04	140'608	85	72'25	614'125
20	4'00	8'000	53	28'09	148'877	86	73'96	636'056
21	4'41	9'261	54	29'16	157'464	87	75'69	658'503
22	4'84	10'648	55	30'25	166'375	88	77'44	681'472
23	5'29	12'167	56	31'36	175'616	89	79'21	704'969
24	5'76	13'824	57	32'49	185'193	90	81'00	729'000
25	6'25	15'625	58	33'64	195'112	91	82'81	753'571
26	6'76	17'576	59	34'81	205'379	92	84'64	778'688
27	7'29	19'683	60	36'00	216'000	93	86'49	804'357
28	7'84	21'952	61	37'21	226'981	94	88'36	830'584
29	8'41	24'389	62	38'44	238'328	95	90'25	857'375
30	9'00	27'000	63	39'69	250 047	96	92'16	884'736
31	9'61	29'791	64	40'96	262'144	97	94'09	912'673
32	10'24	32'768	65	42'25	274'625	98	96'04	941'192
33	10'89	35'937	66	43'56	287'496	99	98'01	970'299

point. Hence, *all numbers that differ only in the position of the decimal point have the same significant part.* For example, .002103, 21.03, 21,030, and 210,300 have the same significant figures 2, 1, 0, and 3, and the same significant part 2103.

SQUARES AND CUBES FOUND FROM TABLE.

21. When the significant part of a given number is contained in one of the columns headed "No." in the table of "Squares and Cubes," the square and cube of the given number can be obtained without any calculation.

EXAMPLE 1.—Find the square and cube of 230.

SOLUTION.—The significant part of the number 230 is 23. In the table we find $23^2 = 529$, and $23^3 = 12,167$. We have

$$230^2 = (23 \times 10)^2 = 23^2 \times 10^2 = 529 \times 100 = 52,900,$$

and

$$230^3 = (23 \times 10)^3 = 23^3 \times 10^3 = 12,167 \times 1,000 = 1,267,000.$$

Thus, $230^2 = 52,900,$
and $230^3 = 1,267,000.$ Ans.

EXAMPLE 2.—Find the square and cube of .045.

SOLUTION.—The significant part of the number .045 is 45. In the table we find $45^2 = 2,025$, and $45^3 = 91,125$. Also, we have

$$.045^2 = \left(\frac{45}{1,000}\right)^2 = \frac{45^2}{1,000^2} = \frac{2,025}{1,000,000} = .002025,$$

and

$$.045^3 = \left(\frac{45}{1,000}\right)^3 = \frac{45^3}{1,000^3} = \frac{91,125}{1,000,000,000} = .000091125.$$

Thus, $.045^2 = .002025,$
and $.045^3 = .000091125.$ Ans.

EXAMPLE 3.—Find the square and cube of 6.4.

SOLUTION.—The significant part of this number is 64. In the table we find $64^2 = 4,096$, and $64^3 = 262,144$. And we have

$$6.4^2 = \left(\frac{64}{10}\right)^2 = \frac{64^2}{10^2} = \frac{4,096}{100} = 40.96,$$

and

$$6.4^3 = \left(\frac{64}{10}\right)^3 = \frac{64^3}{10^3} = \frac{262,144}{1,000} = 262.144.$$

Thus, $6.4^2 = 40.96,$
and $6.4^3 = 262.144.$ Ans.

22. When the significant part of a number is not contained in one of the columns headed "No." in the table, the square and cube of the given number are obtained more easily by multiplication, as explained in Art. **5**, than from the table.

SQUARE ROOTS FOUND FROM TABLE.

23. The method of extracting square roots by means of the table is shown in the examples in the succeeding articles.

24. To extract the square root of any number that has two periods (Art. **14**) in its integral part, we proceed as in the following example:

EXAMPLE.—Find the square root of 9,327.

SOLUTION.—Separating the number into periods of two figures each, we get 93'27, which shows that there are two figures in the integral part of the square root (Art. **14**). Referring to the columns headed "Square," in table of "Squares and Cubes," we find that the given number 93'27 lies between 92'16 ($= 96^2$) and 94'09 ($= 97^2$); therefore, $\sqrt{93'27}$ must lie between 96 and 97. Hence, we know that the integral part of the required square root is 96. The first two figures of the decimal part of the required square root can be found in the following manner. Take the difference of the two square numbers in the table between which the given number falls, and call this the **first difference**; take also the difference between the given number and the smaller of the two square numbers in the table between which the given number lies, and call this the **second difference.** *Divide the second difference by the first difference, and add the quotient to the part of the root already found;* the sum will be a close approximation to the required root.

higher number in table =	9 4 0 9	*given number* =	9 3 2 7
lower number in table =	9 2 1 6	*lower number in table* =	9 2 1 6
first difference =	1 9 3	*second difference* =	1 1 1

```
111.00 | 193
 965   | .575 = quotient
 ----
 1450
 1351
 ----
  990
```

part of root found in table = 96.
quotient = .58
required root = 96.58, nearly.

Hence, correct to three figures, $\sqrt{9,409} = 96.6$. Ans.

25. *To extract the square root of any number, we must first express that number as the product of two factors. One of these factors must be a number having two periods* (Art. **13**) *in its integral part; and the other factor must be* *one of the numbers* 100 $(= 10^2)$, 10,000 $(= 100^2)$, *etc.*, *or one of the numbers* .01 $(= .1^2)$, .0001 $(= .01^2)$, *etc.*

EXAMPLE 1.—Find the square root of 1.56.

SOLUTION.—We have $1.56 = 156 \times .01$. Hence,

$$\sqrt{1.56} = \sqrt{156 \times .01} = \sqrt{156} \times \sqrt{.01} = \sqrt{156} \times .1.$$

Now we have to find the square root of 156, and we shall call 156 the given number. Referring to the table, we find that 156 lies between 144 $(= 12^2)$ and 169 $(= 13^2)$; therefore, $\sqrt{156}$ must lie between 12 and 13.

higher number in table =	1 6 9	*given number* =	1 5 6
lower number in table =	1 4 4	*lower number in table* =	1 4 4
first difference =	2 5	*second difference* =	1 2

```
1 2.0 | 2 5
1 0 0 | .4 8 = quotient.
  2 0 0
  2 0 0
```

part of root found in table = 1 2.
quotient = .4 8
required root = 1 2.4 8, nearly.

Thus, we get $\sqrt{156} = 12.48$, nearly, or 12.5 correct to three figures. Hence,

$$\sqrt{1.56} = \sqrt{156} \times .1 = 12.5 \times .1 = 1.25. \quad \text{Ans.}$$

EXAMPLE 2.—Find the square root of 734,894.

SOLUTION.—We have $734{,}894 = 7{,}348.94 \times 100$.
Hence,

$$\sqrt{734{,}894} = \sqrt{7{,}348.94 \times 100} = \sqrt{7{,}348.94} \times \sqrt{100} = \sqrt{7{,}348.94} \times 10.$$

Now we have to find the square root of 7,348.94, and we shall call 7,348.94 the given number. Referring to the table, we find that 7,348.94 lies between 7,225 $(= 85^2)$ and 7,396 $(= 86^2)$. Therefore, $\sqrt{7{,}348.94}$ lies between 85 and 86.

higher number in table =	7 3 9 6	*given number* =	7 3 4 8.9 4
lower number in table =	7 2 2 5	*lower number in table* =	7 2 2 5
	1 7 1		1 2 3.9 4

```
1 2 3.9 4 | 1 7 1
1 1 9 7   | .7 2 = quotient.
  4 2 4
```

part of root found in table = 8 5.
quotient = .7 2
8 5.7 2

Thus, $\sqrt{7,348.94} = 85.72$, nearly, or 85.7 correct to three figures. Hence,

$$\sqrt{734,894} = \sqrt{7,348.94} \times 10 = 85.7 \times 10 = 857. \quad \text{Ans.}$$

EXAMPLES FOR PRACTICE.

Find the square root of

1. 21,160,000.	Ans. 4,600.
2. 3,169.7.	Ans. 56.3.
3. 781,457.	Ans. 884.
4. 53.29.	Ans. 7.3.

CUBE ROOTS FOUND FROM TABLE.

26. A number that is the cube of another number is called a **perfect cube**; for example, 1.728 is a perfect cube, for it is the cube of 1.2.

27. Of the integers that can be expressed by a single digit, the smallest is 1 and the largest is 9; of the integers that can be expressed by two figures, the smallest is 10 and the largest is 99; of the integers that can be expressed by three figures, the smallest is 100 and the largest is 999; and so on.

28. By multiplication, we find that

$$1^3 = 1, \qquad 9^3 = 729,$$
$$10^3 = 1,000, \qquad 99^3 = 970,299,$$
$$100^3 = 1,000,000, \qquad 999^3 = 997,002,999,$$

and so on.

Therefore, the cube of any number expressed by
1 digit contains either 1, 2, or 3 digits,
2 figures contains either 4, 5, or 6 figures,
3 figures contains either 7, 8, or 9 figures,
and so on.

Hence, we have the following principle:

29. *The number of figures in the cube root of any integer that is a perfect cube can be determined by separating the integer into periods of three figures each, beginning at the right. The number of these periods is equal to the number of figures in the root. The period at the left may contain one, two, or three figures.*

EXAMPLE 1.—The number 1,367,631 is a perfect cube; find the number of figures in its cube root.

SOLUTION.—Beginning at the right, we separate the given integer into periods of three figures each; thus, 1'367'631. Since the number of these periods is 3, the number of figures in the root is 3. Ans.

EXAMPLE 2.—The number 10,808,519,931 is a perfect cube; find the number of figures in its cube root.

SOLUTION.—Beginning at the right, we separate the given integer into periods of three figures each; thus, 10'808'519'931. Since the number of these periods is 4, the number of figures in the root is 4. Ans.

EXAMPLE 3.—The number 704,969 is a perfect cube; find the number of figures in its cube root.

SOLUTION.—Beginning at the right, we separate the given integer into periods of three figures each; thus, 704'969. Since the number of these periods is 2, the number of figures in the root is 2. Ans.

30. *If the integral part of a number is separated into periods of three figures each, beginning at the decimal point, the number of these periods is equal to the number of figures in the integral part of its cube root. The period at the left may contain one, two, or three figures.*

EXAMPLE.—How many figures are in the integral part of the cube root of 985,436.521 ?

SOLUTION.—Beginning at the decimal point, we separate the integral part of the given number into periods of three figures each; thus, 985'436.521. Since there are 2 of these periods, the number of integral figures in the root is 2. Ans.

31. The number of decimal places in the product of three factors is equal to the number of decimal places in the three factors. Thus, the number of decimal places in the product $1.21 \times 7.5 \times 8.923$ is $2 + 1 + 3$, or 6; and the number of decimal places in the product $5.241 \times 5.241 \times 5.241$ is $3 + 3 + 3$, or 9.

Hence we have the following principle:

32. *The number of decimal places in the cube root of a number that is a perfect cube can be found by separating the decimal part of the number into periods of three figures each, beginning at the decimal point. The number of these periods is equal to the number of decimal places in its cube root.*

EXAMPLE 1.—The number 130.323843 is a perfect cube; find the number of figures in the decimal part of its cube root.

SOLUTION.—Beginning at the decimal point, we separate the decimal part of the given number into periods of three figures each; thus, 130.323'843. Since there are 2 of these periods, the number of figures in the decimal part of the root is 2. Ans.

EXAMPLE 2.—The number .416663730709 is a perfect cube; find the number of decimal places in its cube root.

SOLUTION.—Beginning at the decimal point, we separate the decimal part of the given number into periods of three figures each; thus, .416'663'730'709. Since there are 4 of these periods, the number of figures in the decimal part of its cube root is 4. Ans.

33. The method of extracting cube roots by means of the table is shown in the examples in the succeeding articles.

34. To extract the cube root of any number that has two periods (Art. **29**) in its integral part, we proceed as in the following example:

EXAMPLE.—Extract the cube root of 436,759.

SOLUTION.—Separating the given number into periods of three figures each, we have 436'759, which shows that there are two figures in the integral part of its cube root (Art. **30**). Referring to the table, we find that 436'759 lies between 421'875 (= 75^3) and 438'976 (= 76^3). Therefore, $\sqrt[3]{436'759}$ lies between 75 and 76.

higher number in table =	438'976	*given number* =	436759
lower number in table =	421'875	*lower number in table* =	421875
first difference =	17101	*second difference* =	14884

```
14884.00 | 17101
136808   |--------
-------  | .87 = quotient
 120320
 119707
 ------
```

part of root found in table = 75.
quotient = .87
75.87

Thus, $\sqrt[3]{436,759} = 75.87$. Ans.

35. *To extract the cube root of any number, we must first express that number as the product of two factors. One of* *these factors must be a number having two periods* (Art. **30**) *in its integral part; and the other factor must be one of the numbers* 1,000 (= 10^3), 1,000,000 (= 100^3), *etc., or one of the numbers* .001 (= $.1^3$), .000001 (= $.01^3$), *etc.*

EXAMPLE.—Find the cube root of 98,954,672.

SOLUTION.—We have

$$98{,}954{,}672 = 98{,}954.672 \times 1{,}000 = 98{,}954.672 \times 10^3.$$

Hence,

$$\sqrt[3]{98{,}954{,}672} = \sqrt[3]{98{,}954.672 \times 10^3} = \sqrt[3]{98{,}954.672} \times \sqrt[3]{10^3}$$
$$= \sqrt[3]{98{,}954.672} \times 10.$$

Now we have to find the cube root of 98,954.672, and we shall call 98,954.672 the given number. Referring to the table, we find that 98'954.672 lies between 97'336 (= 46^3) and 103'823 (= 47^3). Therefore, $\sqrt[3]{98'954.672}$ lies between 46 and 47.

higher number in table =	103823	*given number* =	98954.672
lower number in table =	97336	*lower number in table* =	97336.
first difference =	6487	*second difference* =	1618.672

```
1 6 1 8.6 7 2 | 6 4 8 7
1 2 9 7 4     |  .2 4 9 = quotient
  3 2 1 2 7
  2 5 9 4 8
    6 1 7 9 2
    5 8 3 8 3
```

part of root found in table = 4 6.
quotient = .2 4 9
required root = 4 6.2 4 9, nearly.

Thus, correct to four figures $\sqrt[3]{98{,}954.672} = 46.25$.

Hence, $\sqrt[3]{98{,}954{,}672} = \sqrt[3]{98{,}954.672} \times 10 = 46.25 \times 10 = 462.5$. Ans.

EXAMPLES FOR PRACTICE.

Find the cube root of

1. 135,754. Ans. 51.4.
2. 275,643,221. Ans. 650.8
3. 846.556. Ans. 9.46.

GEOMETRICAL APPLICATIONS.

36. Lay off the line $A\,C$, Fig. 1, 4 inches long; at C, erect a perpendicular $C\,B$ 3 inches long and draw the line $A\,B$. Then $A\,B$ is 5 inches long and $A\,B\,C$ is a right triangle with the right angle at C. If squares are described on the three sides of this triangle, the square on $A\,C$ will contain 16 square inches, the square on $B\,C$ will contain 9 square inches, and the square on $A\,B$ will contain 25 square inches. Now $25 = 9 + 16$; hence, the square on the hypotenuse $A\,B$ is equal to the sum of the squares on the sides $B\,C$ and $A\,C$.

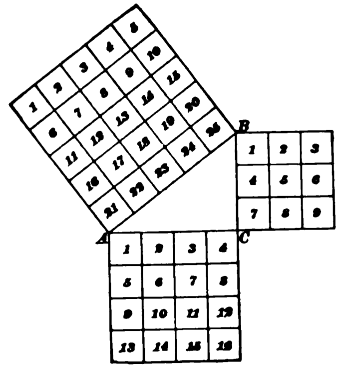


FIG. 1.

This is a particular case of the following very important proposition, which was discovered by the Greek geometer Pythagoras, and is, therefore, called the *Pythagorean proposition.*

37. Pythagorean Proposition.—*In any right triangle the square described on the hypotenuse is equal to the sum of the squares described on the sides.*

Thus, in the right triangle $A\,B\,C$, Fig. 1, the square on the hypotenuse $A\,B$ is equal to the sum of the squares on the sides $A\,C$ and $C\,B$.

EXAMPLE.—The legs of a right triangle are 33 inches and 56 inches, respectively. Find the area of the square described on the hypotenuse.

SOLUTION.—By Art. **3**, the area of the square on the shorter leg is 33^2, or 1,089 square inches; and the area of the square on the longer leg is 56^2, or 3,136 square inches. Hence, the square on the hypotenuse contains 1,089 + 3,136, or 4,225 sq. in. Ans.

38. *In any right triangle, the square described on one leg is equal to the difference between the square described on the hypotenuse and the square described on the other leg.* This is merely another way of stating the Pythagorean proposition (Art. **37**). For, by Art. **36,** the square described on the hypotenuse is equal to the sum of the squares described on the legs; and, therefore, the square described on one leg is equal to the difference between the square described on the hypotenuse and the square described on the other leg.

EXAMPLE.—The hypotenuse of a right-angled triangle is 89 feet and one of the legs is 39 feet. Find the area of the square described on the other leg.

SOLUTION.—By Art. **3,** the area of the square described on the hypotenuse is 89^2, or 7,921 square feet; and the area of the square described on the one leg is 39^2, or 1,521 square feet. Hence, the square described on the other leg contains 7,921 − 1,521, or 6,400 sq. ft. Ans.

39. **I.** *If the sum of the squares on the two shorter sides of a triangle is equal to the square on the longest side, the triangle is a right triangle and the longest side is the hypotenuse.*

II. *If the sum of the squares on the two shorter sides of a triangle is less than the square on the longest side, the triangle is an obtuse triangle and the angle opposite to the longest side is an obtuse angle.*

III. *If the sum of the squares on the two shorter sides of a triangle is greater than the square on the longest side, the triangle is an acute triangle.*

EXAMPLE 1.—In the triangle ABC, AB is 79 inches, BC is 70 inches, and CA is 29 inches. Determine whether the angle C, which is opposite the longest side, is acute, right, or obtuse.

SOLUTION.—We have $70^2 + 29^2 = 4{,}900 + 841 = 5{,}741$, and $79^2 = 6{,}241$. Thus, the sum of the squares on the two shorter sides is less than the square on the longest side. The triangle, therefore, is obtuse (see II), and the angle C is obtuse. Ans.

EXAMPLE 2.—The sides of a triangle measure 65, 72, and 97 inches, respectively. Determine whether the triangle is a right triangle.

SOLUTION.—We have $65^2 + 72^2 = 4{,}225 + 5{,}184 = 9{,}409$, and $97^2 = 9{,}409$. Thus, the sum of the squares on the two shorter sides is equal to the square on the longest side; therefore, the triangle is a right triangle. Ans.

EXAMPLE 3.—The sides of a triangle measure 87 inches, 416 inches, and 415 inches, respectively. Determine whether this is a right, obtuse, or acute triangle.

SOLUTION.—We have $87^2 + 415^2 = 7,569 + 172,225 = 179,794$, and $416^2 = 173,056$. Thus, the sum of the squares on the two shorter sides is greater than the square on the longest side. The triangle, therefore, is acute. Ans.

EXAMPLES FOR PRACTICE.

Find the value of

1. 9.6^2. Ans. 92.16.
2. 830^3. Ans. 571,787,000.
3. 5.5^3. Ans. 166.375.

Solve the following examples:

4. The lengths of the legs of a right triangle are 4.5 and 3.9 inches, respectively. Find the area of the square described on the hypotenuse. Ans. 35.46 sq. in.

5. The lengths of the hypotenuse and one leg of a right triangle are 8.5 and 7.7 inches, respectively. Find the area of the square described on the other leg. Ans. 12.96 sq. in.

6. The lengths of the sides of a triangle are 16, 63, and 67 inches. Determine whether the triangle is right, acute, or obtuse. Ans. Obtuse.

7. The lengths of the sides of a triangle are 104, 153, and 183. Determine whether the triangle is right, acute, or obtuse. Ans. Acute.

40. If two numbers are equal, their square roots must be equal; for example, $25 = 9 + 16$, and, therefore,

$$\sqrt{25} = \sqrt{9 + 16}.$$

Hence, in any equation, the square root of the first member is equal to the square root of the second member.

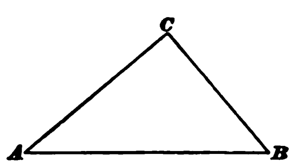


FIG. 2.

41. In the right triangle $A B C$, Fig. 2, the square of the hypotenuse $A B$ is equal to the sum of the squares of the legs $A C$ and $B C$ (Art. **37**). That is,

$$(\text{square of } A B) = (\text{square of } A C) + (\text{square of } B C).$$

By Art. **40,** the square roots of the two members of this equation must be equal. The square root of the first member is AB, and the square root of the second member is

$$\sqrt{(\text{square of } AC) + (\text{square of } BC)}.$$

Hence,

$$AB = \sqrt{(\text{square of } AC) + (\text{square of } BC)}.$$

Thus, we have the following principle:

42. *The hypotenuse of a right triangle is equal to the square root of the sum of the squares of the legs.*

EXAMPLE.—In the right triangle ABC, Fig. 3, find the length of AB.

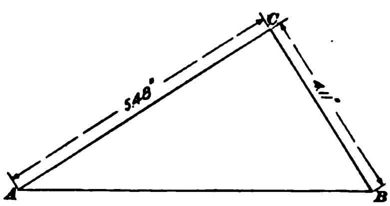


FIG. 3.

SOLUTION.—The square of AC is 5.48^2, or 30.0304 square inches, and the square of BC is 4.11^2, or 16.8921 square inches. Hence, the sum of the squares of the legs is 30.0304 + 16.8921, or 46.9225 square inches. But the hypotenuse AB is equal to the square root of the sum of the squares of the legs AC and BC. Therefore, AB is $\sqrt{46.9225}$, or 6.85 in. Ans.

43. In the right triangle ABC, Fig. 2, the square of the leg AC is equal to the difference between the square of the hypotenuse and the square of the other leg (Art. **38**). That is,

$$(\text{square of } AC) = (\text{square of } AB) - (\text{square of } BC).$$

By Art. **40,** the square roots of the two members of this equation must be equal. The square root of the first member is AC, and the square root of the second member is

$$\sqrt{(\text{square of } AB) - (\text{square of } BC)}.$$

Hence, we have the following principle:

44. *A leg of a right triangle is equal to the square root of the remainder obtained by subtracting the square of the other leg from the square of the hypotenuse.*

EXAMPLE 1.—In a right triangle, the hypotenuse is 4.95 inches and the one leg is 2.97 inches. Find the other leg.

SOLUTION.—The square of the hypotenuse is 4.95^2, or 24.5025 square inches, and the square of the given leg is 2.97^2, or 8.8209 square inches. The remainder obtained by subtracting the square of the given leg from the square of the hypotenuse is $24.5025 - 8.8209$, or 15.6816 square inches. Hence, the required leg is $\sqrt{15.6816}$, or 3.96 in. Ans.

EXAMPLE 2.—The length of each leg of an isosceles triangle is 10 inches, and the length of the base is 12 inches. Find the length of the perpendicular from the vertex to the base.

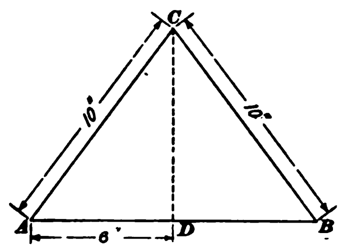


FIG. 4.

SOLUTION.—Let $A\ B\ C$, Fig. 4, be the isosceles triangle. Then, by Art. **65**, Part 3, $A\ D$ is one-half of the base; therefore, $A\ D = 6$ inches. In the right triangle, $A\ C\ D$, the leg $C\ D$ is equal to the square root of the remainder obtained by subtracting the square of the leg $A\ D$ from the square of the hypotenuse $A\ C$. Hence, $C\ D$ is $\sqrt{10^2 - 6^2}$, or $\sqrt{64}$, or 8 in. Ans.

45. Diagonal of a Square.—The diagonal $A\ C$ of the square $A\ B\ C\ D$, Fig. 5, is the hypotenuse of the right triangle $A\ B\ C$. Hence, $A\ C$ is equal to the square root of the sum of the squares of $A\ B$ and $B\ C$. But, since $A\ B$ and $A\ D$ are equal, the sum of their squares is equal to twice the square of $A\ B$. Therefore, $A\ C$ is equal to $\sqrt{2 \times (\text{square of } A\ B)}$. But the square root of the product of two numbers is equal to the product of their square roots (Art. **9**). Thus, $\sqrt{2 \times (\text{square of } A\ B)}$ is equal to $\sqrt{2} \times \sqrt{(\text{square of } A\ B)}$, or $\sqrt{2} \times A\ B$.

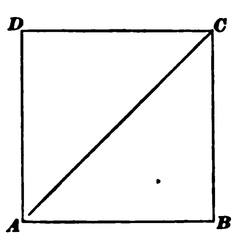


FIG. 5.

Hence, we have

$$A\ C = \sqrt{2} \times A\ B.$$

That is, the *diagonal of a square is equal to $\sqrt{2}$ times its side.*

EXAMPLES FOR PRACTICE.

1. Find the altitude of an isosceles triangle whose equal sides are 17 inches long and whose base is 30 inches long. Ans. 8 in.

2. Each side of an equilateral triangle is 2 inches long. Find its altitude correct to three figures. Ans. 1.73 in.

3. Find the length of the diagonal of a rectangle whose length is 50 inches and whose breadth is 37.5 inches. Ans. 62.5.

4. Given $\sqrt{2} = 1.414$; find the diagonal of a square whose side is 1.176 inches, the side being measured correct to the thousandth part of an inch. Ans. 1.663 in.

ARITHMETIC.

(PART 7.)

FACTORS AND MULTIPLES.

PRIME AND COMPOSITE NUMBERS.

1. In multiplication (Art. **5**, Part 4) and division (Art. **12**, Part 5), any two numbers are regarded as factors of their product. Thus, 2.5 and 2 are two factors whose product is 5. But in treating of factors and multiples, integral numbers only are considered, and 2.5 is not regarded as a factor of 5. The student must remember that in connection with factors and multiples, the word *number* always means an *integral number*.

2. If one integral number when divided by another gives an integral quotient without a remainder, the divisor is said to be a **factor, measure,** or **submultiple** of the dividend, and the dividend is said to be a **multiple** of the divisor. Thus, $46 \div 2 = 23$; therefore, 2 is a factor (or measure) of 46, and 46 is a multiple of 2. Thus, in connection with factors and multiples, the word *factor* has a more restricted sense than it has in multiplication and division. Since integers only are considered as factors and multiples, neither 3.5 nor 2 is a factor of 7, and 7 is not a multiple of 2.

When one number is said to be divisible by another, it is to be understood that the first number is divisible by the second without a remainder unless the contrary is stated.

3. Every number has two factors, namely, the number itself and unity. For example, $\frac{5}{5} = 1$ and $\frac{5}{1} = 5$; that is, 5 is divisible both by 5 and by 1, and, therefore, 5 and 1 are both factors of 5.

4. A number that has no factors except itself and unity is said to be a **prime number,** or, simply, a **prime.** Thus, 5 and 7 are primes.

5. A prime number that is a factor of another number is called a **prime factor** of that number. For example, 5 is a prime factor of 75.

6. It can be shown that every number that is not a prime can be resolved into prime factors, and that this can be done in only one way. For example, $273 = 3 \times 7 \times 13$; therefore, 3, 7, and 13 are prime factors of 273; and it can be proved that there is no other set of prime numbers whose product is 273.

7. Since every number that is not a prime is composed of prime factors, all numbers that are not prime are called **composite numbers.**

8. The multiples of 2 are called **even** numbers; all other numbers are called **odd** numbers. The even numbers are 2, 4, 6, 8, 10, 12, 14, etc., while 1, 3, 5, 7, 9, etc. are odd numbers.

9. The prime numbers less than 100 are: 1, 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89, 97.

10. The prime factors of a number are found by trial; beginning with 2 and taking each prime number in succession, we determine by division which, if any, of them is a factor of the given number.

EXAMPLE 1.—Find the prime factors of 534.

SOLUTION.—

$$\begin{array}{r|l} 2 & 534 \\ \hline 3 & 267 \\ \hline & 89 \end{array}$$

Thus, $534 = 2 \times 3 \times 89$, and each of these factors is a prime number; therefore, the prime factors of 534 are 2, 3, and 89.

That is, $534 = 2 \times 3 \times 89$. Ans.

EXAMPLE 2.—Find the prime factors of 60.

SOLUTION.—

$$\begin{array}{r|l} 2 & 60 \\ \hline 2 & 30 \\ \hline 3 & 15 \\ \hline & \;\;5 \end{array}$$

Therefore, $60 = 2 \times 2 \times 3 \times 5 = 2^2 \times 3 \times 5$. Ans.

11. A number that divides two or more numbers without a remainder is called a **common factor** of those numbers. Thus, 5 is a common factor of 10, 15, 25, and 125, for each of the numbers 10, 15, 25, and 125 is divisible by 5.

EXAMPLES FOR PRACTICE.

Find the prime factors of

1. 581. Ans. 7×83.
2. 117. Ans. $3^2 \times 13$.
3. 388. Ans. $2^2 \times 97$.
4. 234. Ans. $2 \times 3^2 \times 13$.
5. 360. Ans. $2^3 \times 3^2 \times 5$.

CANCELATION.

12. When 45 is divided by 9, the quotient is 5; that is,

$$\frac{45}{9} = 5.$$

If both dividend and divisor are multiplied by 2, we have

$$\frac{2 \times 45}{2 \times 9} = \frac{90}{18},$$

and the quotient, when 90 is divided by 18, is 5; therefore,

$$\frac{2 \times 45}{2 \times 9} = 5.$$

Thus, the quotient is not changed by multiplying both dividend and divisor by 2.

If both dividend and divisor are divided by 3, we have

$$\frac{45 \div 3}{9 \div 3} = \frac{15}{3},$$

and the quotient, when 15 is divided by 3, is 5; therefore,

$$\frac{45 \div 3}{9 \div 3} = 5.$$

Thus, the quotient is unchanged by dividing both dividend and divisor by 3.

These examples illustrate the following important principle:

In any example of division, the quotient is not changed if the dividend and the divisor are both multiplied or both divided by the same number.

13. Suppose it is required to divide 273×895 by 91×895; the quotient can be found by multiplying 895 by 273 and multiplying 895 by 91, and then dividing the former product by the latter. But the quotient can be found more easily by using the principle of Art. **12**; thus, dividing both dividend and divisor by 895, we have

$$\frac{273 \times 895}{91 \times 895} = \frac{273}{91} = 3,$$

which is the required quotient. This result was obtained by *casting out* the common factor 895 from the dividend and divisor.

14. Cancelation is the process of shortening operations in division by casting out the common factors from the dividend and divisor.

EXAMPLE.—Divide $4 \times 45 \times 60$ by 9×24.

SOLUTION.—Place the dividend over the divisor and cancel. Thus,

$$\frac{\overset{1}{\cancel{4}} \times \overset{5}{\cancel{45}} \times \overset{10}{\cancel{60}}}{\underset{1}{\cancel{9}} \times \underset{\underset{1}{\cancel{6}}}{\cancel{24}}} = 50. \quad \text{Ans.}$$

EXPLANATION.—Evidently 4 is a common factor of the 4 in the dividend and the 24 in the divisor; since $4 \div 4 = 1$ and $24 \div 4 = 6$. Strike out the 4 in the dividend and write 1 over it; strike out also the 24 in the divisor and write 6 under it. Thus,

$$\frac{\overset{1}{\cancel{4}} \times 45 \times 60}{9 \times \underset{6}{\cancel{24}}}.$$

Again, $60 \div 6 = 10$ and $6 \div 6 = 1$. Strike out the 60 and write 10 over it; strike out also the 6 and write 1 under it. Thus,

$$\frac{\overset{1}{\cancel{4}} \times 45 \times \overset{10}{\cancel{60}}}{9 \times \underset{\underset{1}{\cancel{6}}}{\cancel{24}}}$$

Again, $45 \div 9 = 5$ and $9 \div 9 = 1$. Strike out 45 and write 5 over it; strike out also 9 and write 1 below it. Thus,

$$\frac{\overset{1}{\cancel{4}} \times \overset{5}{\cancel{45}} \times \overset{10}{\cancel{60}}}{\underset{1}{\cancel{9}} \times \underset{\underset{1}{\cancel{6}}}{\cancel{24}}}.$$

The dividend and the divisor have no remaining common factor except unity, and, therefore, it is impossible to cancel further.

Multiply all the uncanceled factors of the dividend together and divide their product by the product of all the uncanceled factors in the divisor. The result will be the required quotient. The product of all the uncanceled factors in the dividend is $1 \times 5 \times 10 = 50$; and the product of all the uncanceled factors of the divisor is $1 \times 1 = 1$. Hence, the quotient is $50 \div 1 = 50$.

15. Hence, we have the following rule for shortening the work of division by cancelation:

Rule.—I. *Cancel the common factors from the dividend and divisor.*

II. *Then, divide the product of the remaining factors of the dividend by the product of the remaining factors of the divisor, and the result will be the required quotient.*

EXAMPLES FOR PRACTICE.

Divide

1. $14 \times 18 \times 16$ by $7 \times 8 \times 6$. Ans. 12.
2. $164 \times 321 \times 4$ by $82 \times 107 \times 8$. Ans. 3.
3. $880 \times 150 \times 84$ by $11 \times 15 \times 64$. Ans. 1,050.

LEAST COMMON MULTIPLE.

16. A number that is a multiple of two or more numbers is called a **common multiple** of those numbers. The smallest common multiple of two or more numbers is called their **least common multiple,** and is usually denoted by the abbreviation L. C. M. For example, 24 is the L. C. M. of 6 and 8.

17. The L. C. M. of several numbers can be found by the following rule:

Rule.—I. *Having written the numbers in a row, strike out any number that is a factor of any of the other numbers.*

II. *Select any prime that is a factor of two or more of the given numbers, place it at the left of the row, divide it into those numbers of which it is a factor, setting the quotients below these numbers, and bring down those numbers that have not been divided.*

III. *Treat the second row in the same manner as the first, and so continue until a row is reached in which there are no two numbers that have a common factor.*

IV. *The required L. C. M. is the continued product of all the divisors and all the numbers left in the last row.*

EXAMPLE.—Find the L. C. M. of 3, 4, 10, 12, and 16.

SOLUTION.—

$$\begin{array}{r|l} 2 & \not3,\ \not4,\ 10,\ 12,\ 16 \\ \hline 2 & 5,\ 6,\ 8 \\ \hline & 5,\ 3,\ 4 \end{array}$$

Therefore, the L. C. M. $= 2 \times 2 \times 5 \times 3 \times 4 = 240$. Ans.

EXPLANATION.—In the first row, 3 and 4 are stricken out, because each of them is a factor of 12, and it is evident that any multiple of 12 is also a multiple of 3 and 4. Then we select the prime factor 2 and divide each of the remaining numbers 10, 12, and 16 by it. Thus we get the second row $5(= 10 \div 2)$, $6(= 12 \div 2)$, and $8(= 16 \div 2)$. Again, 2 is a factor of the two numbers 6 and 8, which appear in the second row. Since 5, the first number of the second row, is not divisible by 2, the 5 is brought down to the third row (see II of rule). The other two numbers in the third row are $3(= 6 \div 2)$ and $4(= 8 \div 2)$. In the third row there are no two numbers that have a common factor. Therefore, the required L. C. M. is $2 \times 2 \times 5 \times 3 \times 4 = 240$.

CHECK.—If the result is correct, the L. C. M. should be divisible by each of the given numbers; thus,

$$240 \div 3 = 80;\ 240 \div 4 = 60;\ 240 \div 10 = 24;$$
$$240 \div 12 = 20;\ 240 \div 16 = 15.$$

EXAMPLES FOR PRACTICE.

Find the L. C. M. of

1. 9, 12, 18. Ans. 36.
2. 6, 8, 24. Ans. 24.
3. 6, 18, 24. Ans. 72.
4. 3, 4, 9, 16. Ans. 144.

FRACTIONS.

18. When the operation of division is indicated by writing the dividend over the divisor with a horizontal line between them, the expression is called a **common fraction,** or simply a **fraction.** The dividend is called the **numerator** and the divisor is called the **denominator** of the fraction. The numerator and denominator are called the **terms** of the fraction. Thus, $\frac{3}{4}$ is a common fraction whose numerator is 3 and whose denominator is 4; $\frac{11}{8}$ is a common fraction, whose numerator is 11 and whose denominator is 8.

19. A **proper fraction** is one in which the numerator is less than the denominator. An **improper fraction** is one whose numerator is not less than its denominator. Thus, $\frac{3}{4}$, $\frac{5}{8}$, and $\frac{7}{16}$ are proper fractions; but $\frac{4}{4}$, $\frac{9}{8}$, and $\frac{35}{32}$ are improper fractions.

20. Let *A E*, Fig. 1, be a line 1 inch long; and let *A E* be divided into 4 equal parts at the points *B*, *C*, and *D*. Then, *A B* contains 1 of the equal parts, *A C* contains 2 of the equal parts, *A D* contains 3 of the equal parts, and *A E* contains all 4 of the equal parts.

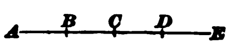


FIG 1.

Therefore,	*A B* is $\frac{1}{4}$ of an inch,
	A C is $\frac{2}{4}$ of an inch,
	A D is $\frac{3}{4}$ of an inch,
and	*A E* is $\frac{4}{4}$ of an inch.

Here an inch is the unit, and the numerical measure of the length of the line *A D* is the fraction $\frac{3}{4}$. In this fraction, the denominator 4 shows that the unit is divided into 4 equal parts, and the numerator 3 shows that the line *A D* contains 3 of these equal parts.

FIG. 2.

Again, let *A I*, Fig. 2, be a line 2 inches long, and let each of the inches *A E* and $E\,I$ be divided into 4 equal parts, as shown in the figure. Then each of the equal parts is $\frac{1}{4}$ inch. Hence,

$A\,F$ is $\frac{5}{4}$ of an inch,
$A\,G$ is $\frac{6}{4}$ of an inch,
$A\,H$ is $\frac{7}{4}$ of an inch,
$A\,I$ is $\frac{8}{4}$ of an inch, or 2 inches.

Here an inch is taken as the unit, and the numerical measure of the length of the line $A\,H$ is the fraction $\frac{7}{4}$. In this fraction, the denominator 4 shows that each unit is divided into 4 equal parts, and the numerator 7 shows that the line $A\,H$ contains 7 of these equal parts.

These illustrations lead to the following statement of the meaning of the numerator and denominator of a fraction:

The denominator of a fraction whose terms are abstract integers shows the number of equal parts into which each unit is divided, and the numerator shows how many of these equal parts is contained in the quantity of which the fraction is the numerical measure.

21. The denominator of a fraction indicates also the names of the equal parts into which each unit is divided. Thus,

$\frac{1}{2}$ is read *one-half*,
$\frac{3}{4}$ is read *three-fourths*,
$\frac{5}{16}$ is read *five-sixteenths*,
$\frac{23}{32}$ is read *twenty-three thirty-seconds*.

22. The **value** of a fraction is the quotient obtained by dividing the numerator by the denominator; thus, $\frac{4}{2} = 2$, and therefore the value of the fraction $\frac{4}{2}$ is 2.

23. A number consisting of an integer and a fraction is called a **mixed number**; thus, $4\frac{2}{3}$ is a mixed number; it is equal to $4 + \frac{2}{3}$, and is read *four and two-thirds*.

24. The decimal fraction .817 may be written in the form $\frac{817}{1,000}$; for, by division, $\frac{817}{1,000}$ is .817 (Art. **8,** Part 5). When .817 is written in the form $\frac{817}{1,000}$, it is said to be written in the **form of a common fraction.**

A decimal fraction may be written in the form of a common fraction by writing the figures of the decimal as numerator, and writing under them for denominator 1 followed by as many zeros as there are decimal places in the decimal.

Thus,

$$.75 = \frac{75}{100};\ .0135 = \frac{0135}{10,000} = \frac{135}{10,000};\ 2.5 = \frac{25}{10}.$$

REDUCTION OF FRACTIONS.

25. Reduction in mathematics is the process of changing the form of an expression without changing its value.

26. *The value of a fraction is unchanged if its terms are both multiplied or both divided by the same number.*

This is merely another way of stating the principle of Art. **12,** that the quotient is unchanged if the dividend and the divisor are both multiplied or both divided by the same number.

This important principle may be illustrated in the following manner. Let AE, Fig. 3, be a line 1 inch in length.

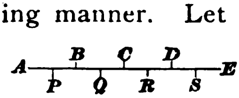


FIG. 3.

Let AE be divided into 4 equal parts in the points B, C, and D. Then, AD contains 3 of these equal parts, and, therefore, $AD = \frac{3}{4}$ of an inch. Let each of the equal parts AB, BC, CD, and DE be divided into 2 equal parts by the points P, Q, R, and S. Then, the whole line AE is divided into 8 equal parts, namely, AP, PB, BQ, QC, CR, RD, DS, and SE; and AD contains 6 of these equal parts. Therefore, $AD = \frac{6}{8}$ of an inch. But we have seen that $AD = \frac{3}{4}$ of an inch. Hence,

$$\frac{3}{4} = \frac{6}{8},$$

or

$$\frac{3}{4} = \frac{3 \times 2}{4 \times 2}.$$

That is, the value of the fraction $\frac{3}{4}$ is not changed by multiplying both terms of the fraction by 2; nor is the value of the fraction $\frac{6}{8}$ changed by dividing both of its terms by 2.

27. *Any fraction may be reduced to an equivalent fraction having a given denominator that is a multiple of the denominator of the given fraction.*

EXAMPLE.—Reduce $\frac{7}{8}$ to a fraction having 32 for its denominator.

SOLUTION.—We have $32 = 8 \times 4$. Multiplying both terms of the given fraction by 4, we get

$$\frac{7}{8} = \frac{7 \times 4}{8 \times 4} = \frac{28}{32}. \quad \text{Ans.}$$

Rule.—*Divide the given denominator by the denominator of the given fraction, and multiply both terms of the given fraction by the quotient.*

28. A fraction is said to be in its **lowest terms** when its numerator and denominator have no common factor.

29. In ordinary cases, a fraction can be reduced to its lowest terms by the following rule:

Rule.—*Resolve the numerator and the denominator into their factors and cancel those factors that are common to both.*

EXAMPLE 1.—Reduce $\frac{24}{32}$ to its lowest terms.

SOLUTION.—

$$\frac{24}{32} = \frac{8 \times 3}{8 \times 4} = \frac{3}{4}. \quad \text{Ans.}$$

EXAMPLE 2.—Express .125 as a common fraction in its lowest terms.

SOLUTION.—By Art. **24**,

$$.125 = \frac{125}{1{,}000} = \frac{5 \times 5 \times 5}{10 \times 10 \times 10} = \frac{1 \times 1 \times 1}{2 \times 2 \times 2} = \frac{1}{8}. \quad \text{Ans.}$$

EXAMPLES FOR PRACTICE.

Solve the following examples:

1. Reduce $\frac{5}{8}$ to 32ds. Ans. $\frac{20}{32}$.
2. Reduce $\frac{6}{16}$ to its lowest terms. Ans. $\frac{3}{8}$.
3. Express .25 as a common fraction in its lowest terms. Ans. $\frac{1}{4}$.
4. Express 1.75 as an improper fraction in its lowest terms. Ans. $\frac{7}{4}$.

HINT.—By Art. **24**,

$$1.75 = \frac{175}{100} = \frac{25 \times 7}{25 \times 4} = \frac{7}{4}.$$

30. *An integer can be reduced to an improper fraction with any given denominator.*

For example, we can reduce 5 to fourths; for each unit contains 4 fourths, and, therefore, 5 units contain 5×4 fourths; thus, $5 = \frac{20}{4}$.

Rule.—*To reduce an integer to an improper fraction having a given denominator, multiply the integer by the given denominator and write the product over the given denominator.*

EXAMPLE.—Reduce 7 to 8ths.

SOLUTION.—

$$7 = \frac{7}{1} = \frac{7 \times 8}{1 \times 8} = \frac{56}{8}. \text{ Ans.}$$

31. *A mixed number can be reduced to an improper fraction having the same denominator as the fractional part of the given number.*

EXAMPLE 1.—Reduce $8\frac{3}{4}$ to 4ths.

SOLUTION.—

$$8\tfrac{3}{4} = 8 + \tfrac{3}{4} = \frac{8}{1} + \frac{3}{4} = \frac{8 \times 4}{1 \times 4} + \frac{3}{4} = \frac{32}{4} + \frac{3}{4} = \frac{35}{4}. \text{ Ans.}$$

Rule.—*To reduce a mixed number to an improper fraction having the same denominator as the fractional part of the given number, multiply the integral part by the denominator of the fractional part; add the product so obtained to the numerator of the fractional part, and place the sum over the denominator.*

EXAMPLE 2.—Reduce $5\frac{3}{8}$ to an improper fraction.

SOLUTION.—Multiplying the integral part by the denominator of the fractional part, we have $5 \times 8 = 40$; adding this to the numerator of the fractional part, we have $40 + 3 = 43$; placing this sum over the denominator of the fractional part, we obtain $5\frac{3}{8} = \frac{43}{8}$. Ans.

EXAMPLES FOR PRACTICE.

Reduce to improper fractions

1. $3\frac{5}{8}$. Ans. $\frac{29}{8}$.
2. $2\frac{1}{16}$. Ans. $\frac{33}{16}$.
3. $3\frac{3}{4}$. Ans. $\frac{15}{4}$.

32. *An improper fraction may be reduced to a whole number or to a mixed number.*

For example, 4 is contained in 13, 3 times and 1 over; therefore, 13 fourths is equal to 3 units and 1 fourth; that is, $\frac{13}{4} = 3 + \frac{1}{4} = 3\frac{1}{4}$.

Rule.—*To reduce an improper fraction to a whole number or to a mixed number, divide the numerator by the denominator; the quotient is the whole number, and the remainder, if there is one, is the numerator of the fraction whose denominator is the same as that of the improper fraction.*

EXAMPLES FOR PRACTICE.

Reduce to integers or to mixed numbers

1. $\frac{37}{8}$. Ans. $4\frac{5}{8}$.
2. $\frac{32}{16}$. Ans. 2.
3. $\frac{9}{4}$. Ans. $2\frac{1}{4}$.
4. $\frac{13}{7}$. Ans. $1\frac{6}{7}$.

33. *A common fraction can be reduced to a decimal.*

EXAMPLE.—Reduce $\frac{7}{32}$ to a decimal.

SOLUTION.—Dividing the numerator by the denominator, we have

```
7.00000 | 32
64      |.21875
 60
 32
 280
 256
  240
  224
   160
   160
```

Therefore, $\frac{7}{32} = .21875$. Ans.

Rule.—I. *To reduce a common fraction to a decimal, divide the numerator by the denominator according to the rule of* Art. **13,** Part 5.

II. *If the resulting decimal contains more decimal places than are necessary, terminate the division when the required number of decimal places has been obtained.* (See Arts. **18** and **19,** Part. 5.)

EXAMPLES FOR PRACTICE.

Reduce to decimals

1. $\frac{1}{8}$. Ans. .125.
2. $\frac{3}{16}$. Ans. .1875.
3. $\frac{1}{5}$. Ans. .2.
4. $\frac{1}{25}$. Ans. .04.
5. $\frac{7}{4}$. Ans. 1.75,

34. *A decimal may be expressed, exactly or approximately as a common fraction having a given denominator.*

EXAMPLE 1.—Express .4375 in 16ths.

SOLUTION.—Since $\frac{16}{16} = 1$, we have

$$.4375 = .4375 \times \frac{16}{16} = \frac{.4375 \times 16}{16}.$$

Multiplying, we get $.4375 \times 16 = 7.$

Therefore, $.4375 = \frac{7}{16}$. Ans.

EXAMPLE 2.—Express .417 in 8ths.

SOLUTION.—Since $\frac{8}{8} = 1$, we have

$$.417 = .417 \times \frac{8}{8} = \frac{.417 \times 8}{8} = \frac{3.336}{8}.$$

Therefore, approximately, $.417 = \frac{3}{8}$. Ans.

EXAMPLE 3.—Reduce .583 to 32ds.

SOLUTION.— $.583 = .583 \times \frac{32}{32} = \frac{18.656}{32}.$

Hence, approximately, $.583 = \frac{19}{32}$. Ans.

Rule.—*To reduce a decimal to an approximately equal common fraction with a given denominator, multiply the decimal by the given denominator, and the integral part of the product is the numerator of the required fraction.*

EXAMPLES FOR PRACTICE.

Express:

1.	.625 in 8ths.	Ans. $\frac{5}{8}$.
2.	.3125 in 16ths.	Ans. $\frac{5}{16}$.
3.	.788 in 32ds.	Ans. $\frac{25}{32}$.
4.	.698 in 24ths.	Ans. $\frac{17}{24}$.

LEAST COMMON DENOMINATORS.

35. Any number of fractions having different denominators can be reduced to equal fractions having a common denominator which is any common multiple of the denominators of the given fractions. For example, $\frac{1}{2}$ and $\frac{1}{3}$ can be reduced to equal fractions whose common denominator is any common multiple of 2 and 3, such as 6, 12, 24, etc.; thus, $\frac{1}{2} = \frac{3}{6}$, $\frac{1}{3} = \frac{2}{6}$; $\frac{1}{2} = \frac{6}{12}$, $\frac{1}{3} = \frac{4}{12}$; $\frac{1}{2} = \frac{12}{24}$, $\frac{1}{3} = \frac{8}{24}$; etc. It is, however, usually most convenient to take the L. C. M. of the denominators of the given fractions as the common denominator; the L. C. M. of the denominators is then called the **least common denominator** of the given fractions, and is denoted by the letters L. C. D.

36. The following is the rule for reducing fractions to equal fractions with their least common denominator:

Rule.—*Find the L. C. M. of the denominators of the given fractions, and reduce the fractions to equal fractions having the L. C. M. for denominator.*

EXAMPLE.—Reduce $\frac{3}{4}$, $\frac{7}{8}$, $\frac{5}{32}$ to fractions having a common denominator.

SOLUTION.—The L. C. M. of 4, 8, and 32 is 32; therefore, 32 is the required L. C. D.

$$\left.\begin{aligned} \frac{3}{4} &= \frac{3\times 8}{4\times 8} = \frac{24}{32}. \\ \frac{7}{8} &= \frac{7\times 4}{8\times 4} = \frac{28}{32}. \\ \frac{5}{32} &= \frac{5}{32}. \end{aligned}\right\} \text{Ans.}$$

EXAMPLES FOR PRACTICE.

Reduce to equal fractions with their least common denominator:

1. $\frac{1}{2}, \frac{3}{4}, \frac{7}{16}$. Ans. $\frac{8}{16}, \frac{12}{16}, \frac{7}{16}$.
2. $\frac{3}{16}, \frac{1}{4}, \frac{31}{32}, \frac{3}{8}$. Ans. $\frac{6}{32}, \frac{8}{32}, \frac{31}{32}, \frac{12}{32}$.
3. $\frac{3}{5}, \frac{5}{8}, \frac{7}{16}$. Ans. $\frac{48}{80}, \frac{50}{80}, \frac{35}{80}$.

ADDITION AND SUBTRACTION OF FRACTIONS.

37. It is evident that, just as

2 inches + 5 inches = 7 inches,

so also 2 sixteenths + 5 sixteenths = 7 sixteenths,

or
$$\frac{2}{16} + \frac{5}{16} = \frac{7}{16}.$$

Thus, the sum of two fractions having a common denominator is found by adding the numerators and writing the sum over the denominators.

Again, 9 inches − 4 inches = 5 inches;

and in like manner,

9 thirty-seconds − 4 thirty-seconds = 5 thirty-seconds,

or
$$\frac{9}{32} - \frac{4}{32} = \frac{5}{32}.$$

Thus, the difference of two fractions with a common denominator is found by writing the difference of the numerators over the denominator.

Hence, we have the following rule for adding or subtracting fractions:

Rule.—*Reduce the fractions to equal fractions with their least common denominator. Add or subtract the numerators and write the result over the common denominator.*

EXAMPLE 1.—Find the sum of $\frac{1}{2}$, $\frac{3}{4}$, and $\frac{5}{8}$.

SOLUTION.—The least common denominator of these fractions is 8. Reducing the given fractions to equivalent fractions with 8 as denominator, we get

$$\frac{1}{2}=\frac{4}{8}$$
$$\frac{3}{4}=\frac{6}{8}$$
$$\frac{5}{8}=\frac{5}{8}$$
$$\frac{15}{8}=1\frac{7}{8}.\quad \text{Ans.}$$

EXAMPLE 2.—Find the sum of $12\frac{3}{4}$, $14\frac{5}{8}$, $7\frac{5}{16}$.

SOLUTION.—The L. C. D. is 16.

$$12\frac{3}{4} = 12\frac{12}{16}$$
$$14\frac{5}{8} = 14\frac{10}{16}$$
$$7\frac{5}{16} = 7\frac{5}{16}$$
$$34\frac{11}{16}\quad \text{Ans.}$$

EXPLANATION.—The sum of the fractions $= \frac{27}{16} = 1\frac{11}{16}$; set down $\frac{11}{16}$ and carry 1 to the units column.

EXAMPLE 3.—From $17\frac{9}{16}$ take $9\frac{15}{32}$.

SOLUTION.—The L. C. D. is 32.

$$\textit{minuend} = 17\frac{9}{16} = 17\frac{18}{32}$$
$$\textit{subtrahend} = 9\frac{15}{32}$$
$$\textit{remainder} = 8\frac{3}{32}\quad \text{Ans.}$$

EXAMPLE 4.—From $9\frac{1}{4}$ take $4\frac{7}{16}$.

SOLUTION.—The L. C. D. is 16.

$$\textit{minuend} = 9\frac{1}{4} = 9\frac{4}{16}$$
$$\textit{subtrahend} = 4\frac{7}{16}$$
$$\textit{remainder} = 4\frac{13}{16}\quad \text{Ans.}$$

EXPLANATION.— $\frac{7}{16}$ cannot be taken from $\frac{4}{16}$; but if the same number is added both to the minuend and to the subtrahend, the remainder is unchanged (Art. **19**, Part 2). Hence, we add $\frac{16}{16}$, or 1, to the minuend, and add 1 to the subtrahend. Adding $\frac{16}{16}$ to $\frac{4}{16}$, we get $\frac{20}{16}$, and $\frac{20}{16} - \frac{7}{16} = \frac{13}{16}$. Adding 1 to 4, we get 5, and $9 - 5 = 4$. Therefore, the remainder is $4\frac{13}{16}$.

38. Shop Method of Adding Fractions.—The shop method of adding fractions by means of one or two rules is very expeditious and reliable, and is applicable to the fractions that most frequently occur in shop work. A similar method can be employed for subtracting fractions.

EXAMPLE 1.—Find the sum of $\frac{3}{4}$, $\frac{7}{8}$, and $1\frac{3}{16}$ by means of one rule.

SOLUTION.—Draw a straight line *A Z*, Fig. 4; on this line, by using the rule, lay off *A B* equal to $\frac{3}{4}$ inch, *B C* equal to $\frac{7}{8}$ inch, and *C D*

FIG. 4.

equal to $1\frac{3}{16}$ inches. Measure the line *A D*, and its length will be found to be $2\frac{13}{16}$ inches. Therefore,

$$\tfrac{3}{4} \text{ inch} + \tfrac{7}{8} \text{ inch} + 1\tfrac{3}{16} \text{ inches} = 2\tfrac{13}{16} \text{ inches};$$

hence, $$\tfrac{3}{4} + \tfrac{7}{8} + 1\tfrac{3}{16} = 2\tfrac{13}{16}. \quad \text{Ans.}$$

EXAMPLE 2.—Find the sum of $\frac{15}{16}$ and $2\frac{3}{4}$ by means of 2 two-foot rules.

SOLUTION.—Place the edges of the two rules together as shown in Fig. 5, so that the $2\frac{3}{4}$-inch mark of the rule *B* is opposite to the $\frac{15}{16}$-inch

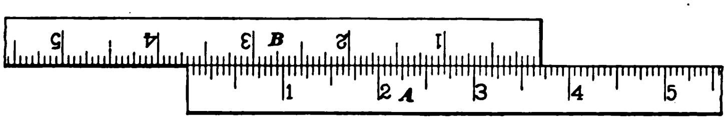


FIG. 5.

mark of the rule *A*; then it will be found that the end of the rule *B* is opposite to the $3\frac{11}{16}$-inch mark of the rule *A*. Therefore,

$$\tfrac{15}{16} \text{ inch} + 2\tfrac{3}{4} \text{ inches} = 3\tfrac{11}{16} \text{ inches};$$

hence, $$\tfrac{15}{16} + 2\tfrac{3}{4} = 3\tfrac{11}{16}. \quad \text{Ans.}$$

EXAMPLES FOR PRACTICE.

Find the value of

1. $2\frac{1}{2} + 3\frac{1}{4} + 1\frac{7}{8}$. Ans. $7\frac{5}{8}$.
2. $7\frac{1}{4} + 8\frac{1}{3} + 5\frac{21}{32}$. Ans. $21\frac{23}{96}$.
3. $9 - 8\frac{3}{16}$. Ans. $\frac{13}{16}$.
4. $30\frac{1}{2} - 13\frac{1}{4}$. Ans. $17\frac{1}{4}$.
5. $27\frac{1}{2} - 12\frac{5}{8}$. Ans. $14\frac{7}{8}$.

MULTIPLICATION OF FRACTIONS.

39. If a line 6 inches long is divided into three equal parts, each of these equal parts contains 2 inches; therefore, $\frac{1}{3}$ of 6 is 2. If $\frac{1}{3}$ is multiplied by 6, we have

$$\frac{1}{3} \times 6 = \frac{1}{3} + \frac{1}{3} + \frac{1}{3} + \frac{1}{3} + \frac{1}{3} + \frac{1}{3} = \frac{6}{3} = 2.$$

Thus, $\frac{1}{3}$ of $6 = \frac{1}{3} \times 6$; for each of the members of this equation is equal to 2. This is a particular case of the following principle:

In multiplication of fractions, the word "of" has the same meaning as the sign of multiplication.

40. There are two cases of multiplication of fractions: (1) when the multiplier is an integer, (2) when the multiplier is a fraction. In the succeeding articles, the number written after the sign of multiplication is the multiplier.

MULTIPLICATION BY AN INTEGER.

41. Suppose it is required to multiply $\frac{3}{7}$ by 5. We have

$$\frac{3}{7} \times 5 = \frac{3}{7} + \frac{3}{7} + \frac{3}{7} + \frac{3}{7} + \frac{3}{7} = \frac{15}{7} = 2\frac{1}{7}. \quad \text{Ans.}$$

Again, if it is required to multiply $\frac{7}{12}$ by 4, we have

$$\frac{7}{12} \times 4 = \frac{7}{12} + \frac{7}{12} + \frac{7}{12} + \frac{7}{12} = \frac{28}{12} = \frac{7}{3} \text{ (by cancelation).}$$

Therefore, $$\frac{7}{12} \times 4 = \frac{7 \times 4}{12} = \frac{7}{3} = 2\frac{1}{3}. \quad \text{Ans.}$$

Hence, we have the following rule:

Rule.—*To multiply a fraction by an integer, multiply the numerator of the fraction by the integer.*

Example.—Multiply $\frac{3}{8}$ by 4.

Solution.—Applying the rule, we have

$$\frac{3}{8} \times 4 = \frac{3 \times 4}{8} = \frac{3}{2} = 1\tfrac{1}{2}. \quad \text{Ans.}$$

42. When the multiplicand is a mixed number, proceed as in the following example:

Example.—Multiply $15\frac{7}{32}$ by 9.

Solution.—

$$\begin{array}{r} 15\frac{7}{32} \\ 9 \\ \hline 136\frac{31}{32}. \end{array} \quad \text{Ans.}$$

Explanation.— $\frac{7}{32} \times 9 = \frac{63}{32} = 1\frac{31}{32}$; set down $\frac{31}{32}$ and carry 1 unit. Then, $15 \times 9 + 1 = 136$.

EXAMPLES FOR PRACTICE.

Multiply

1. $3\frac{5}{16}$ by 7. Ans. $23\frac{3}{16}$.
2. $15\frac{3}{4}$ by 8. Ans. 126.
3. How many inches are there in $\frac{3}{8}$ of 12 inches? Ans. $4\frac{1}{2}$ in.

MULTIPLICATION BY A FRACTION.

43. Let the length of the line AF, Fig. 6, be taken as the unit of length. Let AF be divided into five equal parts by the points B, C, D, and E; and let each of the equal parts AB, BC, CD, DE, and EF be divided into three equal parts, as shown in the figure.

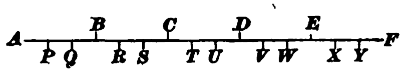


Fig. 6.

Then, the unit AF is divided into five equal parts, of which AC contains two; therefore,

$$AC = \tfrac{2}{5}.$$

Also, $AQ = QR = RC$; that is, AC is divided into three equal parts, of which AR contains two; therefore,

$$AR = \tfrac{2}{3} \text{ of } AC.$$

And we have seen that $AC = \tfrac{2}{5}$; hence,

$$AR = \tfrac{2}{3} \text{ of } \tfrac{2}{5}. \qquad (1)$$

But the whole line AF is divided into fifteen equal parts, of which AR contains four; so that

$$AR = \tfrac{4}{15}. \qquad (2)$$

The value of AR given by equation (1) must be equal to that given by equation (2). Therefore,

$$\frac{2}{3} \text{ of } \frac{2}{5} = \frac{4}{15},$$

or

$$\frac{2}{3} \times \frac{2}{5} = \frac{4}{15} = \frac{2 \times 2}{3 \times 5}.$$

Thus, the product of the fractions $\tfrac{2}{3}$ and $\tfrac{2}{5}$ is the fraction $\frac{4}{15}\left(= \frac{2 \times 2}{3 \times 5}\right)$, whose numerator is the product of their numerators and whose denominator is the product of their denominators.

Hence, we have the following rule for the multiplication of fractions:

Rule.—*To find the product of two fractions, divide the product of their numerators by the product of their denominators.*

Example.—What is the product of $\tfrac{2}{15}$ and $\tfrac{3}{8}$?

Solution.— $$\frac{2}{15} \times \frac{3}{8} = \frac{\overset{1}{\not 2} \times \overset{1}{\not 3}}{\underset{5}{\not{15}} \times \underset{4}{\not 8}} = \frac{1 \times 1}{5 \times 4} = \frac{1}{20}. \quad \text{Ans.}$$

44. **Rule.**—*To find the continued product of several fractions, divide the continued product of their numerators by the continued product of their denominators.*

Example.—What is $\frac{3}{25}$ of $\frac{5}{12}$ of $\frac{5}{8}$?

Solution.—

$$\frac{3}{25} \text{ of } \frac{5}{12} \text{ of } \frac{5}{8} = \frac{3}{25} \times \frac{5}{12} \times \frac{5}{8} = \frac{\not{3} \times \not{5} \times \not{5}}{\not{25} \times \underset{4}{\not{12}} \times 8} = \frac{1}{4 \times 8} = \frac{1}{32}. \text{ Ans.}$$

45. In finding the product of a mixed number and a fraction, it is best to reduce the mixed number to an improper fraction; and in finding the product of two mixed numbers, it is best to reduce both of them to improper fractions. After the reduction, the product is found by the rule of Art. **43.**

Example.—Find the product of $9\frac{3}{4}$ and $5\frac{5}{8}$.

Solution.—We have $9\frac{3}{4} = \frac{39}{4}$, and $5\frac{5}{8} = \frac{45}{8}$.

Therefore, $9\frac{3}{4} \times 5\frac{5}{8} = \frac{39}{4} \times \frac{45}{8} = \frac{39 \times 45}{4 \times 8} = \frac{1,755}{32} = 54\frac{27}{32}$. Ans.

46. A Power of a Fraction.—*A fraction is raised to any power by raising both terms of the fraction to that power.*

Thus, $$\left(\frac{2}{3}\right)^3 = \frac{2}{3} \times \frac{2}{3} \times \frac{2}{3} = \frac{2 \times 2 \times 2}{3 \times 3 \times 3} = \frac{2^3}{3^3}.$$

EXAMPLES FOR PRACTICE.

Perform the following multiplication:

1. $7 \times \frac{3}{32}$. Ans. $\frac{21}{32}$.
2. $\frac{2}{5} \times \frac{5}{16}$. Ans. $\frac{1}{8}$.
3. $\frac{7}{8} \times \frac{5}{16} \times \frac{24}{25}$. Ans. $\frac{21}{80}$.
4. $2\frac{1}{2} \times \frac{3}{25} \times 1\frac{7}{8}$. Ans. $\frac{9}{16}$.

DIVISION OF FRACTIONS.

47. There are two cases of the division of fractions: (1) when the divisor is an integer, (2) when the divisor is a fraction.

DIVISION BY AN INTEGER.

48. Suppose it is required to divide $\frac{9}{16}$ by 3. Dividing by 3 is the same as multiplying by its reciprocal $\frac{1}{3}$ (Art. **10,** Part 5). Therefore,

$$\frac{9}{16} \div 3 = \frac{9}{16} \times \frac{1}{3} = \frac{9}{16 \times 3} = \frac{3}{16}. \quad \text{Ans.}$$

Rule.—*To divide a fraction by an integer, multiply the fraction by the reciprocal of the integer.*

EXAMPLE.—Divide $\frac{3}{8}$ by 4.

SOLUTION.—Applying the rule, we have

$$\frac{3}{8} \div 4 = \frac{3}{8} \times \frac{1}{4} = \frac{3}{8 \times 4} = \frac{3}{32}. \quad \text{Ans.}$$

49. When the dividend is a mixed number, reduce it to an improper fraction before dividing.

EXAMPLE.—Divide $10\frac{1}{2}$ by 4.

SOLUTION.— $10\frac{1}{2} = \frac{21}{2}$.

$$\frac{21}{2} \div 4 = \frac{21}{2} \times \frac{1}{4} = \frac{21}{2 \times 4} = \frac{21}{8} = 2\frac{5}{8}. \quad \text{Ans.}$$

EXAMPLES FOR PRACTICE.

Divide

1. $\frac{3}{8}$ by 3. Ans. $\frac{1}{8}$.
2. $\frac{15}{16}$ by 5. Ans. $\frac{3}{16}$.
3. $\frac{7}{8}$ by 4. Ans. $\frac{7}{32}$.
4. $16\frac{1}{5}$ by 27. Ans. $\frac{3}{5}$.

DIVISION BY A FRACTION.

50. To **invert** a fraction is to make its numerator and denominator change places. Thus, when the fraction $\frac{5}{8}$ is inverted it becomes $\frac{8}{5}$.

51. The product of $\frac{5}{8}$ and $\frac{8}{5}$ is $\frac{5}{8} \times \frac{8}{5} = 1$; and, therefore, $\frac{8}{5}$ is the reciprocal of $\frac{5}{8}$ (Art. **9,** Part 5). Thus, *the reciprocal of a fraction is found by inverting the fraction.*

52. Let it be required to divide $\frac{15}{32}$ by $\frac{3}{8}$. Dividing by $\frac{3}{8}$ is the same as multiplying by its reciprocal $\frac{8}{3}$ (Art. **10**, Part 5), therefore,

$$\frac{15}{32} \div \frac{3}{8} = \frac{15}{32} \times \frac{8}{3} = \frac{5}{4} = 1\tfrac{1}{4}. \quad \text{Ans.}$$

By Art. **51**, the reciprocal of a fraction is found by inverting the fraction; hence, we have the following rule:

Rule.—*To divide by a fraction, invert the divisor and proceed as in multiplication.*

EXAMPLE.—Divide $\frac{3}{4}$ by $\frac{15}{16}$.

SOLUTION.— $\frac{3}{4} \div \frac{15}{16} = \frac{3}{4} \times \frac{16}{15} = \frac{4}{5}$. Ans.

53. Division of Mixed Numbers.—Mixed numbers must be reduced to improper fractions before the division is performed.

EXAMPLE.—Divide $3\frac{3}{4}$ by $7\frac{7}{16}$.

SOLUTION.— $3\frac{3}{4} = \frac{15}{4}$; $7\frac{7}{16} = \frac{119}{16}$.

Therefore, $3\frac{3}{4} \div 7\frac{7}{16} = \frac{15}{4} \div \frac{119}{16} = \frac{15}{4} \times \frac{16}{119} = \frac{60}{119}$. Ans.

EXAMPLES FOR PRACTICE.

Divide

1. $\frac{3}{8}$ by $\frac{9}{10}$. Ans. $\frac{5}{12}$.
2. $\frac{16}{25}$ by $\frac{4}{5}$. Ans. $\frac{4}{5}$.
3. $\frac{11}{16}$ by $\frac{33}{38}$. Ans. $\frac{19}{24}$.
4. $6\frac{1}{2}$ by $3\frac{1}{4}$. Ans. 2.
5. $7\frac{3}{5}$ by $4\frac{3}{4}$. Ans. $1\frac{3}{5}$.

COMPLEX FRACTIONS.

54. A fraction that has a fraction in any of its terms is called a **complex fraction.** For example, $\frac{2\frac{1}{3}}{8}$, $\frac{2}{3\frac{1}{4}}$, and $\frac{3\frac{1}{4}}{5\frac{3}{8}}$ are complex fractions.

55. A complex fraction is equivalent to the quotient obtained by dividing its numerator by its denominator. For example,

$$\frac{3\frac{1}{4}}{4\frac{7}{8}} = 3\tfrac{1}{4} \div 4\tfrac{7}{8} = \frac{13}{4} \div \frac{39}{8} = \frac{13}{4} \times \frac{8}{39} = \frac{2}{3}. \quad \text{Ans.}$$

56. To **simplify** a complex fraction is to reduce it to an equal fraction whose terms are both integral. Thus, in Art. **55**, the complex fraction $\frac{3\frac{1}{4}}{4\frac{7}{8}}$ was simplified by reducing it to the equal fraction $\frac{2}{3}$.

57. Rule.—*To simplify a complex fraction, divide its numerator by its denominator.*

EXAMPLE.—Simplify $\dfrac{2\frac{1}{4} + 1\frac{1}{2}}{2\frac{15}{16} - 1\frac{1}{16}}$.

SOLUTION.—The numerator is $2\frac{1}{4} + 1\frac{1}{2} = 3\frac{3}{4} = \frac{15}{4}$, and the denominator is $2\frac{15}{16} - 1\frac{1}{16} = 1\frac{14}{16} = 1\frac{7}{8} = \frac{15}{8}$.

Therefore, $\dfrac{2\frac{1}{4} + 1\frac{1}{2}}{2\frac{15}{16} - 1\frac{1}{16}} = \dfrac{15}{4} \div \dfrac{15}{8} = \dfrac{15}{4} \times \dfrac{8}{15} = 2.$ Ans.

NOTE.—It is more convenient to write the solution of such an example in the following form:

$$\frac{2\frac{1}{4} + 1\frac{1}{2}}{2\frac{15}{16} - 1\frac{1}{16}} = (2\tfrac{1}{4} + 1\tfrac{1}{2}) \div (2\tfrac{15}{16} - 1\tfrac{1}{16}) = 3\tfrac{3}{4} \div 1\tfrac{7}{8} = \frac{15}{4} \div \frac{15}{8} = \frac{15}{4} \times \frac{8}{15} = 2.$$

58. A heavy line is drawn between the numerator and the denominator of a complex fraction whenever any ambiguity would be likely to result from the use of a light line. Thus, $\dfrac{19}{\frac{3}{8}} = 19 \div \frac{3}{8}$; but $\dfrac{\frac{19}{3}}{8} = \frac{19}{3} \div 8$.

EXAMPLES FOR PRACTICE.

Simplify

1. $\dfrac{13\frac{1}{4}}{19\frac{7}{8}}$. Ans. $\frac{2}{3}$.

2. $\dfrac{2\frac{3}{4} + 3\frac{5}{8}}{6\frac{1}{4} + 4\frac{3}{8}}$. Ans. $\frac{3}{5}$.

3. $\dfrac{3\frac{1}{8} \times 4\frac{1}{4} \times 3\frac{1}{4}}{6\frac{3}{8} \times 8\frac{3}{4} \times 2\frac{1}{3}}$. Ans. $\frac{13}{42}$.

ROOTS OF FRACTIONS.

59. *Any root of a fraction can be found by extracting that root of each term of the fraction separately.*

Thus, $$\sqrt{\frac{9}{25}} = \sqrt{\frac{3^2}{5^2}} = \frac{3}{5};$$

and $$\sqrt[3]{\frac{27}{64}} = \sqrt[3]{\frac{3^3}{4^3}} = \frac{3}{4}.$$

60. When the terms of the fraction are not perfect powers, the root may be obtained by multiplying the terms by such a number as to make the denominator a perfect power. Thus,

$$\sqrt{\frac{2}{3}} = \sqrt{\frac{2 \times 3}{3 \times 3}} = \sqrt{\frac{6}{9}} = \frac{\sqrt{6}}{3} = \frac{2.45}{3}. \quad \text{Ans.}$$

However, a root of a fraction whose terms are not perfect powers is usually extracted by the following rule:

61. Rule.—*To extract any root of a fraction, reduce the fraction to a decimal and extract the root of the decimal.*

EXAMPLE.—Find the square root of $\frac{3}{8}$, correct to two figures.

SOLUTION.— $\frac{3}{8} = .375.$

Therefore, $\sqrt{\frac{3}{8}} = \sqrt{.375} = .61.$ Ans.

GEOMETRICAL APPLICATIONS.

62. Area of a Triangle. — *The area of a triangle is equal to one-half the product of its base and altitude.*

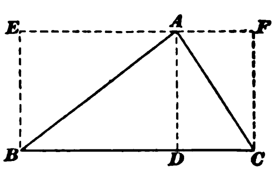


FIG. 7.

Let ABC, Fig. 7, be a triangle whose base is BC and whose altitude is AD. Through A draw EF parallel to BC; through B draw BE perpendicular to BC; and through C draw CF perpendicular to BC.

Then, if the triangle $A E B$ is cut out, it will fit exactly upon the triangle $B D A$; and, therefore, the triangle $A E B$ is equal to the triangle $B D A$. Hence, the triangle $B D A$ is one-half of the rectangle $B D A E$. In like manner, the triangle $D C A$ is one-half of the rectangle $D C F A$. Therefore, the sum of the triangles $B D A$ and $D C A$ is equal to one-half the sum of the rectangles $B D A E$ and $D C F A$; in other words, the triangle $A B C$ is equal to one-half the rectangle $B C F E$. The rectangle $B C F E$ has the same base $B C$ and the same altitude $A D$ as the triangle $A B C$, and the area of the rectangle is equal to the product of its base and altitude (Art. **23**, Part 4). Therefore, the area of the triangle is equal to one-half the product of its base $B C$ and altitude $A D$.

EXAMPLE.—Find the area of the triangle $A B C$, Fig. 8.

SOLUTION.—The base of the triangle is $3\frac{3}{4}$ inches and its altitude is $2\frac{1}{4}$ inches. Therefore, its area is $\frac{1}{2} \times 3\frac{3}{4} \times 2\frac{1}{4}$, or $4\frac{7}{32}$ sq. in. Ans.

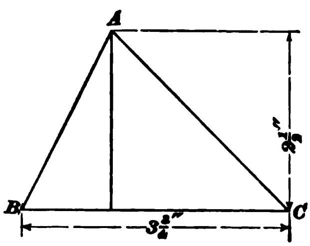


FIG. 8.

63. When the three sides of a triangle are given, its area is found by the following rule:

Rule.—*From half the sum of the three sides, subtract each side separately; find the continued product of the half sum of the sides and the three remainders; the square root of this continued product is equal to the area of the triangle.*

EXAMPLE.—Find the area of a triangle whose sides are 73 inches, 52 inches, and 35 inches, respectively.

SOLUTION.—The sum of the sides is $73 + 52 + 35$, or 160 inches; therefore, half the sum of the sides is $\frac{1}{2} \times 160$, or 80 inches.

Subtracting each side separately from half the sum of the sides, we get the three remainders $80 - 73$, or 7 inches; $80 - 52$, or 28 inches; and $80 - 35$, or 45 inches. Hence, the area of the triangle, in square inches, is $\sqrt{80 \times 7 \times 28 \times 45} = \sqrt{705{,}600}$. Pointing off the number 705,600 into periods of two figures each, we get 70'56'00, which shows that the integral part of the square root contains three figures. From the table of "Squares and Cubes," Part 6, we get $\sqrt{7{,}056} = 84$; therefore, $\sqrt{705{,}600} = 840$. Thus, the area is 840 sq. in. Ans.

64. Area of a Trapezoid.—*The area of a trapezoid is equal to one-half the product of its altitude and the sum of its parallel sides.*

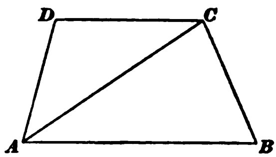


FIG. 9

The diagonal AC divides the trapezoid $ABCD$, Fig. 9, into the two triangles ABC and ADC. The triangle ABC has for its base AB, the longer of the parallel sides of the trapezoid, and its altitude is the same as the altitude of the trapezoid. Hence, by Art. **62,** we have

$$(\text{area of } ABC) = \frac{1}{2} \times AB \times (\text{altitude of trapezoid}).$$

The triangle ACD can be considered as having CD, the shorter of the parallel sides of the trapezoid, for its base and its vertex at A; then its altitude is the same as the altitude of the trapezoid. Therefore, by Art. **62,** we have

$$(\text{area of } ADC) = \frac{1}{2} \times DC \times (\text{altitude of the trapezoid}).$$

But the sum of the areas of the two triangles is the area of the trapezoid; that is,

$$(\text{area of trapezoid } ABCD) = \frac{1}{2} \times AB \times (\text{altitude}) + \frac{1}{2} \times DC \times (\text{altitude}).$$

By Art. **36,** Part 4,

$$(\text{area of } ABCD) = \frac{1}{2} \times (AB + DC) \times (\text{altitude}).$$

This equation when stated in words is the same as the principle stated in italics at the beginning of this article.

EXAMPLE.—Find the area of a trapezoid shown in Fig. 10.

SOLUTION.—By the principle of this article, the area of this trapezoid is equal to $\frac{1}{2} \times (AB + CD) \times$ (altitude); that is, $\frac{1}{2}(4.5 + 2.5) \times 2 = \frac{1}{2} \times 7 \times 2 = 7$. Therefore the area is 7 sq. in. Ans.

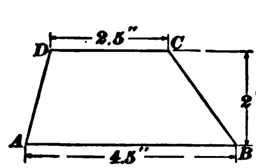


FIG. 10.

65. Area of a Polygon.—*The area of a polygon is found by suitably dividing it into triangles, parallelograms, and trapezoids, and calculating the area of each of these figures. The sum of these partial areas is equal to the area of the polygon.*

EXAMPLE.—Fig. 11 shows the cross-section of an **I** beam; find the area of this section, neglecting the rounding of the corners.

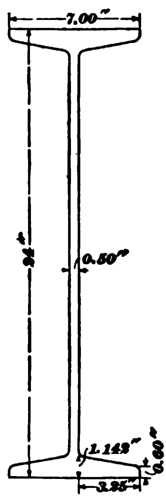


FIG. 11.

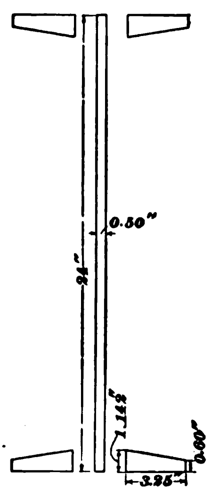


FIG. 12.

SOLUTION.—This section can be divided into a rectangle and four equal trapezoids, as shown in Fig. 12.

The area of the rectangle in square inches =

$$24 \times 0.50 = 12.$$

The area of each of the trapezoids in square inches =

$$\tfrac{1}{2}(1.142 + 0.60) \times 3.25 = \tfrac{1}{2} \times 1.742 \times 3.25.$$

Therefore, the sum of the areas of the four trapezoids in square inches is $4 \times \frac{1}{2} \times 1.742 \times 3.25 = 1.742 \times 6.5 = 11.323$.

Hence, the area of the whole section =

$$(12 + 11.323) \text{ sq. in.} = 23.323 \text{ sq. in.} \quad \text{Ans.}$$

EXAMPLES FOR PRACTICE.

1. Find the area of a triangle whose base is 20.5 feet and whose altitude is 10.2 feet. Ans. 104.55 sq. ft.

2. Find the area of a triangle whose sides are 10 inches, 17 inches, and 21 inches, respectively. Ans. 84 sq. in.

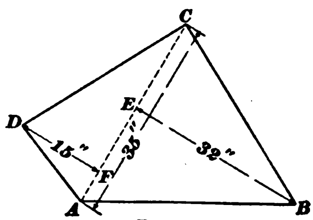


FIG. 13.

3. Find the area of a trapezoid whose parallel bases are 33 feet and 25 feet, and whose altitude is 13 feet. Ans. 377 sq. ft.

4. Find the area of the trapezium shown in Fig. 13. Ans. 822.5 sq. in.

HINT.—Find the area of the triangle *A B C* and of the triangle *A C D*, and the sum of the areas of these two triangles is the area of the trapezium.

DECIMAL EQUIVALENTS.

66. Decimal equivalents for the fractions most commonly used by mechanics are given in the table on the following page. Since the results obtained by the operations necessary in mensuration are usually decimal quantities, it is desirable that a convenient means should be afforded for their conversion into ordinary fractions. By the aid of this table, any decimal may be readily resolved into its corresponding fraction or into a fraction that approaches its real value nearly enough for all practical purposes.

TABLE OF DECIMAL EQUIVALENTS.

Group	Fraction	Decimal
8ths	$\frac{1}{8}$	.125
	$\frac{1}{4}$	.25
	$\frac{3}{8}$	.375
	$\frac{1}{2}$	.5
	$\frac{5}{8}$	.625
	$\frac{3}{4}$	.75
	$\frac{7}{8}$	.875
16ths	$\frac{1}{16}$	.0625
	$\frac{3}{16}$	.1875
	$\frac{5}{16}$	.3125
	$\frac{7}{16}$	.4375
	$\frac{9}{16}$	.5625
	$\frac{11}{16}$	.6875
	$\frac{13}{16}$	.8125
	$\frac{15}{16}$	.9375
32ds	$\frac{1}{32}$	.03125
	$\frac{3}{32}$	.09375
	$\frac{5}{32}$	.15625
	$\frac{7}{32}$	.21875
	$\frac{9}{32}$	.28125
	$\frac{11}{32}$	.34375
	$\frac{13}{32}$	.40625
	$\frac{15}{32}$	.46875
	$\frac{17}{32}$	.53125
	$\frac{19}{32}$	.59375
	$\frac{21}{32}$	.65625
	$\frac{23}{32}$	.71875
	$\frac{25}{32}$	.78125
	$\frac{27}{32}$	.84375
	$\frac{29}{32}$	.90625
	$\frac{31}{32}$	.96875
64ths	$\frac{1}{64}$	.015625
	$\frac{3}{64}$	.046875
	$\frac{5}{64}$	.078125
	$\frac{7}{64}$	.109375
	$\frac{9}{64}$	.140625
	$\frac{11}{64}$	.171875
	$\frac{13}{64}$	.203125
	$\frac{15}{64}$	.234375
	$\frac{17}{64}$	.265625
	$\frac{19}{64}$	.296875
	$\frac{21}{64}$	.328125
	$\frac{23}{64}$	.359375
	$\frac{25}{64}$	.390625
	$\frac{27}{64}$	.421875
	$\frac{29}{64}$	.453125
	$\frac{31}{64}$	.484375
	$\frac{33}{64}$	.515625
	$\frac{35}{64}$	.546875
	$\frac{37}{64}$	.578125
	$\frac{39}{64}$	.609375
	$\frac{41}{64}$	.640625
	$\frac{43}{64}$	.671875
	$\frac{45}{64}$	.703125
	$\frac{47}{64}$	.734375
	$\frac{49}{64}$	.765625
	$\frac{51}{64}$	.796875
	$\frac{53}{64}$	.828125
	$\frac{55}{64}$	.859375
	$\frac{57}{64}$	.890625
	$\frac{59}{64}$	.921875
	$\frac{61}{64}$	.953125
	$\frac{63}{64}$	.984375

ARITHMETIC.

(PART 8.)

RATIO AND PROPORTION.

1. When a number is divided by another number of the same kind, the quotient is called the **ratio** of the dividend to the divisor. Thus, when 12 feet is divided by 3 feet, the quotient $\frac{12 \text{ feet}}{3 \text{ feet}}$ is called the ratio of 12 feet to 3 feet.

2. The ratio of 12 feet to 3 feet is $\frac{12 \text{ feet}}{3 \text{ feet}} = 4$, and the number of times that 12 feet contains 3 feet is $\frac{12 \text{ feet}}{3 \text{ feet}} = 4$; that is, the ratio of 12 feet to 3 feet is the same as the number of times that 12 feet contains 3 feet. Hence, *the ratio of one number to another number of the same kind is the same as the number of times the first number contains the second.*

3. From Art. **2**, the ratio of 12 feet to 3 feet is the answer to the question "How many times does 12 feet contain 3 feet?" Hence, *a ratio is an abstract number* (Art. **1**, Part 1).

4. *A ratio is simply a fraction whose terms are numbers of the same kind.* Thus, the ratio $\frac{12 \text{ feet}}{3 \text{ feet}}$ is a fraction whose terms are both lengths. There can be no ratio between two numbers that are not of the same kind; for example, there can be no ratio of 20 dollars to 6 inches.

5. In finding the ratio of two quantities of the same kind, we must express the two quantities in terms of the same unit. For example, to find the ratio of a length of 3 feet to a length of 5 inches, we must express both lengths in inches, or we must express both lengths in feet. Thus, the ratio of a length of 3 feet to a length of 5 inches is $\frac{3 \times 12 \text{ inches}}{5 \text{ inches}} = \frac{36 \text{ inches}}{5 \text{ inches}} = \frac{36}{5} = 7.2$; or the ratio is $\frac{3 \text{ feet}}{\frac{5}{12} \text{ feet}} = \frac{12 \times 3 \text{ feet}}{5 \text{ feet}} = \frac{36 \text{ feet}}{5 \text{ feet}} = \frac{36}{5} = 7.2$. Both methods give the same result; namely, that a length of 3 feet is 7.2 times as great as a length of 5 inches.

6. When it is desired to compare the magnitudes of two quantities of the same kind, the comparison is usually made by finding the ratio of the first quantity to the second. For example, in comparing the lengths of the lines *A B* and *C D*, Fig. 1, we say that the ratio of *A B* to *C D* is

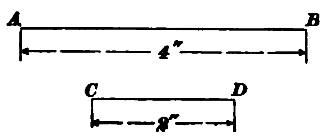


FIG. 1.

$$\frac{4 \text{ inches}}{2 \text{ inches}} = \frac{4}{2} = 2;$$

in other words, the line *A B* is twice as long as the line *C D*.

7. The ratio of the length of the line *A B* to the length of the line *C D*, Fig. 1, may be written in any of the following forms: $\frac{4 \text{ inches}}{2 \text{ inches}}$; 4 inches | 2 inches; 4 inches : 2 inches; or 4 inches ÷ 2 inches.

When this ratio is written in the form 4 inches : 2 inches, it is read *the ratio of four inches to two inches*; when the ratio is written in any of the other forms, it may be read in the same way.

8. Since a ratio is simply a fraction, every principle that was proved for fractions holds equally for ratios. For example, the value of a fraction is not changed by multiplying or dividing both of its terms by the same number (Art. **26,** Part 7); hence, a ratio is not changed by multiplying or dividing both of its terms by the same number. Thus, the fraction $\frac{2}{3}$ = the fraction $\frac{2 \times 5}{3 \times 5}$; and, in like manner, the ratio $\frac{2}{3}$ = the ratio $\frac{2 \times 5}{3 \times 5}$.

9. When the terms of a fraction are interchanged, the fraction is said to be inverted (Art. **50,** Part 7). In like manner, if the terms of a given ratio are interchanged, the resulting ratio is called the **inverse** of the given ratio; thus, the ratio $\frac{3}{2}$ is the inverse of the ratio $\frac{2}{3}$.

10. It is sometimes necessary to determine which of two given ratios is the greater; this is easily done by expressing the ratios as common fractions and reducing these fractions to a common denominator.

EXAMPLE.—Which is greater, the ratio of 20 inches to 7 inches or the ratio of 183 pounds to 64 pounds ?

SOLUTION.—The ratio of 20 inches to 7 inches is $\frac{20 \text{ inches}}{7 \text{ inches}} = \frac{20}{7}$, and the ratio of 183 pounds to 64 pounds is $\frac{183 \text{ pounds}}{64 \text{ pounds}} = \frac{183}{64}$. Reducing the fractions $\frac{20}{7}$ and $\frac{183}{64}$ to a common denominator, we get

$$\frac{20}{7} = \frac{20 \times 64}{7 \times 64} = \frac{1,280}{7 \times 64},$$

and

$$\frac{183}{64} = \frac{183 \times 7}{7 \times 64} = \frac{1,281}{7 \times 64}.$$

Now, $\frac{1,281}{7 \times 64}$ is greater than $\frac{1,280}{7 \times 64}$; and, therefore, the ratio of 183 pounds to 64 pounds is greater than the ratio of 20 inches to 7 inches. Ans.

11. The **specific gravity** of a substance is the ratio of the weight of any volume of that substance to the weight of an equal volume of some other substance taken as the standard. For specific gravities of solids and liquids, distilled water is usually taken as the standard substance. For example, a cubic foot of cast iron weighs 450 pounds, and a cubic foot of water weighs 62.5 pounds, nearly; hence, the specific gravity of cast iron is equal to the ratio of the weight of a cubic foot of cast iron to the weight of a cubic foot of water $= \frac{450 \text{ pounds}}{62.5 \text{ pounds}} = 7.2$. Therefore, the specific gravity of cast iron is 7.2.

EXAMPLE.—A cubic foot of an alloy of lead and zinc weighs 516.25 pounds. What is its specific gravity? Take the weight of a cubic foot of water as 62.5 pounds.

SOLUTION.—Specific gravity of the alloy =

$$\frac{\text{weight of cubic foot of alloy}}{\text{weight of cubic foot of water}} = \frac{516.25 \text{ lb.}}{62.5 \text{ lb.}} = 8.26. \quad \text{Ans.}$$

EXAMPLES FOR PRACTICE.

1. Which is greater 10 : 7 or 16 : 11? Ans. 16 : 11.

2. The weight of a cubic foot of steel is 490 pounds. Find the specific gravity of steel. Ans. 7.84.

3. Which is greater, the ratio of 15 inches to 7 inches or the ratio 135 quarts to 63 quarts? Ans. The ratios are equal.

---

12. When four numbers are such that the ratio of the first to the second is equal to the ratio of the third to the fourth, the four numbers are said to be **proportionals;** and a statement of the equality of the two ratios is called a **proportion.** For example, the four numbers 8 inches, 3 inches, 16 pounds, and 6 pounds are proportionals; for, each of the ratios $\frac{8 \text{ inches}}{3 \text{ inches}}$ and $\frac{16 \text{ pounds}}{6 \text{ pounds}}$ is equal to the fraction $\frac{8}{3}$.

The statement of the equality of the ratios $\frac{8 \text{ inches}}{3 \text{ inches}}$ and $\frac{16 \text{ pounds}}{6 \text{ pounds}}$ may be written in any of the following forms:

$$\frac{8 \text{ inches}}{3 \text{ inches}} = \frac{16 \text{ pounds}}{6 \text{ pounds}}, \quad (1)$$

$$8 \text{ inches} : 3 \text{ inches} = 16 \text{ pounds} : 6 \text{ pounds}, \quad (2)$$

$$8 \text{ inches} : 3 \text{ inches} :: 16 \text{ pounds} : 6 \text{ pounds}. \quad (3)$$

These three equations are merely different ways of writing the same proportion. Equations (1) and (2) are both read *the ratio of 8 inches to 3 inches is equal to the ratio of 16 pounds to 6 pounds;* equation (3) is read *8 inches is to 3 inches as 16 pounds is to 6 pounds.*

In the older English books on arithmetic, a proportion was invariably written in the form of equation (3), but this form is now falling into disuse.

13. When four numbers are in proportion, the first and fourth are called the **extremes,** and the second and third are called the **means.** Thus, in the proportion

$$\frac{8 \text{ inches}}{3 \text{ inches}} = \frac{16 \text{ pounds}}{6 \text{ pounds}},$$

8 inches and 6 pounds are the extremes, and 3 inches and 16 pounds are the means.

14. From the definition of a ratio in Art. **1,** it is evident that when four numbers are proportionals, the first and second must be numbers of the same kind, and the third and fourth also must be numbers of the same kind. Thus, in the example of Art. **13,** the first number, 8 inches, and the second number, 3 inches, are both lengths; while the third number, 16 pounds, and the fourth number, 6 pounds, are both weights.

The ratio $\frac{8 \text{ inches}}{3 \text{ inches}} = \text{the ratio } \frac{8}{3}$,

and the ratio $\frac{16 \text{ pounds}}{6 \text{ pounds}} = \text{the ratio } \frac{16}{6}$;

hence, the proportion $\frac{8 \text{ inches}}{3 \text{ inches}} = \frac{16 \text{ pounds}}{6 \text{ pounds}}$

is equivalent to the proportion $\frac{8}{3} = \frac{16}{6}$.

Therefore, *we may treat the terms of a proportion as if they were abstract numbers.*

15. Test of Proportion.—*In any proportion, the product of the extremes is equal to the product of the means.*

For example, in the proportion

$$\frac{5}{15} = \frac{3}{9},$$

the product of the extremes is 5×9, and the product of the means is 15×3; and these two products are equal, for each of them is equal to 45.

16. When any three of the terms of a proportion are given, the fourth term can be found from the principle of Art. **15.**

EXAMPLE.—Find a quantity that has the same ratio to 18 pounds that 4 inches has to 12 inches.

SOLUTION.—We have to find the first term of the proportion

required quantity : 18 pounds = 4 inches : 12 inches.

The ratio 4 inches to 12 inches is equal to the ratio 4 : 12. Therefore, we have

required quantity : 18 pounds = 4 : 12.

Hence, by Art. **15,**

(required quantity) $\times$ 12 = 18 pounds $\times$ 4.

Therefore,

$$\text{required quantity} = \frac{18 \text{ pounds} \times 4}{12} = 6 \text{ lb.} \quad \text{Ans.}$$

17. When any three terms of a proportion are known, the unknown term can be found by the following rule:

Rule I. *To find an unknown extreme, divide the product of the means by the given extreme.*

Rule II. *To find an unknown mean, divide the product of the extremes by the given mean.*

EXAMPLE 1.—Find the unknown term in the equation

$$\text{unknown term} : 4 = 96 : 24.$$

SOLUTION.—By rule I,

$$\text{unknown term} = \frac{4 \times 96}{24} = 16. \quad \text{Ans.}$$

EXAMPLE 2.—Find the unknown term in the equation

$$4 : 6 = \text{unknown term} : 21.$$

SOLUTION.—By rule II,

$$\text{unknown term} = \frac{4 \times 21}{6} = 14. \quad \text{Ans.}$$

CONTINUED PROPORTION.

18. When three quantities are such that the ratio of the first to the second is equal to the ratio of the second to the third, the three quantities are said to be in **continued proportion**; the second quantity is said to be a **mean proportional** between the first and third; and the third quantity is said to be a **third proportional** to the first and second. For example, the three numbers 36, 18, and 9 are in continued proportion. For, we have $\frac{36}{18} = \frac{18}{9}$; that is, the ratio of the first to the second is equal to the ratio of the second to the third. In this example, 18 is the mean proportional between 36 and 9, and 9 is the third proportional to 36 and 18.

19. When three quantities are in continued proportion, we have

$$(\text{first}) : (\text{second}) = (\text{second}) : (\text{third}).$$

In this proportion, by Art. **15**, the product of the means is equal to the product of the extremes, or

$$(\text{second})^2 = (\text{first}) \times (\text{third}).$$

The square root of the first member of this equation must be equal to the square root of the second member. Hence,

$$(\text{second}) = \sqrt{(\text{first}) \times (\text{third})}.$$

Thus, we have the following principle:

20. *The mean proportional between two numbers is equal to the square root of their product.*

EXAMPLE.—Find a mean proportional between the two numbers 169 and 49.

SOLUTION.—The required mean proportional is equal to the square root of the product of the two given numbers, or $\sqrt{169 \times 49} = 13 \times 7 = 91$. Ans.

21. *When three quantities are in continued proportion, either extreme is equal to the square of the mean divided by the other extreme.*

EXAMPLE.—Find a third proportional to the two numbers 72 and 60.

SOLUTION.—The required number is the last term of the proportion

$$72 : 60 = 60 : \text{required number.}$$

The required number is equal to $\frac{60 \times 60}{72} = 50$. Ans.

UNIT METHOD.

22. In the older books on arithmetic, a large number of problems were solved by proportion; but these problems can be solved much more easily by the **unit method,** which we now proceed to explain by means of examples.

EXAMPLE 1.—If a pump discharging 4 gallons of water per minute can fill a tank in 20 hours, how long will it take a pump discharging 12 gallons per minute to fill the tank?

SOLUTION.—A pump discharging 4 gallons per minute fills the tank in 20 hours. Therefore, a pump discharging 1 gallon per minute fills it in 4×20 hours. Hence, a pump discharging 12 gallons per minute fills it in $\frac{4 \times 20 \text{ hours}}{12} = \frac{20 \text{ hours}}{3} = 6\frac{2}{3}$ hours. Ans.

EXAMPLE 2.—If 4 men earn $65.80 in 7 days, how much can 14 men, paid at the same rate, earn in 12 days?

SOLUTION.— 4 men in 7 days earn $65.80.

Therefore, 1 man in 7 days earns $\frac{\$65.80}{4}$.

Therefore, 1 man in 1 day earns $\frac{\$65.80}{4 \times 7}$.

Therefore, 1 man in 12 days earns $\frac{\$65.80 \times 12}{4 \times 7}$.

Therefore, 14 men in 12 days earn $\frac{\$65.80 \times 12 \times 14}{4 \times 7}$.

Canceling,

14 men in 12 days earn $\$65.80 \times 3 \times 2 = \$65.80 \times 6 = \$394.80$. Ans.

23. The student will notice that in the solution of these examples, in the successive steps, the operations of multiplication and division were merely indicated, and no multiplication or division was performed until the very last, and then the answer was obtained easily by cancelation. In arithmetical calculations, the student should make it an invariable habit to indicate the multiplications and divisions that occur in the successive steps of a solution, and not to perform these operations until the very last. Then, he will probably be able to use the principle of cancelation.

EXAMPLE.—If a block of granite 8 feet long, 5 feet wide, and 3 feet thick weighs 7,200 pounds, what is the weight of a block of granite 12 feet long, 8 feet wide, and 5 feet thick?

SOLUTION.—The first block contains $8 \times 5 \times 3$ cubic feet and the second block contains $12 \times 8 \times 5$ cubic feet.

Therefore, $8 \times 5 \times 3$ cu. ft. weigh 7,200 lb.

Therefore, 1 cu. ft. weighs $\frac{7,200 \text{ lb.}}{8 \times 5 \times 3}$.

Therefore,

$12 \times 8 \times 5$ cu. ft. weigh $\frac{7,200 \times 12 \times 8 \times 5 \text{ lb.}}{8 \times 5 \times 3}$.

Canceling,

$12 \times 8 \times 5$ cu. ft. weigh $7,200 \times 4$ lb. $= 28,800$ lb. Ans.

EXAMPLES FOR PRACTICE.

1. If a pump discharges 90,000 gallons of water in 20 hours, in what time will it discharge 144,000 gallons? Ans. 32 hr.

2. When the barometer stands at 30 inches, the pressure of the atmosphere is 14.7 pounds per square inch. What is the atmospheric pressure per square inch when the barometer stands at 29.5 inches? Give answer correct to three figures. Ans. 14.5.

MENSURATION.

24. Scale Drawings.—One of the most common applications of ratio and proportion is in connection with scale drawings. In many cases, it is impracticable to make a full-size drawing of an object. In order that the reduced drawing may represent the object in its proper proportions, it is drawn to a scale. For example, suppose we wish to make a drawing of an object 4 feet long on a sheet of paper that is only large enough to permit us to make the drawing 1 foot long (one-fourth of the actual length). In order that all the lines of the drawing may represent the corresponding lines of the object in their true proportions, a scale is used that is so graduated that each division representing 1 foot is $\frac{1}{4}$ of a foot in length. Since 3 inches $= \frac{1}{4}$ of a foot, a drawing that represents an object $\frac{1}{4}$ of the real size is said to be made to a scale of *3 inches to the foot.* Similarly, if a drawing is made half size, a length of 6 inches on the scale represents 1 foot of the object; this is called a scale of *6 inches to the foot.* If a drawing is $\frac{1}{8}$ of the true size, then $\frac{1}{8}$ of 12 inches, or $1\frac{1}{2}$ inches on the scale, represents 1 foot, and the scale is said to be $1\frac{1}{2}$ *inches to the foot.*

25. Figures that have the same shape are called **similar figures.**

26. In order that two polygons may be similar, it is manifestly necessary that each angle of the one shall be equal

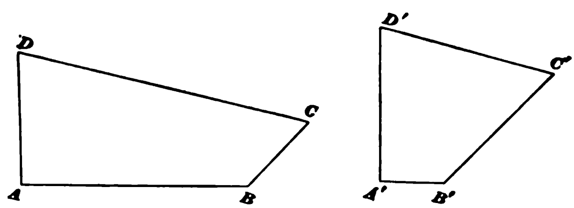


FIG. 2.

to the corresponding angle of the other. But this condition is not sufficient; for in the quadrilaterals $ABCD$ and $A'B'C'D'$, Fig. 2, we have the angle A equal to its corresponding angle A', the angle B equal to its corresponding angle B', the angle C equal to its corresponding angle C', and the angle D equal to its corresponding angle D'; but evidently these two quadrilaterals are not of the same shape.

27. If a drawing of an object is made to a certain scale, the drawing has the same shape as the object, but differs from it in size.

Let the object be the quadrilateral $ABCD$, and let $A'B'C'D'$ be a half-size drawing of the quadrilateral $ABCD$, Fig. 3. Then, the quadrilaterals $ABCD$ and $A'B'C'D'$ have the same shape and are therefore similar figures.

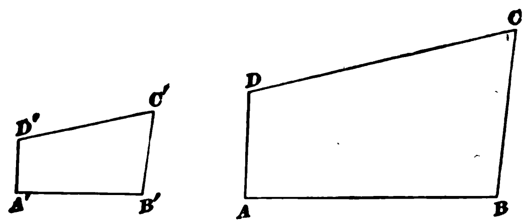


FIG. 3.

Since $A'B'C'D'$ is a half-size drawing of $ABCD$, each side of $A'B'C'D'$ is equal to one-half the corresponding side of $ABCD$; therefore,

$$\frac{A'B'}{AB} = \frac{B'C'}{BC} = \frac{C'D'}{CD} = \frac{D'A'}{DA} = \frac{1}{2}.$$

Also, the corresponding angles of the two quadrilaterals are equal; therefore, if we denote each angle by the letter at its vertex, we have

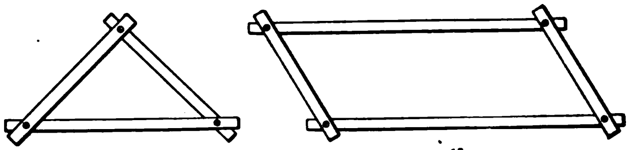
$$A = A',\ B = B',\ C = C',\ D = D'.$$

Hence, we see that there are two conditions necessary that two polygons may be similar; these conditions are:

I. *Their corresponding angles must be equal.*

II. *Their corresponding sides must be proportional.*

28. If a triangular frame is constructed of three bars fastened together by a single pivot at each joint, as shown in (*a*), Fig. 4, it will be found that the frame is rigid, and its shape cannot be altered without bending or breaking the bars. But if a quadrilateral frame is constructed of four bars fastened together by a single pivot at each joint, as



(*a*) (*b*)

FIG. 4.

shown in (*b*), Fig. 4, it will be found that this shape of the frame can be altered freely without bending or breaking any of the bars. Hence, we have the following important principles:

I. *When the three sides of a triangle are given, both the shape and the size of the triangle are fixed.*

II. *In any polygon of more than three sides, when all the sides of the polygon are given, the shape is not fixed.*

Suppose that a carpenter makes the frame of a wooden gate as shown in Fig. 5. Evidently the end *A B* of this gate is supported only by the strength of the joints, and will soon

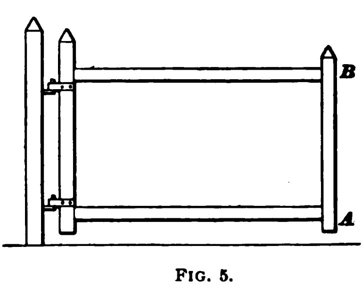


FIG. 5.

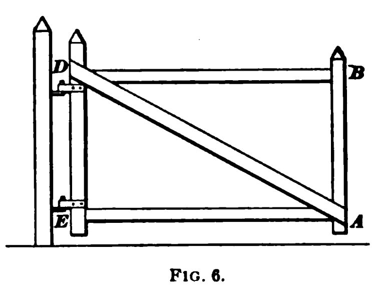


FIG. 6.

sink down until *A* drags on the ground. In order to strengthen this gate, it is necessary to put on a bar, as *A D*, Fig. 6; then, the triangular frame *A D E* is rigid and supports the end *A B* of the gate.

29. In Fig. 7, each side of the triangle $A' B' C'$ is parallel to the corresponding side of the triangle $A B C$; and, therefore, the corresponding angles of these two triangles are equal. It is evident also that the triangles $A B C$ and $A' B' C'$ have the same shape. Hence, we have the following principle:

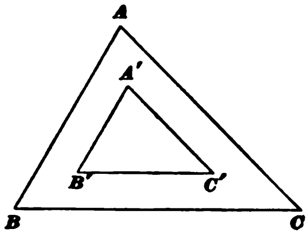


FIG. 7.

If three angles of one triangle are equal respectively to the three angles of another triangle, the two triangles are similar.

30. *From Arts.* **28** *and* **29,** *it appears that the shape of a triangle is fixed if either all its sides or all its angles are given; but the same is not true of any other polygon.*

31. *If the three angles of one triangle are equal, respectively, to the three angles of another triangle, the corresponding sides of the two triangles are proportional.*

For, by Art. **29,** the two triangles are similar, and the corresponding sides of similar figures are proportional (Art. **27**).

EXAMPLE.—In the triangles $A B C$ and $A' B' C'$, Fig. 8, we are given: angle A = angle A', angle B = angle B', and angle C = angle C'; also, the sides $B C$, $C A$, $A B$, and $B' C'$ have the dimensions that are marked on them; find the lengths of the sides $C' A'$ and $A' B'$.

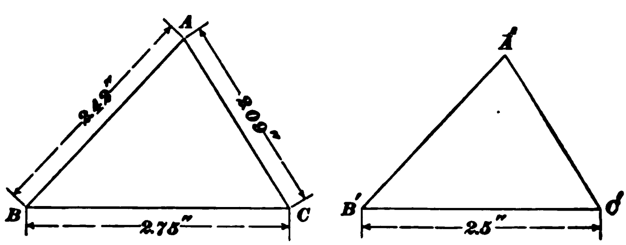


FIG. 8.

SOLUTION.—These two triangles are similar, because their angles are equal. Therefore, by condition II, Art. **27,** we have

$$\frac{A B}{A' B'} = \frac{B C}{B' C'} = \frac{C A}{C' A'}.$$

Putting $AB = 2.42''$, $BC = 2.75''$, $CA = 2.09''$, and $B'C' = 2.5''$, we get

$$\frac{2.42''}{A'B'} = \frac{2.75''}{2.5''} = \frac{2.09''}{C'A'}.$$

Thus we have $$\frac{2.42''}{A'B'} = \frac{2.75''}{2.5''},$$

or $$\frac{2.42''}{A'B'} = \frac{2.75}{2.5}.$$

In this proportion, the product of the means is equal to the product of the extremes (Art. **15**). Therefore,

$$A'B' \times 2.75 = 2.42'' \times 2.5,$$

whence $$A'B' = \frac{2.42'' \times 2.5}{2.75}.$$

Hence, by cancelation, $$A'B' = \frac{2.42''}{1.1} = 2.2''.$$

Also, we have $$\frac{2.75''}{2.5''} = \frac{2.09''}{C'A'},$$

or $$\frac{2.75}{2.5} = \frac{2.09''}{C'A'}.$$

In this proportion, the product of the extremes is equal to the product of the means (Art. **15**). Therefore,

$$2.75 \times C'A' = 2.5 \times 2.09''.$$

Hence, $$C'A' = \frac{2.5 \times 2.09''}{2.75} = \frac{2.09''}{1.1} = 1.9''.$$

Thus, $A'B' = 2.2''$ and $C'A' = 1.9''$. Ans.

32. *If a straight line is drawn parallel to the base of a triangle and cutting the two sides, it divides the sides proportionally.*

Let DE be drawn parallel to the base BC of the triangle ABC, and cutting the sides AB and AC in D and E, as shown in Fig. 9. Then, since DE is parallel to BC, the angle ADE is equal to the angle ABC, and the angle AED is equal to the angle ACB (Art. **34**, Part 3). Also, the angle A is common to the two triangles ABC and ADE. Thus, the three angles of the triangle ABC are equal,

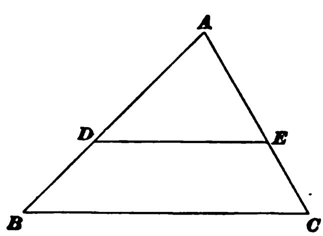


FIG. 9.

respectively, to the three angles of the triangle $A\,D\,E$. Therefore, $A\,B\,C$ and $A\,D\,E$ are similar triangles, and their corresponding sides are proportional (Art. **29**). Thus,

$$\frac{A\,D}{A\,B} = \frac{A\,E}{A\,C}. \qquad (1.)$$

That is, the sides $A\,B$ and $A\,C$ are divided proportionally by the points D and E.

It can also be proved that

$$\frac{D\,B}{A\,B} = \frac{E\,C}{A\,C}, \qquad (2.)$$

and that

$$\frac{A\,D}{D\,B} = \frac{A\,E}{E\,C}. \qquad (3.)$$

EXAMPLE.—In Fig. 10, $D\,E$ is parallel to $B\,C$, and $A\,D$ is one-third of $A\,B$; find the length of $A\,E$.

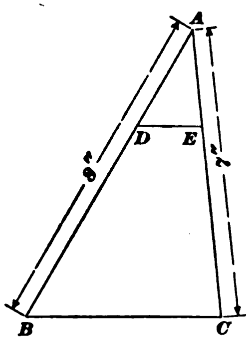


FIG. 10.

SOLUTION.—By equation (1), we have

$$\frac{A\,D}{A\,B} = \frac{A\,E}{A\,C}.$$

But we are given that $A\,D$ is one-third of $A\,B$; therefore,

$$\frac{A\,D}{A\,B} = \frac{1}{3}.$$

Substituting this value for $\frac{A\,D}{A\,B}$ in the preceding equation, we get

$$\frac{1}{3} = \frac{A\,E}{A\,C};$$

therefore, $A\,C = 3 \times A\,E$, (Art. **15**)

which gives $A\,E = \frac{1}{3} \times A\,C = \frac{1}{3} \times 7'' = 2\frac{1}{3}''$. Ans.

33. *If the three sides of one triangle are proportional to the three sides of another triangle, the two triangles are similar.*

EXAMPLE.—Determine whether the two triangles (a) and (b), Fig. 11, are similar.

SOLUTION.—For convenience of reference, we shall letter the vertexes of the triangle (a), A, B, and C. The angle A is opposite to the shortest side of the triangle (a); hence, if the two triangles are similar, the angle in (b) which corresponds to the angle A must be opposite to the shortest side of the triangle (b). Therefore, in the triangle (b) we put the letter A' at the vertex opposite to the side whose length is 2.45". Similarly, the angle B is opposite to the longest side of the triangle (a).

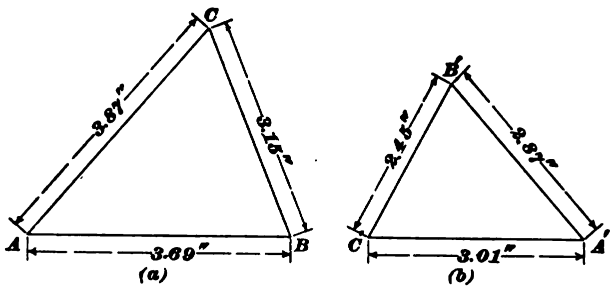


FIG. 11.

Therefore, in the triangle (b), we put the letter B' at the vertex opposite to the longest side; that is, opposite to the side whose length is 3.01". Then, we put the letter C' at the remaining vertex of the triangle (b). If the two triangles are similar, we must have

$$\frac{\text{longest side of } (a)}{\text{longest side of } (b)} = \frac{\text{medium side of } (a)}{\text{medium side of } (b)} = \frac{\text{shortest side of } (a)}{\text{shortest side of } (b)},$$

or

$$\frac{C\,A}{C'\,A'} = \frac{A\,B}{A'\,B'} = \frac{B\,C}{B'\,C'}.$$

Now, if

$$\frac{C\,A}{C'\,A'} = \frac{A\,B}{A'\,B'},$$

we must have $C\,A \times A'\,B' = C'\,A' \times A\,B$. (Art. **15**)

But, $C\,A \times A'\,B' = 3.87 \times 2.87 = 11.1069,$

and $C'\,A' \times A\,B = 3.01 \times 3.69 = 11.1069.$

Therefore, $C\,A \times A'\,B' = C'\,A' \times A\,B,$

and

$$\frac{C\,A}{C'\,A'} = \frac{A\,B}{A'\,B'}.$$

Again, if

$$\frac{C\,A}{C'\,A'} = \frac{B\,C}{B'\,C'},$$

we must have $C\,A \times B'\,C' = C'\,A' \times B\,C.$ (Art. **15**)

But, $C\,A \times B'\,C' = 3.87 \times 2.45 = 9.4815,$

and $C'\,A' \times B\,C = 3.01 \times 3.15 = 9.4815.$

Therefore, $C\,A \times B'\,C' = C'\,A' \times B\,C,$

and

$$\frac{C\,A}{C'\,A'} = \frac{B\,C}{B'\,C'}.$$

Thus, we have proved that each of the ratios $\frac{AB}{A'B'}$ and $\frac{BC}{B'C'}$ is equal to $\frac{CA}{C'A'}$; therefore, we have

$$\frac{CA}{C'A'} = \frac{AB}{A'B'} = \frac{BC}{B'C'};$$

and, consequently, the triangles (*a*) and (*b*) are similar. Ans.

34. *In any right triangle, the perpendicular drawn from the vertex of the right angle to the hypotenuse divides the triangle into two triangles that are similar to the whole triangle and to each other.* Thus, the perpendicular CD, Fig. 12, divides the right triangle ABC into two right triangles ACD and CBD; and the three triangles ABC, ACD, and CBD are similar.

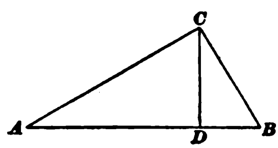


FIG. 12.

35. In the similar triangles ACD and CBD, Fig. 12, we must have

$$\frac{\text{medium side of } ACD}{\text{medium side of } CBD} = \frac{\text{shortest side of } ACD}{\text{shortest side of } CBD},$$

or

$$\frac{AD}{CD} = \frac{CD}{DB}.$$

That is, CD is a mean proportional between AD and DB. Thus, we have the following principle:

36. *In any right triangle, the perpendicular from the vertex of the right angle to the hypotenuse is a mean proportional between the two parts into which it divides the hypotenuse.*

EXAMPLE.—In the right triangle ABC, Fig. 13, find the length of the perpendicular CD, and then find the area of the triangle.

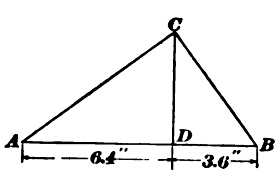


FIG. 13.

SOLUTION.—The perpendicular CD is a mean proportional between the two parts AD and DB into which it divides the hypotenuse. Therefore, by Art. **20**, CD is equal to the square root of the product of AD and DB. Now,

$$\sqrt{3.6 \times 6.4} = \sqrt{\frac{36 \times 64}{100}} = \frac{6 \times 8}{10} = 4.8.$$

Therefore, $CD = 4.8''$.

By Art. **62,** Part 7, the area of the triangle is equal to one-half the product of its base and altitude. The base is 6.4" + 3.6" or 10" and the altitude is 4.8". Therefore, the area is $\frac{1}{2} \times 10 \times 4.8$, or 24 sq. in.

$$\left.\begin{array}{l} CD = 4.8''. \\ \text{Area} = 24 \text{ sq. in.} \end{array}\right\} \text{Ans.}$$

37. By Art. **34,** the right triangles ABC and ACD, Fig. 12, are similar.

Therefore, by Art. **27,**

$$\frac{\text{longest side of } ABC}{\text{longest side of } ACD} = \frac{\text{medium side of } ABC}{\text{medium side of } ACD},$$

or

$$\frac{AB}{AC} = \frac{AC}{AD}.$$

That is, AC is a mean proportional between AB and AD.

By Art. **34,** the triangles ABC and CBD are similar.

Therefore, by Art. **27,**

$$\frac{\text{longest side of } ABC}{\text{longest side of } CBD} = \frac{\text{shortest side of } ABC}{\text{shortest side of } CBD},$$

or

$$\frac{AB}{CB} = \frac{CB}{DB}.$$

That is, CB is a mean proportional between AB and DB.

Hence, we have the following principle:

38. *In any right triangle, either leg is a mean proportional between the whole hypotenuse and the part of the hypotenuse intercepted between that leg and the perpendicular from the vertex of the right angle to the hypotenuse.*

EXAMPLE.—Find the lengths of the legs of the right triangle ABC, Fig. 14, in which CD is the perpendicular from the vertex of the right angle to the hypotenuse.

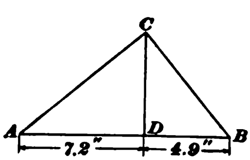


FIG. 14.

SOLUTION.—The hypotenuse AB is $7.2 + 4.9$ or 12.1 inches long. The leg CB is a mean proportional between the hypotenuse AB and the part DB. Therefore, by Art. **20,** CB is equal to the square root of the product of AB and DB. Now,

$$\sqrt{12.1 \times 4.9} = \sqrt{\frac{121 \times 49}{100}} = \frac{11 \times 7}{10} = 7.7.$$

Therefore, CB is 7.7".

The leg AC is a mean proportional between AB and AD; that is, $AC = \sqrt{AB \times AD}$. Now,

$$\sqrt{12.1 \times 7.2} = \sqrt{\frac{121 \times 72}{100}} = \frac{11 \times \sqrt{72}}{10} = \frac{11 \times 8.49}{10} = 9.34.$$

Therefore, $AC = 9.34"$.

$$\left. \begin{aligned} CB &= 7.7". \\ AC &= 9.34". \end{aligned} \right\} \text{Ans.}$$

39. The triangles ABC and $A'B'C'$, Fig. 15, are similar, because their sides are proportional (Art. **33**). The areas of these triangles can be found by the rule of Art. **63,** Part 7. In the triangle ABC, half the sum of the sides BC, CA, and AB is $\frac{1}{2}(21 + 17 + 10)$, or 24 inches.

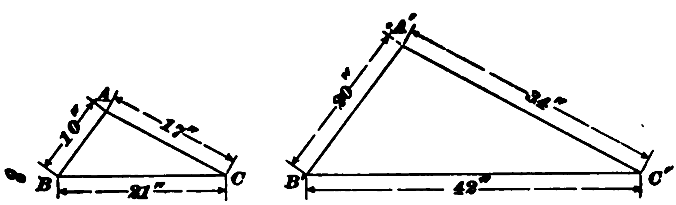


FIG. 15.

Subtracting the sides BC, CA, and AB separately from half the sum of the sides, we get the remainders 3 inches, 7 inches, and 14 inches. Therefore, by Art. **63,** Part 7, the area of the triangle ABC is $\sqrt{24 \times 3 \times 7 \times 14}$, or 84 square inches. Half the sum of the sides of the triangle $A'B'C'$ is $\frac{1}{2}(42 + 34 + 20)$, or 48 inches. Subtracting the sides $B'C'$, $C'A'$, and $A'B'$ separately from half the sum of the sides, we get the three remainders 6 inches, 14 inches, and 28 inches. Therefore, by Art. **63,** Part 7, the area of the triangle $A'B'C'$ is $\sqrt{48 \times 6 \times 14 \times 28}$, or 336 square inches.

Hence,

$$\frac{\text{area of } ABC}{\text{area of } A'B'C'} = \frac{84 \text{ sq. in.}}{336 \text{ sq. in.}} = \frac{1}{4}.$$

But, the ratio of the square of BC to the square of $B'C'$ is

$$\frac{\overline{BC}^2}{\overline{B'C'}^2} = \frac{21^2}{42^2} = \frac{1}{4}.$$

Thus each of the ratios $\frac{\text{area of } A\,B\,C}{\text{area of } A'B'C'}$, and $\frac{\overline{B\,C}^2}{\overline{B'C'}^2}$ is equal to $\frac{1}{4}$. Therefore,

$$\frac{\text{area } A\,B\,C}{\text{area } A'B'C'} = \frac{\overline{B\,C}^2}{\overline{B'C'}^2}.$$

That is, the areas of similar triangles are proportional to the squares of their corresponding sides. This is a particular case of the following general proposition:

40. *The areas of similar figures are proportional to the squares of their corresponding lines.*

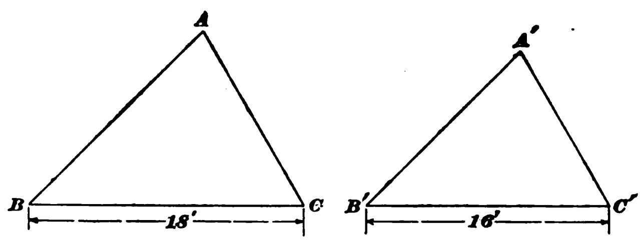


FIG. 16.

EXAMPLE.—In Fig. 16, $A\,B\,C$ and $A'\,B'\,C'$ are similar triangles and the area of the triangle $A'\,B'\,C'$ is 184 square feet; find the area of the triangle $A\,B\,C$.

SOLUTION.—We have

$$\frac{\text{area of } A\,B\,C}{\text{area of } A'\,B'\,C'} = \frac{\overline{B\,C}^2}{\overline{B'\,C'}^2}.$$

Hence,
$$\frac{\text{area of } A\,B\,C}{184} = \frac{18^2}{16^2} = \left(\frac{18}{16}\right)^2 = \left(\frac{9}{8}\right)^2.$$

By Rule I, Art. **17**,

$$\text{area of } A\,B\,C = \left(\frac{9}{8}\right)^2 \times 184 = \frac{9 \times 9 \times 184}{8 \times 8} = 232\tfrac{7}{8}.$$

Thus, the area of $A\,B\,C$ is $232\frac{7}{8}$ sq. ft. Ans.

41. An **equilateral polygon** is a polygon whose sides are all equal. Thus, ABC, Fig. 17, is an equilateral polygon of three sides, or an equilateral triangle, because $BC = CA = AB$.

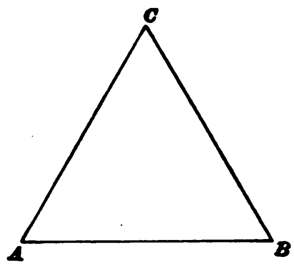


FIG. 17.

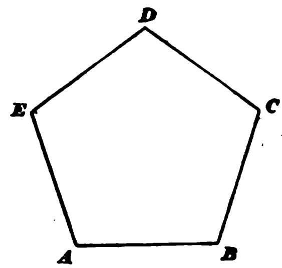


FIG. 18.

42. An **equiangular polygon** is a polygon whose angles are all equal. Thus, $ABCDE$, Fig. 18, is an equiangular polygon of five sides, or an equiangular pentagon, because angle A = angle B = angle C = angle D = angle E.

43. A polygon that is both equilateral and equiangular is called a **regular polygon.**

44. If perpendiculars are erected to all the sides of a regular polygon at their middle points, these perpendiculars meet in a common point, which is called the **center** of the regular polygon. *The lines drawn from the center of a regular polygon to its vertexes are all equal; and also the perpendiculars from the center to the sides of a regular polygon are all equal.*

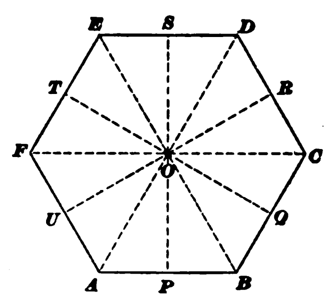


FIG. 19.

Thus, in Fig. 19, O is the center of the regular pentagon $ABCDEF$, and we have

$$OA = OB = OC = OD = OE = OF;$$

also, $$OP = OQ = OR = OS = OT = OU.$$

A circle with O as center and OA as radius will pass through each of the vertexes A, B, C, D, E, and F; and a circle with O as center and OP as radius will pass through each of the points P, Q, R, S, T, and U.

45. *If a triangle is equilateral, it is also equiangular; that is, every equilateral triangle is a regular triangle.*

46. Fig. 20 shows all the regular polygons that have less than thirteen sides.

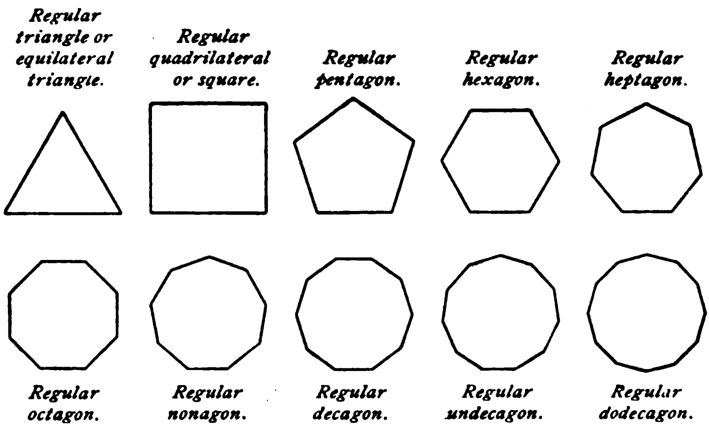


FIG. 20.

47. *Two regular polygons of the same number of sides are similar figures.*

By Art. **40,** the areas of similar figures are proportional to the squares of their corresponding sides; therefore,

$$\frac{\text{area of } A'B'C'D'E'}{\text{area of } ABCDE} = \frac{\overline{A'B'}^2}{\overline{AB}^2} = \frac{2^2}{1}.$$

Hence, by Rule I, Art. **17,**

$$\text{area of } A'B'C'D'E' = 2^2 \times \text{area of } ABCDE.$$

Thus, the area of the regular pentagon $ABCDE$, Fig. 21, is equal to the product obtained by multiplying the square of one of its sides by the area of the similar polygon whose side is equal to the unit of length.

This is a particular case of the following general proposition:

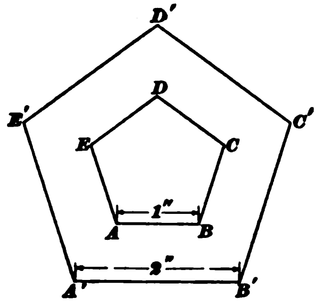


FIG. 21.

48. *The area of any regular polygon is equal to the product obtained by multiplying the square of one of its sides by the area of the similar polygon whose side is equal to the unit of length.*

This proposition enables us to find the area of any regular polygon, if we know the area of a similar polygon each of whose sides is equal to the unit of length.

49. The following table gives the area, correct to four figures, of any regular polygon of less than thirteen sides, when the length of each side of the polygon is equal to the unit of length.

AREAS OF REGULAR POLYGONS.

Number of Sides.	Name.	Area When Side = 1.
3	Triangle	0.433
4	Square	1.000
5	Pentagon	1.720
6	Hexagon	2.598
7	Heptagon	3.634
8	Octagon	4.828
9	Nonagon	6.182
10	Decagon	7.694
11	Undecagon	9.366
12	Dodecagon	11.196

50. The principle of Art. **48** enables us to use the table of Art. **49** to calculate the area of any regular polygon of less than thirteen sides.

EXAMPLE.—The side of a regular octagon is 3 inches; find its area.

SOLUTION.—From Art. **48** the area of the octagon is equal to the product obtained by multiplying the square of its side by the area of the regular octagon whose side is equal to the unit of length. From the table, the area of a regular octagon whose side is 1 inch is 4.828 square inches. Hence, the area of the octagon whose side is 3 inches is $3^2 \times 4.828$ or 43.452 sq. in. Ans.

EXAMPLES FOR PRACTICE.

1. The triangles $A\,B\,C$ and $A'\,B'\,C'$ are similar. Being given $B\,C = 9''$, $C\,A = 7''$, $A\,B = 6''$, and $B'\,C' = 67.5''$; find $C'\,A'$ and $A'\,B'$.
Ans. $C'\,A = 52.5''$ and $A'\,B' = 45''$.

2. The triangles $A\,B\,C$ and $A'\,B'\,C'$ are similar. Being given $B\,C = 13''$, $C\,A = 14''$, $A\,B = 15''$, and $B'\,C' = 19.5''$; find the area of the triangle $A'\,B'\,C'$.
Ans. 189 sq. in.

3. The perpendicular from the vertex of the right angle of a right triangle divides the hypotenuse into parts of 23.04 inches and 1.96 inches. Find (*a*) the length of the perpendicular; (*b*) the lengths of the legs of the triangle.
Ans. { Perpendicular is 6.72". Legs are 24" and 7".

4. Find the area of a regular hexagon, the length of whose side is 4 inches.
Ans. 41.568 sq. in.

ARITHMETIC.

(PART 9.)

MENSURATION.

CIRCLE AND ELLIPSE.

1. *The straight line drawn from the center of a circle to the middle point of any arc bisects the chord of the arc and is perpendicular to the chord.*

In Fig. 1, C is the middle point of the arc $A\,B$ and O is the center of the circle. The straight line $O\,C$ bisects the chord $A\,B$ in the point D, and $O\,C$ is perpendicular to the chord $A\,B$.

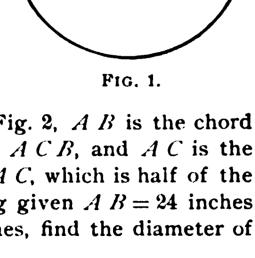


FIG. 1.

2. When the chord of a whole arc and the chord of half of the same arc are given, the diameter of the circle can be found.

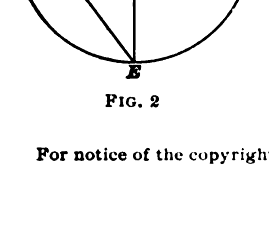


FIG. 2

EXAMPLE.—In Fig. 2, $A\,B$ is the chord of the whole arc $A\,C\,B$, and $A\,C$ is the chord of the arc $A\,C$, which is half of the arc $A\,C\,B$. Being given $A\,B = 24$ inches and $A\,C = 15$ inches, find the diameter of the circle.

SOLUTION.—Let O be the center of the circle and let the diameter $C\,O\,E$ and the line $A\,E$ be drawn. Then, the diameter $C\,O\,E$ bisects the chord $A\,B$ in the point D. Hence, $A\,D = 12$ inches. Therefore, by Art. **44,** Part 6, $C\,D = \sqrt{\overline{A\,C}^2 - \overline{A\,D}^2}$, which is equal to $\sqrt{15^2 - 12^2}$, or $\sqrt{81}$, or 9 inches. By Art. **23**, Part 3, CEA is a right triangle, because CE is a diameter of the circle. Therefore, by Art. **38**, Part 8, AC is a mean proportional between CE and CD. Hence, by Art. **21**, Part 8, CE is equal to the square of AC divided by CD; that is, $\frac{15^2}{9}$, or 25 in. Ans.

3. When a chord of a whole arc and the chord of half of the same arc are given, the chord of a quarter of the same arc can be found.

EXAMPLE.—In Fig. 3, AB is the chord of the whole arc ACB, and AC is the chord of the arc AC, which is half of the arc ACB; being given $AB = 96$ inches and $AC = 60$ inches, it is required to find AF, the chord of the arc AF, which is one-quarter of the arc AB.

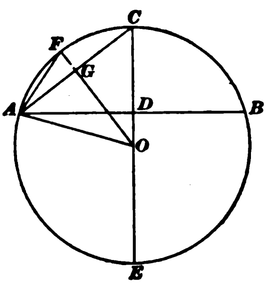


FIG. 3.

SOLUTION.—In the first place, we must find the diameter by the method of Art. **2.** We have $AD = \frac{1}{2} \times AB = 48$ inches. Therefore, by Art. **44**, Part 6, $CD = \sqrt{\overline{AC}^2 - \overline{AD}^2}$, which is equal to $\sqrt{60^2 - 48^2}$, or $\sqrt{1{,}296}$, or 36 inches. By Art. **23**, Part 3, CEA is a right triangle, because CE is the diameter of the circle. Therefore, by Art. **38**, Part 8, AC is a mean proportional between CE and CD. Hence, by Art. **21**, Part 8, CE is equal to the square of AC divided by CD; that is, CE is equal to $\frac{60^2}{36}$, or 100 inches.

In the right triangle OAG, we have $AG = \frac{1}{2} \times AC = \frac{1}{2} \times 60$, or 30 inches, and AO = radius $= \frac{1}{2} \times$ (diameter) $= \frac{1}{2} \times CE = \frac{1}{2} \times 100$, or 50 inches. Therefore, we have $OG = \sqrt{\overline{OA}^2 - \overline{AG}^2}$, which is equal to $\sqrt{50^2 - 30^2}$, or $\sqrt{1{,}600}$, or 40 inches. Now, FG is found by subtracting OG from the radius OF; hence, $FG = OF - OG = (50 - 40)$, or 10 inches. Therefore, by Art. **42**, Part 6, $AF = \sqrt{\overline{AG}^2 + \overline{FG}^2}$, which is equal to $\sqrt{30^2 + 10^2}$, or $\sqrt{1{,}000}$, or 31.623 inches, nearly. Thus, correct to four figures, $AF = 31.62$ in. Ans.

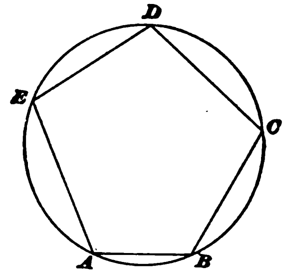


FIG. 4.

4. As already stated in Art. **70**, Part 3, when all the vertexes of a polygon lie on the circumference of a circle, the polygon is said to be **inscribed** in the circle. Thus, the polygon $A\,B\,C\,D\,E$ is inscribed in the circle in Fig. 4; and each side of the pentagon is, therefore, a chord of the circle.

5. ***If an equilateral polygon is inscribed in a circle, it is a regular polygon.***

Thus, if the sides of the inscribed hexagon $A\,B\,C\,D\,E\,F$, Fig. 5, are all equal, the angles must also be equal; and $A\,B\,C\,D\,E\,F$ is a regular hexagon.

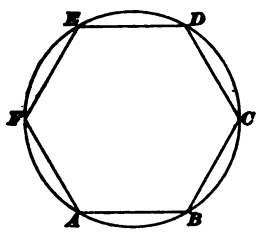


FIG. 5.

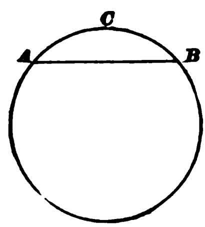


FIG. 6.

6. The shortest line that can be drawn from one point to another is the straight line joining the two points. Therefore, the chord of any circular arc is shorter than the arc; thus, in Fig. 6, the chord $A\,B$ is shorter than the arc $A\,C\,B$.

In Fig. 7, each side of the inscribed polygon $A\,B\,C\,D\,E\,F$ is a chord of the circle, and each of these chords is less than its arc. Therefore, the sum of all the chords is less than the sum of all the arcs. The sum of all the chords is the perimeter of the polygon, and the sum of all the arcs is the circumference of the circle. Hence, *the perimeter of a polygon inscribed in a circle is less than the circumference of the circle.*

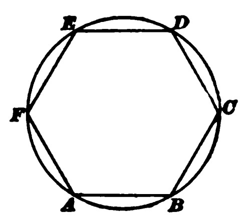


FIG. 7.

7. *The side of a regular hexagon inscribed in a circle is equal to the radius of the circle.*

In Fig. 8, $A\,B\,C\,D\,E\,F$ is a regular hexagon inscribed in a circle; each of the triangles $A\,O\,B$, $B\,O\,C$, $C\,O\,D$, $D\,O\,E$,

EOF, and FOA is an equilateral triangle, and, therefore, each of the sides of the hexagon is equal to the radius of the circle. Thus, we have $AB = BC = CD = DE = EF = FA = OA$.

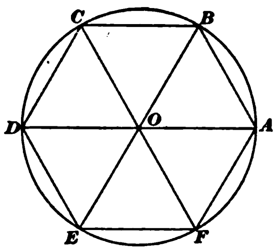


FIG. 8.

8. From Art. 7 we see that the side of a regular hexagon inscribed in a circle is equal to the radius of the circle; and, therefore, the perimeter of the regular inscribed hexagon is equal to six times the radius or three times the diameter. This shows that the circumference of a circle is greater than three times its diameter. In books on geometry it is shown that, correct to six figures, the perimeter of a regular inscribed polygon of 512 sides is equal to 3.14157 times the diameter of the circle. Therefore, the circumference of a circle is greater than 3.14157 times its diameter.

9. A line that meets a circle in one point only is called a **tangent** to the circle; thus, in Fig. 9, AC is a tangent touching the circle at the point B. The point B where the tangent touches the circle is called the **point of contact.**

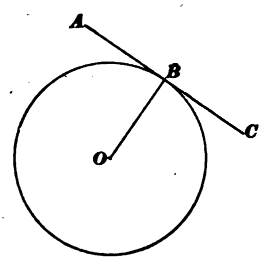


FIG. 9.

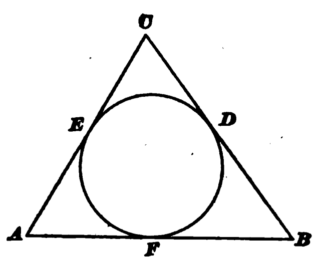


FIG. 10.

A tangent to a circle is perpendicular to the radius drawn to the point of contact. Thus, if O is the center of the circle in Fig. 9, the tangent AC is perpendicular to the radius OB.

10. If each of the sides of a polygon is tangent to a circle, the polygon is said to be **circumscribed** about the circle; thus, in Fig. 10, the triangle $A\,B\,C$ is circumscribed about the circle $D\,E\,F$.

11. If $A\,E$ and $A\,F$, Fig. 11, are two tangents to a circle, the arc $E\,F$ is less than the sum of the two tangents $A\,F$ and $A\,E$.

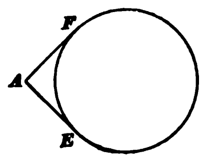


FIG. 11.

In Fig. 12, the arc $E\,F$ is less than the sum of the tangents $A\,E$ and $A\,F$; the arc $F\,D$ is less than the sum of the tangents $B\,F$ and $B\,D$; and the arc $D\,E$ is less than the sum of the tangents $C\,D$ and $C\,E$. Hence, the sum of the three arcs $E\,F$, $F\,D$, and $D\,E$ is less than the sum of the six tangents $A\,E$, $A\,F$, $B\,F$, $B\,D$, $C\,D$, and $C\,E$; that is, the circumference of the circle is less than the perimeter of the triangle $A\,B\,C$.

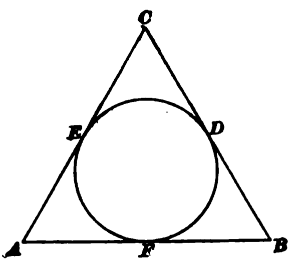


FIG. 12.

Hence, *the circumference of a circle is less than the perimeter of any polygon circumscribed about the circle.*

In books on geometry it is proved that, correct to six figures, the perimeter of a regular circumscribed polygon of 512 sides is equal to 3.14163 times the diameter of the circle.

12. *The ratio of the circumference of a circle to its diameter is equal to 3.1416, correct to five figures.*

By Art. **8,** the circumference is greater than 3.14157 times the diameter, and by Art. **11,** the circumference is less than 3.14163 times the diameter. Therefore, the ratio of the circumference to the diameter is greater than 3.14157 and less than 3.14163. But when five figures only are retained, both 3.14157 and 3.14163 must be written as 3.1416. Therefore, correct to five figures, the ratio

$$\frac{(\textit{circumference})}{(\textit{diameter})} = 3.1416.$$

13. *The circumference of a circle is equal to 3.1416 times its diameter.* That is,

$$(circumference) = 3.1416 \times (diameter).$$

EXAMPLE.—The diameter of a circle is 60 inches; find its circumference.

SOLUTION.—The circumference =

$$3.1416 \times (\text{diameter}) = 3.1416 \times 60 \text{ in.} = 188.496 \text{ in.}$$

Therefore, correct to five figures, the circumference is 188.5 in. Ans.

14. Evidently, the diameter of a circle is equal to the quotient obtained by dividing the circumference by 3.1416. Therefore, by Art. **10,** Part 5, the diameter is equal to the product obtained by multiplying the circumference by the reciprocal of 3.1416. By division we find that the reciprocal of 3.1416 is .31831. Hence, we have the following principle:

15. *The diameter of a circle is equal to the product obtained by multiplying the circumference by .31831.* That is,

$$(diameter) = .31831 \times (circumference).$$

EXAMPLE.—The circumference of a circle is 65.75 inches; find its diameter.

SOLUTION.—The diameter =

$$.31831 \times (\text{circumference}) = .31831 \times 65.75 \text{ in.}$$

Hence, correct to four figures, the diameter is 20.93 in. Ans.

16. *The length of an arc of a circle is to the circumference as the number of degrees in the arc is to 360.* That is,

$$\frac{(length\ of\ arc)}{(circumference)} = \frac{(number\ of\ degrees\ in\ arc)}{360}.$$

EXAMPLE.—An arc of a circle contains 135°; find the ratio of the length of the arc to the circumference.

SOLUTION.—We have

$$\frac{\text{length of arc}}{\text{circumference}} = \frac{135}{360} = \frac{3}{8}.$$

Thus, the ratio of the length of the arc to the circumference = 3 : 8, or the arc is $\frac{3}{8}$ of the circumference. Ans.

17. From Art. **16,** we have

$$\frac{\text{(length of arc)}}{\text{(circumference)}} = \frac{\text{(number of degrees in arc)}}{360}.$$

Applying rule I, Art. **17,** Part 8, to this proportion, we get

$$\text{length of arc} = \frac{\text{(number of degrees in arc) (circumference)}}{360}.$$

Thus we have the following principle:

18. *When the number of degrees in an arc of a circle is given, the length of the arc is found by multiplying the circumference of the circle by the number of degrees in the arc and dividing the product by 360.* That is,

$$\textit{(length of arc)} = \frac{\textit{(number of degrees in arc)} \times \textit{(circumference)}}{360}.$$

EXAMPLE.—In a circle whose diameter is 81 inches; find the length of an arc of 50°.

SOLUTION.—The circumference is equal to 3.1416 × the diameter (Art. **13**). Therefore, the circumference = 3.1416 × 81 inches. The length of the arc is found by multiplying the circumference by the number of degrees in the arc (50) and dividing the product by 360; thus, the length of the arc is $\frac{50 \times 3.1416 \times 81 \text{ in.}}{360}$, which, correct to four figures, is 35.34 in. Ans.

19. From Art. **16,** we have

$$\frac{\text{(length of arc)}}{\text{(circumference)}} = \frac{\text{(number of degrees in arc)}}{360}.$$

Applying rule II of Art. **17,** Part 8, to this proportion, we get

$$\text{(number of degrees in arc)} = \frac{360 \times \text{(length of arc)}}{\text{(circumference)}}.$$

Thus, we have the following principle:

20. *When the length of an arc of a circle is given, the number of degrees in the arc is found by multiplying the* *length of the arc by 360 and dividing the product by the circumference.* That is,

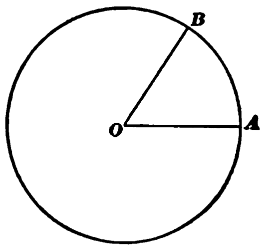


FIG. 13.

$$(\textit{number of degrees in arc}) = \frac{360 \times (\textit{length of arc})}{(\textit{circumference})}.$$

EXAMPLE.—In Fig. 13, the arc $A\,B$ is equal to the radius $O\,A$; find the number of degrees in the central angle $A\,O\,B$.

SOLUTION.—By Art. **15**, Part 3, the number of angular degrees in the angle $A\,O\,B$ is equal to the number of degrees of arc in the arc $A\,B$. But the number of degrees in the arc $A\,B$

$$= \frac{360 \times (\text{length of arc } A\,B)}{(\text{circumference})} = \frac{360 \times (\text{radius})}{3.1416 \times (\text{diameter})}$$

$$= \frac{360 \times (\text{radius})}{3.1416 \times 2 \times (\text{radius})} = \frac{360}{3.1416 \times 2} = 57.3,$$

correct to four figures. Thus, the angle $A\,O\,B$ contains nearly 57.3°. Ans.

21. Let it be required to find the length of the circular arc $A\,B$, Fig. 14, which is less than one-third of the circumference. Bisect the chord $A\,B$ in the point D, and draw $D\,C$ perpendicular to $A\,B$. Then, C is the middle point of the arc $A\,B$. Draw $A\,C$. The straight line $A\,B$ is the chord of the whole arc $A\,B$, and the straight line $A\,C$ is the chord of half the arc. When these two chords are measured, the length of the arc is found by the following rule, which gives the length of an arc less than 120° correct to three figures:

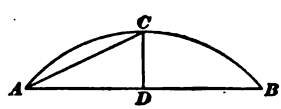


FIG. 14.

Rule.—*From eight times the chord of half the arc, subtract the chord of the whole arc, and divide the remainder by 3. The quotient is the length of the arc, approximately.* That is,

$$(\textit{length of arc } A\,B) = \frac{8 \times A\,C - A\,B}{3}.$$

EXAMPLE.—Find the length of the arc $A B$, Fig. 15.

SOLUTION.—Before we can apply the rule, we must find the length of $A C$, the chord of half the arc. We have $A D = \frac{1}{2}$ of $A B = \frac{1}{2} \times 72$, or 36 inches, and $D C = 8$ inches.

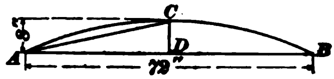


FIG. 15.

By Art. **42,** Part 6,

$$A C = \sqrt{\overline{A D}^2 + \overline{D C}^2} = 36.88''.$$

Hence,

$$(\text{length of arc } A B) = \frac{8 \times A C - A B}{3}$$

$$= \frac{8 \times 36.88'' - 72''}{3} = 74.35''. \text{ Ans.}$$

22. Let it be required to find the length of an arc greater than one-third of the circumference, when the chord of the whole arc and the chord of half the arc are given.

If this problem is solved by the direct application of the rule of Art. **21,** the result is not sufficiently accurate. The problem can readily be solved by the method used in the following example:

EXAMPLE.—In Fig. 16, $A B$ is the chord of the whole arc $A C B$, and $A C$ is the chord of the arc $A C$, which is half of the arc $A C B$; being given $A B = 28.8$ inches and $A C = 24$ inches, find the length of the arc $A B$.

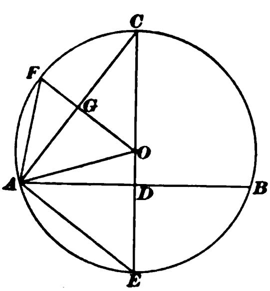


FIG. 16.

SOLUTION.—We have $A D = \frac{1}{2} \times A B = \frac{1}{2} \times 28.8$, or 14.4 inches. Hence, by Art. **44,** Part 6, $C D = \sqrt{\overline{A C}^2 - \overline{A D}^2}$, which is equal to $\sqrt{24^2 - 14.4^2}$, or 19.2 inches. By Art. **23,** Part 3, the triangle $A C E$ is a right triangle, because $C E$ is a diameter of the circle. Therefore, by Art. **38,** Part 8, $A C$ is a mean proportional between $C E$ and $C D$. Hence, by Art. **21,** Part 8, $C E$ is equal to the square of $A C$ divided by $C D$; that is, $C E$ is equal to $\frac{24^2}{19.2}$, or 30 inches.

In the right triangle $O A G$, we have $A G = \frac{1}{2} \times A C = \frac{1}{2} \times 24$, or 12 inches, and $A O$ = radius = $\frac{1}{2} \times$ (diameter) = $\frac{1}{2} \times C E = \frac{1}{2} \times 30$, or 15 inches. Therefore, by Art. **44,** Part 6, we have $O G = \sqrt{\overline{O A}^2 - \overline{A G}^2}$, which is equal to $\sqrt{15^2 - 12^2}$, or 9 inches. Now, FG

is found by subtracting OG from the radius OF; hence, $FG = OF - OG = (15 - 9)$, or 6 inches. Therefore, by Art. **42,** Part 6, $AF = \sqrt{\overline{AG}^2 + \overline{FG}^2}$, which is equal to $\sqrt{12^2 + 6^2}$, or 13.42 inches, nearly. Thus, $AC = 24$ inches = the chord of the whole arc AFC, and $AF = 13.42$ inches = the chord of the arc AF, which is one-half of the arc AFC. Hence, by the rule of Art. **21,** we have

$$\text{arc } AFC = \frac{8 \times AF - AC}{3}$$

$$= \frac{8 \times 13.42'' - 24''}{3} = 27.8''.$$

But,

$$\text{arc } ACB = 2 \times \text{arc } AFC = 2 \times 27.8'' = 55.6''. \quad \text{Ans.}$$

NOTE.—This method gives the length of any arc with sufficient accuracy for practical purposes.

EXAMPLES FOR PRACTICE.

1. What is the circumference of a circle whose diameter is $4\frac{1}{2}$ feet? Give the result correct to four figures. Ans. 14.14 ft.

2. What is the diameter of a circle whose circumference is 131.95 inches? Ans. 42.01 in.

3. What is the length of an arc of 24° in a circle whose radius is 18 inches? Ans. 7.54 in.

4. The chord of a certain arc is 25.88 inches, and the chord of half the arc is 13.05 inches. Find the length of the arc by the rule of Art. **21.** Ans. 26.17 in.

---

23. Let $ABCDEF$, Fig. 17, be a regular polygon circumscribed about a circle; draw the radii to the points of contact of the sides; and join O, the center of the circle, to the vertexes of the polygon by straight lines. The area of the polygon is made up of the areas of the triangles AOB, BOC, COD, DOE, EOF, and FOA. The area of the triangle AOB is equal to one-half the product of its base and altitude (Art. **62,** Part 7). That is,

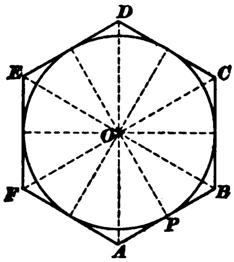


FIG. 17.

$$(\text{area } A\,O\,B) = \tfrac{1}{2}\,A\,B \times O\,P,$$

or

$$(\text{area } A\,O\,B) = \tfrac{1}{2}\,A\,B \times (\text{radius}).$$

Similarly,

$$(\text{area } B\,O\,C) = \tfrac{1}{2}\,B\,C \times (\text{radius}),$$

$$(\text{area } C\,O\,D) = \tfrac{1}{2}\,C\,D \times (\text{radius}),$$

$$(\text{area } D\,O\,E) = \tfrac{1}{2}\,D\,E \times (\text{radius}),$$

$$(\text{area } E\,O\,F) = \tfrac{1}{2}\,E\,F \times (\text{radius}),$$

$$(\text{area } F\,O\,A) = \tfrac{1}{2}\,F\,A \times (\text{radius}).$$

The sum of the first members of these last six equations must be equal to the sum of their second members. The sum of the first members is the area of the polygon, and the sum of the last members is one-half of the sum of the sides of the polygon multiplied by the radius of the circle. Therefore,

$$(\text{area of polygon}) = \tfrac{1}{2}\,(\text{perimeter of polygon}) \times (\text{radius}).$$

This equation is true no matter how great the number of sides of the circumscribed polygon is; and the greater the number of the sides, the more nearly do the area and perimeter of the polygon approach the area and circumference of the circle. Hence,

$$(\text{area of circle}) = \tfrac{1}{2}\,(\text{circumference}) \times (\text{radius}).$$

Now, the circumference of a circle is equal to 3.1416 times its diameter; and, therefore,

$$\tfrac{1}{2}\,(\text{circumference}) = \tfrac{1}{2} \times 3.1416 \times (\text{diameter}),$$

or

$$\tfrac{1}{2}\,(\text{circumference}) = 3.1416 \times (\text{radius}).$$

Substituting $3.1416 \times (\text{radius})$ for $\frac{1}{2}$ (circumference) in the equation

$$(\text{area of circle}) = \tfrac{1}{2}\,(\text{circumference}) \times (\text{radius}),$$

we get

$$(\text{area of circle}) = 3.1416 \times (\text{radius}) \times (\text{radius}),$$

or

$$(\text{area of circle}) = 3.1416 \times (\text{radius})^2.$$

Now, the radius is one-half of the diameter, and, therefore, the square of the radius is equal to one-fourth of the square of the diameter; that is,

$$(\text{radius})^2 = \tfrac{1}{4} \times (\text{diameter})^2.$$

Hence, $(\text{area of circle}) = 3.1416 \times \tfrac{1}{4} \times (\text{diameter})^2,$

or $(\text{area of circle}) = .7854 \times (\text{diameter})^2.$

Hence, we have the following important principle:

24. *The area of a circle is equal to the product obtained by multiplying the square of its radius by 3.1416, or the square of its diameter by .7854.* That is,

$$(\textit{area of circle}) = 3.1416 \times (\textit{radius})^2,$$

or

$$(\textit{area of circle}) = .7854 \times (\textit{diameter})^2.$$

EXAMPLE.—Find the area of a circle whose radius is 14.5 inches.

SOLUTION.—We have

$$(\text{area of circle}) = 3.1416 \times (\text{radius})^2 = 3.1416 \times (14.5)^2 = 660.5214.$$

Hence, correct to four figures, the area of the circle is 660.5 sq. in. Ans.

25. *When the area of a circle is given, its diameter is found by dividing the area by .7854 and extracting the square root of the quotient.* That is,

$$(\textit{diameter}) = \sqrt{\frac{(\textit{area of circle})}{.7854}}.$$

EXAMPLE.—Find the diameter of a circle whose area is 49.76 square inches.

SOLUTION.—We have

$$(\text{diameter}) = \sqrt{\frac{(\text{area of circle})}{.7854}} = \sqrt{\frac{49.76}{.7854}} = \sqrt{63.3563} = 7.96, \text{ nearly.}$$

Hence, correct to three figures, the diameter is 7.96 in. Ans.

26. Two circles that have the same center are said to be **concentric.**

27. The space included between two concentric circles is called a **circular ring**; thus, the space between the two circles $A\ B\ C$ and $D\ E\ F$, Fig. 18, is a circular ring.

28. *The area of a circular ring is found by substracting the area of the smaller circle from the area of the larger circle.*

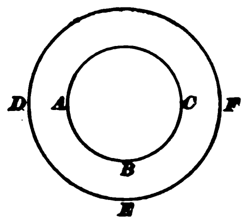


FIG. 18.

EXAMPLE.—Find the area of the circular ring enclosed by two concentric circles whose diameters are 4 inches and 6 inches, respectively.

SOLUTION.—By Art. **24,** we have

$$\text{(area of larger circle)} = .7854 \times 6^2 \text{ sq. in.}$$

$$\text{(area of smaller circle)} = .7854 \times 4^2 \text{ sq. in.}$$

Subtracting, we have the area of circular ring in sq. in.

$$= .7854 \times 6^2 - .7854 \times 4^2 \quad \text{(see Art. 36, Part 4)}$$

$$= .7854 \times (6^2 - 4^2) = .7854 \times 20 = 15.7.$$

Thus, the area of the circular ring, correct to three figures, is 15.7 sq. in. Ans.

29. The space included between an arc of a circle and the two radii drawn to the ends of the arc is called a **sector** of the circle; and the angle included between the two radii is called the **angle of the sector**; thus, $A\ O\ B$ is a sector of the circle in Fig. 19, and the angle $A\ O\ B$ is the angle of the sector $A\ O\ B$.

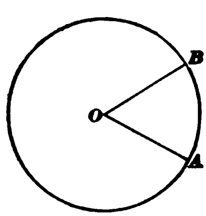


FIG. 19.

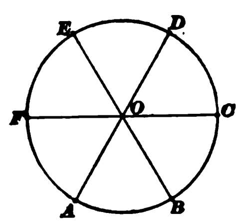


FIG. 20.

30. In Fig. 20, the circumference of the circle is divided into six equal parts by the points A, B, C, D, E, and F. Therefore, the arc $A\ B$ is one-sixth of the circumference.

It is evident that the whole circle is also divided into six equal parts by the radii $O A$, $O B$, $O C$, $O D$, $O E$, and $O F$. Hence, the sector $O A B$ is one-sixth of the circle. Thus, we have

$$\frac{(\text{length of arc } A B)}{(\text{circumference})} = \tfrac{1}{6},$$

and

$$\frac{(\text{area of sector } A O B)}{(\text{area of circle})} = \tfrac{1}{6}.$$

Therefore, the ratio

$$\frac{(\text{length of arc } A B)}{(\text{circumference})} = \frac{(\text{area of sector } A O B)}{(\text{area of circle})},$$

for each of these ratios is equal to $\frac{1}{6}$.

Thus, $$\frac{(\text{area of sector } A O B)}{(\text{area of circle})} = \frac{(\text{length of arc } A B)}{(\text{circumference})}.$$

This is a particular case of the general proposition stated in the next article.

31. If $A O B$ is any sector of a circle,

$$\frac{(\textit{area of sector } A O B)}{(\textit{area of circle})} = \frac{(\textit{length of arc } A B)}{(\textit{circumference})}.$$

32. From Art. **16,** we have

$$\frac{(\text{length of arc } A B)}{(\text{circumference})} = \frac{(\text{number of degrees in arc } A B)}{360}.$$

Substituting

$$\frac{(\text{number of degrees in arc } A B)}{360} \text{ for } \frac{(\text{length of arc } A B)}{(\text{circumference})}.$$

in the equation of Art. **31,** we get

$$\frac{(\text{area of sector } A O B)}{(\text{area of circle})} = \frac{(\text{number of degrees in arc } A B)}{360}.$$

Applying rule I of Art. **17,** Part 8, to this proportion, we get

$$(\text{area of sector } A O B) = \frac{(\text{number of degrees in arc } A B) \times (\text{area of circle})}{360}.$$

Thus, we have the following principle:

33. *The area of a sector of a circle is found by multiplying the area of the circle by the number of degrees in the arc of the sector and dividing the product by 360.* That is,

$$(\textit{area of sector}) = \frac{(\textit{number of degrees in arc of sector}) \times (\textit{area of circle})}{360}.$$

EXAMPLE.—In a circle whose diameter is 8 feet, find the area of a sector whose angle is 30°.

SOLUTION.—The area of the circle in square feet is $.7854 \times 8^2$ (Art. **24**); and the number of degrees in the arc of the sector is equal to the number of degrees in its angle (Art. **15,** Part 3). Hence, the area of the sector in square feet is $\frac{30 \times .7854 \times 8^2}{360} = 4.1888$. Therefore, correct to three figures, the area of the sector is 4.19 sq. ft. Ans.

34. By Art. **31,** we have

$$\frac{(\text{area of sector } A\ O\ B)}{(\text{area of circle})} = \frac{(\text{length of arc } A\ B)}{(\text{circumference})}$$

Substituting $.7854 \times (\text{diameter})^2$ for its equal (area of circle), and $3.1416 \times (\text{diameter})$ for its equal (circumference) in this equation, we get

$$\frac{(\text{area of sector } A\ O\ B)}{.7854 \times (\text{diameter})^2} = \frac{(\text{length of arc } A\ B)}{3.1416 \times (\text{diameter})}.$$

Applying rule I, Art. **17,** Part 8, to this proportion, we get

$$(\text{area of sector } A\ O\ B) = \frac{.7854\ (\text{diameter})^2 \times (\text{length of arc } A\ B)}{3.1416\ (\text{diameter})}.$$

Now,

$$\frac{.7854}{3.1416} = \tfrac{1}{4}, \text{ and } \frac{(\text{diameter})^2}{(\text{diameter})} = (\text{diameter}).$$

Therefore,

$$(\text{area of sector } A\ O\ B) = \tfrac{1}{4}\ (\text{diameter})\ (\text{length of arc } A\ B),$$

or

$$(\text{area of sector } A\ O\ B) = \tfrac{1}{4}\ (\text{length of arc } A\ B)\ (\text{diameter}).$$

By Art. **1**, Part 3, the diameter = 2 × radius. Substituting (2 × radius) for its equal (diameter) in the equation

(area of sector $A\,O\,B$) = $\frac{1}{4}$ (length of arc $A\,B$) (diameter),

we get

(area of sector $A\,O\,B$) = $\frac{1}{4}$ (length of arc $A\,B$) (2 × radius),

or

(area of sector $A\,O\,B$) = $\frac{1}{2}$ (length of arc $A\,B$) (radius).

Thus, we get the following principles:

35. *The area of a sector is equal to one-quarter of the product of its arc and diameter, or the area of a sector is equal to one-half of the product of its arc and radius.* That is,

(*area of sector*) = $\frac{1}{4}$ (*length of arc of sector*) × (*diameter*),

or

(*area of sector*) = $\frac{1}{2}$ (*length of arc of sector*) × (*radius*).

EXAMPLE.—In a circle whose diameter is 25 inches, find the area of a sector whose arc is 22.4 inches long.

SOLUTION.—The area of the sector is $\frac{1}{4}$ × 22.4 × 25, or 140 sq. in. Ans.

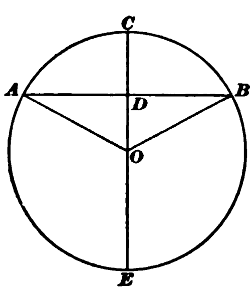


FIG. 21.

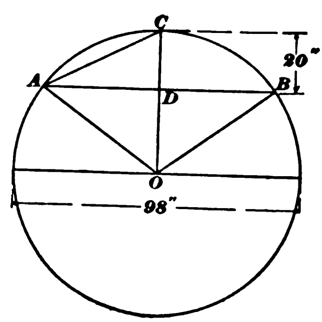


FIG. 22.

36. In Fig. 21, the diameter $C\,E$ passes through C, the middle point of the arc $A\,C\,B$, and, therefore, is perpendicular to the chord $A\,B$ (Art. **1**). The line $D\,C$ is called the **height** of the segment $A\,C\,B$.

37. *The area of the segment $A\,C\,B$, Fig. 21, is found by subtracting the area of the triangle $A\,O\,B$ from the area of the sector $A\,O\,B\,C$.*

EXAMPLE.—Find the area of the segment $A\,B\,C$, Fig. 22.

SOLUTION.—We have

$$O\,C = O\,A = \text{radius} = \tfrac{1}{2} \times 98'' = 49'',$$

and $$O\,D = O\,C - C\,D = 49'' - 20'' = 29''.$$

By Art. **44**, Part 6,

$$A\,D = \sqrt{\overline{A\,O}^2 - \overline{O\,D}^2},$$

which is equal to $\sqrt{49^2 - 29^2}$, or $\sqrt{1{,}560}$, or 39.5 inches;

hence, $$A\,B\,(= 2 \times A\,D) = 79 \text{ inches.}$$

Also, by Art. **42**, Part 6,

$$A\,C = \sqrt{\overline{A\,D}^2 + \overline{D\,C}^2},$$

which is equal to $\sqrt{1{,}560 + 400}$, or $\sqrt{1{,}960}$, or 44.3 inches.

By Art. **21**,

$$(\text{arc } A\,C\,B) = \frac{8 \times A\,C - A\,B}{3} = \frac{8 \times 44.3'' - 79''}{3} = 91.8''.$$

By Art. **35**,

$$(\text{the area of sector } A\,O\,B\,C) = \tfrac{1}{4}\,(\text{arc } A\,C\,B) \times (\text{diameter}),$$

which is equal to $\frac{1}{4} \times 91.8 \times 98$, or 2,249.1 sq. in.

By Art. **62**, Part 7,

$$(\text{the area of triangle } A\,O\,B) = \tfrac{1}{2}\,(\text{base}) \times (\text{altitude}) = \tfrac{1}{2} \times A\,B \times O\,D,$$

which is equal to $\frac{1}{2} \times 79 \times 29$, or 1,145.5 sq. in. Therefore, the area of the segment $A\,C\,B$ is equal to 2,249.1 − 1,145.5, or 1,103.6 square inches, which, correct to three figures, is 1,100 sq. in. Ans.

38. An **ellipse**, Fig. 23, is a plane figure bounded by a line such that the sum of the distances of any point on that line from two fixed points S' and S is always equal to the length of the line $C\,A$, which passes through the fixed points S' and S. Thus, in Fig. 23, $S'P + S\,P = S'Q + S\,Q = S'R + S\,R = C\,A$. The two fixed points S' and S are called the **foci**.

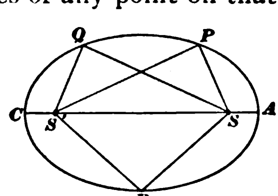


FIG. 23.

39. An ellipse can be constructed conveniently as follows: Tie the ends of a piece of thread together to form a loop; place the loop over two pins fixed at S' and S, Fig. 24; place the point of a pencil in the loop and move the pencil so as to keep the thread always stretched; the curve described by the pencil will be an ellipse having S' and S for foci.

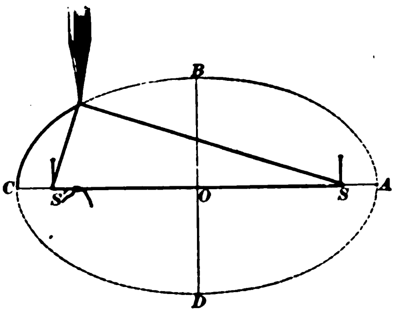


FIG. 24.

The line CA through the foci is called the **transverse,** or **major, axis.** The line BD, which is the perpendicular bisector of CA, is called the **conjugate,** or **minor, axis.** The lines CA and BD are often called the **long diameter** and the **short diameter,** respectively. The lines OA and OB are called the **semi-diameters.** The bounding line $ABCD$ is called the **circumference,** or **perimeter,** of the ellipse.

40. *The circumference of an ellipse is approximately equal to 3.1416 times the square root of half the sum of the squares of the two diameters.*

EXAMPLE.—Find the circumference of an ellipse whose diameters are 14 inches and 10 inches, respectively.

SOLUTION.—The circumference is 3.1416 times the square root of one-half the sum of the squares of the two diameters, which is equal to $3.1416 \times \sqrt{\frac{14^2 + 10^2}{2}}$, or 3.1416×12.17, or 38.233 inches. Therefore, correct to three figures, the circumference is 38.2 in. Ans.

41. *The area of an ellipse is equal to the product of the two semi-diameters multiplied by 3.1416.* That is,

$$(\textit{area of ellipse}) = \left(\frac{\textit{long diameter}}{2}\right) \times \left(\frac{\textit{short diameter}}{2}\right) \times 3.1416.$$

Since, $\frac{\text{long diameter}}{2} \times \frac{\text{short diameter}}{2}$ is equal to

$\frac{1}{4}$ × (long diameter) × (short diameter), by substituting $\frac{1}{4}$ × (long diameter) × (short diameter) for its equal in the above equation, we get

$$\text{(area of ellipse)} = \tfrac{1}{4} \times \text{(long diameter)} \times \text{(short diameter)} \times 3.1416$$
$$= \text{(long diameter)} \times \text{(short diameter)} \times .7854.$$

Hence, we have the following principle:

42. *The area of an ellipse is equal to the product of the two diameters multiplied by .7854.* That is,

(area of ellipse) = (long diameter) × (short diameter) × .7854.

Example.—Find the area of an ellipse whose diameters are 14 inches and 11 inches, respectively.

Solution.—The area is .7854 times the product of the two diameters, which is equal to .7854 × 14 × 11, or 120.95 square inches. Therefore, correct to four figures, the area is 121 sq. in. Ans.

EXAMPLES FOR PRACTICE.

1. Find the area of a circle whose diameter is 2.25 inches. Give the answer correct to three figures. Ans. 3.98 sq. in.

2. Find the area of a circular segment whose height is 11 inches, the radius of the circle being 21 inches. Give the answer correct to two figures. Ans. 290 sq. in.

3. Find the area of a circular sector whose arc contains 21°, the diameter of the circle being 7 feet. Ans. 2.24 sq. ft.

4. Find the area of a circular ring whose diameters are 12 inches and 6 inches, respectively. Ans. 84.82 sq. in.

5. Find the area of an ellipse whose long diameter is 14 inches and whose short diameter is 13 inches. Ans. 142.9 sq. in.

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43. The following table gives the areas and circumferences of circles whose diameters vary from $\frac{1}{64}$ to 100:

AREAS AND CIRCUMFERENCES OF CIRCLES.

Diam.	Circum.	Area.	Diam.	Circum.	Area.
1/64	.0491	.0002	3 7/8	12.1737	11.7933
1/32	.0982	.0008	4	12.5664	12.5664
1/16	.1963	.0031	4 1/8	12.9591	13.3641
1/8	.3927	.0123	4 1/4	13.3518	14.1863
3/16	.5890	.0276	4 3/8	13.7445	15.0330
1/4	.7854	.0491	4 1/2	14.1372	15.9043
5/16	.9817	.0767	4 5/8	14.5299	16.8002
3/8	1.1781	.1104	4 3/4	14.9226	17.7206
7/16	1.3744	.1503	4 7/8	15.3153	18.6655
1/2	1.5708	.1963	5	15.7080	19.6350
9/16	1.7671	.2485	5 1/8	16.1007	20.6290
5/8	1.9635	.3068	5 1/4	16.4934	21.6476
11/16	2.1598	.3712	5 3/8	16.8861	22.6907
3/4	2.3562	.4418	5 1/2	17.2788	23.7583
13/16	2.5525	.5185	5 5/8	17.6715	24.8505
7/8	2.7489	.6013	5 3/4	18.0642	25.9673
15/16	2.9452	.6903	5 7/8	18.4569	27.1086
1	3.1416	.7854	6	18.8496	28.2744
1 1/8	3.5343	.9940	6 1/8	19.2423	29.4648
1 1/4	3.9270	1.2272	6 1/4	19.6350	30.6797
1 3/8	4.3197	1.4849	6 3/8	20.0277	31.9191
1 1/2	4.7124	1.7671	6 1/2	20.4204	33.1831
1 5/8	5.1051	2.0739	6 5/8	20.8131	34.4717
1 3/4	5.4978	2.4053	6 3/4	21.2058	35.7848
1 7/8	5.8905	2.7612	6 7/8	21.5985	37.1224
2	6.2832	3.1416	7	21.9912	38.4846
2 1/8	6.6759	3.5466	7 1/8	22.3839	39.8713
2 1/4	7.0686	3.9761	7 1/4	22.7766	41.2826
2 3/8	7.4613	4.4301	7 3/8	23.1693	42.7184
2 1/2	7.8540	4.9087	7 1/2	23.5620	44.1787
2 5/8	8.2467	5.4119	7 5/8	23.9547	45.6636
2 3/4	8.6394	5.9396	7 3/4	24.3474	47.1731
2 7/8	9.0321	6.4918	7 7/8	24.7401	48.7071
3	9.4248	7.0686	8	25.1328	50.2656
3 1/8	9.8175	7.6699	8 1/8	25.5255	51.8487
3 1/4	10.2102	8.2958	8 1/4	25.9182	53.4563
3 3/8	10.6029	8.9462	8 3/8	26.3109	55.0884
3 1/2	10.9956	9.6211	8 1/2	26.7036	56.7451
3 5/8	11.3883	10.3206	8 5/8	27.0963	58.4264
3 3/4	11.7810	11.0447	8 3/4	27.4890	60.1322

TABLE—(*Continued*).

Diam.	Circum.	Area.	Diam.	Circum.	Area.
8⅞	27.8817	61.8625	13⅞	43.5897	151.202
9	28.2744	63.6174	14	43.9824	153.938
9⅛	28.6671	65.3968	14⅛	44.3751	156.700
9¼	29.0598	67.2008	14¼	44.7678	159.485
9⅜	29.4525	69.0293	14⅜	45.1605	162.296
9½	29.8452	70.8823	14½	45.5532	165.130
9⅝	30.2379	72.7599	14⅝	45.9459	167.990
9¾	30.6306	74.6621	14¾	46.3386	170.874
9⅞	31.0233	76.589	14⅞	46.7313	173.782
10	31.4160	78.540	15	47.1240	176.715
10⅛	31.8087	80.516	15⅛	47.5167	179.673
10¼	32.2014	82.516	15¼	47.9094	182.655
10⅜	32.5941	84.541	15⅜	48.3021	185.661
10½	32.9868	86.590	15½	48.6948	188.692
10⅝	33.3795	88.664	15⅝	49.0875	191.748
10¾	33.7722	90.763	15¾	49.4802	194.828
10⅞	34.1649	92.886	15⅞	49.8729	197.933
11	34.5576	95.033	16	50.2656	201.062
11⅛	34.9503	97.205	16⅛	50.6583	204.216
11¼	35.3430	99.402	16¼	51.0510	207.395
11⅜	35.7357	101.623	16⅜	51.4437	210.598
11½	36.1284	103.869	16½	51.8364	213.825
11⅝	36.5211	106.139	16⅝	52.2291	217.077
11¾	36.9138	108.434	16¾	52.6218	220.354
11⅞	37.3065	110.754	16⅞	53.0145	223.655
12	37.6992	113.098	17	53.4072	226.981
12⅛	38.0919	115.466	17⅛	53.7999	230.331
12¼	38.4846	117.859	17¼	54.1926	233.706
12⅜	38.8773	120.277	17⅜	54.5853	237.105
12½	39.2700	122.719	17½	54.9780	240.529
12⅝	39.6627	125.185	17⅝	55.3707	243.977
12¾	40.0554	127.677	17¾	55.7634	247.450
12⅞	40.4481	130.192	17⅞	56.1561	250.948
13	40.8408	132.733	18	56.5488	254.470
13⅛	41.2335	135.297	18⅛	56.9415	258.016
13¼	41.6262	137.887	18¼	57.3342	261.587
13⅜	42.0189	140.501	18⅜	57.7269	265.183
13½	42.4116	143.139	18½	58.1196	268.803
13⅝	42.8043	145.802	18⅝	58.5123	272.448
13¾	43.1970	148.490	18¾	58.9050	276.117

TABLE—(*Continued*).

Diam.	Circum.	Area.	Diam.	Circum.	Area.
18⅞	59.2977	279.811	23⅞	75.0057	447.690
19	59.6904	283.529	24	75.3984	452.390
19⅛	60.0831	287.272	24⅛	75.7911	457.115
19¼	60.4758	291.040	24¼	76.1838	461.864
19⅜	60.8685	294.832	24⅜	76.5765	466.638
19½	61.2612	298.648	24½	76.9692	471.436
19⅝	61.6539	302.489	24⅝	77.3619	476.259
19¾	62.0466	306.355	24¾	77.7546	481.107
19⅞	62.4393	310.245	24⅞	78.1473	485.979
20	62.8320	314.160	25	78 5400	490.875
20⅛	63.2247	318.099	25⅛	78.9327	495.796
20¼	63.6174	322.063	25¼	79.3254	500.742
20⅜	64.0101	326.051	25⅜	79.7181	505.712
20½	64.4028	330.064	25½	80.1108	510.706
20⅝	64.7955	334.102	25⅝	80.5035	515.726
20¾	65.1882	338.164	25¾	80.8962	520.769
20⅞	65.5809	342.250	25⅞	81.2889	525.838
21	65.9736	346.361	26	81.6816	530.930
21⅛	66.3663	350.497	26⅛	82.0743	536.048
21¼	66.7590	354.657	26¼	82.4670	541.190
21⅜	67.1517	358.842	26⅜	82.8597	546.356
21½	67.5444	363.051	26½	83.2524	551.547
21⅝	67.9371	367.285	26⅝	83.6451	556.763
21¾	68.3298	371.543	26¾	84.0378	562.003
21⅞	68.7225	375.826	26⅞	84.4305	567.267
22	69.1152	380.134	27	84.8232	572.557
22⅛	69.5079	384.466	27⅛	85.2159	577.870
22¼	69.9006	388.822	27¼	85.6086	583.209
22⅜	70.2933	393.203	27⅜	86.0013	588.571
22½	70.6860	397.609	27½	86.3940	593.959
22⅝	71.0787	402.038	27⅝	86.7867	599.371
22¾	71.4714	406.494	27¾	87.1794	604.807
22⅞	71.8641	410.973	27⅞	87.5721	610.268
23	72.2568	415.477	28	87.9648	615.754
23⅛	72.6495	420.004	28⅛	88.3575	621.264
23¼	73.0422	424.558	28¼	88.7502	626.798
23⅜	73.4349	429.135	28⅜	89.1429	632.357
23½	73.8276	433.737	28½	89.5356	637.941
23⅝	74.2203	438.364	28⅝	89.9283	643.549
23¾	74.6130	443.015	28¾	90.3210	649.182

TABLE—(*Continued*).

Diam.	Circum.	Area.	Diam.	Circum.	Area.
28⅞	90.7137	654.840	33⅞	106.422	901.259
29	91.1064	660.521	34	106.814	907.922
29⅛	91.4991	666.228	34⅛	107.207	914.611
29¼	91.8918	671.959	34¼	107.600	921.323
29⅜	92.2845	677.714	34⅜	107.992	928.061
29½	92.6772	683.494	34½	108.385	934.822
29⅝	93.0699	689.299	34⅝	108.778	941.609
29¾	93.4626	695.128	34¾	109.171	948.420
29⅞	93.8553	700.982	34⅞	109.563	955.255
30	94.2480	706.860	35	109.956	962.115
30⅛	94.6407	712.763	35⅛	110.349	969.000
30¼	95.0334	718.690	35¼	110.741	975.909
30⅜	95.4261	724.642	35⅜	111.134	982.842
30½	95.8188	730.618	35½	111.527	989.800
30⅝	96.2115	736.619	35⅝	111.919	996.783
30¾	96.6042	742.645	35¾	112.312	1,003.790
30⅞	96.9969	748.695	35⅞	112.705	1,010.822
31	97.3896	754.769	36	113.098	1,017.878
31⅛	97.7823	760.869	36⅛	113.490	1,024.960
31¼	98.1750	766.992	36¼	113.883	1,032.065
31⅜	98.5677	773.140	36⅜	114.276	1,039.195
31½	98.9604	779.313	36½	114.668	1,046.349
31⅝	99.3531	785.510	36⅝	115.061	1,053.528
31¾	99.7458	791.732	36¾	115.454	1,060.732
31⅞	100.1385	797.979	36⅞	115.846	1,067.960
32	100.5312	804.250	37	116.239	1,075.213
32⅛	100.9239	810.545	37⅛	116.632	1,082.490
32¼	101.3166	816.865	37¼	117.025	1,089.792
32⅜	101.7093	823.210	37⅜	117.417	1,097.118
32½	102.1020	829.579	37½	117.810	1,104.469
32⅝	102.4947	835.972	37⅝	118.203	1,111.844
32¾	102.8874	842.391	37¾	118.595	1,119.244
32⅞	103.280	848.833	37⅞	118.988	1,126.669
33	103.673	855.301	38	119.381	1,134.118
33⅛	104.065	861.792	38⅛	119.773	1,141.591
33¼	104.458	868.309	38¼	120.166	1,149.089
33⅜	104.851	874.850	38⅜	120.559	1,156.612
33½	105.244	881.415	38½	120.952	1,164.159
33⅝	105.636	888.005	38⅝	121.344	1,171.731
33¾	106.029	894.620	38¾	121.737	1,179.327

TABLE—(*Continued*).

Diam.	Circum.	Area.	Diam.	Circum.	Area.
38⅞	122.130	1,186.948	43⅞	137.838	1,511.91
39	122.522	1,194.593	44	138.230	1,520.53
39⅛	122.915	1,202.263	44⅛	138.623	1,529.19
39¼	123.308	1,209.958	44¼	139.016	1,537.86
39⅜	123.700	1,217.677	44⅜	139.408	1,546.56
39½	124.093	1,225.420	44½	139.801	1,555.29
39⅝	124.486	1,233.188	44⅝	140.194	1,564.04
39¾	124.879	1,240.981	44¾	140.587	1,572.81
39⅞	125.271	1,248.798	44⅞	140.979	1,581.61
40	125.664	1,256.640	45	141.372	1,590.43
40⅛	126.057	1,264.510	45⅛	141.765	1,599.28
40¼	126.449	1,272.400	45¼	142.157	1,608.16
40⅜	126.842	1,280.310	45⅜	142.550	1,617.05
40½	127.235	1,288.250	45½	142.943	1,625.97
40⅝	127.627	1,296.220	45⅝	143.335	1,634.92
40¾	128.020	1,304.210	45¾	143.728	1,643.89
40⅞	128.413	1,312.220	45⅞	144.121	1,652.89
41	128.806	1,320.260	46	144.514	1,661.91
41⅛	129.198	1,328.320	46⅛	144.906	1,670.95
41¼	129.591	1,336.410	46¼	145.299	1,680.02
41⅜	129.984	1,344.520	46⅜	145.692	1,689.11
41½	130.376	1,352.660	46½	146.084	1,698.23
41⅝	130.769	1,360.820	46⅝	146.477	1,707.37
41¾	131.162	1,369.000	46¾	146.870	1,716.54
41⅞	131.554	1,377.210	46⅞	147.262	1,725.73
42	131.947	1,385.450	47	147.655	1,734.95
42⅛	132.340	1,393.700	47⅛	148.048	1,744.19
42¼	132.733	1,401.990	47¼	148.441	1,753.45
42⅜	133.125	1,410.300	47⅜	148.833	1,762 74
42½	133.518	1,418.630	47½	149.226	1,772.06
42⅝	133.911	1,426.990	47⅝	149.619	1,781.40
42¾	134.303	1,435.370	47¾	150.011	1,790.76
42⅞	134.696	1,443.770	47⅞	150.404	1,800.15
43	135.089	1,452.200	48	150.797	1,809.56
43⅛	135.481	1,460.660	48⅛	151.189	1,819.00
43¼	135.874	1,469.140	48¼	151.582	1,828.46
43⅜	136.267	1,477.640	48⅜	151.975	1,837.95
43½	136.660	1,486.170	48½	152.368	1,847.46
43⅝	137.052	1,494.730	48⅝	152.760	1,856.99
43¾	137.445	1,503.300	48¾	153.153	1,866.55

TABLE—(*Continued*).

Diam.	Circum.	Area.	Diam.	Circum.	Area.
48⅞	153.546	1,876.14	53⅞	169.254	2,279.64
49	153.938	1,885.75	54	169.646	2,290.23
49⅛	154.331	1,895.38	54⅛	170.039	2,300.84
49¼	154.724	1,905.04	54¼	170.432	2,311.48
49⅜	155.116	1,914.72	54⅜	170.824	2,322.15
49½	155.509	1,924.43	54½	171.217	2,332.83
49⅝	155.902	1,934.16	54⅝	171.610	2,343.55
49¾	156.295	1,943.91	54¾	172.003	2,354.29
49⅞	156.687	1.953.69	54⅞	172.395	2,365.05
50	157.080	1,963.50	55	172.788	2,375.83
50⅛	157.473	1,973.33	55⅛	173.181	2,386.65
50¼	157.865	1,983.18	55¼	173.573	2,397.48
50⅜	158.258	1,993.06	55⅜	173.966	2,408.34
50½	158.651	2,002.97	55½	174.359	2,419.23
50⅝	159.043	2,012.89	55⅝	174.751	2,430.14
50¾	159.436	2,022.85	55¾	175.144	2,441.07
50⅞	159.829	2,032.82	55⅞	175.537	2,452.03
51	160.222	2,042.83	56	175.930	2,463.01
51⅛	160.614	2,052.85	56⅛	176.322	2,474.02
51¼	161.007	2,062.90	56¼	176.715	2,485.05
51⅜	161.400	2,072.98	56⅜	177.108	2,496.11
51½	161.792	2,083.08	56½	177.500	2,507.19
51⅝	162.185	2,093.20	56⅝	177.893	2,518.30
51¾	162.578	2,103.35	56¾	178.286	2,529.43
51⅞	162.970	2,113.52	56⅞	178.678	2,540.58
52	163.363	2,123.72	57	179.071	2,551.76
52⅛	163.756	2,133.94	57⅛	179.464	2,562.97
52¼	164.149	2,144.19	57¼	179.857	2,574.20
52⅜	164.541	2,154.46	57⅜	180.249	2,585.45
52½	164.934	2,164.76	57½	180.642	2,596.73
52⅝	165.327	2,175.08	57⅝	181.035	2,608.03
52¾	165.719	2,185.42	57¾	181.427	2,619.36
52⅞	166.112	2,195.79	57⅞	181.820	2,630.71
53	166.505	2,206.19	58	182.213	2,642.09
53⅛	166.897	2,216.61	58⅛	182.605	2,653.49
53¼	167.290	2,227.05	58¼	182.998	2,664.91
53⅜	167.683	2,237.52	58⅜	183.391	2,676.36
53½	168.076	2,248.01	58½	183.784	2,687.84
53⅝	168.468	2,258.53	58⅝	184.176	2,699.33
53¾	168.861	2,269.07	58¾	184.569	2,710.86

TABLE—(*Continued*).

Diam.	Circum.	Area.	Diam.	Circum.	Area.
58⅞	184.962	2,722.41	63⅞	200.670	3,204.44
59	185.354	2,733.98	64	201.062	3,217.00
59⅛	185.747	2,745.57	64⅛	201.455	3,229.58
59¼	186.140	2,757.20	64¼	201.848	3,242.18
59⅜	186.532	2,768.84	64⅜	202.240	3,254.81
59½	186.925	2,780.51	64½	202.633	3,267.46
59⅝	187.318	2,792.21	64⅝	203.026	3,280.14
59¾	187.711	2,803.93	64¾	203.419	3,292.84
59⅞	188.103	2,815.67	64⅞	203.811	3,305.56
60	188.496	2,827.44	65	204.204	3,318.31
60⅛	188.889	2,839.23	65⅛	204.597	3,331.09
60¼	189.281	2,851.05	65¼	204.989	3,343.89
60⅜	189.674	2,862.89	65⅜	205.382	3,356.71
60½	190.067	2,874.76	65½	205.775	3,369.56
60⅝	190.459	2,886.65	65⅝	206.167	3,382.44
60¾	190.852	2,898.57	65¾	206.560	3,395.33
60⅞	191.245	2,910.51	65⅞	206.953	3,408.26
61	191.638	2,922.47	66	207.346	3,421.20
61⅛	192.030	2,934.46	66⅛	207.738	3,434.17
61¼	192.423	2,946.48	66¼	208.131	3,447.17
61⅜	192.816	2,958.52	66⅜	208.524	3,460.19
61½	193.208	2,970.58	66½	208.916	3,473.24
61⅝	193.601	2,982.67	66⅝	209.309	3,486.30
61¾	193.994	2,994.78	66¾	209.702	3,499.40
61⅞	194.386	3,006.92	66⅞	210.094	3,512.52
62	194.779	3,019.08	67	210.487	3,525.66
62⅛	195.172	3,031.26	67⅛	210.880	3,538.83
62¼	195.565	3,043.47	67¼	211.273	3,552.02
62⅜	195.957	3,055.71	67⅜	211.665	3,565.24
62½	196.350	3,067.97	67½	212.058	3,578.48
62⅝	196.743	3,080.25	67⅝	212.451	3,591.74
62¾	197.135	3,092.56	67¾	212.843	3,605.04
62⅞	197.528	3,104.89	67⅞	213.236	3,618.35
63	197.921	3,117.25	68	213.629	3,631.69
63⅛	198.313	8,129.64	68⅛	214.021	3,645.05
63¼	198.706	3,142.04	68¼	214.414	3,658.44
63⅜	199.099	3,154.47	68⅜	214.807	3,671.86
63½	199.492	3.166.93	68½	215.200	3,685.29
63⅝	199.884	3,179.41	68⅝	215.592	3,698.76
63¾	200.277	3,191.91	68¾	215.985	3,712.24

TABLE—(*Continued*).

Diam.	Circum.	Area.	Diam.	Circum.	Area.
68⅞	216.378	3,725.75	73⅞	232.086	4,286.33
69	216.770	3,739.29	74	232.478	4,300.85
69⅛	217.163	3,752.85	74⅛	232.871	4,315.39
69¼	217.556	3,766.43	74¼	233.264	4,329.96
69⅜	217.948	3,780.04	74⅜	233.656	4,344.55
69½	218.341	3,793.68	74½	234.049	4,359.17
69⅝	218.734	3,807.34	74⅝	234.442	4,373.81
69¾	219.127	3,821.02	74¾	234.835	4,388.47
69⅞	219.519	3,834.73	74⅞	235.227	4,403.16
70	219.912	3,848.46	75	235 620	4,417.87
70⅛	220.305	3,862.22	75⅛	236.013	4,432.61
70¼	220.697	3,876.00	75¼	236.405	4,447.38
70⅜	221.090	3,889.80	75⅜	236.798	4,462.16
70½	221.483	3,903.63	75½	237.191	4,476.98
70⅝	221.875	3,917.49	75⅝	237.583	4,491.81
70¾	222.268	3,931.37	75¾	237.976	4,506.67
70⅞	222.661	3,945.27	75⅞	238.369	4,521.56
71	223.054	3,959.20	76	238.762	4,536.47
71⅛	223.446	3,973.15	76⅛	239.154	4,551.41
71¼	223.839	3,987.13	76¼	239.547	4,566.36
71⅜	224.232	4,001.13	76⅜	239.940	4,581.35
71½	224.624	4,015.16	76½	240.332	4,596.36
71⅝	225.017	4,029.21	76⅝	240.725	4,611.39
71¾	225.410	4,043.29	76¾	241.118	4,626.45
71⅞	225.802	4,057.39	76⅞	241.510	4,641.53
72	226.195	4,071.51	77	241.903	4,656.64
72⅛	226.588	4,085.66	77⅛	242.296	4,671.77
72¼	226.981	4,099.84	77¼	242.689	4,686.92
72⅜	227.373	4,114.04	77⅜	243.081	4,702.10
72½	227.766	4,128.26	77½	243.474	4,717.31
72⅝	228.159	4,142.51	77⅝	243.867	4,732.54
72¾	228.551	4,156.78	77¾	244.259	4,747.79
72⅞	228.944	4,171.08	77⅞	244.652	4,763.07
73	229.337	4,185.40	78	245.045	4,778.37
73⅛	229.729	4,199.74	78⅛	245.437	4,793.70
73¼	230.122	4,214.11	78¼	245.830	4,809.05
73⅜	230.515	4,228.51	78⅜	246.223	4,824.43
73½	230.908	4,242.93	78½	246.616	4,839.83
73⅝	231.300	4,257.37	78⅝	247.008	4,855.26
73¾	231.693	4,271.84	78¾	247.401	4,870.71

TABLE—(*Continued*).

Diam.	Circum.	Area.	Diam.	Circum.	Area.
78⅞	247.794	4,886.18	83⅞	263,502	5,525.30
79	248.186	4,901.68	84	263.894	5,541.78
79⅛	248.579	4,917.21	84⅛	264.287	5,558.29
79¼	248.972	4,932.75	84¼	264.680	5,574.82
79⅜	249.364	4,948.33	84⅜	265.072	5,591.37
79½	249.757	4,963.92	84½	265.465	5,607.95
79⅝	250.150	4,979.55	84⅝	265.858	5,624.56
79¾	250.543	4,995.19	84¾	266.251	5,641.18
79⅞	250.935	5,010.86	84⅞	266.643	5,657.84
80	251.328	5,026.56	85	267.036	5,674.51
80⅛	251.721	5,042.28	85⅛	267.429	5,691.22
80¼	252.113	5,058.03	85¼	267.821	5,707.94
80⅜	252.506	5,073.79	85⅜	268.214	5,724.69
80½	252.899	5,089.59	85½	268.607	5,741.47
80⅝	253.291	5,105.41	85⅝	268.999	5,758.27
80¾	253.684	5.121.25	85¾	269.392	5,775.10
80⅞	254.077	5,137.12	85⅞	269.785	5,791.94
81	254.470	5,153,01	86	270.178	5,808.82
81⅛	254.862	5,168.93	86⅛	270.570	5,825.72
81¼	255.255	5,184.87	86¼	270.963	5,842.64
81⅜	255.648	5,200.83	86⅜	271.356	5,859.59
81½	256.040	5,216.82	86½	271.748	5,876.56
81⅝	256.433	5,232.84	86⅝	272.141	5,893.55
81¾	256.826	5,248.88	86¾	272.534	5,910.58
81⅞	257.218	5,264.94	86⅞	272.926	5,927.62
82	257.611	5,281.03	87	273.319	5,944.69
82⅛	258.004	5,297.14	87⅛	273 712	5,961.79
82¼	258.397	5,313.28	87¼	274.105	5,978.91
82⅜	258.789	5,329.44	87⅜	274.497	5,996.05
82½	259.182	5,345.63	87½	274.890	6,013.22
82⅝	259.575	5,361.84	87⅝	275.283	6,030.41
82¾	259.967	5,378.08	87¾	275.675	6,047.63
82⅞	260.360	5,394.34	87⅞	276.068	6,064.87
83	260.753	5,410.62	88	276.461	6,082.14
83⅛	261.145	5,426.93	88⅛	276.853	6,099.43
83¼	261.538	5,443.26	88¼	277.246	6,116.74
83⅜	261.931	5,459.62	88⅜	277.629	6,134.08
83½	262.324	5,476.01	88½	278.032	6,151.45
83⅝	262.716	5,492.41	88⅝	278.424	6,168.84
83¾	263.109	5,508.84	88¾	278.817	6,186.25

TABLE—(*Continued*).

Diam.	Circum.	Area.	Diam.	Circum	Area.
88⅞	279.210	6,203.69	93⅞	294.918	6,921.35
89	279.602	6,221.15	94	295.310	6,939.79
89⅛	279.995	6,238.64	94⅛	295.703	6,958.26
89¼	280.388	6,256.15	94¼	296.096	6,976.76
89⅜	280.780	6,273.69	94⅜	296.488	6,995.28
89½	281.173	6,291.25	94½	296.881	7,013.82
89⅝	281.566	6,308.84	94⅝	297.274	7,032.39
89¾	281.959	6,326.45	94¾	297.667	7,050.98
89⅞	282.351	6,344.08	94⅞	298.059	7,069.59
90	282.744	6,361.74	95	298.452	7,088.24
90⅛	283.137	6,379.42	95⅛	298.845	7,106.90
90¼	283.529	6,397.13	95¼	299.237	7,125.59
90⅜	283.922	6,414.86	95⅜	299.630	7,144.31
90½	284.315	6,432.62	95½	300.023	7,163.04
90⅝	284.707	6,450.40	95⅝	300.415	7,181.81
90¾	285.100	6,468.21	95¾	300.808	7,200.60
90⅞	285.493	6,486.04	95⅞	301.201	7,219.41
91	285.886	6,503.90	96	301.594	7,238.25
91⅛	286.278	6,521.78	96⅛	301.986	7,257.11
91¼	286.671	6,539.68	96¼	302.379	7,275.99
91⅜	287.064	6,557.61	96⅜	302.772	7,294.91
91½	287.456	6,575.56	96½	303.164	7,313.84
91⅝	287.849	6,593.54	96⅝	303.557	7,332.80
91¾	288.242	6,611.55	96¾	303.950	7,351.79
91⅞	288.634	6,629.57	96⅞	304.342	7,370.79
92	289.027	6,647.63	97	304.735	7,389.83
92⅛	289.420	6,665.70	97⅛	305.128	7,408.89
92¼	289.813	6,683.80	97¼	305.521	7,427.97
92⅜	290.205	6,701.93	97⅜	305.913	7,447.08
92½	290.598	6,720.08	97½	306.306	7,466.21
92⅝	290.991	6,738.25	97⅝	306.699	7,485.37
92¾	291.383	6,756.45	97¾	307.091	7,504.55
92⅞	291.776	6,774.68	97⅞	307.484	7,523.75
93	292.169	6,792.92	98	307.877	7,542.98
93⅛	292.562	6,811.20	98⅛	308.270	7,562.24
93¼	292.954	6,829.49	98¼	308.662	7,581.52
93⅜	293.347	6,847.82	98⅜	309.055	7,600.82
93½	293.740	6,866.16	98½	309.448	7,620.15
93⅝	294.132	6,884.53	98⅝	309.840	7,639.50
93¾	294.525	6,902.93	98¾	310.233	7,658.88

TABLE—(*Concluded*).

Diam.	Circum.	Area.	Diam.	Circum.	Area.
98⅞	310.626	7,678.28	99½	312.589	7,775.66
99	311.018	7,697.71	99⅝	312.982	7,795.21
99⅛	311.411	7,717.16	99¾	313.375	7,814.78
99¼	311.804	7,736.63	99⅞	313.767	7,834.38
99⅜	312.196	7,756.13	100	314.160	7,854.00

ARITHMETIC.

(PART 10.)

MENSURATION.

SPHERE, CYLINDER, AND CONE.

1. A **sphere,** Fig. 1, is a solid bounded by a curved surface every point of which is equally distant from a point within called the **center.** Any straight line drawn from the center to the surface of the sphere is called a **radius** of the sphere; any straight line passing through the center and terminating at both ends by the surface is called a **diameter.** The word *ball* is often used instead of the word *sphere.*

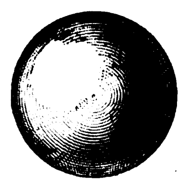

FIG. 1.

2. *The area of the surface of a sphere is equal to 3.1416 times the square of its diameter.* That is,

$$(surface) = 3.1416 \times (diameter)^2.$$

EXAMPLE.—Find the area of the surface of a sphere whose diameter is $12\frac{3}{4}$ inches.

SOLUTION.—We have $12\frac{3}{4} = 12.75$. Therefore, the area of the surface in square inches is $3.1416 \times (12.75)^2$, or 510.7 correct to four figures. Thus, the area is 510.7 sq. in. Ans.

3. If the diameters of two spheres are 4 inches and 3 inches, respectively, the areas of their surfaces are 3.1416×4^2 square inches and 3.1416×3^2 square inches, respectively. Hence, we have

$$\frac{(\text{surface of first sphere})}{(\text{surface of second sphere})} = \frac{3.1416 \times 4^2}{3.1416 \times 3^2} = \frac{4^2}{3^2}.$$

Thus, the surfaces of these two spheres are proportional to the squares of their diameters. This is a particular case of the following general principle:

4. *The surfaces of two spheres are proportional to the squares of their diameters.*

Since two spheres are similar bodies and two spherical surfaces are similar figures, this is merely a special case of the general proposition that the areas of any two similar figures are proportional to the squares of any two corresponding lines of the figures (Art. **40**, Part 8).

5. *When the area of the surface of a sphere is given, its diameter is found by dividing the area by 3.1416 and extracting the square root of the quotient.* That is,

$$(diameter) = \sqrt{\frac{(surface)}{3.1416}}.$$

EXAMPLE.—Find the diameter of a sphere, the area of whose surface is 907.92 square inches.

SOLUTION.—Dividing 907.92 by 3.1416, we get 288.9992, and $\sqrt{288.9992} = 17$ correct to five figures. Hence, the diameter is 17 in. Ans.

6. If we rest a drawing board upon two fixed points P and Q, as shown in Fig. 2, the straight line PQ will lie upon the surface of the drawing board. The surface of the drawing board is a plane surface; therefore, the plane $ABCD$ contains the line PQ.

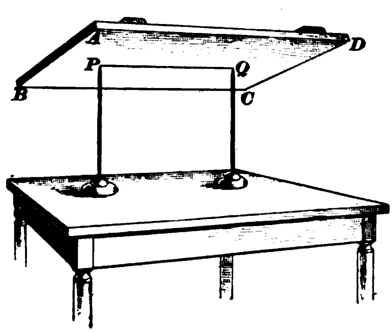

FIG. 2.

While still keeping the board in contact with the points P and Q, we can turn the board round the line PQ as an axis. Thus, we see that the plane $ABCD$ may occupy an unlimited number of positions while still containing the line PQ.

Hence, *the position of a plane is not fixed by the condition that it must contain a given line or pass through two given points.*

7. If we make the drawing board rest upon three fixed points *P*, *Q*, and *R*, Fig. 3, the straight lines *PR* and *PQ* will lie upon the surface of the board. Therefore, the plane surface of the drawing board contains the straight lines *PQ* and *PR*. In this case, the position of this plane is fixed, since it cannot turn about the line *PQ* without ceasing to rest upon the point *R*, nor can it turn about the line *PR* without ceasing to rest upon the point *Q*.

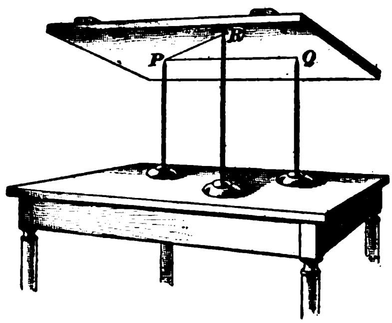

FIG. 3.

Hence, *the position of a plane is fixed by the condition that it must pass through three given points that are not in the same straight line, or by the condition that it must contain two given lines.*

8. From the preceding articles, it is evident that, in referring to a plane, it is necessary to name at least three points in the plane.

9. Let *O P*, Fig. 4, be a straight line meeting the plane *A B C D* in the point *P*. The line *O P* is said to be **perpendicular to the plane** *A B C D* if *O P* is perpendicular to every line through *P* in the plane *A B C D*; thus, the line *O P* is perpendicular to the plane *A B C D* if each of the angles *OPQ*, *OPR*, *OPS*, etc. is a right angle.

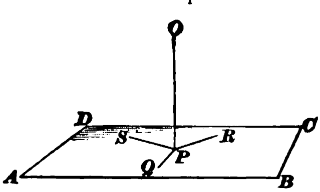

FIG. 4.

10. It can be proved that when OP, Fig. 4, is perpendicular to any two lines drawn through P in the plane $ABCD$, then OP is also perpendicular to every other line drawn through P in the plane $ABCD$. Thus, if each of the angles OPQ and OPR is a right angle, it can be proved that OPS is also a right angle.

Hence, *a straight line that is perpendicular to each of two straight lines at their point of intersection is perpendicular to the plane containing those lines.*

11. Two planes that can never meet, no matter how far they may be extended, are said to be **parallel planes.**

12. *If the same line is perpendicular to each of two planes, the two planes are parallel.* Thus, the planes ABC and $DEFG$, Fig. 5, are parallel if the line PQ is perpendicular to both planes. In such a case, it is usual to say that the triangle ABC is parallel to the rectangle $DEFG$, meaning that the plane containing the triangle is parallel to the plane containing the rectangle.

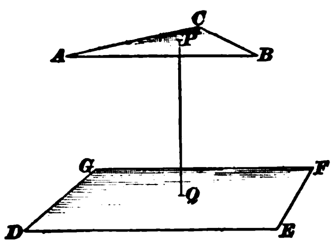

FIG. 5.

13. If a sphere is cut by two parallel planes, as in Fig. 6, the portion of its surface included between the two planes is called a **spherical zone.** The perpendicular distance between the two planes is called the **altitude** of the zone.

FIG. 6.

14. *The area of a spherical zone is equal to the circumference of the sphere multiplied by the altitude of the zone.* That is,

$$(\textit{area of zone}) = \begin{pmatrix}\textit{circumference} \\ \textit{of sphere}\end{pmatrix} \times \begin{pmatrix}\textit{altitude of} \\ \textit{zone}\end{pmatrix}.$$

EXAMPLE.—The circumference of a sphere is 32 inches. The sphere is cut by two parallel planes whose distances from the center of the sphere are 3.5 and 5 inches, respectively. Find the area of the spherical zone between these planes.

SOLUTION.—The perpendicular distance between the planes is $5 - 3.5$, or 1.5 inches. Hence, the altitude of the zone is 1.5 inches. Therefore, the area of the zone is

(circumference of sphere) × (altitude of zone),

which is equal to 32×1.5, or 48 sq. in. Ans.

EXAMPLES FOR PRACTICE.

1. Find the area of the surface of a sphere whose diameter is 6 inches. Give answer correct to five figures. Ans. 113.1 sq. in.

2. In a sphere whose diameter is 4 inches, find the area of a zone whose altitude is 2.5. Ans. 31.42 sq. in.

---

15. *The volume of a sphere is equal to the cube of its diameter multiplied by* $\frac{1}{6} \times 3.1416$, *or* .5236. That is,

$$(\textit{volume}) = \tfrac{1}{6} \times 3.1416 \times (\textit{diameter})^3,$$

or

$$(\textit{volume}) = .5236 \times (\textit{diameter})^3.$$

EXAMPLE.—Find the weight of a cast-iron spherical ball whose diameter is 3.5 inches, taking the weight of a cubic inch of cast iron to be .261 pound.

SOLUTION.—The volume is $.5236 \times (3.5)^3$, or 22.45 cu. in. Hence, the weight is $.261 \times 22.45$, or 5.86 lb. Ans.

16. If the diameters of two spheres are 3 inches and 2 inches, respectively, their volumes are $.5236 \times 3^3$ cubic inches and $.5236 \times 2^3$ cubic inches, respectively. Hence, we have

$$\frac{(\text{volume of first sphere})}{(\text{volume of second sphere})} = \frac{.5236 \times 3^3}{.5236 \times 2^3} = \frac{3^3}{2^3}.$$

That is, the volumes of these two spheres are proportional to the cubes of their diameters. This is a particular case of the general principle stated in Art. **17.**

17. *The volumes of any two spheres are proportional to the cubes of their diameters.*

This is a special case of the general proposition of Art. **18.**

18. *The volumes of any two similar solids are proportional to the cubes of any two corresponding lines of the solids.*

19. Sometimes the dimensions of a solid are given in inches and it is required to find its volume, or **capacity,** in gallons. In the next article we give the rule for finding the number of gallons in a sphere when its diameter is given in inches.

20. *The number of gallons contained in a sphere is equal to the cube of the number of inches in its diameter divided by 441.* That is,

$$\left(\begin{matrix}\textit{number of gallons}\\ \textit{in sphere}\end{matrix}\right) = \frac{1}{441} \times \left(\begin{matrix}\textit{number of inches}\\ \textit{in its diameter}\end{matrix}\right)^3.$$

EXAMPLE.—Find the number of gallons contained in a sphere whose diameter is 10.5 inches.

SOLUTION.—We have

$$(\text{number of gallons in sphere}) = \frac{1}{441} \times (10.5)^3 = 2\tfrac{5}{8}.$$

Thus, the sphere contains $2\tfrac{5}{8}$ gal. Ans.

21. *When the volume of a sphere is given, its diameter is found by dividing the volume by .5236 and extracting the cube root of the quotient.* That is,

$$(\textit{diameter}) = \sqrt[3]{\frac{(\textit{volume})}{.5236}}.$$

22. *When the cubical contents of a sphere is given in gallons, the number of inches in its diameter is found by multiplying the number of gallons contained in the sphere by 441 and extracting the cube root of the product.* That is,

$$(\textit{number of inches in diameter}) = \sqrt[3]{441 \times (\textit{number of gallons})}.$$

EXAMPLE.—A sphere is to contain 21 gallons; find its diameter in inches.

SOLUTION.—We have

$$(\text{number of inches in diameter}) = \sqrt[3]{441 \times 21} = \sqrt[3]{9,261}.$$

Referring to the table of "Squares and Cubes," Part 6, we find $\sqrt[3]{9,261} = 21$. Therefore (number of inches in diameter) = 21.

Hence, the diameter is 21 inches. Ans.

NOTE.—It is remarkable that for a 21-inch sphere, the number of inches in its diameter is the same as the number of gallons it contains. This is not true for any other sphere.

23. The space included between two concentric spherical surfaces is called a **spherical shell.**

24. *The volume of a spherical shell is found by subtracting the volume of the smaller sphere from the volume of the larger sphere.*

EXAMPLES FOR PRACTICE.

1. Find the volume of a sphere whose diameter is 9 inches.
Ans. 381.7 cu. in.

2. Find the volume of a spherical shell whose outer diameter is 7 inches and whose inner diameter is 6 inches. Ans. 66.5 cu. in.

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25. If a rectangle $ABCD$, Fig. 7, is turned through a complete revolution about the side AB as an axis, the crooked line $BCDA$ traces out a surface, and the space enclosed by this surface is called a **right circular cylinder.** While the rectangle is making a complete revolution, the point C describes the circumference of a circle whose center is B, and the point D describes the circumference of a circle whose center is A. The lines BC and AD are equal, because they are opposite sides of a parallelogram (Art. **75,** Part 3). Hence, the circles described by the points C and D are equal. These two equal circles are called

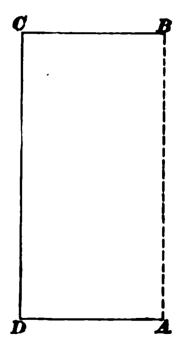


FIG. 7.

the **bases** of the right circular cylinder. The circumference of either of the bases is the circumference of the cylinder, and the radius of either of the bases is the radius of the cylinder.

26. A right circular cylinder is usually called simply a cylinder; whenever the word *cylinder* is used, it always means a right circular cylinder unless the contrary is stated. Fig. 8 shows a right circular cylinder.

FIG. 8.

27. The straight line $A B$, Fig. 7, joining the centers of the bases of a cylinder is called the **axis** of the cylinder.

28. As the rectangle $A B C D$, Fig. 7, turns about the axis $A B$, the lines $B C$ and $A D$, in every position, are perpendicular to the axis $A B$. Hence, *the axis of a cylinder is perpendicular to both its bases.*

The line $A B$ joining the centers of the bases is the **height,** or **altitude,** of the cylinder.

29. *The two bases of a cylinder are parallel;* for, the axis of the cylinder is perpendicular to both bases (Art. **12**).

30. The whole surface of a solid, including its bases, is called its **entire surface.** The surface of a solid, excluding its bases, is called its **convex surface.** Hence, the area of the entire surface of a solid is equal to the sum of the areas of its convex surface and its bases.

31. If a sheet of paper is rolled upon a cylinder and so fitted as to cover its whole convex surface, and then unrolled, the sheet of paper will be in the form of a rectangle whose base is the circumference of the cylinder and whose altitude is the altitude of the cylinder. The area of this rectangle is equal to the product of its base and altitude. Hence, we have the following principle:

32. *The area of the convex surface of a cylinder is equal to the product of its circumference and altitude.* That is,

$$(convex\ surface) = (circumference) \times (altitude).$$

The circumference of a cylinder is equal to 3.1416 times its diameter, or 2×3.1416 times its radius (Art. **13,** Part 9); that is,

$$(\text{circumference}) = 2 \times 3.1416 \times (\text{radius}).$$

Substituting $2 \times 3.1416 \times$ (radius) for (circumference) in the equation

$$(\text{convex surface}) = (\text{circumference}) \times (\text{altitude}),$$

we get

$$(\text{convex surface}) = 2 \times 3.1416 \times (\text{radius}) \times (\text{altitude}).$$

Thus, we have the following principle:

33. *The area of the convex surface of a cylinder is equal to* 2×3.1416 *times the product of its radius and its altitude.* That is,

$$(convex\ surface) = 2 \times 3.1416 \times (radius) \times (altitude).$$

34. The area of the entire surface of a cylinder is equal to the sum of the area of its convex surface and the areas of its two bases (Art. **30**). By Art. **13,** Part 9, the area of the circle which forms one base of a cylinder is equal to $3.1416 \times (\text{radius})^2$. Therefore, the sum of the areas of the two bases is

$$2 \times 3.1416 \times (\text{radius})^2, \text{ or } 2 \times 3.1416 \times (\text{radius}) \times (\text{radius}).$$

Thus,

$$(\text{two bases}) = 2 \times 3.1416 \times (\text{radius}) \times (\text{radius}).$$

By Art. **33,**

$$(\text{convex surface}) = 2 \times 3.1416 \times (\text{radius}) \times (\text{altitude}).$$

Adding the sum of the areas of the bases to the area of the convex surface, we get

$$\begin{aligned}(\text{convex surface}) &+ (\text{two bases}) \\ &= 2 \times 3.1416 \times (\text{radius}) \times (\text{altitude}) \\ &+ 2 \times 3.1416 \times (\text{radius}) \times (\text{radius}).\end{aligned}$$

In the last member of this equation, 2 × 3.1416 × (radius) is multiplied by the altitude and by the radius, and the sum of these two products is the same as the product obtained by multiplying 2 × 3.1416 by the sum of the base and altitude (Art. **36,** Part 4). Hence,

(entire surface) = 2 × 3.1416 × (radius) × (altitude + radius).

Hence, we have the following principle:

35. *The entire surface of a cylinder is equal to 2 × 3.1416 times the product obtained by multiplying its radius by the sum of its altitude and radius.* That is,

(entire surface) = 2 × 3.1416 × (radius) × (altitude + radius).

EXAMPLE.—Find the entire surface of a cylinder whose radius is 6 inches and whose altitude is 14 inches.

SOLUTION.—The area of the entire surface is 2 × 3.1416 × (radius) × (altitude + radius), which is equal to 2 × 3.1416 × 6 × (14 + 6), or 754 sq. in. Ans.

36. *The volume of a cylinder is found by multiplying the area of its base by its altitude.* That is,

(volume) = (area of base) × (altitude).

By Art. **23,** Part 9, we have

(area of base) = 3.1416 × $(\text{radius})^2$.

Substituting 3.1416 × $(\text{radius})^2$ for (area of base) in the equation

(volume) = (area of base) × (altitude),

we get

(volume) = 3.1416 × $(\text{radius})^2$ × (altitude).

By Art. **23,** Part 9, we have

(area of base) = .7854 × $(\text{diameter})^2$.

Substituting .7854 × $(\text{diameter})^2$ for (area of base) in the equation

(volume) = (area of base) × (altitude),

we get

(volume) = .7854 × $(\text{diameter})^2$ × (altitude).

Hence, we have the following proposition:

37. *The volume of a cylinder is equal to 3.1416 times the product obtained by multiplying the square of its radius by its altitude, or the volume of a cylinder is equal to .7854 times the product obtained by multiplying the square of its diameter by its altitude.* That is,

$$(volume) = 3.1416 \times (radius)^2 \times (altitude),$$

or

$$(volume) = .7854 \times (diameter)^2 \times (altitude).$$

EXAMPLE.—Find the volume of a cylinder whose altitude is 24 inches and whose diameter is 18 inches.

SOLUTION.—The volume of the cylinder is $.7854 \times (\text{diameter})^2 \times (\text{altitude})$, which is equal to $.7854 \times 18^2 \times 24$, or 6,107.3 cu. in. Ans.

38. *The number of gallons contained in a cylinder is equal to $\frac{3}{2}$ times the square of the number of inches in its diameter multiplied by the number of inches in its altitude and divided by 441.* That is,

$$(number\ of\ gallons\ in\ cylinder)$$

$$= \frac{3}{2} \times \frac{1}{441} \times \left(\begin{matrix} number\ of\ inches \\ in\ diameter \end{matrix}\right)^2 \times \left(\begin{matrix} number\ of\ inches \\ in\ altitude \end{matrix}\right).$$

EXAMPLE.—Find the number of gallons contained in a cylinder whose diameter is 14 inches and whose altitude is 18 inches.

SOLUTION.—We have

$$(\text{number of gallons in cylinder}) = \frac{3}{2} \times \frac{1}{441} \times 14^2 \times 18 = 12.$$

Thus, the cylinder contains 12 gal. Ans.

39. Cylinders of the sizes given in the table below contain very closely the measures stated, and the table will be found an aid to the student when estimating for cylindrical tanks and vessels of a specified capacity. As far as practicable, dimensions have been chosen that will cut to advantage out of the common sizes of sheet metals carried in stock by the manufacturers.

Diameter. Inches.	Height. Inches.	Capacity. Gallons.	Diameter. Inches.	Height. Inches.	Capacity. Gallons.
$1\frac{3}{4}$	3	$\frac{1}{32}$	21	$26\frac{5}{8}$	40
$2\frac{1}{4}$	$3\frac{5}{8}$	$\frac{1}{16}$	$22\frac{1}{2}$	29	50
$3\frac{1}{16}$	4	$\frac{1}{8}$	$24\frac{1}{2}$	$29\frac{1}{2}$	60
$3\frac{3}{4}$	$5\frac{1}{4}$	$\frac{1}{4}$	29	35	100
$6\frac{1}{4}$	$7\frac{1}{2}$	1	35	48	200
7	12	2	39	58	300
10	15	5	44	61	400
$12\frac{5}{8}$	15	8	$48\frac{1}{4}$	63	500
14	15	10	$51\frac{1}{2}$	67	600
15	$15\frac{3}{4}$	12	55	68	700
16	$17\frac{1}{4}$	15	58	70	800
$17\frac{1}{2}$	$19\frac{3}{16}$	20	59	76	900
20	22	30	60	$81\frac{1}{2}$	1,000

EXAMPLES FOR PRACTICE.

1. Find the convex surface and also the entire surface of a cylinder whose radius is 6 inches and whose altitude is 11 inches.

Ans. { Convex surface = 414.69 sq. in.
Entire surface = 640.89 sq. in.

2. Find the volume of a cylinder whose altitude is 7 inches and whose radius is 3 inches. Ans. 197.92 cu. in.

---

40. If a right triangle ABC, Fig. 9, is turned through a complete revolution about the leg AB, the crooked line BCA traces out a surface, and the space enclosed by this surface is called a **right circular cone.** While the triangle is making a complete revolution, the point C describes the circumference of a circle whose center is A. This circle is called the **base** of the right circular cone.

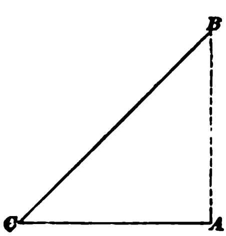


FIG. 9.

41. A right circular cone is usually called simply a cone, and the word *cone* always means a right circular cone, unless the contrary is stated. Fig. 10 shows a right circular cone.

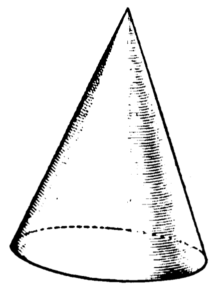


FIG. 10.

42. The line AB, Fig. 9, is called the **axis** of the cone. As the triangle ABC turns about the axis AB, the line AC is always perpendicular to AB. Therefore, *the axis of a cone is perpendicular to its base.*

43. The point B, Fig. 9, is called the **vertex** of the cone; and the line AB is the **height,** or **altitude,** of the cone.

44. Any straight line drawn from the vertex of a cone to the circumference of its base is called the **slant height** of the cone. Thus, BC, Fig. 9, is the slant height.

45. If a sheet of paper is rolled on a cone and so fitted as to cover the whole convex surface, and then unrolled, the sheet of paper will be in the form of the sector of a circle, Fig. 11, whose radius OA is equal to the slant height of the cone and whose arc AB is equal to the circumference of the base of the cone. From Art. **35,** Part 9, the area of a sector is equal to one-half the product of its arc and radius; hence, the area of the sector in Fig. 11 is $\frac{1}{2} \times (\text{arc } AB) \times OA$. Therefore, the area of the convex surface of the cone is equal to $\frac{1}{2} \times (\text{arc } AB) \times (OA)$, or one-half the product of the circumference of the base of the cone and its slant height. Thus we have the following principle:

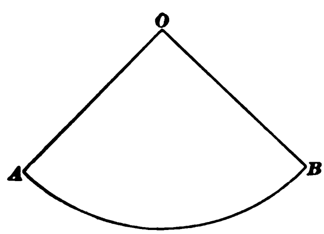


FIG. 11.

46. *The area of the convex surface of a cone is equal to one-half the product of the circumference of its base and its slant height.* That is,

$$(\textit{convex surface}) = \tfrac{1}{2} \times (\textit{circumference of base}) \times (\textit{slant height}).$$

From Art. **13,** Part 9, we have

$$(\text{circumference of base}) = 3.1416 \times (\text{diameter of base}),$$

or

$$(\text{circumference of base}) = 2 \times 3.1416 \times (\text{radius of base}).$$

Substituting $2 \times 3.1416 \times$ (radius of base) for (circumference of base) in the equation

$$\begin{pmatrix}\text{convex}\\ \text{surface}\end{pmatrix} = \tfrac{1}{2} \times (\text{circumference of base}) \times (\text{slant height}),$$

we get

$$\begin{pmatrix}\text{convex}\\ \text{surface}\end{pmatrix} = \tfrac{1}{2} \times 2 \times 3.1416 \times (\text{radius of base}) \times (\text{slant height}),$$

or

$$(\text{convex surface}) = 3.1416 \times (\text{radius of base}) \times (\text{slant height}).$$

Hence, we have the following principle:

47. *The area of the convex surface of a cone is equal to 3.1416 times the product of the radius of its base and its slant height.* That is,

$$(\textit{convex surface}) = 3.1416 \times (\textit{radius of base}) \times (\textit{slant height}).$$

48. The area of the entire surface of a cone is equal to the sum of the areas of its convex surface and its base. By Art. **24,** Part 9, the area of its base is $3.1416 \times (\text{radius})^2$, or $3.1416 \times (\text{radius}) \times (\text{radius})$.

Thus,

$$(\text{area of base}) = 3.1416 \times (\text{radius}) \times (\text{radius}).$$

By Art. **47,**

$$(\text{convex surface}) = 3.1416 \times (\text{radius}) \times (\text{slant height}).$$

Adding the area of the base to the area of the convex surface, we get

(convex surface) + (area of base) = (entire surface)
= 3.1416 × (radius) × (slant height)
+ 3.1416 × (radius) × (radius).

In the last member of this equation, 3.1416 × (radius) is multiplied by the slant height and the radius, and the sum of these products is the same as the product obtained by multiplying 3.1416 × (radius) by the sum of the slant height and the radius (Art. **36,** Part 4). Hence,

(entire surface) = 3.1416 × (radius) × (slant height + radius).

Hence, we have the following principle:

49. *The entire surface of a cone is equal to 3.1416 times the product obtained by multiplying its radius by the sum of its slant height and radius.* That is,

(entire surface) = 3.1416 × (radius) × (slant height + radius).

EXAMPLE.—Find the area of the entire surface of a cone, the diameter of whose base is 6 inches and whose altitude is 9 inches.

SOLUTION.—We have, Fig. 12, the radius $DB = \frac{1}{2}$ of 6, or 3 inches, and the slant height $CB = \sqrt{\overline{CD}^2 + \overline{DB}^2}$, which is equal to $\sqrt{9^2 + 3^2}$, or 9.487 inches. Hence,

(the entire surface)
= 3.1416 × (radius) × (slant height + radius),
which is equal to 3.1416 × 3 × (9.487 + 3), or 117.69 sq. in. Ans.

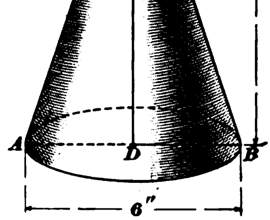


FIG. 12.

50. If a cylinder and a cone are constructed of the same material and have equal bases and altitudes, it will be found that the cylinder weighs three times as much as the cone. Hence, *the volume of a cone is one-third of the volume of a cylinder having the same base and altitude.*

51. *The volume of a cone is equal to one-third of the product of its base and altitude.*

This is evident, for the volume of a cone is one-third of the volume of a cylinder having the same base and altitude, Art. **50,** and the volume of a cylinder is equal to the product of its base and altitude, Art. **36.**

By Art. **37,** the volume of a cylinder is equal to 3.1416 times the product obtained by multiplying the square of its radius by its altitude, or the volume of a cylinder is equal to .7854 times the product obtained by multiplying the square of its diameter by its altitude. Therefore, the volume of a cone is equal to $\frac{1}{3} \times 3.1416$ times the product obtained by multiplying the square of the radius of its base by its altitude, or the volume of a cone is equal to $\frac{1}{3} \times .7854$ times the square of the diameter of its base by its altitude. Thus, we have the following principle:

52. *The volume of a cone is equal to $\frac{1}{3} \times 3.1416$ times the product obtained by multiplying the square of the radius of its base by its altitude, or the volume of a cone is equal to $\frac{1}{3} \times .7854$ times the product obtained by multiplying the square of the diameter of its base by its altitude.* That is,

$$(\textit{volume}) = \tfrac{1}{3} \times 3.1416 \times (\textit{radius of base})^2 \times (\textit{altitude}),$$

or

$$(\textit{volume}) = \tfrac{1}{3} \times .7854 \times (\textit{diameter of base})^2 \times (\textit{altitude}).$$

EXAMPLE.—Find the volume of a cone, the diameter of whose base is 10 inches and whose altitude is 12 inches.

SOLUTION.—We have

$$(\text{volume}) = \tfrac{1}{3} \times .7854 \times (\text{diameter})^2 \times (\text{altitude}),$$

which is equal to $\frac{1}{3} \times .7854 \times 10^2 \times 12$, or 314.16 cu. in. Ans.

53. *The number of gallons contained in a cone is equal to $\frac{1}{2}$ of the square of the number of inches in the diameter of its base multiplied by the number of inches in its altitude and divided by 441.* That is,

$$(\textit{number of gallons in cone})$$

$$= \frac{1}{2} \times \frac{1}{441} \times \left(\begin{matrix}\textit{number of inches} \\ \textit{in diameter of base}\end{matrix}\right)^2 \times \left(\begin{matrix}\textit{number of inches} \\ \textit{in altitude}\end{matrix}\right).$$

Example.—Find the number of gallons in a cone whose altitude is $26\frac{5}{8}$ inches and the diameter of whose base is 21 inches.

Solution.—We have

$$(\text{number of gallons in cone}) = \frac{1}{2} \times \frac{1}{441} \times 21^2 \times 26\tfrac{5}{8} = 13\tfrac{5}{16}.$$

Therefore, the cone contains $13\frac{5}{16}$ gal. Ans.

EXAMPLES FOR PRACTICE.

1. Find the convex surface and the entire surface of a cone whose altitude is 8 inches and the radius of whose base is 6 inches.

Ans. { Convex surface = 188.5 sq. in.
Entire surface = 301.6 sq. in.

2. Find the volume of a cone whose altitude is 15 inches and the radius of whose base is 3 inches. Ans. 141.37 cu. in.

---

54. If a cone is divided into two parts by a plane parallel to its base, the lower part is called a **frustum of a cone,** Fig. 13. The two ends of a frustum are called its **bases.** The perpendicular distance between the two bases is the **altitude** of the frustum. Since the axis of a cone is perpendicular to both bases of the frustum, the altitude of the frustum is equal to the portion of the axis of the cone intercepted between the two bases of the frustum. The **slant height** of the frustum is the difference between the slant height of the whole cone before it was divided and the slant height of the small cone cut off by the dividing plane.

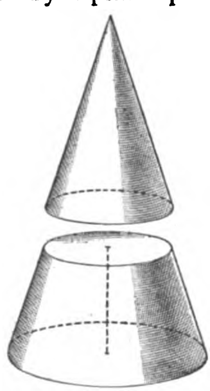


Fig. 13.

55. *The area of the convex surface of a frustum of a cone is equal to 3.1416 times the product obtained by multiplying the sum of the radii of its bases by its slant height.* That is,

$$(\textit{convex surface}) = 3.1416 \times \left(\begin{matrix}\textit{sum of the radii}\\ \textit{of its bases}\end{matrix}\right) \times (\textit{slant height}).$$

EXAMPLE.—Find the area of the entire surface of the frustum of a cone shown in Fig. 14.

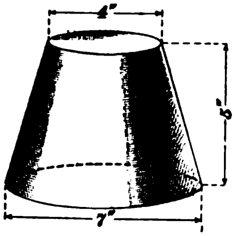


FIG. 14.

SOLUTION.—The area of the entire surface is found by adding the areas of the two bases to the area of the convex surface (Art. **30**). The radius of the smaller base is ½ of 4 inches, or 2 inches; and, therefore, the area of the smaller base is 3.1416×2^2, or 12.5664 square inches. In like manner, the area of the larger base is 3.1416×3.5^2, or 3.1416×12.25, or 38.4846 square inches. Hence, the sum of the areas of the two bases is (12.5664 + 38.4846), or 51.051 square inches.

To find the area of the convex surface, it is necessary to find the slant height of the frustum. For this purpose, we take a section through the axis of the cone as shown in Fig. 15. In Fig. 15, $AD = 3.5$ inches and $BD = 2$ inches; hence, $AB = AD - BD = (3.5 - 2)$, or 1.5 inches. Therefore, in the right triangle ABC, we have $AB = 1.5"$ and $BC = 5"$. Hence, by Art. **42**, Part 6, we have the slant height AC is equal to $\sqrt{1.5^2 + 5^2}$, or $\sqrt{27.25}$, or 5.2202 inches. The area of the convex surface is equal to 3.1416 × (sum of the radii of its bases) × (slant height), which is equal to $3.1416 \times (2 + 3.5) \times 5.2202$, or, correct to five figures, 90.199 square inches. Therefore, the entire area is equal to (90.199 + 51.051), or 141.25 sq. in. Ans.

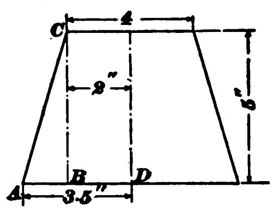


FIG. 15.

56. Rule.—*To find the volume of a frustum of a cone, add together the square of the radius of the greater base, the square of the radius of the smaller base, and the product of these two radii. Then multiply this sum by ⅓ × 3 1416 times the altitude of the frustum, and the product will be the volume.*

EXAMPLE.—The diameters of the bases of a frustum of a cone are 18 inches and 14 inches, respectively, and its altitude is 12 inches; find its volume.

SOLUTION.—The radius of the greater base is ½ × 18 or 9 inches, and the radius of the smaller base is ½ × 14, or 7 inches. Adding together the square of the radius of the greater base, the square of the radius of the smaller base, and the product of these two radii, we get the sum $9^2 + 7^2 + 9 \times 7 = 193$. To find the volume we must multiply this sum by ⅓ × 3.1416 times the altitude. Hence, the volume is ⅓ × 3.1416 × 12 × 193, or, correct to four figures, 2,425 cu. in. Ans.

EXAMPLES FOR PRACTICE.

1. The diameters of the bases of the frustum of a cone are 12 inches and 8 inches, respectively, and its slant height is 10 inches. Find the convex surface and the entire surface of the frustum.

Ans. { Convex surface = 314.16 sq. in.
Entire surface = 477.52 sq. in.

2. The radii of the bases of a frustum of a cone are 8 inches and 3 inches, respectively, and its altitude is 12 inches. Find its volume.

Ans. 1,218.9. cu. in.

PARABOLOID AND SPHEROIDS.

57. The convex surface of a cone is called a **conical surface.** Any straight line drawn from the vertex of a cone to a point in the circumference of its base is called an **element** of the conical surface; thus, in Fig. 16, each of the lines BC, BD, and BE is an element of the conical surface.

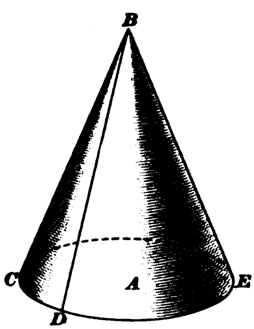


FIG. 16.

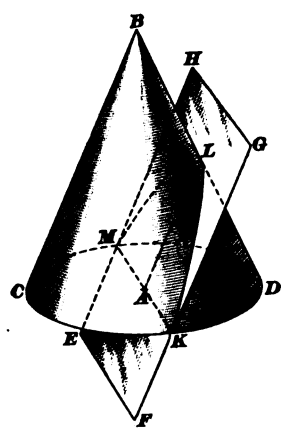


FIG. 17.

58. If a cone is cut by a plane that is parallel to one of the elements of the conical surface, the intersection of the cone and the plane is a figure called a **parabola.** Thus, in Fig. 17, the cone is cut by the plane $EFGH$, which is parallel to the element $C B$ of the conical surface; hence, the figure $K L M$ is a parabola. Fig. 18 shows an accurately drawn parabola; the line $L A$ is the perpendicular bisector of the line $M K$; the line $L A$ is called the **center line** of the parabola, and it divides the parabola into two parts that are equal in every respect. The line $M K$ is the **base** and $L A$ is the **altitude** of the parabola.

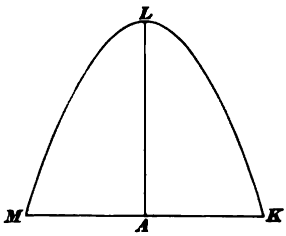


FIG. 18.

59. *The area of a parabola is equal to $\frac{2}{3}$ of the product of the base and the altitude.* That is,

$$(area) = \frac{2}{3} \times (base) \times (altitude).$$

60. If the crooked line $A K L$, Fig. 18, is turned through a complete revolution about the line $L A$ as an axis, it traces out a surface, and the figure enclosed by this surface is a **paraboloid.** The base of the paraboloid is a circle whose center is A and whose radius is $A K$. Fig. 19 shows a paraboloid.

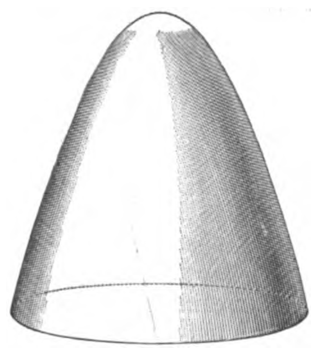


FIG. 19.

61. *The volume of a paraboloid is equal to $\frac{1}{2} \times 3.1416$ times the product obtained by multiplying the square of the radius of its base by its altitude; or the volume of a paraboloid is equal to $\frac{1}{2} \times .7854$ times the product obtained by multiplying the square of the diameter of its base by its altitude.* That is,

$$(volume) = \frac{1}{2} \times 3.1416 \times (radius\ of\ base)^2 \times (altitude),$$

or

$$(volume) = \frac{1}{2} \times .7854 \times (diameter\ of\ base)^2 \times (altitude).$$

EXAMPLE.—The diameter of the base of a paraboloid is 14 inches and its altitude is 15 inches. Find its volume.

SOLUTION.—We have

$$(\text{volume}) = \tfrac{1}{2} \times .7854 \times (\text{diameter of base})^2 \times (\text{altitude}),$$

which is equal to $\frac{1}{2} \times .7854 \times 14^2 \times 15$, or 1,155 cu. in. Ans.

62. *The number of gallons contained in a paraboloid is equal to $\frac{3}{4}$ of the square of the number of inches in the diameter of its base multiplied by the number of inches in its altitude and divided by 441.* That is,

$$(\textit{number of gallons in paraboloid})$$
$$= \frac{3}{4} \times \frac{1}{441} \times \begin{pmatrix}\textit{number of inches}\\ \textit{in diameter of base}\end{pmatrix}^2 \times \begin{pmatrix}\textit{number of inches}\\ \textit{in altitude}\end{pmatrix}.$$

EXAMPLE.—Find the capacity in gallons of a paraboloid, the diameter of whose base is 28 inches and whose altitude is 30 inches.

SOLUTION.—We have

$$(\text{number of gallons in paraboloid})$$
$$= \frac{3}{4} \times \frac{1}{441} \times \begin{pmatrix}\text{number of inches}\\ \text{in diameter of base}\end{pmatrix}^2 \times \begin{pmatrix}\text{number of inches}\\ \text{in altitude}\end{pmatrix},$$

which is equal to $\frac{3}{4} \times \frac{1}{441} \times 28^2 \times 30$, or 40. Hence, the number of gallons in this paraboloid is 40. Ans.

63. The line CA is the long diameter and the line BD is the short diameter of the ellipse in Fig. 20. If the curved line DAB is turned about the line DB as an axis, it traces out a surface, and the space enclosed by this surface is called an **oblate spheroid.** The line DB is the **axis of revolution** of the oblate spheroid. Fig. 21 shows an oblate spheroid.

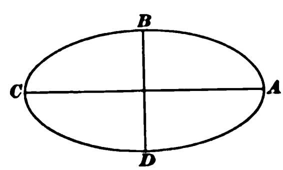
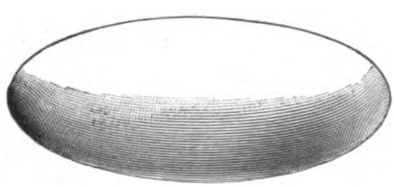


FIG. 20.

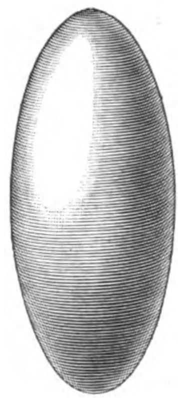
If the curved line ABC is turned about the line CA as an axis, it traces out a surface, and the space enclosed by this surface is called a **prolate spheroid.** The line CA is the axis of revolution of the prolate spheroid. Fig. 22 shows a prolate spheroid.



Oblate Spheroid

FIG. 21.

It is important to remember that an oblate spheroid is formed by turning an ellipse about its short diameter, and that a prolate spheroid is formed by turning an ellipse about its long diameter. Hence, *the axis of revolution of an oblate spheroid is the short diameter of the ellipse, and the axis of revolution of a prolate spheroid is the long diameter of an ellipse.*



Prolate Spheroid

FIG. 22.

64. *The surface of an oblate spheroid is approximately equal to 3.1416 times the product obtained by multiplying the square root of half the sum of the squares of the diameters by the long diameter.* That is,

$$(surface) = 3.1416 \times \sqrt{\frac{(long\ diameter)^2 + (short\ diameter)^2}{2}} \times (long\ diameter).$$

EXAMPLE.—Find the surface of an oblate spheroid whose diameters are 7 and 17 inches, respectively.

SOLUTION.—We have

$$(\text{surface}) = 3.1416\sqrt{\frac{(\text{long diameter})^2 + (\text{short diameter})^2}{2}} \times \binom{\text{long}}{\text{diameter}},$$

which is equal to $3.1416\sqrt{\frac{17^2 + 7^2}{2}} \times 17$, or 694.29 square inches.

Hence, correct to three figures, the surface is 694 sq. in. Ans.

65. *The surface of a prolate spheroid is approximately equal to 3.1416 times the product obtained by multiplying the square root of half the sum of the squares of the diameters by the short diameter.* That is,

$$(surface) = 3.1416 \times \sqrt{\frac{(long\ diameter)^2 + (short\ diameter)^2}{2}} \times (short\ diameter).$$

EXAMPLE.—Find the area of a prolate spheroid whose diameters are 7 and 17 inches, respectively.

SOLUTION.—We have

$$(\text{surface}) = 3.1416 \times \sqrt{\frac{(\text{long diameter})^2 + (\text{short diameter})^2}{2}} \times \left(\begin{matrix}\text{short}\\ \text{diameter}\end{matrix}\right)$$

which is equal to $3.1416 \times \sqrt{\frac{17^2 + 7^2}{2}} \times 7$, or 285.89 square inches. Hence, correct to three figures, the surface is 286 sq. in. Ans.

66. *The volume of an oblate spheroid is equal to* $\frac{1}{6} \times 3.1416$ *times the product obtained by multiplying the square of the long diameter by the short diameter.* That is,

$$(volume) = \tfrac{1}{6} \times 3.1416 \times (long\ diameter)^2 \times (short\ diameter).$$

EXAMPLE.—Find the volume of an oblate spheroid whose diameters are 16 and 12 inches, respectively.

SOLUTION.—We have

$$(\text{volume}) = \tfrac{1}{6} \times 3.1416 \times (\text{long diameter})^2 \times (\text{short diameter}),$$

which is equal to $\frac{1}{6} \times 3.1416 \times 16^2 \times 12$, or 1,608.4992 cubic inches. Hence, correct to four figures, the volume is 1,608 cu. in. Ans.

67. *The volume of a prolate spheroid is equal to* $\frac{1}{6} \times 3.1416$ *times the product obtained by multiplying the square of the short diameter by the long diameter.* That is,

$$(volume) = \tfrac{1}{6} \times 3.1416 \times (short\ diameter)^2 \times (long\ diameter).$$

EXAMPLE.—Find the volume of a prolate spheroid whose diameters are 10 and 13 inches, respectively.

SOLUTION.—We have

$$(\text{volume}) = \tfrac{1}{6} \times 3.1416 \times (\text{short diameter})^2 \times (\text{long diameter}),$$

which is equal to $\frac{1}{6} \times 3.1416 \times 10^2 \times 13$, or 680.68 cubic inches. Hence, correct to three figures, the volume is 681 cu. in. Ans.

EXAMPLES FOR PRACTICE.

1. The diameter of the base of a paraboloid is 10 inches and its altitude is 12 inches. Find its volume. Ans. 471.24 cu. in.

2. Find the volume of a prolate spheroid whose diameters are 6 and 10 inches, respectively. Ans. 188.5 cu. in.

COMPARISON OF VOLUMES.

68. The following comparisons of the volumes of solids are very important because they relieve the student of the necessity for remembering a large number of independent rules.

69. *The volume of a sphere is equal to two-thirds of the volume of a cylinder having its diameter and altitude each equal to the diameter of the sphere.*

70. *The volume of a cone is equal to one-third of the volume of a cylinder having the same base and altitude as the cone.*

71. *The volume of a paraboloid is equal to one-half of the volume of a cylinder having the same base and altitude as the paraboloid.*

72. *The volume of a spheroid is equal to two-thirds of the volume of a cylinder whose diameter is equal to the axis of revolution of the spheroid and whose altitude is equal to the other diameter of the spheroid.*

ARITHMETIC.

(PART 11.)

MENSURATION.

PRISM, PYRAMID, AND WEDGE.

1. A **prism** is a solid whose ends are equal parallel polygons having their corresponding sides parallel, and whose sides are parallelograms. The parallel polygons are called the **bases** of the prism; and the perpendicular distance between the bases is the **altitude** of the prism. The lines in which the sides of the prism intersect are called the **edges** of the prism. Fig. 1 shows a prism whose bases are the equal triangles $A\ B\ C$ and $A'\ B'\ C'$. The sides of this prism are the three parallelograms $B\ B'\ A'\ A$, $B\ B'\ C'C$, and $A\ A'\ C'\ C$; and its edges are the lines $A\ A'$, $B\ B'$, and $C\ C'$.

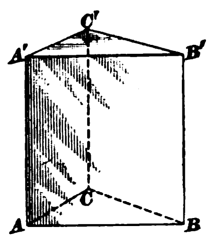
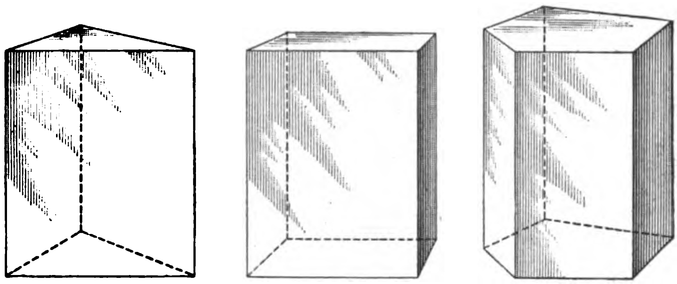


FIG. 1.

2. A prism whose bases are triangles is called a **triangular prism**; a prism whose bases are quadrilaterals is called a **quadrangular prism**; and so on.

3. A prism whose bases are parallelograms is called a **parallelopipedon.**

4. A **right** prism is a prism whose edges are perpendicular to its bases. Fig. 2 shows three right prisms.



Right triangular prism. Right quadrangular prism. Right pentagonal prism.

FIG. 2.

5. The **altitude** of a right prism is equal to the length of any of its edges.

6. A right prism whose bases are rectangles is the same as a rectangular solid (Art. **28,** Part 4).

A right prism whose bases are regular polygons is called a **regular prism.**

7. The line joining the centers of the bases of a regular prism is perpendicular to both bases and is called the **axis** of the prism. The length of the axis intercepted between the two bases of a regular prism is equal to the altitude of the prism.

Any prism that is not a regular prism is called an **irregular prism.** There is no line that is called the axis of an irregular prism.

8. If a sheet of paper is rolled upon a right prism and so fitted as to cover the whole of its convex surface, and then unrolled, the sheet of paper will be in the form of a rectangle whose base is the perimeter of the base of the right prism and whose altitude is the altitude of the prism. The area of this rectangle is found by multiplying its base by its altitude; therefore, the area of the convex surface of any right prism is found by multiplying the perimeter of its base by its altitude. Hence, we have the following proposition:

9. *The area of the convex surface of a prism is equal to the product of the perimeter of its base and its altitude.* That is,

$$(convex\ surface) = (perimeter\ of\ base) \times (altitude).$$

EXAMPLE.—How many square feet of zinc will be required to line a closed tank having the interior dimensions 10 feet × 8 feet × 3 feet?

SOLUTION.—The perimeter of the base is 2 × 10 + 2 × 8, or 36 feet. Hence, the convex surface is 36 × 3, or 108 square feet. The area of one of the bases is 10 × 8, or 80 square feet, and the area of the two bases is 2 × 80, or 160 square feet. The entire inside surface is equal to the convex surface plus the area of the two bases; hence, the entire surface is 108 + 160, or 268 square feet. Thus, it will require 268 sq. ft. of zinc to line the tank. Ans.

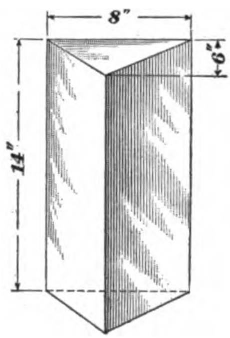


FIG. 3.

10. *The volume of any right prism is found by multiplying the area of its base by its altitude.* That is,

$$(volume) = (area\ of\ base) \times (altitude).$$

EXAMPLE.—Find the volume of the right triangular prism shown in Fig. 3.

SOLUTION. — The bases of the prism are equal triangles whose bases are 8 inches and whose altitudes are 6 inches. By Art. **62**, Part 7, the area of one of these triangles is ½ × 8 × 6, or 24 square inches. Hence, the volume of the prism is 24 × 14, or 336 cu. in. Ans.

11. In many cases, the volume of a solid can be found by dividing the given solid into two or more parts each of which is a right prism. In other cases it is easier to find the volume by regarding the given solid as equivalent to the difference between two rectangular prisms. For example, let it be required to find the volume of the irregular box shown in Fig. 4. The volume of the box is equivalent to the difference between the volume of a prism whose

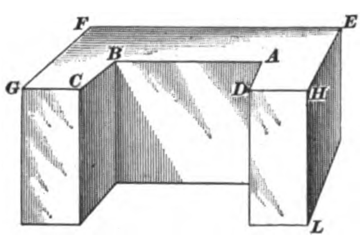


FIG. 4.

cross-section is $EFGH$ and the volume of a prism whose cross-section is $ABCD$. The volume of the prism whose cross-section is $EFGH$ is, by Art. **10,**

$$(\text{area of base } EFGH) \times (\text{altitude } HL).$$

The volume of the prism whose cross-section is $ABCD$ is

$$(\text{area of base } ABCD) \times (\text{altitude } HL).$$

Hence, (volume of box)

$$= (\text{area of base } EFGH) \times (\text{altitude } HL)$$
$$- (\text{area of base } ABCD) \times (\text{altitude } HL),$$

or, by Art. **36,** Part 4,

$$(\text{volume of box})$$
$$= (\text{area of base } EFGH - \text{area of base } ABCD)$$
$$\times (\text{altitude } HL).$$

EXAMPLE.—An iron casting has the form and dimensions shown in Fig. 5. Find the weight of this casting, a cubic inch of cast iron weighing .261 pound.

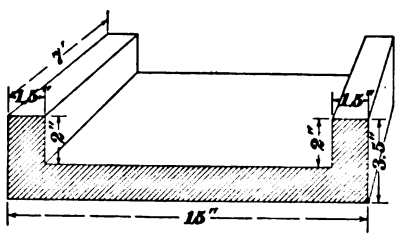


FIG. 5.

SOLUTION.—The area of the whole cross-section if the casting were solid throughout is 15 × 3.5, or 52.5 square inches. The inside width of the channel is 15 − 2 × 1.5, or 12 inches. Hence, the area of a cross-section of the channel is 12 × 2, or 24 square inches. The length of the casting is 7 feet = 7 × 12, or 84 inches. Hence,

$$(\text{volume of casting}) = (52.5 - 24) \times 84 = 2{,}394.$$

Hence, the volume of the casting is 2,394 cubic inches. Since each cubic inch weighs .261 pound, the weight of the casting is 2,394 × .261, or 624.834 lb. Ans.

EXAMPLES FOR PRACTICE.

1. Find the volume of a rectangular solid whose length, breadth, and thickness are 5.5 feet, 4 feet, and 3.2 feet. Ans. 70.4 cu. ft.

2. Find the volume of a triangular prism whose length is 10 feet and one side of the equilateral end is 2.4 feet. Ans. 24.94 cu. ft.

12. A **pyramid** is a solid whose base is a polygon and whose sides are triangles meeting in a common point. The common point in which the triangular sides meet is called the **vertex** of the pyramid. Fig. 6 shows a quadrangular pyramid whose base is the quadrilateral $ABCD$, whose sides are the four triangles PAB, PBC, PCD, and PDA, and whose vertex is the point P.

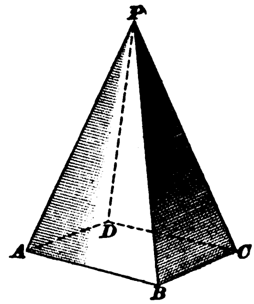


FIG. 6.

13. The **altitude** of a pyramid is the length of the perpendicular drawn from the vertex to the base.

14. A **regular pyramid** is a pyramid whose base is a regular polygon and whose vertex lies on the perpendicular drawn from the base at its center. The perpendicular drawn from the vertex of a regular pyramid to the base is called the **axis** of the regular pyramid. The part of the axis intercepted between the vertex and the base of a regular pyramid is equal to the altitude. Any pyramid that is not a regular pyramid is called an **irregular pyramid.**

15. *In a regular pyramid the sides are all equal isosceles triangles.* Thus, in the regular pentagonal pyramid shown in Fig. 7, each of the sides PAB, PBC, PCD, PDE, and PEA is an isosceles triangle, and these five triangles are all equal. Also $PA = PB = PC = PD = PE$.

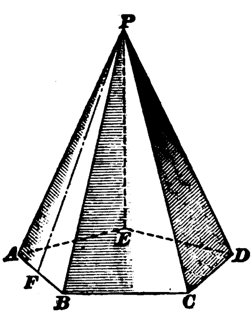


FIG. 7.

16. *The altitudes of the isosceles triangles that form the sides of a regular pyramid are all equal.* The altitude of any of these isosceles triangles is called the **slant height** of the regular pyramid. Thus, in Fig. 7, if PF is drawn from the vertex P perpendicular to the line AB, PF is the slant height of this regular pyramid.

In an irregular pyramid the altitudes of the triangular sides are not all equal; and there is no line that is called the slant height of an irregular pyramid.

17. By Art. **62**, Part 7, the area of the triangle PAB, Fig. 7, is equal to one-half of the product of the base AB and the altitude PF. Since all the triangular sides of the pyramid are equal, the convex surface of this pyramid is one-half of five times the product of AB and PF. But, five times AB is equal to the perimeter of the base and PF is equal to the slant height of the pyramid. Hence, we have the following rule:

18. *The area of the convex surface of any regular pyramid is equal to one-half of the product of the perimeter of the base and the slant height.* That is,

(*convex surface*) = $\frac{1}{2}$ × (*perimeter of base*) × (*slant height*).

EXAMPLE.—Find (*a*) the area of the convex surface and (*b*) the entire area of a regular pyramid whose altitude is 60 inches and whose base is a regular polygon of six sides, each 4.6 inches long (Fig. 8).

SOLUTION.—Let O be the center of the base. Then, OAB is an equilateral triangle each of whose sides is 4.6" long (Art. **7**, Part 9). Hence, in the right triangle OAG, we have $OA = 4.6"$, and $AG = 2.3"$. Whence, $OG = \sqrt{\overline{OA}^2 - \overline{AG}^2} = 3.98"$, correct to three figures. Whence, PG = slant height $= \sqrt{\overline{PO}^2 + \overline{OG}^2}$. Putting $PO = 60"$ and $OG = 3.98"$, we find that the number of inches in PG is $\sqrt{60^2 + 3.98^2} = 60.1$, correct to three figures. The number of sides of the base is six and each side is 4.6"; therefore, the perimeter of the base is 6×4.6 or 27.6 inches. Hence, the convex surface is $\frac{1}{2} \times 27.6 \times 60.1$, or 829 square inches, correct to three figures.

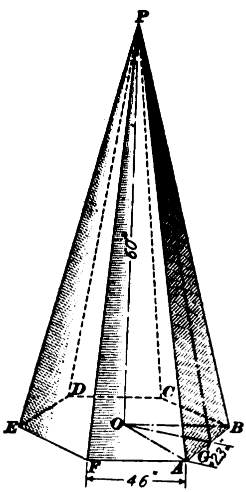


FIG. 8.

The base of the pyramid is made up of six equilateral triangles whose bases are each 4.6 inches and whose altitudes are each 3.98 inches. Hence, the area of the base is $6 \times \frac{1}{2} \times 4.6 \times 3.98$, or 54.9 square inches, correct to three figures. Therefore, the entire surface of the pyramid is 829 + 54.9, or 883.9 square inches.

Ans. { 829 sq. in. = convex surface.
883.9 sq. in. = entire surface.

19. If it is required to find the area of the convex surface of an irregular pyramid, the area of each of the irregular sides must be calculated separately; the sum of these areas will be the required area of the convex surface.

20. *The volume of any pyramid is equal to one-third of the product of the area of its base and its altitude.* That is,

$$(volume) = \frac{1}{3} \times (area\ of\ base) \times (altitude).$$

EXAMPLE.—Find the volume of a pyramid whose altitude is 9 inches and whose base is a square 5 in. × 5 in.

SOLUTION.—The area of the base is 5 × 5, or 25 square inches. Hence, the volume is $\frac{1}{3} \times 25 \times 9$, or 75 cu. in. Ans.

21. *The volume of a pyramid is equal to one-third of the volume of a prism having the same base and altitude.*

22. If a pyramid is divided into two parts by a plane parallel to its base, the lower part is called a **frustum** of a pyramid. See Fig. 9. The two ends of a frustum are called its **bases,** and the perpendicular distance between the two bases is the **altitude** of the frustum.

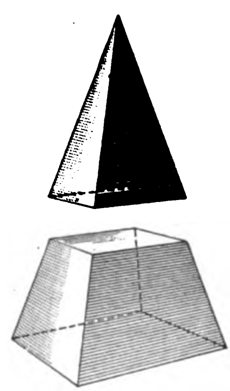


FIG. 9.

23. *The volume of a frustum of a pyramid is equal to one-third of the product obtained by multiplying the altitude by the sum of the areas of the two bases and the square root of the product of the areas of the two bases.* That is,

$$(volume) = \frac{1}{3} \times (altitude) \times \left(\begin{array}{c} upper\ base + lower\ base \\ + \sqrt{upper\ base \times lower\ base} \end{array} \right).$$

EXAMPLE.—Find the volume of a frustum of a square pyramid of which the larger base is 2.5 feet square, the smaller base is 1 foot square, and the altitude is 16 feet.

SOLUTION.—The area of the larger base is 2.5 × 2.5, or 6.25 square feet, the area of the smaller base is 1 × 1, or 1 square foot, and the square root of their product is $\sqrt{6.25 \times 1}$ or 2.5 square feet. Hence, the volume of the frustum is $(6.25 + 1 + 2.5) \times \frac{1}{3} \times 16$, or 52 cu. ft. Ans.

24. A **wedge** is a solid having plane surfaces, of which the base is a parallelogram, the ends are triangles, and the sides are trapezoids or parallelograms meeting in a line parallel to two of the sides of the base. Fig. 10 shows a wedge, of which the base is the parallelogram $ABCD$, the ends are the triangles ADE and BCF, and the sides are the trapezoids $ABFE$ and $DCFE$. The sides $ABFE$ and $DCFE$ meet in the line EF, which is parallel to AB and CD, two of the sides of the base. Thus, the lines EF, AB, and CD are the three **parallel edges.**

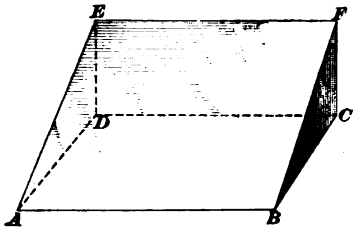


FIG. 10.

If perpendiculars are drawn from several points in the line EF to the plane $ABCD$, these perpendiculars are all equal, and the length of any one of them is called the **length** of the wedge.

The perpendicular distance between the lines AB and CD is the **width of the base.**

25. *The volume of a wedge is equal to one-sixth of the continued product of the width of the base, the length of the wedge, and the sum of the three parallel edges.* That is,

$$(volume) = \frac{1}{6} \times (width\ of\ base) \times (length) \times \begin{pmatrix} sum\ of\ three \\ parallel\ edges \end{pmatrix}.$$

EXAMPLE.—Find the volume of a wedge of which the length is 8 inches, the width of the base is 1.5 inches, and the lengths of the three parallel edges are 3 inches, 3 inches, and 2 inches, respectively.

SOLUTION.—The sum of the three parallel edges is $3 + 3 + 2$, or 8 inches. Hence, the volume is $\frac{1}{6} \times 1.5 \times 8 \times 8$, or 16 cu. in. Ans.

26. A wedge whose ends are parallel and whose base is a rectangle is a right triangular prism, and its volume is best found by the rule of Art. **10.**

27. A **prismoid** is a solid whose bases are polygons in parallel planes and whose faces are triangles or quadrilaterals. Thus, the solid shown in Fig. 11 is a prismoid; its bases are the pentagon $ABCDE$ and the quadrilateral $FGHI$, which lie in parallel planes; and its faces are the triangle GBC and the quadrilaterals $GCDH$, $HDEI$, $IEAF$, and $FABG$.

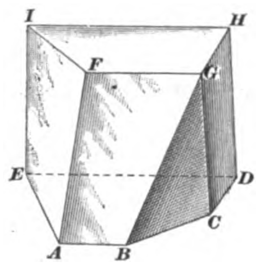


FIG. 11.

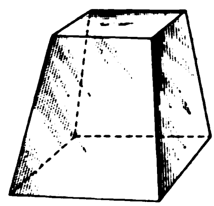


FIG. 12.

28. If a wedge is divided into two parts by a plane parallel to its base, the lower part is called a **right prismoid.** Thus, Fig. 12 is a right prismoid.

29. The perpendicular distance between the bases of a prismoid is called its **altitude.**

30. If a prismoid is cut by a plane parallel to its bases and equidistant from the two bases, the polygon so formed is called the **middle section** of the prismoid. Thus, the polygon $PQRS$ is the middle section of the prismoid shown in Fig. 13.

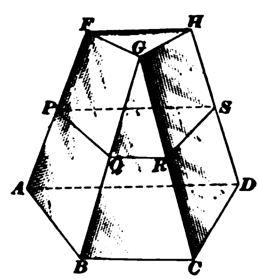


FIG. 13.

31. *The lengths of the sides of the middle section of a prismoid are equal, respectively, to one-half the sum of the corresponding sides of the bases.* Thus, in Fig. 13, we have $PQ = \frac{1}{2}(AB + FG)$, $QR = \frac{1}{2}BC$, $RS = \frac{1}{2}(GH + CD)$, and $SP = \frac{1}{2}(HF + DA)$.

32. The area of the middle section of a prismoid is not in general equal to one-half the sum of the areas of the bases.

33. *The volume of a prismoid is equal to one-sixth of the product obtained by multiplying the altitude by the sum of the area of the lower base, the area of the upper base, and four times the area of the middle section.* That is,

$$(volume) = \frac{1}{6} \times (altitude) \times \left(\begin{matrix} lower \\ base \end{matrix} + \begin{matrix} upper \\ base \end{matrix} + 4 \times \begin{matrix} middle \\ section \end{matrix}\right).$$

EXAMPLE.—Find the volume of the prismoid shown in Fig. 14. The altitude of the prismoid is 14 inches.

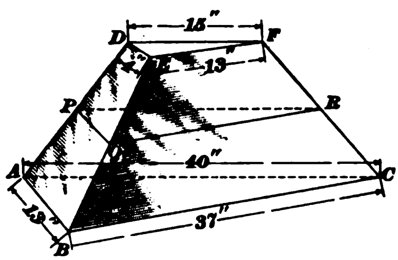


FIG. 14.

SOLUTION.—Let PQR be the middle section. Then, by Art. **31**, we have

$$PQ = \tfrac{1}{2}(AB + DE) = \tfrac{1}{2}(13'' + 4'') = 8.5''.$$
$$QR = \tfrac{1}{2}(BC + EF) = \tfrac{1}{2}(37'' + 13'') = 25''.$$
$$RP = \tfrac{1}{2}(AC + DF) = \tfrac{1}{2}(40'' + 15'') = 27.5''.$$

The areas of the triangles ABC, DEF, and PQR are calculated by the rule of Art. **63**, Part 7, which gives area of $ABC = 240$ square inches, area of $DEF = 24$ square inches, and area of $PQR = 105.2$ square inches, nearly. Hence, the volume is equal to

$\frac{1}{6} \times$ (altitude) $\times$ (lower base + upper base + 4 $\times$ middle section),

which is equal to $\frac{1}{6} \times 14 \times (240 + 24 + 4 \times 105.2)$, or 1,597.9 cu. in., nearly. Ans.

34. The principle of Art. **33,** called the **prismoidal formula,** is used very much in practice. It may be used to calculate the volume of a prism, pyramid, cylinder, cone, frustum of a pyramid, frustum of a cone, wedge, sphere, or segments of a sphere in addition to the irregularly shaped bodies to which it is usually applied. For a pyramid, cone, and wedge, the area of the upper base and the dimensions of the upper base are 0; for a sphere the areas of the upper and lower bases are 0; for a prism and a cylinder the areas of the two bases and of the middle section are all equal.

The prismoidal formula gives a good approximation to the volume of a cylindrical ungula, which is a section of a cylinder formed by a plane cutting the base and the convex surface of the cylinder. Thus, Fig. 15 shows a cylindrical ungula.

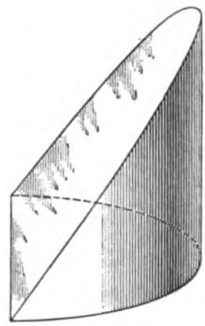


FIG. 15.

The prismoidal formula, when applied to the frustum of a pyramid, saves the labor of extracting a square root.

EXAMPLE. — Find by the prismoidal formula the volume of the frustum of the pyramid given in the example of Art. **23.**

SOLUTION.—The area of the larger base is 2.5 × 2.5, or 6.25 square feet; the area of the smaller base is 1 × 1, or 1 square foot. The middle section is a square whose side is one-half the sum of the side of the upper and lower base; that is, $\frac{1}{2}$ of (2.5 + 1), or 1.75 feet. The area of the middle section is 1.75^2, or 3.0625 square feet. The volume of the frustum is $\frac{1}{6} \times 16 \times (6.25 + 1 + 4 \times 3.0625)$, or $\frac{1}{6} \times 16 \times 19.50$, or 52 cu. ft. Ans.

35. By comparing Art. **51,** Part 10, and Art. **20** the student will see that the volumes of a cone and a pyramid are both equal to one-third of the product of the area of the base and altitude. Similarly, the volume of any solid whose base is a plane figure and which tapers to a point like a cone or pyramid is equal to one-third of the product of its base and altitude.

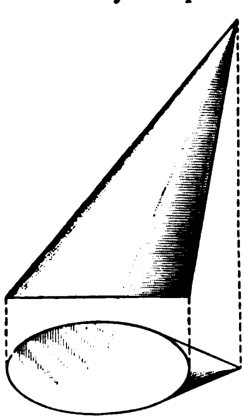


FIG. 16.

EXAMPLE.—Find the volume of the solid shown in Fig. 16. The upper part of the figure is a side view of the object sitting on its base, and the lower part of the figure is a view of the elliptic base. The base is an ellipse whose long diameter is 8 inches and whose short diameter is 6 inches, and the altitude is 7.5 inches.

SOLUTION.—By Art. **42,** Part 9, the area of the base is .7854 times the product of the two diameters; that is,

$$\text{area of base} = .7854 \times 8 \times 6.$$

The volume is equal to one-third the product of the area of the base and altitude; that is,

$$\text{volume} = \tfrac{1}{3} \times (.7854 \times 8 \times 6) \times 7.5 = .7854 \times 8 \times 2 \times 7.5 = 94.248.$$

Hence, the volume is 94.248 cu. in. Ans.

EXAMPLES FOR PRACTICE.

1. Steel weighs .28 pound per cubic inch. Find the weight of a steel wedge whose base is a 3" × 1½" rectangle and whose length is 8 inches. Ans. 5.04 lb.

2. Find the volume of the frustum of a square pyramid of which the larger base is a 15" square, the smaller base is a 14" square, and the altitude is 3 inches. Ans. 631 cu. in.

3. Find the volume of a triangular pyramid of which the altitude is 4 inches and the base is an equilateral triangle having each side 3 inches long. Ans. 5.2 cu. in.

4. Find the volume of a right prismoid whose bases are rectangles, which measure 10 inches by 8 inches and 8 inches by 6 inches, and the height 40 inches. Ans. 2,533⅓ cu. in.

PERCENTAGE.

36. A certain specimen of solder weighs 10 pounds and contains 6 pounds of lead. In this specimen of solder, the ratio of the weight of the lead to the whole weight of the solder is

$$\frac{6 \text{ pounds}}{10 \text{ pounds}} = \frac{6}{10} = \frac{60}{100};$$

that is, the weight of the lead is 60 hundredths of the whole weight of the solder. Thus, the ratio of the weight of the lead to the whole weight of the solder is expressed by the number of hundredths that the weight of the lead is of the whole weight of the solder.

Hence, *the ratio of any quantity to another quantity of the same kind may be expressed by the number of hundredths that the first quantity is of the second.*

37. When the ratio of one quantity to another quantity of the same kind is expressed by the *number* of hundredths that the first quantity is of the second, this *number* is called the **per cent.** that the first quantity is of the second; the second quantity is called the **base**; and the first quantity is called the **percentage.**

In the specimen of solder referred to in Art. **36,** the weight of the lead is 60 hundredths, or 60 per cent., of the whole weight of the solder; the whole weight of the solder, 10 pounds, is the base; and the weight of the lead, 6 pounds, is the percentage.

EXAMPLE.—Find the weight of tin in 150 pounds of solder of which 50 per cent., by weight, is tin.

SOLUTION.—The weight of the tin is 50 per cent., or 50 hundredths, of the whole weight of the solder. Therefore, the weight of the tin is

$$\frac{50}{100} \text{ of } 150 \text{ lb.} = \frac{50}{100} \times 150 \text{ lb.} = 75 \text{ lb.} \quad \text{Ans.}$$

38. The sign % means per cent.; thus, 6% is read *6 per cent.*, and means 6 hundredths; 12½% is read *12½ per cent.*, and means 12½ hundredths.

Hence,

$$1\% = \frac{1}{100} = .01,$$

$$2\% = \frac{2}{100} = .02,$$

$$3\% = \frac{3}{100} = .03,$$

$$12\tfrac{1}{2}\% = \frac{12\frac{1}{2}}{100} = .125,$$

and so on.

EXAMPLE.—What is 12½% of 300 gallons?

SOLUTION.— $12\frac{1}{2}\%$ of 300 gal. is $\frac{12\frac{1}{2}}{100}$ of 300 gal. $= \frac{12\frac{1}{2}}{100} \times 300$ gal. $= 12\frac{1}{2} \times 3$ gal. $= 37\frac{1}{2}$ gal. Ans.

39. When we say that a certain specimen of solder contains 60 per cent., by weight, of lead, we mean that

60 hundredths of the weight of the solder is lead; or, that every pound of the solder contains .60 pound of lead. Hence, we say that the **rate** at which this solder contains lead is .60, meaning that every pound of solder contains .60 pound of lead. The weight of the lead in any quantity of this solder can be found by multiplying the weight of the solder by the rate .60.

40. In the example of Art. **39**, the weight of the quantity of solder is the base, and the weight of lead contained in the solder is the percentage. We saw that the weight of lead is found by multiplying the weight of the quantity of solder by the rate. Hence, we have the following important principle:

41. *The percentage is equal to the product of the base and the rate.* That is,

$$(percentage) = (base) \times (rate).$$

42. *The rate is equal to the quotient obtained by dividing the percentage by the base.* That is,

$$(rate) = \frac{(percentage)}{(base)}.$$

This result is obtained by dividing both members of the equation in Art. **41** by "*base.*"

43. *The base is equal to the quotient obtained by dividing the percentage by the rate.* That is,

$$(base) = \frac{(percentage)}{(rate)}.$$

This result is obtained by dividing both members of the equation in Art. **41** by "*rate.*"

44. If a specimen of solder contains 60 per cent., by weight, of lead, it is evident that every hundred pounds of the solder contains 60 pounds of lead. Hence, we may say that the **rate per cent.** at which this solder contains lead is 60, meaning that every hundred pounds of solder contains 60 pounds of lead.

The weight of lead in any quantity of this solder can be found by multiplying the weight of the quantity of solder by the *rate per cent.*, 60, and dividing the product by one hundred.

45. In the example of Art. **44,** the weight of the quantity of solder is the base and the weight of lead contained in the solder is the percentage. We saw that the weight of the lead is found by multiplying the weight of the quantity of solder by the rate per cent. and dividing the product by one hundred. Hence, we have the following principle:

46. *The percentage is equal to the product of the base and rate per cent. divided by one hundred.* That is,

$$(percentage) = \frac{(base) \times (rate\ per\ cent.)}{100}.$$

47. *The rate per cent. is equal to one hundred times the rate.* That is,

$$(rate\ per\ cent.) = 100 \times (rate).$$

Thus, in the example considered in Arts. **39** and **44,** the rate is .60 and the rate per cent. is 60; here, the rate per cent., 60, is one hundred times the rate, .60.

EXAMPLE.—If the rate is .025, what is the rate per cent.?

SOLUTION.—The rate per cent. is one hundred times the rate; therefore, the rate per cent. is $100 \times .025 = 2.5$, or $2\frac{1}{2}$. Ans.

48. *The rate is equal to the rate per cent. divided by one hundred.* That is,

$$(rate) = \frac{(rate\ per\ cent.)}{100}.$$

EXAMPLE.—If the rate per cent. is $37\frac{1}{2}$, what is the rate?

SOLUTION.—The rate is equal to the rate per cent. divided by one hundred; therefore, the rate is $37\frac{1}{2} \div 100 = 37.5 \div 100 = .375$. Ans.

49. In calculations, it is always better to use the rate than to use the rate per cent.; when the rate per cent. is given, find the rate by Art. **48,** and then use the rate in calculations.

50. The **amount** is the sum of the base and the percentage. That is,

$$(\text{amount}) = (\text{base} + \text{percentage}).$$

By Art. **41,** (percentage) = (base) × (rate); substituting (base) × (rate) for (percentage) in the equation

$$(\text{amount}) = (\text{base} + \text{percentage}),$$

we get $$(\text{amount}) = (\text{base} + \text{base} \times \text{rate}).$$

Hence, by Art. **36,** Part 4,

$$(\text{amount}) = (\text{base}) \times (1 + \text{rate}).$$

Hence, we have the following rule:

51. Rule.—*To find the amount, multiply the base by the sum obtained by adding the rate to unity.* That is,

$$(\textit{amount}) = (\textit{base}) \times (1 + \textit{rate}).$$

EXAMPLE.—The employes in a factory received an increase of 10% on the first of January, 1901. What is the wages per week in 1901 of a man who in 1900 received $20.00 per week?

SOLUTION.—We have

rate = 10% = .1;
base = man's wages per week in 1900 = $20.00.

Hence, we have

(amount) = (base) × (1 + rate) = $20 × (1 + .1) = $20 × 1.1 = $22.00. Ans.

52. The **difference** is the remainder obtained by subtracting the percentage from the base. That is,

$$(\text{difference}) = (\text{base} - \text{percentage}).$$

By Art. **41,** (percentage) = (base) × (rate); substituting (base) × (rate) for (percentage) in the equation

$$(\text{difference}) = (\text{base} - \text{percentage}),$$

we get $$(\text{difference}) = (\text{base} - \text{base} \times \text{rate}).$$

Therefore, by Art. **36**, Part 4,

(difference) = (base) × (1 − rate).

Hence, we have the following rule:

53. Rule.—*To find the difference, multiply the base by the remainder obtained by subtracting the rate from unity.* That is,

(difference) = (base) (1 − rate).

EXAMPLE.—A reduction of 10% in the wages paid in a certain factory goes into effect on a certain date. What wages will be received after the reduction by a man who before the reduction received $22.00 per week?

SOLUTION.—We have

rate = 10% = .1;
base = man's wages before the reduction = $22.00;
difference = man's wages after the reduction.

Hence, we have

(difference) = (base) × (1 − rate) = $22 × (1 − .1) = $22 × .9 = $19.80. Ans.

EXAMPLES FOR PRACTICE.

1. What is 9% of 881? Ans. 79.29.

2. What per cent. of 450 is 90? Ans. 20.

3. 54 is 6% of a certain number; find the number. Ans. 900.

4. A man in making 150 pounds of solder uses 53⅓% of lead and 46⅔% of tin; how many pounds of each does he use?

Ans. { 80 lb. of lead. 70 lb. of tin.

TRADE DISCOUNTS.

54. Trade discounts are reductions made by manufacturers, jobbers, or merchants from their list or catalogue prices. The selling price of an article is found by subtracting the discount from the list price. The selling price is sometimes called the **net** price.

EXAMPLE.—The list price of an article is $62.50 and a discount of 40% is allowed. Find the discount and the net, or selling, price.

SOLUTION.—The list price is the base, the discount is the percentage, and the selling price is the difference (Arts. **37** and **52**). Hence, by Art. **41**, we have

$$(\text{discount}) = (\text{list price}) \times (\text{rate})$$
$$= \$62.50 \times .40 = \$25.00.$$

By Art. **53**, we have

$$(\text{selling price}) = (\text{list price}) \times (1 - \text{rate})$$
$$= \$62.50 \times (1 - .40) = \$62.50 \times .60 = \$37.50.$$

Thus,
$$\left.\begin{array}{r}\text{discount} = \$25.00,\\ \text{selling price} = \$37.50.\end{array}\right\} \text{Ans.}$$

From this example we derive the rules given in the next two articles.

55. **Rule.**—*To find the discount, multiply the list price by the rate.* That is,

$$(\textit{discount}) = (\textit{list price}) \times (\textit{rate}).$$

EXAMPLE.—An article listed at $36 is sold at 12½ per cent. discount; what is the discount?

SOLUTION.—We have

$$(\text{discount}) = (\text{list price}) \times (\text{rate})$$
$$= \$36 \times .12\tfrac{1}{2} = \$4.50.$$

Hence, the discount is $4.50. Ans.

56. **Rule.**—*To find the selling price, multiply the list price by the remainder obtained by subtracting the rate from unity.* That is,

$$(\textit{selling price}) = (\textit{list price})\ (1 - \textit{rate}).$$

EXAMPLE.—On a bill of goods amounting to $720.00, a discount of 30% is allowed. Find the selling price.

SOLUTION.—We have

$$(\text{selling price}) = (\text{list price}) \times (1 - \text{rate})$$
$$= \$720 \times (1 - .30) = \$720 \times .70 = \$504.00.$$

Hence, the selling price is $504. Ans.

57. Frequently more than one discount is allowed on the same bill of goods. The discounts are usually called *per cent. off.*

EXAMPLE 1.—On a bill of goods amounting to $360.00, 30% and 10% off is allowed. Find the selling price.

SOLUTION.—This example can be solved by the rule of Art. **56.** The price after deducting the 30% off is

$$(\text{list price}) \times (1 - \text{rate}) = \$360.00 \times (1 - .30).$$

From this last price, a discount of 10% is to be deducted. By the rule of Art. **56,** the price after deducting this second discount is

$$\$360.00 \times (1 - .30) \times (1 - .10) = \$360.00 \times .70 \times .90 = \$226.80. \quad \text{Ans.}$$

EXAMPLE 2.—On a bill of goods amounting to $360.00, 10% and 30% off is allowed. Find the selling price.

SOLUTION.—The price after deducting the 10% discount is $\$360.00 \times (1 - .10)$. From this last price a discount of 30% must be deducted. Hence, the selling price is $\$360.00 \times (1 - .10) \times (1 - .30) = \$360.00 \times .90 \times .70 = \$226.80$. Therefore, the selling price is $226.80. Ans.

From comparison of these two examples it is evident that it is a matter of indifference whether the 10% or the 30% discount is deducted first. Hence, we have the following principle:

58. *When two or more discounts are allowed, the order of deducting the discounts is immaterial.*

59. Rule.—*To find the selling price of an article on which two or more discounts are allowed, multiply the list price by the continued product of the remainders obtained by subtracting each rate from unity.*

EXAMPLE.—A bill of goods for $550.00 is marked "Less 30%, 20%, 10%." Find the selling price.

SOLUTION.—By the rule, we have

$$\$550 \times (1 - .30) \times (1 - .20) \times (1 - .10) = \$550 \times .70 \times .80 \times .90$$
$$= \$550 \times .504 = \$277.20.$$

Hence, the selling price is $277.20. Ans.

EXAMPLES FOR PRACTICE.

1. On a bill of goods for $360, a discount of 25% is allowed. Find (*a*) the discount; (*b*) the selling price.

Ans. { (*a*) Discount = $90.
(*b*) Selling price = $270. }

2. A bill of goods for $450 is marked less 40%, 20%, 10%. Find the selling price. Ans. $194.40.

ARITHMETIC.

(PART 12.)

DENOMINATE NUMBERS.

SIMPLE AND COMPOUND NUMBERS.

1. The word *denominate* means *having a special name.* A concrete number is called a **denominate number** because it contains the special name of the concrete unit. Thus, 1 foot is a denominate number, because the special name *foot* is joined to the abstract number 1.

2. It is convenient to measure large quantities with a large unit and to measure small quantities with a small unit. For example, it is convenient to measure the distance between two cities in miles and to measure the length of a page of this book in inches; but it would be highly inconvenient to measure the distance between two cities in inches or to measure the length of this page in miles. Hence, custom has established different units for measuring quantities of the same kind.

3. In civilized countries, laws have been enacted defining the units to be used in measuring the several kinds of quantities. A unit fixed and preserved by the authority of a government is called a **standard unit.** For example, the British parliament has decreed that the standard unit for measuring length shall be the **yard,** which is defined to be the distance between two marks on a certain metal bar, at a definite temperature. This bar is carefully preserved at the Standards Office of the British Board of Trade. The standard yard of the United States is the same as the British standard yard. The standard unit for measuring weight is the **pound weight,** which is the weight of a certain piece of platinum preserved in the Standards Office of the British Board of Trade. Standard units have also been defined for measuring quantities of other kinds.

4. From each standard unit, other *subsidiary units* have been derived; these subsidiary units, as a rule, are either multiples or submultiples of the standard unit. Thus, the foot and the mile are subsidiary units of length; the foot is a submultiple of the standard unit of length, since a yard contains 3 feet, and the mile is a multiple of the standard unit of length, since a mile contains 1,760 yards.

5. A list showing the relative magnitudes of the units used for measuring quantities of the same kind is called a **table.**

6. When several units are used for measuring quantities of the same kind, these units are said to be of different **denominations;** that is, of different names. Any unit is said to be of **higher denomination** than any smaller unit and of **lower denomination** than any larger unit. Thus, a *foot* is of higher denomination than an *inch* and of lower denomination than a *yard.*

7. Sometimes two or more different units of the same kind are used in measuring the same quantity. For example, if the length of a string is measured with a yardstick, its length may be found to be *five yards two feet and six inches.*

Thus, the length of the string is expressed in terms of three different units of the same kind.

8. The expression of a quantity in terms of a single unit is called a **simple number,** and the expression of a quantity in different units of the same kind is called a **compound number.** For example, if a piece of string is fifteen inches long, its length may be expressed as 15 inches or as 1 foot 3 inches; the expression 15 inches is a simple number and the expression 1 foot 3 inches is a compound number.

9. A table in which each unit is equal to ten units of the next lower denomination is called a **decimal table**; for example, the table of United States money is a decimal table.

10. A table that is not a decimal table is called a **non-decimal table**; for example, the table of linear measure based on the yard as a standard unit is a non-decimal table.

11. A set of decimal tables of measures is called a **decimal system.** The only complete decimal system of measures that has ever been extensively used is the **metric system,** which is rapidly coming into general use. The metric system will be explained later.

12. In a decimal table, the names of the several subsidiary units are used in writing and in speaking, but a single unit, usually the standard unit, is used in calculations. For example, in writing or speaking of United States money, we use the words *dime* and *mill;* but in calculations, United States money is usually expressed in dollars and we calculate with dollars exactly in the same way as with abstract numbers written in ordinary Arabic notation. This is one of the great advantages of a decimal system of measures.

13. We shall give first the tables of the measures in common use in the United States and then the tables of the metric system.

MEASURES IN COMMON USE.

LINEAR MEASURE.

14. As previously stated, the standard unit of length is the yard; the other common units of length, except the rod, are either multiples or submultiples of the yard. The following table gives the relative magnitudes of the common linear units:

TABLE OF LINEAR MEASURE.

12	inches (*in.*)	= 1 foot	*ft.*
3	feet	= 1 yard	*yd.*
$5\frac{1}{2}$	yards	= 1 rod	*rd.*
320	rods	= 1 mile	*mi.*

in.		ft.		yd.		rd.		mi.
12	=	1						
36	=	3	=	1				
198	=	$16\frac{1}{2}$	=	$5\frac{1}{2}$	=	1		
63,360	=	5,280	=	1,760	=	320	=	1

SQUARE MEASURE.

15. The most convenient unit of area is a square whose side is equal to the unit of length (Art. **21,** Part 4). When an inch is used as the unit of length, the unit of area is a square inch; when a foot is used as the unit of length, the unit of area is the square foot. Thus, to each linear unit there is a corresponding unit of area.

The relative magnitudes of the units of area can be found from the relative magnitudes of the units of length. For, from the table of linear measure, a foot contains 12 inches; therefore, by Art. **3,** Part 6, a square foot contains 12^2, or 144 square inches. In like manner, a square yard contains 3^2, or 9 square feet; and so on.

The unit of area known as an *acre* is exceptional; an acre is not a square whose side is equal to any linear unit. A square whose side measures 208.71 feet contains almost exactly an acre.

The following table gives the relative magnitudes of the common square units:

TABLE OF SQUARE MEASURE.

144	square inches (*sq. in.*)	=	1 square foot	*sq. ft.*
9	square feet	=	1 square yard	*sq. yd.*
30¼	square yards	=	1 square rod	*sq. rd.*
160	square rods	=	1 acre	*A.*
640	acres	=	1 square mile	*sq. mi.*

sq. in.	sq. ft.	sq. yd.	sq. rd.	A.	sq. mi.
144 =	1				
1,296 =	9 =	1			
39,204 =	272¼ =	30¼ =	1		
6,272,640 =	43,560 =	4,840 =	160 =	1	
4,014,489,600 =	27,878,400 =	3,097,600 =	102,400 =	640 =	1

CUBIC MEASURE.

16. The most convenient unit of volume is the cube, each of whose edges is equal to the unit of length (Art. **30,** Part 4). When an inch is the unit of length, the unit of volume is a cubic inch; when a foot is the unit of length, the unit of volume is a cubic foot. Thus, there is a unit of volume corresponding to each unit of length.

The relative magnitudes of the units of volume can be found from the relative magnitudes of the units of length. Thus, from the table of linear measure, a foot contains 12 inches; and, therefore, by Art. **3,** Part 6, a cubic foot contains 12^3, or 1,728 cubic inches.

TABLE OF CUBIC MEASURE.

1,728 cubic inches (*cu. in.*)	=	1 cubic foot	*cu. ft.*
27 cubic feet	=	1 cubic yard	*cu. yd.*

cu. in.	cu. ft.	cu. yd.
1,728 =	1	
46,656 =	27 =	1

LIQUID MEASURE.

17. Liquid measure is used for measuring liquids. In the United States, the standard unit of liquid measure is the **wine gallon,** which contains 231 cubic inches.

TABLE OF LIQUID MEASURE.

4 gills (*gi.*)	=	1 pint	*pt.*
2 pints	=	1 quart	*qt.*
4 quarts	=	1 gallon	*gal.*
$31\frac{1}{2}$ gallons	=	1 barrel	*bbl.*
2 barrels / 63 gallons	=	1 hogshead	*hhd.*

gi.	pt.	qt.	gal.	bbl.	hhd.
4 =	1				
8 =	2 =	1			
32 =	8 =	4 =	1		
1,008 =	252 =	126 =	$31\frac{1}{2}$ =	1	
2,016 =	504 =	252 =	63 =	2 =	1

1 gallon = 231 cubic inches.
1 liquid quart = $57\frac{3}{4}$ cubic inches.

APOTHECARIES' FLUID MEASURE.

18. Apothecaries' fluid measure is used in prescribing and compounding medicines in the fluid state. The apothecaries' gallon is the same as the wine gallon. When the symbols are written instead of the abbreviations, they are placed before the figures; thus, Cong. 2 O.7 f ℥ 12, means 2 gallons 7 pints 12 fluid ounces.

TABLE OF APOTHECARIES' FLUID MEASURE.

60 minims, or drops, (♏)	=	1 fluid dram	*f* ʒ.
8 fluid drams	=	1 fluid ounce	*f* ℥.
16 fluid ounces	=	1 pint	*O.*
8 pints	=	1 gallon	*Cong.*

DRY MEASURE.

19. Dry measure is used in measuring such commodities as fruits, vegetables, and grain. The standard unit is the Winchester bushel, which contains 2,150.42 cubic inches, or very nearly 9.31 wine gallons. A box 14 inches long, 12.8 inches deep, and 12 inches wide (all inside measurements) contains almost exactly a bushel.

TABLE OF DRY MEASURE.

2 pints (*pt.*)	=	1 quart	*qt.*
8 quarts	=	1 peck	*pk.*
4 pecks	=	1 bushel	*bu.*

pt.		qt.		pk.		bu.
2	=	1				
16	=	8	=	1		
64	=	32	=	4	=	1

1 dry quart = $67\frac{1}{5}$ cu. in.

AVOIRDUPOIS WEIGHT.

20. In all ordinary transactions, the standard unit of weight is the pound avoirdupois, which is the weight of a piece of platinum preserved in the Standards Office of the British Board of Trade. The pound contains 7,000 grains (*gr.*).

Originally, in the avoirdupois table, 112 pounds made 1 hundredweight, and 20 hundredweight, or 2,240 pounds, made 1 ton. The hundredweight of 112 pounds and the ton of 2,240 pounds are still used in Great Britain; the ton of 2,240 pounds is used also by the United States Custom Houses and by wholesale dealers in coal, iron, and iron ore. With these exceptions, the hundredweight of 112 pounds and the ton of 2,240 pounds are no longer used. Ordinarily, a hundredweight consists of a hundred pounds and a ton consists of 2,000 pounds. In all calculations, a hundredweight is to be taken as 100 pounds and a ton as 2,000 pounds, unless the contrary is stated. The ton of 2,240 pounds is called the *long ton*.

TABLE OF AVOIRDUPOIS WEIGHT.

16 ounces (*oz.*)	=	1 pound	*lb.*
100 pounds	=	1 hundredweight	*cwt.*
20 hundredweight / 2,000 pounds	=	1 ton	*T.*

gr.	oz.	lb.	cwt.	T.
7,000 =	16 =	1		
	1,600 =	100 =	1	
	32,000 =	2,000 =	20 =	1

LONG TON TABLE.

16 ounces (*oz.*)	=	1 pound	*lb.*
28 pounds	=	1 quarter	*qr.*
4 quarters	=	1 hundredweight	*cwt.*
20 hundredweight / 2,240 pounds	=	1 ton	*T.*

gr.	oz.	lb.	qr.	cwt.	T.
7,000 =	16 =	1			
	448 =	28 =	1		
	1,792 =	112 =	4 =	1	
	35,840 =	2,240 =	80 =	20 =	1

TROY WEIGHT.

21. Troy weight is used for weighing gold, silver, and precious stones. The standard unit is the troy pound, which is the weight of a piece of platinum preserved in the United States Mint at Philadelphia. The troy pound contains 5,760 grains, and a grain troy is the same as a grain avoirdupois.

TABLE OF TROY WEIGHT.

24 grains (*gr.*)	=	1 pennyweight	*pwt.*
20 pennyweights	=	1 ounce	*oz.*
12 ounces	=	1 pound	*lb.*

gr.	pwt.	oz.	lb.
24 =	1		
480 =	20 =	1	
5,760 =	240 =	12 =	1

APOTHECARIES' WEIGHT.

22. Apothecaries' weight is used by physicians in prescribing and by druggists in compounding medicines. Medicines are bought and sold by avoirdupois weight.

The apothecaries' pound is the same as the troy pound, and apothecaries' weight differs from troy weight only in the subdivisions of the ounce.

When the symbols are used instead of the abbreviations, they are placed before the figures; thus, ʒ 5 ℈ 2 means 5 drams 2 scruples.

TABLE OF APOTHECARIES' WEIGHT.

20 grains (*gr.*)	=	1 scruple	*sc.* or ℈.
3 scruples	=	1 dram	*dr.* or ʒ.
8 drams	=	1 ounce	*oz.* or ℥.
12 ounces	=	1 pound	*lb.* or ℔.

gr.	℈	ʒ	℥	℔
20 =	1			
60 =	3 =	1		
480 =	24 =	8 =	1	
5,760 =	288 =	96 =	12 =	1

MEASURES OF ANGLES OR ARCS.

23. The ordinary method of measuring angles and arcs is fully explained in Arts. **7** to **14,** Part 3.

TABLE OF ANGULAR MEASURE.

60 seconds (″)	=	1 minute	′
60 minutes	=	1 degree	°
90 degrees	=	1 right angle	∟
360 degrees of arc	=	1 circumference.	

$$60'' = 1'$$
$$3{,}600'' = 60' = 1^\circ$$
$$1{,}296{,}000'' = 21{,}600' = 360^\circ$$

MEASURES OF TIME.

24. The principal unit of time is the second. A minute contains 60 seconds, an hour contains 60 minutes, and a day contains 24 hours, or 86,400 seconds. The ordinary civil day commences at 12 o'clock, midnight.

TABLE OF MEASURES OF TIME.

60 seconds (*sec.*)	=	1 minute	*min.*
60 minutes	=	1 hour	*hr.*
24 hours	=	1 day	*da.*
7 days	=	1 week	*wk.*
4 weeks	=	1 month	*mo.*
12 months	=	1 year	*yr.*
100 years	=	1 century	*C.*

25. The time in which the earth makes a complete revolution about the sun is called a **solar** year. The solar year contains 365¼ days, nearly.

Since the seasons and all the operations that depend on the seasons are regulated by the sun, it is very desirable that the civil year should agree as closely as possible with the solar year. But it is also essential that the civil year should contain an exact number of days. For the purpose of fulfilling these two conditions, it has been arranged that the ordinary civil year shall contain 365 days, while certain years shall have 366 days. A year containing 366 days is called a **leap year.**

When the number expressing a year is divisible by 4 and is not divisible by 100, that year is a leap year. When the number expressing a year is divisible by 100, that year is not a leap year unless the number is also divisible by 400. Thus, the years 1888, 1892, and 1896 were leap years, because the numbers 1888, 1892, and 1896 are divisible by 4 and are not divisible by 100. The year 1900 was not a leap year, because the number 1900 is divisible by 100 and is not divisible by 400. The year 2000 will be a leap year, because the number is divisible by 400.

26. The following is a list of months, in regular order, with the number of days that each contains.

		Days.			*Days.*
1.	January (Jan.)	31	7.	July	31
2.	February (Feb.)	28	8.	August (Aug.)	31
3.	March (Mar.)	31	9.	September (Sept.)	30
4.	April (Apr.)	30	10.	October (Oct.)	31
5.	May	31	11.	November (Nov.)	30
6.	June	30	12.	December (Dec.)	31

In leap years, one day is added to February, giving it 29 days. The following lines will assist the student in remembering the number of days in each month.

"Thirty days hath September,
April, June, and November;
All the rest have thirty-one,
Save February, which alone
Hath but twenty-eight, in fine,
But leap year gives it twenty-nine."

In many business transactions, the year is regarded as 360 days, or 12 months of 30 days each.

MEASURES OF VALUE.

27. In commercial transactions, value is measured by money. The standard unit of money in the United States is the **gold dollar.**

TABLE OF UNITED STATES MONEY.

10 mills (*m*)	=	1 cent	*ct.*
10 cents	=	1 dime	*d.*
10 dimes	=	1 dollar	*$.*
10 dollars	=	1 eagle	*E.*

m.		ct.		d.		$.		E.
10	=	1						
100	=	10	=	1				
1,000	=	100	=	10	=	1		
10,000	=	1,000	=	100	=	10	=	1

OPERATIONS WITH DENOMINATE NUMBERS.

REDUCTION.

28. The process of changing the denomination in which a denominate number is expressed is called **reduction.** If the change is from a higher denomination to a lower, the process is called *reduction descending;* if the change is from a lower denomination to a higher, the process is called *reduction ascending.*

REDUCTION DESCENDING.

29. There are 12 inches in a foot; therefore, in 5 feet there are 5 × 12, or 60 inches. Thus, the denominate number *5 feet* is reduced to the equivalent denominate number 60 inches by multiplying the abstract number 5 by the abstract number 12. Hence, we have the following rule:

30. Rule.—*To reduce a given simple denominate number of a higher denomination to an equivalent simple denominate number of lower denomination, multiply the number of units in the given denominate number by the number of units of the lower denomination that make one unit of the higher denomination.*

EXAMPLE.—Reduce 215 yards to feet.

SOLUTION.— 215 yd. = 215 × 3 ft. = 645 ft. Ans.

31. It is frequently necessary to reduce a compound number to a simple denominate number. Suppose it is required to reduce 7 yards 2 feet 6 inches to a simple number; the easiest way is to reduce this number to inches. Now, 7 yards is equal to 7 × 3, or 21 feet; therefore, 7 yards 2 feet = 21 feet + 2 feet = 23 feet. Again, 23 feet is equal to 23 × 12, or 276 inches; therefore, 23 feet 6 inches = 276 inches + 6 inches = 282 inches. Thus, 7 yards 2 feet 6 inches is equal to 282 inches. This reduction is usually written in the following form:

```
yd.   ft.   in.
  7    2     6
  3
---
 21
  2
---
 23  ft.
 12
---
276
  6
---
282  in.
```

From this example we derive the following rule:

32. Rule.—*To reduce a given compound number to a simple denominate number, multiply the number of units of the highest denomination in the given number by the number of units of the second highest denomination that makes one unit of the highest denomination, and to this product add the number of units of the second highest denomination in the given number. Proceed in this way until the given compound number is reduced to a simple denominate number.*

EXAMPLE.—Reduce 2 sq. yd. 6 sq. ft. 112 sq. in. to square inches.

SOLUTION.—

```
sq. yd.   sq. ft.   sq. in.
     2       6        112
     9
  ----
    18
     6
  ----
    24   sq. ft.
   144
  ----
    96
   96
  24
  ----
  3456
   112
  ----
  3568   sq. in.   Ans.
```

33. When it is required to find the area or the volume of a figure whose dimensions are given in compound numbers, it is best to reduce the compound numbers to simple numbers.

EXAMPLE.—Find the area of a rectangle whose sides are 4 ft. 3 in. and 3 ft. 7 in.

SOLUTION.—We have

4 ft. 3 in. = 51 in.

and 3 ft. 7 in. = 43 in.

By the rule of Art. **23**, Part 4, the area of this rectangle is 51 × 43, or 2,193 sq. in. Ans.

EXAMPLES FOR PRACTICE.

Reduce

1.	8 lb. 4 oz. to ounces.	Ans. 132 oz.
2.	24 gal. 1 pt. to pints.	Ans. 193 pt.
3.	27 sq. yd. 8 sq. ft. to square feet.	Ans. 251 sq. ft.
4.	8 yd. 2 ft. 1 in. to inches.	Ans. 313 in.

REDUCTION ASCENDING.

34. There are 3 feet in a yard; therefore, in 18 feet there are 18 ÷ 3, or 6 yards. Thus, the denominate number *18 feet* is reduced to the equivalent denominate number *6 yards* by dividing the abstract number 18 by the abstract number 3. Hence, we derive the following rule:

35. Rule.—*To reduce a given simple denominate number of lower denomination to an equivalent simple denominate number of higher denomination, divide the number of units in the given denominate number by the number of units of the lower denomination that makes one unit of the higher denomination.*

EXAMPLE 1.—Reduce 80 ounces to pounds.

SOLUTION.—

$$\begin{array}{r|l} & \text{oz.} \\ 16 & 80 \\ \hline & 5 \text{ lb. Ans.} \end{array}$$

EXAMPLE 2.—Reduce 27 inches to feet.

SOLUTION.—

$$\begin{array}{r|l} & \text{in.} \\ 12 & 27 \\ \hline & 2.25 \text{ ft. Ans.} \end{array}$$

EXAMPLE 3.—The diameter of a cylinder is 54 in. and the length is 8 ft. 3 in. Find its volume in cubic feet.

SOLUTION.—We have the radius $= \frac{1}{2} \times 54$, or 27 in.; and the length 8 ft. 3 in. = 99 in. By Art. **37**, Part 10, we have (volume) $= 3.1416 \times (\text{radius})^2 \times (\text{altitude})$, which is equal to $(3.1416 \times 27^2 \times 99)$ cu. in. Since 1,728 cu. in. = 1 cu. ft., we have volume in cu. ft.
$$= \frac{3.1416 \times 27^2 \times 99}{1{,}728} = \frac{3.1416 \times 27 \times 27 \times 99}{1{,}728} = \frac{.3927 \times 27 \times 99}{8} = 131.21.$$
Hence, the volume is 131.21 cu. ft. Ans.

36. Sometimes it is desired to reduce a simple denominate number to a compound number. Suppose it is required to reduce 98 inches to a compound number; $98 \div 12 = 8$ and 2 over; therefore, 98 inches is equal to 8 feet 2 inches. From this example the following rule is derived:

37. Rule.—*To reduce a given simple denominate number to a compound number, divide the number of units in the given number by the number of units of that denomination that makes one unit of the next higher denomination. The remainder, if there is one, is of the same denomination as the given number, and the quotient is of the next higher denomination. Divide this quotient by the number of units of its denomination that makes one unit of the next higher denomination, and so on.*

EXAMPLE.—Reduce 236 inches to a compound number.

SOLUTION.—

```
12 | 136 in.
 3 | 11 ft. + 4 in.
      3 yd. + 2 ft.
```

Therefore, 136 in. = 3 yd. 2 ft. 4 in. Ans.

EXAMPLES FOR PRACTICE.

Solve the following examples:

1. Reduce 231 in. to feet. Ans. 19.25 ft.
2. Reduce 10,548 sq. in. to square feet. Ans. 73.25 sq. ft.

Reduce the following to compound numbers:

3. 129 pt. Ans. 16 gal. 0 qt. 1 pt.
4. 235 oz. Ans. 14 lb. 11 oz.
5. 157 in. Ans. 4 yd. 1 ft. 1 in.
6. 789 sq. in. Ans. 5 sq. ft. 69 sq. in.

ADDITION OF COMPOUND NUMBERS.

38. Addition of compound numbers is similar to addition of simple numbers.

EXAMPLE.—Find the sum of 6 yd. 2 ft. 3 in.; 5 yd. 1 ft. 7 in.; 4 yd. 2 ft. 9 in.; and 1 yd. 2 ft. 2 in.

SOLUTION.—

yd.	ft.	in.
6	2	3
5	1	7
4	2	9
1	2	2
18	2	9

or 18 yd. 2 ft. 9 in. Ans.

EXPLANATION.—The numbers are written so that units of the same denomination are in the same column. Beginning to add with the right-hand column, we have $2+9+7+3=21$; hence, the sum of the numbers in the right-hand column is 21 in., or 1 ft. 9 in. (Art. **36**). We write the 9 under the column headed "in.," and carry the 1 to the column headed "ft." Adding the 1 that is carried to the sum of the numbers in the second column, we get $1+2+2+1+2=8$; therefore, in the sum we have 8 ft., or 2 yd. 2 ft. We write 2 under the column headed "ft." and carry 2 to the column headed "yd." Adding the 2 that is carried to the sum of the numbers in the third column, we get $2+1+4+5+6=18$; thus, in the sum we have 18 yd.

39. From the preceding example we derive the following rule for adding compound numbers:

Rule.—*Write the numbers so that units of the same denomination shall be in the same column. Begin to add at the right-hand column, which contains the units of the lowest denomination.*

If the sum of the numbers in the right-hand column is less than the number of units of the lowest denomination that makes one unit of the next higher denomination, write the sum in the right-hand column and proceed to add the next column.

If the sum of the numbers in the right-hand column is as great as the number of units of the lowest denomination that makes one unit of the next higher denomination, divide the sum by the number of units of the lowest denomination that makes one unit of the next higher denomination; write the remainder under the right-hand column and carry the quotient to the next column. Continue in this manner until the left-hand column is added.

EXAMPLE.—Find the sum of 4 lb. 13 oz.; 1 lb. 11 oz.; 17 lb. 14 oz.; and 3 lb. 15 oz.

SOLUTION.—

lb.	oz.
4	13
1	11
17	14
3	15
28	5

or 28 lb. 5 oz. Ans.

EXAMPLES FOR PRACTICE.

Find the sum of

1. 3 sq. ft. 79 sq. in.; 2 sq. ft. 135 sq. in.; 1 sq. ft. 99 sq. in.
Ans. 8 sq. ft. 25 sq. in.

2. 5 lb. 15 oz.; 2 lb. 14 oz.; 3 lb. 10 oz.; 4 lb. 9 oz. Ans. 17 lb.

3. 3 ft. 7 in.; 2 ft. 5 in.; 3 ft. 4 in; 7 ft. 9 in.; 8 ft. 3 in.
Ans. 25 ft. 4 in.

SUBTRACTION OF COMPOUND NUMBERS.

40. Subtraction of compound numbers is similar to subtraction of simple numbers.

EXAMPLE.—From 9 yd. 1 ft. 10 in. take 4 yd. 2 ft. 9 in.

SOLUTION.—

yd.	ft.	in.
9	1	10
4	2	9
4	2	1

or 4 yd. 2 ft. 1 in. Ans.

EXPLANATION.—The numbers are written so that units of the same denomination are in the same column. Beginning to subtract at the right-hand column, we have $10 - 9 = 1$; hence, the difference of the numbers in the right-hand column is 1 in. We cannot take 2 from 1; therefore, we add 1 yd. or 3 ft. both to the minuend and to the subtrahend (Art. **19,** Part 2). Adding 3 ft. to 1 ft., we get 4 ft., and we have $4 - 2 = 2$; thus, in the remainder, we have 2 ft. Adding 1 yd. to 4 yd., we get 5 yd., and $9 - 5 = 4$; thus, in the remainder, we have 4 yd.

41. From the preceding example we derive the following rule for subtracting one compound number from another:

Rule.—*Write the subtrahend under the minuend, so that units of the same denomination will be in the same column.*

Begin to subtract at the right-hand column, which contains the units of the lowest denomination. Subtract, if possible, the number of units of each denomination in the subtrahend from the number of units of the same denomination in the minuend. But, if the number of units of any denomination in the minuend is less than the number of units of that denomination in the subtrahend, add to the number of units of that denomination in the minuend the number of units of that denomination which makes one unit of the next higher denomination. To balance this addition to the minuend, add one unit of the next higher denomination to the number of units of the next higher denomination in the subtrahend.

EXAMPLE 1.—From 9 bu. 2 pk. 2 qt. take 3 bu. 3 pk. 3 qt. 1 pt.

SOLUTION.—

bu.	pk.	qt.	pt.
9	2	2	0
3	3	3	1
5	2	6	1

or 5 bu. 2 pk. 6 qt. 1 pt. Ans.

EXAMPLE 2.—In the triangle *A B C*, the angle *A* is 34° 40′ 30″ and the angle *B* is 45° 21′ 45″. Find the angle *C*.

SOLUTION.—By Art. **47**, Part 3, the sum of the three angles A, B, and C is 180°; hence, the angle C is found by subtracting the sum of the angles A and B from 180°.

$A =$	34°	40′	30″	
$B =$	45°	21′	45″	
$A + B =$	80°	2′	15″	
	180°			
	80°	2′	15″	
$C =$	99°	57′	45″	Ans.

EXAMPLES FOR PRACTICE.

1. From 5 lb. 14 oz. take 2 lb. 15 oz. Ans. 2 lb. 15 oz.
2. From 8 yd. 1 ft. 10 in. take 5 yd. 2 ft. 7 in. Ans. 2 yd. 2 ft. 3 in.
3. In the triangle $A B C$, $A = 60°\ 35'$, $C = 75°$, 40′. Find B. Ans. $B = 43°\ 45'$.
4. From 20 sq. yd. 7 sq. ft. 33 sq. in. take 15 sq. yd. 8 sq. ft. 55 sq. in. Ans. 4 sq. yd. 7 sq. ft. 122 sq. in.

MULTIPLICATION OF COMPOUND NUMBERS.

42. The following example illustrates the method of multiplying compound numbers:

EXAMPLE.—Multiply 3 yd. 2 ft. 5 in. by 4.

SOLUTION.—

yd.	ft.	in.
3	2	5
		4
15	0	8

or 15 yd. 0 ft. 8 in. Ans.

EXPLANATION.—We have 4 × 5 in. = 20 in. = 1 ft. 8 in.; we write 8 in the column headed "in." and carry 1 ft. to the next column. Then we have 4 × 2 ft. + 1 ft. = 9 ft. = 3 yd. 0 ft.; we write 0 in the column headed "ft." and carry 3 yd. to the next column. Then, 4 × 3 yd. + 3 yd. = 15 yd.

43. The following is the rule for multiplication of compound numbers:

Rule.—*Multiply the number of units of the lowest denomination by the multiplier. If the product is less than the number of units of the lowest denomination that makes one unit of the next higher denomination, write the product in the right-hand column. If the product is as great as the number of units of the lowest denomination that makes one unit of the next higher denomination, divide the product by the number of units of the lowest denomination that makes one unit of the next higher denomination; carry the quotient to the next column and write the remainder in the right-hand column. Then multiply the number of units of the second lowest denomination by the multiplier and to this product add the quotient, if any, carried from the first column, and so on.*

EXAMPLE.—Multiply 7 lb. 3 oz. by 13.

SOLUTION.—

lb.	oz.
7	3
	13
93	7

or 93 lb. 7 oz. Ans.

EXAMPLES FOR PRACTICE.

Multiply

1. 2 ft. 6 in. by 11. Ans. 27 ft. 6 in.
2. 3 sq. ft. 72 sq. in. by 7. Ans. 24 sq. ft. 72 sq. in.
3. 4 lb. 15 oz. by 21. Ans. 103 lb. 11 oz.

DIVISION OF COMPOUND NUMBERS.

DIVIDING BY AN ABSTRACT NUMBER.

44. The following example illustrates the method of dividing a compound number by an abstract number:

EXAMPLE.—Divide 49 lb. 11 oz. by 8.

SOLUTION.—

	lb.	oz.
8	49	11
	6	$3\frac{3}{8}$

or 6 lb. $3\frac{3}{8}$ oz. Ans.

Explanation.—Dividing 49 lb. by 8, we get the quotient 6 lb. and the remainder 1 lb. Carrying this 1 lb. to the next column, we get 1 lb. + 11 oz. = 16 oz. + 11 oz. = 27 oz. and 27 oz. ÷ 8 = $3\frac{3}{8}$ oz.

45. The following is the rule for dividing a compound number by an abstract number:

Rule.—*Begin to divide at the highest denomination of the dividend. Divide the number of units of the highest denomination by the divisor; write the quotient under the highest denomination of the dividend; multiply the remainder by the number of units of the second highest denomination that makes one unit of the highest denomination, and add the product to the number of units of the second highest denomination. Divide the sum so obtained by the divisor, and write the quotient under the second highest denomination of the dividend. Continue in this way until the lowest denomination is reached.*

Example.—Divide 9 sq. yd. 7 sq. ft. 101 sq. in. by 4.

Solution.—

	sq. yd.	sq. ft.	sq. in.
4	9	7	101
	2	4	$25\frac{1}{4}$

or 2 sq. yd. 4 sq. ft. $25\frac{1}{4}$ sq. in. Ans.

EXAMPLES FOR PRACTICE.

Divide

1. 1 cwt. 7 lb. 15 oz. by 3. Ans. 35 lb. $15\frac{2}{3}$ oz.
2. 7 ft. .5 in. by 8. Ans. $11\frac{1}{8}$ in.
3. 19 sq. ft. 75 sq. in. by 11. Ans. 1 sq. ft. $111\frac{6}{11}$ sq. in.

DIVIDING BY A CONCRETE NUMBER.

46. When a compound number has to be divided by a concrete number, the two numbers should be reduced to simple numbers of the same denomination.

Example.—How often is 3 ft. 3 in. contained in 7 yd. 1 ft. 9 in.?

Solution.—We have 3 ft. 3 in. = 39 in. and 7 yd. 1 ft. 9 in. = 273 in.

Therefore, $\frac{7 \text{ yd. } 1 \text{ ft. } 9 \text{ in.}}{3 \text{ ft. } 3 \text{ in.}} = \frac{273 \text{ in.}}{39 \text{ in.}} = 7.$ Ans.

EXAMPLES FOR PRACTICE.

Divide

1. 2 yd. 2 ft. 2 in. by 2 ft. 4 in. Ans. $8\frac{1}{4}$.
2. 4 sq. ft. 56 sq. in. by 1 sq. ft. 14 sq. in. Ans. 4.

THE METRIC SYSTEM.

47. In 1875 an international convention was agreed upon by seventeen of the principal governments of the world, including the government of the United States, for the purpose of establishing and maintaining at the common expense an International Bureau of Weights and Measures.

The first business of this international convention was to construct a unit of length and a unit of weight. The unit of length agreed upon by the convention is called the **standard meter,** and is the distance, at a certain temperature, between two marks on a metal bar that is preserved in the International Bureau of Weights and Measures at Paris. The unit of weight agreed upon by the convention is called the **standard kilogram,** and is the weight, under proper conditions, of a piece of metal that is preserved in the International Bureau of Weights and Measures.

The standard meter and the standard kilogram are both made of an alloy of platinum with 10 per cent. of iridium.

48. The international convention constructed copies of the standard meter and the standard kilogram, which were distributed by lot among the contributing governments in the year 1899. The United States Government obtained meters Nos. 21 and 27, and kilograms Nos. 4 and 20.

49. The hundredth part of a meter is called a **centimeter,** and the tenth part of a meter is called a **decimeter.**

50. The thousandth part of a kilogram is called a **gram.** The standard kilogram was constructed so that a gram is equal to the weight of 1 cubic centimeter of pure water at the temperature of its greatest density, which is 39.2° F. or 4° C.

51. The system of measures based on the standard meter and the standard kilogram is called the **metric system.**

52. In the metric system the principal unit of capacity is the **liter** (pronounced lee'ter), which is equal to 1 cubic decimeter. The liter is used for both dry and liquid measure.

53. The names of the subsidiary units in the metric system are derived from the words *meter*, *gram*, and *liter*. For this reason the meter, gram, and liter are called **principal** units. The meter is a standard unit (Art. **3**) as well as a principal unit, because the meter is the unit of length preserved in the International Bureau; on the other hand, the gram is a principal unit but not a standard unit, for the standard unit of weight is the kilogram.

The names of multiples of the principal units are formed by prefixing to the names of the principal units the Greek words *deka* (10), *hekto* (100), *kilo* (1,000), and *myria* (10,000). Thus we have

1 dek'a me'ter	=	10 meters.
1 hek'to me'ter	=	100 meters.
1 kil'o meter	=	1,000 meters.
1 myr'ia meter	=	10,000 meters.

The names of the submultiples of the principal units are formed by prefixing to the names of the principal units the Latin words *deci* ($\frac{1}{10}$), *centi* ($\frac{1}{100}$), and *milli* ($\frac{1}{1000}$). Thus we have

1 dec'i me'ter	=	$\frac{1}{10}$ meter.
1 cent'i me'ter	=	$\frac{1}{100}$ meter.
1 mil'li me'ter	=	$\frac{1}{1000}$ meter.

54. In all the tables of the metric system, except the tables of square and cubic measure, each unit is equal to ten units of the next lower denomination. In square measure, each unit is equal to one hundred units of the next lower denomination. In cubic measure, each unit is equal to one thousand units of the next lower denomination.

55. It will be noticed on reference to the following tables that the abbreviations of the names of the multiples of the principal units begin with a *capital letter*, and the abbreviations of the names of the submultiples of the principal units begin with a *small letter*.

In the following tables, the names in common use are printed in **black-face** type.

LINEAR MEASURE.

56. The standard meter contains 39.37043 inches.

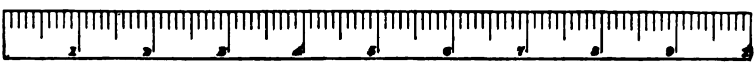


FIG. 1.

Fig. 1 shows a decimeter divided into centimeters and millimeters.

TABLE OF LINEAR MEASURE.

Submultiples	1 **millimeter** (*mm.*)	= .001 of a meter.
	1 **centimeter** (*cm.*)	= .01 of a meter.
	1 decimeter (*dm.*)	= .1 of a meter.
	1 **meter** (*m.*)	= principal unit.
Multiples	1 dekameter (*Dm.*)	= 10 meters.
	1 hektometer (*Hm.*)	= 100 meters.
	1 **kilometer** (*Km.*)	= 1,000 meters.
	1 myriameter (*Mm.*)	= 10,000 meters.

SQUARE MEASURE.

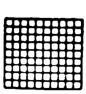


FIG. 2.

57. The principal unit of square measure is the square meter. Fig. 2 shows a square centimeter divided into square millimeters.

TABLE OF SQUARE MEASURE.

Submultiples	1 square millimeter ($mm.^2$)	=	.000001 of a square meter.
	1 **square centimeter** ($cm.^2$)	=	.0001 of a square meter.
	1 square decimeter ($dm.^2$)	=	.01 of a square meter.
	1 **square meter** ($m.^2$)	=	principal unit.
Multiples	1 square dekameter ($Dm.^2$)	=	100 square meters.
	1 square hektometer ($Hm.^2$)	=	10,000 square meters.
	1 **square kilometer** ($Km.^2$)	=	1,000,000 square meters.
	1 square myriameter ($Mm.^2$)	=	100,000,000 square meters.

58. The square dekameter is also called an **are**; and, since 100 $Dm.^2 = 1\ Hm.^2$, a square hektometer is called a **hektare.** The are and the hektare are used in measuring land.

CUBIC MEASURE.

59. The principal unit of cubic measure is the cubic meter. Fig. 3 shows a cubic centimeter.

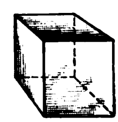


FIG. 3.

TABLE OF CUBIC MEASURE.

Submultiples	1 **cubic millimeter** ($mm.^3$)	=	.000000001 of a cubic meter.
	1 **cubic centimeter** ($cm.^3$)	=	.000001 of a cubic meter.
	1 cubic decimeter ($dm.^3$)	=	.001 of a cubic meter.
	1 **cubic meter** ($m.^3$)	=	principal unit.
Multiples	1 cubic dekameter ($Dm.^3$)	=	1,000 cubic meters.
	1 cubic hektometer ($Hm.^3$)	=	1,000,000 cubic meters.
	1 cubic kilometer ($Km.^3$)	=	1,000,000,000 cubic meters.

60. The cubic meter is also called a **stere,** and is used in measuring wood.

MEASURES OF CAPACITY.

61. The principal unit for measuring capacity is the liter, which is equal to 1 cubic decimeter.

TABLE OF MEASURES OF CAPACITY.

Submultiples	1 milliliter (*ml.*)	= .001 of a liter.
	1 centiliter (*cl.*)	= .01 of a liter.
	1 deciliter (*dl.*)	= .1 of a liter.
	1 **liter** (*e.*)	= principal unit.
Multiples	1 dekaliter (*Dl.*)	= 10 liters.
	1 hektoliter (*Hl.*)	= 100 liters.
	1 kiloliter (*Kl.*)	= 1,000 liters.

MEASURES OF WEIGHT.

62. The principal unit of weight is the gram. The

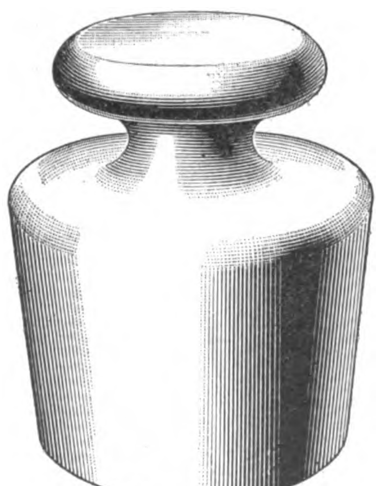


FIG. 4.

kilogram is the standard unit of weight and is equal to 2.20462 pounds avoirdupois. Fig. 4 shows a kilogram.

TABLE OF MEASURES OF WEIGHT.

Submultiples	1 **milligram** (*mg.*)	= .001 of a gram.
	1 centigram (*cg.*)	= .01 of a gram.
	1 decigram (*dg.*)	= .1 of a gram.
	1 **gram** (*g.*)	= principal unit.
Multiples	1 dekagram (*Dg.*)	= 10 grams.
	1 hektogram (*Hg.*)	= 100 grams.
	1 **kilogram** (*Kg.*)	= 1,000 grams.
	1 **metric ton** (*T.*)	= 1,000 kilograms.

63. A kilogram is frequently called a **kilo,** and is then denoted by the abbreviation *K.* instead of *Kg.*

64. As stated in Art. **50,** the kilogram was constructed so that 1 cubic centimeter of pure water at 4° C. weighs 1 gram. Hence, we have

One cubic centimeter of water weighs one gram.
One liter of water weighs one kilogram.
One cubic meter of water weighs one metric ton.

NOTATION AND REDUCTION.

65. In the metric system any simple or compound number can easily be expressed as a simple number in terms of the proper principal unit.

EXAMPLE 1.—Express 8 kilograms in grams.

SOLUTION.—From the table of measures of weight, we have

$$1\text{ Kg.} = 1{,}000\text{ g.}$$

Therefore, $8\text{ Kg.} = 8 \times 1{,}000\text{ g.} = 8{,}000\text{ g.}$ Ans.

EXAMPLE 2.—Express 5 dekaliters 5 deciliters 2 centiliters in liters.

SOLUTION.—From the table of measures of capacity, we have

$$1\text{ Dl.} = 10\text{ l.}$$
$$1\text{ dl.} = .1\text{ l.}$$
$$1\text{ cl.} = .01\text{ l.}$$

$$\begin{aligned} & & & \text{l.} \\ 5\text{ Dl.} &= 5 \times 10\text{ l.} &=& 50. \\ 5\text{ dl.} &= 5 \times .1\text{ l.} &=& .5 \\ 2\text{ cl.} &= 2 \times .01\text{ l.} &=& .02 \\ \hline & & & 50.52 \end{aligned}$$

Hence, 5 dekaliters 5 deciliters 2 centiliters = 50.52 l. Ans.

EXAMPLE 3.—Express 3 kilometers 1 centimeter 2 millimeters in meters.

SOLUTION.—From the table of linear measure, we have

$$1 \text{ Km.} = 1{,}000 \text{ m.}$$
$$1 \text{ cm.} = .01 \text{ m.}$$
$$1 \text{ mm.} = .001 \text{ m.}$$

Therefore,
$$\begin{aligned} & & & \text{m.} \\ 3 \text{ Km.} &= 3 \times 1{,}000 \text{ m.} &=& \ 3\,0\,0\,0. \\ & 1 \text{ cm.} &=& \qquad .0\,1 \\ 2 \text{ mm.} &= 2 \times \ .001 \text{ m.} &=& \qquad \underline{.0\,0\,2} \\ & & & 3\,0\,0\,0.0\,1\,2 \end{aligned}$$

Hence, 3 kilometers 1 centimeter 2 millimeters = 3,000.012 m. Ans.

EXAMPLE 4.—Express 3 square dekameters and 5 square decimeters in square meters.

SOLUTION.—From the table of square measure, we have

$$1 \text{ Dm.}^2 = 100 \text{ m.}^2$$
$$1 \text{ dm.}^2 = .01 \text{ m.}^2$$

Therefore,
$$\begin{aligned} & & & \text{m.}^2 \\ 3 \text{ Dm.} &= 3 \times 100 \text{ m.}^2 &=& \ 3\,0\,0. \\ 5 \text{ dm.} &= 5 \times .01 \text{ m.}^2 &=& \quad \underline{.0\,5} \\ & & & 3\,0\,0.0\,5 \end{aligned}$$

Hence, 3 square dekameters and 5 square decimeters = 300.05 m.^{2}. Ans.

66. A metric number expressed in terms of a principal unit can easily be expressed in terms of any specified subsidiary unit.

EXAMPLE 1.—Express 7.65 cubic meters in cubic centimeters.

SOLUTION.—From the table of cubic measure, we have

$$.000001 \text{ m.}^3 = 1 \text{ cm.}^3$$

Therefore, $1 \text{ m.}^3 = 1{,}000{,}000 \text{ cm.}^3$

Hence, $7.65 \text{ m.}^3 = 7.65 \times 1{,}000{,}000 \text{ cm.}^3 = 7{,}650{,}000 \text{ cm.}^3$. Ans.

EXAMPLE 2.—Express 24.32 grams in hektograms.

SOLUTION.—From the table of measures of weight, we have

$$100 \text{ g.} = 1 \text{ Hg.}$$

Therefore, $1 \text{ g.} = .01 \text{ Hg.}$

Hence, $24.32 \text{ g.} = 24.32 \times .01 \text{ Hg.} = .2432 \text{ Hg.}$ Ans.

67. The following is the rule for reducing a given simple number expressed in terms of one subsidiary unit to an equivalent number expressed in terms of any specified subsidiary unit.

Rule.—I. *Reduce the given number to an equivalent number expressed in terms of the principal unit by the method of* Art. **65.**

II. *Reduce the number expressed in terms of the principal unit to an equivalent number expressed in terms of the specified subsidiary unit by the method of* Art. **66.**

EXAMPLE 1.—Reduce .073 dekameters to centimeters.

SOLUTION.—From the table of linear measure, we have

1 Dm. = 10 m.

Therefore, .073 Dm. = .073 × 10 m. = .73 m.

From table, .01 m. = 1 cm.

Therefore, 1 m. = 100 cm.

Hence, .73 m. = .73 × 100 cm. = 73 cm. Ans.

EXAMPLE 2.—Reduce 8.5 kiloliters to hektoliters.

SOLUTION.—From the table of measures of capacity, we have

1 Kl. = 1,000 l.

Therefore, 8.5 Kl. = 8.5 × 1,000 l. = 8,500 l.

From table, 100 l. = 1 Hl.

Therefore, 1 l. = .01 Hl.

Hence, 8,500 l. = 8,500 × .01 Hl. = 85 Hl. Ans.

68. Measures of capacity can readily be changed to cubic measures and cubic measures can be changed to measures of capacity.

EXAMPLE 1.—Change 1.2 hektoliters to cubic centimeters.

SOLUTION.—We have

1.2 Hl. = 120 l.,

and 1 l. = 1 dm.^{3} = 1,000 cm.^{3}.

Therefore,

1.2 Hl. = 120 × 1,000 cm.^{3} = 120,000 cm.^{3}. Ans.

EXAMPLE 2.—Change 1.5 cubic meters to liters.

SOLUTION.—We have

$$1.5 \text{ m.}^3 = 1.5 \times 1{,}000 \text{ dm.}^3 = 1{,}500 \text{ dm.}^3,$$

and $1 \text{ dm.}^3 = 1 \text{ l.}$

Therefore, $1.5 \text{ m.}^3 = 1{,}500 \text{ dm.}^3 = 1{,}500 \text{ l.}$ Ans.

69. Since 1 cubic centimeter of water weighs 1 gram, the weight of any volume of water and the volume of any weight of water can be found at once.

EXAMPLE 1.—What is the weight of 37 cubic millimeters of water?

SOLUTION.— $37 \text{ mm.}^3 = \frac{1}{1000} \times 37 \text{ cm.}^3 = .037 \text{ cm.}^3$.

Therefore, the weight of 37 mm.³ of water $= .037 \times 1$ g. $= .037$ g. Ans.

EXAMPLE 2.—Find the volume occupied by 1.74 hectograms of water.

SOLUTION.—We have

$$1.74 \text{ Hg.} = 1.74 \times 100 \text{ g.} = 174 \text{ g.}$$

Hence, the volume occupied by 1.74 Hg. of water is 174 times the volume occupied by 1 gram; therefore, the required volume is $174 \times 1 \text{ cm.}^3 = 174 \text{ cm.}^3$. Ans.

EXAMPLES FOR PRACTICE.

Reduce

1. 8 hectometers to centimeters. Ans. 80,000 cm.
2. 4.54 decimeters to kilometers. Ans. .000454 Km.
3. 97.56 grams to kilograms. Ans. .09756 Kg.
4. Find the weight in kilograms of the water in a tank 2 m. × 1.8 m. × 1.2 m. Ans. 4,320 Kg.
5. How many liters are there in a box 1.4 m. × 1.3 m. × 1.1 m.? Ans. 2,002 l.

COMPARISON OF METRIC AND COMMON MEASURES.

70. When the metric system is used alone, all measurements and calculations are made in that system and there is no necessity for reducing from the metric system to the common system. But when the two systems are used together, it is necessary to reduce from one system to the other. The following table of equivalents can be used to reduce any quantity from one system to the other. The table is to be used for reference only, and is not to be committed to memory.

The equivalents given in the following table are correct to five figures:

TABLE OF EQUIVALENTS.

1 m.	= 39.370 in.	1 in.	= 2.5400 cm.
	= 1.0936 yd.	1 yd.	= .91439 m.
1 Km.	= .62138 mi.	1 mi.	= 1.6093 Km.
1 $m.^2$	= 1,550.0 sq. in.	1 sq. in.	= 6.4515 $cm.^2$.
	= 1.1960 sq. yd.	1 sq. yd.	= .83611 $m.^2$.
1 $cm.^3$	= .061025 cu. in.	1 cu. in.	= 16.387 $cm.^3$.
1 $m.^3$	= 1.3080 cu. yd.	1 cu. yd.	= .76453 $m.^3$.
1 l.	= 1.0567 liquid qt.	1 liquid qt.	= .94633 l.
	= .90810 dry qt.	1 dry qt.	= 1.1012 l.
1 g.	= 15.432 gr.	1 gr.	= .06480 g.
1 Kg.	= 2.2046 lb.	1 oz.	= 28.350 g.
		1 lb.	= .45359 Kg.

71. In solving the following examples, the student will take the equivalents from the table:

EXAMPLE 1.—Reduce 2 ft. 6¼ in. to meters.

SOLUTION.—We have

2 ft. 6¼ in. = 30¼ in. = 30.25 in.

From the table, 1 in. = 2.54 cm.

Therefore,

30.25 in. = 30.25 × 2.54 cm. = 76.835 cm. = .76835 m. Ans.

EXAMPLE 2.—Reduce 27.5 Kg. to pounds.

SOLUTION.—From the table,

$$1 \text{ Kg.} = 2.2046 \text{ lb.}$$

Therefore, $27.5 \text{ Kg.} = 27.5 \times 2.2046 \text{ lb.}$

Hence, correct to five figures, 27.5 Kg. = 60.627 lb. Ans.

EXAMPLES FOR PRACTICE.

Reduce

1. 6 sq. ft. 72 sq. in. to square meters. Ans. .60386 $m.^2$.
2. 23.5 gal. to liters. Ans. 88.955 l.
3. 4 cwt. to kilograms. Ans. 181.44 Kg.

INSTRUMENTAL DRAWING.

1. Instrumental drawing is the art of making drawings by means of instruments. This art is used to advantage in making the drawings needed by the mechanic, for the use of instruments gives accuracy to the work of representing objects. But, before we can use an instrument, we must know the instrument and the purposes for which it is used.

INSTRUMENTS AND THEIR USES.

PAPER.

2. The best drawing paper is made of linen rags, there being two standard varieties—**bristol board** and **Whatman's paper.** Both are white, strong, and do not shrink much after being wetted. Bristol board is always very smooth and hard, and is used for very accurate or very fine drawings, especially when they are to be inked in and photoengraved. All patent-office drawings are made on this paper. It is made in various thicknesses, which are denoted by the names one-sheet, two-sheet, three-sheet, etc.

Three-sheet, the kind most generally used, is not quite $\frac{1}{32}$ inch thick.

3. Whatman's paper is made with two kinds of surface: *hot-pressed*, a surface nearly as smooth as bristol board, and *cold-pressed*, a surface even-grained and rather rough. Both bristol board and Whatman's paper are more expensive than is necessary for this Course.

The Technical Supply Company furnishes a good paper. It takes ink well and will withstand considerable erasing. It is cheaper than bristol board or Whatman's and will serve for all the purposes of this Course.

4. Manila paper is of a buff color and is made in many varieties and qualities. It is put up in rolls of various widths and is generally called **detail** paper. It is not suitable for water coloring, because it shrinks too much, and wetting it makes its surface rough. It is cheap, and is commonly used for making sketches and pencil drawings that are to be traced. The student should provide himself with some of this paper upon which to practice.

DRAWING BOARD.

5. In instrumental drawing, the paper is always tacked or glued upon a rectangular board, called a **drawing board.**

Drawing boards are made with a flat, smooth, true surface, or face, on which the drawing paper is fastened. The left edge (as the draftsman faces the board) is made perfectly straight and true, to serve as a guide for the **T** square, which is described later.

Although it is convenient and preferable that the board be exactly rectangular, that is, that its four edges should be perfectly straight and its four corners perfectly square, this is not required, as the **T** square is seldom used on any edge but the left-hand one. If this edge is perfectly straight and smooth, that is all that is strictly necessary.

6. Fig. 1 shows a top and an edge view of a drawing board. In the drawing board proper, or middle part, the grain of the wood runs lengthwise; in the end pieces, or cleats, the grain runs crosswise, or at right angles to the grain of the middle part. This middle part should be white pine or some similar soft wood; the end pieces may be either soft or hard wood. In small drawing boards, the end pieces *e f* and *g h*, Fig. 1, should be dovetailed into the middle part, as shown by the edge view.

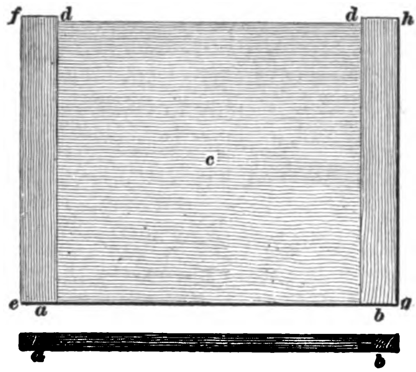


FIG. 1.

The end pieces must not be glued or fastened on the middle part their full length, but may be fastened at one end, as at *a* and *b*, and the remainder held in place by the dovetails only, so that the middle part *c* may be free to shrink, as indicated at *d*, *d*, or swell out, as the case may be. In either case, *c* will slide on the dovetail. The object of this construction is to keep the board from being distorted by the expansion and contraction due to dampness, changes of temperature, etc.

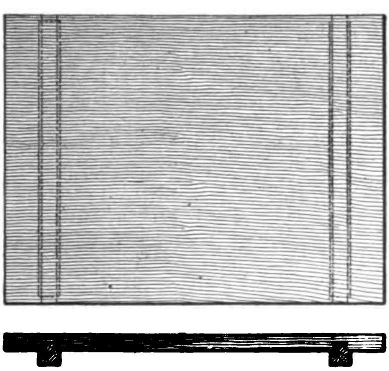


FIG. 2.

7. In large drawing boards, it is best to put the cleats or ledges underneath, as shown in Fig. 2, instead of putting on end pieces. These cleats should be a little shorter than the width of the board, so that they will not project when the board shrinks. They should be fastened in the middle only.

For this Course a board 16″ × 21″ is required, but the student may use a larger one, if he prefers.

8. The paper is fastened on the drawing board by means of **thumbtacks,** Fig. 3, which are small pegs having a sharp point and a large flat head. When fastening a sheet of drawing paper on the drawing board, care must be taken to stretch it evenly, so that it will have no wrinkles. To do this, proceed as follows: Lay the paper on the drawing board with the edges parallel to, and equally distant from, the sides. Insert a thumbtack in the upper right-hand corner, about ¼ inch from the edge of the paper, and press it in until the head bears evenly on the paper all around. Take another thumbtack in the right hand, and place the left hand on the paper near the upper right-hand corner. Slide the hand lightly toward the lower left-hand corner, and, holding the paper there, press in the thumbtack, as before. Lay the left hand on the middle of the sheet, slide it very lightly toward the upper left-hand corner, and insert another tack. The fourth tack is inserted in the same way as the third, except that the left hand is slid from the center to the lower right-hand corner. If the paper is wrinkled or loose, it shows that it has been unevenly stretched, and the preceding operation must be repeated until the sheet lies flat and smoothly on the board.

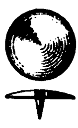


FIG. 3.

9. When damp weather sets in after a spell of dry weather, the paper usually swells and becomes loose and wavy. In such cases a tack may be put in the middle of each edge of the sheet, after gently and evenly flattening the paper from the center to the middle of each edge. The tacks in the corners are then taken out and reinserted a little to one side of their former positions, after flattening the sheet evenly toward each corner. Putting the four tacks in the middle of the sides first will keep the drawing in the same position on the board, which is very important when a **T** square has been used.

LEAD PENCILS, INK, AND ERASERS.

10. **Lead pencils** are made in various forms—triangular, hexagonal, round, etc. Hexagonal ones are the best to use. Triangular pencils are clumsy, and round ones roll off the board too easily.

11. For drawing purposes, pencils of the best quality should be obtained; although they cost more, they last longer and work better. In this as in almost every other case, cheapness is no economy.

FIG. 4.

12. A soft pencil should never be used for drawing, because it quickly becomes dull, which makes it impossible to draw fine lines and keep the paper clean. The degree of hardness of the pencil depends on the nature of the paper used. A pencil that would seem quite hard when used on ordinary writing pads would be too soft if used on Whatman's paper.

For detail papers, Faber's 4 H will answer very well. For drawings on Whatman's or The Technical Supply Company's paper, Faber's 6 H, or Dixon's H—No. 4, 125, pencils may be used. It is often convenient to use case leads in a pencil holder. These holders resemble an ordinary hexagonal pencil, being of the same length, thickness, and color as one that has not been sharpened, Fig. 4. A lead can be fixed in each end; either a hard one in one end and a soft one in the other, or a chisel point in one end and a round point in the other; both these arrangements will prove convenient, the softer pencil being used for writing the dimension figures. The use of a pencil holder has the advantage of saving the time and trouble necessary for sharpening the pencil.

13. The **point of the drawing pencil** should be made chisel like, as shown in Fig. 5, which gives both a front and a side view of a point so shaped. The whole length of the sharpened part should be about 1 inch and the exposed lead about ¼ to ⅜ inch. With such a point as this, fine, narrow lines can be drawn, and although the point is thin, yet it has a certain amount of breadth and so will not readily become blunt. Use a knife to remove the wood, then sharpen the lead flat by rubbing it against a fine file or a piece of fine emery cloth or sandpaper that has been glued or tacked upon a flat stick. Grind the lead to a sharp edge, like a knife blade, and round the corners very slightly, as shown in the figure.

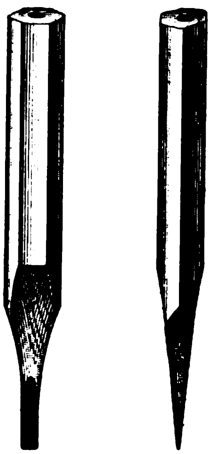


FIG. 5.

The lead for the compasses should be sharpened with a flat edge, but should have its width narrower than that shown in Fig. 5. The lead should be so fixed that the flat sides are in line with the circumference of the circle that is being drawn; or, in other words, the flat sides should be at right angles to the radius of the circle.

14. India ink is the name of the ink used for drawings. The best kind is sold in cakes or sticks, and is for this reason called **stick ink.** It is dissolved by rubbing it upon a saucer containing a small quantity of water. Rough saucers made of slate and provided with air-tight glass covers are the best to use. It takes some time to prepare ink in this manner. The ink may be tried with an ordinary writing pen, and should not be used before it makes a jet-black line; but care must be taken not to get the ink too thick, as it will get sticky, run with difficulty, and clog the pen. Always keep the dissolved ink covered.

Although stick ink is the best for fine drawings (such as are made in the government offices), **prepared** or **liquid ink** is generally preferred for working drawings. It is sold in bottles. The Technical Supply Company's superior waterproof India ink is one of the best of liquid inks. A great advantage attending the use of liquid ink is that all lines on drawings are of equal blackness. Not only black ink, but also colored inks of different shades can be obtained in this prepared form. There is attached to the cork of each bottle a quill for putting the ink in the drawing pen.

We strongly recommend that the student ink his drawings, but accept drawings in pencil.

15. Pencil erasers, or **rubbers,** are made of india rubber. They are used for rubbing out pencil lines. Many varieties are made. The "velvet" rubber and Tower's "multiplex" are good kinds, but almost any other kind will answer the purpose well.

16. Ink erasers consist of ordinary rubber mixed with powdered pumice stone. If an erasing knife is used at all, great care is required, or the surface of the paper will be cut up too much. For erasing lines from a tracing it is better to use only an ink eraser. When a knife is used, it should be used to remove only the upper films of ink, the rest being rubbed out with the eraser. The paper should always be smoothed with the rubber after the knife has been used.

17. Sponge rubbers are very soft loose-grained black pieces of rubber intended for cleaning a drawing after it has been inked in. The soft part of a loaf of bread about two days old answers the same purpose, and is one of the best things obtainable; it removes the dirt without erasing the ink lines.

RULERS.

18. Rulers are flat, thin, narrow pieces of wood, steel, ebony, or other close- and even-grained material, which have one or two perfectly straight edges. They are used for guiding the pencil or pen in drawing straight lines. If made of wood, they must be straight-grained, otherwise they will not keep straight.

The best rulers are made of two kinds of wood, a central part of rather hard wood, such as maple or mahogany, and narrow edges of very hard wood, such as ebony. Steel rulers keep their shape very well, but are not so easy to handle as wood rulers, and soon become rusty, unless nickel-plated. Then, too, the reflection of the light from the polished surface makes it difficult to draw a line at precisely the point desired, and they soil the drawing more than wood rulers. Hard rubber wears better than wood, but, unless frequently washed, it soils the drawing; it breaks easily and does not always keep its shape.

A transparent ruler made of celluloid, or with celluloid edges, is very convenient, as the ends of lines under it can be seen and other lines drawn to meet them.

It is preferable that the edge of a ruler should not be exactly flat, but a little rounded, so that the point of the pen will not slide along the bottom of the edge, for in the latter case a slight motion of the ruler will cause it to slide over the line and make a blot. Besides, when the pen is run close to the bottom of the edge of the ruler, the ink often runs over, adheres to the ruler, and blots the paper.

19. Besides the white drawing paper on which the student is to draw the plates to be sent in for examination, he should have some manila paper on which to practice. He should keep a piece tacked on the board, and draw on it, in pencil, the exercises for practice given in the following articles, whether these exercises refer to figures contained in the plates with which he is furnished or not. When the drawing called for is to be made on the examination sheet to be sent in, the student will be so informed.

20. To draw a straight line by means of a ruler.

The ruler is placed on the paper with its edge where the line is to be drawn, and is held down by the fingers of the left hand. The body of the pencil is held directly over the edge of the ruler, with the flat side of the point against the edge, so that the line will be thin. The little finger of the right hand rests on the ruler, and is slid along ahead of the pencil. The pencil should be held in an upright position perpendicular to the board, and should be kept in this position while the line is being drawn. No part of the arm or hand (except the little finger) should touch the paper or ruler while the line is being drawn.

21. To test a ruler.

Every new ruler should be tested before it is used, and old ones should be tested occasionally. The simplest way to test a ruler is to place the edge of the ruler about 8 inches from the eye and look at it in the direction of its length. Large inequalities can thus be detected and the places smoothed down with sandpaper or with a very fine file. However, unless the draftsman is a good mechanic, he is advised not to try to fix a defective ruler or any other instrument, but to return it to the manufacturer or seller and have it exchanged.

A more accurate method of testing a ruler is shown in Fig. 6. Draw a line with the ruler, and then turn the latter over, as shown, so that the points *d* and *e* will remain as before, while *b* and *c* will fall on *b′* and *c′*, respectively. Retrace the line with the ruler in this new position. If the edge of the ruler is not perfectly straight, the irregularities will show twice as large, as at *a*. If the ruler has beveled edges, extreme care must be taken when drawing the second line.

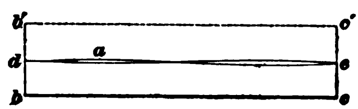


FIG. 6.

22. To draw a straight line through two given points by means of a ruler.

First, suppose the line joining the points to be horizontal, or nearly so, as at (*a*), Fig. 7. Place the ruler on the paper a short distance below the points, and then slide it toward them; when it is close to them, set the edge directly over *one* of the points. Then, holding the edge over this point (not too firmly), swing the ruler around until it is against the other point; if, now, the edge of the ruler has moved slightly away from the first point, readjust the ruler until

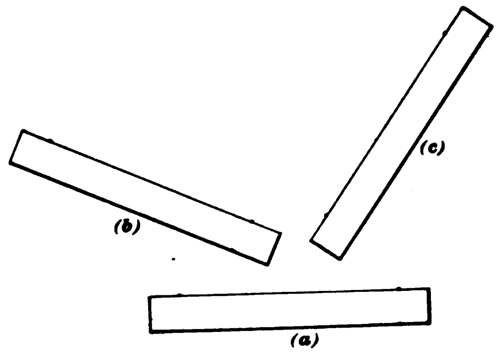


FIG. 7.

the edge is exactly in line with both points. The line can now be drawn. If the points are so situated that the line joining them will be in an inclined position, as at (*c*), Fig. 7, place the ruler to the right of them; if as at (*b*), place it on their left.

23. Long lines may be drawn by joining two rulers together, as shown in Fig. 8. The line is to pass through *a* and *b*. The two rulers are placed alongside each other so that they will overlap for a distance *c d* of about 6 or 8 inches.

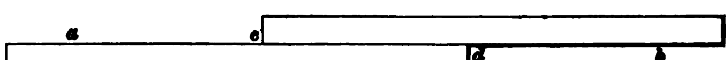


FIG. 8.

Then, holding them so that the edges will be in perfect contact along *c d*, one edge is placed on *a* and the other moved until it passes through *b*. Then *a c* and *b d* are drawn, the rulers removed, and *c d* is drawn with one of them; or *a c* is drawn first, the rulers removed, and *b c* is drawn with one of them.

EXERCISE.

Draw several lines with the ruler, first laying the ruler anywhere on the paper, then select points, two for each line, through which the lines are to pass.

DIFFERENT STYLES OF LINES.

24. Full lines, whether straight or curved, are those that are drawn continuously, or uninterruptedly, *a*, *b*, *c*, and *d*, Fig. 9. They are made of various thicknesses, according to requirements. **Half-dark,** or **medium,** lines, as *c*, and **dark,** or **heavy,** lines, as *d*, are used for **shade lines,** that is, for lines that separate surfaces supposed to be in the light from those supposed to be in the dark. For the other lines of

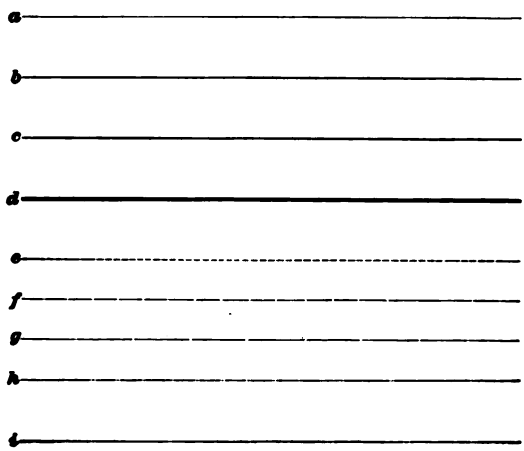


FIG. 9.

the outline of a figure, thin, sharp, and well-defined lines, as *b*, should be used. Shade lines are not always used; they will not be used in this Course, and the student should make his drawings with all lines of the same thickness, making them thin, but not so much so that they will be indistinct. The thickness of the line *b* is a good average thickness for drawings of ordinary size.

25. A **dotted line,** as *c*, Fig. 9, is one composed of very short dashes, called "dots," separated by very small intervals. Dotted lines are generally used for drawing those parts of an object that are not seen, but whose position it is desirable to indicate; also, to indicate the object in another position.

26. A **dash line,** or **broken line,** as *f*, Fig. 9, is one composed of long dashes. It is used on drawings to indicate **guide,** or **construction, lines,** that is, lines that do not belong to the main figure, but aid only in its construction. With a little longer dashes, as *g*, broken lines are used for **dimension lines,** that is, to mark distances, the number written on the line referring to the distance between its two ends. For dimension lines, however, faint, full lines, as *a*, Fig. 9, are often used. The use of dimension lines is illustrated in Fig. 10. The full lines form a triangle, which is the figure whose dimensions are to be given; *a m* and *c n* are equal lines, both perpendicular to *a c*; they are called **extension lines.** The long-dash line *m n* is the dimension line for *a c*, and the number $2\frac{3}{4}''$ written on it expresses the length of *a c*. Similarly for the other sides. The extension lines should be short (about $\frac{3}{16}$ inch or $\frac{1}{4}$ inch), unless there are other lines in the way that will confuse the figure. It is advisable, when practicable, to make them long enough so that the dimension lines will not be confused with other lines. A very short extension line may be made full, but *very light;* longer ones should be made broken, or broken and dotted, the dashes being short and fine.

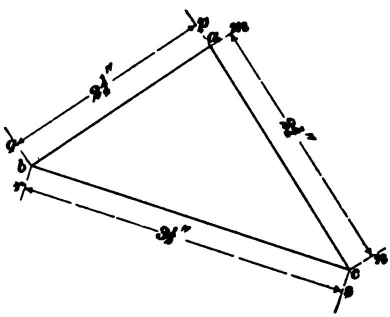


FIG. 10.

27. A **dash-and-dot** or a **broken-and-dotted** line, *h* and and *i*, Fig. 9, consists of long dashes with one or more dots between each dash and the following one. Dash-and-dot lines are used for various purposes: they are sometimes used for construction lines, the same as broken lines; for **center lines,** that is, lines passing through the center of a figure, to which distances are referred; and for **section planes,** that is, to indicate the direction in which an object is cut, when this is necessary for some special purpose.

28. There is no uniformity of practice with respect to the use of broken and broken-and-dotted lines; thus, we have seen that construction lines are made either long-dash or broken-and-dotted; dimension lines, too, are made sometimes like *g*, Fig. 9, sometimes like *h*, and sometimes like *a*. Practice, however, as to the use of the full line *b* and the dotted line *e* seems to be universally as stated above—the full line for the *visible* outlines of the figure, and the dotted line for the *invisible*, or *hidden*, outlines of the figure, when it is desirable to show the position of the latter. Special uses of the different kinds of lines will be introduced and explained as they come up in the working out of this Course.

29. The preceding styles of lines are not generally used while drawings are being made in pencil. The pencil lines are drawn full, and, when the drawing is inked, the proper styles of lines are made with the pen. The full pencil lines used as construction lines should be very light, so that they can be easily erased, as they are seldom inked in working drawings. Also those lines that are to be inked as either dotted or broken lines should be light, but center lines and lines that are to be full when inked should be made with a good pressure on the pencil, without, however, making them heavy. If the pencil drawing is not to be inked nor traced, but is to be used by itself, the pencil lines should be drawn according to the purposes they serve, although in this case both dimension and construction lines may be made full, but *very light*.

Light pencil lines are also used for marking the direction of a line before its length is laid off. Suppose that a line

4 inches long is to be drawn through a certain point and in a certain direction, say horizontal. In this case a very light horizontal line is drawn through the point and made about $4\frac{1}{2}$ inches, as nearly as can be judged by the eye; then the distance of 4 inches is laid off from the given point along the line, the remaining part of the line erased, and the pencil run over the 4-inch portion so as to make it of ordinary thickness.

EXERCISE.

Draw several lines of each of the kinds represented in Fig. 9. In drawing the "dotted" lines, remember that the so-called "dots" are really *very short dashes.*

T SQUARES.

30. A **T square,** Fig. 11, is a ruler with a fixed cross-piece *a a*, called the **head,** at one end; the ruler proper *b* is called the **blade.** The head has one straight edge *c c* next to the blade, which is below the blade when the latter is in the working position, as shown in the side view Fig. 11 (*b*).

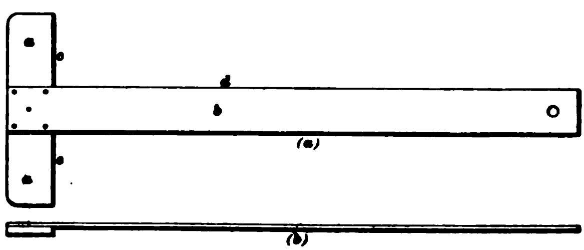


FIG. 11.

The head is made to slide along the edges of a drawing board so as to guide the blade. The head is usually made of hard wood, occasionally of metal, and is fastened to the blade in such a position that its straight edge *c c* is at right angles to the edges of the blade. When the **T** square is more than

42 inches long, the blade is tapered in width, and in that case only the upper edge *d* is square with the edge *c c*.

31. The **T** square serves as a guide for drawing parallel and perpendicular lines. Although the head is usually made at right angles to the blade, this condition is not necessary. In order that the lines may be parallel, all that is necessary is that both the edge of the board and the edge of the head of the **T** square should be perfectly straight. Sometimes the head is made adjustable, so that, by sliding it along the edge of the board, the blade may move parallel to any required direction or line.

Unless you are sure that the edges of the drawing board are exactly true or square, never use the **T** square on more than one edge of the board. The board should be placed so that the straight end piece is at the left of the draftsman. When, however, the head is adjustable, the **T** square may be adjusted and used on any edge, provided the latter is straight.

For this Course, a **T** square with a fixed head and having a length of blade of 20 to 24 inches is required.

32. To draw lines by means of a T square.

After tacking the paper on the drawing board, as already described, lay the blade of the **T** square on the paper, with the head down and outside of the left-hand edge of the board, Fig. 12. Hold the head in the middle with the left hand, and press it lightly against the edge of the board. Then slide the head along the edge of the board, keeping a light pressure against it, until the straight edge of the blade is where the straight line is required. The figure illustrates the position of the **T** square for drawing the horizontal diameter of a given circle.

33. When the line is short and near the left-hand edge of the board, it may be drawn without taking the left hand off the head; but for long lines and for lines near the free end of the blade, the latter is pressed down with the right hand as soon as it has been moved into the required place, and the left hand is then placed on the blade to help hold it down close to the paper while the line is being drawn with the right hand.

34. When the head of the **T** square slides along one side of the board, all lines drawn along the edge of the blade are parallel to one another. If the head is fixed, only horizontal or vertical lines can be drawn along the blade. It is, however, advisable to use the fixed-head **T** square to draw nothing but horizontal lines, and to place the head always

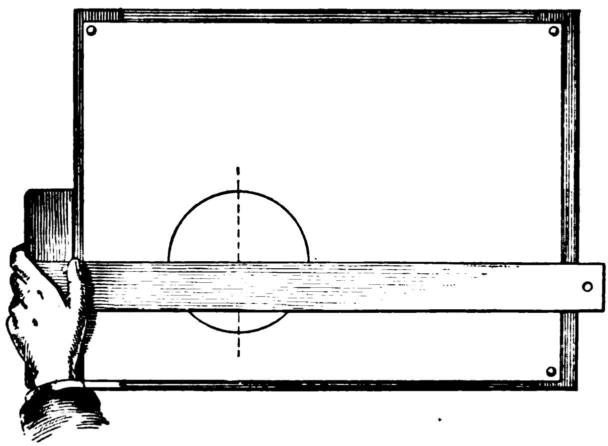


FIG. 12.

against the left-hand edge of the board. Vertical lines are then drawn by means of the triangles. By placing one of the legs of the triangle against the edge of the **T** square and sliding the triangle along the blade of the **T** square, the other leg of the triangle may be used as a guide to draw a vertical line through any point.

If the **T** square has an adjustable head, the blade may be turned about its joint with the head until it takes any desired position; the head is then clamped and the adjusted **T** square used as before; in this case all lines drawn along the edge of the blade will be parallel to the adjusted direction. The adjustable-head **T** square, however, is neither accurate nor reliable, and is but little used.

TRIANGLES.

35. Triangles, or **set squares,** are used for drawing perpendiculars, angles, and parallels. They consist of three straight edges, of which two are at right angles to each

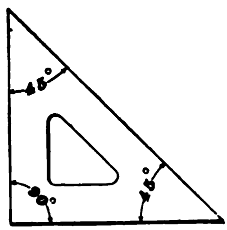


FIG. 13.

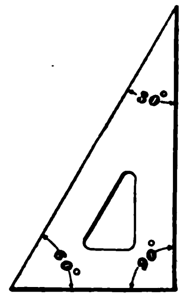


FIG. 14.

other. The third side usually makes either angles of 45° with the other two sides, Fig. 13, or it makes an angle of 30° with one side and an angle of 60° with the other, Fig. 14.

36. Triangles are made of the same materials as rulers. Those made of celluloid are convenient to use, as they are nearly transparent.

For this Course, two triangles, 45° and 60°, with the longest sides about 8 and 10 inches, respectively, are required. Larger ones may be used, however.

37. To draw, by means of the triangles, a perpendicular to a line at a given point.

Let *a b*, Fig. 15, be the given line, and *c* the given point, which, in this particular problem, lies on the line. First, place one triangle so that the hypotenuse will coincide with the line, as shown by *m n p*. Place one edge of the other triangle, or of a ruler, along *m n*, as shown by *q r s*. Hold *q r s* (or the ruler) firmly in place and slide *m n p* along *q r* a short distance, until it takes the position *m' n' p'*. Hold *m' n' p'* firmly, and take the other triangle *q r s* and put it in the position *q' r' s'*, with one leg in contact with *m′ p′*. Any line drawn along *q′ r′* will be perpendicular to *a b*, and, by sliding *q′ r′ s′* along *m′ p′*, the edge *q′ r′* may be made to pass through any required point, as *c*.

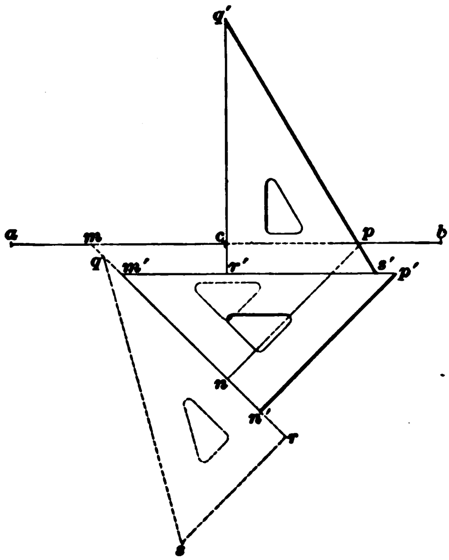


FIG. 15.

This method has the disadvantage that it is necessary to change the triangle *q r s* from its position in contact with one leg of *m′ n′ p′*, to a new position; in making this change, there is danger of letting the triangle *m′ n′ p′* slip and get out of position. Either of the methods given in the next two articles is, therefore, to be preferred whenever it can be used conveniently.

38. In Fig. 16, *a b* is the given line to which it is required to draw a perpendicular through the point *c*. Place one triangle, as *A*, so that one leg lies on the given line, while the other leg is a little to one side of the given point, as shown by the dotted triangle *1–2–3*. Place the other triangle *B* along the hypotenuse *1–3* of *A*, as shown by the triangle *4–5–6*. Now slide the triangle *A* along the side of *B* until the leg *2–3* lies on the point through which the perpendicular is to be drawn, as shown by the triangle *1′–2′–3′*. A line drawn with the leg *2′–3′* as a guide will be the required perpendicular.

This method can be used both when the given point is on the line and when the given point is off the line.

39. Still another method of drawing a perpendicular to a line is the following: Let *a b*, Fig. 17, be the line.

Place one triangle in the position *m n p*, so that its hypotenuse will lie along the given line. Place a ruler (or the

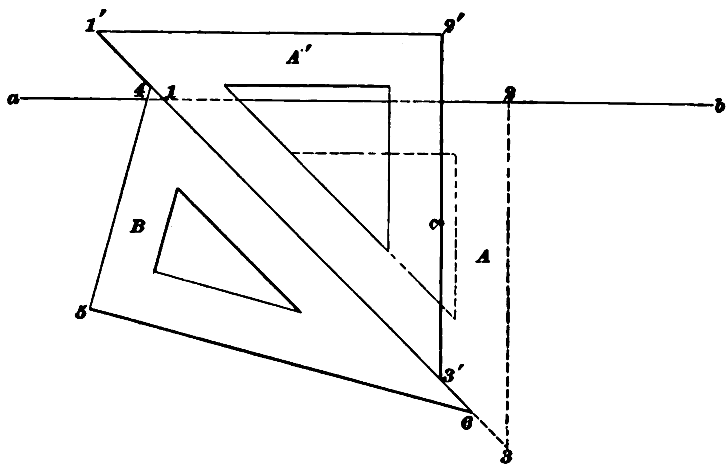


FIG. 16.

other triangle) against *m n*, as shown. Hold the ruler fast, and turn the triangle so that the side *p n* will be against the

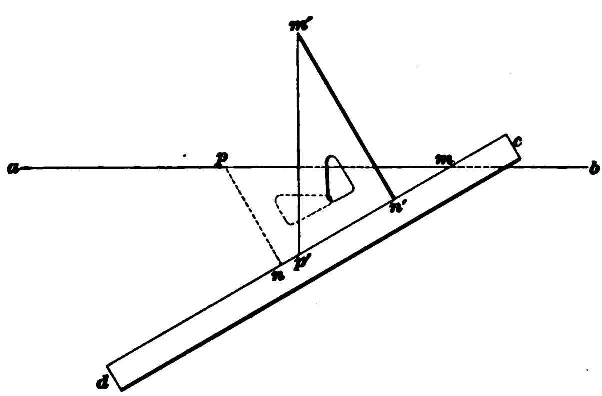


FIG. 17.

ruler. In this new position the triangle may be slid up or down the ruler, and any line drawn along *p' m'* will be perpendicular to *a b*.

40. To draw, by means of the triangles, a line parallel to a given line.

Let *a b*, Fig. 15, be the given line. Place one of the triangles in the position *m n p*, so that one of its sides (not necessarily the hypotenuse) will fall on the line. Place the ruler or the other triangle against either *m n* or *n p*, and, holding it fast, slide *m n p* up or down until *m p* passes through the required point. Any line drawn along *m p*, in any position of the triangle, will be parallel to *a b*. If a series of lines parallel to *a b* are to be drawn through different points, *m p* is made to pass through the different points successively by sliding the triangle *m n p* along the ruler.

41. To draw, by means of the triangles, lines making angles of 45°, 30°, or 60° with a given line.

There are several ways of doing this, all similar to those used for the drawing of perpendiculars. The following method is the most convenient:

Let *a b*, Fig. 18, be the line with which the required line is to make an angle of 30°. Place the 45° triangle so that one of its edges (preferably the hypotenuse) will coincide with *a b*, and then slide it down along the edge of the other

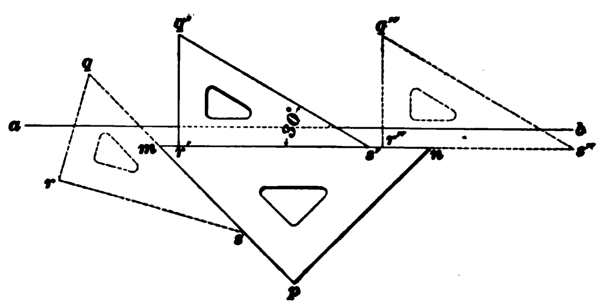


FIG. 18.

triangle until it takes the position *m n p*. Holding it firmly in this position, place the other triangle as shown by *q' r' s'*, that is, with the vertex of the right angle and that of the 30° angle against *m n*. By sliding *q' r' s'* along *m n*, we may make $q's'$ pass through any required point. Any line drawn along $q's'$, in any of the positions of $q'r's'$, will make an angle of 30° with ab. If it is desired to have the angle opening toward the right, $q'r's'$ must be turned over about $q'r'$; in other words, it should be so placed on mn that r' will be at the right of s'.

An angle of 60° may be drawn in a similar manner by placing the triangle $q'r's'$ so that r' and q' will be on mn. The method of drawing a 45° angle is the same, but in this case the 60° triangle will have to be used in the position mnp, and the 45° triangle in the position $q'r's'$.

Notice that it is not necessary that both r' and s' should be on mn; in the case shown in the figure, the upper triangle may have the position $q''r''s''$, *so long as the overlapping portion* nr'' *of the two triangles is in perfect contact.* The case shown in the figure, however, is an extreme one; it is difficult, in this case, to keep the two triangles in the right positions, and it is, therefore, advisable never to move the upper triangle so much that the part ns'' extending beyond the lower triangle is greater than half the length of the side.

42. Any angle that is the sum or difference, or a combination of sums and differences, of the angles (90°, 60°, 45°, 30°) of the triangles can be easily drawn. Thus, in Fig. 19 we obtain the line ad, making an angle of 75° with ab, by first drawing ac, making a 30° angle with ab, and then ad, making a 45° angle with ac. In like manner, by drawing ae, making a 60° angle with ad, we obtain an angle of 135°.

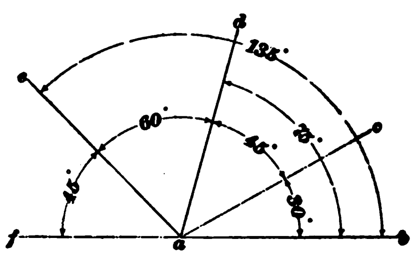


FIG. 19.

When, however, the required angle is greater than 90°, we may construct it more easily by subtracting it from 180°, and constructing an angle equal to the difference on the opposite end of the given line. Thus, in the preceding example, the difference between 180° and 135° is 45°. By drawing *a e*, making an angle of 45° with *a f*, which is the prolongation of *b a*, we get the angle *e a b* equal to 135°.

43. Testing the Triangles. — As triangles are constantly used in practice, they should be carefully tested. To test the edges, proceed as in testing a ruler. To test the right angle, lay a ruler (one that has previously been

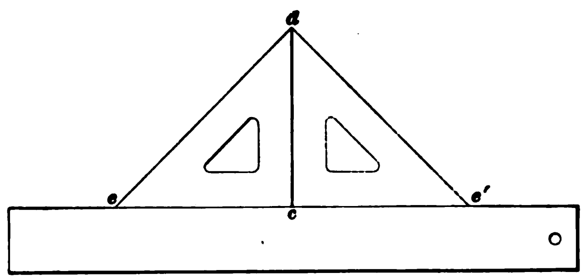


FIG. 20.

tested and found true should be used) on the paper, Fig. 20, then place the triangle in the position *e c d*, so that one of the legs, as *e c*, will rest against the edge of the ruler, and draw a line along *c d*. Turn the triangle over, as shown

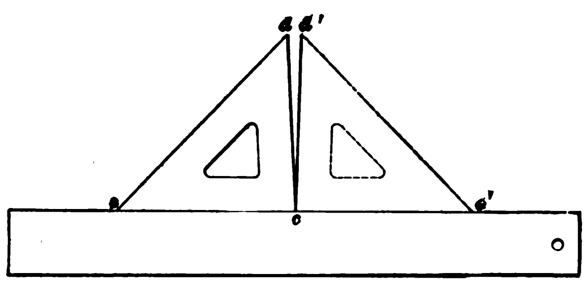


FIG. 21.

at *d c e'*, so that the vertex of the right angle will be at *c*, as before, and see if the edge *c d* follows exactly the line first drawn. If it does, as in Fig. 20, the angle of the triangle is a right angle. If the new position of the edge does not coincide with the line *c d*, but lies to the right of it, as at *c d'*, Fig. 21, the angle is too small, that is, it is an acute angle whose difference from a right angle is *one-half* the angle *d c d'*. If the edge in its new position slants toward the left of *c d*, the angle is too large; it is an obtuse angle.

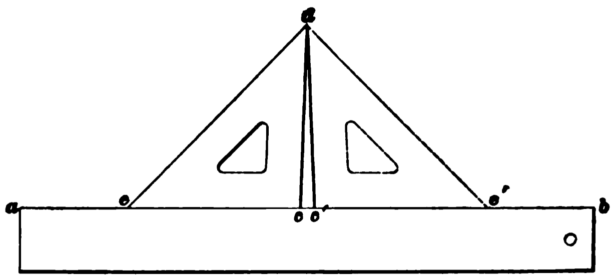


FIG. 22.

Slide the triangle along *a b*, Fig. 22, toward the right, until the end of the edge of the triangle is on the end *d* of the line *c d*, in which position the vertex of the "right" angle will be at *c'*, to the right of *c*; if a line is drawn from *d* to *c'*, the angle *c d c'* will be twice the difference between the

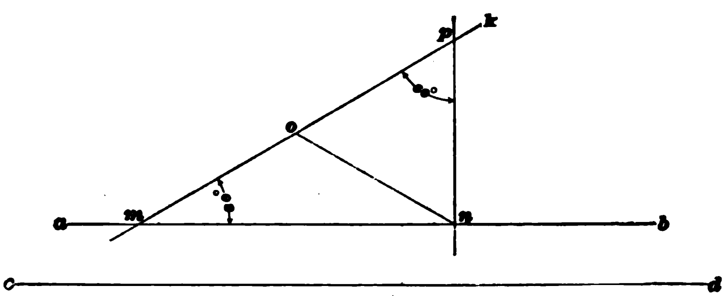


FIG. 23.

angle of the triangle and a right angle. The best thing to do when the triangle is not true is to get another, or to get a carpenter or cabinetmaker to remedy the defect.

To test the 30° and the 60° triangles, first test the right angle as described above. Then draw a line *a b*, Fig. 23, with a ruler, and a short distance below (say ½ inch) draw a line *c d* parallel to it. Lay the triangle on *c d*, so that the vertex of the right angle and that of the 30° angle will be on *c d*, and draw *m k* along the hypotenuse; lay off from *m* toward *k* a convenient distance, say 3 or 4 inches, to *p*. Place the triangle in the same position as before, with the vertex of the right angle and that of the 30° angle on *c d*, and slide it toward the left, until the short side passes through *p*, and draw *p n*, which will be perpendicular to *a b*. If the angles of the triangle are exactly 30° and 60°, respectively, the line *p n* is equal to one-half of *m p*. After drawing *m p* and *p n*, locate *o*, the mid-point of *m p*, and draw *n o*. If the triangle is true, *n p*, *p o*, *n o*, and *o m* are all equal.

To test the 45° triangle, first test the right angle. Then, using the 45° triangle instead of the 60° triangle, draw *m p* and *p n*, as described in connection with Fig. 23. If the triangle is true, *m p* will equal *p n*.

44. To draw lines at 45°, 30°, or 60° with a horizontal or with a vertical line.

The method is substantially the same as that given in Art. **41**, but very much simplified. Let *a b*, Fig. 24, be a

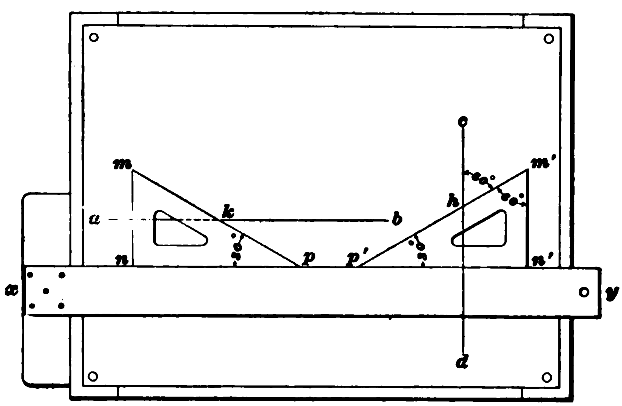


FIG. 24.

horizontal line, and *k* a point of it at which an angle of 30° with *a b* is required. Place the **T** square in its working position and slide it to any position, as *x y*, below *a b*. Place the 30° triangle against the upper edge, as shown, and slide it until *p m* passes through *k*. Draw a line along *m p* and in the direction from *m* to *p*; then the angle *a k m* will be equal to 30°. The angle *p k b* is also an angle of 30°. If the triangle be reversed, the 30° angle will also be reversed, and will lie in the position *m' p' n'* instead of *m p n*.

Now let *c d* be a vertical line and *h* a point at which an angle of 60° is to be made. Place the triangle as shown, that is, so that its shorter leg adjacent to the 60° angle will be vertical; slide it until *m' p'* passes through *h*, then draw a line along *m' p'*, and in the direction from *p'* to *m'*. The angle *c h m'* will be an angle of 60°. By placing the triangle so that the leg *m n* will lie against the **T** square, lines making angles of 60° with the horizontal or 30° with the vertical may be drawn.

Lines making angles of 45° with either the horizontal or the vertical may be drawn by placing one of the two equal legs of the 45° triangle against the **T** square.

EXERCISES.

1. **Draw several horizontal lines, and then a series of vertical lines.**

2. **Draw a horizontal line, and at any point of it draw two lines making with it angles of 30°, one toward the left, one toward the right.**

3. **Draw a vertical line, and at any point of it draw a line perpendicular to it; a line making with it an angle of 45°, above the perpendicular and to the right of the vertical; and a line making with the same vertical an angle of 60°, below the perpendicular and to the left of the vertical.**

SCALES.

45. When the outlines of the drawing of an object are of the same size as the outlines of the object itself, the drawing is said to be **full size**. Usually, however, the lines on the drawing are either shorter or longer (generally the former) than the lines on the object. The relation between the lengths of the lines on the drawing and those on the

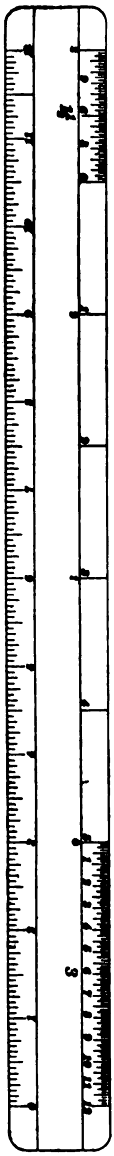


FIG. 25.

object is called the **scale** of the drawing. Thus, if a rectangular piece of metal 8″ × 14″ is represented by a rectangle 4″ × 7″, the scale is ½, and every line on the drawing is one-half as long as a corresponding line on the object. Or, we may say that the scale is so many inches to the foot, meaning that so many inches (or parts of an inch) on the drawing represent 1 foot on the object. In such cases, should it be desired to find from the drawing the true length of a line on the object, divide the length of the line (in inches) on the drawing by the scale; the result will be the length in feet of the line on the object. Thus, if a line on a drawing measures 10½ inches, and the scale is ½ inch to the foot, the true length of the line on the object is $10\frac{1}{2} \div \frac{1}{2} = 21$ feet; were the scale 3 inches to the foot, the actual length of the line on the object would be $10\frac{1}{2} \div 3 = 3\frac{1}{2}$ feet.

46. The word **scale** is also applied to any graduated piece of wood, ivory, metal, or other suitable material used for measuring lines on drawings and for laying out and taking off distances. Scales are made in various forms, commonly either flat, as shown in Fig. 25, or triangular, as shown in Fig. 26.

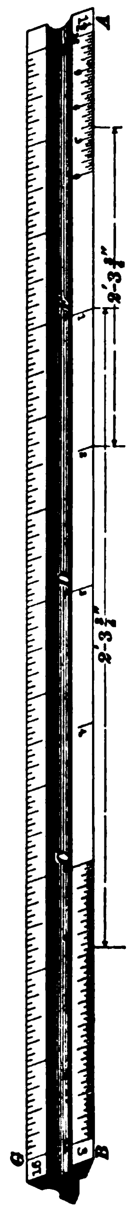


FIG. 26.

47. Graduations.—The most familiar method of graduating scales is that seen on the common 1-foot and 2-foot rules, in which 1 foot is divided into 12 equal parts, or inches, and each inch is subdivided into halves, fourths, eighths, and sixteenths.

Fig. 26 shows a common method of graduating the triangular scales used by draftsmen. The scale at *G* is full size; that is, this edge of the scale is divided into inches and sixteenths, in the same manner as a common 2-foot rule. On the side *B* is shown a scale of 3 inches to the foot. The first 3-inch division from *B* to *C* is subdivided into 12 parts representing inches, each of which is then divided into proportional fractions of an inch. From *C* to *D*, *D* to *E*, and *E* to *F*, the scale is marked in its main divisions of 3 inches each. From *A* to *B*, the scale is independently divided into spaces of $1\frac{1}{2}$ inches each, to form the scale of $1\frac{1}{2}$ inches to the foot, the divisions on the latter occurring on and between the marks for the 3-inch scale. The other sides and edge of the instrument are divided into scales of 1 inch and $\frac{1}{2}$ inch, $\frac{3}{4}$ inch and $\frac{3}{8}$ inch, $\frac{1}{4}$ inch and $\frac{1}{8}$ inch, and $\frac{3}{16}$ inch and $\frac{3}{32}$ inch, respectively, making, with the full-size scale at *G*, eleven scales in all. It will be observed that the numbers of the feet of these scales do not start at the end of the instrument, but at the first division from the end. Thus, on a scale of 3 inches to the foot, the zero mark is placed at *C* and the first foot is measured to *D*. This is done so that the feet and inches may be laid off independently with one reading of the scale.

48. Scale Drawing.—For making a drawing to scale, any graduated rule may be used; if, for instance, a line 4 feet 6 inches ($4\frac{1}{2}$ feet) is to be drawn to a scale of 3 inches to the foot, it is only necessary to multiply $4\frac{1}{2}$ by 3, which gives $13\frac{1}{2}$ inches as the reduced length of the line; then this length may be measured with an ordinary rule. But it saves much time and labor to use a scale that is actually graduated to the required scale of the drawing; that is, so divided that any number of feet and inches may be read off the scale, without any calculation. An illustration will show how these scales are used. Fig. 27 represents part of a scale of $\frac{3}{4}$ inch to the foot. Each of the distances, *0-1*, *1-2*, *2-3*, etc., is equal to $\frac{3}{4}$ inch, and represents, therefore, 1 foot. The graduated space at the left is also equal to $\frac{3}{4}$ inch, but is divided into 12 parts, each part representing 1 inch. Each of these inch divisions is then divided into 2 equal parts to represent half inches.

To lay out a distance of, say, 3 feet $10\frac{1}{2}$ inches from a given point *b'*, draw a line *a' b'* in the given direction and passing through *b'*. Lay the scale on the line so that the division marked 3 on the scale will coincide with the point *b'*;

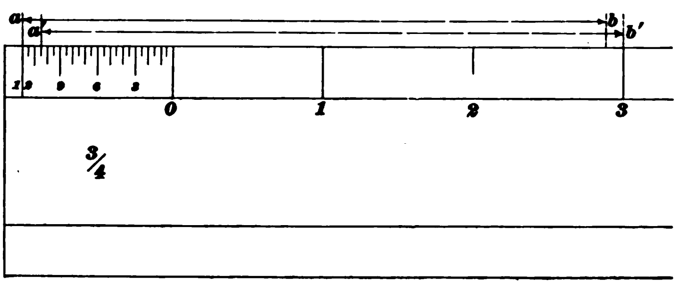


FIG. 27.

then, counting from 0 to the left, lay off a distance equal to $10\frac{1}{2}$ of the small divisions. The line *a' b'* thus obtained will be the required line.

If, on the contrary, the line is already drawn, and, knowing the scale of the drawing, the actual length of the line is required, proceed as follows: Place the end of the graduated division (that is, the end marked *12*) on the end of the line, as shown at *a b*, and notice between what divisions the other end falls; in this case, *b* falls between *2* and *3*; then move the scale to the left, so as to make *3* coincide with *b*; this gives 3 as the number of feet; the number opposite the other end of the line, on the graduated division, will then give the number of inches.

The student should have a flat or a triangular scale with scales of 1, $1\frac{1}{2}$, and 3 inches to the foot.

EXERCISES.

1. Draw, in any convenient direction, a line representing a distance of 3 feet 6 inches, to a scale of $\frac{3}{4}$ inch to the foot. At one extremity draw a perpendicular to it, and at the other extremity draw a line making an angle of 60° with the given line and produce it to meet the perpendicular.

2. Take any convenient point within the triangle formed by the preceding construction, and through it draw lines parallel, respectively, to each of the three lines first drawn.

3. Through each of the vertexes of the triangle of Exercise 1 draw parallels to the opposite side, and prolong these parallels until they meet, forming another triangle.

NOTE.—In general problems and exercises, lines are (unless otherwise stated) supposed to be of indefinite length, and extend in both directions from any of their points.

4. Measure the sides of the larger triangle and compare them with the sides of the smaller triangle. If the drawing has been done correctly, each side of the larger triangle is twice as long as the corresponding parallel side of the smaller triangle.

5. Draw a line to represent a distance of 2 feet 8 inches to a scale of $\frac{3}{4}$ inch to the foot. At one extremity draw a dotted line perpendicular to it, making it 1 foot long to the same scale. At the other extremity draw a dotted perpendicular line of the same length as the first one, and connect the ends of the two perpendiculars by a long-dash, or dimension, line.

6. Draw a line of any convenient length, and at one extremity draw lines making angles of 15°, 75°, 105°, 120°, and 165° with it. (Notice that 15° is the difference between 45° and 30°; 105°, the sum of 60° and 45°; 120°, the sum of 60° and 60°, or of 90° and 30°; 165°, the difference between 180° and 15°.)

7. Draw a line nearly horizontal, and at four different points draw lines making with it angles of 30°.

DRAWING PENS.

49. For the inking of drawings, a special instrument called a **ruling, drawing,** or **right-line, pen** is used. It consists of a handle *a b*, Fig. 28, made of wood, ivory, rubber, or any other suitable material, and two steel blades *a c* with rounded points, whose distance apart can be adjusted by a setscrew. The distance between the ends of the blades determines the width of the line drawn. The blades should be of the same length and have well-rounded and thin (but not sharp) ends. They should be slightly curved, but not too much.

The pens most used have a joint at *a*, so that one of the blades *a c*, Fig. 28, can be turned open for cleaning or sharpening. Compasses also have this arrangement. Many draftsmen, however, prefer pens having both blades fixed, as they are more rigid than those with a hinged blade.

FIG. 28.

The student that wishes to ink his drawings may buy a 5½-inch pen with or without a hinged blade. We recommend him to get one with a hinged blade.

50. To Sharpen the Drawing Pen.—When the ruling, or compass, pen becomes badly worn, it must be sharpened. For this purpose a fine oilstone should be used. If an oilstone is to be purchased, a small, flat, close-grained stone should be obtained, those having a triangular section being preferable, as the narrow edge can be used on the inside of the blades in case the latter are not made to swing apart so as to permit the use of a thicker edge.

The first step in sharpening is to screw the blades together, and, holding the pen perpendicular to the oilstone, to draw it back and forth over the stone, changing the slope of the pen from downwards and to the right, to downwards and to the left, for each movement of the pen to the right and left. The object of this is to bring the blades to exactly the same length and shape and to round them nicely at the point.

This process, of course, makes the edges even duller than before. To sharpen, separate the points by means of the screw, and rub one of the blades to and from the operator in a straight line, giving the pen a slight twisting motion at the same time, and holding it at an angle of about 15° with the face of the stone. Repeat the process for the other blade.

To be in good condition the edges should be fairly sharp and smooth, but not sharp enough to cut the paper. *All the sharpening must be done on the outside of the blades.* The inside of the blades should be rubbed on the stone only enough to remove any burr that may have been formed. Anything more than this will be likely to injure the pen. The whole operation must be done very carefully, bearing on lightly, as it is easy to spoil a pen in the process. Examine the points frequently, and keep at work until the pen will draw both *fine* lines and *smooth* heavy lines.

51. Before using a drawing pen, see that the ends are in proper shape and condition. *Never screw the blades together, and always wipe them clean immediately after using.*

The pen should be held as nearly perpendicular to the board as possible, so that both blades will bear equally on the paper when drawing. See Fig. 29. Only when both blades rest on

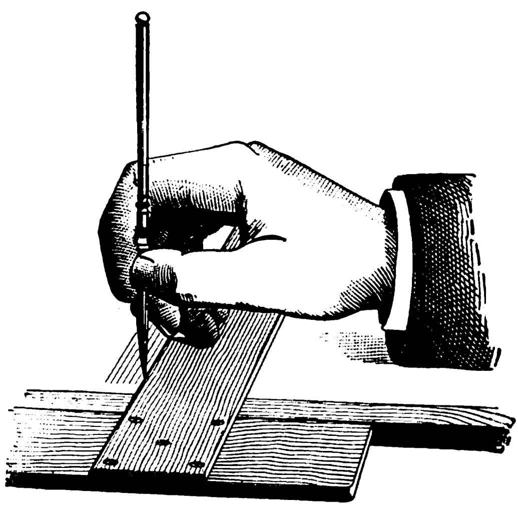


FIG. 29.

the paper, is it possible to draw smooth lines. The flat side of the blade should be against the edge of the ruler, and the head of the adjusting screw should preferably (although not necessarily) be on the outside. Too much pressure should not be exerted, as this will cut the paper and wear out the pen in a short time. The pen should be handled as if it were a very light object. The pen must not be pressed against the edge of the ruler, for when the pen is pressed against the ruler, the blades close together and make the line uneven. The edge of the ruler should serve only as a guide.

52. Too much ink should not be put between the blades, otherwise it may run out and make a blot. Put the ink between the blades above the points, so that no ink will be on the outside of the blades. For this purpose an ordinary writing pen may be used. Dip the writing pen into the ink and then pass the end of it between the blades of the drawing pen. Bottles of liquid ink have a short quill fixed in the stopper, which is very convenient for filling the pen. Never hold the pen over the drawing while it is being filled. When using liquid ink, do not put the cork on the table or board after filling the pen, but replace it in the bottle.

A new pen, as a rule, does not take the ink well and should be moistened with the tongue before using, or, better still, be well washed in clean water.

53. By tightening or loosening the setscrew between the blades, lines of different thicknesses may be drawn. For every pen there is a certain thickness beyond which it is not advisable to go, as the ink may run over and spoil the drawing. Otherwise, the ordinary thickness of lines depends on the size of the drawing. When an extra heavy line is to be drawn, as for a heavy border, it is made by drawing a series of heavy lines each below and along the edge of the preceding one.

54. After inking in a drawing, all pencil lines and marks should be rubbed out. For taking out ink lines an ink eraser may be used, or a scratching knife with a curved blade. If the latter is used, only the upper films of ink should be scratched out, and the rest taken out with the rubber.

EXERCISE.

If the student wishes to learn inking, he may begin by drawing several pencil lines of different lengths and in different positions, and then inking them in, making the ink lines of different thicknesses, some full, some dotted, and some broken.

COMPASSES.

55. Compasses are instruments used for drawing arcs and circles. They consist of two parts, called **legs,** jointed together at one end *a*, as shown at (*a*), Fig. 30. This joint should be adjustable, so that it can be made stiff enough to prevent the legs closing when the compasses are being

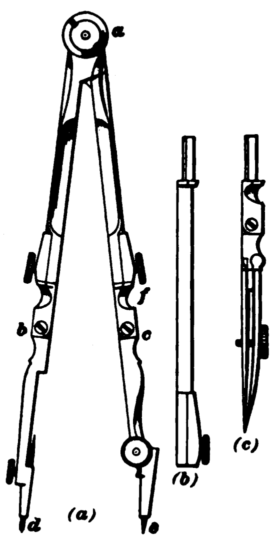


FIG. 30.

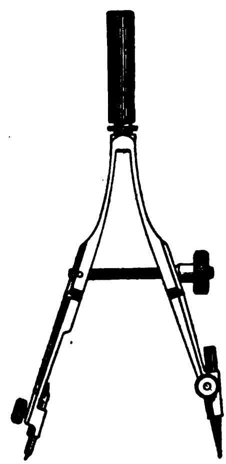


FIG. 31.

used, and yet not so stiff as to make it difficult to open or close the legs with one hand. When the joint gets too loose, it may be tightened with the small screwdriver that always accompanies a case of instruments.

The joints *b* and *c* at about the middle of each leg are for the purpose of bending the legs, so that the points of the compasses may always be kept nearly perpendicular to the paper. The leg *b d* ends in a sharp point, which is fixed on the paper. The other leg, carrying the pencil or the pen, turns about this fixed point, which is the center of the circle described. In cheap instruments the point *d* is made solid with the leg, that is, the leg is simply sharpened to a point; but all good instruments have a separate **needle-point** arrangement. The leg *b d*, instead of being sharpened to a point, is provided at the end with a groove, into which a fine needle or a small rod ending in a sharp needle-like point *d* can be put and fastened by means of a setscrew.

The other leg is fitted with a socket, into which may be inserted either a pen, Fig. 30 (*c*), or a pencil point. In Fig. 30 (*a*), the leg holds the pencil point *f e*.

56. A **lengthening bar** is shown at Fig. 30 (*b*); it is used for making larger circles than can be drawn with the ordinary parts of the compasses. It is nothing but an auxiliary long leg having a peg, or tenon, at one end, that fits into the socket at *f*, and a socket at the other end into which the ordinary pen or pencil point can be fitted, thus increasing the length of the moving or describing leg.

For this Course a pair of compasses is required; they should be 5 inches to 6 inches long when closed and have needle points.

57. Spring compasses, Fig. 31, or **spring bows,** are small compasses used for describing small circles up to about 1 inch or 1¼ inches in diameter. They are not strictly needed in this Course, but the student is advised to buy a pair.

58. Beam compasses are used for describing circles of very large radii. The instrument consists of a beam *a*, Fig. 32, along which the two pieces *b* and *c* slide; these pieces are provided with clamp screws by which they can be fixed on the beam. The piece *b* carries a needle point, and at *c* either a pencil holder or a pen may be inserted. By fixing *b* and sliding *c* along the beam, the instrument may be set to any radius.

Although beam compasses are very useful to the professional draftsman, they are not required in this Course.

59. Machine oil should never be used for the purpose of lubricating the joints of compasses. The makers put a little tallow or beeswax in the joints and this lasts for years. Machine oil makes the joints move too freely, or if they are tightened, it will cause them to move too stiffly,

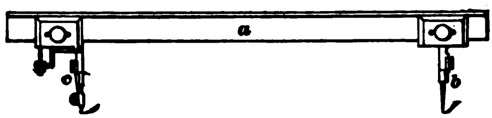


FIG. 32.

or in a jerky manner, instead of with a smooth and steady action. The points of dividers or of compasses that are not fitted with needle points should not be so fine and sharp as to pierce deeply into the paper. They must be just sharp enough to keep from slipping.

60. To describe circles with the pencil compasses, the point of the lead should be set a little shorter than the needle point. For ordinary compasses, the lead should be sharpened with a flat edge. For spring bows, a round point is perhaps as good as a flat one. If a flat point is used, it should be made narrow and in all cases *the lead should be so fastened that when circles are struck in either direction, but one line will be drawn with the same radius and center.*

61. The tendency of beginners is to use both hands in opening and closing the compasses; this should be avoided and the student should learn from the first to use the compasses with one hand. To open or close the compasses with one hand, hold them, as shown in Fig. 33, with the needle-point leg resting between the thumb and fourth finger, and the other leg between the middle finger and the forefinger. The point of the lead must be held a little above the paper in opening or closing the compasses, while the needle point is fixed at the point that is to be used as the center of the circle. It is very often necessary to describe a circle of which the center is at a given point, and the circumference passes through another given point. To describe this circle,

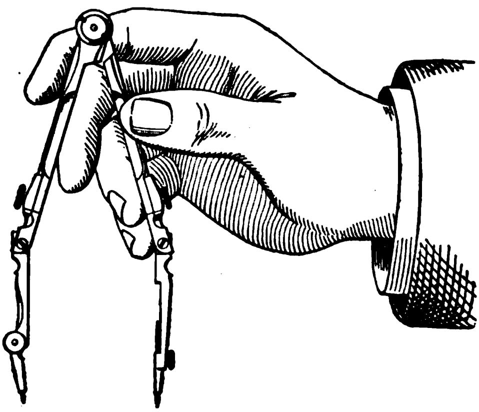


FIG. 33.

it is necessary to locate the needle point of the compasses at the given center and the pencil point at the given point through which the circumference is to pass. For this purpose hold the compasses as shown in Fig. 33; incline them until the hand is steadied by the under side of it resting on the paper, and place the needle point at the given center; then the pencil point can be easily moved out or in until it is at the other given point. Having thus located the points of the compasses at the given points, shift the hand, and in drawing the circle hold the compasses lightly

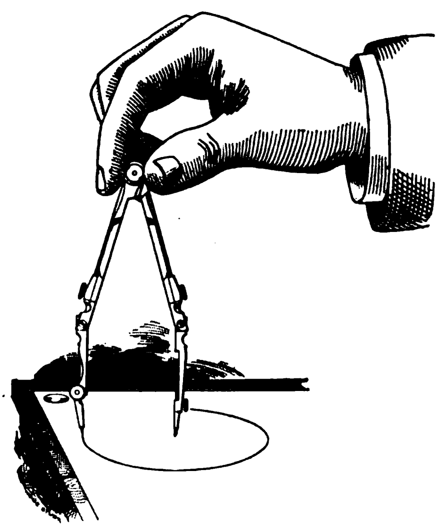


FIG. 34.

at the top, between the thumb and forefinger, or thumb, forefinger, and middle finger, as in Fig. 34. When the lengthening bar is used, both hands may be used to adjust the compasses.

Before drawing a circle, the accuracy of the adjustment should be tested by holding the compasses with the right hand and letting the lead point down at the point through which the circumference is to pass and describing a very faint and very small arc extending on both sides of this point, noticing carefully whether the arc passes exactly through the point; if it does not, readjust the compasses.

62. In describing a circle, the leg with the needle point should be kept nearly upright, leaning but a little toward the points successively occupied by the pencil point. As little pressure as possible should be exerted on the needle point, otherwise the needle will make a hole in the paper. Keep the needle point adjusted so that it will be perpendicular to the paper, when drawing circles, and *do not bear upon it*. Nor must too much pressure be exerted on the pencil point, or it will wear away too quickly, besides making a hollow in the paper, which will show badly in case a circle or arc has to be rubbed out. Another effect of too much pressure on the pencil is that the compasses will spring open more or less, and thus make an imperfect circle. This is a most important matter in connection with the describing of circles and should be carefully borne in mind.

In order that both the needle and pencil points may be nearly perpendicular to the paper, before adjusting the compasses, the legs should be bent at their joints to make them perpendicular to the paper. If necessary, these joints must be set a second time before finally adjusting the compasses accurately. This applies especially when the pen is used in the compasses for inking in. See that the ends of both blades of the pen touch the paper when the pen is open about $\frac{1}{32}$ inch; then it will make sharp, clear, even lines.

In drawing *concentric* circles, that is, circles having the same center, it is necessary to keep bending the legs in proportion as the radii are increased, so as to keep them approximately perpendicular to the paper.

EXERCISES.

1. Draw several circles of different radii, some with different centers, some with the same center, beginning with circles of small radii, say ½ inch, and gradually increasing the radii.

2. Draw a horizontal line and on it lay off eight successive half inches. Set the needle point at the 2-inch mark, and, with the pencil point at the end of the line, describe a circle. Keeping the same center, describe circles through the 3½-inch, the 3-inch, and the 2½-inch marks. Still keeping the same center, describe another series of circles, beginning at a point ½ inch from the center and increasing the radii by half inches.

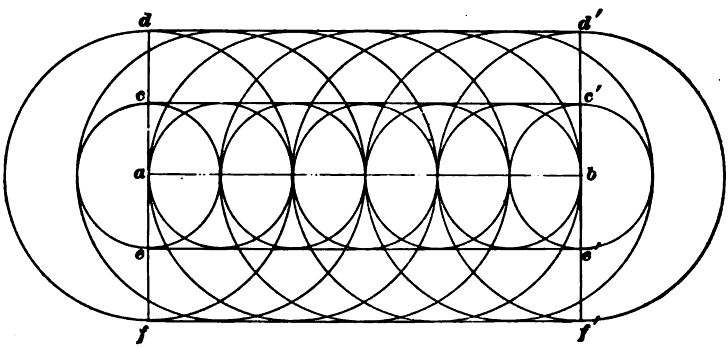


FIG. 35.

3. Draw a line *a b*, Fig. 35, 3 inches long and divide it into six equal parts. From each of the points of division, including the first and the last, describe two circles with radii equal to ½ inch and 1 inch, respectively. Describe the ½-inch circles first, then the others. Through *a* and *b* draw *d f* and *d' f'* perpendicular to *a b*, intersecting the circles described from *a* and *b*, respectively, at *c*, *d*, *e*, and *f*, and *c'*, *d'*, etc. Draw *d d'*, *c c'*, etc. and see if these lines are *tangent* to the 1-inch and ½-inch circles, respectively; that is, see if they *just touch* these circles, without going inside. Also, see that those circles that pass through the same center do not overlap, but just touch each other.

4. The student may, if he wishes, remove the pencil point of the compasses, put in the pen, and ink in the preceding figures.

DIVIDERS.

ıders are similar to compasses, except that they ...olid points and no joints in their legs, Fig. 36. ...e used for laying off distances on a drawing or for ...ng straight lines or circles into equal parts. Some ...ıders, called **hair-spring dividers,** have a spring adjust...ment in one leg for opening or closing the legs a very small amount, thus enabling the draftsman to make the final adjustment very accurately.

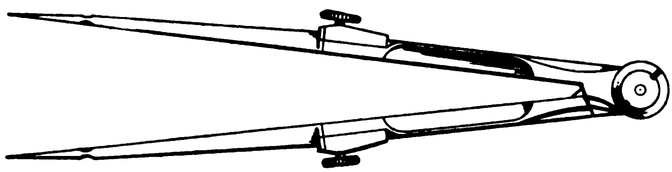


FIG. 36.

The points of the dividers should be *very* sharp, so that they will not punch holes in the paper larger than is absolutely necessary to be seen. Compasses are sometimes furnished with two steel divider points, besides the pen and pencil points, so that the instrument may be used either as compasses or dividers.

The student requires a pair of dividers about the same size as the compasses; 5-inch dividers will be large enough. The spring adjustment is very convenient, but not strictly necessary.

64. To transfer a given distance or length from one line to another.

This may be done by means of a graduated rule or scale. The distance to be transferred is first ascertained by simply laying the scale along the line, so that its zero end will coincide with one of the ends of the line, and noticing the division on which the other end of the line falls; the scale is then placed on the second line, with its zero where the transferred length is to begin, and a mark made at the point where the division of the scale observed before falls. Usually, however, no division of the scale will coincide exactly with the end of the line. In this case, lay the edge of a strip of paper along the line and mark the extremities of the latter, whose length may thus be transferred anywhere. Long lines are transferred in this manner or by means of a beam compass. For transferring short lines, dividers are the most convenient instruments to use.

65. In measuring a line with the dividers, one of the points of the instrument is placed at one end of the line and kept there by a steady but light pressure, so that the point will neither move nor cut a hole in the paper; then the instrument is opened until the point of the other leg reaches exactly to the other end of the line. The dividers should be held upright, and should be opened and closed in the same manner as the compasses. They should not be opened wider than about a right angle; if this is not enough to include the whole line to be measured or transferred, use one of the methods already mentioned, or the line may be spaced into a number of equal parts, and transfer the same number of these equal parts. In using the dividers to space a line or circle into a number of equal parts, hold the dividers at the top between the thumb and forefinger, as when using the compasses, and step off the spaces, turning the instrument alternately to the right and left. If the line or circle does not space exactly, vary the distance between the divider points and try again; so continue until it is spaced equally. When spacing in this manner, great care must be exercised not to press the divider points into the paper; for, if the points enter the paper, the spacing can never be done accurately. The student should satisfy himself of the truth of this statement by actual trial.

When several lengths are to be transferred from one line to another, a convenient way is to mark them all at the same time on a strip of paper placed along the line, and transfer them all together.

66. It must be borne in mind that the preceding rules apply to distances represented by lines in a drawing, the exact lengths of the lines not being known. But, if the exact length to be transferred is known, the measuring scale should be used. Thus, if we know that a given line represents a distance of exactly $3\frac{1}{2}$ inches, and wish to draw a line of the same length, it is better *not* to *transfer* the length from one line to another, but to draw and measure each line independently.

67. To bisect a given straight line.

There are several ways of doing this:

1. *By a Graduated Rule or Scale.*—Lay the scale with its zero mark (this is not necessary, but convenient) on one end of the line, and read off the division at the other end; divide the number of divisions by 2, and lay off the resulting length from the zero mark. For instance, let *a b*,

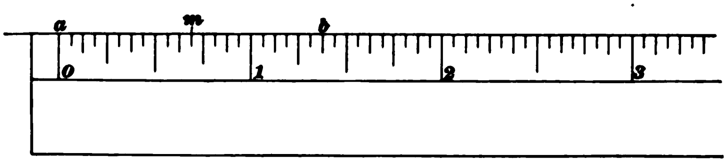


FIG. 37.

Fig. 37, be the line to be divided; by laying the scale on it, with its zero on *a*, it is seen that *b* lies at a distance of $1\frac{3}{8}$ inches, or $\frac{22}{16}$ inches, from *a*; one-half of this is $\frac{11}{16}$, which laid off from *a* determines the middle point *m* of the line.

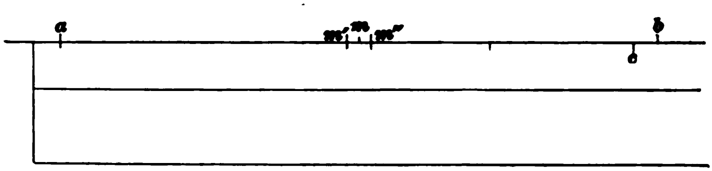


FIG. 38.

If the scale is not graduated to sixteenths, lay off from *a* $\frac{5}{8}$ inch, which is equal to $\frac{10}{16}$ inch, and mark, by eye, the point *m* midway between the $\frac{5}{8}$ division and the $\frac{6}{8}$ division of the scale.

If the point *b* does not coincide with any division, or if the division is not even (divisible by 2), the general way of proceeding is as follows: Place the scale on the line as before, with its zero on *a*, Fig. 38; note the point *c*, which is the even division of the scale nearest *b*; mark the middle point *m'* of the line *a c*; slide the scale to the right until *c* coincides with *b*, when the division of the scale that was at *m'* will be at *m"*; mark, by eye, the point *m*, half-way between *m'* and *m"*; this will be the middle point of *a b*.

2. *By Folding a Strip of Paper.*—Lay the edge of a strip of paper along the line, and mark the ends of the line on it; by folding it in the middle a double strip is obtained equal in length to one-half the line. This method is not accurate, and should not be used except when no other method can be employed.

3. *By a Pair of Dividers.*—Let *a b*, Fig. 39, be the distance to be divided into halves. Fixing one point of the dividers at *a*, open the instrument until the other point reaches to a point *c* about the middle of the line; make a faint mark *c* with the free point. Keeping the instrument at the same opening, lay off *a c* from *b* to *d*; the middle point *m* of *c d* will be the middle point of the line. The point *m* is marked by the eye, and then tested by opening the dividers from *a* to *m* and seeing if, on swinging them about *m*, the point of the dividers that was at *a* falls on *b*.

FIG. 39.

68. To divide a straight line into four, eight, sixteen, etc. equal parts.

First bisect the line, then bisect each division until the required number of divisions is obtained.

69. To divide a line into any number of equal parts.

There are two methods:

1. *By a Scale.*—Measure the line; if the number of divisions is divisible by the required number of parts, each

part is at once found by performing the division. Otherwise, proceed as in 1 of Art. **67**; that is, take the next lower division on the scale that is divisible by the required number of parts; divide the number at this division of the scale by the number of parts into which the line is to be divided; lay off a distance equal to the quotient, and add to it one of the small parts obtained by dividing, by eye, the distance from the end of the line to the division of the scale used, into the required number of equal parts. Thus, if it is required to divide a line $6\frac{5}{8}$ inches long into nine equal parts, proceed as follows: Lay the scale on the line, with its zero end coinciding with one end of the line, and if the scale is divided into sixteenths, note that the other end of the line is at the $6\frac{10}{16}$ division of the scale. Now the $6\frac{3}{16}$, or $\frac{99}{16}$, division is the nearest division of the scale that is exactly divisible by 9. The distance between this $6\frac{3}{16}$ division of the scale and the end of the line is $6\frac{10}{16}'' - 6\frac{3}{16}''$, or $\frac{7}{16}''$. Dividing, by eye, $\frac{7}{16}''$ into nine equal parts, we see that each of the nine equal parts of the line is $\frac{1}{9}$ of $\frac{99}{16}'' + \frac{1}{9}$ of $\frac{7}{16}''$ long. That is, that each part is a little less than $\frac{12}{16}''$ long.

2. *By a Pair of Dividers.*—Let $a\,b$, Fig. 40, be a line to be divided into five equal parts. Estimate, by eye, about what one-fifth of the line is, say the distance from a to c, and take that distance in the dividers; step off five of these divisions on the line. If the point marked at the fifth step is not b but another point, as h, estimate one-fifth of the

FIG. 40.

distance $h\,b$ and open or close the dividers a distance equal to this one-fifth and try again. Proceed in this way until the line is divided into exactly five equal parts. When the point h falls beyond b, close the dividers one-fifth of $h\,b$, and when the point h falls short of b, open the dividers one-fifth of $h\,b$.

EXERCISES.

1. Draw lines of different lengths and divide them into halves, both with the scale and with the dividers.

2. Draw a horizontal line to represent a distance of 2 feet 9 inches to a scale of $1\frac{1}{2}$ inches to the foot. At each extremity make an angle of 30°, the one toward the right, the other toward the left, producing the two sides until they meet. Divide one of these two sides into three equal parts with the dividers, and the other into three equal parts by means of the scale. See if the divisions on one side are equal to those on the other, as they should be.

3. Draw a right-angled triangle of any convenient size; divide the three sides into halves and join the points of division. See, by means of the triangle, if each of the lines thus drawn between the middle points of two sides is parallel to the third side of the triangle, as it should be. Compare the lengths of the parallel sides of the four small triangles with each other and with the side of the large triangle that is parallel to them. What kind of triangles are the small triangles? (To draw a right-angled triangle, draw two lines perpendicular to each other, as a horizontal and a vertical, and join any point on the one to any point on the other.)

4. Draw any triangle, calling the vertexes *a*, *b*, and *c*. Divide *a b* into eight equal parts, and through the points of division draw parallels to *b c*. If the work has been done properly, the distances cut off by these parallels on *a c* will be equal *to one another* (not to the parts on *a b*).

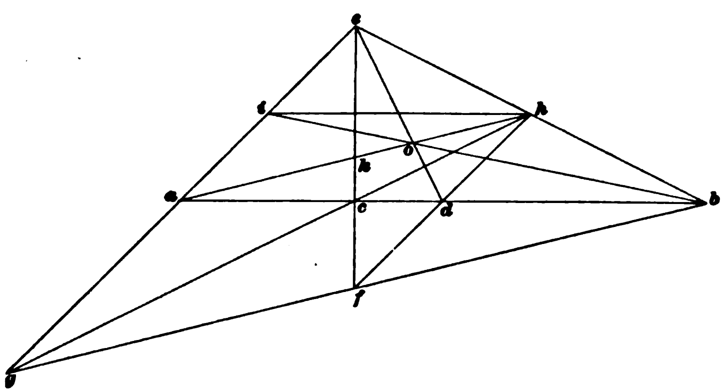


FIG. 41.

5. Draw a horizontal line *a b*, Fig. 41, of a length equal to $4\frac{15}{16}$ inches and divide it into six equal parts. Lay off two of these parts from *a* to *c*, and one from *c* to *d*, marking these points by very faint marks. See if *d* is exactly in the middle of *a b*, as it should be. Use the dividers

for checks of this kind. At *c* draw a perpendicular to *a b* and take *c e* equal to $1\frac{9}{16}$ inches. Divide *e c* into four equal parts, and from *c* lay off one of those parts to *k*, and below *a b* two parts to *f*. Draw *e b* and *e a*; produce *e a* to *g*, making *a g* equal to *e a*. Mark the middle points *h* and *i* of *e b* and *e a*. Draw *g b*; if the work is correct, this line will pass through *f*. Draw *e d* and divide it into three equal parts, and lay off one of them from *d* to *o*. Draw *b i* and *a h*; these lines should pass through *o*, and, besides, *o h* should be one-third of *a h*, and *o i* one-third of *b i*; also, *a h* should pass through *k*. Similarly, the line *g h* should pass through *c*, and the line *f h* through *d*. See if *i h* is equal and parallel to *a d*, and *a h* equal and parallel to *g f*.

6. Draw a horizontal line *a b*, Fig. 42, equal to $4\frac{1}{16}$ inches. Divide it (with the dividers) into eight equal parts, as shown. Through *b* draw *b y*, making an angle of 30° with *a b*; and through *a* draw *a c* perpendicular to *b y*. Through *g*, the third point of division on *a b*, draw *g h* parallel to *a c*, and through the fourth point of division, *d*, draw *d x* so that the angle *x d a* is 60°. If the construction is correct, *d x* will

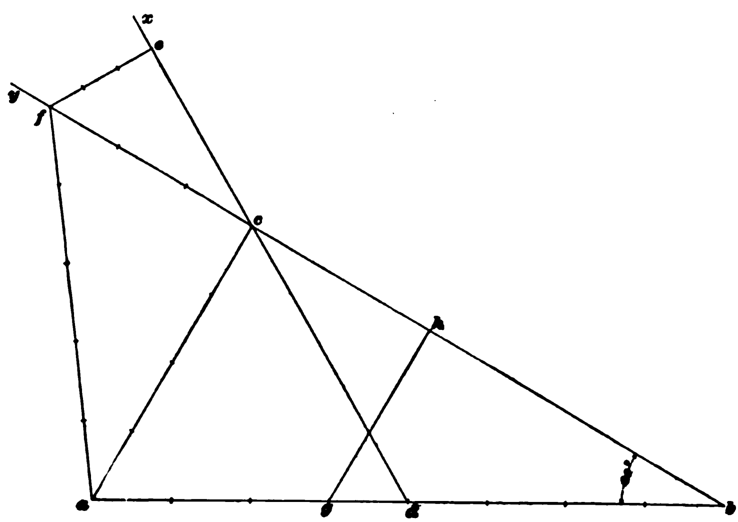


FIG. 42.

pass through *c*. On *d x* take *c e* equal to *c h*, and draw *e f* perpendicular to *d x* at *e*. Draw *f a*. Divide *a c* into quarters, *a f* into fifths, *f c* into thirds, and see if each of the parts thus obtained is equal to each of the eight divisions of *a b*, as it should be. Divide *f e* into thirds, and see if each of the parts thus obtained is just one-half of any of the divisions obtained before for the other lines. Also, see if the three lines *a c*, *c d*, and *d a* are equal, as they should be.

IRREGULAR CURVES.

70. Irregular curves are thin, flat pieces of wood, celluloid, or hard rubber cut out in various shapes, with curved outlines of different forms and curvatures. Figs. 43 and 44 show two curves of this kind. They are used as guides for drawing curves other than arcs of circles.

FIG. 43.

Before the irregular curve can be used, a series of points through which the curve is to pass, must be found by some method. The line is then drawn through these points by using such parts of the irregular curve as will pass through several points at once, the curve being shifted from time to time as required. It is usually difficult to draw a smooth

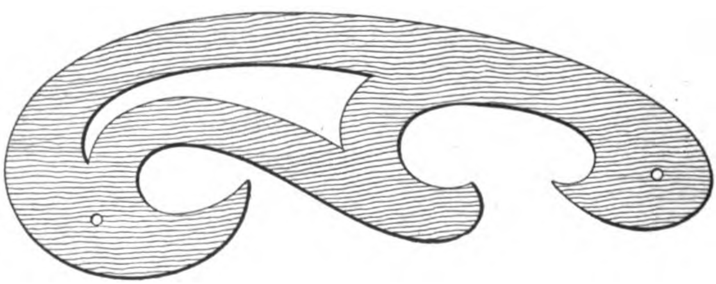


FIG. 44.

continuous curve. The tendency is to make it curve out too much between the points, thus giving it a wavy appearance, or else to cause it to change its direction abruptly where the different parts join, making angles at these points. These defects may be avoided for the most part by always fitting the curve to at least three points, and, when moving it to a new position, setting it so that it will coincide with part of the line already drawn.

71. To construct an ellipse whose axes are given.

Draw the two axes *a b* and *c d*, Fig. 45, perpendicular bisectors of each other. From *c*, with a radius equal to *a o*, describe arcs cutting *a b* at *f* and *g*. These are the foci of the ellipse and are the points at which the pins are set for the loop construction of an ellipse. With the pins set at these points construct the ellipse, according to the method given in *Arithmetic*.

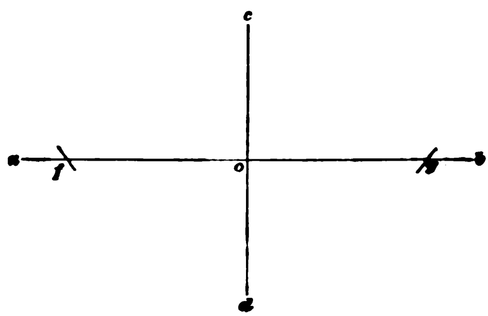


FIG. 45.

The method of constructing an ellipse given in *Arithmetic* is very convenient for drawing large ellipses, especially on boards, on the ground, etc. When the ellipse is very long and narrow, however, it is better to construct it approximately by circular arcs, as explained in another section of this Course, or "by points," in the following manner.

72. Take a piece of stiff paper *m n*, Fig. 46, and mark on it, starting from any point *o'*, the distances *o' c'* and *o' a'*, equal, respectively, to half the minor and half the major axis. Place the strip so that *c'* will fall on any point of the major axis, and turn it until *a'* falls exactly on the minor axis, as shown in Fig. 47; then the position *o'* will determine one point of the ellipse; make a small pencil

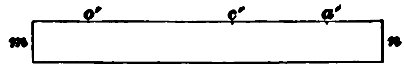


FIG. 46.

mark (a cross or a dot) at that point, and proceed to find other points by placing *a'* and *c'* on the axes in different

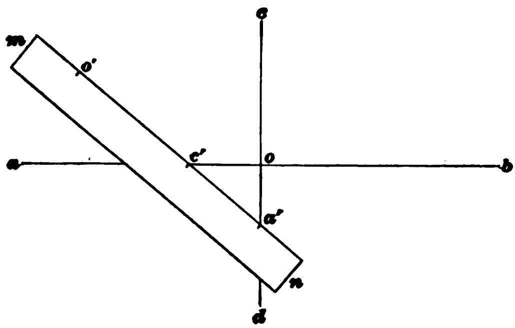


FIG. 47.

positions. Having found a sufficient number of points (the more the better) trace the curve through them, either freehand or with an irregular curve.

The use of the irregular curve is illustrated in Fig. 48, where *a*, *b*, *c*, *d*, etc. are points through which a curve is to be drawn. It is advisable to sketch in the curve freehand

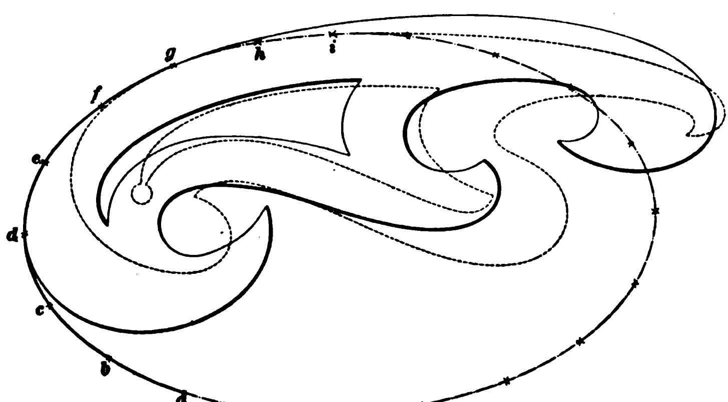


FIG. 48.

first, so as to have a general idea of its direction at every point. Lay the curve in a position like that shown by the full lines, so that *at least three consecutive points*, as *d*, *e*, and *f*, will lie against the edge of the curve; this is accomplished by sliding the curve along and trying different parts of it until the required position is obtained. If possible, more than three points should be obtained lying against the curve, but three are enough; in Fig. 48, there are four points, *d*, *e*, *f*, and *g*. Now draw, by moving the pencil along the edge, a portion of the curve from a point a little to the right of *d*, to a point near *g*, between *f* and *g*. Shift the irregular curve so that its edge will pass through *g*, *h*, and *i*, and coincide for some distance with the last part of the line just drawn. Then draw another part from the end of the one first drawn to a point near *i*, between *h* and *i*. Readjust the curve for the succeeding points and proceed as before.

73. When inking a curved line with the irregular curve, great care should be taken that the blades of the pen are constantly kept flat against the edge of the curve; that is, the pen should be at every point in the position it would have if a straight line were to be drawn just touching the curve at that point.

Another thing that must be carefully done is the joining of two successive portions of the curved line. As explained before, the edge of the curve should always coincide for a short distance with the part previously drawn; this short distance should be retraced, and the motion of the pen *continued* through the next portion, and so on.

EXERCISES.

1. Draw, by means of a string, an ellipse having axes of 3½ and 2½ inches, respectively.

2. Draw an ellipse of the same dimensions by points.

3. Draw three *concentric* (having the same center) ellipses, the axes of the outer one being 5 and 4 inches, those of the middle one 4½ and 3½ inches, and those of the inner one 4 and 3 inches. Draw them in pencil by means of a string.

4. Draw, by points, an ellipse having axes of 6 inches and 1 inch, using the irregular curve.

PROTRACTORS.

74. A **protractor** is an instrument used for measuring and laying off angles. It consists commonly of a semicircular plate made of metal, horn, or paper, the circumference of which is divided into 180 equal parts, or **degrees.** In large protractors, each degree is subdivided into halves and sometimes into quarters. Fig. 49 shows a common form of protractor; the point *c* on the line *a b* through the zero marks is at the center of the semicircle.

75. To lay off an angle by means of a protractor.

Let *c*, Fig. 49, be a point of the line *a b* from which an angle of $63\frac{1}{2}°$ is to be laid off. Place the center of the protractor exactly on the point *c*, and make the line *d e* of the protractor (which passes through the *zero* points, or from one *zero* mark to the opposite 180° mark) coincide with the

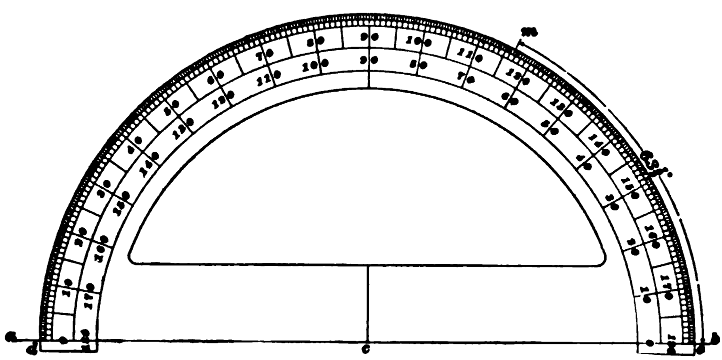


FIG. 49.

given line *a b*. If the instrument is graduated to read half-degrees, take either from *e* or from *d*, according to the desired direction, the required $63\frac{1}{2}°$, and make a small mark there, as shown at *m*. If the protractor does not read to half-degrees, observe the 63° and the 64° divisions, and, judging by the eye, mark the point *m* midway between them. If the line *a b* is so short that its ends do not project beyond the protractor, and the latter is not transparent, produce *a b* as much as is necessary. A line from *c* to *m* will make with *a b* the required angle of 63½°.

76. To divide a circular arc into equal parts.

Let *a b*, Fig. 50, be the arc to be divided into two equal parts. Draw the two radii *o a* and *o b* and produce them. Lay the protractor with its center at *o* and its zero on *a*, or *o a* produced, as at *c*; observe the division of the protractor falling on *o b*, or *o b* produced, as at *d*; if this is an even number, divide it by two, and mark off the resulting number of divisions from *c* to *n*; remove the protractor, lay the ruler or the triangle so that its edge will pass through *o* and *n*, and mark the point *m*, where it crosses the arc; then will *m* be the middle point of *a b*. If the number of divisions is not even, proceed as for the division of a straight line. Generally, however, the best way is to take the dividers and proceed by trial, as in the case of straight lines. This method has the advantage in that it is not necessary to know the center of the arc to be divided in order to use the dividers. Even when the protractor is used, the divisions should be tested with the dividers.

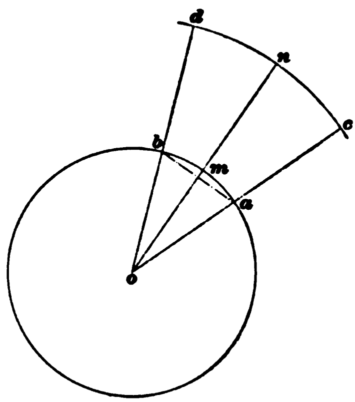


FIG. 50.

In a similar manner arcs are divided into any number of equal parts. For very large arcs or for whole circles, it is advisable to use the protractor for a first approximate value, and then complete the operation with the dividers.

When the center of the circle is known and the arc is to be divided into halves, draw the chord *a b*, and from *o*, with the triangles, draw *o m* perpendicular to *a b*; this perpendicular will pass through the middle point *m* of the arc. Likewise, the arc *a b* may be divided into quarters by bisecting $a\,m$ and $b\,m$, then into eighths, etc. Always test divisions with dividers.

There are some *special* cases in which the division of an arc can be accomplished by means of the triangles, as explained below.

77. To divide the circumference of a circle into two, four, or eight equal parts.

Any diameter, as $a\,b$, Fig. 51, will divide the circumference into halves. A diameter $c\,d$ perpendicular to the first will divide the circumference into quarters. Finally, through the center o draw $g\,h$ and $e\,f$, at 45° with $a\,b$ and in the directions shown; then will each of the arcs $g\,c$, $c\,e$, $e\,b$, etc. be one-eighth of the circumference.

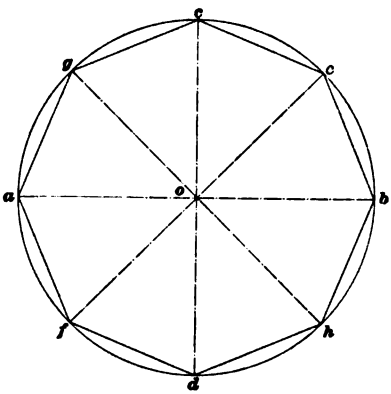


FIG. 51.

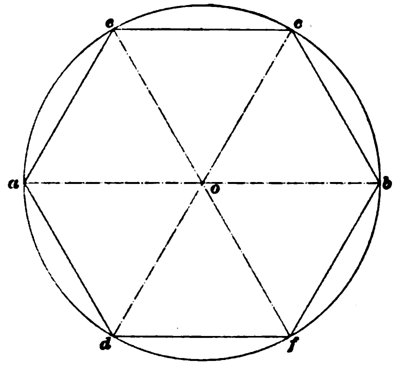


FIG. 52.

78. To divide the circumference of a circle into six equal parts.

There are two methods:

Either open the dividers equal to the radius of the circle and step off the circumference, beginning at any point. The end of the sixth space should fall exactly on the beginning of the first; if it does not, the dividers must have moved or the work has not been carefully done.

Or, draw a diameter *a b*, Fig. 52, and through *o* draw the lines *c d* and *e f*, making the angles *b o c* and *a o e* equal to 60°. Then each of the arcs *a e*, *e c*, etc. will be one-sixth of the circumference; for each of the arcs is 60°, or $\frac{1}{6}$ of 360°.

79. Regular inscribed polygons are constructed by joining the points of division of an equally divided circle. Thus, in Fig. 51 we have a regular inscribed octagon, and in Fig. 52 a regular inscribed hexagon. To inscribe a regular polygon of any number of sides in a given circle, it is only necessary to divide the circumference of the circle into the required number of parts and join the points of division.

EXERCISES.

1. Draw an arc *a b*, Fig. 53, with a radius of 5 inches, making it a little smaller than a quadrant, and divide it into two equal parts. Divide each of the halves thus found into thirds. Produce the arc to *c*, making *b c* equal to *a b* (which is done by stepping off the *chord a b* from *b* to *c*), and divide *b c* into sixths. See if one of the latter divisions goes exactly twelve times from *a* to *c*, and is equal to each of the six divisions in *a b*.

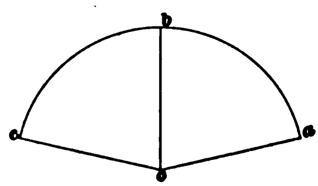


FIG. 53.

2. On a 4-inch circle (one having a diameter equal to 4 inches), lay off, from any convenient point, an arc of 46°, using the protractor. Divide this arc into four equal parts, first by means of the protractor, then by means of perpendiculars to the chords.

3. Draw a circle having a radius of 6 inches, and inscribe in it a square and a regular octagon.

4. Draw a circle having a *diameter* of 2 inches, and inscribe in it a regular hexagon and an equilateral triangle. (The latter is obtained from the hexagon by joining alternate points, such as *c* and *f*, *f* and *a*, *a* and *c*, Fig. 52.)

5. Draw a circle having a *diameter* of 3 inches, and in it inscribe a regular five-sided and a regular ten-sided polygon.

LETTERING.

80. Working Drawings.—Since it is desirable in many cases to designate certain portions of a drawing in a particular way, a conventional freehand alphabet is useful. Such an alphabet, together with the numerals, is shown in Fig. 54, and should be used on drawings in preference to

FIG. 54.

written characters. Until the student has obtained sufficient practice in lettering, he should draw horizontal lines in pencil to serve as a guide for the tops and bottoms of the letters (see Fig. 55). The outside lines should be $\frac{3}{32}$ inch apart for capitals, and the two lower lines $\frac{1}{16}$ inch apart for the small letters. The letters should be made to extend fully up to the top and down to the bottom of the guide lines. They must not fall short of the guide lines nor extend beyond them. Failure to observe this point will cause the lettering to look ragged, as in the second word in Fig. 55.

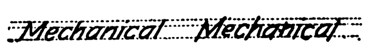


FIG. 55.

81. The letters and figures should have the slant indicated in Fig. 56, that is, three parts of base to eight of height. This inclination, as will be seen from Fig. 54, is given to the general direction of the letter, and in some cases applies to the center line of the letter rather than to its straight lines. It is very important that all the letters have the same inclination. For example, by referring to Fig. 57, it will

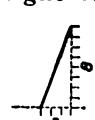


FIG. 56.

be seen that the backs of letters like *B*, *E*, *l*, *g*, *d*, *i*, *t*, etc. are parallel and slant the same way. This is also true of both sides of letters like *H*, *M*, *n*, *u*, *h*, *y*, etc. To aid in keeping the slant uniform, draw parallel, slanting lines across the guide lines, as in Fig. 57, and, in lettering, make the backs or sides parallel with these.

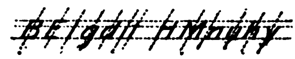


FIG. 57.

The lower-case letters—that is, the small letters—should be accurately spaced, and as little distance as possible allowed between the letters. Letters that extend above the line, as *b*, *d*, *f*, *h*, *k*, and *l*, should reach the height of capitals, while those that extend below the line, as *f*, *g*, *j*, *p*, *q*, and *y*, may be slightly shortened. A little practice in the formation of these letters will render the student quite expert in their use. A common sharp-pointed steel pen is used for this kind of lettering.

82. No particular stress is laid on the lettering in this Course, but the student will find it decidedly advantageous to be able to letter neatly and rapidly, and for any efforts that may be expended in this direction he will be well repaid by the satisfactory appearance presented in a neatly lettered drawing.

83. The alphabet shown in Fig. 58, called the **block letter,** is to be used for the large headings or titles of plates, as shown on the copy plates. The letters and figures are to be made $\frac{5}{16}$ inch high and $\frac{1}{4}$ inch wide, except *M*, which is $\frac{5}{16}$ inch wide, and *W*, which is $\frac{3}{8}$ inch wide. The thickness of all the lines forming the letters is $\frac{1}{16}$ inch, measured horizontally. The distance between any two letters of a word is $\frac{1}{16}$ inch, except where *A* follows *P* or *F*; where *V*, *W*, or *Y* follows *L*; where *J* follows *F*, *P*, *T*, *V*, *W*, or *Y*; where *A* either precedes or follows *T*, *V*, *W*, or *Y*. When such combinations occur, the upper and lower lateral extensions of adjacent letters should be in the same vertical line. The tops of these letters should be $\frac{3}{8}$ inch below the upper edge of the drawing, and the words should be so located as to leave the same distance on both sides of the title.

Since these letters are composed of straight lines, they can be made with the **T** square and triangle. In lettering the title of the drawing plates, the student should draw six horizontal lines in lead pencil, to represent the thickness of the letters at the top, center, and bottom; then, by using the triangle, he should draw in the width of the letters and

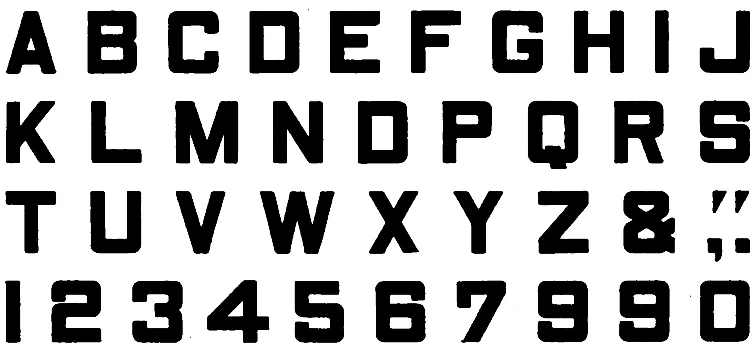


FIG. 58.

the spaces between them in lead pencil. Having the letters all laid out, he can very easily ink them in. Use the ruling pen for inking in the straight outlines of the letters, and a steel writing pen for rounding the corners and filling in the thickness of the lines. It is well to ink in all the perpendicular lines first, next the horizontal lines, and then the oblique lines.

84. For the convenience of students that do not desire to acquire proficiency in lettering, a copy of each plate title will be found printed on the reduced copies of the accompanying plates. If the student desires, he may cut out these titles and paste them in the proper position on the plates, although it is recommended that they be drawn and inked in, since the practice thus obtained will be valuable in future work.

PLATES.

DRAWING PLATE, TITLE: STRAIGHT-LINE FIGURES.

GENERAL INSTRUCTIONS.

85. The student is now prepared to draw his first examination plate, for doing which he should carefully follow the instructions given below. Copies of this and the second plate accompany the text, but to a reduced scale. The student should draw his figures larger, as directed, although in the same relative positions as those in the reduced plate.

86. Take a sheet of drawing paper 15″ × 20″, and fasten it to the drawing board in the manner previously described. A sheet of this size is used in order that the thumbtack holes may be cut out when the plate is finished; the extra margin is also very convenient for testing the pen, in order to see whether the ink is flowing well and whether the lines are of the proper thickness.

The size of each plate over all is to be 14″ × 18″, thus leaving a ½-inch margin on each side that is to be cut off. The plate is to have a border line ½ inch from each edge, thus making the size of the sheet on which the drawing is made 13″ × 17″.

Whenever any dimensions are specified, they should be laid off as accurately as possible. All drawings should be made as neatly as possible, and the penciling entirely finished before any part of it is inked in. Great care should be taken in distributing the different views, parts, details, etc. on the drawing, so that when the drawing is completed, one view will not be so near to another as to mar the appearance of the drawing. The hands should be perfectly clean, and should not touch the paper except when necessary. No lines should be erased except when *absolutely* necessary; for, whenever a line has once been erased, the dirt flying around in the air and constantly falling on the drawing will stick to any spot where an erasure has been made and render it very difficult, if not impossible, to entirely remove it. For this reason, all construction lines that are to be removed or that are liable to be changed should be drawn lightly, that the finish of the paper may not be destroyed when they are being erased.

87. For convenience of reference, points are marked by letters on the reduced plate, but the student need not put these on his paper. However, when the student does put any letters on his plate, he should take great care to make them correctly, as nothing mars the appearance of a drawing so much as poor lettering. In any case, the lettering should not be done until the drawing is finished.

88. To prepare the sheet, first draw a very light horizontal line the full length of the sheet, about ½ inch below its upper edge, as illustrated on the reduced plate by the line *a b*. Draw this line with the T square, holding its head against the left edge of the drawing board. Then draw a vertical line *c d* about 1 inch from the left-hand edge of the paper, beginning at the line *a b* and running down to *d* on the lower edge of the paper. Draw this vertical line with the triangle by first drawing a vertical line through *c* as far down as the triangle will reach, and then producing it by moving the T square down and using the triangle from the end of the line first drawn. Set off ½ inch, 17 inches, and ½ inch, successively on *c b*, from *c* to *m*, *m* to *e*, and *e* to *f*, respectively; and ½ inch, 13 inches, and ½ inch, successively, on *c d* from *c* downwards to *g*, *h*, and *i*. Draw horizontal lines from *g*, *h*, and *i* across the sheet, and vertical lines from *m*, *e*, and *f* to make a border line and outline, as shown. Rub out the portions *m j*, *g j*, and the corresponding ones at the other corners. From *g* downwards set off ⅜ inch and $\frac{11}{16}$ inch, and draw two horizontal guide lines for the title. The title should be made in block letters. It should not, however, be printed until all the figures are finished. The figures are to be drawn in pencil first, and after *all* have been drawn, the student may ink them, if he wishes; but, as already stated, he is not required to do so in this Course.

FIG. 1.

89. To draw six horizontal lines, of the kinds shown, the upper one to be 1 inch below the upper border line, the others to be $\frac{1}{8}$ inch from one another, and all to extend to a distance of $\frac{1}{2}$ inch from each of the side border lines.

From the upper end of the left-hand border line set off $\frac{1}{2}$ inch to the right, and from the right-hand border line set off $\frac{1}{2}$ inch to the left; draw through the points thus marked two very faint vertical lines extending downwards about $1\frac{3}{4}$ inches from the upper border line. Upon one of these lines, set off from the upper border line downwards 1 inch, $1\frac{1}{8}$ inches, $1\frac{2}{8}$ inches, $1\frac{3}{8}$ inches, $1\frac{4}{8}$ inches, and $1\frac{5}{8}$ inches, marking these distances by very faint marks; a sharp-pointed instrument, like a needle, is best for doing this. Setting the edge of the **T** square successively on the six points thus marked off, draw the six horizontal lines, extending each horizontal line from one to the other of the two faint vertical lines.

FIG. 2.

90. To draw an upright oblong 3 inches high and 1 inch wide, at a distance of $\frac{1}{2}$ inch to the right of the left-hand border and $\frac{3}{4}$ inch below the preceding figure.

Draw a very faint vertical line along the left extremities of the parallel lines just drawn; from the lowest parallel line measure ¾ inch downwards on this vertical line; through the point thus found draw a faint horizontal line extending to the right a little over 1 inch; on this horizontal line mark points at distances of ½ inch and 1½ inches from the border; the horizontal line between these two points will be the upper side of the oblong. Now draw two vertical lines through the ends of this side, making them a little over 3 inches long, and mark a point on each line 3 inches from the upper side; through this point draw a horizontal line, whose intersection with the other vertical line will complete the oblong. Erase all construction lines, that is, those that do not belong to the oblong. Draw the extension and dimension lines as shown, making the former about ⅛ inch long.

91. If a point, as *p*, Fig. 59, is to be at a given distance from a vertical line *c d* and at a given distance below (or above) a horizontal line *a b*, the point is located as follows: Through any point *e* on *c d* draw a faint horizontal line *e f*, on which lay off *e o* equal to the distance that *p* is to be from the vertical line (to the right in this case); through *o* draw the faint vertical line *o g*, meeting *a b* at *g*, and from *g* lay off *g p* equal to the distance that *p* is to be from the horizontal line. In practice, it is not necessary to draw the lines full length, but only those portions that, judging by the eye, will be needed.

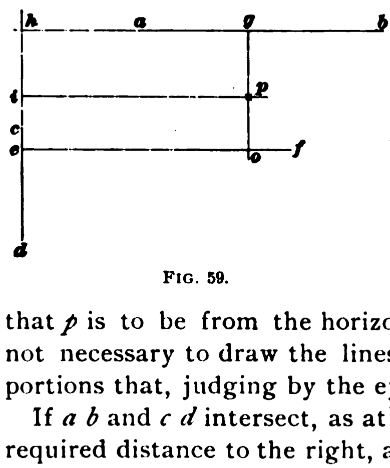


FIG. 59.

If *a b* and *c d* intersect, as at *h*, measure *h g* equal to the required distance to the right, and at *g* draw *g p* equal to the required distance below *a b*; or measure first the latter distance from *h* to *i*, and draw the horizontal *i p*, making it equal to the required distance to the right. In all cases of this kind the lines are drawn a little longer than required, so that it will not be necessary to produce them.

FIG. 3.

92. To draw a hollow square 3 inches square outside and 1 inch square inside, and at a distance of 1 inch to the right of the preceding figure and $\frac{3}{4}$ inch below the lower line of Fig. 1.

The upper left-hand corner of the square will be $\frac{1}{2}'' + 1'' + 1'' = 2\frac{1}{2}''$ to the right of the left-hand border line and $\frac{3}{4}$ inch below the lower line of Fig. 1; locate this point as described in Art. **91,** and through this point draw a horizontal and a vertical line, making them each 3 inches long. Then complete the square. Through the middle point of the sides draw two center lines, one horizontal and one vertical, as shown. From the center lay off, on these lines, four distances, each equal to $\frac{1}{2}$ inch. Through the points thus marked draw vertical and horizontal lines, respectively. Their points of intersection will determine the four angles of the inner square. Draw the dimension and the extension lines as shown, or in any other convenient positions. Erase all construction lines, except the center lines, which are useful for reference.

The outside square might have been constructed as follows: First locate the center of the square, which is 4 inches $(2\frac{1}{2}'' + 1\frac{1}{2}'')$ to the right of the left-hand border line and $3\frac{7}{8}$ inches $(\frac{11}{16}'' + \frac{5}{16}'' + \frac{5}{8}'' + \frac{3}{4}'' + 1\frac{1}{2}'')$ from the upper border line. Then draw the center lines; that is, lines perpendicular to each other through the center. From the center lay off on these lines four distances each equal to $1\frac{1}{2}$ inches and complete the square by drawing through the points thus marked vertical and horizontal lines, respectively. This is the usual method followed in practice where distances are almost always referred to and measured from center lines.

FIG. 4.

93. To draw the eye of a horizontal "fret" $3'' \times 4''$ and 1 inch thick, the drawing to be $1\frac{1}{8}$ inches from Fig. 3 and $\frac{3}{4}$ inch below Fig. 1.

Locate the point k $2\frac{1}{2}'' + 3'' + 1\frac{1}{8}'' = 6\frac{5}{8}''$ from the left-hand border line and $\frac{3}{4}$ inch below the lower line of Fig. 1. Draw km horizontal and 4 inches long, and kc vertical and 3 inches long; then draw mg and cg. Divide cg into four equal parts, and ck into three equal parts. Through h and i draw faint horizontal lines, and through d, e, and f draw faint vertical lines. Now draw in strong lines the required figure, as shown, over the parts of the faint guide lines, and erase all the remaining portions of the guide lines.

FIG. 5.

94. To inscribe a square diamond in a 3-inch square, the right-hand side of the latter to be $2\frac{3}{4}$ inches from the right-hand border line and the upper side to be $\frac{3}{4}$ inch below Fig. 1.

The upper right-hand corner of the square is $\frac{3}{4}$ inch below the lower line of Fig. 1 and $2\frac{3}{4}$ inches from the right-hand border line. Locate this corner and complete the square as for Fig. 3. Divide each side into two equal parts (that is, bisect each side), thus obtaining the points e, d, c, d', and join these points by straight lines, as shown. If the work has been done correctly, the angles that the sides of the diamond make with the sides of the square should be 45°. If this is found not to be the case, the student should go over his work again, and if he is sure that he has done everything correctly, and the angles are still wrong, he should test his instruments.

It will be noticed that the upper lines of the four figures just drawn are at the same distance below the upper border line; hence, if the upper line of any one of these figures be produced in either direction, it ought to coincide with the upper lines of the other figures. This is also true of the lower lines. The student should test his work by producing these lines.

FIG. 6.

95. To draw an upright diamond 3 inches high and having each upright angle equal to 60°, the center of the diamond to be $1\frac{5}{8}$ inches from the right-hand border line and the upper vertex $\frac{3}{4}$ inch below Fig. 1.

Locate the center *a* $\frac{3}{4}'' + 1\frac{1}{2}'' = 2\frac{1}{4}''$ below the lower line of Fig. 1 and $1\frac{5}{8}$ inches from the right-hand border line. Through *a* draw the center lines *b c* and *m n*, and lay off *a b* and *a c* equal to $3'' \div 2 = 1\frac{1}{2}''$. From *c* draw *c m*, making an angle of 30° with a vertical line or 60° with a horizontal line; slide the triangle along the T square and draw *b n* from *b* and parallel to *c m*; turn the triangle over, and from *c* draw a line at 30° with the vertical, and from *b* a parallel to the line just drawn. These two lines should pass through *n* and *m*, respectively.

FIG. 7.

96. To draw a horizontal diamond with horizontal angles of 60° and sides of $2\frac{1}{4}$ inches, the horizontal center line to be $5\frac{13}{16}$ inches above the lower border line, and the left-hand corner of the diamond $\frac{1}{2}$ inch from the left-hand border line.

Locate the point *a* and draw the horizontal center line *a b*. Draw *a c* and *a d*, making angles of 30° with *a b*, in the directions shown, and lay off on *a c* and *a d* $2\frac{1}{4}$ inches, locating the points *c* and *d*. From *c* and *d* draw lines making angles of 30° with a horizontal line, and in the directions shown. They should meet on the center line *a b*, and the center line *c d* should be vertical; test with triangle and T square.

FIG. 8.

97. To draw two equilateral triangles in the positions shown, each side of the outer one to be $2\frac{3}{4}$ inches long, the base *a b* to be $4\frac{5}{8}$ inches above the lower border

line and *a*, the left end of the base, to be 4¾ inches from the left-hand border.

Locate *a*; draw *a b* horizontal and 2¾ inches long; at *a* and *b* draw lines making angles of 60° with *a b*, in the directions shown, and meeting at *d*; bisect *a b* at *c*, and draw *c e* and *c f*, each making angles of 60° with *a b*, the one to the right, the other to the left; draw *e f*. If the construction is correct, *f e* will be horizontal (test with **T** square); *f* will be the middle point of *a d*, and *e* of *b d*.

FIG. 9.

98. To draw a regular hexagon with a horizontal base, each side to be 1¼ inches long, the left-hand end of the base to be 4¾ inches above the lower border line and 8½ inches to the right of the left-hand border line.

Locate *a* and draw *a b* equal to 1¼ inches; at *a* and *b* draw lines making angles of 60° with a horizontal line, in the directions shown, and take *a c* and *b d* each equal to 1¼ inches; at *c* and *d* draw lines at 60° with the horizontal, but changing the directions, as shown, and take *c e* and *d f* each equal to 1¼ inches; draw *e f*. Slide the **T** square up to *e f*, and see if the latter line is horizontal, as it should be. Also see if the verticals through *a* and *b* pass through *e* and *f*, respectively.

Another way of constructing the figure is as follows: Draw *a b*, *a c*, and *b d* as before, and vertical lines through *a* and *b*; through *c* and *d* draw 60° lines, as before, stopping them at their intersections with the verticals through *a* and *b*, respectively, which will be at *e* and *f*; then draw *e f*, which should be horizontal.

FIG. 10.

99. To draw a regular six-pointed star 2½ inches high, its center to be 4¾ inches to the left of the right-hand border line and 5¾ inches above the lower border line.

Locate the center *a*; draw the center line *m n* and take *a b* and *a c* each equal to 1¼ inches (half the height of the star); draw *a d* and *a e* making angles of 60° with the vertical (or 30° with the horizontal) in the directions shown, and produce them in both directions; take *a f*, *a g*, *a h*, and *a i* each equal to *a b*, that is, 1¼ inches; draw *b i* and *b h*, *c f* and *c g*, *g f* and *h i*; make the heavy lines as shown, and erase all construction lines, but not the center line *m n*. See if the verticals through *h* and *i* pass through *g* and *f*, respectively, as they should.

FIG. 11.

100. To draw a hollow regular octagon standing on its base, each side of the outer octagon to be 1 inch long and each side of the inner octagon to be ½ inch long. The point *a* of the base to be 4⅝ inches above the lower border line and 2¼ inches to the left of the right-hand border line.

Locate *a*; draw the horizontal base *a b*, and make it 1 inch long; draw *a c* and *b d* at angles of 45° to the horizontal, each 1 inch long; from *c* and *d* draw verticals *c j* and *d m* each 1 inch long; from *j* and *m* draw *j k* and *m l*, at 45° to the horizontal, and produce them until they meet the verticals through *a* and *b* at *k* and *l*, respectively; draw *k l*, which should be horizontal and 1 inch long. Draw the construction lines *a l*, *b k*, etc., all of which should intersect at the same point *o*. Bisect *a b* at *n*, and from *n* set off *n e* and *n f* each equal to one-half the side of the inner octagon (¼ inch in this case); draw verticals through *e* and *f*, meeting *a o* and *b o* at *g* and *h*, respectively; draw *g h*, which should be horizontal and equal to ½ inch; from *g* and *h* draw 45° lines to *p* and *q* on *o c* and *o d*, respectively; from *p* and *q* draw the verticals *p r* and *q u*, then the 45° lines *r s* and *u t*, and finally *s t*, which should be horizontal and equal to ½ inch. The points *k* and *l*, *s* and *t* should be on the verticals through *a* and *b*, *e* and *f*, respectively.

FIG. 12.

101. To draw, to a scale of 1 inch to the foot, a piece of pavement consisting of regular hexagonal tiles, the side of each tile being 6 inches long.

Locate the point *a* at a distance of ½ inch above the lower border line, and ½ inch to the right of the left-hand border line. Draw *a l* and *a g*, the former horizontal, the latter vertical. Since the scale is 1 inch to the foot, every line on the drawing will be one-twelfth of the length of a corresponding line on the object, and each side of the tiles must be made 6 ÷ 12, or ½ inch long on the drawing. Lay the scale on *a l* with the zero on *a*, and mark off ¼ inch to *c*, ¾ inch to *d*, 1¼ inches to *e*, 1¾ inches to *f*, etc., so that each of the distances *c d*, *d e*, *e f*, etc. will be ½ inch; any number of such distances may be taken, according to the length of pavement to be drawn; in this case there are seven equal distances, besides *a c*, which is one-half the length of the side of a tile. From *a*, and including *a c*, take on *a l* as many of the preceding distances as there are tiles in the first vertical row, or four in this case; through *f*, the end of the fourth space, draw the 60° line *f j*, meeting *a g* at *j*; draw the 60° line *c i*; take the distance *a i* by the dividers and set it off from *j* to *g*; through *g* draw a horizontal line and through *l* a vertical line, meeting the former at *h*. Through each of the points *d*, *e*, *f*, etc. draw two 60° lines, one to the right and one to the left, and produce them to meet either *g h* or one of the end vertical lines; from each of the points thus determined on *a g* and *l h* draw 60° lines to meet *g h*, as shown; the points thus found on *g h* should coincide with the extremities of the lines drawn from corresponding points on *a l*: thus, the point *n*, where the 60° degree line from *j* meets *g h*, should be the same as the point where the 60° line from *m* meets *g h*. Through the intersections *o*, *p*, etc. draw horizontal lines, which should pass through all the corresponding intersections at the right, as *q*, *r*, *s*, etc., *t*, *u*, etc. The heavy lines may be drawn as the work just explained is done, or, afterwards, by laying the **T** square or the triangles (the former for the horizontal lines) along the different lines in succession and retracing in heavy lines those parts that are to make the sides of the tiles. In drawing the 60° lines, the edge of the **T** square is placed a little below the figure, and the 60° triangle slid along, making its edge pass through the different points *j*, *k*, *i*, *c*, *d*, etc. in succession. A little practice will enable the draftsman to dispense with many auxiliary lines; thus, when drawing the horizontal sides on *k v*, it is only necessary to draw *t u*, then skip one intersection and join the next two, *x* and *y*, etc. The beginner, however, should draw all the lines indicated above, as this is a good test of his accuracy. *Do not erase the construction lines of this figure.*

FIG. 13.

102. To draw, to a scale of $\frac{3}{4}$ inch to the foot, a piece of pavement consisting of squares and regular octagons, each side being 6 inches long.

Locate *a* $\frac{1}{2}$ inch from the lower border line and 1 inch to the right of *l h* of Fig. 12. Draw a horizontal line from *a* to the right, producing it to the left about $\frac{1}{2}$ inch. The scale being $\frac{3}{4}$ inch to the foot, every line on the drawing will be $\frac{3}{4} \div 12 = \frac{1}{16}$ its actual length, and 6 inches, or half a foot, will be represented by $6 \div 16 = \frac{6}{16}$, or $\frac{3}{8}$ inch on the drawing. Lay off $\frac{3}{8}$ inch from *a* to *b*; draw the 45° lines *a c* and *b d* each equal to the side (6 inches, to scale) of the octagon and square. Draw vertical lines through *d* and *c*, meeting *a f* at *g* and *e*, respectively. Take the distance *e g* on the dividers and step it off from *g* to *h*, from *h* to *i*, etc. as many times as there are octagons in a horizontal row (five in this case). Similarly, step off the same distance from *e* to *k*, from *k* to *l*, etc. as many times as there are octagons in a vertical row (four in this case). Draw vertical lines through *g*, *h*, etc. and horizontal lines through *k*, *l*, etc. Take the distance *g b* on the dividers and lay it off from *g* to *p*, from *h* to *q* and *h* to *r*, from *i* to *s* and *i* to *t*, etc.; also, from *k* to *u* and *k* to *v*, from *l* to *x* and *l* to *w*, etc. From each of the points thus found draw 45° lines, as shown, and also from the points of intersection of these lines with *no* and *fo* (*f* and *o* being the extremities of the equal horizontal construction lines through *a* and *n*, respectively). Complete the figure as in the preceding case. Erase the construction lines from that part of the figure to the right of *i*, and leave in the others.

FIG. 14.

103. To draw, to a scale of $1\frac{1}{2}$ inches to the foot, a lozenge molding 2 feet high, the width of each band being 3 inches, and the inclination of the bands, 60° to the horizontal.

This is called a **lozenge molding** from the fact that the inner figures are all equal-sided parallelograms, or lozenges.

Locate the point *a* at a distance of $1\frac{1}{4}$ inches from the line *fo* of Fig. 13, and $\frac{1}{2}$ inch above the lower border line. Every line on the drawing will be $1\frac{1}{2} \div 12 = \frac{1}{8}$ of its actual length and 24 inches (2 feet) $\div 8 = 3'' =$ height of molding to the scale used. Likewise, the width of each band will be $3 \div 8$, or $\frac{3}{8}$ inch on the drawing. Draw *ac* vertical and make it 3 inches long. Draw the horizontal lines *ab* and *cd* of any convenient length, so that several lozenges can be shown; make it of such a length that *d* shall be $\frac{1}{2}$ inch from the right-hand border line, and draw the vertical construction line *bd*. On *ca* take a point *e* near *c* (in the figure *e* is about $\frac{7}{16}$ inch from *c*), and draw the 60° line *ef*, producing it a little in the direction *eg*; at any point *g* of *ef* produced draw a perpendicular to *eg*, on which lay off *gh* equal to $\frac{3}{8}$ inch, the width of the bands. Draw the vertical *fi* through *f* and the 60° line *hj* through *h*, meeting *fi* at *j*; draw the horizontal *jl*; take distance *fj* on the dividers and set it off from *i* to *k*; draw the horizontal *km* and the 60° lines *kx* and *iv*. Now draw the 60° lines *fn* and *jo*, *kp* and *iq*, *ps* and *qr*, *nt* and *ou*, etc., always drawing first those lines that go through, as *jo*, *fn*, etc.

Care must be taken, especially if the figure is inked, that the intercepted lines, such as *kp* and *iq*, stop exactly on the crossing lines, such as *jo* and *fn*. A figure with imperfect intersections, as in Fig. 60, where some lines project beyond

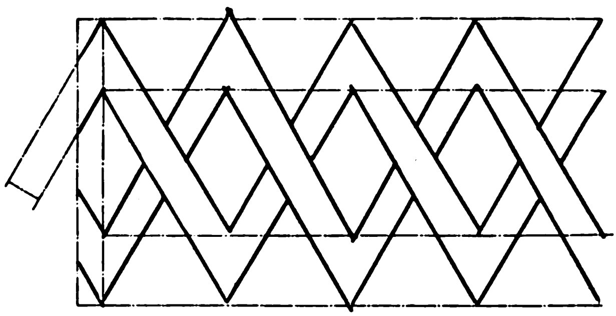


FIG. 60.

or do not reach the points where they should end, has an exceedingly poor appearance, and at once gives one an unfavorable idea of the draftsman's skill.

FINISHING THE PLATE.

104. Having drawn all the figures in pencil, the student should rub out all superfluous lines (except where he has been directed to preserve them, as in Figs. 12 and 13) and clean the paper with a soft rubber. He should then letter the figures and put on the title, either drawing it or pasting it between the guide lines at the top of the sheet, and finally write the date, his name, and class letter and number as shown on the printed plate.

The student may ink in his drawings, if he wishes to do so, by going over the pencil lines with the ruling pen, always using India ink. The letters may be made with a common pen, but with India ink. *Ordinary writing ink is never to be used on a drawing.*

The paper should be cut out along the outer lines *cf*, *ci*, etc. while tacked on the board. For this purpose a sharp knife should be used. The knife may be moved along the edge of the T square, but care should be taken to turn the edge of the knife a little (very little) outwards, so as not to cut the blade of the T square. **Paper cutters** are very convenient for this purpose; they are cheap and can be used on the T square without any danger of cutting it.

HOW THE WORK SHOULD BE SENT.

105. Do not send all the plates together, but send one plate to us at a time. Thus, after you finish Plate I, send it to us and then start on Plate II. In the meantime we will return Plate I to you. On receiving Plate I from us, you should carefully note all corrections and suggestions that may be sent with it, and observe them when drawing the succeeding plates. On no account send us Plate II until you have received back Plate I. On finishing Plate II, send it to us and start on the study of *Geometrical Drawing*. Do this with all the drawing plates in the Course.

It is very essential that you strictly comply with these directions; since, otherwise, it will be impossible for us to point out your mistakes to you. This procedure should be strictly adhered to while you are drawing the first plates of the Course—it will enable you to make rapid progress. Do not be discouraged if there are a large number of corrections on your early plates; we are merely pointing out ways in which the drawing or lettering can be improved, so that your later plates may be as nearly perfect as they can be made. No one can attain proficiency unless the work is criticised, and we are doing our best to help you to succeed. We should not be doing our duty if we did not point out the defects. The *number* of corrections is no indication of our appreciation of the merits of the drawing.

On all plates that you send to us, write your name and address in full in lead pencil on the back of the plates. This should in no case be omitted, as delay in the return of your work will otherwise surely occur.

DRAWING PLATE, TITLE: CURVED-LINE FIGURES.

106. The student is now to draw Plate II, for which he will require comparatively few explanations. The student will prepare his paper for his second plate according to the directions given for Plate I.

FIG. 1.

107. To draw a series of concentric circles.

The center *i* is to be $4\frac{5}{8}$ inches below the upper border line and 4 inches to the right of the left-hand border line. Draw a faint horizontal line through *i*, and with the zero of the scale on *i*, mark off $\frac{3}{8}$ inch to *h*, $\frac{1}{2}$ inch to *g*, $1\frac{3}{8}$ inches to *f*, $1\frac{1}{2}$ inches to *e*, $2\frac{3}{8}$ inches to *d*, $2\frac{1}{2}$ inches to *c*, $3\frac{3}{8}$ inches to *b*, and $3\frac{1}{2}$ inches to *a*; then describe the circles as shown, being careful, when drawing the large circles, to bend the legs of the compasses so that they will be nearly perpendicular to the paper.

FIG. 2.

108. To draw a series of circles somewhat similar to those given in Exercise 3, Art. 62.

The radius of the small circles is (on the student's plate, to which all the dimensions here given refer) $\frac{11}{16}$ inch, and that of the large circles $\frac{11}{8}$, or $1\frac{3}{8}$, inches. The center *a* of the first small circle is $4\frac{9}{16}$ inches from the right-hand border line and $2\frac{1}{2}$ inches from the upper border line. The center of the first large circle is at *b*, $\frac{11}{16}$ inch from *a*; that of the second small circle is at *c*, $\frac{11}{16}$ inch from *b*, etc. To lay off these distances take a distance of $\frac{11}{16}$ inch on the dividers and step it off, along a horizontal line through *a*, from *a* to *b*, from *b* to *c*, etc. Be careful that the circles touch one another on the horizontal line, as shown, without crossing.

FIG. 3.

109. To draw a regular hexagon, each of whose sides is $1\frac{5}{8}$ inches long, its center being $1\frac{7}{8}$ inches from the right-hand border line and $2\frac{1}{2}$ inches from the upper border line.

To construct this figure, locate the center *c*, and, with a radius equal to $1\frac{5}{8}$ inches, describe a circle. Lay the T square so that it will pass through *c*, and mark *a* and *b* in line with it; move the T square down and set the 60° triangle in position to draw a 60° line through *c*, either to the right or to the left; but, instead of drawing the line, mark only the points *d* and *f*, or *e* and *g*, where it crosses the circle. Turn the triangle over, so that the 60° angle will face the other way, and mark the corresponding points. Join the six points thus marked.

FIG. 4.

110. To draw six semi-circumferences, the three at the right having the same center *d*, and the three at the left the same center *e*, the corresponding extremities being connected by horizontal tangents.

Locate *d* at a distance of $6\frac{1}{2}$ inches below the upper border line and $5\frac{3}{4}$ inches from the right-hand border line. Through *d* draw a horizontal center line, and on it take a point *e* at a distance of $2\frac{1}{16}$ inches from *d*. Draw vertical lines through *d* and *e*. From *d* and *e*, with a radius of $\frac{1}{4}$ inch, describe two semi-circumferences, stopping at the vertical lines, as shown. From the same centers describe two semi-circumferences with a radius of $\frac{3}{4}$ inch and two with a radius of $1\frac{1}{4}$ inches. Join the corresponding ends, as shown.

When inking this figure, or any other containing curved lines, ink the curved lines first, as, in case there is any inaccuracy, it is easier to adjust a straight line than a curved line. Thus, if, when inking the *horizontal* line *h f*, it is found that it does not pass exactly through the ends of the semi-circumferences, a line that is *not* horizontal may be drawn between those two points, and if the error is small, it will not show. But if the horizontal lines *h f* and *i g* are drawn first and the drawing is not accurate, the semi-circumferences cannot be easily adjusted so as to make them pass through the ends of the horizontal lines and be tangent to them. The preceding "adjustment," however, *is to be used only in cases of very small imperfections and when time is pressing. All drawings should be made as accurate as possible.*

FIG. 5.

111. To draw a 4-lobe figure 3½ inches high and 3½ inches wide, the diameter of each corner circle being 1¼ inches.

Locate *e* at a distance of 2¼ inches from the right-hand border line and 6½ inches below the upper border line. Through *e* draw the two center lines *a b* and *c d*, the former horizontal, the latter vertical. From the height of the figure, 3½ inches, take the diameter of the corner circle, or 1¼ inches, which leaves 2¼ inches. Construct the square *j k l m*, having its sides equal to 2¼ inches and its center at *e*; this is done by simply laying off *e f*, *e g*, *e h*, *e i*, each equal to half the side of the square, and drawing vertical and horizontal lines, as shown. From *j*, *k*, *l*, and *m* as centers, with a radius of ⅝ inch (= ½ of 1¼ inches), describe circular arcs going from one vertical to one horizontal side of the square, on the outside; from *g*, *h*, *f*, and *i*, with a radius of ½ inch (which is the difference between half the side of the square and the radius of the corner circles) describe semi-circumferences, as shown. The latter should be tangent to the corner circles at their points of intersection with the horizontal and vertical lines, respectively. Draw the dimension and extension lines as shown, or in any other convenient positions.

FIG. 6.

112. To draw the end of a cylinder, or pipe, cut square across.

When an object is cut this way, the end surface exposed by the cut is called a **cross-section,** and is usually indicated by **cross-section lines,** as shown.

The *diameter* of the outside circle is 3½ inches and the thickness of the material is ⅞ inch; the inside *diameter* is, therefore, 1¾ inches. (Although for drawing a circle we must use its radius, in practice it is customary to give the diameter instead of the radius.)

Locate the center at a distance of 2¼ inches from the left-hand border line and 2¼ inches from the lower border line, and draw the two circles. Then draw the cross-section lines at 45° to the horizontal. In doing this, the T square is set so that its edge will be a little lower than the lowest part of the outside circle. Then the 45° triangle is laid against the T square with its long side in the direction required (to the right in this case), and slid along until only a very small arc of the circle projects beyond its long side; the first line is then drawn and the triangle slid to the right a very small distance, the second line is then drawn and the triangle slid to the right the same distance as before, and so on.

Great care should be taken that all the lines are at the same distance apart and that they stop at the proper place, without falling short or projecting beyond those lines whereon they are to end. In case a space is found too large or too small, the next one should not be made of the proper size, but a little smaller (or larger) than the defective one, thus correcting the error very gradually. The lines should be a little finer than the outline of the figure, but not too fine.

FIG. 7.

113. To draw another illustration of cross-section lining, the cross-section lines having an angle of 30° to the horizontal.

To draw the outline of the figure, locate the center *a* at a distance of $5\frac{1}{2}$ inches to the right of the left-hand border and $2\frac{1}{4}$ inches above the lower border line. Draw two center lines through *a*, one horizontal and one vertical. Take *a b* and *a c* each equal to $\frac{7}{8}$ inch, and draw horizontals through *c* and *b*; from *b* and *c* as centers, with a radius of 1 inch, describe semi-circumferences as shown, connecting their ends by vertical tangents *g e* and *h f*. Take a distance of $\frac{1}{2}$ inch on the dividers and set it off from *e* to *i*, from *g* to *j*, etc. With *i*, *j*, etc. as centers, and with a radius of $\frac{1}{2}$ inch, describe arcs of circles, stopping where shown. Now draw the cross-section lines, using the 30° triangle.

FIG. 8.

114. To draw two concentric ellipses, the axes of the outside one being, respectively, 5 inches and $3\frac{1}{2}$ inches, and those of the inside one 4 inches and $2\frac{1}{2}$ inches.

The center is to be located $9\frac{1}{2}$ inches to the right of the left-hand border line and $2\frac{1}{4}$ inches above the lower border line. First draw the axes, then locate the foci, and then construct the ellipse by means of a string.

FIG. 9.

115. To draw a niche 7 feet 6 inches wide and 7 feet 7 inches high, consisting of two upright sides, each 5 feet high, and an elliptical top 2 feet 7 inches deep.

The drawing is to be made to a scale of $\frac{1}{2}$ inch to the foot, thus making each line on the drawing $\frac{1}{2} \div 12 = \frac{1}{24}$ the actual length. Locate the lower right-hand corner of the niche ½ inch from the right-hand border line and ½ inch above the lower border line, and draw the horizontal base of the niche equal to 7 feet 6 inches, using the half-inch scale. To the same scale lay off the two upright sides, each equal to 5 feet, and connect them by a horizontal construction line. Bisect this line at *a* and draw the center line *a b*, laying off *a b* equal to 2 feet 7 inches. Having *b*, which is the extremity of the minor axis of the ellipse, and *c d*, which is the major axis, construct the half ellipse by a string or by points. Draw the extension and dimension lines as shown.

Rub out all construction lines, write title, name, etc.; cut out the plate and send it in for examination.

GEOMETRICAL DRAWING.

1. In the study of his arithmetic, the student has learned many geometrical definitions and principles. Now in **geometrical drawing,** he is shown how to make certain drawings which are based on those geometrical principles.

2. The principles of geometry are based on the *reason*, unaided by, or at least independent of, any such considerations as measurement, accuracy of construction, etc. These principles are, therefore, *absolutely true* and the student may be sure that the correctness of his result depends only on the accuracy of his drawing. If the result is incorrect, the fault lies in the drawing of the given parts, and, in no case, is it to be attributed to the principles on which the construction is based, because they are true.

3. The problems (and their solutions) herein given are to be studied and practiced in the following manner: First of all, read through the statement of the problem and also the entire solution, at the same time referring back to the various principles therein applied, so as to more readily grasp the proof. Then, draw the parts given in the course of the solution, completing each step in the construction before proceeding to the next.

As regards the exercises to be drawn on the examination sheets, it is advisable to first work out each problem on a spare sheet of paper, and when the solution is thoroughly understood and all incidental errors in the construction rectified, the problem may be drawn on the examination sheet, the result being a neat and inviting set of drawings. No reference letters are to be placed on these sheets.

PLATES.

PLATE I.

PREPARATION OF PLATES.

4. The student should again read over the instructions given in *Instrumental Drawing*, and with these general instructions in mind he should carefully follow the instructions given here.

5. Take a sheet of drawing paper 15 inches wide and 20 inches long and fasten it to the board. On this paper draw the outlines of the size of the plate, 14″ × 18″, and draw the border line all around $\frac{1}{2}$ inch from the edge of the outline, leaving the space inside for the drawing 13″ × 17″. When the word *drawing* is used hereafter, it refers only to the space inside the border lines and the objects drawn upon it. To understand clearly what follows, refer to the folding sheet at the end of this section. On it is drawn Plate I to a reduced scale. Divide the drawing into two equal parts by means of a faint horizontal line. This line is shown dotted in the reduced copy of Plate I. Divide each of these halves into three equal parts, as shown by the dotted vertical lines; this divides the drawing into six rectangular spaces. *These division lines are not to be inked in, but must be erased when the plate is completed.* On the first five plates, space for the lettering must be taken into account. For each of the six equal spaces, the lettering will take up one or two lines. The height of all capital letters on these plates will be $\frac{3}{32}$ inch, and of the small letters, $\frac{1}{16}$ inch. The distance between any two lines of lettering will be $\frac{1}{8}$ inch. The distance between the tops of the letters on the first line of lettering and the top line of the equal divisions of the drawing is to be $\frac{7}{16}$ inch; and the space between the bottoms of the letters and the topmost point of the figure represented on the drawing within one of these six divisions must be not less than $\frac{1}{2}$ inch. This makes a very neat arrangement, if the figure is so placed that the outermost points of the bounding lines are equally distant from the sides of one of the equal rectangular spaces. Consequently, if there is one line of lettering, no point of the figure drawn should come nearer than $\frac{7}{16}'' + \frac{3}{32}'' + \frac{1}{2}'' = 1\frac{1}{32}''$ to the top line of the space within which it is represented; or, if there are two lines of lettering, nearer than $\frac{7}{16}'' + \frac{3}{32}'' + \frac{1}{8}'' + \frac{3}{32}'' + \frac{1}{2}'' = 1\frac{1}{4}''$. The letter heading for each figure on the first five plates is printed in heavy-faced type in this section, at the beginning of the directions explaining each problem. The student must judge for himself by the length of the heading whether it will take up one line or two, and must make due allowance for the space it takes up. This is necessary, because the lettering should never be done until the rest of the drawing is entirely finished and inked in. The space for the lettering is to be left at the top of each of the rectangles, even when the student does not do the lettering. Again the student's attention is called to the fact that in this Course the lettering is optional with him.

PROBLEM 1.

6. To bisect a straight line.

See Fig. 1 and Problem 1 of Plate I. Locate the point *A* $1\frac{1}{16}$ inches from the left side and 3 inches from the lower side of the first of the six rectangles into which the drawing sheet has been divided. Through the point *A* draw the straight line *A B* perpendicular to the left side, making *A B* $3\frac{1}{2}$ inches long. Let *A B* be the given line which it is required to bisect. Set the compasses, by estimate, so that the distance

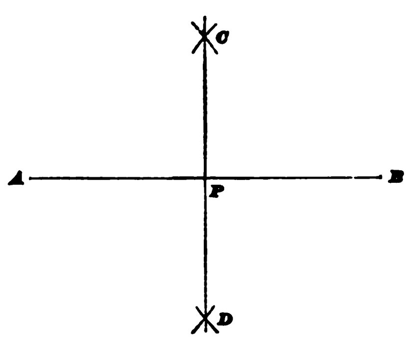


FIG. 1.

between the points shall be, as nearly as possible, equal to two-thirds of the line *A B*. With this radius and the point *A* as center, describe an arc of a circle on each side of the given line; with the other extremity *B* as center and the same radius, describe another arc intersecting the first in the points *C* and *D*. Join *C* and *D* by the line *C D*, and the point *P*, where it intersects *A B*, is the middle point of *A B*; that is, $AP = PB$. To locate the points *C* and *D* it is not necessary to draw the whole arc from *A* as center, but only such parts of it as the student thinks pass through the points *C* and *D*.

Since *C D* is perpendicular to *A B*, this construction also gives a *perpendicular to a straight line at its middle point.*

PROBLEM 2.

7. To draw a perpendicular to a straight line from a given point in that line.

NOTE.—As there are two cases of this problem, requiring two figures on the plate, the line of letters will be run clear across both figures, as shown in Plate I.

Case I.—*When the point is at or near the middle of the line.*

See Fig. 2 and Problem 2, Case I, of Plate I. Locate the point *A* $1\frac{1}{16}$ inches from the left side and $1\frac{1}{4}$ inches from the lower side of the second rectangle. Through the point *A* draw *A B* perpendicular to the left side, making *A B* $3\frac{1}{2}$ inches long. On the line *A B*, and near the middle of it, take any point *P*. Let *A B* be the given line, to which a perpendicular is to be drawn through the point *P*. With *P* as center and any

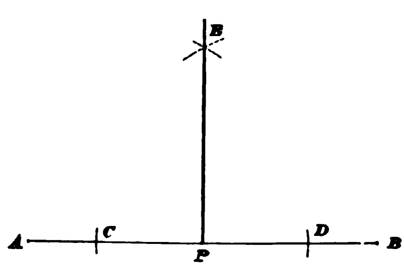


FIG. 2.

radius, as PD, describe two short arcs cutting AB in the points C and D. With C and D as centers and any convenient radius greater than PD, describe two arcs intersecting in E. Draw the straight line PE, which will be the required perpendicular to the given line AB at the given point P.

Case II.—*When the point is near the end of the line.*

See Fig. 3 and Problem 2, Case II, of Plate I. Locate the point A $1\frac{1}{16}$ inches from the left side and $1\frac{1}{4}$ inches from the lower side of the third rectangle. Through the point A draw AB $3\frac{1}{2}$ inches long perpendicular to the left side. On the line AB and near the end B, take any point P. Let AB be the given line to which a perpendicular is to be drawn through the point P. With any point O as center and a radius OP, describe an arc cutting AB in P and D. Draw the straight line DO, and prolong it until it intersects the arc in the point C. Draw the straight line CP, which will be the required perpendicular to the given line AB at the given point P.

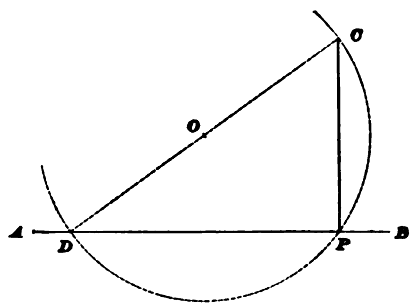


FIG. 3.

PROBLEM 3.

8. To draw a perpendicular to a straight line from a point without it.

Case I.—*When the point lies nearly over the middle of the line.*

See Fig. 4 and Problem 3, Case I, of Plate I. In the fourth rectangle of Plate I, locate the point A $1\frac{1}{16}$ inches from the left side and $2\frac{7}{8}$ inches from the lower side.

Draw $A B$ $3\frac{1}{2}$ inches long perpendicular to the left side. Over the middle part of the line take any point P. Let $A B$ be the given line to which a perpendicular is to be drawn from the point P. With P as center and any radius $P D$, greater than the distance from P to $A B$, describe an arc cutting $A B$ in C and D. With C and D as centers and a radius equal to about two-thirds of $C D$, describe two short arcs intersecting in E. Draw the straight line PE and it will be the required perpendicular to the given line $A B$ from the given point P.

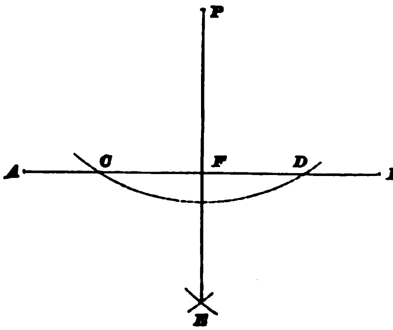


FIG. 4.

Case II.—*When the point lies nearly over one end of the line.*

See Fig. 5 and Problem 3, Case II, of Plate I. In the fifth rectangle of Plate I, draw $A B$ $3\frac{1}{2}$ inches long as directed in Case I. Take any point P nearly over the end B of the line $A B$. Let $A B$ be the given line to which a perpendicular is to be drawn from the point P. With any point C on the line $A B$ as center and the distance $C P$ as a radius, describe an arc $P E D$ cutting $A B$ in E. With E as center and the distance $E P$ as a radius, describe an arc cutting the arc PE in D. Draw the straight line PD, which is the required perpendicular to the given line $A B$ from the given point P. This construction is based on the fact that a radius drawn to the middle point of a chord is perpendicular to the chord.

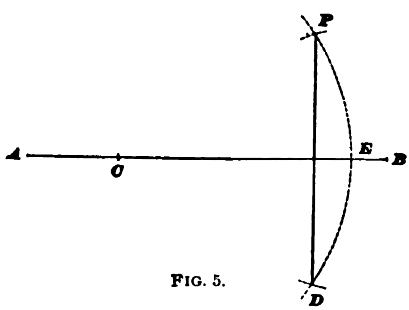


FIG. 5.

PROBLEM 4.

9. Through a given point, to draw a straight line parallel to a given straight line.

See Fig. 6 and Problem 4 of Plate I. Locate the point *A* $2\frac{1}{8}$ inches above the lower side and $1\frac{1}{16}$ inches to the right of the left side of the sixth rectangle of Plate I, and draw *A B* perpendicular to the left side of the rectangle, making *A B* $3\frac{1}{2}$ inches long. Locate *P* $1\frac{1}{2}$ inches above the line *A B*. Let *P* be the given point through which a line is to be drawn parallel to *A B*. With *P* as center and any radius longer than the distance from *P* to *A B*, describe an arc *C D* intersecting *A B* in *D*. With *D* as center and the same radius, describe the arc *PE*, cutting *A B* in *E*. With *D* as center and a radius equal to the chord of the arc *P E*, describe an arc intersecting *C D* in *C*. Draw the straight line *PC*, which will be the required parallel to *A B* through *P*.

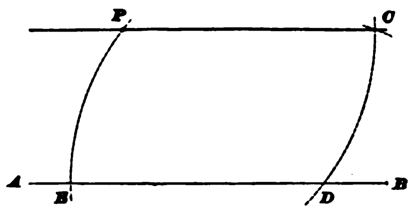


FIG. 6.

10. These four problems form Plate I. They should be carefully and accurately drawn in with lead-pencil lines; when all are so drawn, the student if he wishes may ink them. It will be noticed that on Plate I, and Figs. 1 to 6, the given lines are *light*, the required lines *heavy*, and the construction lines, which, in a practical working drawing, would be left out, are *light dotted*. This system must also be followed in the plates that are to follow. A single glance enables one to see at once the reason for drawing the figure, and the eye is directed immediately to the required line.

In the plates, accuracy and neatness are the main things for which the student should strive. He should be certain that the lines are of *precisely* the length that is specified in the description. When drawing a line through two points, be sure that the line goes through the points; if it does not pass *exactly* through the points, erase it and draw it over again. If a line is supposed to end at some particular point, make it end there—do not let it extend beyond or fall short. Thus, in Fig. 6, if the line *P C* does not pass through the points *P* and *C*, it is not parallel to *A B*. By paying careful attention to these points, the student saves himself a great deal of trouble in the future. *Do not hurry your work.*

11. The instructions for inking and lettering are here given for those who desire to ink and letter their drawings. First ink in all the light lines and light dotted lines (which have the same thickness); then ink in the heavy required lines after the pen has been readjusted. Now do the lettering (first read carefully the articles in regard to lettering both in this section and in *Instrumental Drawing*), and finally draw the heavy border lines, which should be thicker than any other line on the drawing. The word "Plate" and its number should be printed at the top of the sheet, outside the border lines and midway of its length, as shown.

Every student must put his name, followed by the word "Class," and after this his Course letter and *class number* in the lower right-hand corner below the border line. Thus, John Smith, Class Y 4529. The date on which the drawing was completed should be placed in the lower left-hand corner below the border line. *All this lettering is to be in capitals $\frac{3}{32}$ inch high.* Erase the division lines and clean the drawing by rubbing very gently with the eraser. Care must be exercised when doing this or the inked lines will also be erased. *If any part of a line has been erased or weakened, it must be redrawn.* Then write with the lead pencil your name and address in full on the back of your drawing, after which put your drawing in the empty tube that was sent you and send it to the School.

HINTS FOR PLATE I.

12. *Do not forget to make a distinction between the width of the given and required lines, nor forget to make the construction lines dotted.*

When drawing dotted lines, take pains to have the dots and spaces uniform in length. Make the spaces only about one-third the length of the dots.

Try to get the work accurate. The constructions must be accurate, and all lines or figures should be drawn of the length or size stated in the descriptions. To this end, work carefully and keep the pencil leads very sharp, so that the lines will be fine.

The lettering, if done, is fully as important as the drawing, and should be done in the neatest possible manner.

All reference letters like A, B, C, etc., as shown in Fig. 1, are not to be put on the plates.

Do not neglect to trim the plates to the required size. Do not punch large holes in the paper with the dividers or compasses. Remember that the division lines are to be erased—not inked in.

PLATE II.

13. Draw the division lines in the same manner as described for Plate I. The following five problems, Nos. 5 to 9, inclusive, are to be drawn in regular order, as was done in Plate I with problems from 1 to 4. The letter headings are given in heavy-faced type after the problem number.

The location of the figure in the rectangle is left to the student, which should be done before he begins his drawing. A good way to do this is as follows : Take the length of the main horizontal line of the figure from the width of one of the rectangles and divide the difference by 2, the quotient will tell you how far from the left side of the rectangle you are to begin. In determining the distance from the upper or lower side of the rectangle at which you are to begin, the letter heading must be considered. That is, the tops of the letters must be $\frac{7}{16}$ inch from the upper side of the rectangle and $\frac{3}{32}$ inch must be left for each line of letters and a space of $\frac{1}{8}$ inch between each line. Thus, in

Problem 1, Plate I there is one line of letters; the base line of these letters is $\frac{7}{16}''$ (space) + $\frac{3}{32}''$ (line of letters), or $\frac{17}{32}''$ from the upper side of the rectangle. In each of the other problems of Plate I there are two lines of letters, and the base of the second line of letters is $\frac{7}{16}''$ (space) + $\frac{3}{32}''$ (first line of letters) + $\frac{1}{8}''$ (space) + $\frac{3}{32}''$ (second line of letters), or $\frac{3}{4}''$ from the upper line of the rectangle. As each rectangle is $6\frac{1}{2}$ inches long, this leaves a space in which to put the figure, when there is one line of letters, $6\frac{1}{2}'' - \frac{17}{32}''$, or $5\frac{31}{32}''$ long, and a space, when there are two lines of letters $6\frac{1}{2}'' - \frac{3}{4}''$, or $5\frac{3}{4}''$ long. If the figure extends equally on each side of a horizontal line, divide the length of space just found by 2 and the quotient is the distance above the lower side at which to begin. When the figure does not extend an equal distance on each side of a horizontal line, the student must judge for himself, from the description of the figure, where he is to begin, always taking into account, that no line of the figure is to come nearer than $\frac{1}{2}$ inch to the base line of the letter heading.

The student is advised to draw each figure on a separate sheet of paper before he tries to draw it on his examination plate. This will make him familiar with the construction, give him practice in drawing, and help him to locate the figure on the plate. When drawing, whether on a practice sheet or plate, always be careful, painstaking, and neat, for slovenly habits in drawing, as in everything else, are to be avoided.

PROBLEM 5.

14. To bisect a given angle.

Case I.—*When the sides intersect within the limits of the drawing.*

See Fig. 7. In the first rectangle of Plate II draw the angle *A O B*, making the sides $3\frac{1}{2}$ inches long. Let *A O B* be the angle to be bisected. With the vertex O as center and with a radius equal to about one-half OA, describe an arc intersecting OA in D and OB in E. With D and E as centers and a radius equal to about two-thirds the chord DE, describe two arcs intersecting at C. Draw the straight line OC, which is the bisector of the angle AOB; that is, the angle $AOC =$ the angle COB.

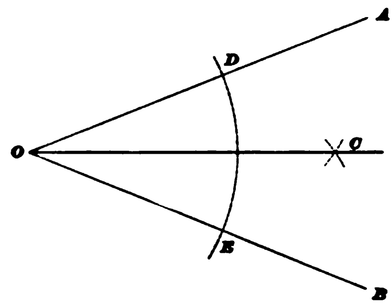


FIG. 7.

This construction is based on the fact that a line joining the center of a circle to the middle point of a chord bisects the angle at the center.

NOTE.—Since the letter heading in this problem is very short, it will be better to place it over each of the two cases separately, instead of running it over the division line, as was done with the long headings of the two cases in Plate I. Put Case I and Case II under the heading, as in the previous plate.

Case II.—*When the sides do not intersect within the limits of the drawing.*

See Fig. 8. In the second rectangle of Plate II draw two lines AB and CD, each $3\frac{1}{2}$ inches long, and inclined toward

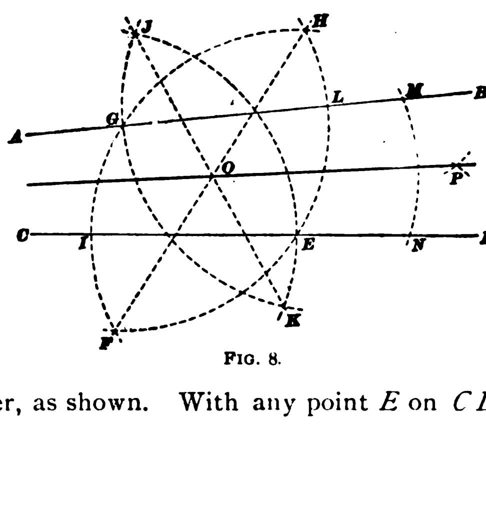


FIG. 8.

each other, as shown. With any point E on CD as center and any convenient radius, describe arc $FIGH$; with G as center and same radius, describe arc $HLEF$, intersecting $FIGH$ in H and F. With L as center and same radius, describe arc KGJ; with I as center and same radius, describe arc JEK, intersecting KGJ in K and J. Draw HF and JK; they intersect at O, a point on the bisecting line. With O as center and the same, or any convenient, radius, describe an arc intersecting AB and CD in M and N. With M and N as centers and any radius greater than one-half MN, describe arcs intersecting at P. A line drawn through O and P is the required bisecting line.

PROBLEM 6.

15. To divide a given straight line into any required number of equal parts.

See Fig. 9. In the third rectangle of Plate II, draw the line AB $3\frac{7}{16}$ inches long. Let AB be a given line that it is required to divide into eight equal parts. Through one extremity A of the line, draw the straight line AC of indefinite length, making an acute angle with AB. Set the dividers as nearly one-eighth of the given line AB as it is possible to do by guess. With the dividers set thus, space off on AC eight equal distances, as AK, KI, IH, etc. Join the points C and B by the straight line CB, and through the points D, E, F, G, etc. draw lines parallel to CB and designate the points where these parallels cut AB as L, M, N, O, P, etc. Then the given line is divided into the eight equal parts BL, LM, MN, NO, OP, etc. Proceed in a similar way for any number of equal parts into which AB is to be divided. Step off the line AB with the dividers and see that the parts are equal.

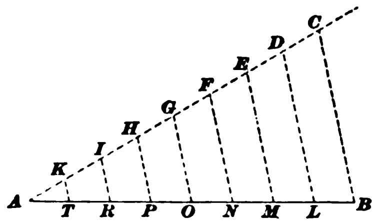


FIG. 9.

This construction is based on the principle that if a straight line is drawn parallel to the base of a triangle and cutting the two sides, it divides the sides proportionally. Since KT is parallel to IR, the base of the triangle AIR, we have

$$\frac{AK}{KI} = \frac{AT}{TR}.$$

But AK was made equal to KI; and, therefore,

since $$\frac{AK}{KI} = 1, \quad \frac{AT}{TR} = 1,$$

or $$AT = TR.$$

Since, IR is parallel to HP, the base of the triangle AHP, we have

$$\frac{AR}{RP} = \frac{AI}{IH}.$$

But, $$AI = 2IH$$

and, therefore, $$\frac{AI}{IH} = 2.$$

Hence, $$\frac{AR}{RP} = 2,$$

or $$AR = 2RP.$$

But, since $$AT = TR,$$

we have $$AR = 2AT.$$

Therefore, $$2AT = 2RP,$$

or $$AT = RP.$$

Thus, we have proved that the three parts AT, TR, and RP are equal. Similarly, we can prove that AT is equal to each of the remaining parts PO, ON, NM, ML, and LB. Thus, the construction is proved correct. The proof of the construction is given in this case in order that the student may see the method and thus be able to work out proofs for himself whenever he wishes to do so.

NOTE.—In a drawing when it is necessary to locate a point by the intersection of two lines, it is very desirable that the two lines should intersect nearly perpendicularly, as in Fig. 10 (*a*); and, if possible, the student should avoid locating a point by the intersection of two lines that intersect at a very acute angle, as in Fig. 10 (*b*). It is for

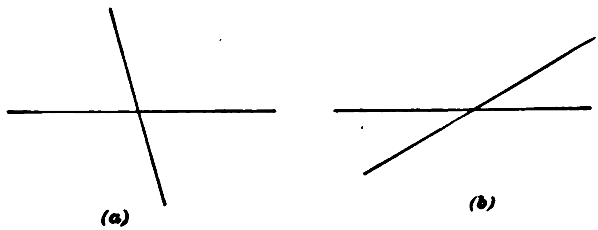


FIG. 10.

this reason that, in Problem 6, the student is directed to make the angle *B A C* an acute angle, and to take *A K*, as nearly as possible, equal to one-eighth of *A B*. When these two precautions are taken, each of the points *T*, *R*, *P*, etc. is located by a good clear intersection of two lines that intersect nearly perpendicularly. By making the angle at *A* obtuse, the student can easily see the value of this note.

PROBLEM 7.

16. Through a given point on a given straight line, to draw a straight line making with the given line an angle equal to a given angle.

See Fig. 11. In the fourth rectangle of Plate II draw the line *A B* 3½ inches long. Let *A B* be the given straight line, *P* the given point, and *M O N* the given angle. It is required to draw through *P* a straight line making with the line *A B* an angle equal to the given angle *M O N*. With the vertex *O* as center and any convenient radius less than *O M*, describe an arc *E F* cutting *O M* in *E* and *O N* in *F*. With *P* as center and the same radius, describe an arc *H D* cutting the line *A B* in *D*. With *D* as center and a radius equal to the chord of the arc *E F*,

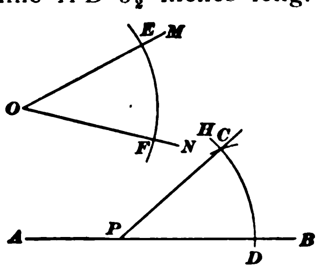


FIG. 11.

describe an arc cutting HD in C. Draw the line CP, and this line makes an angle at P with the line AB equal to the given angle MON; that is, the angle CPB is equal to the angle MON.

PROBLEM 8.

17. To draw an equilateral triangle, one side being given.

See Fig. 12. In the fifth rectangle of Plate II draw AB $2\frac{1}{2}$ inches long. Let AB be the given side. With A as center and a radius equal to AB, describe an arc; with B as center and the same radius, describe an arc cutting the former arc in C. Draw the straight lines CA and CB, and CAB is the required equilateral triangle.

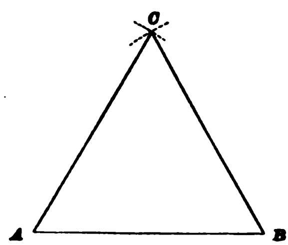


FIG. 12.

PROBLEM 9.

18. The altitude of an equilateral triangle being given, to draw the triangle.

See Fig. 13. In the sixth rectangle of Plate II draw AB $2\frac{1}{4}$ inches long. Let AB be the given altitude. Through the points A and B draw the parallel lines CD and EF perpendicular to AB. With A as center and any radius less than AB, describe the semicircle GH, intersecting CD in G and H. With G and H as centers and the same radius, describe arcs cutting the semicircle in I

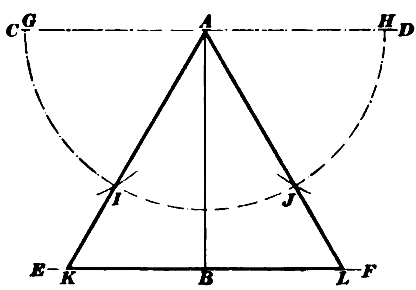


FIG. 13.

and J. Draw $A\,I$ and $A\,J$, and prolong them to meet $E\,F$ in K and L. Then, $A\,K\,L$ is the required equilateral triangle.

This problem finishes Plate II. The directions for inking in, lettering, etc. are the same as for Plate I.

PLATE III.

19. This plate is to be divided like Plates I and II, and the six following problems are to be drawn in the six rectangles.

PROBLEM 10.

20. Two sides and the included angle of a triangle being given, to construct the triangle.

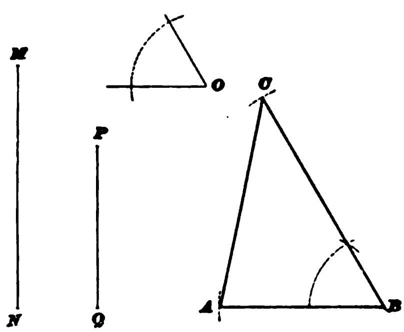


FIG. 14.

See Fig. 14. Let $M\,N$ $2\frac{1}{4}$ inches long and $P\,Q$ $1\frac{7}{8}$ inches long be the given sides, and let O be the given angle. Draw $A\,B$ and make it equal in length to $P\,Q$. Make the angle $C\,B\,A$ equal to the given angle O, and make $C\,B$ equal in length to the line $M\,N$. Draw $C\,A$, and $C\,A\,B$ is the required triangle.

PROBLEM 11.

21. To construct a triangle when the three sides are given.

See Fig. 15. Let $M\,N$ $3\frac{1}{2}$ inches long, $O\,P$ 4 inches long, and $Q\,R$ $2\frac{1}{2}$ inches long be the given sides. Draw $A\,B$ and make it equal in length to MN. With B as center and a radius equal to OP, describe an arc; with A as center and a radius equal to QR, describe an arc cutting the former arc in C. Draw the straight lines AC and BC. Then ABC is the required triangle.

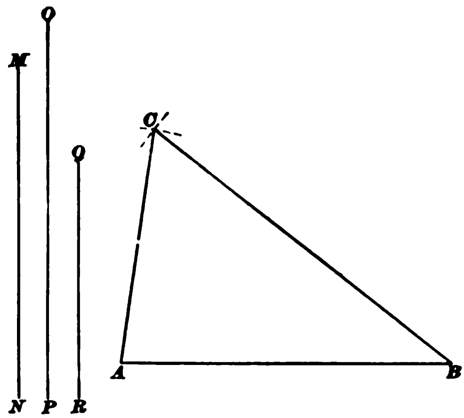


FIG. 15.

This construction is very convenient when a given triangle is to be drawn in a different position or is to be transferred from a drawing to sheet metal, as then the compasses can be set directly from the given triangle.

PROBLEM 12.

22. To draw a parallelogram when the sides and one of the angles are given.

See Fig. 16. Make the given sides MN $2\frac{1}{2}$ inches long and PQ $1\frac{1}{8}$ inches long. Let O be the given angle. Draw AB equal to MN, and draw BC, making an angle with AB equal to the given angle O. Make BC equal to PQ. With C as center and a radius equal to MN, describe an arc. With A as center and a radius equal to PQ, describe an arc intersecting the arc described from C in D. Draw AD and CD, and $ABCD$ is the required parallelogram.

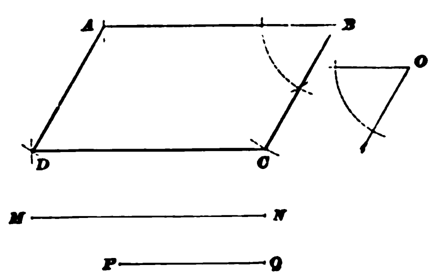


FIG. 16.

PROBLEM 13.

23. To construct a trapezium similar to a given trapezium.

NOTE.—This problem may be used in more than one way. We may copy the figure in exactly the same size as it is given, as is frequently done in pattern drafting when we transfer a figure from a drawing to sheet metal; or, we may reduce or enlarge the figure, as in detailing or in making reproductions of drawings to different scales. For convenience, since we wish to have the problem on the plate of a proportionate size, we shall enlarge the given trapezium.

See Fig. 17. The student is required to copy this figure in exactly the same size on his plate and then to draw a figure similar to it but having the side corresponding to *A D* $4\frac{7}{32}$ inches long. The diagonal *A C* divides the trapezium into the two triangles *A B C* and *A D C*, and the problem of copying the trapezium has resolved itself into copying two triangles, the three sides of each being given.

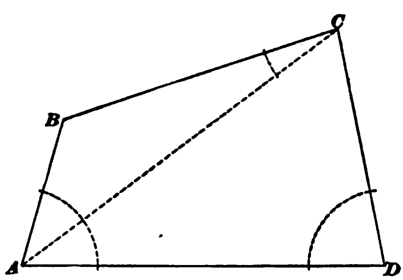


FIG. 17.

Since both the trapeziums are to be drawn in the one rectangle, the point *A* is located $3\frac{5}{8}$ inches from the lower side and $\frac{1}{2}$ inch from the left side. The line corresponding to *A D* is the same length as *A D*; that is, $1\frac{7}{8}$ inches. Now draw the triangle *A D C* according to Art. **21**, and then draw the triangle *A C B* in a similar manner. In this copy it is not necessary to draw the diagonal *A C*.

To draw the enlarged figure, draw a line $4\frac{7}{32}$ inches long, which is to correspond to *A D*. Then draw angles to this line equal to the angles *C A D* and *A D C* of the given figure and produce the sides until they meet in *C*. The triangle *A B C* may now be constructed by drawing angles on *A C* equal to the angles *B A C* and *B C A*, and producing the sides until they meet in *B*. The result will be a trapezium similar to the given trapezium but having the side *A D* $4\frac{7}{32}$ inches long.

Since $4\frac{7}{32}'' \div 1\frac{7}{8}'' = 2\frac{1}{4}$, each line of the larger trapezium is two and one-fourth times as long as the corresponding line of the smaller trapezium. The student can test the accuracy of his drawing by measuring the lines of the larger trapezium and seeing if each one is two and one-fourth times as long as the corresponding line of the smaller trapezium.

24. The problem of Art. **23** admits of practical application in copying irregular figures of any number of sides, it being necessary only to divide the figure, by diagonals, into triangles, and to proceed as in the above construction.

When the figure which is to be copied is small, the method used in drawing the small figure is most convenient; that is, draw the triangles according to Art. **21.** When the figure is so large that the lengths of the lines cannot be taken directly by the compasses, use the method shown in drawing the larger trapezium.

PROBLEM 14.

25. An arc and its radius being given, to find the center.

See Fig. 18. With a radius $1\frac{3}{4}$ inches long draw any arc, as *A B*. Let *A B* be the arc, and *M N*, $1\frac{3}{4}$ inches long, the radius. With *M N* as a radius and any point *C* in the given arc as center, describe an arc. With any other point *D* in the given arc as a center and the same radius, describe an arc intersecting the first in *O*. Then *O* is the required center.

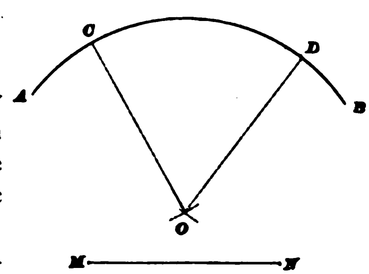


FIG. 18.

A I.—25

PROBLEM 15.

26. To pass a circumference through any three points not in the same straight line.

See Fig. 19. Let *A*, *B*, and *C*, any three points not in a straight line, be the given points. With *A* and *B* as centers and a radius equal to about two-thirds the distance *A B*, describe arcs intersecting each other in *K* and *I*. With *B* and *C* as centers and a radius equal to about two-thirds *B C*, describe arcs intersecting each other in *D* and *E*. Through *I* and *K* and through *D* and *E* draw lines intersecting at *O*. With *O* as center and *O A* as a radius, describe a circle. It will pass through *A*, *B*, and *C*, and is, therefore, the required circle.

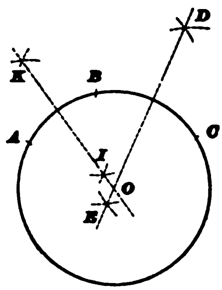


FIG. 19.

This problem completes Plate III. The instructions for finishing this plate are the same as for Plates I and II.

PLATE IV.

27. The problems on this plate are more difficult than any on the preceding plates and will require very careful construction. All the sides of each polygon of the last four problems must be of exactly the same length, so that they will space around evenly with the dividers. The figures should not be inked in until the pencil construction is done accurately. The preliminary directions for this plate are the same as for the preceding ones.

PROBLEM 16.

28. To inscribe a circle in a given triangle; that is, to describe a circle to which each side of the triangle shall be tangent.

See Fig. 20. Draw the line *A B* 3½ inches long. Make *A C* and *C B* each 4 inches in length. Let *A B C* be the given triangle in which it is required to inscribe a circle. Bisect the angles *C A B* and *A B C*, by Art. **14**, producing the bisectors until they intersect in *D*.

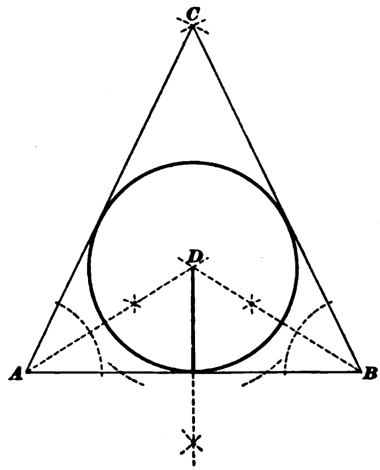


FIG. 20.

From the point *D* draw a perpendicular to the line *A B*, using the method shown in Case I, Art. **8**. With a radius equal to the distance from *D* to the line *A B* and with *D* as center, describe a circle. The sides *A B*, *B C*, and *A C* will be tangent to this circle.

When it is desired to circumscribe a circle about a triangle, use the vertexes of the triangle as the three given points through which a circle is to be drawn and proceed as in Art. **26**.

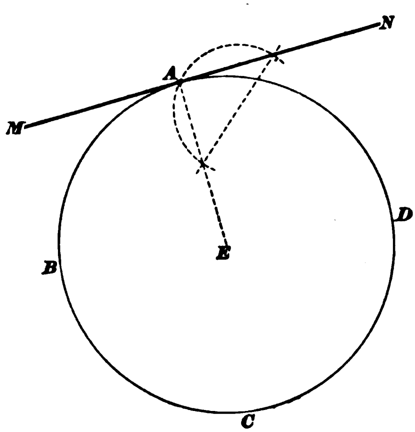


FIG. 21.

PROBLEM 17.

29. At a given point on a circle, to draw a tangent to the circle.

See Fig. 21. Describe a circle 3½ inches in diameter. Let *A* be the given point on the circle through which it is desired to draw a tangent to the circle. Draw the radius *A E*, and by Case II, Art. **7**, erect a perpendicular *M N* to *A E*, producing it in both directions from the point *A*. The line *MN* passes through the point *A* and is the desired tangent to the circle *A B C D*.

PROBLEM 18.

30. To inscribe a square in a given circle.

See Fig. 22. Describe the circle *A B C D* $3\frac{1}{2}$ inches in diameter. Let *A B C D* be the given circle in which it is required to inscribe a square. Draw two diameters, *A C* and *D B*, at right angles to each other. Draw the lines *A B*, *B C*, *C D*, and *D A* joining the points of intersection of these diameters with the circumference of the circle, and thus form the sides of the required square.

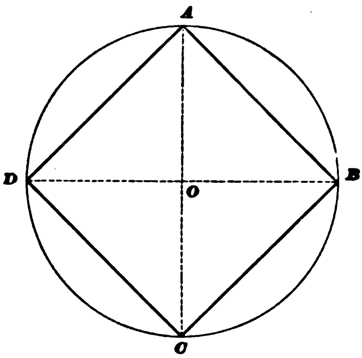


FIG. 22.

PROBLEM 19.

31. To inscribe a regular hexagon in a given circle.

See Fig. 23. With *O* as center and a radius equal to $1\frac{3}{4}$ inches, describe the circle *A B C D E F*. Through *O* draw any diameter *A D*; with *A* as center and a radius equal to the radius of the circle, describe arcs cutting the given circle in the points *B* and *F*; also with *D* as center and the same radius, describe arcs cutting the given circle in *C* and *E*. Draw the straight lines *A B*, *B C*, *C D*, *D E*, *E F*, and *F A*; then *A B C D E F* is the required hexagon.

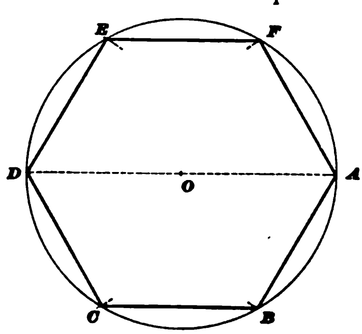


FIG. 23.

PROBLEM 20.

32. To inscribe a regular pentagon in a given circle.

See Fig. 24. With *O* as center and with the compasses set to 1¾ inches, describe the circle *A B C D*. Let *A B C D* be the given circle in which it is required to inscribe a regular pentagon. Draw the two diameters *A C* and *D B* perpendicular to each other. Bisect the radius *O B*, and get the point *I*. With *I* as center and the distance *I A* as a radius, describe an arc cutting *D O* at *J*. With *A* as center and the distance *A J* as a radius, describe an arc cutting the circumference of the given circle at *H*. The chord *A H* is one side of the pentagon. With the dividers set equal to *A H*, step off the circumference; if the end of the fifth division coincides exactly with the beginning of the first, the work has been done correctly, and by joining the points marked by the dividers the pentagon is complete. If the circumference does not space exactly, there is an error in the work and it must be done over.

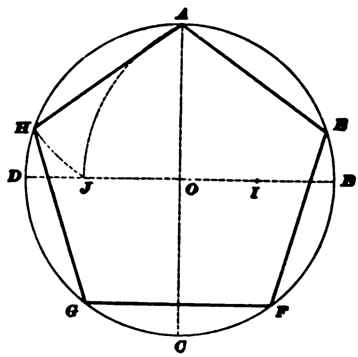


FIG. 24.

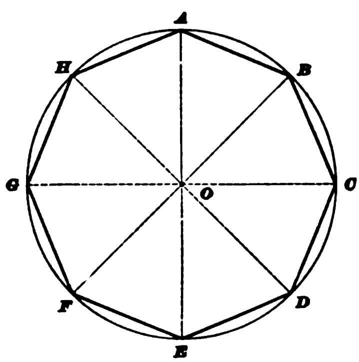


FIG. 25.

PROBLEM 21.

33. To inscribe a regular octagon in a given circle.

See Fig. 25. With *O* as center, with the compasses set to 1¾ inches, describe the circle *A B C D E F G H*. Let this be the given circle in which it is required to inscribe a regular octagon. Draw the two diameters *A E* and *G C* perpendicular to each other. The points *A C E G* divide the circumference into four equal arcs. Bisect one of the chords of the equal arcs, as *A G*, and let the perpendicular bisector of the chord *A G* meet the circumference at *H*. Draw the diameter *H O D*. Bisect another of the chords, as *A C*, and let the perpendicular bisector of the chord meet the arc at *B*; then draw the diameter *B O F*. Straight lines drawn from *A* to *B*, from *B* to *C*, etc. will form the required octagon.

This completes Plate IV; the instructions for finishing this plate are the same as for the preceding plates.

PLATE V.

34. The preliminary directions for preparing this plate are the same as for the preceding plates.

PROBLEM 22.

35. To inscribe a regular polygon of any number of sides in a given circle.

See Fig. 26. With *O* as center and a radius equal to $1\frac{3}{4}$ inches, describe the circle *A 7 C D*. Let *A 7 C D* be the given circle in which it is required to inscribe a regular polygon of any number of sides. Draw the two diameters *D 7* and *A C* perpendicular to each other. Divide the diameter *D 7* into as many equal parts as the polygon has sides. We have chosen to inscribe a heptagon, so that the diameter is divided into seven equal parts. Prolong the diameter *A C* and make *3′ A* equal to three-fourths of the radius *O A*. Through *3′*

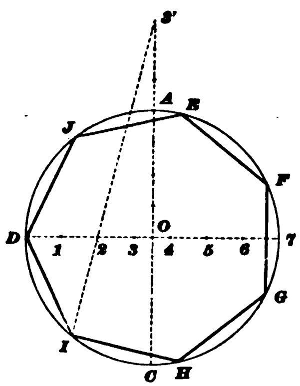


FIG. 26.

and *2*, the second division from *D* on the diameter *D 7*, draw the line *3′ I* cutting the circumference at *I*. Draw the chord *D I*, and it is one side of the required polygon. With the dividers set equal to *D I*, step off the circumference. The end of the seventh division should coincide exactly with the beginning of the first; if it does not, the work has not been done correctly. By joining the ends of the divisions marked by the dividers, the regular heptagon will be inscribed.

The draftsman frequently solves this problem by "spacing." This method requires practice, as the spacing may require several adjustments of the dividers, but it should be practiced by the student, as the pattern draftsman must be expert in the use of tools.

PROBLEM 23.

36. To draw a circle of a given radius, to which both sides of a given angle shall be tangent.

NOTE.—As there are two cases of this problem, requiring two figures on the plate, the line of letters will be run clear across both figures, as shown in Problems 2 and 3 of Plate I.

Case I.—*When the given angle is a right angle.*

See Fig. 27. Draw the lines *B C* and *A C* perpendicular to each other, making them 3½ inches long. Draw *M N*, the given radius, 1½ inches long. Let *M N* be the given radius and *A B C* the right angle whose sides *A B* and *B C* are to be tangent to the circle. With the compasses set equal to the given radius and with *B* as center, describe arcs cutting *A B*

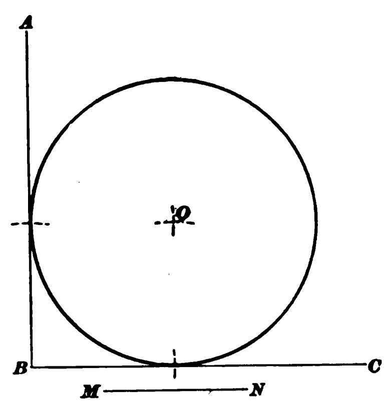


FIG. 27.

and $B\ C$. Now, using the intersection of these arcs with the sides of the angle $A\ B\ C$ as centers and with the same radius, describe arcs intersecting at O. With O as center and the same radius, describe a circle, and this circle is the one to which the sides $A\ B$ and $B\ C$ of the given angle will be tangent.

Case II.—*When the given angle is not a right angle.*

See Fig. 28. Draw $A\ B\ C$ an acute angle of 60° by using the **T** square and triangle, and make each side of the angle $3\frac{1}{2}$ inches long. Draw $M\ N$ $1\frac{1}{2}$ inches long. Let $A\ B\ C$ be the given angle whose sides $A\ B$ and $B\ C$ are to be tangent to a circle whose radius is $M N$. At a distance equal to $M\ N$ ($= 1\frac{1}{2}$ inches) from $A\ B$, draw a line parallel to $A\ B$, and at the same distance from $B\ C$ draw a line parallel to $B\ C$. The point O, the intersection of these parallel lines, is the center of the required circle, to which each side of the angle $A\ B\ C$ is tangent.

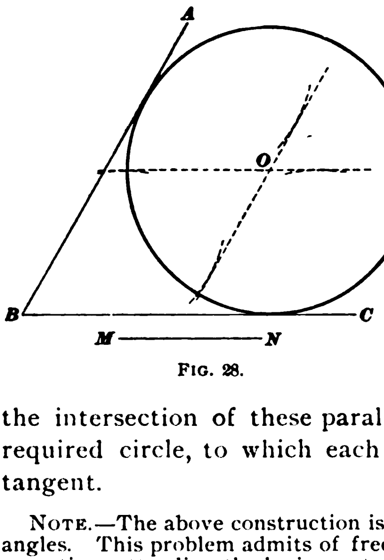


FIG. 28.

NOTE.—The above construction is also applied in the case of obtuse angles. This problem admits of frequent application in the practical operations attending the laying out of work.

PROBLEM 24.

37. To describe a circle of a given radius, to which a given arc and a given line shall be tangent.

See Fig. 29. Describe the arc $A\ B$ with a radius of $1\frac{1}{2}$ inches and a center located at a point C $2\frac{1}{2}$ inches above the left end of the given line $D\ E$, which is drawn 4 inches long. Draw $M\ N$ $1\frac{3}{4}$ inches long. Let $A\ B$ be the given arc and DE the given line, tangent to each of which a circle with a radius equal to MN is to be drawn. From the point C draw a line of indefinite length cutting the arc AB in the point F. From the point F set off on this line the distance FG equal to MN. With a radius CG and C as center, describe an arc. At a distance equal to MN (= 1¾ inches), draw a line parallel to DE. The intersection O of the parallel line thus drawn with the arc drawn through G is the center of the required circle, which is then to be described with the given radius MN.

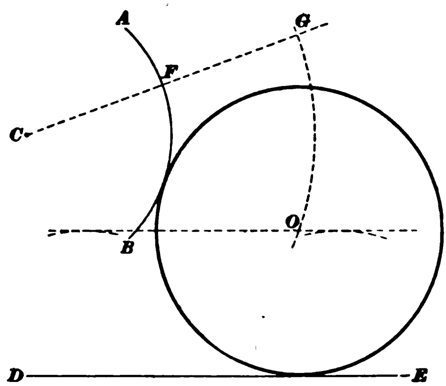


FIG. 29.

NOTE.—In the problem as above stated, the center of the given arc is located at C. It frequently happens that this center is not given, in which case it is first necessary to find the location of the center.

38. The problem of drawing a line parallel to a given line and at a given distance from it is of frequent occurrence in practical work. The method of laying off on a

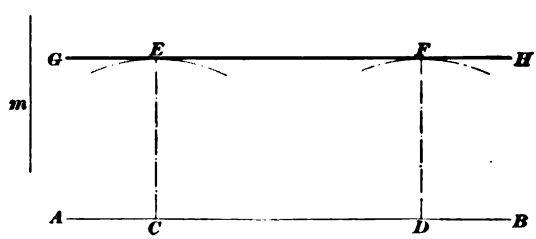


FIG. 30.

perpendicular to the given line the given distance and then drawing a line through the point thus found parallel to the given line is very accurate. But since a tangent to a circle is perpendicular to a radius drawn to the point of tangency, there is a shorter method which, when the distance between the parallel lines is not too great, gives good results.

Suppose it is required to draw a line parallel to *A B* and at a distance from it equal to *m*, Fig. 30. Take as centers any points, as *C* and *D*, at or near the ends of the line *A B*, and with a radius equal to *m* describe short arcs. Then set the ruler or T square so that the edge just touches these arcs, and draw the line *G H*, which is the parallel required. The parallels in Case II, Problem 23, and in Problem 24 are easily drawn by this method. The drawing of parallels by this method is not a problem to be put on the drawing plate.

PROBLEM 25.

39. The side of a regular polygon being given, to construct the polygon.

See Fig. 31. Draw *A C* 1½ inches long and let it be required to draw any regular polygon, say an octagon, on the line *A C*. Produce *A C* to *B*, making *B C* equal to *A C*. From *C* as center with a radius equal to *C A*, describe the semicircle *A 1 2 3 4 5 6 7 B* and divide it into as many equal parts as there are sides in the required polygon (in this case eight). To divide this semicircle into eight equal parts, the student may step it off with the

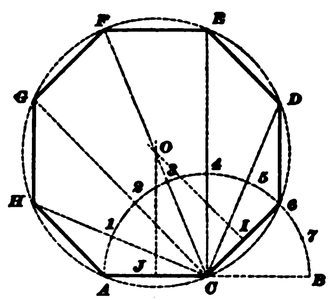


FIG. 31.

dividers by Art. **76**, *Instrumental Drawing*, or he may draw on a separate sheet a circle with a radius equal to *A C* and then inscribe in this circle a regular polygon of sixteen sides. Then he can set the dividers to the length of one of these sides, and with them so set, space off at once the semicircle *A 1 2 3 4 5 6 7 B*. From the point *C* and through *6*, the second division from *B*, draw the straight line *C 6*.

Draw the perpendicular bisectors of *A C* and *C 6*, and produce them until they meet, as at *O*. From *O* as center and with *O C* as a radius, describe the circle *C A H G F E D*-*6*. From *C* and through the points *1*, *2*, *3*, *4*, *5* in the semicircle, draw lines *C 1*, *C 2*, *C 3*, *C 4*, and *C 5*, meeting the circumference in the points *H*, *G*, *F*, *E*, and *D*. Joining the points *6* and *D*, *D* and *E*, *E* and *F*, etc. by straight lines, will complete the required polygon.

PROBLEM 26.

40. To find an arc of a circle having a known radius, which shall be equal in length to a given straight line.

NOTE.—There is no exact method, but the following approximate method is close enough for all practical purposes, when the required arc does not exceed $\frac{1}{6}$ of the circumference; that is, when the given line is not longer than the known radius.

Problems 26 and 27 are both to be put in one rectangle. To do this put the figures on the same horizontal line and leave space for the lettering for both problems at the top of the rectangle.

See Fig. 32. Draw *A C* 2 inches long. At *A* erect the perpendicular *A O*, and make it equal in length to the given radius, say 3 inches long. With *O A* as a radius and *O* as center, describe the arc *A E*. Let *A C* be the given line and let it be required to find on *A E* an arc equal to *A C*. Divide the given line *A C* into four equal parts, *A D* being the first of these parts, counting from *A*. With *D* as center and a radius *D C*, describe the arc *C B* intersecting *A E* in *B*. The length of the arc *A B* very nearly equals the length of the straight line *A C*.

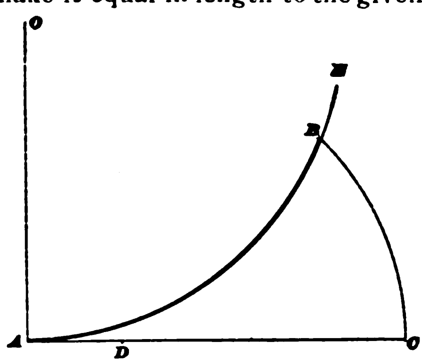


FIG. 32.

PROBLEM 27.

41. An arc of a circle being given, to find a straight line of the same length.

This is also an approximate method, but close enough for practical purposes, when the arc does not exceed $\frac{1}{6}$ of the circumference.

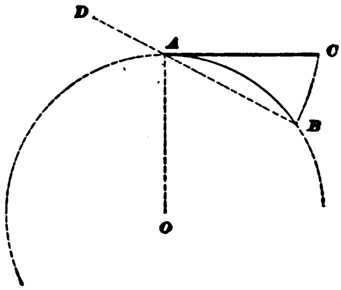


FIG. 33.

See Fig. 33. Let $A B$ be the given arc. For this problem, choose the arc so that the radius will not exceed $1\frac{1}{4}$ inches. Find the center O of the arc, and draw the radius $O A$. At A draw $A C$ perpendicular to the radius (it is, of course, tangent to the arc). Draw the chord $A B$ and prolong it to D, so that $A D = \frac{1}{2}$ the chord $A B$. With D as center and a radius $D B$, describe the arc $B C$ cutting $A C$ in C. $A C$ will be very nearly equal to the arc $A B$.

This problem completes Plate V. The instructions for finishing the plate are the same as for the preceding ones.

PLATE VI.

42. Plate VI is to be prepared in the same manner as the preceding plates.

PROBLEM 28.

43. To draw an oval.

See Fig. 34. Draw the line $A B$ $2\frac{3}{4}$ inches long. On $A B$ as a diameter describe a circle $A C B G$. Through the center O draw a line perpendicular to $A B$ cutting the circumference $A C B G$ in C. Draw the straight lines $B C$ and

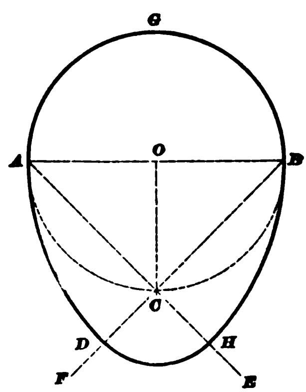


FIG. 34.

$A C$ and produce them to F and E. With B and A as centers and the diameter $A B$ as a radius, describe arcs terminating in D and H, the points of intersection of these arcs with $B F$ and $A E$. With C as center and $C D$ as a radius, describe the arc $D H$. The curve $A D H B G$ is the required oval.

PROBLEM 29.

44. To draw an ellipse, the diameters being given.

See Fig. 35. Draw $A C$ $3\frac{1}{2}$ inches long and $B D$ $2\frac{1}{4}$ inches long, perpendicular bisectors of each other. Then $D O = O B$ and $A O = O C$. Let $A C$ be the long diameter, or major axis, and $B D$ the short diameter, or minor axis, of the ellipse which is to be constructed. With O as center and $O A$ as a radius, describe a circle; with the same center and $O D$ as a radius, describe another circle. Draw at random a number of radii of the larger circle, such as $O r$, $O s$, $O t$, etc. These radial lines will divide the smaller circle into the same number of parts that the larger one has been divided into. Through the points r', s', t', etc., where these radii cut the smaller circle, draw horizontal lines $r' 1$, $s' 2$, $t' 3$, etc., and through the points r, s, t, etc. on the larger circle draw vertical lines $r\,1$, $s\,2$, $t\,3$, etc.; the points 1, 2, 3, etc. where the vertical and horizontal lines meet are points on the ellipse. Trace a curve through the points thus found by placing an irregular curve on the drawing in such a manner that one of its bounding lines will pass

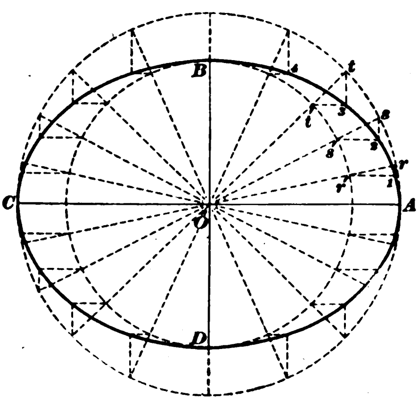


FIG. 35.

through three or more points, judging with the eye whether the curve so traced bulges out too much or is too flat. Then adjust the curve again, so that its bounding line will pass through several more points, and so on, until the curve is completed. Care should be taken to make all changes in curvature as gradual as possible, and all curves drawn in this manner should be drawn in pencil before being inked in. It requires considerable practice to be able to draw a good curved line in this manner by means of an irregular curve, and the general appearance of a curve thus drawn depends a great deal on the student's taste and the accuracy of his eye.

PROBLEM 30.

45. To draw an ellipse by circular arcs.

This is not a true ellipse, but is very convenient for many purposes.

See Fig. 36. Draw $A\,B$ and $C\,D$ in the same manner and the same length as $B\,D$ and $A\,C$ were drawn in Problem 29. On the major axis $A\,B$, set off $A\,a = C\,D$, the minor axis, and divide $a\,B$ into three equal parts. With O as center and a radius equal to the length of two of these parts, describe arcs cutting $A\,B$ in d and d'. Upon $d\,d'$ as a side, construct two equilateral triangles $d\,b\,d'$ and $d\,b'\,d'$. With b as center and a radius equal to $b\,D$, describe the arc $g\,D\,f$ intersecting $b\,d$ and $b\,d'$ in f and g. With the same radius and b' as a center,

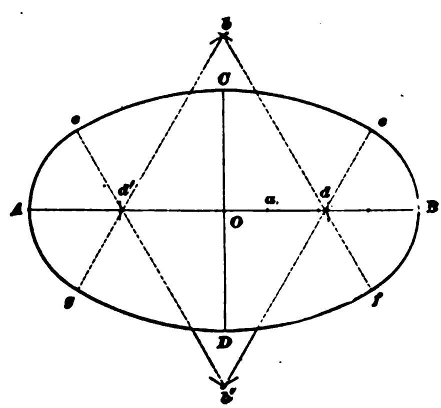


FIG. 36.

describe the arc *c C e* intersecting *b' d'* and *b' d* in *c* and *e*. With *A* and *B* as centers and a radius equal to the chord of the arc *A c* or the arc *B e*, describe arcs cutting *A B* very near to *d'* and *d*. With the points of intersection of these arcs with *A B* as centers and the same radius, describe the arcs *c A g* and *e B f*. These four arcs form a good approximation to an ellipse.

PROBLEM 31.

46. To draw an ellipse within a given rectangle.

The sides of the rectangle correspond in length, respectively, to the major and minor axes of the ellipse.

See Fig. 37. Draw the sides *A B* and *D C* of the given rectangle *A B C D* $3\frac{1}{2}$ inches long, and the sides *A D* and *B C* $2\frac{1}{4}$ inches long. Draw the center lines *E F* and *G H*, and observe that they divide the rectangle into four equal parts.

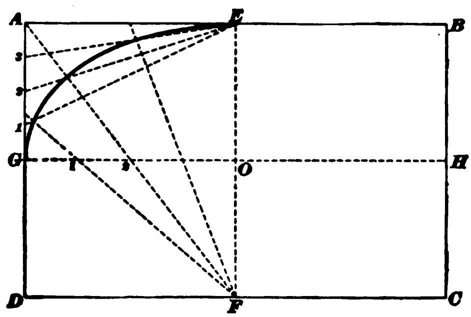


FIG. 37.

Divide the lines *G A* and *G O* into the same number of equal parts, as *G–1*, *1–2*, etc. From the point *E* draw lines to the intermediate points of division on *A G*. From point *F* draw lines through the intermediate points on the line *G O*, producing them until they intersect the lines drawn from *E*. Now, commencing at *G*, trace a line with the irregular curve through the points where the lines from *F* meet the corresponding lines from *E*; that is, where the line *F 1* meets the line *E 1* etc. This will be one-quarter of the required ellipse, and the same construction applied to the balance of the figure will complete the ellipse. The student will complete the entire figure on Plate VI.

PROBLEM 32.

47. In a given ellipse, to find centers by which an approximate figure may be constructed by arcs.

See Fig. 38. Draw an ellipse of the same dimensions as the one used in Art. **46**, using the method given in that article. Draw the major and minor axes, as shown in Fig. 38. The construction of this problem is somewhat similar to that given in Problem 31, but is applied in an inverse way. At *A* erect *A E* perpendicular to *A O* and equal in length to *B O*. Divide *A E* into a number of equal parts, as in Problem 31. Draw lines connecting these points with *B*, thus, by the intersection of these lines with the ellipse, establishing the points *F*, *G*, and *H*. The spaces *A F*, *F G*, *G H*, and *H B* on the ellipse are now to be treated as arcs of circles, the centers of which are to be found by drawing the respective chords and drawing perpendicular bisectors to them in a manner similar to that used in Art. **6.**

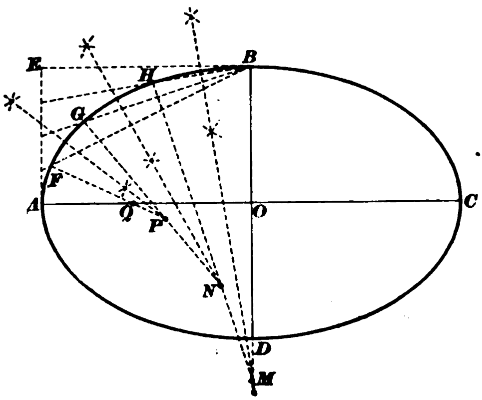


FIG. 38.

In order, however, to provide definite data from which to locate the centers, it is necessary first to produce the perpendicular erected at the middle point of the chord *H B* until it intersects the line *B D*, which, in the case of ellipses of certain proportions, may have to be extended beyond the line of the ellipse, as shown in Fig. 38. This establishes the point *M*, which is the center for the arc *H B*, as well as for the corresponding arc from *B* toward *C*. Draw *H M* and bisect the chord *G H*. Erect a perpendicular at the middle point of *G H*, producing it until it intersects *H M* at *N*.

The point N is the second center for the figure. The chord $F\,G$ is to be bisected in a similar manner and the perpendicular erected, thus locating the center P, as shown in Fig. 38. At the intersection of $F\,P$ with the line $A\,C$, establish the point Q. The points M, N, P, and Q are the centers from which one-quarter of a figure nearly approaching the ellipse $A\;B\;C\;D$ may be constructed by arcs described with the radii $M\,H$, $N\,G$, $P\,F$, and $Q\,A$.

NOTE.—The figure thus constructed will not be an exact ellipse, but for many practical purposes, such as in the construction of elliptical articles, both of flaring and straight work, in the sheet-metal trades, it is a very convenient method. An adaptation of this will frequently save working out patterns by triangulation.

PROBLEM 33.

48. To draw a parabola, the axis and longest double ordinate being given.

Any line, $B\,A$ or $A\,C$, drawn perpendicular to $O\,A$, and whose length is included between $O\,A$ and the curve, is

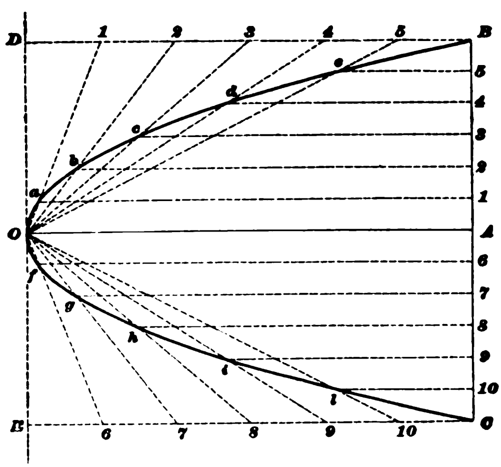


FIG. 39.

called an **ordinate.** Any line, as $B\,C$, both of whose extremities rest on the curve, is called a **double ordinate.**

See Fig. 39. Draw the axis *O A* 3½ inches long and the longest double ordinate *B C* 3 inches long, making *B A* equal to *A C*. Draw *D E* through the other extremity of the axis and perpendicular to it; also draw *B D* and *C E* perpendicular to *B C* and intersecting *D E* in *D* and *E*. Divide *D B* and *A B* into the same number of equal parts, as shown (in this case six); through the vertex *O* draw *O 1*, *O 2*, etc. to the points of division on *D B*, and through the corresponding points *1*, *2*, etc. on *A B* draw lines parallel to the axis. The points of intersection of these lines, *a*, *b*, *c*, etc., are points on the curve, through which it may be traced. In a similar manner, draw the lower half *O f g h i l C* of the curve.

This completes Plate VI. The instructions for finishing this plate are the same as for the preceding ones.

PLATE VII.

49. On this plate there are five problems instead of six. It should, however, be divided into six equal parts, or divisions, as were the previous plates. The two right-hand divisions are used for the last figure, which is too large to put in one.

PROBLEM 34.

50. Within a given circle, to draw any number of semicircles tangent to the given circle, the diameters of the semicircles forming a regular polygon.

See Fig. 40. With a radius 1¾ inches long describe the circle *A B C D*. Let it be required to draw within this circle three semicircles tangent to this circle, the diameters of the semicircles forming a regular triangle. Draw the two diameters *A C* and *B D* perpendicular to each other. Beginning at *D*, divide the circumference of the circle into six equal parts (= twice the number of semicircles to be drawn). Let *E*, *F*, *B*, *G*, *H*, and *D* be the points of division of the circumference. Draw the diameters *E G* and *H F*. Draw the line *D A* intersecting *E G* at *I*. Then with *O* as center and *O I* as a radius, describe a circle intersecting *O B* in *K* and *O H* in *M*. Draw the line *I K* intersecting *O F* in *J*; draw the line *K M* intersecting *O G* in *L*; and draw the line *M I* intersecting *O D* in *N*. Now with *N*, *J*, and *L* as centers and *N D* as a radius, describe the semicircles *M D I*, *I F K*, and *K G M*. These semicircles are each tangent to the given circle and their diameters form the equilateral triangle *I K M*.

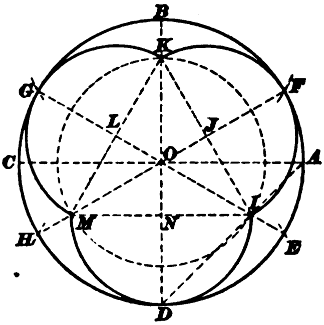


FIG. 40.

PROBLEM 35.

51. To divide a circle into any number of parts that are equal in area.

See Fig. 41. With *O* as center and a radius 1¾ inches long, describe a circle, which is to be divided into four parts equal in area. Draw any diameter *A B*. Divide *A B* into eight equal parts (= twice the number of parts into which the circle is to be divided). Let the points of division be *1*, *2*, *3*, *O*, *5*, *6*, *7*. With the points *1* and *7* as centers and *1 A* as a radius, describe semicircles on opposite sides of the diameter *AB*. With the points *2* and *6* as centers and *2 A* as a radius, describe semicircles on opposite sides of the diameter *A B*. These semicircles should join each other at the point *O*. With the points *3* and *5* as centers and *3 A* as a radius, describe semicircles

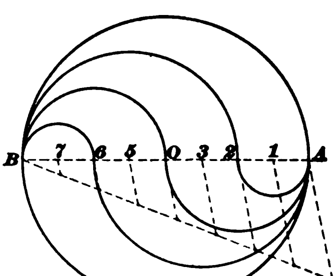


FIG. 41.

on opposite sides of the diameter; the semicircle with *3* as center should join the one with *7* as center at the point *6*, and the semicircle with *5* as center should join the one with *1* as center at the point *2*. The parts of the circle mapped out by these semicircles are equal in area.

PROBLEM 36.

52. To divide a circle into concentric circular rings that are equal in area.

See Fig. 42. With O as center and a radius $1\frac{3}{4}$ inches long, describe a circle. Let it be required to divide this circle into four concentric rings which are equal to each other in area. Draw the diameter $A\ B$. Divide $O\ B$ into four equal parts (= as many parts as there are to be rings). Let the points of division be *1*, *2*, and *3*. On $O\ B$ as diameter, describe a semicircle. At the points *1*, *2*, and *3*, erect perpendiculars to the radius $O\ B$, meeting the semicircle just drawn in the points C, D, and E. With O as center and radii equal to $O\ E$, $O\ D$, and $O\ C$, describe circles. These circles will divide the given circle into four rings that are equal in area.

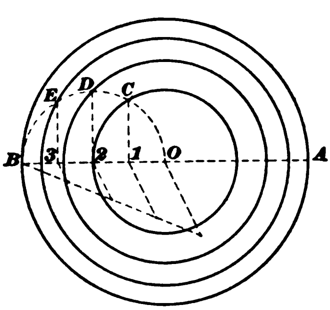


FIG. 42.

As an exercise the student can measure the radii of the different circles and find the area of the circular rings, and thus prove his work.

PROBLEM 37.

53. A chord and a point on the arc of a circle, or three points on the arc being given, to describe the arc by points, that is, without locating the center.

See Fig. 43. Draw *A B* 3 inches long and making an angle of about 15° with the horizontal. Take a point *C* nearer the point *B* than the point *A* and distant $\frac{3}{4}$ inch from the line *A B*. Draw *A C* and *B C*. With *A* and *B* as centers and the same radius, describe arcs, as *D L* and *F K*. On the arc *D L* mark the point *E*, so that the angle *C A E* is equal to the angle *A B C*, and on the arc *F K* mark the point *G*, so that the angle *G B C* is equal to the angle *C A B*. Divide the arcs *D E* and *F G* into the same number of equal parts, in this case five, as we wish to locate four points between the points *A* and *B* on the arc which is to be drawn. The points of division of the arc *F G* are numbered *1, 2, 3, 4* from *G*, and the points of division on the arc *D E* are numbered *1, 2, 3, 4* from *D*. Through *A* and the points *1, 2, 3, 4* on *D E* draw lines, and through *B* and the points *1, 2, 3, 4* on *F G* draw lines. Then the points where the lines through *B* meet the corresponding lines through *A* are points on the arc. That is, the point where the line *A 1* meets the line *B 1* is a point on the arc. With the irregular curve trace the arc through these points and the points *A*, *B*, and *C*. To get a point on the arc below the point *B*, lay off on *G K* the point *H*, making *G H* equal to *G 1*, and draw *H B*; then lay off from *D* on the arc *D L* a distance equal to *D 1* and through this point and *A* draw a line; the point where this line meets the line *H B* is on the arc.

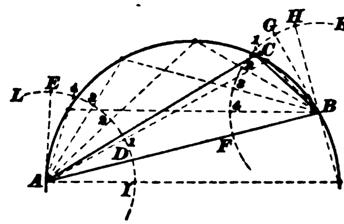


FIG. 43.

PROBLEM 38.

54. The **helix** is a curve formed by a point moving around a cylinder and at the same time advancing along its length a certain distance; this forms the winding curved line shown in Fig. 44. The center line *A O*, drawn through the cylinder, is called the **axis** of the helix. All lines perpendicular to the axis and terminated by the helix are of the same length, being equal to the radius of the cylinder. The distance *B 12* that the point advances lengthwise during one revolution is called the **pitch.**

55. To draw a helix, the pitch and the diameter being given.

See Fig. 44. As mentioned before, this figure occupies two spaces of the plate. The diameter of the cylinder is $3\frac{1}{2}$ inches, the pitch of the helix is 2 inches, and a turn and a half of the helix is to be shown. The rectangle *F B E D* is a side view of the cylinder, and the circle *1' 2' 3' 4'*, etc. is a bottom view. It will be noticed that one-half of a turn of the helix is shown dotted; this is because that part of it is on the other side of the cylinder, and cannot be seen. Lines that are hidden are drawn dotted. Draw the axis *O A* in the center of the space. Draw *F D* $3\frac{1}{2}$ inches long and 4 inches from the top border line; on it construct a rectangle whose height *F B* = 3 inches. Take the center *O* of the circle

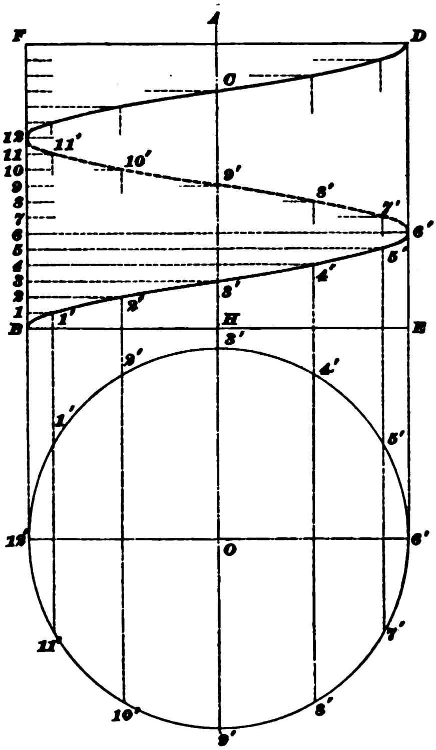


FIG. 44.

$2\frac{3}{4}$ inches below the point *H* on the axis *A O*, and describe a circle having a diameter of $3\frac{1}{2}$ inches, equal to the diameter of the cylinder. Lay off the pitch from *B* to *12* equal to

2 inches, and divide it into a convenient number of equal parts (in this case 12), and divide the circle into the same number of equal parts, beginning at one extremity of the diameter *12′ O 6′*, drawn parallel to *B E*. At the point *1′* on the circle divisions, erect *1′–1′* perpendicular to *B E*; through the point *1* of the pitch divisions draw *1–1′* parallel to *B E*, intersecting the perpendicular in *1′*, which is a point on the helix. Through the point *2′* erect a perpendicular *2′–2′*, intersecting *2–2′* in *2′*, which is another point on the helix. So proceed until the point *6* is reached; from here on, until the point *12* of the helix is reached, the curve will be dotted. It will be noticed that the points of division *7′*, *8′*, *9′*, *10′*, and *11′* on the circle are directly opposite the points *5′*, *4′*, *3′*, *2′*, and *1′*; hence, it was not necessary to draw the lower half of the circle, since the point *5′* could have been the starting point, and the operation could have been conducted backwards to find the points on the dotted upper half of the helix. The other full-curved line of the helix can be drawn in exactly the same manner as the first half.

PLATE VIII.

56. There are but four problems on this plate, and the drawing is, therefore, to be divided into four equal parts, by means of a horizontal and a single vertical line. These are to be treated in the same manner as the division lines of the preceding plate.

These problems are more in the nature of technical construction problems than those which have been given for the preceding plates, and they illustrate in a simple way methods of reaching conclusions which the student will be called on to exercise in his later work.

PROBLEM 39.

57. To find points through which two indefinite arcs having a common center may be drawn.

The corresponding chords of a small portion of the arcs, as well as the perpendicular or radial distance between the chords, are given.

See Fig. 45. Let *A B* and *C D* be the given chords. Make *A B* $1\frac{1}{2}$ inches long and *C D* $1\frac{1}{4}$ inches long, the radial distance *E F* between them being $1\frac{3}{4}$ inches. Since a radius drawn to a chord at its middle point is perpendicular to the chord, *E* and *F* being the middle points of their respective chords, it is apparent that the center for describing these arcs is located somewhere on the line *E F* when prolonged in the direction of *F*. The distance to this center being greater than it is practicable to reach with the ordinary drawing instruments, it is the purpose of this problem to give a convenient method of locating points through which a continuation of the arcs, of which *A B* and *C D* are the chords, may be described by means of the irregular curve. Draw *A C* and *B D*. With *D* as a center and a radius *D A*, describe the arc *A P*, as shown in Fig. 45. With *B* as a center and *B A* as a radius, describe an arc intersecting the arc *A P* at *P*. With *B* as a center and *B C* as a radius, describe an arc *C Q*. With *D* as center and *D C* as a radius, describe an arc intersecting the arc just drawn at *Q*. Then *P* is a point on the circle of which *A B* is a chord, and *Q* is a point on the circle of which *C D* is a chord. Draw *B P*, *D Q*, and *P Q*. More points are to be located in a similar manner, the trapezoid *B P Q D* being treated in the same way as *A B D C*. The figure is to be extended until three additional points for each arc have been located, after which one curve is to be drawn, by means of the irregular curve, through the points *A*, *B*, *P*, etc., and one through the points *C*, *D*, *Q*, etc.

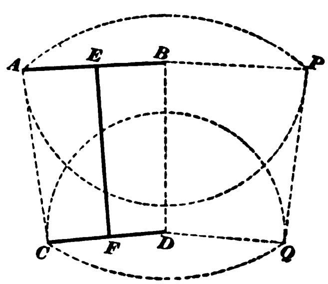


FIG. 45.

PROBLEM 40.

58. To draw a spiral.

See Fig. 46. Draw the line *A C* 4½ inches long. Locate the center *o* by bisecting *A C*, and with a radius *o A* describe the circle *A B C D*. Also, describe the small circle *E F* with a radius of ¼ inch. This circle is called the **eye** of the spiral, and may, in this construction, be of any proportionate diameter. Divide the circle *A B C D* into twelve equal parts, and to the points of division draw radial lines from *o*, as shown in Fig. 46. Divide *A E* into as many equal parts as the spiral is to have revolutions, in this case, two. Divide each of these divisions, *A G* and *G E*, into as many equal parts as there are radial lines. Now, with the compasses, using *o* as a center and *o–1* as a radius, describe *1–1'*; with *o–2* as a radius, describe *2–2'*; and with *o–3* as a radius, describe *3–3'*; etc., carrying each succeeding division on the line *A E* to the next radial line, as shown in Fig. 46.

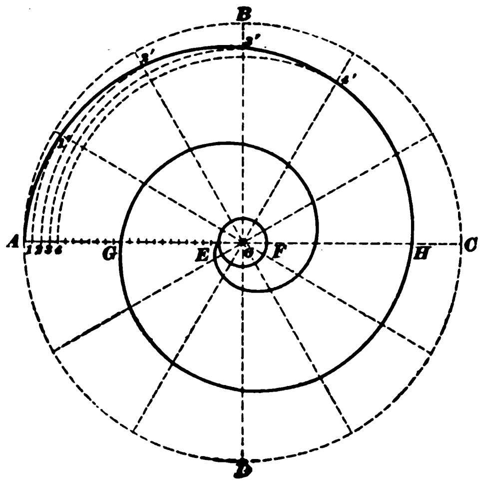


FIG. 46.

In the above construction, the circle *A B C D* was divided into twelve parts, but the student will observe that, in the case of large spirals, it will be necessary to have the points through which the curve of the spiral is traced nearer together, so that the curve may be more easily traced. This may be done by dividing the circle into a *larger* number of parts and drawing more radial lines, but care must be taken to see that the distances on the line *A E*, Fig. 46, between the beginning and the ending of each revolution, as at *A G* and *G E*, are divided into the same number of equal parts as there are radial lines.

It is possible to draw a curve very nearly resembling that of the spiral *A 1′ 2′ 3′*, etc. by a series of arcs, which may be described as follows: With a radius *o 1* and with *A* and *1′*, respectively, as centers, describe arcs intersecting each other near the point *o*, and then use the intersection of these arcs as a center from which to describe the arc *A 1′*. The arc *1′–2′* may then be described from a center found in the same way, using the points *1′* and *2′* as centers and the distance *o 2* as a radius.

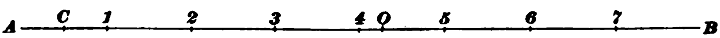


FIG. 47.

In a case where the total width of the spiral (*A H*, Fig. 46) is given, as well as the number of revolutions, the construction is as follows: In Fig. 47, *A B*, the width of the spiral is divided into twice as many equal parts as there are revolutions in the required spiral. In Fig. 47 the construction is for a spiral having four revolutions; *A B* is therefore divided into eight equal parts. Bisect *A 1* at *C* and bisect *C B* at *O*, at which point is located the center of the eye of the spiral, and its radius is equal to the distance from *O* to that division point which is removed from *A* the same number of spaces as the spiral has revolutions; thus, *O 4* is the radius of the eye. The spiral may now be completed by the same construction as in Problem 40.

The student may construct this figure as an exercise, but is not required to complete the figure on Plate VIII, it being merely an adaptation of Problem 40 which may frequently be used in practical work.

PROBLEM 41.

59. The **classic** or **Ionic volute,** Fig. 48, unlike that of the spiral of Problem 40, is constructed by quadrants, the centers of which are established by several methods. It is of frequent occurrence as an architectural ornament, and is constructed to certain proportions, which should be carefully observed. It is used principally as a part of a capital, but the metal worker is often called on to use the ornament as a finish to certain moldings, the upper members of which are carried around the curve, gradually diminishing in size, until they disappear altogether in the eye. The method of adapting the proportion of the curves is as follows: In Fig. 48, *A B*, the height of the volute, is represented by the number 13, *B C* by 11, and *C D* by 9. *E F* is one-half of *A B*, and *E D* is one-half of the sum of *C D* and *E F*.

60. To draw a classic or Ionic volute.

See Fig. 48. Draw *A B* 4½ inches long and divide it into thirteen equal parts. Draw *B C* at right angles to *A B*

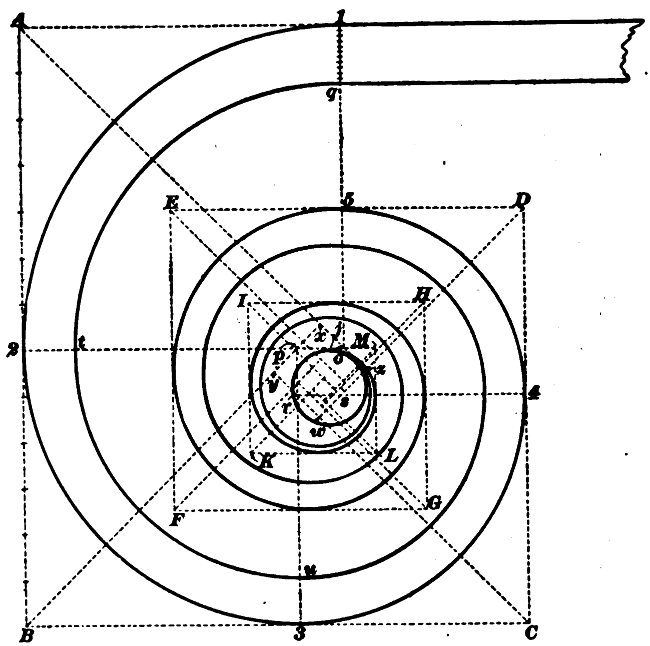


FIG. 48.

and make the length equal to eleven of the spaces on *A B*. Draw also *C D*, as in Fig. 48, making its length equal to nine of the spaces on *A B*. As noted in Art. **59,** *E F* is one-half of *A B*, while *D E* is one-half of $(CD + EF = 9 + 6\frac{1}{2})$, or $7\frac{3}{4}$ spaces. This completes the part of the construction lines necessary for one revolution, and the similar lines of the next and succeeding revolutions are always one-half of their corresponding lines in the preceding revolution. Thus, *E F* is one-half of *A B*, *F G* one-half of *B C*, etc.

To find the centers from which the quadrants are described, draw lines with the 45° triangle, as shown, from *A*, *B*, *C*, and *D*. The intersection of these four lines at the center forms the square *w y x z*. Bisect the side *x z* at *o*, and

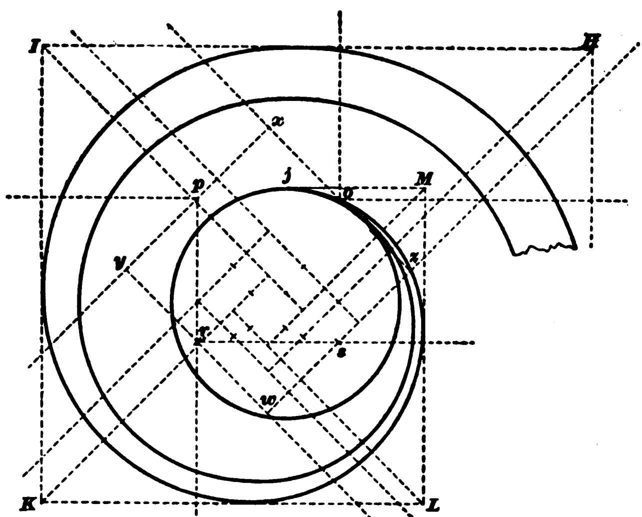


FIG. 49.

with *o* as center describe the arc *1–2* with the radius *o–1*. Bisect the line *x y* at *p*, and with *p* as center describe the arc *2–3* with the radius *p 2*; and then bisect the lines *y w* and *w z* in *r* and *s*, and describe the arcs *3–4* and *4–5* with the radii *r 3* and *s 4*. This completes the curve of the first revolution.

The second and third revolutions are described in a similar manner, using *E*, *F*, *G*, and *H* and *I*, *K*, *L*, and *M* as the points from which to draw the lines to form the small squares at the center.

Find the radius of the eye of the volute by dividing the distance from *A* to *B* into 16 equal parts. The eye is then made by describing, with a radius equal to one of these parts, a circle that is tangent to the end of the final revolution at *j*. The band *q* is drawn by setting off its primary dimension on the radial line *o 1*, as at *q*, and then dividing the distance *q 1* into eleven equal parts. The width of the band at *2* will then be ten of these parts, and at *3* will be nine parts, and so forth all the way around. To describe the quadrants, place one leg of the compasses at *o* and set the pencil point for the radius *o q*; then, with a center on the line *o t* and a radius *o q*, describe the quadrant *q t*. Then, with a center on the line *p u* and a radius *p t*, describe the arc *t u*, and so on to the completion of the volute.

As a help to the student, the eye of the volute is shown enlarged in Fig. 49, the lettering of the points being the same as in Fig. 48.

PROBLEM 42.

61. To draw a scroll of a given height, resembling a volute.

This problem illustrates a method of obtaining a simple scroll much used as a terminal finish.

See Fig. 50. Draw *A B*, the given height, $4\frac{1}{2}$ inches long, and set off from *A*, at *C*, 1 inch for the width of the band. Make *A D* equal to one-third of *A B*. Bisect the line *A B* at *E*. Now divide *A D* into eight equal parts, and set off the point *F* from *E* toward *B*, Fig. 50, at a distance equal to one of these parts. At *F* erect a perpendicular to *A B*, and intersect this at *G* with a line drawn from *A*, making an angle of 45° with *A B*. Complete the rectangle *G H B F*, and using the 45° triangle, draw from *H* a line intersecting FG at J. Complete the square $FJKM$, and after dividing FM into four equal parts, on two of these parts construct the smaller square, as shown in Fig. 50.

The quadrants may now be described and the figure completed as follows: With F as center and radii FA and FC, describe the quadrants from A and C. With J as center, describe quadrants tangent to those first drawn, continuing the operation by using K and M as centers for the successive quadrants, the outer curve only being carried beyond the point O. The completion of the figure is made by using the points P, Q, and R at the angles of the smaller square and describing the tangent arcs shown in Fig. 50.

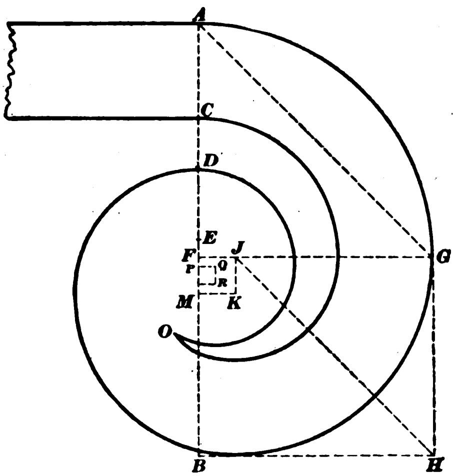


FIG. 50.

TRACINGS.

62. In actual practice in the drawing room, it is necessary to have more than one copy of a drawing. It would be very expensive to make a finished drawing every time an extra copy was wanted, and to avoid this, tracings and blueprints are made. Any number of blueprint copies can be made from the same tracing. In this Course, no drawings are to be duplicated, but as the practical man has need for duplicates, the method of tracing and blueprinting is here shown.

A complete pencil drawing is made first; then, instead of inking in as heretofore, a piece of tracing paper or tracing cloth of the same size as the pencil drawing is fastened to the board over the original drawing. The tracing paper or cloth being almost transparent, the lines of the drawing can be readily seen through it, and the drawing is inked in on the tracing paper or cloth in the same manner as if inking in a finished drawing.

Tracing paper is but little used. It is easily torn, and cannot be preserved as well as tracing cloth. **Tracing cloth** is fine linen treated in a way that makes it transparent. The two sides of the tracing cloth are known as the glazed side and the dull side; they are also known as the front and the back. The glazed side, or front, is covered with a preparation that gives it a very smooth polished surface; the back, or dull side, has very much the appearance of a piece of ordinary linen cloth. Either side may be used for drawing upon, but when the glazed side is used, care must be taken to remove all dirt and grease, otherwise the ink will not flow well from the pen. This can be done by sprinkling powdered chalk or pipe clay upon the tracing cloth; then, take a soft rag of some kind—cotton flannel or chamois skin—and rub it all over the tracing cloth, being sure to rub the powder over every spot. Finally, dust the rag and remove as much of the powder from the cloth as can be gotten off by rubbing with the rag. The finer the powder is the better. It is not usual to put powder on the dull side, but it improves it to do so. The glazed side takes ink much better than the dull side, the finished drawing looks better and will not soil so easily, and it is also easier to erase a line that has been drawn on this side. Pencil lines can be satisfactorily drawn on the dull side, and if it is desired to photograph the drawing, it is better to draw on this side. The draftsman uses either side, according to the work he is doing and to suit his individual taste, but if the glazed side is used, *it must be chalked.* The tracings are drawn in a manner similiar to the finished drawings, the center lines, section lines, etc. being drawn exactly as described.

BLUEPRINTING.

63. Blueprinting is the process of duplicating a tracing by means of the action of light upon a sensitized paper. The following solution is much used for sensitizing the paper: Dissolve 2 ounces of citrate of iron and ammonia in 8 ounces of water; also, 1½ ounces of red prussiate of potash in 8 ounces of water. Keep the solutions separate and in dark-colored bottles in a dark place where the light cannot reach them. Better results will be obtained if ½ an ounce of gum arabic is dissolved in each solution.

When ready to prepare the paper, mix equal portions of the two solutions, and be particularly careful not to allow any more light to strike the mixture than is absolutely necessary to see by. For this reason it is necessary to have a dark room to work in. There must be in this room a tray or sink of some kind that will hold water; it should be larger than the blueprint and about 6 inches deep. There should also be a flat board large enough to cover the tray or sink. If the sink is lined with zinc or galvanized iron, so much the better. There must be an arrangement like a towel rack to hang the prints on while they are drying. For the want of a better name, this arrangement will be called a *print rack*. The paper used for blueprinting should be a good, smooth, white paper, and may be purchased of any dealer in drawing materials. Cut it into sheets a little larger than the tracing, so as to leave an edge around it when the tracing is placed upon it. Place eight or ten of these sheets upon the flat board before mentioned, taking care to spread flatly one above another, so that the edges do not overlap. Secure the sheets to the board by driving a brad or small wire nail through the two upper corners sufficiently far into the board to hold the weight of the papers when the board is placed in a vertical position. Lay the board on the edges of the sink, so that one edge is against the wall and the board is inclined so as to make an angle of about 60° with the horizontal. Darken the room as much as possible, and obtain what light may be necessary from a lamp or gas jet, which should be turned down very low. With a wide camel's-hair brush or a fine sponge, spread the solution just prepared over the top sheet of paper. Be sure to cover every spot, and do not get too much on the paper. Distribute it as evenly as possible over the paper, in much the same manner that the finishing coat of varnish would be put on by a painter. Remove the sheet by pulling on the lower edge, tearing it from the nail that holds it, and place it in a drawer where it can lie flat and be kept from the light. Treat the next sheet, and each succeeding sheet, in exactly the same manner, until the required number of sheets has been prepared.

Unless a large number of prints is constantly used, it is cheaper to buy the paper already prepared. It can be bought in rolls of 10 yards or more, of any width, or in sheets already cut and ready for use. There is very little, if anything, saved in preparing the paper, and better results are usually obtained from the commercial sensitized paper, since the manufacturers have machines for applying the solution and are able to distribute it very evenly.

64. In Figs. 51 and 52 are shown two views of a printing frame that is well adapted to sheets not over 17″ × 21″. The frame is placed face downwards, and the back *A* is removed by unhooking the brass spring clips *B*, *B* and lifting it out. The tracing is laid upon the glass *C*, with the ***inked*** side touching the glass. A sheet of the prepared paper, perfectly dry, is laid upon the tracing with the yellow (sensitized) side downwards. The paper and tracing are smoothed out so as to lie perfectly flat upon the glass, the cover *A* is replaced, and the brass spring clips *B*, *B* are sprung under the plates *D*, so that the back cannot fall out. While all this is being done, the paper should be kept from the light as much as possible. The frame is now placed where the sun can shine upon it, and adjusted as shown in Fig. 52, so that the sun's rays will fall upon it as nearly at right angles as possible. According to the conditions of the sky—whether clear or cloudy—and the time of the year, the print must be exposed from 3 to 15 minutes. The tray, or sink, already mentioned, should be filled to a depth of about 2 inches with clear water (rain water if possible). The print having been exposed the proper length of time, the frame is carried into a dark part of the room, the cover removed, and the print (prepared paper) taken out. Now place it on the water with the yellow side down, and be sure that the water touches every part of it. Let it soak while putting the next print in the frame. Be sure that the hands

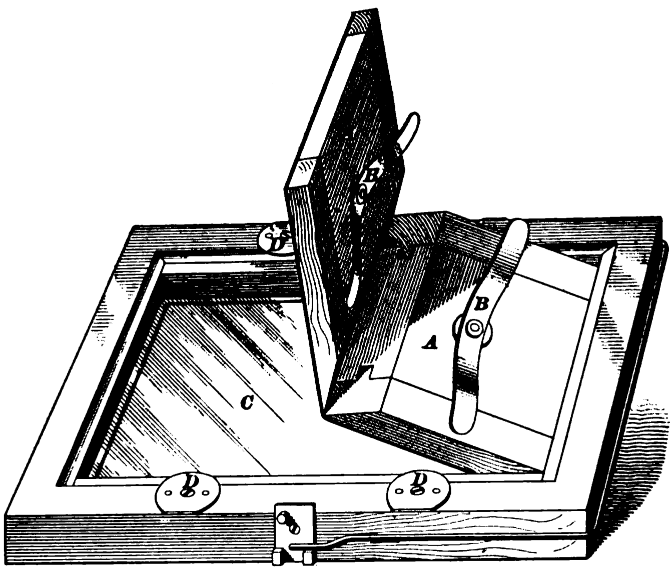


FIG. 51.

are dry before touching the next print. The first print having soaked a short time (about 10 minutes), take hold of two of its opposite corners and lift it slowly out of the water. Dip it back again and pull out as before. Repeat this a number of times, until the paper appears to get no bluer; then hang it by two of its corners to dry on the print rack previously mentioned. If there are any dark-purple or bronze-colored spots on the prints, it indicates that the prints were not washed thoroughly on those spots. If these spots are well washed before the print is dried, they will disappear.

65. It is best to judge of the proper time of exposure to the light by the color of the strip of print projecting beyond the edge of the tracing. To obtain the exact shade of the projecting edge, take a strip of paper about 12 or 14 inches long and 3 or 4 inches wide. Divide it into, say, 12 equal parts by lead-pencil marks, and, with the lead pencil, number each part 1, 2, 3, etc. Sensitize this side of the paper,

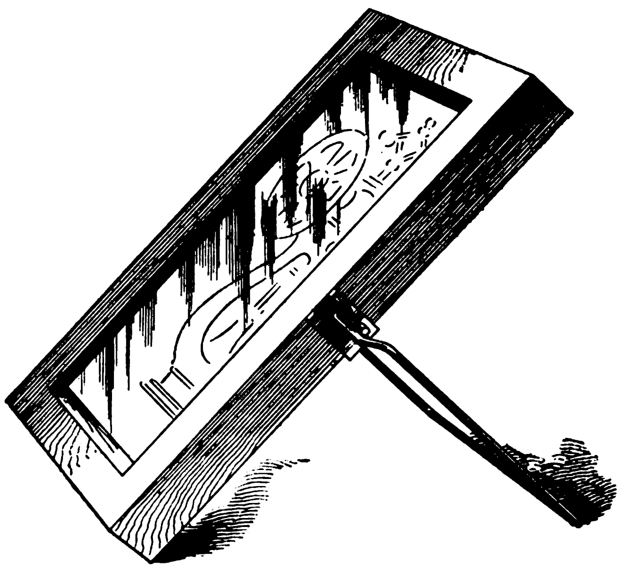


FIG. 52.

and, after it has been properly dried, place it in the print frame with the sensitized side and the marks and figures against the glass. Expose the whole strip to the light for one minute; then cover the part of the strip marked 1 with a thin board or anything that will prevent the light from striking the part covered. At the end of the second minute, cover parts 2 and 1; at the end of the third minute, parts 3, 2, and 1, etc. When twelve minutes are up, part 1 will have been exposed one minute; part 2, two minutes, etc., part 12 having been exposed twelve minutes. Remove the frame to a dark part of the room, and tear the strip so

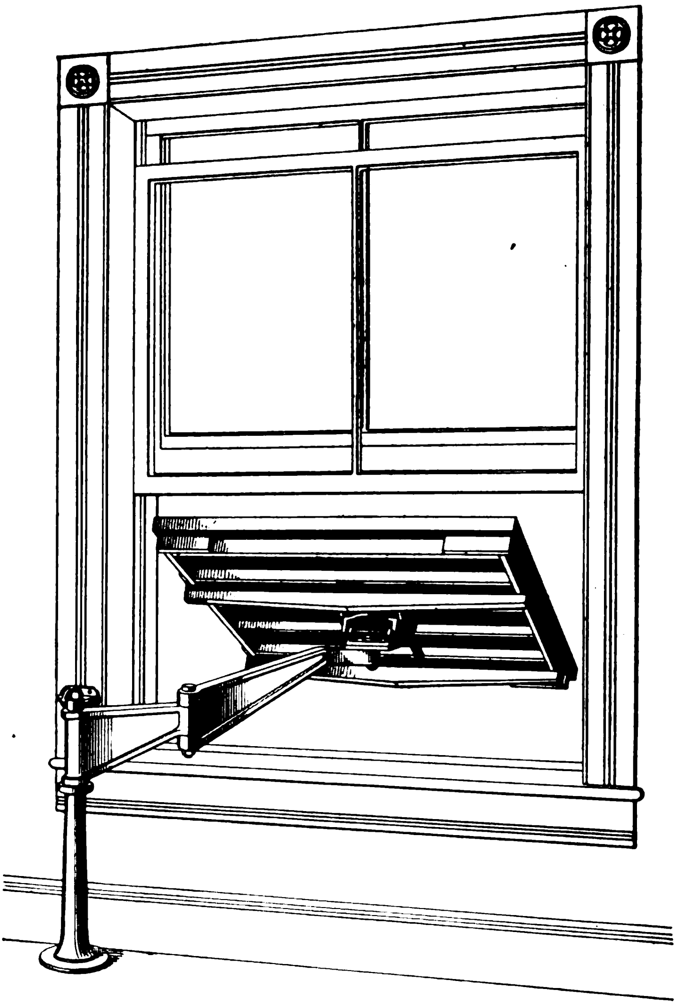


FIG. 53.

as to divide it into two strips of the same length and about half the original width. Wash one of the strips as before described, and when it has dried, select a good rich shade of

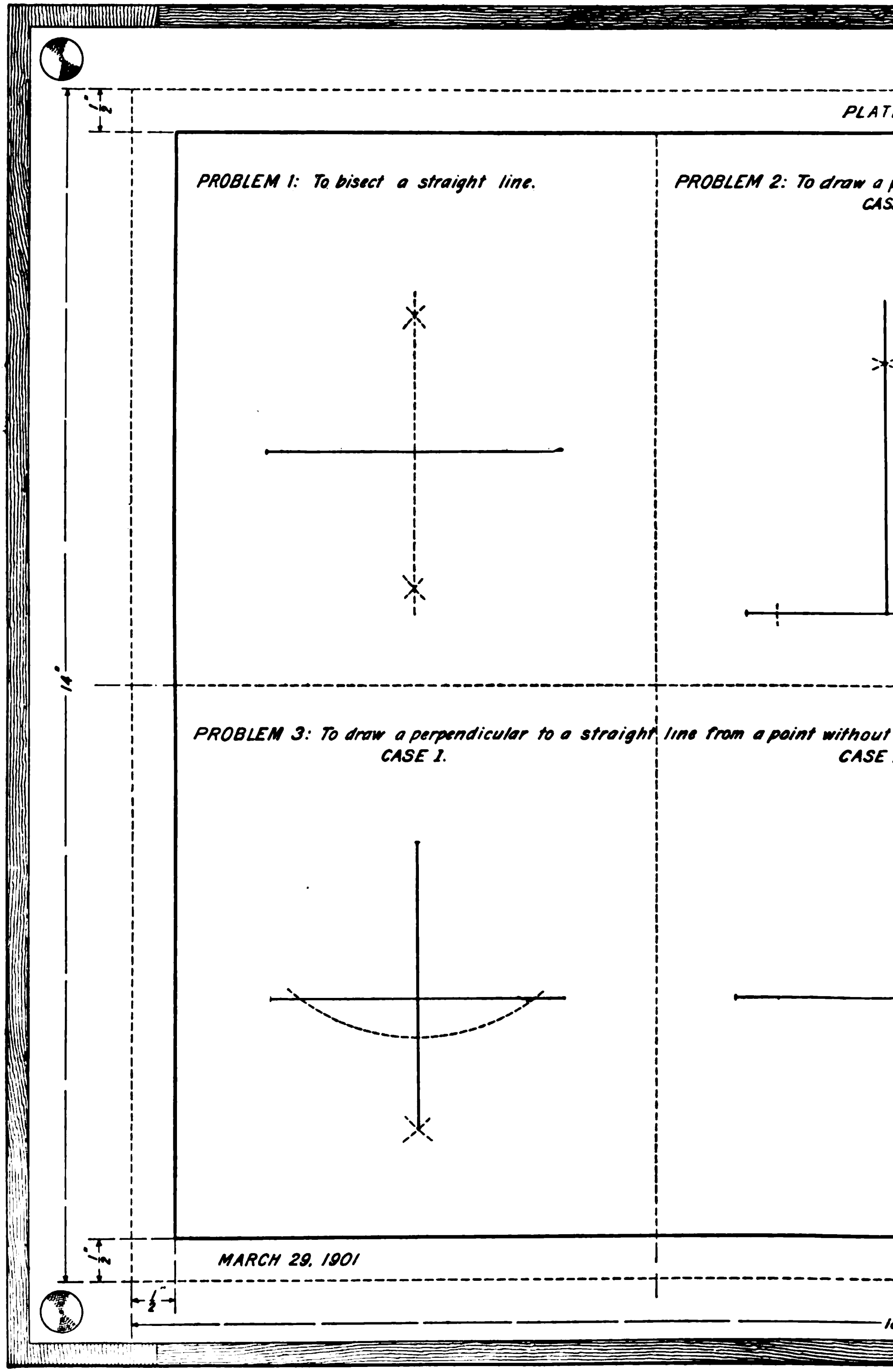
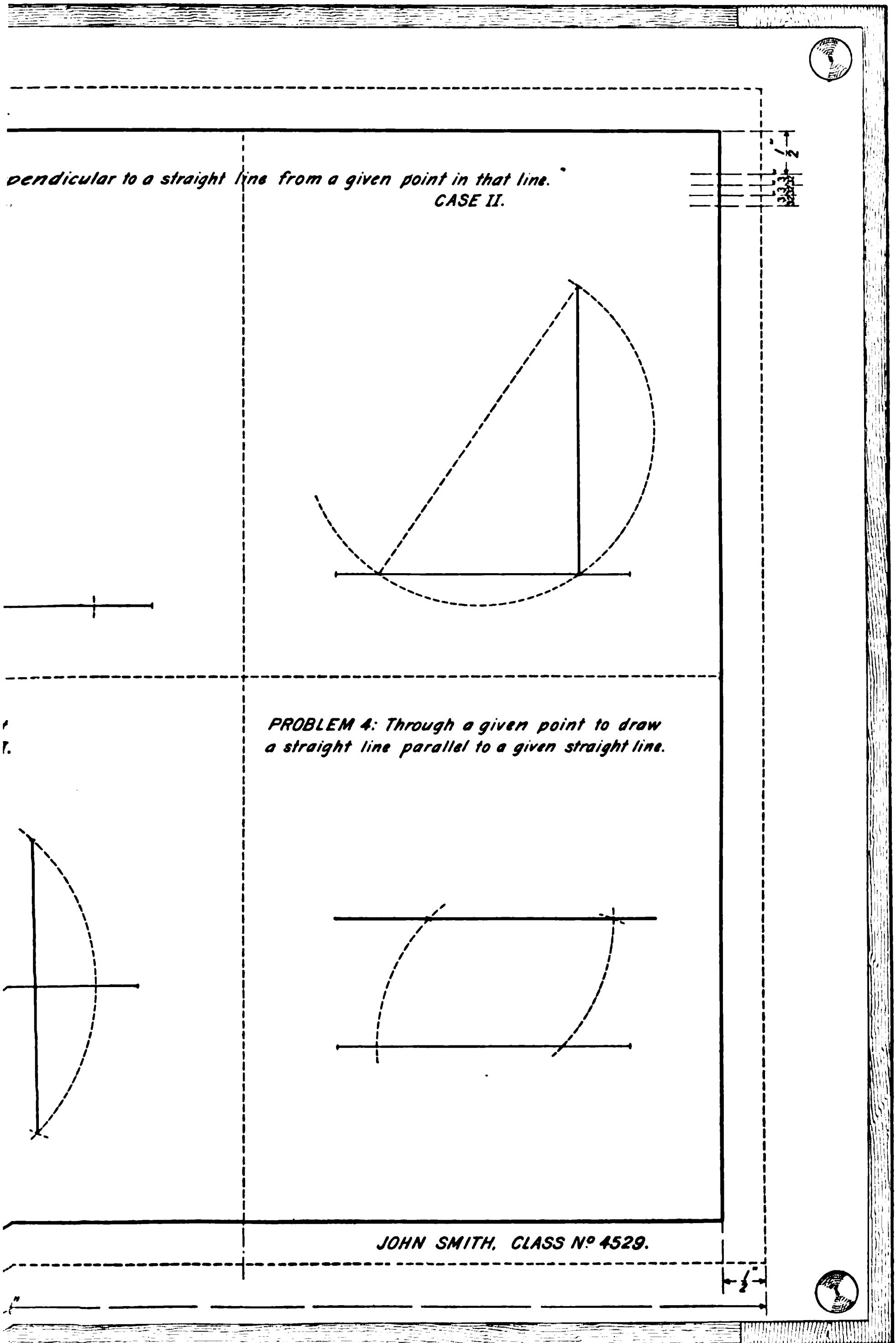


PLATE
PROBLEM 1: To bisect a straight line.
PROBLEM 2: To draw a
PROBLEM 3: To draw a perpendicular to a straight line from a point without
CASE 1.
CASE 1
MARCH 29, 1901
14"
1/2"
1/2"
18


pendicular to a straight line from a given point in that line.
CASE II.
PROBLEM 4: Through a given point to draw a straight line parallel to a given straight line.
JOHN SMITH, CLASS Nº 4529.

blue, neither too light nor too dark; notice the number of the part chosen, and it will indicate the length of time that the print was exposed. Examine carefully the corresponding part of the other strip, and the correct color of the edge of the print projecting beyond the tracing is determined. All prints should be exposed until this color is reached, no matter how long or how short the time may be; then they should be immediately taken out and washed.

In Fig. 53 is shown a patented frame which can be shoved out of the window and adjusted to any angle. When not in use, it can be folded up against the wall and occupies but little space. It is made in different sizes, from 16″ × 24″ to 48″ × 72″. It is one of the best frames in the market, and is placed in such a position relatively to the window that the window can be lowered to the top of the main arm, when it is desired to keep out the cold during the winter.

A SERIES

OF

QUESTIONS AND EXAMPLES

Relating to the Subjects Treated of in This Volume.

It will be noticed that the questions and examples contained in the following pages are divided into sections corresponding to the sections of the text of the preceding pages, so that each section has a headline which is the same as the headline of the section to which the questions refer. No attempt should be made to answer any questions or to work any examples until the corresponding part of the text has been carefully studied.

ARITHMETIC.

(PART 1.)

EXAMINATION QUESTIONS.

(1) Write each of the following numbers in words: (*a*) 708; (*b*) 2,905; (*c*) 10,420.

(2) Write each of the following numbers in figures: (*a*) Seven thousand six hundred; (*b*) eighty-one thousand four hundred two; (*c*) three thousand four.

(3) Write each of the following numbers in words: (*a*) .31; (*b*) .026; (*c*) .205.

(4) Write each of the following numbers in figures: (*a*) Four hundredths; (*b*) twenty-five thousandths; (*c*) seven hundred three thousandths.

(5) Using the dollar mark ($), write each of the following sums of money in figures: (*a*) Twelve dollars and fifty cents; (*b*) eleven dollars and five cents; (*c*) two dollars fourteen cents and six mills.

(6) Write each of the following sums of money in words: (*a*) $18.02; (*b*) $7.25; (*c*) $17.268.

(7) Write each of the following numbers in figures: (*a*) Three hundred four and seventeen hundredths; (*b*) seventy and two hundred four thousandths; (*c*) two hundred and four tenths.

(8) Write each of the following numbers in words: (*a*) 208.07; (*b*) 30.102; (*c*) 25.209.

(9) Give the name of the lowest place in each of the following numbers: (*a*) 21.07; (*b*) 105.263; (*c*) .0009.

(10) Give the name of the highest place in each of the following numbers: (*a*) 204.1; (*b*) 30,046; (*c*) 625.09.

ARITHMETIC.

(PART 2.)

EXAMINATION QUESTIONS.

(1) Add the following numbers:

(*a*) 52, 47, 31, 26, 33, and 40.

(*b*) 65, 90, 62, 9, 71, and 33.

Ans. { (*a*) 229. (*b*) 330.

(2) Add the following numbers:

(*a*) 1,375, 9,402, 8,976, 3,201, 3,004, and 226.

(*b*) 3,049, 685, 72, 17,148, 10,838, and 9.

Ans. { (*a*) 26,184. (*b*) 31,801.

(3) Add the following numbers:

(*a*) 52.38, 367.4, .172, 6.0053, and 9,005.079.

(*b*) 30.709, .80063, 96.40, .21, and 93.905.

(*c*) 4.37805, .04459, and 600.2.

Ans. { (*a*) 9,431.0363. (*b*) 222.02463. (*c*) 604.62264.

(4) (*a*) From 395 take 263.

(*b*) From 18,576 take 13,246.

Ans. { (*a*) 132. (*b*) 5,330.

(5) (*a*) From 13,093 take 9,566.

(*b*) From 10,056 take 9,879.

Ans. { (*a*) 3,527. (*b*) 177.

(6) (*a*) From 14.5389 take 5.87.
(*b*) From 8.425 take 5.3875.

Ans. { (*a*) 8.6689.
(*b*) 3.0375.

(7) A man owes $1,000; he pays $329.25 the first year, and $438.50 the second year. How much does he still owe? Ans. $232.25.

(8) A man has in his pocketbook on Monday morning $96.42. During the six days of the week he spends the following amounts: $5.27, $2.48, $18.05, $0.94, $11.89, $31.16. On counting his money at the end of the week, he finds that he has only $16.50 left. How much has been lost or stolen? Ans. $10.13.

(9) The sum of two numbers is 37.65; one of the numbers is 20.89. Find the difference between the two numbers. Ans. 4.13.

(10) The difference between two numbers is 25.32; the greater of the two numbers is 36.45. Find the sum of the two numbers. Ans. 47.58.

ARITHMETIC.

(PART 8.)

EXAMINATION QUESTIONS.

(1) A circular arc contains 230.5°; how many degrees are there in the rest of the circumference? Ans. 129.5°.

(2) There are two angles on one side of a straight line; one of the angles contains 99.6°. How many degrees are there in the other angle? Ans. 80.4°.

(3) There are five angles about a point; four are as follows: 23.6°, 48.2°, 106.9°, and 98.7°. How many degrees are there in the fifth angle? Ans. 82.6°.

(4) Two angles of a triangle contain 75° and 46°, respectively. How many degrees are there in the other angle? Ans. 59°.

(5) Each of the angles at the base of an isosceles triangle contains 68°. How many degrees are there in the angle at the vertex? Ans. 44°.

(6) One of the acute angles of a right triangle contains 36°. How many degrees are there in the other acute angle? Ans. 54°.

(7) The legs of an isosceles triangle are each 4.5 feet and the base is 5.6 feet. Find the length of the perimeter. Ans. 14.6 ft.

(8) The perimeter of a triangle is 129.5 inches; two of the sides are 45 inches and 54.6 inches. Find the length of the third side. Ans. 29.9 in.

(9) Three of the angles of a quadrilateral are 93°, 59°, and 87°. Find the fourth angle. Ans. 121°.

(10) The perimeter of a quadrilateral is 226 feet; three of the sides are: 95 feet, 27.6 feet, and 46.2 feet. Find the fourth side. Ans. 57.2 ft.

ARITHMETIC.

(PART 4.)

EXAMINATION QUESTIONS.

(1) Multiply

(*a*) 9,287 by 306.
(*b*) 565 by 94.

Ans. { (*a*) 2,841,822. (*b*) 53,110.

(2) Multiply

(*a*) 4.36 by 2.19.
(*b*) 7.43 by .067.

Ans. { (*a*) 9.5484. (*b*) .49781.

(3) Multiply .7854 by

(*a*) 10, (*b*) 100, (*c*) .1, (*d*) .01.

Ans. { (*a*) 7.854. (*b*) 78.54. (*c*) .07854. (*d*) .007854.

(4) The base of a rectangle is 6.3 inches and one side is 4.5 inches. Find the area. Ans. 28.35 sq. in.

(5) The base of a parallelogram is 7.92 feet and the altitude is 4.07 feet. Find the area. Ans. 32.2344 sq. ft.

(6) A solid is 6 feet by 5 feet by 4 feet; what is its volume? Ans. 120 cu. ft.

(7) A man earns $9.75 a week, how much will he earn in 7 weeks? Ans. $68.25.

(8) How many gallons will a tank hold that is 7 feet long, 4 feet wide, and 3 feet deep? Allow 7.5 gallons to the cubic foot. Ans. 630 gal.

(9) A factory employs 315 men, 95 women, and 57 children. Each man receives $9 a week, each woman $4.50 a week, and each child $2 a week. What is the total amount of wages paid by the factory each week? Ans. $3,376.50

(10) Find the value of $.31831 \times 26 + .31831 \times 35.3 + .31831 \times 21.36 + .31831 \times 17.34$. Ans. 31.831.

ARITHMETIC.

(PART 5.)

EXAMINATION QUESTIONS.

(1) Divide 589,824 by 576. Ans. 1,024.

(2) Divide 38.566 by 5.5. Ans. 7.012.

(3) Divide 52,218 by 9.67. Ans. 5,400.

(4) Find the value of 178.92 ÷ 2.31 correct to five figures. Ans. 77.455.

(5) A cubic inch of water weighs .03617 of a pound. What is the volume of a body of water that weighs 58 pounds? Find the answer correct to four figures. Ans. 1,604 cu. in.

(6) The product of two numbers is 9,614 and one of the numbers is 79.13; find the other number correct to four figures. Ans. 121.5.

(7) Divide 1.4142 by (*a*) 10, (*b*) 100, (*c*) .1, (*d*) .01.

Ans. (*a*) .14142. (*b*) .014142. (*c*) 14.142. (*d*) 141.42.

(8) A man bought 2 dozen wash boilers and paid $15.60 for them. How much did each boiler cost? Ans. $.65.

(9) The area of a rectangle is 1,950.88 square feet and the length of the side is 35.6 feet. What is the length of the base? Ans. 54.8 ft.

(10) The volume of a rectangular solid is 756 cubic feet. It is 9 feet thick and 7 feet high; how long is it? Ans. 12 ft.

ARITHMETIC.

(PART 6.)

EXAMINATION QUESTIONS.

(1) Extract the square root of 49.4209. Ans. 7.03.

(2) (*a*) Extract the square root of .021675. Give answer correct to three figures. Ans. .147

(*b*) Find, by means of the table, the square root of .021675 correct to three figures. Ans. .147.

(3) Find, by means of the table, the cube root of 279.89564. Give answer correct to three figures. Ans. 6.54.

(4) The legs of a right triangle are 35 inches and 84 inches. Find (*a*) the area of the square on the hypotenuse; (*b*) the length of the hypotenuse.

Ans. { (*a*) 8,281 sq. in.
(*b*) 91 in.

(5) Find the square root of 1,024 × 4,225. Ans. 2,080.

(6) The hypotenuse of a right triangle is 122 inches and one leg is 22 inches. Find (*a*) the area of the square on the other leg; (*b*) the length of the other leg.

Ans. { (*a*) 14,400 sq. in.
(*b*) 120 in.

(7) Each of the legs of an isosceles triangle is 143 inches and the base is 264 inches. Find the altitude. Ans. 55 in.

(8) The side of a square is 9.5 inches; find its area.
Ans. 90.25 sq. in.

(9) The edge of a cube is 9.3 inches; find its volume.
Ans. 804.357 cu. in.

(10) The side of a square is 5 inches; find the length of its diagonal. Ans. 7.071 in.

ARITHMETIC.

(PART 7.)

EXAMINATION QUESTIONS.

(1) Find the prime factors of each of the following numbers:

(*a*) 16; (*b*) 18; (*c*) 42; (*d*) 693.

(2) Solve the following problems by cancelation:

(*a*) Divide $4 \times 8 \times 12$ by $2 \times 4 \times 6$.
(*b*) Divide $6 \times 5 \times 10$ by $2 \times 3 \times 5$.
(*c*) Divide $13 \times 9 \times 10$ by $39 \times 2 \times 5$.

Ans. { (*a*) 8. (*b*) 10. (*c*) 3.

(3) Express each of the following decimals as a common fraction in its lowest terms: (*a*) .75; (*b*) .375; (*c*) .625; (*d*) .5625.

Ans. { (*a*) $\frac{3}{4}$. (*b*) $\frac{3}{8}$. (*c*) $\frac{5}{8}$. (*d*) $\frac{9}{16}$.

(4) Reduce the following fractions to equivalent fractions having their least common denominator: $\frac{5}{6}$, $\frac{3}{5}$, $\frac{2}{3}$, and $\frac{9}{10}$.

(5) Find the sum of: (*a*) $\frac{1}{2}$, $\frac{1}{3}$, $\frac{3}{4}$, and $\frac{5}{6}$; (*b*) $\frac{1}{5}$, $2\frac{1}{2}$, $4\frac{1}{4}$, and $3\frac{1}{8}$; (*c*) $21\frac{2}{3}$, $16\frac{4}{5}$, $25\frac{3}{4}$, $7\frac{5}{12}$, $37\frac{5}{6}$, and $5\frac{3}{10}$.

Ans. { (*a*) $2\frac{5}{12}$. (*b*) $10\frac{3}{40}$. (*c*) $114\frac{23}{30}$.

(6) From

(*a*) $\frac{8}{9}$ take $\frac{5}{8}$; (*b*) $231\frac{3}{4}$ take $219\frac{4}{5}$.

Ans. { (*a*) $\frac{19}{72}$. (*b*) $11\frac{19}{20}$.

§ 7

(7) Perform the following operations: (*a*) $17\frac{4}{5} \times 8$; (*b*) $4\frac{3}{4} \times \frac{8}{9}$; (*c*) $\frac{9}{10} \div 12$; (*d*) $\frac{15}{16} \div \frac{5}{8}$; (*e*) $43\frac{1}{3} \div 4\frac{1}{6}$.

Ans. (*a*) $142\frac{2}{5}$.
(*b*) $4\frac{2}{9}$.
(*c*) $\frac{3}{40}$.
(*d*) $1\frac{1}{2}$.
(*e*) $10\frac{2}{5}$.

(8) Extract the square root of $\frac{625}{1089}$. Ans. $\frac{25}{33}$.

(9) The sides of a triangle are 13 inches, 14 inches, and 15 inches; find its area. Ans. 84 sq. in.

(10) Find the area of a triangle whose base is 45.6 inches and whose altitude is 22.5 inches. Ans. 513 sq. in.

ARITHMETIC.

(PART 8.)

EXAMINATION QUESTIONS.

(1) Which is the greater, 36 : 45 or 54 : 63? Ans. 54 : 63.

(2) A flask holds 27 ounces of water; what is the weight of the petroleum that it will hold? The specific gravity of petroleum is .7. Ans. 18.9 oz.

(3) Find the value of the unknown number in each of the following equations:

(*a*) unknown number : 6 = 5 : 15.

(*b*) 8 : 24 = unknown number : 45.

(*c*) $\frac{3}{4}$: $2\frac{1}{4}$ = $\frac{3}{8}$: unknown number.

Ans. (*a*) 2. (*b*) 15. (*c*) $1\frac{1}{8}$.

(4) The weights of equal volumes of iron and water are in the ratio of 7.216 to 1; a cubic foot of water weighs 1,000 ounces, what does a cubic foot of iron weigh? Ans. 7,216 oz.

(5) If a block of granite 5 feet long, 4 feet wide, and 3 feet thick weighs 9,945 pounds, what is the weight of a block 10 feet long, 8 feet wide, and 5 feet thick?

Ans. 66,300 lb.

(6) Find a quantity that has the same ratio to 14 ounces that 3 quarts has to 28 quarts. Ans. $1\frac{1}{2}$ oz.

(7) If a pole 8 feet high casts a shadow 9 feet long, how long a shadow would be cast by a pole 64 feet high?

Ans. 72 ft.

(8) The triangles $A\ B\ C$ and $A'\ B'\ C'$ are similar. The side $B\ C$ is 2.5 inches and the area of the triangle $A\ B\ C$ is 3.3 square inches. The side $B'\ C'$ is 7.5 inches. Find the area of the triangle $A'\ B'\ C'$. Ans. 29.7 sq. in.

(9) The triangles $A\ B\ C$ and $A'\ B'\ C'$ are similar. Being given $A\ B = 11.9$ inches, $B\ C = 15$ inches, $C\ A = 24.1$ inches, and $A'\ B' = 47.6$ inches, find the lengths of the sides $B'\ C'$ and $C'\ A'$.

Ans. $\begin{cases} B'\ C' = 60 \text{ in.} \\ C'\ A' = 96.4 \text{ in.} \end{cases}$

(10) Find the area of a regular octagon whose side is 5 inches? Ans. 120.7 sq. in.

ARITHMETIC.

(PART 9.)

EXAMINATION QUESTIONS.

(1) The chord of an arc is 24 inches; the chord of half the same arc is 15 inches. Find the diameter of the circle.
Ans. 25 in.

(2) The diameter of a circle is 25 inches; find the circumference.
Ans. 78.54 in.

(3) The diameter of a circle is 25 inches; find the length of an arc of 60°.
Ans. 13.09 in.

(4) The chord of the whole arc is 15 inches, and the chord of half the arc is 8 inches; find the length of the arc.
Ans. $16\frac{1}{3}$ in.

(5) The radius of a circle is 12.5 inches; find the area.
Ans. 490.88 sq. in.

(6) Find the area of a circular ring whose outer and inner diameters are 13 inches and 7 inches, respectively.
Ans. 94.25 sq. in.

(7) The radius of a circle is 20 inches. Find the area of a sector of the circle, the arc of which contains 60°.
Ans. 209.44 sq. in.

(8) The diameter of the circle is 20 inches, the chord of the whole arc is 14.14 inches, and the chord of half this arc is 7.65 inches. Find (*a*) the area of the sector formed by

radii drawn to the ends of the 14.14-inch chord; (*b*) the area of the segment enclosed between the 14.14-inch chord and its arc. Give answers correct to four figures.

Ans. { (*a*) 78.43 sq. in.
(*b*) 28.45 sq. in.

(9) The diameters of an ellipse are 24 and 20 inches; find the circumference. Ans. 69.4 in.

(10) The diameters of an ellipse are 12 and 9 inches; find the area. Ans. 84.82 sq. in.

ARITHMETIC.

(PART 10.)

EXAMINATION QUESTIONS.

(1) Find the area of the surface of a sphere whose diameter is 15 inches. Give the answer correct to four figures.
Ans. 706.9 in.

(2) The diameter of a sphere is 15 inches. The sphere is cut by two parallel planes whose distances from the center of the sphere are 4 and 6.5 inches, respectively. Find the area of the spherical zone between these planes.
Ans. 117.8 sq. in.

(3) Find the volume of a sphere 28 inches in diameter. Give answer correct to four figures. Ans. 11,490 cu. in.

(4) Find the entire surface of a cylinder 5 inches in diameter and 8 inches long. Give answer correct to three figures. Ans. 165 sq. in.

(5) A closed cylindrical flour bin 1 foot in diameter and 3 feet high is made of tin. (*a*) How many square feet of tin is required, no allowance being made for lap? (*b*) What is the volume of the bin? Give answers correct to two figures.
Ans. { (*a*) 11 sq. ft. / (*b*) 2.4 cu. ft.

(6) How much canvas will make a conical tent 12 feet in diameter at the base and 11 feet high? Give answer correct to three figures. Ans. 236 sq. ft.

(7) The radius of the base of a cone is 3.5 inches and its altitude is 21 inches. Find its volume correct to three figures. Ans. 269 cu. in.

(8) The radii of the bases of a frustum of a cone are 15 inches and 10 inches, respectively, and its altitude is 20 inches; find its volume. Ans. 9,948 cu. in.

(9) The diameter of the base of a paraboloid is 3 feet and its altitude is 5 feet; find its volume. Give answer correct to three figures. Ans. 17.7 cu. ft.

(10) Find the volume of an oblate spheroid whose diameters are 15 and 10 inches, respectively. Ans. 1,178.1 cu. in.

ARITHMETIC.

(PART 11.)

EXAMINATION QUESTIONS.

(1) How many square feet of galvanized iron will be required for an open tank 3 feet square and 2.5 feet deep?

Ans. 39 sq. ft.

(2) (*a*) How many cubic feet of water will this tank hold? (*b*) How many gallons will it hold, allowing 7½ gallons to the cubic foot?

Ans. { (*a*) 22.5 cu. ft.
(*b*) 168.75 gal. }

(3) Find the cubic contents of a hexagonal prism 12 inches long, each side of the base being 1 inch long.

Ans. 31.2 cu. in.

(4) What is the volume of a triangular pyramid, each side of the base being 6 inches and the altitude 8 inches? Give answer correct to two figures. Ans. 42 cu. in.

(5) Find the volume of a wedge of which the length is 9 inches, the width of the base is 2 inches, and the lengths of the three parallel edges are 6 inches, 6 inches, and 4 inches, respectively. Ans. 48 cu. in.

(6) Find, by the prismoidal formula, the volume of the frustum of a square pyramid of which the lower base is 4 feet square and the upper base is 2 feet square; the altitude of the frustum is 5 feet. Ans. 46⅔ cu. ft.

(7) A man who received $2.50 a day had his wages raised 20%. What does he now receive? Ans. $3.00.

(8) What per cent. of 15 tons of coal have I used when I have 9¼ tons left? Ans. 38⅓%.

(9) A town having 6,375 inhabitants increased in population 28% in five years. How many inhabitants were there after the increase? Ans. 8,160.

(10) A piano listed at $500 was sold after 40%, 20%, and 10% discounts were given. What was the selling price of the piano? Ans. $216.

ARITHMETIC.

(PART 12.)

EXAMINATION QUESTIONS.

(1) Reduce

(*a*) 8 yd. 2 ft. 8 in. to a simple number. Ans. 320 in.

(*b*) 2 sq. yd. 7 sq. ft. 98 sq in. to a simple number.
Ans. 3,698 sq. in.

(2) Find the volume of a rectangular box 6 ft. 9 in. long, 5 ft. 6 in. wide, and 4 ft. 3 in. deep.
Ans. 272,646 cu. in.

(3) Reduce

(*a*) 272,646 cubic inches to higher denominations.
Ans. 5 cu. yd. 22 cu. ft. 1,350 cu. in.

(*b*) 2,000 ounces to pounds avoirdupois. Ans. 125 lb.

(4) Find the surface of the box described in question 2. Give answer in square feet and square inches.
Ans. 178 sq. ft. 54 sq. in.

(5) A book is 8 inches long, 6 inches wide, and $1\frac{1}{4}$ inches thick. Find the depth of a box 3 feet 4 inches long, 2 feet 6 inches wide, that it may hold 400 books. Ans. 1 ft. 8 in.

(6) How much lead is required to line an open cistern 4 ft. 6 in. long, 2 ft. 8 in. wide, and 3 ft. 6 in. deep?
Ans. 62 sq. ft. 24 sq. in.

(7) Add 3 ft. 6 in., 17 ft. $4\frac{1}{2}$ in., 4 ft., 3 ft. $3\frac{3}{4}$ in., $5\frac{3}{4}$ in., and 2 ft. 2 in. Ans. 30 ft. 10 in.

(8) Reduce 12.5 grams to (*a*) kilograms; (*b*) milligrams.

Ans. { (*a*) .0125 Kg.
(*b*) 12,500 mg.

(9) (*a*) How many cubic centimeters are there in 13.5 kiloliters? (*b*) What is the weight in kilograms of 13.5 kiloliters of water?

Ans. { (*a*) 13,500,000 $cm.^3$
(*b*) 13,500 Kg.

(10) Reduce 6.5 liters to liquid quarts.

Ans. 6.86855 qt.

A KEY

TO ALL THE

QUESTIONS AND EXAMPLES

INCLUDED IN THE

EXAMINATION QUESTIONS ON ARITHMETIC.

It will be noticed that the Key is divided into sections which correspond to the sections in the Examination Questions on Arithmetic. The answers and solutions are so numbered as to be similar to the numbers before the questions to which they refer.

ARITHMETIC.

(PART 1.)

(1) (*a*) Seven hundred eight.

(*b*) Two thousand nine hundred five, or twenty-nine hundred five.

(*c*) Ten thousand four hundred twenty.

(2) (*a*) Seven thousand six hundred is written 7,600.

(*b*) Eighty-one thousand four hundred two is written 81,402.

(*c*) Three thousand four is written 3,004.

(3) (*a*) Thirty-one hundredths.

(*b*) Twenty-six thousandths.

(*c*) Two hundred five thousandths.

(4) (*a*) Four hundredths is written .04.

(*b*) Twenty-five thousandths is written .025.

(*c*) Seven hundred three thousandths is written .703.

(5) (*a*) Twelve dollars and fifty cents is written $12.50.

(*b*) Eleven dollars and five cents is written $11.05.

(*c*) Two dollars fourteen cents and six mills is written $2.146.

(6) (*a*) Eighteen dollars and two cents.

(*b*) Seven dollars and twenty-five cents.

(*c*) Seventeen dollars twenty-six cents and eight mills.

(7) (*a*) Three hundred four and seventeen hundredths is written 304.17.

(*b*) Seventy and two hundred four thousandths is written 70.204.

(*c*) Two hundred and four tenths is written 200.4.

(8) (*a*) Two hundred eight and seven hundredths.

(*b*) Thirty and one hundred two thousandths.

(*c*) Twenty-five and two hundred nine thousandths.

(9) (*a*) Hundredths.

(*b*) Thousandths.

(*c*) Ten-thousandths.

(10) (*a*) Hundreds.

(*b*) Ten-thousands.

(*c*) Hundreds.

ARITHMETIC.

(PART 2.)

(1) (*a*) In addition, write the numbers in columns, placing units under units, tens under tens, etc. Add the column at the right, naming only the result of each addition; thus, 0, 3, 9, 10, 17, 19, which is 1 ten and 9 units. Write 9 units under the units column and carry the 1 ten for the tens column. Add the tens column, beginning with the 1 carried from the sum of the units column and naming only the results of each addition; thus, 1, 5, 8, 10, 13, 17, 22, which is 2 hundreds and 2 tens. Write the 2 tens under the tens column and the 2 hundreds in the hundreds order of the result.

```
 52
 47
 31
 26
 33
 40
----
229  Ans.
```

(*b*)

```
 65
 90
 62
  9
 71
 33
----
330  Ans.
```

(2) (*a*)

```
 1375
 9402
 8976
 3201
 3004
  226
-----
26184  Ans.
```

(*b*)

```
 3049
  685
   72
17148
10838
    9
-----
31801  Ans.
```

(3)

(*a*)	(*b*)	(*c*)
5 2.3 8	3 0.7 0 9	4.3 7 8 0 5
3 6 7.4	.8 0 0 6 3	.0 4 4 5 9
.1 7 2	9 6.4 0	6 0 0.2
6.0 0 5 3	.2 1	6 0 4.6 2 2 6 4 Ans.
9 0 0 5.0 7 9	9 3.9 0 5	
9 4 3 1.0 3 6 3 Ans.	2 2 2.0 2 4 6 3 Ans.	

(4) (*a*) In subtraction, write the subtrahend under the minuend and begin to subtract at the right-hand column. Thus, we have $5 - 3 = 2$; $9 - 6 = 3$; and $3 - 2 = 1$.

3 9 5
2 6 3
1 3 2 Ans.

(*b*)
1 8 5 7 6
1 3 2 4 6
5 3 3 0 Ans.

(5) (*a*) We cannot take 6 from 3, so we add 10 to 3 and have $13 - 6 = 7$. As we have added 10 to the minuend, to balance this, we add 1 ten to the tens of the subtrahend, and we have $9 - 7 = 2$. We cannot take 5 from 0, so we add ten to the 0 and have $10 - 5 = 5$, and to balance the ten added to the 0, we add 1 to the 9 of the subtrahend and have $13 - 10 = 3$.

1 3 0 9 3
9 5 6 6
3 5 2 7 Ans.

(*b*)
1 0 0 5 6
9 8 7 9
1 7 7 Ans.

(6)

(*a*)	(*b*)
1 4.5 3 8 9	8.4 2 5
5.8 7	5.3 8 7 5
8.6 6 8 9 Ans.	3.0 3 7 5

(7)
$ 3 2 9.2 5 = amount paid the 1st year.
4 3 8.5 0 = amount paid the 2d year.
$ 7 6 7.7 5 = whole amount paid.

$ 1 0 0 0 = amount of debt.
7 6 7.7 5 = amount paid.
$ 2 3 2.2 5 = amount he still owes. Ans.

(8)

```
  $5.27
   2.48
  18.05
   0.94
  11.89
  31.16
 ------
 $69.79 = amount spent during the week.

 $96.42 = amount in pocketbook Monday morning.
  69.79
 ------
 $26.63 = amount he should have had at the end of the week.

 $26.63
  16.50 = amount he had at the end of the week.
 ------
 $10.13 = the amount that was lost or stolen.   Ans.
```

(9)

```
 37.65 = sum of two numbers.
 20.89 = one of the numbers.
 -----
 16.76 = the other of the two numbers.

 20.89 = greater of the two numbers.
 16.76 = smaller of the two numbers.
 -----
  4.13 = difference between the two numbers.   Ans.
```

(10)

```
 36.45 = greater of two numbers.
 25.32 = difference between the two numbers.
 -----
 11.13 = smaller of the two numbers.

 36.45 = greater of two numbers.
 11.13 = smaller of two numbers.
 -----
 47.58 = sum of the two numbers.   Ans.
```

ARITHMETIC.

(PART 3.)

(1) $360° = $ the degrees in a whole circumference.
$230.5° = $ the degrees in the circular arc.
$129.5° = $ the degrees in the rest of the circumference. Ans.

(2) $180° = $ the degrees on one side of a straight line.
$99.6° = $ the degrees in one of the angles.
$80.4° = $ the degrees in the other angle. Ans.

(3) $23.6° + 48.2° + 106.9° + 98.7° = 277.4° = $ the degrees in the four angles.

$360° = $ the degrees about a point.
$277.4°$
$82.6° = $ the degrees in the fifth angle. Ans.

(4) $75° + 46° = 121° = $ the degrees in the two angles.

$180° = $ the degrees in the three angles of a triangle.
$121°$
$59° = $ the degrees in the third angle of the triangle. Ans.

(5) $68° + 68° = 136° = $ the degrees in the base angles of the triangle.

$180° = $ degrees in the three angles of the triangle.
$136°$
$44° = $ degrees in the angle at the vertex of the triangle. Ans.

(6) $90° = $ degrees in the acute angles of a right triangle.
$36° = $ degrees in one of the acute angles.
$54° = $ degrees in the other acute angle. Ans.

(7) 4.5 ft. + 4.5 ft. + 5.6 ft. = 14.6 ft. = the feet in the perimeter of the triangle. Ans.

(8) 45 in. + 54.6 in. = 99.6 in.

129.5 in. = perimeter.
99.6 in. = length of two sides.
29.9 in. = length of the third side. Ans.

(9) 93° + 59° + 87° = 239°.

360° = degrees in the four angles of a quadrilateral.
239° = degrees in three of the angles.
121° = degrees in the fourth angle. Ans.

(10) 95 ft. + 27.6 ft. + 46.2 ft. = 168.8 ft.

226 ft. = feet in the perimeter of the quadrilateral.
168.8 ft. = feet in three of the sides.
57.2 ft. = feet in the fourth side. Ans.

ARITHMETIC.

(PART 4.)

(1) (*a*) Write the multiplier under the multiplicand, placing the right-hand digit of the multiplier under the right-hand digit of the multiplicand, as shown. Multiplying the first digit at the right of the multiplicand, or 7, by the multiplier 6, the result is 42 units, 4 tens and 2 units. Write the 2 units in units place in the product and reserve the tens to add to the product of the tens by the 6. Multiplying the second digit of the multiplicand by the multiplier 6, we have 48 tens; 48 tens + 4 tens reserved = 52 tens, or 5 hundreds and 2 tens. Write the 2 tens and reserve the 5 hundreds. 6 × 2 hundreds = 12 hundreds; 12 hundreds + 5 hundreds = 17 hundreds, or 1 thousand and 7 hundreds. Write the 7 hundreds in the hundreds place in the product and reserve the 1 thousand. 6 × 9 thousands = 54 thousands; 54 thousands + 1 thousand = 55 thousands. As the next figure of the multiplier is 0, write a 0 under the tens figure of the first partial product. Then multiply 9287 by 3 and write the product, 27861, under the first partial product, as shown, with the right-hand digit, 1, under the multiplier 3, thus placing the 1 next to the 0. Add the partial products; their sum, 2,841,822, is the product of 9,287 × 306.

```
   9287
    306
  -----
  55722
 278610
-------
2841822  Ans.
```

(*b*)

```
  565
   94
 ----
 2260
5085
-----
53110  Ans.
```

(2) (*a*)

```
  4.36
  2.19
  ----
  3924
  436
 872
------
9.5484
```

As there are two decimal places in the multiplier and two in the multiplicand, we point off 2 + 2, or 4, decimal places in the product.

(*b*)

```
  7.4 3
  .0 6 7
 -------
  5 2 0 1
 4 4 5 8
 -------
 .4 9 7 8 1
```

As there are three decimal places in the multiplier and two in the multiplicand, we point off 3 + 2, or 5, decimal places in the product.

(3) By Art. **18,** we have

(*a*) $.7854 \times 10 = 7.854$. Ans.

(*b*) $.7854 \times 100 = 78.54$. Ans.

By Art. **19,** we have

(*c*) $.7854 \times .1 = .07854$. Ans.

(*d*) $.7854 \times .01 = .007854$. Ans.

(4) By Art. **23,** the area of the rectangle in square inches is $6.3 \times 4.5 = 28.35$. Therefore, the area is 28.35 sq. in. Ans.

(5) By Art. **25,** the area of the parallelogram in square feet is $7\ 92 \times 4.07 = 32.2344$. Therefore, the area is 32.2344 sq. ft. Ans.

(6) By Art. **32,** the volume of the solid in cubic feet is $6 \times 5 \times 4 = 120$. Therefore, the volume is 120 cu. ft. Ans.

(7) The man earns $9.75 in 1 week; in 7 weeks he will earn 7 × $9.75 = $68.25. Ans.

(8) By Art. **32,** the volume of the tank in cubic feet is $7 \times 4 \times 3 = 84$. If a tank 1 cubic foot in volume holds 7.5 gallons, a tank whose volume is 84 cubic feet will hold 84 × 7.5, or 630 gallons. Therefore, the tank will hold 630 gal. Ans.

(9) 315 men, who receive $9 each, receive 315 × $9 = $2,835.
95 women, who receive $4.50 each, receive 95 × $4.50 = $427.50.
57 children, who receive $2 each, receive 57 × $2 = $114.

```
$2,8 3 5.    = amount paid to the men.
   4 2 7.5 0 = amount paid to the women.
   1 1 4.    = amount paid to the children.
-------------
$3,3 7 6.5 0 = total amount paid.  Ans.
```

(10) By Art. **36,**

$$.31831 \times 26 + .31831 \times 35.3 + .31831 \times 21.36 + .31831 \times 17.34$$

$$= .31831\,(26 + 35.3 + 21.36 + 17.34).$$
$$= .31831 \times 100 = 31.831. \quad \text{Ans.}$$

ARITHMETIC.

(PART 5.)

```
5 8 9 8 2 4 | 5 7 6
5 7 6       | 1 0 2 4  Ans.
  1 3 8 2
  1 1 5 2
    2 3 0 4
    2 3 0 4
```

(1) The divisor 576 is contained in the first three figures of the dividend, 589, one time. Place the 1 as the first figure of the quotient. Multiply 576 by 1 and subtract the product 576 from 589. Annex the next figure of the dividend, 8, to the remainder 13. Now, 576 is not contained in 138, so place a 0 in the quotient and annex the next figure of the dividend, 2, to the 138. 576 is contained in 1,382 two times. Place the 2 as the next figure of the quotient. Multiply 576 by 2 and subtract the product 1,152 from 1,382. To the remainder 230 annex the next figure of the dividend, 4. 576 is contained in 2,304 four times. Place the 4 as the next figure of the quotient. Multiply 576 by 4 and subtract the product 2,304 from 2,304; there is no remainder.

(2)

```
3 8.5 6 6 0 | 5.5
3 8 5       | 7.0 1 2  Ans.
      6 6
      5 5
      1 1 0
      1 1 0
```

There are 4 decimal places including the annexed cipher in the dividend and 1 decimal place in the divisor, so we set off 4 − 1, or 3, decimal places in the quotient.

(3)

```
5 2 2 1 8.0 0 | 9.6 7
4 8 3 5       |   5 4 0 0.  Ans.
  3 8 6 8
  3 8 6 8
          0 0
```

There are 2 decimal places in the divisor and no decimal places in the dividend, so we annex two ciphers to the dividend after the decimal point and divide as if the dividend and divisor were whole numbers.

(4)
```
178.92|2.31
1617  |77.4545.
 1722
 1617
  1050
   924
   1260
   1155
    1050
     924
     1260
```
The sixth figure is 5, so we increase the fifth figure by unity, when we retain only five figures. Hence, correct to five figures, the quotient is 77.455. Ans.

(5) If 1 cubic inch of water weighs .03617 pound, it will take as many cubic inches of water to weigh 58 pounds as .03617 is contained times in 58, or 1,604 cu. in. Ans.

```
58.00000|.03617
3617    |1603.5,
21830
21702
  12800
  10851
   19490
```
which, correct to four figures, is 1,604.

(6) Dividing the product 9,614 by one of the numbers, 79.13, the other number is the quotient, which, correct to four figures, is 121.5. Ans.

```
9614.00|79.13
7913   |121.49,
17010
15826
 11840
  7913
  39270
  31652
   76180
```
or, correct to four figures, 121.5.

(7) By Art. **8,** we have

(*a*) 1.4142 ÷ 10 = .14142.

(*b*) 1.4142 ÷ 100 = .014142.

By Art. **11,** we have

(*c*) 1.4142 ÷ .1 = 14.142.

(*d*) 1.4142 ÷ .01 = 141.42.

(8) The man bought 2 dozen, or 2 × 12, or 24, wash boilers. If 24 wash boilers cost $15.60, one wash boiler cost as many dollars as 24 is contained times in $15.60, which is $15.60 ÷ 24 = $.65. Ans.

```
15.60 | 24
144   | .65
  120
  120
```

(9) The base is as many feet long as 35.6 is contained times in 1,950.88, or 54.8 ft. Ans.

```
1950.88 | 35.6
1780    | 54.8
 1708
 1424
  2848
  2848
```

(10) The volume of a rectangular solid is equal to the continued product of its length, height, and thickness. Therefore, the length must be equal to the volume divided by the product of the height and thickness. The product of the height and thickness is 7 × 9 = 63. And the volume 756 divided by 63 is 756 ÷ 63 = 12. Hence, the length of the solid is 12 ft. Ans.

ARITHMETIC.

(PART 6.)

(1)

trial divisor.	*complete divisor.*	*root.*
		49.42'09 | 7.03 Ans.
		49
1400	1403	4209
		4209

Begin at the decimal point, and counting left and right, separate the number into periods of two figures each. The greatest number whose square is contained in the left-hand period, 49, is 7; therefore 7 is the first figure of the root. Subtracting 49 ($= 7^2$) from the left-hand period, there is no remainder. Since this is the only integral period, insert the decimal point after the 7 in the root. Bringing down the next period, we have 42 as the first partial dividend. The double of 7 is 14 ($= 2 \times 7$), and annexing a cipher, we get 140, the first trial divisor. 140 is not contained in 42, and we put a 0 as the next figure of the root. Bringing down the next period 09, we get 4,209, the second partial dividend. The second trial divisor, 1,400, is contained in the second partial dividend, 4,209, 3 times, hence, 3 is the next figure of the root. Adding 3 to the trial divisor 1,400, we get the complete divisor 1,403. Subtracting 4,209 ($= 3 \times 1,403$) from 4,209, there is no remainder.

(2) (*a*)

trial divisor.	*complete divisor.*	*root.*
		.02'16'75 | .147 Ans.
		1
20	24	116
		96
280	287	2075
		2009

(*b*) We have $.021675 = 216.75 \div 10{,}000.$

Hence, $$\sqrt{.021675} = \frac{\sqrt{216.75}}{\sqrt{10{,}000}} = \frac{\sqrt{216.75}}{100}.$$

higher number in table = 225 *given number* = 216.75
lower number in table = 196 *lower number in table* = 196
first difference = 29 *second difference* = 20.75

```
20.75 |29
203   |.7 = quotient
  45
```

part of root found in table = 14.
quotient = .7
required root = 14.7.

Hence, $$\sqrt{.021675} = \frac{14.7}{100} = .147. \quad \text{Ans.}$$

(3) We have $279.89564 = 279{,}895.64 \div 1{,}000.$

Hence, $$\sqrt[3]{279.89564} = \frac{\sqrt[3]{279{,}895.64}}{\sqrt[3]{1{,}000}} = \frac{\sqrt[3]{279{,}895.64}}{10}$$

higher number in table = 287496 *given number* = 279895.64
lower number in table = 274625 *lower number in table* = 274625.
first difference = 12871 *second difference* = 5270.64

```
5270.64 |12871
51484   |.4 = quotient
 12224
```

part of root found in table = 65.
quotient = .4
required root = 65.4

Hence, $$\sqrt[3]{279.89564} = \frac{65.4}{10} = 6.54. \quad \text{Ans.}$$

(4) (*a*) By Art. **37**, we have

area of square on hypotenuse = $(35^2 + 84^2)$ square inches
= (1,225 + 7,056) square inches
= 8,281 sq. in. Ans.

(*b*) By Art. **42**, we have

hypotenuse = $\sqrt{35^2 + 84^2}$ inches
= $\sqrt{8{,}281}$ inches (from table)
= 91 in. Ans.

(5) By Art. **9**, we have

$$\sqrt{1{,}024 \times 4{,}225} = \sqrt{1{,}024} \times \sqrt{4{,}225}.$$

From the table, $\sqrt{1{,}024} = 32$ and $\sqrt{4{,}225} = 65$.

Hence $\sqrt{1{,}024 \times 4{,}225} = 32 \times 65 = 2{,}080$. Ans.

(6) (*a*) By Art. **38**, we have

$$\textit{area of square on leg} = (122^2 - 22^2) \text{ square inches}$$
$$= 14{,}400 \text{ sq. in. Ans.}$$

(*b*) By Art. **44**, we have

$$\begin{aligned} leg &= \sqrt{122^2 - 22^2} \text{ inches} \\ &= \sqrt{14{,}400} \text{ inches} \\ &= 120 \text{ in. Ans.} \end{aligned}$$

(7) By Art. **65**, Part 3, the perpendicular from the vertex of an isosceles triangle bisects the base. One-half the base = 264 inches ÷ 2 = 132 inches. Hence, the altitude of the isosceles triangle is one leg of a right triangle whose hypotenuse is 143 inches and whose other leg is 132 inches. By Art. **44**,

$$\begin{aligned} altitude &= \sqrt{143^2 - 132^2} \text{ inches} \\ &= \sqrt{20{,}449 - 17{,}424} \text{ inches} \\ &= \sqrt{3{,}025} \text{ inches} \\ &= 55 \text{ in. Ans.} \end{aligned}$$

(8) The area of the square = 9.5×9.5, or 90.25 sq. in. Ans.

(9) The volume of the cube = $9.3 \times 9.3 \times 9.3$, or 804.357 cu. in. Ans.

(10) By Art. **45**, we have

$$\begin{aligned} diagonal &= \sqrt{2} \times 5 \text{ inches} \\ &= 1.4142 \times 5 \text{ inches} \\ &= 7.071 \text{ in. Ans.} \end{aligned}$$

ARITHMETIC.

(PART 7.)

(1) (*a*) $2\,)\,\underline{16}$, $2\,)\,\underline{8}$, $2\,)\,\underline{4}$, 2

(*b*) $2\,)\,\underline{18}$, $3\,)\,\underline{9}$, 3

(*c*) $2\,)\,\underline{42}$, $3\,)\,\underline{21}$, 7

(*d*) $3\,)\,\underline{693}$, $3\,)\,\underline{231}$, $7\,)\,\underline{77}$, 11

$$\left.\begin{array}{ll} (a) & 16 = 2 \times 2 \times 2 \times 2 = 2^4. \\ (b) & 18 = 2 \times 3 \times 3 = 2 \times 3^2. \\ (c) & 42 = 2 \times 3 \times 7. \\ (d) & 693 = 3 \times 3 \times 7 \times 11 = 3^2 \times 7 \times 11. \end{array}\right\} \text{Ans.}$$

(2) See Arts. **14** and **15**.

(*a*) $$\frac{\not{4} \times 8 \times \overset{2}{\not{12}}}{\not{2} \times \not{4} \times \not{6}} = 8. \quad \text{Ans.}$$

(*b*) $$\frac{\overset{3}{\not{6}} \times \not{5} \times 10}{\not{2} \times \not{3} \times \not{5}} = 10. \quad \text{Ans.}$$

(*c*) $$\frac{\not{13} \times \overset{3}{\not{9}} \times \overset{5}{\not{10}}}{\underset{3}{\not{39}} \times \not{2} \times \not{5}} = 3. \quad \text{Ans.}$$

(3) See Art. **24**.

(*a*) $$.75 = \frac{75}{100} = \frac{3 \times 25}{4 \times 25} = \frac{3}{4}. \quad \text{Ans.}$$

(*b*) $$.375 = \frac{375}{1,000} = \frac{3 \times 5 \times 25}{8 \times 5 \times 25} = \frac{3}{8}. \quad \text{Ans.}$$

(*c*) $.625 = \frac{625}{1,000} = \frac{5 \times 5 \times 25}{5 \times 8 \times 25} = \frac{5}{8}.$ Ans.

(*d*) $.5625 = \frac{5625}{10,000} = \frac{5 \times 5 \times 9 \times 25}{5 \times 5 \times 8 \times 25 \times 2} = \frac{9}{16}.$ Ans.

NOTE.—The common fractions can be taken directly from the table of Decimal Equivalents, Art. **66.**

(4) Place the denominators in a row. Strike out 5, as it is a factor of 10; and 3, as it is a factor of 6. Divide by 2, which divides both 6 and 10. The second row is 3 and 5, and these numbers are primes. Hence, the L. C. D. is $2 \times 3 \times 5$, or 30. To reduce the fractions to fractions having 30 as denominator, multiply both terms of each fraction by some number that will make the denominator 30. Thus,

$$\begin{array}{r|l} 2 & 6,\ \not{5},\ \not{3},\ 10 \\ \hline & 3, \quad\quad 5 \end{array}$$

$$\left.\begin{aligned} \frac{5}{6} &= \frac{5 \times 5}{6 \times 5} = \frac{25}{30}. \\ \frac{3}{5} &= \frac{3 \times 6}{5 \times 6} = \frac{18}{30}. \\ \frac{2}{3} &= \frac{2 \times 10}{3 \times 10} = \frac{20}{30}. \\ \frac{9}{10} &= \frac{9 \times 3}{10 \times 3} = \frac{27}{30}. \end{aligned}\right\} \text{Ans.}$$

(5) (*a*) The L. C. D. is 12.

$$\tfrac{1}{2} = \tfrac{6}{12};\ \tfrac{1}{3} = \tfrac{4}{12};\ \tfrac{3}{4} = \tfrac{9}{12};\ \tfrac{5}{6} = \tfrac{10}{12}.$$

$$\tfrac{6}{12} + \tfrac{4}{12} + \tfrac{9}{12} + \tfrac{10}{12} = \tfrac{29}{12} = 2\tfrac{5}{12}. \quad \text{Ans.}$$

(*b*) The L. C. D. is 40.

$$\begin{aligned} \tfrac{1}{5} &= \tfrac{8}{40} \\ 2\tfrac{1}{2} &= 2\tfrac{20}{40} \\ 4\tfrac{1}{4} &= 4\tfrac{10}{40} \\ 3\tfrac{1}{8} &= 3\tfrac{5}{40} \\ \hline & 9\tfrac{43}{40} = 10\tfrac{3}{40}. \quad \text{Ans.} \end{aligned}$$

(*c*) The L. C. D. is 60.

$$\begin{aligned} 21\tfrac{2}{3} &= 21\tfrac{40}{60} \\ 16\tfrac{4}{5} &= 16\tfrac{48}{60} \\ 25\tfrac{3}{4} &= 25\tfrac{45}{60} \\ 7\tfrac{5}{12} &= 7\tfrac{25}{60} \\ 37\tfrac{5}{6} &= 37\tfrac{50}{60} \\ 5\tfrac{3}{10} &= 5\tfrac{18}{60} \\ \hline & 111\tfrac{226}{60} = 114\tfrac{46}{60}. \quad \text{Ans.} \end{aligned}$$

(6) (*a*)

$$\begin{aligned} &\textit{minuend} &= \tfrac{8}{9} &= \tfrac{64}{72} \\ &\textit{subtrahend} &= \tfrac{5}{8} &= \tfrac{45}{72} \\ \hline &\textit{remainder} && = \tfrac{19}{72}. \quad \text{Ans.} \end{aligned}$$

(*b*)

$$\begin{aligned} &\textit{minuend} &= 231\tfrac{3}{4} &= 231\tfrac{15}{20} \\ &\textit{subtrahend} &= 219\tfrac{4}{5} &= 219\tfrac{16}{20} \\ \hline &\textit{remainder} &= & \quad 11\tfrac{19}{20}. \quad \text{Ans.} \end{aligned}$$

We cannot take $\frac{16}{20}$ from $\frac{15}{20}$, so we add $1 = \frac{20}{20}$ to the $\frac{15}{20}$ and have $\frac{35}{20} - \frac{16}{20} = \frac{19}{20}$. As 1 has been added to the minuend, it must be balanced by adding 1 to the subtrahend, Art. **19**, Part 2. $219 + 1 = 220$ and $231 - 220 = 11$.

(7) (*a*) $\frac{4}{5} \times 8 = \frac{32}{5} = 6\frac{2}{5}$; set down $\frac{2}{5}$ and carry 6 units. Then, $17 \times 8 + 6 = 142$.

$$\begin{array}{r} 17\frac{4}{5} \\ 8 \\ \hline 142\frac{2}{5}. \end{array} \quad \text{Ans.}$$

(*b*) See Art. **45.** Reduce $4\frac{3}{4}$ to an improper fraction; thus, $4\frac{3}{4} = \frac{19}{4}$.

Then,
$$\frac{19}{4} \times \frac{8}{9} = \frac{19 \times 8}{4 \times 9} = \frac{38}{9} = 4\frac{2}{9}. \quad \text{Ans.}$$

(*c*) By Art. **48,** $\frac{9}{10} \div 12 = \frac{9}{10} \times \frac{1}{12} = \frac{3}{40}$. Ans.

(*d*) By Art. **52,** $\frac{15}{16} \div \frac{5}{8} = \frac{15}{16} \times \frac{8}{5} = \frac{3}{2} = 1\frac{1}{2}$. Ans.

(*e*) $43\frac{1}{3} = \frac{130}{3}$; $4\frac{1}{6} = \frac{25}{6}$.

Therefore,
$$43\frac{1}{3} \div 4\frac{1}{6} = \frac{130}{3} \div \frac{25}{6} = \frac{130}{3} \times \frac{6}{25} = \frac{52}{5} = 10\frac{2}{5}. \quad \text{Ans.}$$

(8) By Art. **59,**
$$\sqrt{\frac{625}{1089}} = \frac{\sqrt{625}}{\sqrt{1089}} = \frac{25}{33}. \quad \text{Ans.}$$

(9) See Art. **63.** The sum of the sides is $13 + 14 + 15$, or 42 inches; therefore, half the sum of the sides is $\frac{1}{2} \times 42$, or 21 inches. Subtracting each side separately from half the sum of the sides, we get the three remainders $21 - 13$, or 8 inches; $21 - 14$, or 7 inches; and $21 - 15$, or 6 inches. Hence, the area of the triangle in square inches is $\sqrt{21 \times 8 \times 7 \times 6} = \sqrt{7,056}$. From the table of Squares and Cubes, Part 6, we get $\sqrt{7,056} = 84$. Thus, the area is 84 sq. in. Ans.

(10) By Art. **62,** the area of the triangle is $\frac{1}{2} \times 45.6 \times 22.5$, or 513 sq. in. Ans.

ARITHMETIC.

(PART 8.)

(1) $36 : 45 = \frac{36}{45} = \frac{4}{5}$. $54 : 63 = \frac{54}{63} = \frac{6}{7}$.

Reducing the fractions $\frac{4}{5}$ and $\frac{6}{7}$ to a common denominator, we get

$$\frac{4}{5} = \frac{4 \times 7}{5 \times 7} = \frac{28}{35}.$$

$$\frac{6}{7} = \frac{6 \times 5}{7 \times 5} = \frac{30}{35}.$$

Now, $\frac{30}{35}$ is greater than $\frac{28}{35}$; and, therefore, the ratio 54 to 63 is greater than the ratio 36 to 45. Ans.

(2) The specific gravity of water is 1. Therefore, we have

$$\frac{27 \text{ ounces}}{\text{weight of the petroleum}} = \frac{1}{.7}.$$

By rule II, Art. **17**,

$$\text{weight of petroleum} = \frac{27 \text{ ounces} \times .7}{1} = 18.9 \text{ oz. Ans.}$$

(3) (*a*) By rule I, Art. **17**, we have

$$\text{unknown number} = \frac{6 \times 5}{15} = 2. \text{ Ans.}$$

(*b*) By rule II, Art. **17**,

$$\text{unknown number} = \frac{8 \times 45}{24} = 15. \text{ Ans.}$$

(*c*) By rule I, Art. **17**, we have

$$\text{unknown number} = \frac{2\frac{1}{4} \times \frac{3}{5}}{\frac{3}{4}} = \frac{\frac{9}{4} \times \frac{3}{5}}{\frac{3}{4}} = \frac{27}{20} \div \frac{3}{4} = \frac{27}{20} \times \frac{4}{3} = \frac{9}{5} = 1\frac{4}{5}. \text{ Ans.}$$

(4) We have

$$\frac{\text{weight of a cubic foot of water}}{\text{weight of a cubic foot of iron}} = \frac{1}{7.216}.$$

Therefore, $$\frac{1{,}000 \text{ ounces}}{\text{weight of a cubic foot of iron}} = \frac{1}{7.216}.$$

By rule II, Art. **17,** we have

$$\text{weight of cubic foot of iron} = \frac{1{,}000 \text{ ounces} \times 7.216}{1} = 7{,}216 \text{ oz. Ans.}$$

(5) The one block contains $5 \times 4 \times 3$ cu. ft. and the other block contains $10 \times 8 \times 5$ cu. ft.

Therefore, $5 \times 4 \times 3$ cu. ft. weigh 9,945 lb.

Therefore, 1 cu. ft. weighs $\frac{9{,}945 \text{ lb.}}{5 \times 4 \times 3}$.

Therefore, $10 \times 8 \times 5$ cu. ft. weigh $\frac{9{,}945 \times 10 \times 8 \times 5 \text{ lb.}}{5 \times 4 \times 3}$.

Canceling, $10 \times 8 \times 5$ cu. ft. weigh 66,300 lb. Ans.

(6) The proportion stated is

$$\frac{\text{unknown number of ounces}}{14 \text{ ounces}} = \frac{3 \text{ quarts}}{28 \text{ quarts}} = \frac{3}{28}.$$

By rule I, Art. **17,** we have

$$\text{unknown number of ounces} = \frac{14 \text{ ounces} \times 3}{28} = 1\tfrac{1}{2} \text{ oz. Ans.}$$

(7) In this problem, the heights of the poles are to each other as their shadows. Therefore,

$$\frac{8}{64} = \frac{9}{\text{unknown number}}.$$

By rule I, Art. **17,** we have

$$\text{unknown number} = \frac{9 \times 64}{8} = 72.$$

Therefore, a pole 64 feet high casts a shadow 72 ft. long. Ans.

(8) By Art. **40,** we have

$$\frac{\text{area of } A\,B\,C}{\text{area of } A'\,B'\,C'} = \frac{2.5^2}{7.5^2} = \left(\frac{2.5}{7.5}\right)^2 = \left(\frac{1}{3}\right)^2 = \tfrac{1}{9}.$$

Therefore, $$\frac{3.3 \text{ square inches}}{\text{area of } A'\,B'\,C'} = \tfrac{1}{9}.$$

Therefore, area of $A'\,B'\,C' = 9 \times 3.3$, or 29.7 sq. in. Ans.

(9) By Art. **27,** we have

$$\frac{11.9 \text{ inches}}{47.6 \text{ inches}} = \frac{15 \text{ inches}}{B'C'} = \frac{24.1 \text{ inches}}{C'A'}.$$

By rule I, Art. **17,** we have

$$B'C' = \frac{15 \text{ inches} \times 47.6 \text{ inches}}{11.9 \text{ inches}} = 60 \text{ inches}.$$

Again,

$$C'A' = \frac{24.1 \text{ inches} \times 47.6 \text{ inches}}{11.9 \text{ inches}} = 96.4 \text{ inches}.$$

$$\left.\begin{aligned} B'C' &= 60 \text{ in.} \\ C'A' &= 96.4 \text{ in.} \end{aligned}\right\} \text{Ans.}$$

(10) From the table of Art. **49,** the area of a regular octagon whose side is 1 inch is 4.828 square inches. Therefore, by Art. **48,** the area of a regular octagon whose side is 5 inches is $5^2 \times 4.828$, or 120.7 sq. in. Ans.

ARITHMETIC.

(PART 9.)

(1) One-half the chord is $\frac{1}{2} \times 24$, or 12 inches. The perpendicular distance from the mid-point of the chord to the circumference is, therefore, $\sqrt{15^2 - 12^2}$, or $\sqrt{81}$, or 9 inches. Then, the chord of half the arc is a mean proportional between the diameter of the circle and the 9 inches, the height of the segment. Hence, the diameter is equal to $\frac{15^2}{9}$, or 25 in. Ans.

(2) By Art. **13**, we have

$$\text{circumference} = 3.1416 \times 25 \text{ inches} = 78.54 \text{ in.} \quad \text{Ans.}$$

(3) The circumference $= 3.1416 \times 25$, or 78.54 inches. Therefore, by Art. **18**, we have

$$\text{length of arc} = \frac{60 \times 78.54 \text{ inches}}{360} = 13.09 \text{ in.} \quad \text{Ans.}$$

(4) By the rule of Art. **21**, we have

$$\text{length of arc} = \frac{8 \times 8 \text{ inches} - 15 \text{ inches}}{3}$$

$$= \frac{49 \text{ inches}}{3} = 16\tfrac{1}{3} \text{ in.} \quad \text{Ans.}$$

(5) By Art. **24**, we have

$$\text{area} = 3.1416 \times 12.5^2 \text{ square inches}$$
$$= 490.875 \text{ square inches.}$$

Hence, correct to five figures, the area is 490.88 sq. in. Ans.

(6) area of larger circle $= .7854 \times 13^2$ sq. in.,
area of smaller circle $= .7854 \times 7^2$ sq. in.

Hence, area of circular ring in square inches

$$= .7854 \times 13^2 - .7854 \times 7^2$$
$$= .7854\,(13^2 - 7^2) = .7854 \times 120 = 94.248.$$

Thus, the area of the circular ring, correct to four figures, is 94.25 sq. in. Ans.

(7) The area of the circle is 3.1416×20^2 square inches. Then, by Art. **33,** we have

$$\text{area of sector} = \frac{60 \times 3.1416 \times 20^2 \text{ square inches}}{360}$$
$$= 209.44 \text{ sq. in. Ans.}$$

(8) (*a*) By Art. **21,** we have

$$\text{length of arc of sector} = \frac{8 \times 7.65 \text{ inches} - 14.14 \text{ inches}}{3}$$
$$= 15.68\tfrac{2}{3} \text{ inches.}$$

Then, by Art. **35,** we have

$$\text{area of sector} = \tfrac{1}{2} \times 15.68\tfrac{2}{3} \text{ inches} \times 20 \text{ inches}$$
$$= 78.43 \text{ sq. in. Ans.}$$

(*b*) One-half of 14.14 inches = 7.07 inches, and one-half of 20 inches = 10 inches. Then, the altitude of the triangle formed by two radii and the chord 14.14 inches long is $\sqrt{10^2 - 7.07^2}$, or 7.07 inches. Hence, the area of the triangle is $\frac{1}{2} \times 14.14 \times 7.07$, or 49.98 square inches, correct to four figures. By Art. **37,** the area of the segment is (78.43 − 49.98), or 28.45 sq. in. Ans.

(9) By Art. **40,** we have circumference $= 3.1416 \times \sqrt{\dfrac{24^2 + 20^2}{2}}$, or 3.1416×22.1, or 69.4 inches, correct to three figures. Therefore, the circumference is 69.4 in. Ans.

(10) By Art. **42,** the area is .7854 times the product of the two diameters, which is equal to $.7854 \times 12 \times 9$, or 84.8232 square inches. Therefore, correct to four figures, the area is 84.82 sq. in. Ans.

ARITHMETIC.

(PART 10.)

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(1) By Art. **2,** the area of the surface in square inches is 3.1416 × 15^2, or 706.86. Thus, the area correct to four figures is 706.9 sq. in. Ans.

(2) The circumference of the sphere is 3.1416 × 15, or 47.124 inches. The perpendicular distance between the planes is (6.5 − 4), or 2.5 inches. Hence, the altitude of the zone is 2.5 inches. Therefore, by Art. **14,** the area of the zone is 47.124 × 2.5, or 117.81 square inches. Thus, correct to four figures the area is 117.8 sq. in. Ans.

(3) By Art. **15,** the volume of the sphere in cubic inches is .5236 × 28^3, or .5236 × 21,952, or 11,494.0672. Thus, correct to four figures, the volume is 11,490 cu. in. Ans.

(4) The radius of the cylinder is ½ × 5, or 2.5 inches. By Art. **35,** the entire surface of the cylinder in square inches is

2 × 3.1416 × 2.5 × (8 + 2.5), or 2 × 3.1416 × 2.5 × 10.5, or 164.934.

Thus, correct to three figures, the entire surface of the cylinder is 165 sq. in. Ans.

(5) (*a*) The number of square feet of tin required is the number of square feet in the entire surface of the cylinder. The radius of the bin is ½ × 1, or .5 feet. By Art. **35,** the entire surface of the bin in square feet is

2 × 3.1416 × .5 × (3 + .5), or 2 × 3.1416 × .5 × 3.5, or 10.9956.

Thus, the entire surface of the bin and the tin required, correct to two figures, is 11 sq. ft. Ans.

(*b*) The area of the base of the bin in square feet is .7854 × 1^2, or .7854. By Art. **36,** the volume of the bin, in cubic feet, is .7854 × 3, or 2.3562. Thus, correct to two figures, the volume of the bin is 2.4 cu. ft. Ans.

§ 10

(6) The number of square feet of canvas required is the same as the number of square feet in the convex surface of the cone. The radius of the base is $\frac{1}{2} \times 12$, or 6 feet. The slant height of the cone is $\sqrt{11^2 + 6^2}$, or $\sqrt{157}$, or 12.53 feet, correct to four figures. Then, by Art. **47,** the area of the convex surface is $3.1416 \times 6 \times 12.53$, or 236.185488 square feet. Thus, correct to three figures, the canvas required is 236 sq. ft. Ans.

(7) By Art. **52,** the volume of the cone is $\frac{1}{3} \times 3.1416 \times 3.5^2 \times 21$, or 269.3922 cubic inches. Thus, correct to three figures, the volume is 269 cu. in. Ans.

(8) According to rule Art. **56,** adding together the square of the radius of the greater base, the square of the radius of the smaller base, and the product of these two radii, we get the sum $15^2 + 10^2 + 15 \times 10 = 475$. To find the volume, multiply this sum by $\frac{1}{3} \times 3.1416$ times the altitude. Hence, the volume is $\frac{1}{3} \times 3.1416 \times 20 \times 475$, or, correct to four figures, 9,948 cu. in. Ans.

(9) By Art. **61,** the volume of the paraboloid in cubic feet is $\frac{1}{2} \times .7854 \times 3^2 \times 5$, or 17.6715. Thus, correct to three figures, the volume is 17.7 cu. ft. Ans.

(10) By Art. **66,** the volume of the oblate spheroid is $\frac{4}{3} \times 3.1416 \times 15^2 \times 10$, or 1,178.1 cu. in. Ans.

ARITHMETIC.

(PART 11.)

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(1) The perimeter of the base is 4×3, or 12 feet. Hence, the convex surface is 12×2.5, or 30 square feet. The area of the bottom of the tank is 3×3, or 9 square feet. The galvanized iron required for the tank is, therefore, $30 + 9$, or 39 sq. ft. Ans.

(2) (*a*) By Art. **10,** the volume of the tank is equal to the area of the base multiplied by its depth, which is equal to $3^2 \times 2.5$, or 22.5 cu. ft. Ans.

(*b*) If 1 cubic foot holds $7\frac{1}{2}$ gallons, 22.5 cubic feet will hold $22.5 \times 7\frac{1}{2}$, or 168.75 gal. Ans.

(3) To find the area of the base of the prism, divide the base into 6 equilateral triangles with vertexes at the center of the hexagon. Each side of these triangles is 1 inch long (Art. **7,** Part 9). The area of each of these triangles is found by Art. **63,** Part 7. Thus, the sum of the three sides is $1 + 1 + 1$, or 3. One-half the sum is $\frac{1}{2} \times 3$, or 1.5; subtracting each side from the half sum, the difference in each case is .5. Then we have area $= \sqrt{1.5 \times .5 \times .5 \times .5}$, or $\sqrt{.1875}$, or .433 square inch. The area of the base is $6 \times .433$, or 2.598 square inches. The volume of the prism is, therefore, 2.598×12, or, correct to three figures, 31.2 cu. in. Ans.

(4) The area of the base of the pyramid is found, by Art. **63,** Part 7, to be 15.6 square inches, correct to three figures. By Art. **20,** the volume of the pyramid is $\frac{1}{3} \times 8 \times 15.6$, or, correct to two figures, 42 cu. in. Ans.

(5) By Art. **25,** the volume of the wedge is $\frac{1}{6} \times 2 \times 9 \times (6 + 6 + 4)$, or $\frac{1}{6} \times 2 \times 9 \times 16$, or 48 cu. in. Ans.

(6) The area of the larger base is 4×4, or 16 square feet; the area of the smaller base is 2×2, or 4 square feet. The middle section is a square whose side is one-half the sum of the sides of the upper and lower base; that is, $\frac{1}{2}$ of $(4 + 2)$, or 3 feet. The area of the middle section is 3×3, or 9 square feet. The volume of the frustum is

$$\tfrac{1}{6} \times 5 \times (16 + 4 + 4 \times 9), \text{ or } 46\tfrac{2}{3} \text{ cu. ft.} \quad \text{Ans.}$$

(7) We have

rate = 20% = .20.
base = wages before the raise = $2.50.

Hence, the amount = $2.50 × (1 + .20) = $3.00. Therefore, the man's wages after the raise is $3.00 per day. Ans.

(8) There has been used $15 - 9\frac{1}{4}$, or $5\frac{3}{4}$ tons. The base is the amount of coal at first and the percentage is the amount of coal used. By Art. **42**, the rate is equal to percentage divided by the base; that is, the rate is equal to $5\frac{3}{4} \div 15 = .38\frac{1}{3}$. Therefore, $38\frac{1}{3}$% of the coal has been used. Ans.

(9) We have

rate = 28% = .28.
base = population at beginning of the five years = 6,375.

Therefore, the amount = $6,375 \times (1 + .28) = 8,160$. Therefore, the population after the five years was 8,160. Ans.

(10) By the rule, Art. **59**, we have

$$\$500 \times (1 - .40)(1 - .20)(1 - .10) = \$500 \times .60 \times .80 \times .90$$
$$= \$500 \times .432 = \$216.$$

Hence, the selling price is $216. Ans.

ARITHMETIC.

(PART 12.)

(1) (*a*) Since there are 3 feet in 1 yard, in 8 yards there are 3 × 8, or 24 feet. 24 feet + 2 feet = 26 feet. There are 12 inches in 1 foot; therefore, in 26 feet there are 12 × 26, or 312 inches. 312 inches + 8 inches = 320 inches. Hence, 8 yards 2 feet 8 inches is equal to 320 inches.

```
yd.
  8
  3
 --
 24
  2
 --
 26 ft.
 12
---
312
  8
---
320 in.   Ans.
```

(*b*) Since there are 9 square feet in 1 square yard, in 2 square yards there are 9 × 2, or 18 square feet. 18 square feet + 7 square feet = 25 square feet. There are 144 square inches in 1 square foot; in 25 square feet there are 144 × 25, or 3,600 square inches. 3,600 square inches + 98 square inches = 3,698 square inches.

```
sq. yd.
     2
     9
    --
    18
     7
    --
    25 sq. ft.
   144
   ---
   100
  100
  25
  ----
  3600
    98
  ----
  3698 sq. in.   Ans.
```

(2) We have

6 feet 9 inches = 81 inches,
5 feet 6 inches = 66 inches,
4 feet 3 inches = 51 inches.

Hence, by Art. **32,** Part 4, the volume of the box is 81 × 66 × 51, or 272,646 cu. in. Ans.

```
    81
    66
   ---
   486
  486
  ----
  5346
    51
  ----
  5346
 26730
 ------
 272646.
```

(3) (*a*) Since there are 1,728 cubic inches in 1 cubic foot, in 272,646 cubic inches there are 272,646 ÷ 1,728, or 157 cubic feet and 1,350 cubic inches remaining. There are 27 cubic feet in 1 cubic yard; in 157 cubic feet there are 157 ÷ 27, or 5 cubic yards and 22 cubic feet remaining. Hence, 272,646 cubic inches = 5 cu. yd. 22 cu. ft. 1,350 cu. in. Ans.

```
272646 | 1728
1728   |  157
----
 9984
 8640
 -----
 13446
 12096
 -----
  1350

  157 | 27
  135 |  5
  ---
   22
```

(*b*) Since there are 16 ounces in 1 pound, in 2,000 ounces there are 2,000 ÷ 16, or 125 lb. Ans.

```
2000 | 16
16   | 125
--
 40
 32
 --
  80
  80
  --
```

(4) From the solution of example 2, the box is 81 inches long, 66 inches wide, and 51 inches deep.

Area of sides = 2 (81 × 51), or 8,262 square inches.
Area of ends = 2 (66 × 51), or 6,732 square inches.
Area of top and bottom = 2 (81 × 66), or 10,692 square inches.
Total area = 25,686 square inches = 178 sq. ft. 54 sq. in. Ans.

(5) By Art. **32,** Part 4, each book contains $8 \times 6 \times 1\frac{1}{4}$, or 60 cubic inches. To pack 400 books, it takes 400 × 60, or 2,400 cubic inches. 3 feet 4 inches = 40 inches and 2 feet 6 inches = 30 inches. A box

40 inches long and 30 inches wide, in order to contain 2,400 cubic inches, must be $\frac{2,400}{40 \times 30}$, or 20 inches deep. Hence, the depth of the box must be 20 inches, or 1 ft. 8 in. Ans.

(6) 4 feet 6 inches = 54 inches,
2 feet 8 inches = 32 inches,
3 feet 6 inches = 42 inches.

Area of ends = 2 (32 × 42), or 2,688 square inches.
Area of sides = 2 (42 × 54), or 4,536 square inches.
Area of bottom = 32 × 54, or 1,728 square inches.
Total area = 8,952 square inches = 62 square feet 24 square inches.

Therefore, it will require 62 sq. ft. 24 sq. in. of lead to line the tank. Ans.

(7)

ft.	in.	
3	6	
17	4½	
4		
3	3¾	
	5¾	
2	2	
30	10	Ans.

The sum of the inches is 22 inches, which is equal to 1 foot 10 inches. Put the 10 under the inches and add the 1 to the sum of the feet. The sum of the feet is then 30 feet.

(8) (*a*) From the table, Art. **62,**

1 Kg. = 1,000 g.

Therefore, 1 g. = .001 Kg.

Hence, 12.5 g. = 12.5 × .001 Kg. = .0125 Kg. Ans.

(*b*) From the table, Art. **62,**

1 mg. = .001 g.

Therefore, 1 g. = 1,000 mg.

Hence, 12.5 g. = 12.5 × 1,000 mg. = 12,500 mg. Ans.

(9) (*a*) From the table, Art. **61,**

1 Kl. = 1,000 l.

Hence, 13.5 Kl. = 13.5 × 1,000 l. = 13,500 l.

But, 1 l. = 1 $dm.^3$; therefore, 13,500 l. = 13,500 $dm.^3$.

From the table, Art. **59,**

$$1 \text{ dm.}^3 = .001 \text{ m.}^3.$$

Therefore, $13{,}500 \text{ dm.}^3 = 13{,}500 \times .001 \text{ m.}^3 = 13.5 \text{ m.}^3$.

But, $1 \text{ cm.}^3 = .000001 \text{ m.}^3$.

Therefore, $1 \text{ m.}^3 = 1{,}000{,}000 \text{ cm.}^3$.

Hence, $13.5 \text{ m.}^3 = 13.5 \times 1{,}000{,}000 \text{ cm.}^3 = 13{,}500{,}000 \text{ cm.}^3$. Ans.

(*b*) From (*a*), 13.5 Kl. contains 13,500,000 cm.^{3} But 1 cm.^{3} weighs 1 g. (Art. **50**). Hence, 13.5 Kl. of water weigh 13,500,000 g.

From the table, Art. **62,**

$$1 \text{ Kg.} = 1{,}000 \text{ g.}$$

Therefore, $13{,}500{,}000 \text{ g.} = \frac{13{,}500{,}000}{1{,}000} \text{ Kg.} = 13{,}500 \text{ Kg.}$ Ans.

(10) From the table, Art. **70,**

$$1 \text{ l.} = 1.0567 \text{ liquid qt.}$$

Hence, $6.5 \text{ l.} = 6.5 \times 1.0567$, or 6.86855 liquid qt. Ans.

INDEX.

NOTE.—All items in this index refer first to the section (see the Preface) and then to the page of the section. Thus, "Divisor 5 1" means that divisor will be found on page 1 of section 5.

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