THEORY AND DESIGN

OF

REINFORCED CONCRETE ARCHES

A Treatise

For Engineers and Technical Students

BY

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PREFACE.

Of all the problems in bridge designing the analysis of the elastic arch is by far the most difficult. The works which have heretofore appeared on this subject are either so mathematically abstruse, or leave so much to the reader to demonstrate for himself, that they are of little value to the practical engineer or to the technical student whose mathematical training has not been of exceptional order. The entire absence in technical literature of a work obviating these unfortunate features has been brought so forcibly to the attention of the author that he has felt justified in undertaking the present work.

It was felt that the graphical method of analysis would be preferred by both engineers and technical students to the longer and more involved mathematical method of analysis.

Every principle involved in the graphical treatment is explained thoroughly and in detail in the theoretical portion of the work. There are no missing steps in the necessary mathematical analysis of the theory as set forth in the present treatise.

The author has in preparation a companion volume to the present one, which will treat of hinged arches and unsymmetrical arches.

The author wishes to express his indebtedness to the classic work of Professor Henry T. Eddy, and to the works of Professor William Cain. He also desires to express his sincere appreciation to Mr. Roy Malony, B. S. in C. E., for making the drawings for the plates; to Mr. Avery F. Crounse, B. S. E., for checking the stress calculations; to Mr. Charles F. Smith, bridge engineer (graduate of Engineering Department, U. S. Military Academy, West Point), for the final reading of the manuscript; and to Mr. A. D. Butler, B. S. in C. E., for critical reading of proofs.

The author is especially grateful to the publishers for the splendid aid they have extended him and for the exceptional care they have taken in the production of this work.

Through the courtesy of the Concrete Steel Engineering Company of New York the author is able to include in the present work the specifications of that company for reinforced concrete bridges.

Although the "specifications for reinforced concrete structures, as embodied in the building ordinance in the city of St. Louis, which is a report of the special committee on reinforced concrete of the Engineers' Club of St. Louis," would seem to have no place in a work on the theory and design of reinforced concrete arches, yet the fact that many purchasers of the book will doubtless have concrete building work to do would seem to make it advisable to include them.

A. R.

Spokane, Washington, September, 1908.

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CHAPTER I.

THEORY OF THE ELASTIC ARCH.

1. **Force Factors.**—Let Fig. 1 represent a small section of an elastic arch subjected to direct and bending stresses and let

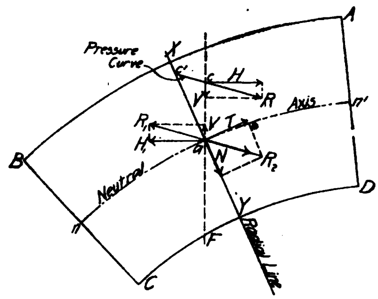


Fig. 1.

the line $c\ R$ be a portion of the entire pressure curve which it is assumed has been found. We may regard the pressure curve as a system of force lines acting at the vertical load lines as $c\ F$ in the figure. Let the magnitude and the direction of the force line acting at this particular section be given by $c\ R$, which is a portion of the pressure curve. For brevity we will term this force R. Let $\Delta\ S$ equal the length $n\ n'$ along the neutral axis and included between the two normal planes $A\ D$ and $B\ C$, that make an angle a before deformation and an angle a' after deformation. The point a on $n\ n'$ is taken as the mid-point, i. e., $n\ a = n'\ a = \frac{1}{2}\ \Delta\ S$.

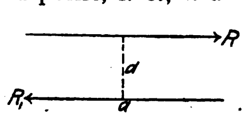


Fig. 2.

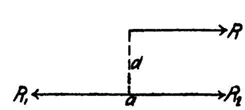


Fig. 3.

Referring now to Fig. 2, it is clear that the two equal and parallel forces R and R_1 constitute a couple whose only tendency is toward rotation. A couple can have no resultant. The moment m of this couple is $m = d\ R = d\ R_1$ for $R = R_1$.

It is evident that we can substitute for the force R (Fig. 3) an equal and parallel force R_2 and a couple $d\ R_1 = m$, if $R = R_1$

$-R_2$, for the two equal and opposite forces R_1 and R_2 acting at a point a, neutralize each other and hence have no actual effect on the entire system of forces.

Let us now substitute for the force R acting at point c (Fig. 1), a force or thrust R_2 acting at point a on the neutral axis and equal to R, and a couple $m = d\,R_1$ where $R = R_1 = R_2$. This substitution will in no manner affect the condition of equilibrium of the system. We have, therefore, substituted for the single force R (Fig. 1), a thrust R_2 and a couple $m = d\,R_1$ acting at a point a of the neutral axis along the infinitesimal radial slice XY through point a. The distance d is the perpendicular distance from the pressure curve to point a of the neutral axis.

The thrust R_2 (Fig. 1) may be resolved into two components, one N, normal to the neutral axis at point a, and the other T, tangential to the neutral axis at point a. The normal component N is effective in producing shear only. It is similar to the shearing stress in a simple beam. It is generally small and may be disregarded. Professor Howe, in his "Treatise on Arches," gives this matter some consideration and the reader is referred to his work for discussion of this subject.

The so-called direct thrust is given by the tangential component T. The tangential component T tends to shorten the length of the section, and therefore the length of the entire arch.

The forces R and R_1 constituting the couple m produce a bending moment. They may be resolved, as in Fig. 1, into horizontal and vertical components. It is seen from Fig. 1 that the vertical components V act in opposite directions and are of equal magnitude; hence they neutralize each other. There remains, therefore, a couple M made up of the two equal horizontal forces H and H_1 whose lever arm is $a\,c$, the vertical distance from the pressure curve to the neutral axis.

The force H is known as the horizontal thrust. It is a constant for all sections of the arch.

The moment of the couple M which is made up of the two equal horizontal components or thrusts H and H_1 is equal to the product of the lever arm $a\,c$ by either of the two equal forces H or H_1.

Hence the bending moment M is given by the equation:

$$M = H.\,a\,c \quad \ldots\ldots\ldots\ldots \quad (1)$$

The moment M is the factor which is most effective in producing deformation of the arch.

If, however, we had resolved the forces R and R_1 of the couple into components normal and tangential to the neutral axis, we would have a case absolutely analogous to the method of resolution explained above with this difference, that it would now be the two normal components of magnitude N which would neutralize each other. These components would act along the radial line XY. Also instead of a couple made up of the two forces H and H_1 we would have a couple constituted from the two equal tangential forces T, having a lever arm $a\,c'$ measured along the radial line from point a on the neutral axis, to point c' on the pressure curve, and since the forces T are tangential components the lever arm $a\,c'$ would satisfy the condition of being perpendicular to the forces. This tangential couple would produce a moment M' which is equal to $T.\,a\,c'$. It is evident that the moment M' must equal the moment M, because both are ultimately derived from the same original forces R and R_1.

Hence,

$$M' = T.\,a\,c' = M = H.\,a\,c.$$

It follows that

$$\frac{M}{T} = a\,c' \quad \ldots\ldots\ldots\ldots \quad (2)$$

where $a\,c'$ is the radial distance from the pressure curve to the neutral axis.

2—The Deformation of an Elastic Arch due to Bending Moments.—Let $A\ B\ C\ D$, Fig. 4, be an exceedingly small portion of an elastic arch subjected to the action of a bending moment M. The action of the direct thrust T (the tangential component of Fig. 1) is not considered, for, as previously stated, T produces no change in curvature of the arch.

Suppose that the force R, located by the pressure curve, acts below the neutral axis, then the intrados will experience the greatest compression. In order to emphasize this fact, in passing, we may state it as follows:

When pressure curve (R) is below neutral axis then greatest compression is at intrados, and central angle a is increased.

When pressure curve (R) is above neutral axis then greatest compression is at extrados and central angle a is decreased.

Then under the above supposition of R being below the neutral axis the central angle a may be regarded to increase to some value a'. Let the angle $O' \, n \, O$ (Fig. 4) be designated as angle $\Delta \, a$. It then follows that,

$$a' = a + \Delta \, a$$

$$\text{Hence } \Delta \, a = a' - a \ldots\ldots\ldots\ldots\ldots\ldots (3)$$

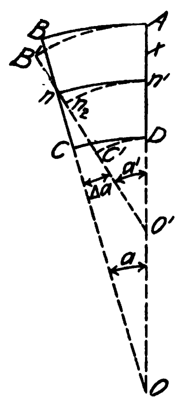


Fig. 4.

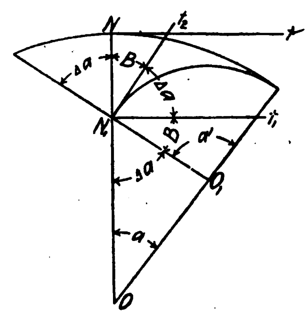


Fig. 5.

Let $N \, t$, Fig. 5, be the tangent before change of curvature, and let $N_1 \, t_1$ be parallel to it. Also let $N_1 \, t_2$ be the direction of the tangent after deformation, then the change of inclination of the tangent due to deformation is given by the angle $t_2 \, N_1 \, t_1$. That angle $t_2 \, N_1 \, t_1 = \Delta \, a$ is at once evident from Fig. 5.

Regard

Angle $\Delta \, a$ as + when M is +, i. e., when R (pressure curve) acts above neutral axis.

Angle $\Delta \, a$ as $-$ when M is $-$, i. e., when R acts below neutral axis.

The action is again supposed to take place at the mid-point of $n \, n'$.

Let the distance $n' \, A = x$ (Fig. 4) be the distance from the neutral axis of any fiber whose area is A. For fibers above the neutral axis, x is regarded as positive (+); for fibers below the neutral axis x is regarded as negative (—).

$$A\,B = \frac{2\,\pi}{360}\,a\,(O\,n' + x)$$

$$A\,B' = \frac{2\,\pi}{360}\,a'\,(O'\,n' + x)$$

As the factor $\frac{2\,\pi}{360}$ is constant it can have no effect upon the realtive change in length of $A\,B$ due to deformation of the arch; consequently it will be neglected and the above equations become:

$$A\,B = a\,(O\,n' + x) = a\,(O\,n') + a\,x \quad \ldots\ldots\ldots\ldots (4)$$

$$A\,B' = a'\,(O'\,n' + x) = a'\,(O'\,n') + a'\,x \quad \ldots\ldots\ldots\ldots (5)$$

As the angle a decreases the corresponding radius increases and the converse is also true; hence, since the change of the angle a must be slight because the section considered is assumed very small, it follows that for practical purposes $a\,(O\,n') = a'\,(O'\,n')$.

Let $\Delta\,S = a\,(O\,n')$, then equations (4) and (5) become,

$A\,B = \Delta\,S + a\,x$ before deformation

$A\,B' = \Delta\,S + a'x$ after deformation

Subtracting the first of the above equations from the second we get,

$$A\,B' - A\,B = a'\,x - a\,x = x\,(a' - a),$$

which is then the change in length due to deformation.

From equation (3) $x\,\Delta\,a = x\,(a' - a)$ hence,

$$A\,B' - A\,B = x\,\Delta\,a = \text{change in length} \quad \ldots\ldots\ldots (6)$$

Since,

the unit stress on any fiber =

$$\frac{\text{elongation of fiber}}{\text{original length of fiber} \times \text{modulus of elasticity}}$$

then if

a = area of any fiber

c = unit stress on the concrete

s = unit stress on the steel

E_c = modulus of elasticity of concrete

E_s = modulus of elasticity of steel

$n = \frac{E_s}{E_c}$

we find the unit stress on the fiber of concrete to be given by the equation,

$$c = \frac{x \Delta a}{\Delta S + x a} E_c$$

Since the above equation represents the stress on one unit of the fiber, if the total number of units of area is represented by a, we must multiply both sides of the above equation by a to get the *total stress on the fiber of concrete.*

Hence,

$$\text{Total stress on the fiber of concrete} = c\,a = \frac{x \Delta a}{\Delta S + x a} a E_c \ldots (7)$$

Similarly,

$$\text{Total stress on the fiber of steel} = s\,a = \frac{x \Delta a}{\Delta S + x a} a E_s \ldots (8)$$

Since the value $x\,a$ in the denominators is small compared with the value ΔS, it may be neglected, therefore,

$$c\,a = \frac{x \Delta a}{\Delta S} a E_c \ldots\ldots\ldots\ldots\ldots\ldots (9)$$

$$s\,a = \frac{x \Delta a}{\Delta S} a E_s \ldots\ldots\ldots\ldots\ldots\ldots (10)$$

The combined stress on concrete and steel on the entire section due to the bending moment only is the sum Σ of the separate stresses on the individual fibers constituting the section, hence,

$$\Sigma\,(c\,a) + \Sigma\,(s\,a) = \Sigma \frac{x \Delta a}{\Delta S} a E_c + \Sigma \frac{x \Delta a}{\Delta S} a E_s$$

Since Δa and ΔS are constants for any particular section, and since E_c and E_s are constants which depend upon the kind of the material it follows that they can be placed outside of the summation sign, therefore,

$$\Sigma\,(c\,a) + \Sigma\,(s\,a) = \frac{\Delta a E_c}{\Delta S} \Sigma\,(x\,a) + \frac{\Delta a E_s}{\Delta S} \Sigma\,(x\,a) \quad (11)$$

It is clear that the moment of any fiber about the neutral axis is equal to the total force or stress on the fiber multiplied by its distance from the neutral axis, hence, since the total force or stress on a fiber of concrete is given by $c\,a$ and similarly for the total force on the steel we have the expression $s\,a$, then since the distance from the neutral axis to the fiber is x, we obtain for the moment on a fiber of concrete, the value $(c\,a)\,x$, and for the moment on a fiber of steel, the value $(s\,a)\,x$.

This gives for the total bending moment M of concrete and steel

$\Sigma\,(x.c\,a) + \Sigma\,(x.\,s\,a)$ which also equals the total resisting moment.

Comparing this expression with equation (11) we find that

$$M = \Sigma\,(x.c\,a) + \Sigma\,(x.\,s\,a)$$
$$= \frac{\Delta\,a\,E_c}{\Delta\,S}\,\Sigma\,(x^2\,a) + \frac{\Delta\,a\,E_s}{\Delta\,S}\,\Sigma\,(x^2\,a) \qquad (12)$$

The moment of inertia I of any cross section whose radius of gyration is x, is equal to the sum Σ of all the particles composing the cross-section multiplied by the square of the radius of gyration.

Hence $\Sigma\,(x^2\,a)$ is the moment of inertia of either the concrete or the steel, depending upon which may be referred to.

If I_c = moment of inertia of concrete

I_s = moment of inertia of steel

then substituting these values for $\Sigma\,(x^2\,a)$ in equation (12) we obtain

$$M = \frac{\Delta\,a\,E_c}{\Delta\,S}\,I_c + \frac{\Delta\,a\,E_s}{\Delta\,S}\,I_s$$

and therefore,

$$M = \frac{\Delta\,a}{\Delta\,S}\,(E_c\,I_c + E_s\,I_s)$$

Hence since $E_s = n\,E_c$

$$\Delta\,a = \frac{M\,\Delta\,S}{E_c\,(I_c + n\,I_s)} = \text{change of inclination of the tangent.} \qquad (13)$$

Certain assumptions and requirements must now be made concerning equation (13) if it is to be applicable to graphical analysis:—

1st.—That the value of $\Delta\,a$ given by equation (13) holds for appreciable lengths $\Delta\,S$ of the neutral axis.

2d.—That the value of M is regarded as a constant quantity for each particular section considered.

3d.—That the value of M is taken at mid-point of $\Delta\,S$.

4th.—That the values of both I_c and I_s are taken at the mid-point of $\Delta\,S$.

In equation (13) $\Delta\,a$ represents the change of inclination of the tangent for a minute section $\Delta\,S$ (infinitesimal).

To find the total change of inclination θ for a finite length of section S we must take the sum of all the infinitesimal changes of $\Delta\,a$, hence

$$\theta = \Sigma \frac{M \Delta S}{E_c (I_c + n I_s)}$$

but $\Sigma(\Delta S) = S$, then if M_m is the moment at the middle of S and I_c and I_s are respectively the moments of inertia of concrete and steel both taken at the mid-point of S, it follows that:

$$\theta = \frac{M_m S}{E_c (I_c + n I_s)} \quad \text{..........(14)}$$

3. **Three Conditions to be Fulfilled for an Arch Fixed at the Ends and Having no Hinges.**—Let $A\ B\ C$, Fig. 6, represent the neutral axis of an arch before deformation. Let point a be the mid-point of a section of length S along the neutral axis $A\ B\ C$.

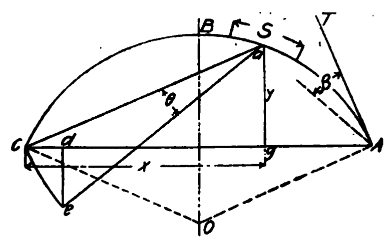


Fig. 6.

Upon loading the arch either with its own weight or with a live load or both, deformation of the neutral axis takes place. The end tangents to the portion S of the neutral axis change their inclination (see Fig. 5). The total amount of this change θ is given by Equation 14, Art. 2, and is equal to

$$\theta = \frac{M_m S}{E_c (I_c + n I_s)}$$

If the end point C of the neutral axis of the arch be temporarily regarded as free to move under the application of a deforming load applied to section S at point a, then, since the change of inclination of the end tangent at section S due to the action of the deforming load is θ, point C must move about point a as an axis through an infinitesimal arc $C\ e$ commensurate with the angle of deformation. Since $C\ e$ and θ are both very small it follows that $C\ e$ may be regarded as perpendicular to $a\ C$. If point C is taken as the origin of coordinates with $C\ A$ as the axis of X then we may designate the coordinates of point a as x and y. Let $e\ d$ be drawn perpendicular to $C\ A$.

The triangles $e\,C\,d$ and $a\,g\,C$ are similar for angle $C\,e\,d =$ angle $a\,C\,g$.
Hence

$$C\,d : e\,C = y : a\,C \text{ or}$$

$$C\,d = \frac{e\,C}{a\,C}\,y$$

But $\frac{e\,C}{a\,C} = \tan\ \theta$. Since θ is very small we can replace the tan θ by the angle θ or arc θ itself. Hence

Horizontal displacement $C\,d = \theta\,y$ ....(15)

Similarly

$$e\,d : e\,C = x : a\,C \text{ and}$$

vertical displacement $e\,d = \frac{e\,C}{a\,C}\,x = \theta\,x$ .... (16)

To obtain an absolutely correct result it would be necessary to subdivide section S into infinitesimal portions $\Delta\,S$. The angle of deformation $\Delta\,a$ at the end of each elementary portion $\Delta\,S$ must then be obtained. The coordinates x and y for these infinitesimal portions $\Delta\,S$ would then have values greater and less than the coordinates for the mid-point a of section S.

In order to make the analysis possible, therefore, we must assume that the quantities M_m, I_c, I_s, x and y, are all taken at the mid-point a of the section S which is under consideration. As a consequence of this assumption it follows that the horizontal and vertical displacements of the end-point C of the neutral axis are, for practical purposes, represented by equations (15) and (16).

If a summation of the horizontal and vertical displacements contributed by the individual sections S which constitute the neutral axis of the arch due to their deformation be made we obtain respectively the total horizontal and vertical displacement which must equal $\Sigma\ (\theta\,y)$, $\Sigma\ (\theta\,x)$.

Suppose that the tangent $T\,A$ at point A move through some small angle β then the vertical displacement $e\,d$ will in a manner analogous to equation (16) be equal to $\beta\,x$. For point A, the value of x is $C\,A$; hence $\beta\,x = \beta.\,C\,A$. The horizontal displacement $C\,d$ will similarly equal $\beta.\,y$, but since y for point $A = 0$, it follows that the horizontal displacement $\beta.y$ must equal 0.

Now since by equation (14), Art. 2, θ is given by the expression

$$\theta = \frac{M_m S}{E_c (I_c + n I_s)}$$

it follows that

$$\Sigma(\theta y) = \Sigma \frac{M_m S y}{E_c (I_c + n I_s)} \quad \ldots\ldots (17)$$

and

$$\Sigma(\theta x) = \Sigma \frac{M_m S x}{E_c (I_c + n I_s)} + \beta . C A \quad \ldots\ldots (18)$$

where $\beta . C A$ is the additional vertical displacement due to a motion of the tangent $T A$ through an angle β.

At points A and C it is seen also that the total change of inclination of the tangents is:—

$$\Sigma(\theta) = \Sigma \frac{M_m S}{E_c (I_c + n I_s)} \quad \ldots\ldots (19)$$

If an arch is properly designed there should be no deflections or deformations due to the action of the dead load and live load. Hence it follows that the three equations (equations 17, 18 and 19) which represent all the possible deformations of an arch without hinges must equal zero. Hence the following three conditions:—

1. $\Sigma(\theta)$, the angle of deformation or the change of inclination of the tangents must $= 0$.
2. $\Sigma(\theta y)$, the horizontal displacement, being the change in length of the span must $= 0$.
3. $\Sigma(\theta x)$, the vertical displacement, being the vertical deflection of one springing line with respect to the other must $= 0$.

When the arch is of reinforced concrete the materials involved are concrete whose modulus of elasticity is E_c, and steel whose modulus of elasticity is E_s, then the above conditions are represented by equations (19), (17) and (18) respectively. Hence the conditions become:—

$$1. \quad \Sigma(\theta) = \Sigma \frac{M_m S}{E_c (I_c + n I_s)} = 0 \quad \ldots\ldots (20)$$

$$2. \quad \Sigma(\theta y) = \Sigma \frac{M_m S y}{E_c (I_c + n I_s)} = 0 \quad \ldots\ldots (21)$$

$$3. \quad \Sigma(\theta x) = \Sigma \frac{M_m S x}{E_c (I_c + n I_s)} + \beta . C A = 0 \quad \ldots (22)$$

All the dimensions in the above equations are in feet, and M is expressed in foot pounds.

Furthermore, x and y are the ordinates of the mid-point of section S with the origin at the springing line.

Let H = horizontal thrust; *constant* for every section of the arch ring for any given loading.

$a\,c$ = vertical distance of the mid-point of section S from the neutral axis to the pressure curve.

Then we have equation (1), Art. 1,

$$M_m = H . a\,c$$

Substituting this value of M_m in the previous equations we obtain:

$$\Sigma \frac{H . a\,c . S}{E_c\,(I_c + n\,I_s)} = 0 \quad \ldots\ldots\ldots\ldots \quad (23)$$

$$\Sigma \frac{H . a\,c . S . y}{E_c\,(I_c + n\,I_s)} = 0 \quad \ldots\ldots\ldots\ldots \quad (24)$$

$$\Sigma \frac{H . a\,c . S . x}{E_c\,(I_c + n\,I_s)} + \beta . C\,A = 0 \quad \ldots\ldots\ldots\ldots \quad (25)$$

Now for any given arch E_c must be regarded as constant. Furthermore E_s must be regarded as constant and hence,

$$n = \frac{E_s}{E_c} \text{ is a constant.}$$

Therefore, if we so proportion an arch that

$$\frac{S}{I_c + n\,I_s} \text{ is a constant,}$$

then since H is a constant, and E_c is a constant, it follows that the value

$$\frac{H . S}{E_c\,(I_c + n\,I_s)} \text{ is a constant.}$$

In equations (23), (24) and (25) the above value being a constant, can be placed outside the summation sign, reducing the equations to the form:

(Condition 1.) $\Sigma\,(a\,c) = 0$ .... (26)

(Condition 2.) $\Sigma\,(ac.\,y) = 0$ .... (27)

(Condition 3.) $\Sigma\,(ac.\,x) = 0$ .... (28)

In equation (25) the term $\beta . C\,A$ must $= 0$ for if there is to be no deformation of the arch, there must be no vertical displacement of the end tangent $T\,A$. Hence the angle β must $= 0$ which reduces the entire expression $\beta . C\,A$ to 0. Consequently the second term $\beta . C A$ of equation (25) vanishes and we obtain its simplified form, under the above assumptions, in equation (28).

It will generally be sufficiently exact to suppose that E_c is a constant for the entire arch, and that $n I_s$ is constant, for the value I_s cannot be determined until the size and disposition of the reinforcement is known. Hence the expression

$$\frac{S}{I_c + n I_s} \text{ becomes } \frac{S}{I_c + \text{constant}}$$

Therefore if the arch ring is so designed that $\frac{S}{I_c}$ in the above expression is also constant we will be sufficiently close for practical purposes.

But $I_c = \frac{bD^3}{12}$ where

b = thickness of slice of ring considered, generally 1 foot.

D = radial depth of section S at mid-point a.

Hence $\frac{b}{12}$ is a constant; therefore to make $\frac{S}{I_c}$ constant it is only necessary to make $\frac{S}{D^3}$ constant.

Referring again to equations (20),(21)and(22), if $\frac{S}{E_c(I_c+n I_s)}$ is constant these equations can be restated so that the three conditions become:

(1st.) $\Sigma(\theta) = \Sigma(M_m) = 0$ .... (29)

(2d.) $\Sigma(\theta y) = \Sigma(M_m y) = 0$ .... (30)

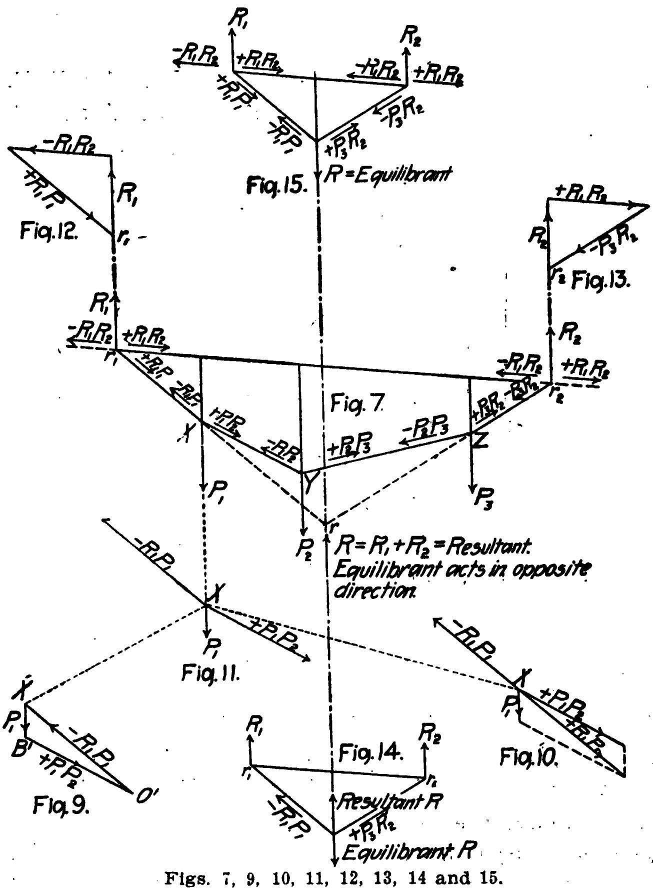
and since $\beta . C A$ is again $= 0$

(3d.) $\Sigma(\theta x) = \Sigma(M_m x) = 0$ .... (31)

4. **Theory of the Equilibrium Polygon.**—Assume that the parallel forces (Fig. 7) R_1, P_1, P_2, P_3, and R_2 are in equilibrium and that they are applied at the points r_1, X, Y, Z, and r_2, respectively. Let their intensities be represented in Fig. 8 to some arbitrary scale of a certain number of pounds to the inch, so that the lengths R_1, P_1, P_2, P_3 and R_2, along the line $A C$ are proportionate to the intensities in pounds. To the extremities of these lengths along $A C$ draw rays from some point O not on the line $A C$. Then in Fig. 7 begin at some point r_1 and draw

$r_1 X$ parallel to ray $R_1 P_1$
$X Y$ parallell to ray $P_1 P_2$
$Y Z$ parallel to ray $P_2 P_3$
$Z r_2$ parallel to ray $P_3 R_2$

We will now demonstrate that line $r_1 r_2$ (Fig. 7) will be parallel to the ray $R_1 R_2$ of Fig. 8.



Figs. 7, 9, 10, 11, 12, 13, 14 and 15.

Suppose in Fig. 7, we apply along the lines $r_1 X$, $X Y$, $Y Z$ and $Z r_2$, two equal forces, opposite in direction, and each equal to a force measured by the magnitude of the ray (Fig. 8) to which the respective lines are parallel. Thus in Fig. 7

Apply along $r_1 X$, forces $+ R_1 P_1$ and $- R_1 P_1$ each = force represented by ray $R_1 P_1$ (Fig. 8).

Apply along $X\ Y$, forces $+\ P_1\ P_2$ and $-\ P_1\ P_2$ each = force represented by ray $P_1\ P_2$ (Fig. 8).

Apply along $Y\ Z$, forces $+\ P_2\ P_3$ and $-\ P_2\ P_3$ each = force represented by ray $P_2\ P_3$ (Fig. 8).

Apply along $Z\ r_2$, forces $+\ P_3\ R_2$ and $-\ P_3\ R_2$ each = force represented by ray $P_3\ R_2$ (Fig. 8).

This does not in any way affect the equilibrium of the system. There is added, however,

at point X (Fig. 7) the two forces $-\ R_1\ P_1$ and $+\ P_1\ P_2$

at point Y (Fig. 7) the two forces $-\ P_1\ P_2$ and $+\ P_2\ P_3$

at point Z (Fig. 7) the two forces $-\ P_2\ P_3$ and $+\ P_3\ R_2$

We shall now prove that by this mathematical device complete equilibrium is established at points X, Y and Z.

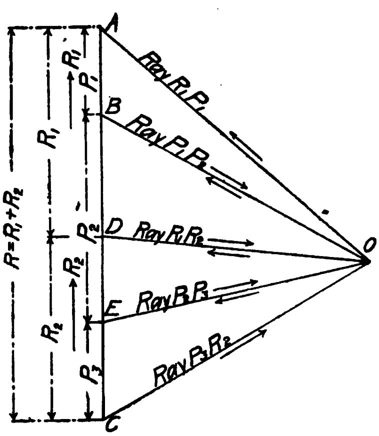


Fig. 8.

Observe that at point X (Figs. 7 and 11) we now have three forces acting,

force P_1 parallel and proportionate to P_1 (Fig. 8)

force $-\ R_1\ P_1$ parallel and proportionate to $R_1\ P_1$ (Fig 8)

force $+\ P_1\ P_2$ parallel and proportionate to $P_1\ P_2$ (Fig. 8)

In the above the − sign signifies that the force acts to the left, and the + sign signifies that the force acts to the right.

Beginning at point X (Fig. 9) we can construct to the same scale used in Fig. 7, a triangle $X\ B'\ O'$ similar to the triangle $A\ B\ O$ of Fig. 8, for the respective sides are proportional and parallel. It is a well known mathematico-physical law that if a system of coplanar forces progress rotationally in their direction, (like the hands of a clock or vice versa) and form a closed polygon then the system of forces must be in equiblibrium.

That this is true for the triangle in question is evident from the graphical construction in Fig. 10, where the resultant $+ R_1 P_1$ of the two given forces P_1 and $+ P_1 P_2$ is exactly balanced by the anti-resultant or equilibrant $- R_1 P_1$ which was the third given force assumed to act at point X.

It is therefore evident that the above mentioned condition for equilibrium is satisfied in Fig. 9. Hence equilibrium exists at point X (Fig. 7) provided that force $+ P_1 P_2$ acts toward the right and force $- R_1 P_1$ acts toward the left which was originally assumed.

A similar demonstration holds good for points Y and Z. Hence the primary forces P_1, P_2 and P_3 (Fig. 7) must be in equilibrium. Consequently the complete system must be in equilibrium. This necessitates that equilibrium must exist at points r_1 and r_2.

At points X, Y and Z we have taken care of forces P_1, $- R_1 P_1$, $+ P_1 P_2$; P_2, $- P_1 P_2$, $+ P_2 P_3$; and P_3, $- P_2 P_3$, $+ P_3 R_2$ respectively; consequently there remains,

at point r_1, forces R_1 and $+ R_1 P_1$
at point r_2, forces R_2 and $- P_3 R_2$.

The resultant of the two forces R_1 and $+ R_1 P_1$ acting at r_1 must act toward the right, and from Fig. 8 we see that it must be equal in magnitude to ray $R_1 R_2$ for this ray completes the triangle $A O D$. This resultant, therefore, is $+ R_1 R_2$ acting toward the right. For equilibrium at point r_1 (Fig. 7) we must apply an equilibrating force $- R_1 R_2$ equal to $+ R_1 R_2$ in magnitude and opposite in direction. We then obtain the closed triangle of Fig 12, which satisfies the condition for equilibrium.

Similarly the resultant at r_2 must act toward the left, and again we see from Fig. 8 that it must be equal in magnitude to ray $R_1 R_2$ for this ray also completes the triangle $D O C$. The equilibrating force at r_2 will therefore be $+ R_1 R_2$ and the condition of equilibrium will be satisfied for point r_2 as seen by Fig. 13.

It is evident that $+ R_1 R_2$ and $- R_1 R_2$ must act along the same straight line $r_1 r_2$ (Fig. 7) otherwise the condition of equilibrium for the entire polygon $r_1 r r_2$ (Fig. 7) will not hold good.

Hence, since the equilibrants at r_1 and r_2 must be in the same straight line and also parallel to the ray $R_1 R_2$ of Fig. 8, it follows the system represented by Fig. 7 cannot be in a condition of equilibrium unless the closing line $r_1 r_2$ of the equilibrium polygon $r_1 X Y Z r_2$ is parallel to ray $R_1 R_2$ of Fig. 8.

In the previous discussion we have assumed that the two reactions R_1 and R_2 were known. In the actual problem of graphical analysis the loads P_1, P_2, P_3 etc., are ascertainable directly and the reactions R_1 and R_2 must be found by graphical analysis.

It has been proved that *the closing line $r_1 r_2$ of an equilibrium polygon must be parallel to the ray $R_1 R_2$ which divides the sum of the loads P_1, P_2 and P_3, etc., into the two reactions R_1 and R_2.* The converse of this must also be true, hence,

If a ray diagram be constructed (Fig. 8) and a ray $R_1 R_2$ be drawn parallel to the closing line $r_1 r_2$ (Fig. 7) of the equilibrium polygon, then the ray $R_1 R_2$ will divide the force or load line $A C$ into two parts, $A D$ and $C D$ which will respectively constitute the reactions R_1 and *R_2.*

The resultant R of the forces $R_1 P_1$ and $P_3 R_2$ is seen from Fig. 8 to be represented in magnitude and direction by line $A C$.

If the equilibrium triangle $r_1 r r_2$ be constructed (Figs. 7 and 14) the resultant R will act from point r, the point of intersection of $R_1 P_1$ and $P_3 R_2$, and it must be parallel to the various loads.

If the reactions R_1 and R_2 acting at r_1 and r_2 respectively be considered as acting upwards then the resultant R will act upwards. Hence the equilibrant R must act downwards.

By supplying in the equilibrium triangle $r_1 r r_2$ (Fig. 15) the opposed forces similarly to the above procedure the fact that equilibrium exists can still further be demonstrated.

It is seen from Fig. 8 that

$$R_1 + R_2 + (-P_1 - P_2 - P_3) = 0 \text{ or}$$

$$R_1 + R_2 = P_1 + P_2 + P_3 \quad \text{(32)}$$

5. **Properties of the Equilibrium Polygon.**—In Figs. 16 and 17 we have exaggerated the case of Figs. 7 and 8 in order to more clearly show the geometrical constructions and properties of the polygon. Assume for Figs. 16 and 17 the action of

the same forces and in the same position as in Figs. 7 and 8. Suppose that a vertical plane $g\,h$ is passed through point g, any point on the equilibrium polygon $r_1\,X\,Y\,Z\,r_2$, and that the portion to the right of this plane, i. e., portion $gh\,r_2$, is removed. This in no way affects the conditions existing at points r_1, X and Y, hence equilibrium must exist at these points for we

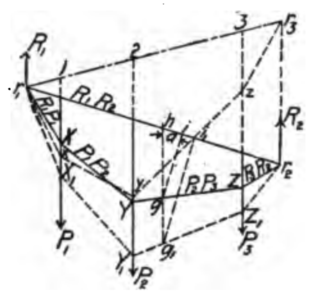


Fig. 16.

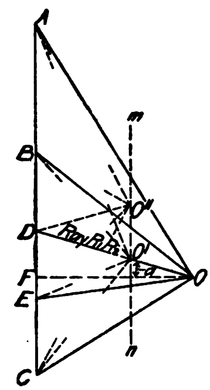


Fig. 17.

still imagine the opposed forces of Fig. 7 to be acting. If $\Sigma\,(M)$ represents the sum of the moments of a given number of forces acting about any point in their plane, then if equilibrium is to exist,

$$\Sigma\,(M) = 0$$

Since the opposed equal forces along $r_1\,X$ (i. e. $+\,R_1\,P_1$, and $-\,R_1\,P_1$) and along $X\,Y$ (i. e. $+\,P_1\,P_2$ and $-\,P_1\,P_2$) neutralize

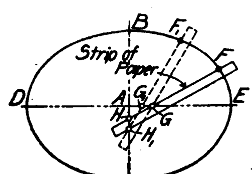


Fig. 18.

each other; then there remains only the forces R_1, P_1, P_2, and $+\,P_2\,P_3$ along $Y\,Z$, and $+\,R_1\,R_2$ along $r_1\,r_2$. This is evident from Fig. 7.

Let us designate the sum of the moment of R_1, P_1, and P_2 about point g by M_1. From Fig. 19 it is clear that $M_1 = R_1\,d_0 - (P_1\,d_1 + P_2\,d_2)$. The moment of force $+\,P_2\,P_3$ about point g

is $= 0$ for the force passes through point g, therefore the lever arm of force $+ P_2 P_3$ is $=0$.

The moment of force $+ R_1 R_2$ along $r_1 r_2$ about point g is $= + R_1 R_2 (g\,i)$ where $g\,i$ is the perpendicular from point g (Fig. 16) to the line of direction of the force $R_1 R_2$.

The moment of force $R_1 R_2$ about point g acts in a clockwise direction, while M_1 acts counter-clock-wise. Consequently,

$$\Sigma (M) = M_1 + P_2 P_3 (0) - R_1 R_2 (g\,i) = 0$$
$$= M_1 - R_1 R_2 (g\,i) = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (33)$$

Erect a perpendicular $O\,F$ from the pole O (Fig. 17) to the force line $A\,C$. Let the length $O\,F$ thus drawn be designated by H which we will hereafter term the pole-distance $= H$.

From the triangle $g\,h\,i$ (Fig. 16), we see that

$$\cos a = \frac{g\,i}{g\,h} \text{ therefore } g\,i = g\,h\,.\cos a$$

If we substitute in the term $- R_1 R_2 (g\,i)$ of equation (33) the value of $g\,i = g\,h\,.\cos a$, we obtain

$$\Sigma (M) = M_1 - R_1 R_2\,.\,g\,h \cos a \quad \ldots\ldots\ldots\ldots\ldots\ldots (34)$$

In triangle $O\,F\,D$ (Fig. 17) it is seen that angle $D\,O\,F$ is equal to angle a of triangle $g\,h\,i$ (Fig. 16).

Line $g\,i$ is perpendicular to $r_1 r_2$ (Fig. 16) which is parallel to line $O\,D$ (Fig. 17); hence $g\,i$ is perpendicular to $O\,D$. Also $g\,h$ (Fig. 16) being drawn vertical must be perpendicular to $O\,F$ (Fig. 17) which is drawn horizontal. Therefore *angle* $h\,g\,i$ = *angle* $D\,O\,F$ = *angle* a. The triangles being right triangles are similar.

From triangle $O\,F\,D$ (Fig. 17) we obtain the relation:

$$\cos a = \frac{O\,F}{O\,D} = \frac{H}{R_1 R_2},$$

for $O\,F$ is the *pole-distance* H, and $O\,D$ is the ray $R_1 R_2$ equal in magnitude and direction to $R_1 R_2$ along line $r_1 r_2$ of Fig. 16.

Hence

$$H = R_1 R_2\,.\,\cos a$$

Substituting this value of $R_1 R_2\,.\,\cos a = H$ in equation (34) we obtain

$$\Sigma M = M_1 - H\,.\,g\,h = 0$$

Hence

$$M_1 = H\,.\,g\,h \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (35)$$

We can state these facts in the form of the following theorems:

I. *In any equilibrium polygon, the moment of all the vertical forces on one side of a vertical section through any point g is equal to the pole-distance H multiplied by the vertical ordinate g h through the point.* (The value H must be measured to the force scale while the length $g\,h$ must be measured to the distance scale.)

II. *In any equilibrium polygon, the moment of all the vertical forces on one side of a vertical section through any point g is equal to the pole distance H multiplied by the vertical ordinate g h through the point even if the moments of the vertical forces are taken about any other point besides g along the line g h.* (This is at once evident since the length of the lever arm from the forces to any point whatever on the line of $g\,h$ must be the same for the direction of the action of the forces is assumed vertical, hence they are parallel to the vertical line $g\,h$.)

III. *For any number of equilibrium polygons having the same system of forces and consequently having the same force line and closing line the product H . g h is a constant quantity. Therefore if H is increased, g h must be decreased in the same ratio and vice versa.* (This can be demonstrated by assuming a new pole O' (Fig. 17) on the ray $R_1\,R_2$ since this ray must be parallel to the closing line and constructing the polygon $r_1\,X_1\,Y_1\,Z_1\,r_2$. It is evident that $g_1\,h$ has increased and H decreased. That the ratio is the same can be proved geometrically.)

If the pole O' could not readily have been located on ray $R_1\,R_2$ (Fig. 17) but was instead located at O'' along the perpendicular $m\,n$ through O', then on constructing the equilibrium polygon $r_1\,x\,y\,z\,r_3$ we find r_3 located above r_2 when O'' is above O' and the reverse is also true. But the vertical ordinates through points *1*, *2* and *3* for polygons $r_1\,x\,y\,z\,r_3$ and $r_1\,X_1\,Y_1\,Z_1\,r_2$ will be respectively equal. Hence by successively taking the lengths $x\,1$, $y\,2$, $z\,3$ and laying them off from the original closing line $r_1\,r_2$ we can determine the location of the points X_1, Y_1 and Z_1 or the polygon corresponding to the pole O'.

Let d_0, d_1, d_2, respectively (Fig. 19) be the perpendicular distances from the line of action of forces R_1, P_1 and P_2 to the vertical line $g\,h$ extended. Then from equation (35) it follows that,

$$M_1 = R_1\,d_0 - (P_1\,d_1 + P_2\,d_2) = H \,.\, g\,h \;\ldots (36)$$

Extend $r_1\,X$ until it intersects $g\,h$ extended to point r. Triangle $r_1\,h\,r$ (Fig. 19) is similar to triangle $O\,D\,A$ (Fig. 20) for the corresponding sides are parallel. Draw line $r_1\,k$ (Fig. 19) perpendicular to $g\,h$ extended. Triangle $r_1\,k\,h$ will then b similar to triangle $O\,F\,D$ for homologous sides are parallel. Now regard $r_1\,k$ as the altitude of triangle $r_1\,h\,r$ and $O\,F$ (or H) as the altitude of triangle $O\,D\,A$. Since these two triangles are similar it follows that their altitudes and bases are in proportion, hence since $r_1\,k = d_0$,

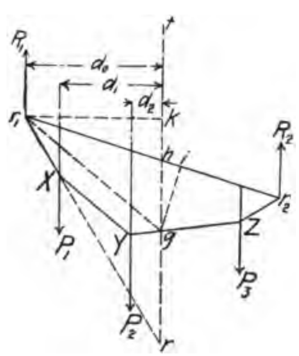


Fig. 19.

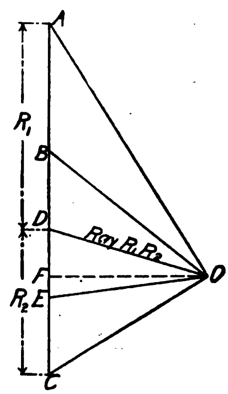


Fig. 20.

$H : R_1 = d_0 : h\,r$, therefore

$H \,.\, h\,r = R_1\,d_0$. Substituting this value of $R_1\,d_0$ in equation (36) we obtain,

$H \,.\, h\,r - (P_1\,d_1 + P_2\,d_2) = H \,.\, g\,h$, therefore

$H \,.\, h\,r - H \,.\, g\,h = P_1\,d_1 + P_2\,d_2$ which gives

$H\,(h\,r - g\,h) = (P_1\,d_1 + P_2\,d_2)$

but by Fig. 19 $(h\,r - g\,h) = g\,r$ whence we obtain

$$H\,(g\,r) = P_1\,d_1 + P_2\,d_2 \;\ldots\ldots\ldots\ldots (37)$$

If then we desire to find the sum of a series of products of the type Pd in an equilibrium polygon we simply construct the small polygon $r_1\,X\,Y\,g$ (Fig. 19) with $r_1\,g$ as closing line. Whatever may be the number of forces to one side of $g\,h$ that enter into the consideration (in this case P_1 and P_2) they are drawn to some force scale along the line $A\,C$ (Fig. 20) and then the polygon $r_1\,X\,Y\,g$ can easily be constructed. The length $g\,r$ is then found by continuing $r_1 X$ and $g h$ to point r, their intersection. Then the value of $P_1 d_1 + P_2 d_2$ will be $H (g r)$ if P_1, P_2 and H are measured to the same scale of force, and d_1, d_2 and $g r$ to the same scale of distance.

6. Theory of the Pressure Curve.—Suppose that there exists in the force system of Fig. 21, equilibrium between the

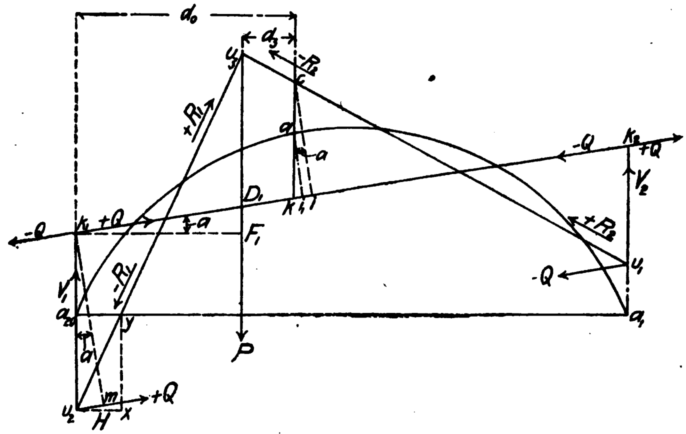


Fig. 21.

force P and the forces V_1 and Q at point u_2; and V_2 and Q at u_1. Forces Q at u_1 and u_2 are assumed parallel.

By reference to Figs. 14 and 15 it is seen that the resultant of V_1 and Q at u_2 and the resultant of V_2 and Q at u_1 must intersect the line of action of force P in the point u_3. Along the force line A C (Fig. 22) take a distance A C on the scale of force equal to force P. Draw A O (R_2) parallel to $u_1 u_3$ and O C (R_1) parallel to $u_3 u_2$. The resultant of V_1 and Q at u_2 then $= R_1$ and the resultant of V_2 and Q at $u_1 = R_2$. If the line O D (Fig. 22) is drawn parallel to the parallel forces Q of Fig. 21, then the magnitudes of the forces acting at u_2 and u_1 will be given by the parts of the diagram of Fig. 22.

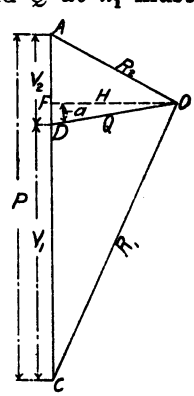


Fig. 22.

Let the line $k_1 k_2$ be drawn in Fig. 21 parallel to O D (Q). Lines $u_2 k_1$ and $u_1 k_2$ are parallel to P. Equilibrium will not be disturbed by the addition of two equal and opposite forces $\pm Q$ at each of the points k_1 and k_2. The forces $+Q$ and $-Q$ between points k_1 and k_2, however, neutralize each other and leave the couple $-Q$, $+Q$ at u_2 and similarly the couple $+Q$, $-Q$ at u_1. In the triangle $F_1\ k_1\ D_1$ let the angle $F_1\ k_1\ D_1$ be designated by a when $k_1\ F_1$ is drawn perpendicular to P. Then if $k_1\ m$ is drawn perpendicular to Q we have the angle $u_2\ k_1\ m$ of triangle $u_2\ k_1\ m$ equal to angle a for the sides of these angles are perpendicular each to each.

The magnitude of the moment of the couple at u_2 is clearly $Q.\ k_1\ m$ but $k_1\ m = k_1\ u_2\ .\ cos\ a$, hence

Moment of couple at $u_2 = Q\ (k_1\ u_2)\ .\ cos\ a$ .....(38)

Now triangle $u_2\ k_1\ m$ (Fig. 21) is clearly similar to triangle $D\ O\ F$ (Fig. 22) and angle $D\ O\ F$ = angle a.

$$Cos\ a = \frac{O\ F}{O\ D} = \frac{H,}{Q} \text{ hence } Q\ .\ cos\ a = H$$

Substituting this value of $Q\ .\ cos\ a$ in equation (38) we obtain,

Moment of couple at $u_2 = H\ (k_1\ u_2)$ ....(39)

similarly Moment of couple at $u_1 = H\ (k_2\ u_1)$ where H is the pole-distance of the force polygon.

If we imagine the opposed forces $-R_1$ and $+R_1$ acting along $u_2\ u_3$ and the opposed forces $-R_2$ and $+R_2$ acting along $u_1\ u_3$ then we find that equilibrium exists at points k_1, u_2, u_3, u_1 and k_2. This is evident without further discussion from Art. 4.

Let us assume that the forces to the right of the vertical line $k\ c$ are removed. This will not disturb equilibrium at points u_2, k_1 and u_3. By Art. 5 the sum of the moments $\Sigma\ M$ about any point u_3 must then be $= 0$.

If we remove the opposed forces $-R_1$ and $+R_1$ acting along $u_2\ u_3$ there remain forces P, V_1, couple $-Q + Q$; force $+Q$ at k_1; and force $-R_2$ along $u_3\ c$.

Let the lever arms of V_1 and P about point c be d_0 and d_3 respectively and observing that the lever arm for force $-R_2 = 0$ we obtain

Moment V_1 about point $c = +V_1\ d_0$.

Moment couple at u_2 about point $c = -H\ (k_1\ u_2)$ (see equation 39)

Moment P about point $c = -P\ d_3$

Moment $-R_2$ about point $c = -R_2\ (0) = 0$

Moment $+Q$ about point $c = -[+Q\ (c\ i)]$

Hence by Art. 5

$$\Sigma (M) = V_1 d_0 - H (k_1 u_2) - P d_3 - Q (c i) = 0$$

In Figs. 21 and 22 the triangles $c k i$ and $O D F$ are similar (Line $k c$ is parallel to P and $c i$ is perpendicular to $k_1 k_2$).

In triangle $c k i$,

$$c i = (k c) . \cos a .$$

Substituting this value of $c i$ in the expression $- Q (c i)$ we obtain $- Q (k c) \cos a$.

But $O D$ of triangle $O D F$ is equal to Q.

Hence $Q \cos a$ (triangle $O D F$) is equal to H.

The expression, therefore, reduces to $- H (k c)$. Hence, since $- Q . c i = - H . (k c)$,

$$\Sigma (M) = V_1 d_0 - H (k_1 u_2) - P d_3 - H (k c) = 0$$

$$\text{or} \quad V_1 d_0 - H (k_1 u_2) - P d_3 = H (k c)$$

$$\text{Let} \quad V_1 d_0 - H (k_1 u_2) - P d_3 = M_c$$

where M_c means the sum of the moments about point c of the forces acting to the left of $k c$. Then

$$M_c = H (k c) \ldots\ldots (40)$$

If line $k c$ is located to the left of force P, equation (40) still holds with this exception that the term $P d_3$ vanishes if we still assume that all the forces to the right of line $k c$ are removed.

If point c is above the line $k_1 k_2$ the moment of force $+ Q$ at k_1 is counter-clock-wise; while if point c is below line $k_1 k_2$ the moment of force $+ Q$ at k_1 is clock-wise. Hence in the latter case M_c is opposite in sign to what it is in the former case, for $M_c = Q (c i) = H (k c)$. The equilibrium polgyon for the forces discussed is, evidently, $k_1 u_2 u_3 u_1 k_2$. We can now state the following theorem:

IV. *If c be any point on the equilibrium polygon whose closing line is $k_1 k_2$, then M_c the moment at the point c is equal to H, the pole-distance multiplied by the vertical ordinate $k c$ from point c to the closing line $k_1 k_2$.*

Since $k c = \frac{M_c}{H}$, any change in the moment M_c (or in the end moment) will cause a change in value $k c$, hence in the location of the closing line $k_1 k_2$. Clearly the magnitude of this change in location of $k_1 k_2$ is measured by the change in $k c$ and hence by $\frac{M_c}{H}$. Now if the moments of the two couples in the same plane are equal then the couples may be equal even if the forces constituting the couple are unequal and their direction not parallel. Hence if it so happened that the forces of the end couple at u_2 are not coincident with $+Q$ and $-Q$, we can replace this couple by another one having an equal moment but whose forces are coincident with $+Q$ and $-Q$. Although, for simplicity, but a single force P has been used in the above demonstration yet the reasoning, evidently, holds for any number of forces.

Suppose that $a_{20}\,a\,a_1$ (Fig. 21) is the neutral line of an arch rib acted upon by a single load or force P and that each arch rib is fixed at the ends. Let the actual reactions be represented by R_1 which then must act at point u_2 in the direction $u_2\,u_3$, and R_2 which similarly acts at u_1 in the direction $u_1\,u_3$. Resolve reaction R_1 along $u_2\ u_3$ into two components $u_2\,x = H$, $x\,y = V_1$, where H acts horizontally and V_1 vertically. Similarly resolve reaction R_2 at u_1 into the horizontal component H and the vertical component V_2. As V_1 acts through a_{20} the moment of V_1 about a_{20} must $= 0$. The moment of H, however, about $a_{20} = H\,(a_{20}u_2)$. Similarly the moment of V_2 about $a_1 = 0$, while the moment of H at a_1 about $a_1 = H(a_1u_1)$.

If an arch is hinged at the points a_{20} and a_1 then there can be no bending moment. The reactions must then pass through the center lines of the hinges, hence through points a_{20} and a_1.

Again, we will apply the two opposed forces $-Q$ and $+Q$ at k_1 parallel to $+Q$ at u_2, and also $-Q$ and $+Q$ at k_2 parallel to $-Q$ at u_1, these groups of opposed forces thus acting along the line $k_1\,k_2$. As seen before, equilibrium is not disturbed by this but we do cause a transfer of force $+Q$ from u_2 to k_1 and the addition of a couple $-Q,\ +Q = -H\ (k_1\,u_2)$. Similarly at the right we transfer force $-Q$ from u_1 to k_2 and add the couple $+Q, -Q = -H\ (k_2\,u_1)$.

From Art. 5 and Theorem II of the same article, if we take moments about point a along the line $k\,c$ for the forces acting to the left of $k\,c$ we obtain just as before,

$$\Sigma\,(M) = V_1\,d_0 - H\,(k_1\,u_2) - P\,d_3 - H\,(k\,a) = 0$$

For the moment of $+Q$ about point $a = -[+Q\,(a\,i_1)]$ where $a\,i_1$ is the perpendicular distance from a to $k_1\,k_2$. This makes triangle $a\,k\,i_1$, $c\,k\,i$ and $O\,D\,F$ similar, and angle $k\,a\,i_1$ = angle a. Hence

$a\, i_1 = k\, a\, (cos\ a)$, therefore,

$-[+ Q\, (a\, i_1)\,] = -[+ Q\, (k\, a)\, cos\ a]$, but from triangle $O\, D\, F$,

$Q\ cos\ a = H$, whence

$$-[+ Q\, (k\, a)\, cos\ a] = -H\, (k\, a)$$

But since $M_c = V_1\, d_0 - H\, (k_1\, u_2) - P\, d_3$, the previous equation becomes

$$\Sigma\, (M) = M_c - H\, (k\, a) = H (k\, c) - H (k\, a) = H (k\, c - k\, a) \quad (41)$$

But $(k\, c - k\, a) = a\, c$; hence this may be stated as follows:

$$\Sigma\, (M) = H\, (a\, c) \quad \ldots \ldots \quad (42)$$

Given an arch rib acting as a girder and subjected to the action of the force P and having end moments at a_{20} *and* a_1 *respectively equal to* $H\, (k_1\, u_2)$ *and* $H\, (k_2\, u_1)$ *where H is the horizontal thrust of the arch equal to the pole-distance; then if the equilibrium polygon* $k_1\, u_2\, u_3\, u_1\, k_2$ *satisfies the condition that the moment about point c,* $M_c = H\, (k\, c)$ *it follows that M the bending moment about point a is* $= H\, (a\, c)$.

Since the displacement of the pressure curve from the neutral axis of the arch is commensurate with the deforming powers of the loads acting, it follows that the bending moment at any point whatever of the arch must be equal to the algebraic difference of the bending moments at points c and a on the pressure curve and neutral axis respectively and also located on the same vertical ordinate which passes through the point in question. Hence the following theorem:

V. *At any point whatever of an arch the bending moment M is equal to the horizontal thrust H multiplied by the distance from the pressure curve c to the neutral axis a of the arch. The values of k c and k a are regarded as positive when measured above the closing line k k and negative when below. The bending moment M is positive when a c is positive, and negative when a c is negative.*

Still further, to emphasize the truth of the above, let point a (Fig. 23) be the mid-point of the portion $n\, n'$ of the neutral axis included in a section S. Let $a\, c$ be a vertical ordinate from point a to the pressure curve $b\, c$. Draw $a\, e$ perpendicular to $b\, c$ and draw $a\, b$ as a horizontal component of $b\, c$. If R be the magnitude of the resultant force acting along the pressure curve $b\, c$ and H be the magnitude of its horizontal component,

i. e., the horizontal thrust, then from Fig. 23, we have M the moment of R about point $a = R\,(a\,e)$. But $\frac{R}{H} = \frac{a\,c}{a\,e}$ hence $R\,(a\,e) = H\,(a\,c)$, therefore $M = H\,(a\,c)$.

The neutral axis itself may be regarded as an equilibrium polygon, having a pole-distance H equal to the horizontal thrust, a closing line $k_1\,k_2$ determined by some well defined system of loading. For this polygon it is clear that $M = M_c - M_a$ and that $H\,(k\,a)$ is the amount of the external forces acting at a. This is true under the condition that the resultants at the abutments act at points u_2 and u_1.

For an arch fixed at the ends we must impose the further restrictions of the three conditions given by equations (29), (30) and (31).

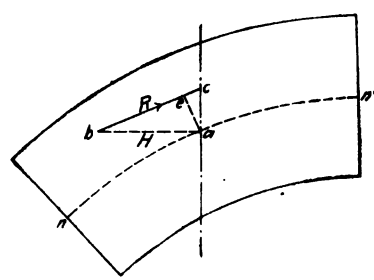


Fig. 23.

First. $\Sigma\,(M\ ) = 0$ . . . . (43)
Second. $\Sigma\,(My) = 0$ . . . . (44)
Third. $\Sigma\,(Mx) = 0$ . . . . (45)

where M is assumed to act at the mid-point of section S.

If the end of moments $H\,(k_1\,u_2)$ and $H\,(k_2\,u_1)$ are of sufficient magnitude to fix the direction of the tangents to the neutral axis of an arch acting as a girder and subjected to the same loading as the arch; then if the difference of vertical displacement of the two ends is $= 0$ we obtain for the summation of the moments of such an arch girder

$\Sigma\,(M_c) = 0$ and $\Sigma\,(M_c\,x) = 0$ . . . . (46)

From $\Sigma\,(M_c)$ take equation (43), this gives

$$\Sigma\,(M_c) - \Sigma\,(M) = 0.$$

But since $M = M_c - M_a$ we obtain

$$\Sigma\,(M_c) - \Sigma\,(M) = \Sigma\,(M_c) - \Sigma\,(M_c - M_a) = 0$$
$$= \Sigma\,(M_c) - \Sigma\,(M_c) - \Sigma\,(M_a) = 0,$$

hence

$\Sigma\,(M_a) = 0$ . . . . (47)

Subtracting $\Sigma\ (M\ x)$ equation (45) from $\Sigma\ (M_c\ x) = 0$, we obtain,

$$\Sigma\ (M_c\ x) - \Sigma\ (M\ x) = \Sigma\ (M_c\ x) - \Sigma\ (M_c - M_a)\ x = 0$$
$$= \Sigma\ (M_c\ x) - \Sigma\ (M_c\ x) - \Sigma\ (M_a\ x) = 0$$

or

$$\Sigma\ (M_a\ x) = 0 \qquad \ldots\ldots\ldots\ldots (48)$$

That the above demonstration for a single load P can be extended to any number of loads ought to be clear.

To Prof. Henry T. Eddy belongs the credit of having first enunciated the principle underlying the graphical theory of the elastic arch. It is, therefore, fitting to bring this article to a close by a quotation from his classic "Researches in Graphical Statics" published in 1878:

"If in any arch that equilibrium polygon (due to the weights) be constructed which has the same horizontal thrust as the arch actually exerts; and if its closing line be drawn from consideration of the conditions imposed by the supports, etc., and if furthermore the curve of the arch itself be regarded as another equilibrium polygon due to some system of loading not given, and its closing line be also found from the same considerations respecting supports, etc., then, when these two polygons are placed so that these closing lines coincide and their areas partially cover each other, the ordinates intercepted between these two polygons are proportional to the real bending moments acting in the arch."—("Researches in Graphical Statics," page 12.)

7. **Development of Fiber Stress Equations.**—The fundamental equation connecting stress and deformation is

$$f = \frac{d}{l} \cdot E \qquad \ldots\ldots\ldots\ldots (49)$$

where

f = unit stress on any fiber

d = deformation of fiber due to tension or compression (may be either elongation or compression of fiber)

l = original length of fiber

$$E = \text{modulus of elasticity} = \frac{\text{stress per unit section}}{\text{amount of deformation}}$$

When we state that the modulus of elasticity E_s of steel is 30,000,000 we mean, if a unit stress (a pound per square inch) is applied, either in tension or compression, the section will be stretched or compressed an amount equal to its length l divided by 30,000,000. Similarly if the modulus of elasticity E_c of concrete is given as 1,500,000 this means that a unit stress (a pound per square inch) applied either in tension or compression will stretch or compress the section an amount equal to its length l divided by 1,500,000.

With these particular values for E_s and E_c,

$$n = \frac{E_s}{E_c} = \frac{30,000,000}{1,500,000} = 20$$

If

A_c = area section of concrete
A_s = area section of steel
E_c = modulus of elasticity of concrete
E_s = modulus of elasticity of steel
F_c = total stress in the concrete
F_s = total stress in the steel
f_c = unit stress in concrete (pounds per square inch)
f_s = unit stress in steel (pounds per square inch)

Then

$F_c = f_c A_c$; $F_s = f_s A_s$, dividing F_s by F_c

$$\frac{F_s}{F_c} = \frac{f_s A_s}{f_c A_c} \qquad (50)$$

If $E_s = 30,000,000$ and $E_c = 1,500,000$, then under the action of a unit stress of one pound to the square inch applied to both the steel and the concrete of same fiber length l, the amount of deformation of the steel is $\frac{l}{30,000,000}$ and for concrete $\frac{l}{1,500,000}$; i. e., the steel is deformed $\frac{1}{20}$ of the amount of the deformation of the concrete, that is

$$d_s : d_c = E_c : E_s \qquad (51)$$

where d_s and d_c are the deformation of the same length l of steel and concrete respectively, due to the application of a unit of stress (one pound per square inch).

If the steel and the concrete are in contact there would be sliding of the one over the other. Furthermore if the steel in a steel-concrete member had been properly proportioned as to permissible fiber stress the concrete might, nevertheless, be strained beyond its ultimate resistance. Consequently it is a matter of ultimate importance in the designing of steel-concrete members that the fiber stresses assumed for the steel and concrete shall be in the same ratio as their moduli of elasticity for then the deformation in both the steel and the concrete will be the same.

We will therefore assume that

$\frac{f_s}{f_c} = \frac{E_s}{E_c} = n.$ Substituting this value in equation (50) we obtain,

$$\frac{F_s}{F_c} = \frac{E_s A_s}{E_c A_c} \qquad (52)$$

In Art. 1 we have seen that in any section S of an arch (Figs. 1 and 4) the forces acting may ultimately be resolved into a couple m and a thrust R_2 acting at the point a. We have also seen that the thrust R_2 may be resolved into the components N normal to the neutral axis at a and T, tangential to the neutral axis at a. The normal component N produces shear and is generally so small it will be neglected in the following calculations. The tangential component T known as the direct thrust is effective in shortening the length of section S and consequently of the entire arch.

The direct thrust T can produce but one type of effect, i. e., a shortening of the section S while the couple m can be effective in two distinct and opposite ways depending upon whether the rotation is clock-wise or counter-clock-wise. Hence the total effect of the fiber stresses in a section S can be represented by the equation,

Total effect of fiber stress = Effect of thrust ± Effect of couple. (53)

Now

$$F_c = f_c A_c; \; F_s = f_s A_s; \text{ and } \frac{f_s}{f_c} = \frac{E_s}{E_c} = n;$$

Hence

$$f_s = n f_c.$$

Let

f = stress on concrete in pounds per square foot, then $f = 144 f_c$ and

$n f$ = stress on steel in pounds per square foot,

and let the areas A_c of concrete and A_s of steel be given in square feet, then the combined total stress on concrete and steel will be

$$f A_c + n f A_s = f (A_c + n A_s) \qquad (54)$$

In the above we have virtually substituted for the area of the steel bars A_s their equivalent $n A_s$ in concrete thus obtaining a homogeneous mass composed only of the one substance concrete. We shall now prove that the neutral axis of such a substituted homogeneous section passes through the center of gravity of the section.

Let the forces $a\,D$, $a\,C$ and $a\,B$ acting at point a of the neutral axis $n\,n'$ (Fig. 24) have such magnitudes and directions that the equilibrium triangle $a\,B\,C$ can be constructed. This would mean that equilibrium exists at point a on $n\,n'$ due to the action of the forces.

Continue $a\,B$ to H making $a\,H = a\,B$. Construct the parallelogram $a\,D\,H\,C$. Let it be required to find the effect of the components of the forces along any sectional plane, preferably radial, as $v\,v_1$.

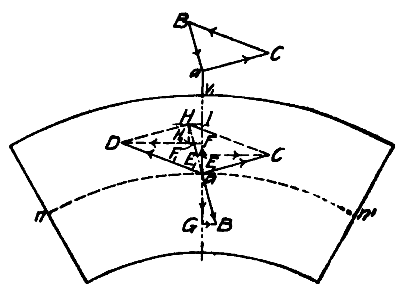


Fig. 24.

Draw $B\,G$, $C\,E$, $D\,F$ and $H\,I$ perpendicular to $v\,v_1$. The triangles $a\,C\,E_1$ and $H\,D\,F_1$ are equal, hence $a\,E_1 = H\,F_1$. In the triangle $a\,H\,I$ if $a\,E_1 = H\,F_1$ then $a\,E = I\,F$.

$a\,F + I\,F = a\,I = a\,G$ and since, $a\,E = I\,F$

$a\,F + a\,E = a\,G$, but

Since $v\,v_1$ is a radial section, components along $v\,v_1$ will be normal components of section S of arch. Hence

$a\,E$ is the normal component of force $a\,C$ in section S of arch

$a\,F$ is the normal component of force $a\,D$ in section S of arch

$a\,G$ is the normal component of force $a\,B$ in section S of arch

But $a\,E$ and $a\,F$ act in directions opposite to $a\,G$, hence

$a\,F + a\,E - a\,G = 0.$

Therefore the sum of the *normal components* at point $a = 0$.

Now draw $H\,H_1$ parallel to $v\,v_1$ and perpendicular to $D\,F$. Triangles $a\,C\,E$ and $H\,D\,H_1$ are then equal and $E\,C = H_1\,D$. Also $H\,I = B\,G = H_1\,F$, therefore

$E\,C + B\,G = H_1\,D + H\,I = D\,F$

Hence the sum of the perpendicular components in section S at point $a = 0$.

From Art. 2, equation 12 and referring to Fig. 4, we see that the total bending moment M for concrete and steel is

$$M = \Sigma\,(x.\,c\,a) + \Sigma\,(x.\,s\,a)$$

where

c = unit stress on the concrete
s = unit stress on the steel
x = distance of fiber from neutral axis
a = area of fiber

Now if for our unit stresses we adopt pounds per square foot, then c becomes f for concrete, and if we are to have no sliding between steel and concrete, the stress for the steel becomes $n\,f$. In terms of these units we obtain

$$M = \Sigma\,(x.\,f\,a) + \Sigma\,(x.\,n\,f\,a) \text{ or}$$
$$M = \Sigma\,(x\,a.\,f) + \Sigma\,(x\,a.\,n\,f)$$

Since the direct thrust T at point a is neutralized by the stresses which cause uniform shortening of the fibers constituting the section, it follows that the stresses due to M and (as has been shown) the normal components of these stresses must neutralize each other, hence

$$\Sigma\,(x\,a.\,f) + \Sigma\,(x\,a.\,n\,f) = 0$$

therefore

$$\Sigma\,(x\,a) = 0$$

This means that if x is measured above or below the the neutral axis from a to all the little differential areas, a constituting the total area along the sectional plane considered, then $\Sigma\,(x) = 0$. This cannot be true unless point a itself is the center of gravity of the sectional plane considered. Hence it follows that *the neutral axis must pass through the center of gravity of the revised homogeneous section.*

In Fig. 25 let

D = depth of arch in feet
D' = depth of steel rib in feet, center to center
x_c = distance in feet from lowest outside fiber of section to neutral axis
x_s = distance in feet from lowest outside fiber of section to center of gravity of steel rib
x = distance in feet from lowest outside fiber to center of gravity of revised homogeneous section.

We have from equation (54) the total stress on concrete and steel given by

$$f A_c + n f A_s.$$

Taking moments about the lowest outside fiber $A B$ of section we obtain,

$x_c f A_c + n x_s f A_s = f(x_c A_c + n x_s A_s)$, but since the neutral axis must pass through the center of gravity of the system x_c must $= \frac{D}{2}$ hence the above expression becomes

$$f\left(A_c \frac{D}{2} + n A_s x_s\right) \quad \ldots\ldots (55)$$

Hence equation (55) must be equivalent to the moment obtained for the revised homogeneous section having x for a lever arm, therefore

$$x f (A_c + n A_s) = f\left(A_c \frac{D}{2} + n A_s x_s\right)$$

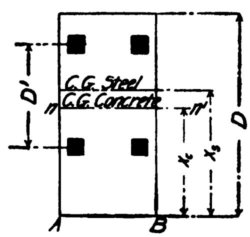


Fig. 25.

But since equation (55) represents the moment of the resultant of those stresses which produce a shortening of the section it must be exactly equal to and opposed by the direct thrust T acting at point a. Consequently the moment of T about $A B$ (Fig. 25) must be given by $x f (A_c + n A_s)$ where x is the length of the lever arm of T about $A B$. Therefore, it is evident that

$$T = f(A_c + n A_s) \text{ or}$$

$$f = \frac{T}{A_c + n A_s} \quad \ldots\ldots (56)$$

where f is the stress on the concrete in pounds per square foot due to the action of the direct thrust T. Hence

$$n f = n \frac{T}{A_c + n A_s} \quad \ldots\ldots (57)$$

which gives us the stress on the steel in pounds per square foot due to the action of the direct thrust T.

Referring to equations (9) and (10), Art. 2, we find

$$c\,a = \frac{x_1\,\Delta\,a}{\Delta\,S}\,a\,E_c$$

$$s\,a = \frac{x_2\,\Delta\,a}{\Delta\,S}\,a\,E_s$$

where x_1 = distance from neutral axis to extreme fiber of concrete and x_2 = distance from neutral axis to center of gravity of steel bar.

Let the units adopted be the pound and the square foot then for c, the unit stress on the concrete we substitute f_c (pounds per square foot), and for s, the unit stress on the steel we substitute f_s (pounds per square foot).

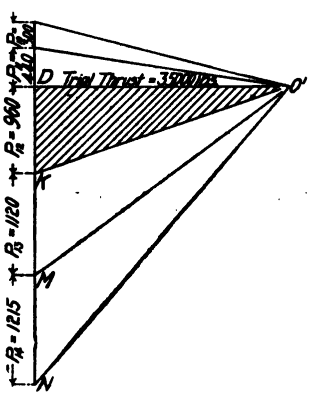


Fig. 26.

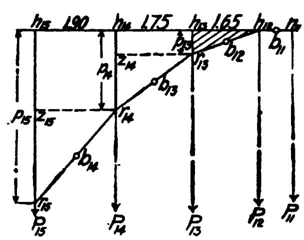


Fig. 27.

These equations, after eliminating a from both, become

$$f_c = \frac{x_1\,\Delta\,a}{\Delta\,S}\,E_c \qquad (58)$$

$$f_s = \frac{x_2\,\Delta\,a}{\Delta\,S}\,n\,E_c \qquad (59)$$

for it has been assumed that $E_s = n\,E_c$.

Multiplying each member of equations (58) and (59) by the corresponding member of equation (13), Art. 2, i. e., by

$$\Delta\,a = \frac{M\,\Delta\,S}{E_c\,(I_c + n\,I_s)} \text{ we obtain,}$$

$$f_c\,.\,\Delta\,a = \frac{x_1\,\Delta\,a\,.\,E_c\,M\,\Delta\,S}{\Delta\,S\,E_c\,(I_c + n\,I_s)} \text{ hence}$$

$$f_c = \frac{M\,x_1}{I_c + n\,I_s} \qquad (60)$$

Similarly

$$f_s = \frac{M\,x_2\,n}{I_c + n\,I_s} \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(61)$$

These values given by equations (60) and (61) are respectively the fiber stresses in the concrete and steel due to the action of the couple m.

The total fiber stress in concrete S_c is therefore from equation (53) equal to eq. (56) $\pm$ eq. (60). Similarly the total fiber stress in steel S_s is equal to eq. (57) $\pm$ eq. (61). Therefore,

$$S_c = \frac{T}{A_c + n\,A_s} \pm \frac{M\,x_1}{I_c + n\,I_s} \quad\ldots\ldots\ldots\ldots\ldots(62)$$

$$S_s = \left(\frac{T}{A_c + n\,A_s} \pm \frac{M\,x_2}{I_c + n\,I_s}\right) n \quad\ldots\ldots\ldots\ldots(63)$$

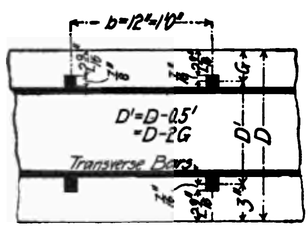


Fig. 28.

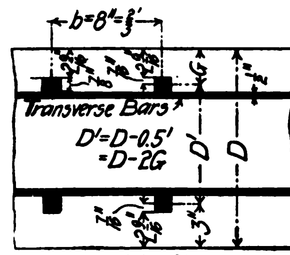


Fig. 29.

In Figs. 28 and 29 let

D = radial depth in feet of arch ring at point a measured from outside to outside of concrete.

G = distance in feet from center of gravity of steel bar to extreme fiber of concrete.

D' = distance in feet between centers of gravity of two steel bars constituting a rib.

Then,

$$x_1 = \frac{D}{2} \text{ and } x_2 = \frac{D'}{2} \text{ (From Fig. 28)}$$

$$I_c = \frac{D^3}{12};\ I_s = \tfrac{1}{4} A_s (D')^2 \text{ (This from the principles of mechanics.)}$$

$$\text{Hence, } I_s = A_s \left(\frac{D'}{2}\right)^2 = A_s\,x_2^2$$

$$I_c + n\,I_s = \tfrac{1}{12} D^3 + n\,A_s\,x_2^2 \quad\ldots\ldots\ldots\ldots\ldots(64)$$

Substituting this value in equations (62) and (63) we obtain,

$$S_c = \frac{T}{A_c + n\,A_s} \pm \frac{M\,x_1}{\frac{1}{12}\,D^3 + n\,A_s\,x_2^2} \quad \ldots\ldots\ldots\ldots (65)$$

$$S_s = \left(\frac{T}{A_c + n\,A_s} \pm \frac{M\,x_2}{\frac{1}{12}\,D^3 + n\,A_s\,x_2^2}\right) n \quad \ldots\ldots (66)$$

8. **Effects Due to Changes in Temperature.**—It is customary to assume as the mean temperature for an arch that temperature at which it is completed, the supposition being that then no stresses exist due to temperature. With a rise or fall in temperature from this *mean* temperature, stresses tending to change the length of the span will develop. This tendency to change the length of the span of an arch fixed at the ends will be resisted by the abutments, thus developing there either a horizontal thrust or a horizontal tension.

Suppose in diagram of Fig. 41, page 84, that a horizontal force H_t acts at point k at the left of the arch. Again resort to the device of the two opposed forces acting at point O at the left of springing line of the neutral axis, each equal to H_t and acting in opposite horizontal directions. Equilibrium is not disturbed by the addition of these two equal but opposed forces at point O. But a transfer of the horizontal force H_t from k to O has thereby been brought about and furthermore there has been added to the system a couple $+H_t, -H_t$ having a lever arm $k\,O$ and therefore $= H_t . k\,O$.

That the stresses brought into play by a change in the length of the span (due to variation in temperature in this case) create a couple is evident from considering the nature of the effects produced along a radial section at the abutments. That the fibers at the extrados and intrados are differently affected and in precisely such a way as when under the action of rotational forces, should be apparent.

If the neutral axis of the arch be regarded as a force polygon then the stresses due to variation in temperature will always act along the closing line $k\,k$ of the force polygon $O\,a\,O_1$.

This is clear from the previous decisions concerning the equilibrium polygon and force polygon.

Since we are assuming a change in the length of the span there will be a horizontal displacement, hence only the 1st and the 3rd conditions given in Article 3 can be fulfilled.

These three conditions, as already given by equations (20) (21) and (22), are

1st. $\Sigma (\theta) = 0$ (Change of inclination of tangents,

2d. $\Sigma (\theta y) = 0$ (Change in length of span)

3d. $\Sigma (\theta x) = 0$ (Deflection one springing line with respect to the other)

Referring also to equations (40), (41) and (42) of Art. 6) we find that

$$\Sigma (M_c) = \Sigma H (k\, c) = 0$$
$$\Sigma (M_a) = \Sigma H (k\, a) = 0$$
$$\Sigma (M) = \Sigma H (a\, c) = 0$$

and since H is constant

$$\Sigma (k\, c) = 0, \text{ and therefore } (k\, c\, .\, x) = 0 \quad \text{(67)}$$
$$\Sigma (k\, a) = 0, \text{ and therefore } (k\, a\, .\, x) = 0 \quad \text{(68)}$$
$$\Sigma (a\, c) = 0, \text{ and therefore } (a\, c\, .\, x) = 0 \quad \text{(69)}$$

Also from Art. 3, equation (27)

$$\Sigma (a\, c\, .\, y) = 0$$

Since conditions (1) and (3) stated above are fulfilled there remains merely the evaluation of the term $\Sigma (\theta y)$.

Now by Art. 3, equation (21), the change in the length of the span is given by

$$\Sigma (\theta\, y) = \Sigma \frac{M\, Sy}{E_c\, (I_c + n\, I_s)}$$

Let

L = length of span of neutral axis in feet

$\pm t^\circ$ = number of degrees of change in temperature

e = coefficient of expansion of section

H_t = horizontal thrust due to change in temperature

Then since the change in length of the span is equal to $L\, e\, t^\circ$ it follows that

$$L\, e\, t^\circ = \Sigma \frac{M\, Sy}{E_c\, (I_c + n\, I_s)} \quad \text{(70)}$$

If the horizontal force is due to changes in the temperature it must be represented by H_t and from the preceding

$$M = H_t\, (k\, a)$$

Substituting this value for M in equation (70) we obtain

$$L\, e\, t^\circ = H_t \frac{S}{E_c\, (I_c + n\, I_s)} \Sigma (k\, a . y)$$

for $H_t \dfrac{S}{E_c\, (I_c + n\, I_s)}$ is constant and can therefore be placed outside of the summation sign. If the arch is symmetrical the summation may be extended over one-half of the span only which necessitates that the result be multiplied by 2.

Assuming that the summation is over one-half of the span only, we have for the total span

$$L\,e\,t^{\circ} = 2\,H_t \frac{S}{E_c\,(I_c + n\,I_s)}\,\Sigma\,(k\,a\,.\,y) \ldots\ldots\ldots (71)$$

If $\Sigma\,(k\,a)$ is to $= 0$ the algebraic sum of the ordinates from points a of the neutral axis to the line $k\,k$ must $= 0$, assuming that the ordinates which are above line $k\,k$ are positive; and below, negative.

Line $k\,k$ must therefore be placed at such a distance above $O\,O_1$, Figs. 31, 32 and 40, that its ordinate e_1 is equal to the mean length of the ordinates from line $O\,O_1$ to points a on the neutral axis. These latter ordinates to points a will hereafter be designated as of the type y.

If

N = number of ordinates of type y.
y = length of individual ordinates from line $O\,O_1$ to points a of neutral axis.
e_1 = length of mean ordinate.

then

$$e_1 = \frac{\Sigma\,(y)}{N}$$

Since $k\,a = y - e_1$

$$\Sigma\,(k\,a\,.\,y) = \Sigma\,(y - e_1)\,y = \Sigma\,(y^2) - e_1\,\Sigma\,(y) \ldots . (72)$$

Substituting this value of $\Sigma\,(k\,a\,.\,y)$ in equation (71) we find

$$L\,e\,t^{\circ} = 2\,H_t \frac{S\,[\Sigma\,(y^2) - e_1\,\Sigma\,(y)]}{E_c\,(I_c + n\,I_s)}$$

Solving this expression for H_t we have

$$H_t = \frac{E_c\,.\,L\,e\,t^{\circ}}{2\,[\Sigma\,(y^2) - e_1\,\Sigma\,(y)]} \times \frac{I_c + n\,I_s}{S} \ldots\ldots\ldots (73)$$

After having obtained the value of H_t as above we construct, at point a of the section considered, a tangent T_t to the neutral axis (hence perpendicular to the radial line of the neutral axis drawn to point a) making this one side of a right triangle $a\,D\,E$ as shown in Fig. 30. The hypotenuse $a\,E$ of this right triangle $a\,D\,E$ is then made equal and parallel to the horizontal temperature thrust H_t. Completing the right triangle we then get $D\,E$, the normal component for the section.

In triangle $a\ D\ E$,

H_t = horizontal temperature thrust (constant for the entire arch)

T_t = tangential temperature component for section considered.

N_t = normal temperature component for section considered.

In Fig. 30, line $a\ E$ extended is perpendicular to $R\ C$ the center line of the arch. Also $a\ D$ is perpendicular to the radial line $a\ R$. Hence angle $E\ a\ D$ = central angle $a\ R\ C = \phi$.

The tangential temperature thrust T_t can therefore be found (see Table IX) from the following relation:

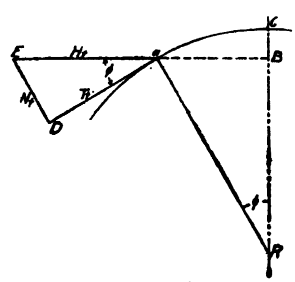


Fig. 30.

$$\cos\ \phi = \frac{a\ D}{a\ E} = \frac{T_t}{H_t},\ \text{hence}$$

$$T_t = H_t \cos\ \phi \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (74)$$

Similarly

$$N_t = H_t \sin\ \phi \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (75)$$

The bending moment M_t due to changes in temperature is from preceding developments given by

$$M_t = H_t\ (k\ a) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (76)$$

Substituting the values T_t and M_t for T and M respectively in equations (65) and (66), we obtain,

Fiber Stress in Concrete S_{ct} Due to Change in Temperature

$$S_{ct} = \frac{T_t}{A_c + n\ A_s} \pm \frac{M_t\ x_1}{\frac{1}{12}\ D^3 + n\ A_s\ x_2^2} \quad \ldots\ldots\ldots\ldots (77)$$

Fiber Stress in Steel S_{st} Due to Change in Temperature

$$S_{st} = \left(\frac{T_t}{A_c + n\ A_s} \pm \frac{M_t\ x_2}{\frac{1}{12}\ D^3 + n\ A_s\ x_2^2}\right) n \quad \ldots\ldots (78)$$

A rise of $t°$ in temperature increases the length of the span of a free arch by $L\ e\ t°$ feet if L is in feet. Similarly a fall of t° decreases the length of the span by the same amount. If the abutments are stable, allowance should be made for this variation in length.

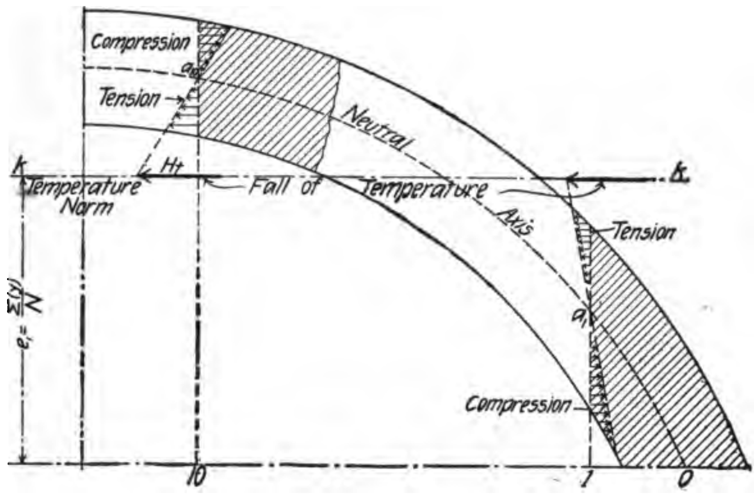


Fig. 31.

	-Ordinates of Type ka		+Ordinates of Type ka	
	Extrados	Intrados.	Extrados	Intrados
Rise	Compression	Tension	Tension	Compression
Fall	Tension	Compression	Compression	Tension

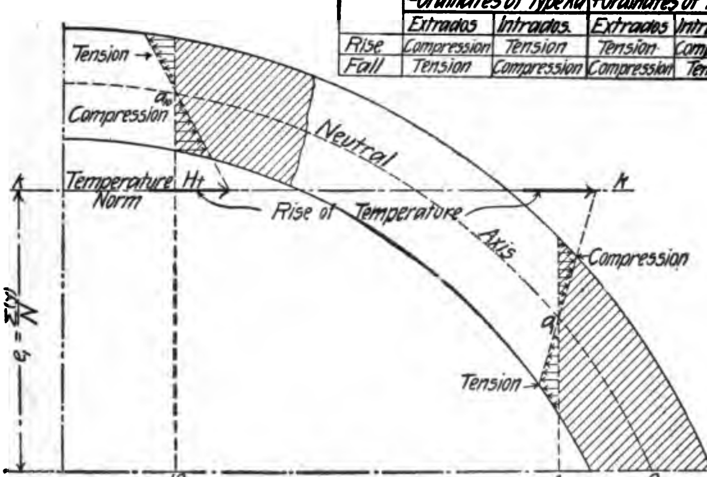


Fig. 32.

In order to bring out the nature of the fiber stresses due to variations in temperature we show in Fig. 31 the effects produced at two typical points a_1 and a_{10} due to a fall in temperature, and similarly in Fig. 32 the effects produced by a rise in temperature. The table which accompanies these two figures explains itself. For both figures we have imagined that the portion of the arch extending to the left has been removed. We substitute for the action of this removed portion the single force H_t acting along the closing line *k k*. For a fall of temperature, Fig. 31, the force H_t acts toward the left, its effect being of the nature of a rotation of each arch filament through its corresponding point *a* on the neutral axis as a center. For a rise in temperature, Fig. 32, H_t acts toward the right producing a complete reversal of effects.

CHAPTER II.

DESIGN OF A REINFORCED CONCRETE ARCH.

9. **Loading.**—*Dead Load.* Actual weight of material in the structure. Computed on a basis of 150 lbs. per cu. ft. for concrete and 120 lbs. per cu. ft. for backing above arch (ballast or filling). Pavement, 150 lbs. per sq. ft. on a basis of 12 in. in depth. Rails, 60 lbs. per lin. ft. of track. The ratio $\frac{120}{150} = \frac{4}{5}$; consequently, for convenience in estimating the quantity of material in the structure, we can draw a line of reduced backing, Fig. 39, at a distance above the extrados at the crown equal to $\frac{4}{5}$ of crown distance between surface of roadway and extrados curve. Then we can multiply every cubic foot of material below the line of reduced backing by the single constant 150 to find the total dead load. This gives somewhat more than the actual load, for the actual line of reduced backing is not a straight line but a curve approaching the extrados curve at the springing line more than the straight line.

To find graphically the point of application of the load, that is the center of gravity of a section included between two vertical lines, $E\,M$ and $D\,B$, Fig. 39, and the portion of the reduced backing $E\,D$ and the intrados curve $M\,B$; proceed as follows:

On $E\,M$ extended take $E\,F = B\,D$.
On $D\,B$ extended take $B\,A = E\,M$.

Connect A and F. Bisect $E\,M$ at X. Bisect $B\,D$ at Y. Connect X and Y. The point of intersection of $X\,Y$ and $A\,F$ is the center of gravity of the section which has been assumed, without appreciable error, to be a trapezoid.

Live Load. The assumptions for live load will depend entirely upon the service for which the structure is intended.

Type A.—For bridges and subways carrying railroad or electric car traffic, the company for which the structure is designed will specify the loads that must be assumed. The load on each track shall be assumed to be distributed over a width equal to the distance from center to center of tracks. In the calculations the greatest equivalent load per lineal foot that will ever be sustained by the arch span with all tracks loaded must be employed.

Type B.—For bridges and subways carrying highway traffic the uniformly distributed live load shall be from 125 lbs. to 200 lbs. per sq. ft. of the roadway and sidewalks, depending upon the conditions which must be provided for in the particular design. A live load of 200 lbs. per sq. ft. provides for a 15-ton steam roller.

The following concentrated loads may be used:

(1) For city and suburban bridges, 15-ton steam roller 11 ft. between axles, 6 tons on forward wheel 4 ft. wide, and 4.5 tons on each of the two rear wheels 5 ft. between centers and 20 ins. wide.

(2) For country bridges **5** tons on four wheels 8 ft. between axles, and 6 ft. gage.

To elucidate the scheme for loading for *Type A*, suppose that the loading adopted for a particular railroad bridge is 5,000 lbs. per lin. ft. of track distributed over $16\frac{2}{3}$ ft., together with an excess at the head of the train of 50,000 lbs.

Since all the calculations consider merely the portion of an arch and backing included between two vertical longitudinal planes 1 ft. apart, we must find the number of pounds per lineal foot for a width of 1 ft, i. e., $5{,}000 \div 16\frac{2}{3} = 300$ lbs. per lin. ft. for longitudinal arch rib, 1 ft. wide. For maximum or nearly maximum bending moments this live load of 300 lbs. per lin. ft. is placed over one-half of the span, say the left half. To obtain the live load concentration at the center of gravity at any particular section we multiply its horizontal length by 300. The 50,000 lbs. excess at the head of the train may then be assumed to be concentrated at points 10, 9 and 8. For a rib 1 ft. wide the total amount of this excess load will be $50{,}000 \div 16\frac{2}{3} = 3{,}000$ lbs. If this be distributed equally at points 10, 9 and 8, we have at each of these points an excess loading of 1,000 lbs.

For more detailed information concerning methods of loading, the reader is referred to "De Pontibus" by J. A. L. Waddell.

10. **Proportioning of an Arch.**—For proportioning arches the following formulas are prescribed:

(*a*) *Crown Thickness:*

(F. F. Weld's Formula.)

Let d = depth of arch at crown in inches.
L = the clear span in feet.
w = live load in pounds per square foot uniformly distributed.
p = weight of dead load above the crown of the arch per square foot in pounds.

then,
$$d = \sqrt{L} + 0.1\,L + 0.005\,w + 0.0025\,p \quad \ldots\ldots (79)$$

(D. B. Luten's Formula.)

Let t = thickness of crown in inches.
S = span in feet.
r = rise from springing to crown in feet.
F = fill over crown of extrados in feet.
L_l = live load (uniform) in pounds per square foot.
L_m = moving load that will be concentrated on single track or single roadway, over entire span in tons of 2,000 lbs.

then,
$$t = \frac{3\,S^2\,(r + 3\,F)}{4{,}000\,r - S^2} + \left[\frac{L_l\,S^2}{30{,}000\,r} + \frac{L_m(S + 5\,r)}{150\,r}\right] + 4. \quad (80)$$

Use Greater of these.

(*b*) *Construction of Trial Arch.*

Mr. Daniel B. Luten has devised a method for constructing the extrados and intrados curves of an arch which is quite satisfactory in many cases. It has the disadvantage, however, that if the neutral axis is to be an arc of a circle, which is very desirable, it may often happen that the distance from a point on the neutral axis to the extrados curve may not equal the distance from the same point to the intrados curve, the distance being measured radially. The author suggests the following method which he believes obviates this difficulty:

Having calculated the crown thickness by either or both of the above formulas and having decided upon the thickness $b\,c$ in some integral number of inches, lay off the distance on $F\,c$ continued (see Fig. 33) where $F\,c$ is the rise of the intrados curve.

Construct the semi-ellipse $S\,E\,e\,c\,S_1$ passing through points S and c. For method of constructing an ellipse see, Fig. 18, page 17.

On $S\,S_1$, the line joining the springings, take

$$S\,V = \tfrac{1}{10}\,L$$

where L = length of span.

Erect a perpendicular at V to $S\,S_1$ until it intersects the ellipse in point E.

Draw line $S\,c$. Bisect angle $S\,c\,F$ by line $K\,c$. Through point E erect a perpendicular $B\,K$ to line $K\,c$. Through

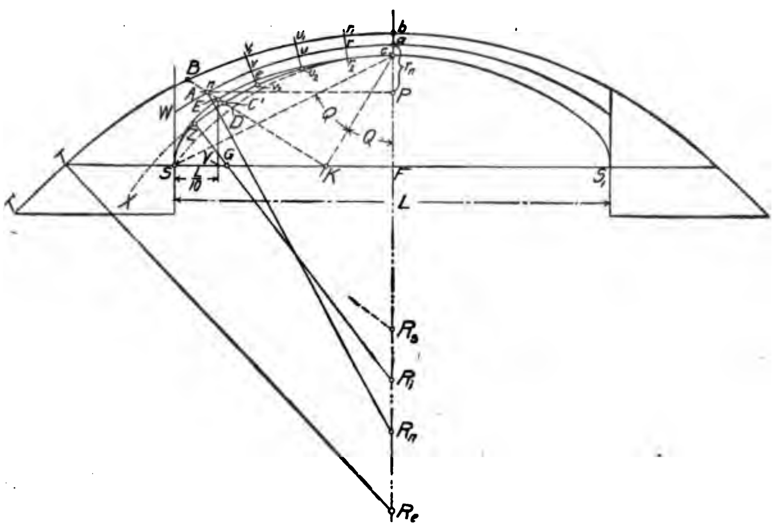


Fig. 33.

points S and c pass an arc of a circle, center at R_s on line $c\,F$ extended. (Line $c\,F$ is perpendicular to $S\,S_1$). Along the line $B\,K$ there is intercepted the distance $E\,D$ between the ellipse and the arc just drawn. Bisect $E\,D$ in point C'. Take on line $B\,K$, the length $C'A = A\,B = c\,b$ the crown thickness. Bisect $c\,b$ at a. Draw $A\,P$ perpendicular to $F\,c$.

Let

$P\,a = r_n$ and

$2\,A\,P = l$ then

$$R_n = \frac{(r_n)^2 + (l^2 + 4)}{2r_n} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (81)$$

where R_n = radius of neutral axis, and r_n = rise of neutral axis.

From equation (81) it is seen that

$$r_n = \frac{2 R_n \pm \sqrt{4R_n^2 - l^2}}{2}.$$

With length R_n as a radius and point R_n as a center describe an arc of a circle passing through points a and A. Line $A\,a$ is then the neutral axis.

Through points B and b pass an arc of a circle whose radius R_e may be found from equation (81) by substituting the corresponding rise and span. Continue arc $B\,b$ to point T on line $S\,F$ extended. From T draw a tangent $T\,T'$ which is therefore perpendicular to $T\,R_e$. This determines the extrados curve.

Divide the distance $A\,a$ along the neutral axis into four equal parts finding points v, u and r. Through these points draw radial lines to point R_n. From the points v, u and r take $v\,v_2 = v\,v_1$; $u\,u_2 = u\,u_1$; and $r\,r_2 = r\,r_1$. Measure all these distances radially.

Find one or more arcs of circles passing through points c, r_2, u_2, v_2 and C'. Proceed from line $F\,b$ toward the springing. In order that the multi-centered curve shall not be distorted, when an arc of different curvature is joined to another and preceding arc their tangents must coincide at the point of juncture. Consequently the radius of any succeeding arc, in passing from crown to springing, must have its center on the last and bounding radius of the preceding arc. Thus in Fig. 33, point G, the center of arc $S\,Z$ is on radial line $G\,R_1$ of preceding arc.

When the springing is approached, continue the last arc indefinitely to X. On $S\,F$ find a point G which will, with point R_1, locate line $G\,R_1$, so that $G\,S = G\,Z$. With G as a center draw arc of circle $S\,Z$ becoming tangent to $S\,W$ at S. This determines the intrados curve.

11. **Proportioning of Backing.**—The following give the amount of backing required:

Bridge of Type A.—In railroad bridges the fill or ballast between the extrados curve of the arch and the bottom of the ties should be at least 3 ft., and preferably 5 to 6 ft., depending upon the nature of the service requirements.

Bridge of Type B.—In highway bridges carrying electric cars and in country bridges the distance between the maximum surface elevation of roadway and the extrados curve at the crown may be determined as follows:

If

t = thickness of crown

F = distance between maximum elevation of roadway and extrados curve at crown, then

$F = 0.9\,t$ ....(82)

This gives the amount of backing required.

12. **Amount of Steel Reinforcement.**—If

A_c = area of concrete in cross-section of crown,

A_s = area of steel in same units (square feet)

then

A_s shall not be less than $0.006\,A_c$

A_s shall not generally be greater than $0.02\,A_c$

13. **Properties of Concrete.**—The coefficient of expansion of concrete per degree Fahrenheit is, according to

Clark..................................	0.00000795
W. D. Pence..........................	0.00000540
Rae & Dougherty (average)............	0.00000608
	3)0.00001943
Average coefficient of expansion concrete	0.00000648

The modulus of elasticity of burnt clay concrete = 1,500,000 lbs.

The modulus of elasticity of all other concrete = 2,000,000 lbs.

The safe adhesion to iron or steel = 60 to 100 lbs. per sq. in.

14. **Properties of Steel.**—The coefficient of expansion of steel per degree Fahrenheit is, according to

Kent.........0.00000648 to 0.00000686; average	= 0.00000667
U. S. Gov't...0.00000617 to 0.00000676; average	= 0.00000646
	2)0.00001313
Average coefficient of expansion of steel........	= 0.00000656

Ultimate strength, 58,000 to 66,000 lbs. per sq. in.

Elastic limit, 55 per cent. of the ultimate strength.

Modulus of elasticity = 30,000,000.

Safe working stress using safety factor of 4 = 15,000 lbs. per sq. in.

Safe working stress using safety factor of 5 = 12,000 lbs. per sq. in.

Safe working stress using safety factor of 6 = 10,000 lbs. per sq. in.

15. **Conditions of Calculations.**—Modulus of elasticity of concrete, 1,500,000 lbs., and modulus of elasticity of steel, 30,000,000 lbs.

Maximum compression allowed on concrete in arches of Type A, exclusive of temperature stresses, 400 lbs. per sq. in.; including stresses due to 40°F. variation in temperature, 500 lbs. per sq. in. Slabs, girders, beams, floors, walls and posts in subways and girder bridges shall have a safety factor of 5 in one month.

Maximum compression allowed on concrete in arches of Type B, exclusive of temperature stresses, 500 lbs. per sq. in.; including stresses due to 40°F. variation in temperature, 600 lbs. per sq. in. Slabs, girders, beams, floors, walls and posts in subways and girder bridges shall have a safety factor of 4 in one month.

Maximum tension allowed on concrete in arches, exclusive of temperature stresses, 50 lbs. per sq. in. (Best practice is to allow no tension for concrete.) In arches, including stresses due to 40°F. variation in temperature, 75 lbs. per sq. in. In slabs, girders, beams, floors, walls and posts, 0 lbs. per sq. in.

Maximum shear allowed on concrete, 75 lbs. per sq. in.

Maximum stress allowed on steel in arches:

Safe working stress, safety factor 4 = 15,000 lbs. per sq. in.

Safe working stress, safety factor 5 = 12,000 lbs. per sq. in.

Safe working stress, safety factor 6 = 10 000 lbs. per sq. in.

This assumes an average ultimate strength of 60,000 lbs.

In a true combination design of steel and concrete $f_s \div f_c = 20$ for values of E_s and E_c assumed, hence the stress on the steel should not exceed 20 f_c. Hence if the maximum stress for concrete (exclusive of temperature) is 500 lbs. per sq. in., then the maximum allowable stress on the steel is 10,000 lbs. per sq. in., giving a safety factor of 6.

In slabs, girders, beams, floors, walls, subjected to transverse stress, the steel should be assumed to take the entire tensile stress without aid from the concrete, and shall have an area sufficient to equal the compressive strength of concrete composed of 1 part Portland cement, 3 parts sand and 6 parts of broken stone, of the age of six months.

In walls and posts subjected to compression only, no allowance shall be made for the strength of the imbedded steel, which is to be used merely as a precaution against cracks due to shrinkage or changes in temperature.

16. **Proportioning of Concrete in Bridge Members.**—The proportions shall be as follows:

	Cement.	Sand.	Stone.
Arches	1	2	4
Spandrel walls	1	3	5
Spandrel walls, preferably	1	2	4
Piers and abutments	1	3	6
Piers and abutments, preferably	1	3	5
Sidewalks and pavement	1	3	5
Railings and balustrades	1	2	4
Wing and retaining walls	1	3	6
Wing and retaining walls, preferably	1	3	5

17. **Design of an Arch Ring for a Highway Bridge.**—Data: Full-barrel arch ring; Clear span of intrados curve = 70 ft.; rise of intrados curve = 15 ft.; Thickness at crown of arch = 20 ins. = 1.66 ft.; live load = 200 lbs. per sq. ft. on roadway and sidewalk; dead load = 150 lbs. per cu. ft. for concrete; dead load = 120 lbs. per cu. ft. for backing.

Order of Procedure.—(*a*) Construct trial arch. (Method of Art. 10). Mark lengths of all radii on drawing. Find, by scaling, rise of neutral axis. These quantities will be needed later in the work.

(*b*) Proportion arch so that

$$\frac{S}{I_c + n\,I_s} = constant. \quad \text{(See Art. 3.)}$$

(*c*) Location of points a and loads P.

(*d*) Construction of equilibrium polygon. (1) Graphical method. (2) Mathematical method.

(*e*) Determination of position of true closing line of equilibrium polygon. Method I. Method II.

(*f*) Determination of position of true closing line of neutral axis regarded as an equilibrium polygon. Conditional equations pertaining to true closing line of neutral axis.

(a) $\Sigma (M) = 0$ hence $\Sigma (k\,a) = 0$

(b) $\Sigma (M\,x) = 0$ hence $\Sigma (k\,a\,.\,x) = 0$

(g) Development of pressure curve of arch by superposition of equilibrium polygon on arch.

(1) Conditional equation (30), Art. 3, must be satisfied, $\Sigma (M\,y) = 0$. (2) Determination of true pole-distance H. (3) Construction of pressure curve.

We will now consider, in detail, the individual steps in the "Order of Procedure:"

(a) *Construct Trial Arch.* (1) Determine first the thickness at the crown by both Weld's and Luten's formulas.

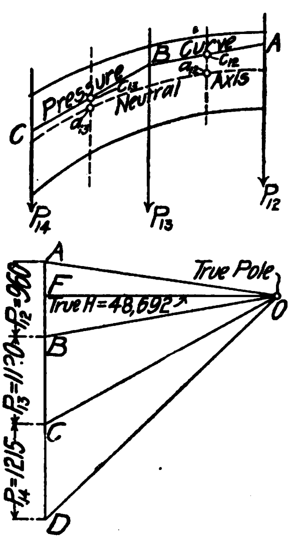


Fig. 34.

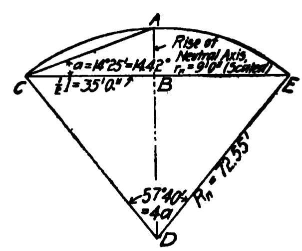


Fig. 35.

Adopt an integral number of inches for crown thickness. Judgment and experience may often dictate a modification in thickness obtained by the use of these formulas. (2) Construct trial arch by method given in Art. 10 and shown by Figs. 33, 34 and 38. Place such data as lengths of all radii, rise of intrados and neutral axis, value of subtended angles, etc., on the drawing. Endeavor to get as large a rise for the neutral axis as the conditions will possibly allow.

(b) *Proportion Arch so that* $\frac{S}{I_c + n\,I_s}$ *= constant.* The reason for this step is set forth in Art. 3. Since a trial arch [by step (a)] has been adopted, the proportioning of the arch reduces to a mere subdivision of the neutral axis (Fig. 40) into lengths S_1, S_2, S_3.........S_{20}, measured along the neutral axis so that

$$\frac{S_1}{I_c + n\,I_s} = \frac{S_2}{I_c + n\,I_s} = \frac{S_3}{I_c + n\,I_s} = \ldots = \textit{Constant.}$$

By equation (64), Art. 7

$$I_c + n\,I_s = \tfrac{1}{12}\,D^3 + n\,A_s\,x_2^2$$

but from Fig. 28 $x_2 = \frac{D'}{2}$; $(x_2)^2 = \left(\frac{D'}{2}\right)^2$

$$D' = D - 2\,G,\ \text{hence}\ \frac{D'}{2} = \tfrac{1}{2}\,(D - 2G) = \left(\frac{D}{2} - G\right)$$

Therefore

$$\left(\frac{D'}{2}\right)^2 = \left(\frac{D}{2} - G\right)^2$$

Since $n = 20$, we have

$$I_c + n\,I_s = \tfrac{1}{12}\,D^3 + 20\,A_s\left(\frac{D}{2} - G\right)^2$$

Hence

$$\frac{S}{\tfrac{1}{12}\,D^3 + 20\,A_s\left(\frac{D}{2} - G\right)^2} = \textit{constant} \quad \ldots\ldots\ldots\ldots (82)$$

To facilitate this work, which is somewhat laborious, the depth of the arch ring (in feet) for every foot of length of one-half of the arch from springing line to the crown may be tabulated as in Table II.

As a safeguard against obtaining too many divisions S it is advisable that the following equation involving divisions S_{20} and S_{11} be solved for S_{11};

$$\frac{S_{20}}{\tfrac{1}{12}\,D_{20}^3 + 20\,A_s\left(\frac{D_{20}}{2} - G\right)^2} = \frac{S_{11}}{\tfrac{1}{12}\,D_{11}^3 + 20\,A_s\left(\frac{D_{11}}{2} - G\right)^2}$$

Substitute in the above equation

$S_{20} = 13\ ft.$ (Trial assumption.)

Find depth D_{20} of arch at mid-point of S_{20}, i. e., at a point 6.5 ft. from springing. In Table II. find depth at 6 ft. = 3.5 ft. and at 7 ft. = 3.4 ft. By interpolation depth at 6.5 ft. = 3.45 ft. = D_{20} for S_{20}.

For S_{11}, $D_{11} = 20$ ins. = 1.66 ft. This is not strictly true for D_{11} is to be measured at mid-point of S_{11}, but S_{11} being an unknown quantity this will be impossible. Yet the change of curvature will be so slight for a short division like S_{11} at the crown that the error is negligible.

For the solution of the above equation between S_{20} and S_{11}, there remains the evaluation of A_s.

Suppose it had been decided to use $\frac{7}{8}$ in. corrugated bars for reinforcement. Area of one $\frac{7}{8}$ in. bar = 0.77 sq. in. = 0.00534 sq. ft.

One steel rib is composed of two bars in the same vertical longitudinal plane, one at the extrados and the other at the intrados. Let

A = cross sectional area in square feet of two bars constituting a rib

b = distance in feet between centers of ribs

A_s = area of steel included between two vertical longitudinal planes 1 ft. apart

Then

$$A_s = \frac{A}{b}$$

Now $A = 2\ (0.00534) = 0.01068$ sq. ft.

Suppose the total width of the arch is 58 ft., then the area of a cross-section at the crown where the depth is 20 ins. = 1.66 ft. is 96.28 sq. ft.

Let us assume as a trial amount of steel, 1 per cent of the cross-sectional area at the crown, hence

Total assumed steel area $A_t = 0.01 \times 96.28 = 0.9628$ sq. ft.

Number of ribs $= A_t \div A = \dfrac{0.9628}{0.01068} = 89$

Distance in feet between centers of ribs = total width of arch ÷ by number of ribs,

Hence $b = \dfrac{58}{89} = 0.65. = \frac{2}{3}$ ft.

(Reduce to nearest integral number of inches, i. e., 8 ins. when dimension is placed on drawing.)

This gives us the disposition of reinforcement shown in Fig. 29,

Hence

$$A_s = \frac{A}{b} = \frac{0.01068}{\frac{2}{3}} = 0.016 \quad \text{(Constant for this arch.)}$$

Since $n = 20$, then $n\,A_s = 20\ (0.016) = 0.32$ (Constant.)

The next thing which must be determined is the distance from the center of gravity of a steel bar to the extreme outer fiber of the concrete.

From equations (60), (61) and succeeding, it is evident that the maximum fiber stresses are partially dependent upon the value of I_s and since I_s (the moment of inertia of the steel) appears in the denominator it will be advantageous to have its value as great as possible consistent with practical requirements.

Suppose we adopt a minimum distance of 2 ins. between outermost fiber of concrete to nearest fiber of steel.

Let us then adopt in this particular problem a distance G of 3 ins. $= 0.25$ ft. between center of gravity of steel and outermost fiber of concrete. This gives $2\frac{9}{16}$ ins. between outermost fibers of concrete and nearest fiber of steel, as shown in Fig. 29.

The above equations may now be put in the form

$$\frac{S_{20}}{\frac{1}{12} D_{20}^3 + 0.32 \left(\frac{D_{20}}{2} - 0.25\right)^2} = \frac{S_{11}}{\frac{1}{12} D_{11}^3 + 0.32 \left(\frac{D_{11}}{2} - 0.25\right)^2}$$

In the above all the quantities except S_{11} are known and the equation can be solved for S_{11}. Any value of S_{20} which makes S_{11} less than 0.75 ft. should generally be discarded.

When S_{20} has been so determined that S_{11} is not too small, then trial values for S_{19}, S_{18}, etc., must be assumed and the corresponding depths D of the arch at their respective midpoints must be found and substituted in the general expression

$$\frac{S}{\frac{1}{12} D^3 + 0.32 \left(\frac{D}{2} - 0.25\right)^2}$$

until the values found for all the assumed sections S give the same constant quantity.

If the arch is symmetrical about the crown the work need only be carried out for one-half the arch. Ultimately the sum of the S values for one-half of the arch must equal one-half of the length of the neutral axis if the arch is symmetrical. Hence

$$S_{20} + S_{19} + S_{18} \ldots . + S_{11} = \tfrac{1}{2} \textit{ length of neutral axis} \quad (84)$$

Method of Finding Length of Neutral Axis.—Find the rise of the neutral axis by scaling from large diagram, Fig. 39.

Let r_n = rise neutral axis = 9 ft.

l = span neutral axis = 70 ft.

Then in Fig. 35

$$tan\ a = \frac{r_n}{\frac{1}{2}l} = \frac{9}{35} = 0.2571$$

From a table of natural tangents we find nearest angle to this value to be 14° 25′, therefore

angle a = 14° 25′ = 14.42°

The entire angle subtending the neutral axis is equal to $4\ a$ = 57° 40′.

Using formula of equation (81) we find the radius of the neutral axis R_n = 72.55 ft.

Let

N = length neutral axis (length of arc $C\ A\ E$) (Fig. 35)

C = length circumference whose radius is R_n.

then

$$N = \frac{4a}{360}\ C = \frac{4a}{360}\ 2\ \pi\ R_n$$

Hence

$$N = \frac{a}{45}\ \pi\ R_n \qquad \dots\dots\dots\dots (85)$$

For $R_n = 72.55$, $a = 14.42°$, N = 73.04 ft.

or $\frac{1}{2}\ N$ = 36.52 ft.

The above method gives the most correct subdivision of the neutral axis of the arch, but on account of the labor involved it is rarely resorted to; the shorter method of making $\frac{S}{D^3}$ constant, as explained in Art. 3, being generally used. This is sufficiently accurate for most purposes. The work in detail is shown in Tables I and II. Five trials were necessary before proper subdivision of the neutral axis was effected. In this case $\frac{S}{D^3} = 0.334$.

Not less than twenty subdivisions of the neutral axis should be used. In Table I., by a slight readjustment of the lengths in the fourth trial, the sum was made = $\frac{1}{2}$ length of the neutral axis, i. e., = 36.52 ft.

(*c*) *Location of Points a and Loads P.*—(1) Location of Points a. Beginning at one of the springing lines, Fig. 40, (say at the left) bisect in succession the lengths S_{20}, S_{19}, S_{18},

TABLE I.—SUBDIVISION OF NEUTRAL AXIS.

Division.	First Trial.		Second Trial.		Third Trial.		Fourth Trial.		Final Trial.		Distance from Springing Line to Centers of Divisions Measured Along Neutral Axis.
	Length.	Summation.	Length.	Summation.	Length.	Summation.	Length.	Summation.	Length.	Summation.	
S_{20}	12.00	12.00	13.00	13.00	14.00	14.00	13.50	13.50	13.51	13.51	6.75
S_{19}	5.51	17.51	5.44	18.44	5.47	19.47	5.30	18.80	5.17	18.68	16.10
S_{18}	3.40	20.91	3.44	21.88	3.67	23.14	3.54	22.34	3.71	22.39	20.54
S_{17}	2.78	23.69	2.55	24.43	2.92	26.06	2.88	25.22	2.92	25.31	23.85
S_{16}	2.41	26.10	2.40	26.83	2.32	28.38	2.30	27.52	2.40	27.71	26.51
S_{15}	2.24	28.34	1.93	28.76	2.04	30.42	1.96	29.48	2.02	29.73	28.72
S_{14}	2.17	30.51	1.81	30.57	1.97	32.39	1.86	31.34	1.86	31.59	30.66
S_{13}	2.07	32.58	1.81	32.38	1.87	34.26	1.77	33.11	1.73	33.32	32.46
S_{12}	1.92	34.50	1.66	34.04	1.72	35.98	1.63	34.74	1.64	34.96	34.14
S_{11}	1.25	35.75	1.39	35.43	1.57	37.55	1.43	36.17	1.56	36.52	35.74
	35.75		35.43		37.55		36.17		36.52		

TABLE II.

Distance Along Neutral Axis From Springing Line in Feet.	Depth of Arch Ring in Feet.	Distance Along Neutral Axis From Springing Line in Feet.	Depth of Arch Ring in Feet.
1	4.5	19	2.3
2	4.1	20	2.25
3	3.9	21	2.2
4	3.7	22	2.15
5	3.6	23	2.1
6	3.5	24	2.05
7	3.4	25	2.00
8	3.3	26	1.95
9	3.2	27	1.90
10	3.1	28	1.85
11	3.0	29	1.82
12	2.9	30	1.79
13	2.8	31	1.76
14	2.7	32	1.74
15	2.6	33	1.72
16	2.5	34	1.70
17	2.45	35	1.68
18	2.4	36	1.66

Point.	Depth of Arch at Mid-point of S in Feet. D	D^3	S	$\frac{S}{D^3} = C$
1	3.43	40.35	13.51	0.334
2	2.49	15.44	5.17	0.334
3	2.23	11.09	3.71	0.334
4	2.06	8.74	2.92	0.334
5	1.93	7.19	2.40	0.334
6	1.82	6.03	2.02	0.334
7	1.77	5.55	1.86	0.334
8	1.73	5.18	1.73	0.334
9	1.70	4.91	1.64	0.334
10	1.67	4.66	1.56	0.334

S_{17}......S_3, S_2 and S_1 along the neutral axis. Label the respective mid-points of the section a_{20}, a_{19}, a_{18}, a_{17}.......a_3, a_2, and a_1.

Through points a_{20}.......a_1 draw vertical lines as in Figs. 39 and 40. Since the arch is assumed symmetrical, the center line $A\ B$ (Fig. 39) bisects the length $a_{11}\ a_{10}$. Using these two parts as independent sections we have between the vertical lines from a_{20} to a_1, twenty sections leaving a portion of the arch from a_{20} to the left springing and a similar portion from a_1 to the right springing. It should be remarked that the points a_{20}. . . . . a_1 are the points at which the depths D, given in Table II., were measured for the corresponding sections S. If then we use the vertical lines through the points a as boundary lines for our load sections, the radial depth d at the boundary of each section will satisfy the condition that $\frac{S}{D^3} = a$ *constant*.

A load section will then consist of the volume inclosed between two longitudinal vertical planes 1 ft. apart and the two transverse vertical planes which pass through any two adjoining points a; and between the plane of the reduced backing and the suface of the intrados, i. e., the soffit. This is true in every case except at the crown where the transverse vertical plane through the center produces two sections instead of one, i. e., the section from the center line to a_{11} and the section from the center line to a_{10}. By this means, as will immediately be apparent, we can locate the load on each side and near the center line and yet avoid the undesirable location of a load at the center line.

(2) Location of Loads P.—We will now concentrate the weight included in each load section at the center of gravity of the corresponding section thus obtaining the location of the load or force lines P.

As an illustration of the method employed, take the load section $M\,B\,D\,E$ (Fig. 39). Regard this section as a trapezoid, i. e., regard line $M\,B$ as a straight line and proceed as in Art. 9.

If in the trapezoid $M\,B\,D\,E$,

$a = E\,M$ (the long side)

$b = D\,B$ (the short side)

$c =$ altitude of the trapezoid ($E\,D$ in $M\,B\,D\,E$)

Then

Area trapezoid $= \frac{1}{2}(a + b)\,c$.

Since the transverse thickness of all the load sections is 1 ft.,

Volume trapezoid $= \frac{1}{2}(a + b)\,c$.

If

$W =$ dead load per load section.

$W_1 =$ live load per load section.

then

$$W = \tfrac{1}{2}\ (a + b)\ c\ (150) \text{ hence}$$

$$W = 75\ (a + b)\ c \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (86)$$

The live load distributed on any load section as at $E\,D$, Fig. 39, must equal the area of a rectangle having $E\,D$ for one side and 1 ft. for the other (for the longitudinal vertical planes are 1 ft. apart) multiplied by the live load (uniform) assumed per square foot, hence if w_1 is the live load in pounds per square foot,

$$W_1 = w_1 \,.\, c \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (87)$$

For the present case where $w_1 = 200$ lbs. per sq. ft. $W_1 = 200\ c$.

In Table III the entire work is shown in detail. In column 4 if 75 is used instead of 150 then $(a + b)$ must be used instead of $\frac{1}{2}\ (a + b)$ in column 1.

To produce maximum moments we concentrate the live load on one-half of the arch—the left half in this case.

In the last column of Table III we give the total summation for live load and dead load.

As we approach the crown the trapezoidal load sections approach the rectangular form and, therefore, it will be close enough to locate the loads P for such sections at the mid-points of c, the altitudes of the sections.

The location of the load or force lines P as well as their magnitudes having been determined their effects at points a, the mid-points of the sections S must be determined.

The development of the entire theory depends upon this fact that the effects are to be ascertained for these mid-points a (see Art. 1 and the preceding chapter on theory) hence the error must not be made of making calculations for points along the load and force lines.

To the left of point a_{20} and to the right of a_1 we have load sections whose force lines $P\,E$, at the left and right respectively, should be determined as above.

(*d*) *Construction of Equilibrium polygon.*—(1) Graphical method. (2) Mathematical method.

(1) We will first consider the purely graphical method.

TABLE III.—CALCULATION OF LOADS P.

Division.	1 Trapezoid. Long Side. (a)	Short Side. (b)	½ (a+b)	2 Altitude Trapezoid. (c)	3 Vol. Cu. Ft. Section 1 Ft. Thick.	4 Dead Load at 150 Lbs. per Cu. Ft. (col. 3×150)	5 Length of Division. (c)	6 Area Section 1 Ft. Wide. Sq. Ft.	7 Live Load at 200 Lbs. per Sq. Ft. (col. 6×200)	Total D.L. + L.L. (col. 4+col. 7)
PE Right	13.5	9.9	11.70	6.0	70.2	10,530	6.0	6.0	....	10,530
P_1	9.9	6.1	8.00	9.0	72.0	10,800	9.0	9.0	....	10,800
P_2	6.1	4.8	5.45	4.4	23.98	3,597	4.4	4.4	....	3,597
P_3	4.8	4.1	4.45	3.2	14.24	2,136	3.2	3.2	....	2,136
P_4	4.1	3.7	3.9	2.6	10.1	1,515	2.6	2.6	....	1,515
P_5	3.7	3.4	3.55	2.2	7.8	1,170	2.2	2.2	....	1,170
P_6	3.4	3.2	3.30	1.9	6.3	945	1.9	1.9	....	945
P_7	3.2	3.1	3.15	1.8	5.7	855	1.8	1.8	....	855
P_8	3.1	3.0	3.05	1.7	5.2	780	1.7	1.7	....	780
P_9	3.0	2.9	2.95	1.5	4.4	660	1.5	1.5	....	660
P_{10}	2.9	2.9	2.90	0.7	2.0	300	0.7	0.7	....	300
P_{11}	2.9	2.9	2.90	0.7	2.0	300	0.7	0.7	140	440
P_{12}	3.0	2.9	2.95	1.5	4.4	660	1.5	1.5	300	960
P_{13}	3.1	3.0	3.05	1.7	5.2	780	1.7	1.7	340	1,120
P_{14}	3.2	3.1	3.15	1.8	5.7	855	1.8	1.8	360	1,215
P_{15}	3.4	3.2	3.30	1.9	6.3	945	1.9	1.9	380	1,325
P_{16}	3.7	3.4	3.55	2.2	7.8	1,170	2.2	2.2	440	1,610
P_{17}	4.1	3.7	3.90	2.6	10.1	1,515	2.6	2.6	520	2,035
P_{18}	4.8	4.1	4.45	3.2	14.24	2,136	3.2	3.2	640	2,776
P_{19}	6.1	4.8	5.45	4.4	23.98	3,597	4.4	4.4	880	4,477
P_{20}	9.9	6.1	8.00	9.0	72.0	10,800	9.0	9.0	1,800	12,600
PE Left	13.5	9.9	11.70	6.0	70.2	10,530	6.0	6.0	1,200	11,730
							Total	D.L. + L.	L., lbs =	73,576

(*a*) Assume trial horizontal thrust H_1, (Fig. 40) to act between two loads near the crown, say P_{11} and P_{12}. This necessitates that the ray H_1 in the ray diagram must be a horizontal line $O'D$ perpendicular to the load line $A\ C$ at point D where load P_{11} ends and load P_{12} begins. Point D gives us a starting point on the load line $A\ C$ from which to lay off the loads P in succession to some arbitrarily selected scale of a certain number of pounds to the inch; in this instance, 5,000 lbs. = 1 in.

The loads for the portion of the arch to the right must be scaled on $A\ C$ toward A (above $O'\ D$) and those to the left, below $O'\ D$.

Lay off, to scale, load $P_{11} = 440$ lbs. above $O'\ D$, $P_{10} = 300$ lbs., also above $O'\ D$, etc., $P_1 = 10{,}800$ lbs., and $P\ E$ at the right = 10,530 lbs.

Similarly, lay off to scale, load $P_{12} = 960$ lbs. below $O'\ D$, $P_{13} = 1{,}120$ lbs., etc.; $P_{20} = 12{,}600$ lbs., and $P\ E$ at the left = 11,730 lbs.

(*b*) Assume a trial pole-distance (trial horizontal thrust H_1) equal to about ½ sum total of all the loads, i. e., ½ of 73,546, or in round numbers 35,000 lbs. By using about one-half the value of the total load for trial H_1 the angle at O' becomes nearly a right angle which is desirable. Lay off on line $D\ O'$ a length equal to 35,000 lbs. to the scale of 5,000 lbs. per inch, hence make $D\ O' = 7$ ins. thus determining the position of the trial pole O'.

In the equilibrium polygon the line of the polygon which is drawn between the loads P_{11} and P_{12} must be parallel to the ray of the ray diagram which is drawn to point D where load P_{11} ends and P_{12} begins; therefore the polygon line between P_{11} and P_{12} must be parallel to ray $O'\ D$ or to the trial horizontal line H_1. Hence it follows that this side of the polygon, i. e., between forces P_{11} and P_{12} must be a horizontal line. Draw therefore between P_{11} and P_{12} the line $h\ h$ which we extend to both sides as in Fig. 40. Where $h\ h$ intersects load line P_{12} begin another side of the polygon. Make it parallel to the ray drawn from O' to the point where P_{12} ends and P_{13} begins on the load line $A\ C$ of the ray diagram. Where this last found side of the polygon intersects P_{13} begin another side and make it parallel to the ray which passes to the point of junction of P_{13} and

P_{14} of the ray diagram. Proceed in this way for all the sides of the polygon until $P E$ at the left is reached where polygon side $B' C$ must be parallel to the outside ray $O' C$. Similarly for the right hand side of the polygon.

The equilibrium polygon may be drawn in the reverse order. Beginning at point C of the load line $A C$, let the outside ray $C O'$ itself be coincident with the side of the polygon thus obtaining $C B'$. From B' draw $B' F$ parallel to ray $B O'$ etc., continuing until the load line P_{12} is reached when the side of the polygon between P_{12} and P_{11} is drawn horizontally as $h h$.

(2) Mathematical method.—In Figs. 26 and 27, page 33, we have an exaggerated case of the equilibrium polygon of Fig. 40. Line $h_{11} h_{15}$ is a portion of the horizontal line $h h$ of Fig. 40, and it is parallel to $O' D$ of Fig. 26.

Also

$h_{12} r_{13}$ is parallel to ray $O' K$
$r_{13} r_{14}$ is parallel to ray $O' M$
$r_{14} r_{15}$ is parallel to ray $O' N$

Triangle $h_{12} h_{13} r_{13}$ is similar to triangle $O' D K$
Triangle $r_{13} z_{14} r_{14}$ is similar to triangle $O' D M$
Triangle $r_{14} z_{15} r_{15}$ is similar to triangle $O' D N$
Hence since $O' D = H_1 = 35{,}000$

$$D K = P_{12} = 960$$
$$D M = D K + K M = P_{12} + P_{13} = 960 + 1120 = 2080$$
$$D N = D K + K M + M N = P_{12} + P_{13} + P_{14} = 960 + 1120 + 1215 = 3295$$

we obtain the following relations:

$$P_{12} : H_1 = h_{13} r_{13} : h_{13} h_{12} ;$$
$$h_{13} r_{13} = \frac{P_{12} (h_{13} h_{12})}{H_1}$$
$$(P_{12} + P_{13}) : H_1 = z_{14} r_{14} : h_{14} h_{13};$$
$$z_{14} r_{14} = \frac{(P_{12} + P_{13}) (h_{14} h_{13})}{H_1}$$
$$(P_{12}+P_{13}+P_{14}) . H_1 = z_{15} r_{15} : h_{15} h_{14};$$
$$z_{15} r_{15} = \frac{(P_{12}+P_{13}+P_{14}) (h_{15} h_{14})}{H_1}$$

Let

$$p_{13} = h_{13} r_{13}$$
$$p_{14} = p_{13} + z_{14} r_{14}$$
$$p_{15} = p_{14} + z_{15} r_{15}$$

then

$$p_{13} = \frac{P_{12}\,(h_{13}\,h_{12})}{H_1} = \frac{960\,(1.65)}{35{,}000} = 0.045$$

$$p_{14} = p_{13} + \frac{(P_{12} + P_{13})\,(h_{14}\,h_{13})}{H_1} = 0.045 + \frac{2080(1.75)}{35{,}000} =$$

$$0.045 + 0.104 = 0.149$$

$$p_{15} = p_{14} + \frac{(P_{12} + P_{13} + P_{14})\,(h_{15}\,h_{14})}{H_1} = 0.149 + \frac{3295\,(1.90)}{35{,}000} =$$

$$0.149 + 0.179 = 0.328$$

Therefore for any value in general as p_n, after the first we have the following general equation:

$$p_n = p_{n-1} + \frac{(P_{12} + \ldots + P_{n-1})}{H_1}\,(h_n\,h_{n-1}) \quad \ldots\ldots (88)$$

The work may conveniently be arranged in the form of a table, as shown below, for the above calculated points:

Load line.	$z\,r$	p
13	0.045	0.045
14	0.104	0.149
15	0.179	0.328

Thus p_{14} is obtained by adding to p_{13} the $z\,r$ value for the load line 14; and p_{15} is similarly obtained by adding to p_{14} the $z\,r$ values for load line 15, etc., indefinitely.

Since the lengths $h_{12}\,h_{13}$, $h_{13}\,h_{14}$, etc., are in feet the values p_{13}, p_{14}, etc., are in feet measured to the same scale. An engineer's scale is preferable for this work as it reads in tenths.

From the line $h\,h$ on the proper load line lay off the distances p_{13}, p_{14}, etc., calculated as above.

This mathematical method offers a splendid check on the purely graphical method given above.

(*e*) *Determination of Position of True Closing Line of Equilibrium Polygon.*—

(1) Conditional Equations Pertaining to True Closing Line.

(*a*) $\Sigma\,(M) = 0$ hence $\Sigma\,(v\,m) = \Sigma\,(b\,v)$

(*b*) $\Sigma\,(M\,x) = 0$. Resultant of $\Sigma\,(b\,v)$ must equal and coincide with resultant of $\Sigma\,(v\,m)$

(2) Determination of Magnitude of Resultant R.
(3) Determination of Location of Resultant R.
(4) Location of True Closing Line to Equilibrium Polygon so that the sum of its ordinates to $v_{20}\,v_1 = R$, acting r feet to left of Center Line. (a) Location and Magnitude of Trial T. (b) Location and Magnitude of Trial T_1. (c) Determination of Position of True Closing Line. Method I. Method II.

(1) With the equilibrium polygon drawn by the above methods extend vertical lines from a_{20}, a_{19}, ... a_1 on the neutral axis of the arch until they intersect the polygon in points b_{20}, b_{19}, ... b_1 and the line $v_{20}\,v_1$ (which should be drawn) in points v_{19}, v_{18}, ... v_2.

From Art. 6, we have seen that the moment M_c about any point c of the equilibrium polygon is

$M_c = H\ (k\ c)$ where $k\ c$ is the distance from point c to the closing line $k\ k$. Furthermore by equation (46),

$\Sigma\ (M_c) = 0$, and $\Sigma\ (M_c\,x) = 0$.

$\Sigma\ (M_c) = 0$, corresponds to Condition 1, equation (29). $\Sigma\ (M_c\,x) = 0$, corresponds to Condition 3, equation (31).

Since the expression $M_c = H . k\ c$ is one in which H is constant, it follows that M_c must be proportional to $k\ c$, i. e., to the distance from the points c on the equilibrium polygon to the closing line $k\ k$. If, then, $\Sigma\ (M_c)$ is to equal 0, the sum of the ordinates measured above and below the closing line $k\ k$ must equal 0 for H is constant.

In the equilibrium polygon, Fig. 40, let us assume the location of a trial closing line $n\ n_1$.

If this were the true closing line $m\ m_1$ of the polygon, the algebraic sum of the ordinates of type $b\ m$, i. e., $-v_{20}\,m - b_{19}\,m_1$. ..... $+ b_{17}\,m_{17} + b_{12}\,m_{12}$ ..... $- b_2\,m_2 - v_1\,m_1$ must $= 0$. Ordinates measured above $m\ m_1$ to points b of the polygon being regarded as positive and ordinates measured below $m\ m_1$ being regarded as negative.

This may be expressed mathematically by

$$\Sigma\ (\pm\ b\ m) = \Sigma\ (-\ b\ m)\ +\ \Sigma\ (+\ b\ m) = 0$$

or

$$-\ \Sigma\ (+\ b\ m)\ +\ \Sigma\ (-\ b\ m) = 0 \quad \text{................} (89)$$

The ordinates of the equilibrium polygon from line $v_{20}\,v_1$ to points b are of the type $b\,v$. Beginning, then, with the identity

$$\Sigma\,(b\,v) = \Sigma\,(b\,v)$$

let us add to the left hand member of this identity the value given in equation (89). This will in no way effect the value of the identity for the left hand member of equation (89) = 0.

Hence we obtain

$$\Sigma\,(b\,v) - \Sigma\,(+\,b\,m) + \Sigma\,(-\,b\,m) = \Sigma\,(b\,v).$$

But the left hand member in the last result constitutes the quadrilateral $v_{20}\,m\,m_1\,v_1$, for we have subtracted from the $b\,v$ ordinates of the equilibrium polygon the positive values of $b\,m$ (those above the line $m\,m_1$) and we have added the negative values of $b\,m$ (those below the line $m\,m_1$). Therefore

$$\Sigma\,(v\,m) = \Sigma\,(b\,v) \qquad (90)$$

This makes $\Sigma\,(M_c) = 0$ and Condition 1, equation (29) is satisfied.

In other words the ordinates of the quadrilateral $v_{20}\,m\,m_1\,v_1$ = the ordinates of the equilibrium polygon $v_{20}\,b\,v_1$.

Equation (90) may be written

$$\Sigma\,(b\,v) - \Sigma\,(v\,m) = 0.$$

But since $\Sigma\,(\pm\,b\,m) = 0$ this can be given the form

$$\Sigma\,(b\,v) - \Sigma\,(v\,m) = \Sigma\,(\pm\,b\,m) \qquad (91)$$

That is, if we subtract from the ordinates $b\,v$ of the equilibrium polygon the ordinates $v\,m$ of the quadrilateral we have left the ordinates of the type $\pm\,b\,m$. This follows the fact that

$$\Sigma\,(b\,v) - \Sigma\,(+\,b\,m) + \Sigma\,(-\,b\,m) = \Sigma\,(v\,m)$$

Subtracting each of these members from the corresponding members of the identity $\Sigma\,(b\,v) = \Sigma\,(b\,v)$ we obtain

$$\Sigma\,(b\,v) - [\Sigma\,(b\,v) - \Sigma\,(+\,b\,m) + \Sigma\,(-\,b\,m)] = \Sigma\,(b\,v) - \Sigma\,(v\,m)$$

Reducing this into the form

$$+\,\Sigma\,(+\,b\,m) - \Sigma\,(-\,b\,m) = \Sigma\,(b\,v) - \Sigma\,(v\,m)$$

it is evident at once that

$$\Sigma\,(\pm\,b\,m) = \Sigma\,(b\,v) - \Sigma\,(v\,m) = 0$$

which is equation (91) above.

Now since $\Sigma\,(M_c\,x)$ must = 0, if we introduce the value x throughout every term of the preceding expression we still have an equation, hence

$$\Sigma\ (b\,m\ .\ x) = \Sigma\ (b\,v\ .\ x) - \Sigma\ (v\,m\ .\ x) = 0$$

That is,

$$\Sigma\ (b\,v\ .\ x) = \Sigma\ (v\,m\ .\ x) \qquad (92)$$

If the ordinates $b\,v$ and $v\,m$ are regarded as forces and the lengths x as the lever arms measured from the abutment of the arch then the sums of their moments must be equal.

$\Sigma\ (b\,v)$ must $=$ the resultant of $\Sigma\ (v\,m)$ for

$$\Sigma\ (b\,v) = \Sigma\ (v\,m).$$

It must therefore follow that the

Resultant of $\Sigma\ (b\,v) =$ *Resultant of* $\Sigma\ (v\,m)$ ..... (93)

This is evident since $\Sigma\ (b\,v) = \Sigma\ (v\,m)$.

Furthermore the resultant of $\Sigma\ (b\,v)$ must coincide in position with the resultant of $\Sigma\ (v\,m)$ because the ordinates regarded as forces are located along the same verticals and consequently must have the same length of lever arms about the abutment for any particular vertical selected. $\Sigma\ (M_c\,x)$ will then $= 0$ and Condition 3, equation (31), will be satisfied.

We may state the above as follows:

(*a*) To satisfy Condition (1) that $\Sigma\ (M_c) = 0$, it is necessary that $\Sigma\ (v\,m) = \Sigma\ (b\,v)$.

(*b*) To satisfy Condition (3) that $\Sigma\ (M\,x) = 0$, the resultant of $\Sigma\ (b\,v)$ must coincide with the resultant of $\Sigma\ (v\,m)$.

(2) Determination of Magnitude of Resultant R. Let R be the resultant of the $b\,v$ ordinates regarded as forces, then R must equal their sum, hence

$$R = \Sigma\ (b\,v) = \Sigma\ (v\,m) \qquad (94)$$

The sum of the $b\,v$ ordinates may be found by scaling from the diagram, using the scale of distance adopted for the drawing. It is conveniently found also by marking off the $b\,v$ lengths in succession on a long strip of paper and scaling the total lengths for them all.

In Table IV, columns 3 and 4, the values of the ordinates $v\,b$ are given and their sum $\Sigma\ (b\,v) = R = 137.10$.

The resultant $R = 137.10$ must act to the left of the center line for the live load is distributed over the left half, hence making the loading for the left side greater than for the right side. The reverse will be true if the live load acts on the right half of the arch.

TABLE IV.—CALCULATIONS FOR LOCATION OF RESULTANT R.

Points on Neutral Line.	1	2	3	4	Equilibrium Polygon.	
	y Ordinate to Neutral Axis of Arch.	y^2	$v\ b$ Ordinates From Line $v_{20}\ v_1$ to Equilibrium Polygon.	$v\ b$ Ordinates From Line $v_1\ v_{20}$ to Equilibrium Polygon.	z Lever Arm About Center Line $A\ B$.	Moments About Center Line $A\ B$.
a_1	$y_1 = 2.90$	8.41	$v_{20}b_{20} = 0.00$	$v_1\ b_1 = 0.00$	$z_1 = 29.05$	$(v_{20}b_{20} - v_1\ b_1)\ z_1 = 0.00$
a_2	$y_2 = 6.15$	37.82	$v_{19}b_{19} = 5.30$	$v_2\ b_2 = 4.70$	$z_2 = 20.05$	$(v_{19}b_{19} - v_2\ b_2)\ z_2 = 12.03$
a_3	$y_3 = 7.25$	52.56	$v_{18}b_{18} = 6.85$	$v_3\ b_3 = 6.15$	$z_3 = 15.70$	$(v_{18}b_{18} - v_3\ b_3)\ z_3 = 10.99$
a_4	$y_4 = 7.90$	62.41	$v_{17}b_{17} = 7.60$	$v_4\ b_4 = 6.95$	$z_4 = 12.55$	$(v_{17}b_{17} - v_4\ b_4)\ z_4 = 8.16$
a_5	$y_5 = 8.30$	68.89	$v_{16}b_{16} = 8.10$	$v_5\ b_5 = 7.50$	$z_5 = 9.75$	$(v_{16}b_{16} - v_5\ b_5)\ z_5 = 5.85$
a_6	$y_6 = 8.60$	73.96	$v_{15}b_{15} = 8.40$	$v_6\ b_6 = 7.85$	$z_6 = 7.75$	$(v_{15}b_{15} - v_6\ b_6)\ z_6 = 4.26$
a_7	$y_7 = 8.75$	76.56	$v_{14}b_{14} = 8.50$	$v_7\ b_7 = 8.10$	$z_7 = 5.80$	$(v_{14}b_{14} - v_7\ b_7)\ z_7 = 2.32$
a_8	$y_8 = 8.80$	77.44	$v_{13}b_{13} = 8.60$	$v_8\ b_8 = 8.30$	$z_8 = 4.00$	$(v_{13}b_{13} - v_8\ b_8)\ z_8 = 1.20$
a_9	$y_9 = 8.95$	80.10	$v_{12}b_{12} = 8.60$	$v_9\ b_9 = 8.45$	$z_9 = 2.25$	$(v_{12}b_{12} - v_9\ b_9)\ z_9 = 0.34$
a_{10}	$y_{10} = 9.00$	81.00	$v_{11}b_{11} = 8.60$	$v_{10}b_{10} = 8.55$	$z_{10} = 0.70$	$(v_{11}b_{11} - v_{10}b_{10})\ z_{10} = 0.03$
	76.60	619.15	70.55	66.55		Sum = 45.18
	2	2				
	153.20	$1238.30 = \Sigma(y^2)$	$R = \Sigma(v \cdot b) = 70.55 + 66.55$		$r = \frac{\text{Sum}}{R} = \frac{45.18}{137.10} = 0.329$ feet	
	$\Sigma(y) = 153.20$		$R = 137.10$		that R acts to the left of center line $A\ B$	

(3) Determination of Location of Resultant R.—Let us now take moments about the center line of the ordinates $b\ v$ regarded as forces. The lever arm for $v_{20}\ b_{20}$ will equal the lever arm for $v_1\ b_1$; let it be designated by z_1. The moments will have opposite signs being on opposite sides of the center line. We obtain for these two forces:

$$(v_{20}\ b_{20})\ z_1 - (v_1\ b_1)\ z_1 = 0$$

since $v_{20}\ b_{20} = 0 = v_1\ b_1$.

Similarly for the symmetrically located pair, $v_{19} b_{19}$, $v_2\ b_2$ having lever arms each equal to z_2 we obtain,

$$(v_{19}\ b_{19})\ z_2 - (v_2\ b_2)\ z_2 = 12.03.$$

In the same manner the other terms may be obtained, as shown in detail in Table IV.

Taking the sum of all these terms we obtain the sum total of all the moments of all the $v\,b$ ordinates about the center line. Let this sum total of moments $v\,b$ be designated as $\Sigma\ (v\,b\ .\ z)$ then

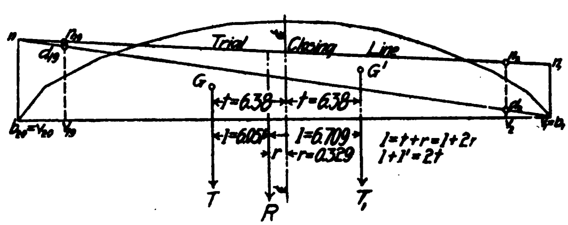


Fig. 36.

$$\Sigma\ (v\,b\ .\ z) = R\ .\ r \quad \ldots\ldots\ldots\ldots (95)$$

and

$$r = \frac{\Sigma\ (v\,b\ .\ z)}{R} \quad \ldots\ldots\ldots\ldots (96)$$

where r is the distance from the center line that the resultant R acts.

But since $\Sigma\ (v\,b\ .\ z) = 45.18$ (by Table IV),

hence

$$r = \frac{\Sigma\ (v\,b\ .\ z)}{R} = \frac{45.18}{137.10} = 0.329.$$

Therefore the resultant R acts 0.329 ft. to the left of the center line. See Fig. 36.

(4) Location of True Closing Line to Equilibrium Polygon so that Sum of Its Ordinates to $v_{20}\ v_1 = R$, acting r feet to Left of Center Line.

(*a*) Location and Magnitude of *Trial T*.

(*b*) Location and Magnitude of *Trial* T_1.

(*c*) Determination of Position of True Closing Line. Method I. Method II.

(*a*) Having drawn the trial closing line $n\, n_1$ for the equilibrium polygon draw the straight line $n\, v_1$ dividing the quadrilateral $v_{20}\, n\, n_1\, v_1$ into the two triangles $v_{20}\, n\, v_1$ and $n\, v_1 n_1$.

The lengths $v_{20}\, n$ and $v_1\, n_1$ are arbitrarily chosen.

Let,

Resultant of ordinates of triangle $v_{20}\, n\, v_1$ = Trial T.

Resultant of ordinates of triangle $n\, v_1\, n_1$ = Trial T_1.

Resultant of ordinates of quadrilateral $v_{20}\, n\, n_1\, v_1 = R_t$

It is evident from Fig. 40 and also from Fig. 36, that

Sum of ordinates quadrilateral $v_{20}\, n\, n_1\, v_1$ = sum of ordinates of triangle $v_{20}\, n\, v_1$ + sum of ordinates of triangle $n\, v_1\, n_1$.

Hence

$$R_t = Trial\ T + Trial\ T_1 \qquad (97)$$

The ordinates of the triangle $v_{20}\, n\, v_1$ are of the type $v\, d$. The resultant of these ordinates treated as forces = Trial T

Hence

$$\Sigma\ (v\, d) = Trial\ T \qquad (98)$$

By Table V,

$$\Sigma\ (v\, d) = 69.55 = Trial\ T.$$

We can locate the position of Trial T by the same process as was used for the location of *R*.

Hence taking moments about the center line of the ordinates $v\, d$ of triangle $v_{20}\, n\, v_1$ and using the same lever arms z as for *R* we obtain,

$$\Sigma\ (v\, d\ .\ z) = 444.14$$

As the sum of the $v\, d$ ordinates of triangle $v_{20}\, n\, v_1$ to the left of the center line is greater than the sum to the right, Trial T must act to the left of the center line.

Let t = distance that Trial T acts to left of center line, then

$$\Sigma\ (v\, d\ .\ z) = Trial\ T\ .\ t \qquad (99)$$

hence

$$t = \frac{\Sigma\ (v\, d\ .\ z)}{Trial\ T} \qquad (100)$$

or

$$t = \frac{444.14}{69.55} = 6.38 \quad \text{See Table V.}$$

TABLE V.—CALCULATIONS FOR LOCATION OF TRIAL T AND TRIAL T_1.

vd	vd	Triangle $v_{20}\,n\,v_1$.		Triangle $v_1\,n\,n_1$.	
Ordinates of Type vd of Triangle $v_{20}\,n\,v_1$.	Ordinates of Type vd of Triangle $v_{20}\,n\,v_1$.	z Lever Arm About AB.	Moments About Center Line AB.	nd Ordinates of Type nd of Triangle $v_1\,n\,n_1$.	
$v_{20}n = 6.95$	$v_1\,d_1 = 0.00$	$z_1 = 29.05$	$(v_{20}n - v_1\,d_1)\,z_1 = (6.95)\,29.05 = 201.89$	$v_1\,n_1 = 6.15$	$n\,d_{20} = 0.00$
$v_{19}d_{19} = 5.85$	$v_2\,d_2 = 1.10$	$z_2 = 20.05$	$(v_{19}d_{19} - v_2\,d_2)\,z_2 = (4.75)\,20.05 = 95.24$	$n_2\,d_2 = 5.20$	$n_{19}d_{19} = 0.95$
$v_{18}d_{18} = 5.35$	$v_3\,d_3 = 1.55$	$z_3 = 15.70$	$(v_{18}d_{18} - v_3\,d_3)\,z_3 = (3.80)\,15.70 = 59.66$	$n_3\,d_3 = 4.75$	$n_{18}d_{18} = 1.45$
$v_{17}d_{17} = 5.00$	$v_4\,d_4 = 2.00$	$z_4 = 12.55$	$(v_{17}d_{17} - v_4\,d_4)\,z_4 = (3.00)\,12.55 = 37.65$	$n_4\,d_4 = 4.40$	$n_{17}d_{17} = 1.75$
$v_{16}d_{16} = 4.70$	$v_5\,d_5 = 2.35$	$z_5 = 9.75$	$(v_{16}d_{16} - v_5\,d_5)\,z_5 = (2.35)\,9.75 = 22.91$	$n_5\,d_5 = 4.15$	$n_{16}d_{16} = 2.05$
$v_{15}d_{15} = 4.40$	$v_6\,d_6 = 2.55$	$z_6 = 7.75$	$(v_{15}d_{15} - v_6\,d_6)\,z_6 = (1.85)\,7.75 = 14.34$	$n_6\,d_6 = 3.90$	$n_{15}d_{15} = 2.25$
$v_{14}d_{14} = 4.10$	$v_7\,d_7 = 2.80$	$z_7 = 5.80$	$(v_{14}d_{14} - v_7\,d_7)\,z_7 = (1.30)\,5.80 = 7.54$	$n_7\,d_7 = 3.75$	$n_{14}d_{14} = 2.50$
$v_{13}d_{13} = 3.90$	$v_8\,d_8 = 3.00$	$z_8 = 4.00$	$(v_{13}d_{13} - v_8\,d_8)\,z_8 = (.90)\,4.00 = 3.60$	$n_8\,d_8 = 3.50$	$n_{13}d_{13} = 2.70$
$v_{12}d_{12} = 3.75$	$v_9\,d_9 = 3.20$	$z_9 = 2.25$	$(v_{12}d_{12} - v_9\,d_9)\,z_9 = (.55)\,2.25 = 1.24$	$n_9\,d_9 = 3.30$	$n_{12}d_{12} = 2.85$
$v_{11}d_{11} = 3.55$	$v_{10}d_{10} = 3.45$	$z_{10} = 0.70$	$(v_{11}d_{11} - v_{10}d_{10})\,z_{10} = (.10)\,0.70 = 0.07$	$n_{10}d_{10} = 3.10$	$n_{11}d_{11} = 3.00$
$\Sigma\,(vd) = 69.55$ = Trial T		Sum = 444.14 $= \frac{\text{Sum}}{\text{Trial } T} = \frac{444.14}{69.55} = 6.38$ **Trial T acts 6.38 ft. to the left of center line $A\,B$.** $t = r + l$; $6.38 = 0.329 + l$ $l + l' = 2t$; $l = 6.051$; $l' = 6.709$		$\Sigma(nd) = 61.70$ = **Trial T_1** **Trial T_1 acts 6.38 ft. to right of center line $A\,B$.**	

(*b*) The location and magnitude of Trial T_1 may be obtained in a similar manner.

The ordinates of triangle $n\ v_1\ n_1$ are of the type $n\ d$.

As above,

$$\Sigma\ (n\ d\ .\ z) = Trial\ T_1\ .\ t' \qquad (101)$$

and

$$t' = \frac{\Sigma\ (n\ d\ .\ z)}{Trial\ T_1} \qquad (102)$$

where t' is the distance that T_1 acts from the center line.

We now wish to prove that

$$t' = t.$$

Suppose in the triangle $n\ v_1\ n_1$ we move the point n_1 to any other point m_1 along the vertical line $v_1\ n_1$.

We shall then prove:

1st. *That the changing of the position of n_1 along the vertical through n_1 in the triangle $n\ v_1\ n_1$ does not change the distance t' that the resultant acts from the center line; that is, that t' is a constant.*

Let us now consider the triangle $n\ v_1\ n_1$ and the imaginary triangle $n\ v_1\ m_1$ whose side $n\ m_1$ is not shown, having the common vertex n. The ordinates of the triangle $n\ v_1\ n_1$ are of the type $n\ d$, while the ordinates of the triangle $n\ v_1\ m_1$ are of the type $m\ d$.

By equation (102)

$$\frac{\Sigma\ (n\ d\ .\ z)}{Trial\ T_1} = t'$$

In triangle $n\ v_1\ m_1$ a similar relation holds;

$$\frac{\Sigma\ (m\ d\ .\ z)}{Trial\ T_m} = t_m \qquad (103)$$

where Trial T_m = the resultant of the forces of triangle $n\ v_1\ m_1$ and t_m is the distance that it acts from the center line.

It is evident from the triangle that,

$$\frac{Ordinates\ of\ Type\ m\ d}{Ordinates\ of\ Type\ n\ d} = a\ constant = S \qquad (104)$$

Hence any $m\ d = S$ times the corresponding $n\ d$

$$m\ d = S\ (n\ d) \qquad (105)$$

Substituting this last expression in equation (103);

$$\frac{\Sigma\ (m\ d\ .\ z)}{\Sigma\ (m\ d)} = \frac{\Sigma\ S\ (n\ d\ .\ z)}{\Sigma\ S\ (n\ d)} = \frac{S\ \Sigma\ (n\ d\ .\ z)}{S\ \Sigma\ (n\ d)}$$

for the constant S can be placed outside the summation sign, and eliminating S, we have

$$\frac{\Sigma\,(m\,d\,.\,z)}{\Sigma\,(m\,d)} = \frac{\Sigma\,(n\,d\,.\,z)}{\Sigma\,(n\,d)} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (106)$$

from which it follows at once that

$$t' = t_m \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (107)$$

Hence for any position of n_1 as at m_1 the value t' is a constant.

2nd. It remains to show that $t' = t$ where t is the distance that the resultant of triangle $v_{20}\,n\,v_1$ acts from the center line.

Suppose that n_1 is moved to a point N making $v_1\,N = v_{20}\,n$. This will make triangle $n\,v_1\,N$ equal to triangle $v_{20}\,n\,v_1$. The corresponding ordinates become equal; and the sum of the ordinates of the two triangles consequently become equal.

Therefore the resultant Trial T of $v_{20}\,n\,v_1$ is equal to the resultant Trial T_n of $n\,v_1\,N$. Hence

$$\Sigma\,(v\,d\,.\,z) = \Sigma\,(N\,d\,.\,z) = Trial\ T\,.\,t$$

and $\Sigma\,(N\,d)$ acts t units from the center line.

But in the 1st proof we have shown that no matter where N (or n_1) may be located on $n_1\,v_1$ the resultant acts at a constant distance t' from the center line.

By the 2d proof we have shown that in the particular triangle $n\,v_1\,N$ the resultant acts t units from the center line. Hence t must be the value of the constant distance for any and all triangles whatever like $n\,v_1\,n_1$, $n\,v_1\,N$ and $n\,v_1\,m_1$ having a vertex at n. Therefore,

$$t' = t_m = t \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (108)$$

By a similar line of reasoning for triangle $v_{20}\,n\,v_1$ if n is moved to any position whatever, as m, along line $v_{20}\,n$ then although the value of the resultant Trial T is changed in magnitude yet its location at a distance t from the center line remains unchanged and t is a constant.

If we suppose, therefore, that the true position of n is at m and the true position of n_1 is at m_1 we do not in any way change the location of the new resultants, True T and True T_1, thus produced, as compared with the position of the old resultants Trial T and Trial T_1.

Therefore if $m\,m_1$ is the true closing line of the equilibrium polygon the resultant True T of the triangle $v_{20}\,m\,v_1$ as well as the resultant True T_1 of the triangle $m\,v_1\,m_1$ must each act t

units from the center line. Also the resultant R of the quadrilateral $v_{20}\, m\, m_1\, v_1$ is given by the expression

$$R = True\ T + True\ T_1.$$

In our present problem

$$t = 6.38$$

(*c*) *Method I.* (Determination of position of True Closing Line.)

In Fig. 37, let point n of triangle $v_{20}\, n\, v_1$ be a point on the trial closing line and let point m of triangle $v_{20}\, m\, v_1$ be on the true closing line, then

$$Trial\ T = \Sigma\ (ordinates\ triangle\ v_{20}\, n\, v_1)$$
$$True\ T = \Sigma\ (ordinates\ triangle\ v_{20}\, m\, v_1)$$

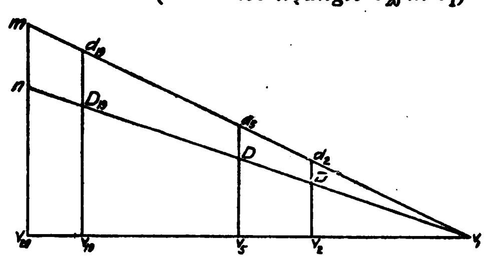


Fig. 37.

It is clear that

$$\frac{v_{20}\, m}{v_{20}\, n} = \frac{v_{20}\, d_{20}}{v_{20}\, D_{20}} = \frac{v_{19}\, d_{19}}{v_{19}\, D_{19}} = \frac{v_{18}\, d_{18}}{v_{18}\, D_{18}} = \dots = \frac{v_3\, d_3}{v_3\, D_3} = \frac{v_2\, d_2}{v_2\, D_2} = \frac{v_1\, d_1}{v_1\, D_1}$$

By a theorem in proportion it follows that

$$\frac{v_{20}\, m}{v_{20}\, n} = \frac{v_{20}\, d_{20} + v_{19}\, d_{19} + v_{18}\, d_{18} + \dots + v_3\, d_3 + v_2\, d_2 + v_1\, d_1}{v_{20}\, D_{20} + v_{19}\, D_{19} + v_{18}\, D_{18} + \dots + v_3\, D_3 + v_2\, D_2 + v_1\, D_1} =$$

$$\frac{\Sigma\ (ordinates\ triangle\ v_{20}\ m\ v_1)}{\Sigma\ (ordinates\ triangle\ v_{20}\ n\ v_1)} \quad \dots\dots\dots\dots \quad (109)$$

hence

$$\frac{v_{20}\, m}{v_{20}\, n} = \frac{True\ T}{Trial\ T} \quad \dots\dots\dots\dots \quad (110)$$

therefore

$$v_{20}\, m = \frac{True\ T}{Trial\ T} \cdot v_{20}\, n \quad \dots\dots\dots\dots \quad (111)$$

If, in a like manner, m_1 of triangle $n\, v_1\, m_1$ is a point on the true closing line $m\, m_1$ (Fig. 40) then

$$v_1\, m_1 = \frac{True\ T_1}{Trial\ T_1} v_1\, n_1 \quad \dots\dots\dots\dots \quad (112)$$

where True T_1 is the sum of the ordinates of triangle $n\, v_1\, m_1$, that is, their resultant.

We must now determine the value of True T and True T_1

In Fig. 36 True T will act along the same ordinate that Trial T does, and True T_1 will act along the same ordinate that Trial T_1 does. This has been shown above.

It should be recalled that

$$R = True\ T + True\ T_1 = E \text{ (the equilibrant)} \quad \ldots (113)$$

If we take moments about the ordinate of Trial T_1, and remembering that $l + l' = 2t$, we obtain

$$True\ T\ (l + l') = True\ T\ (2t) = Rl'$$

hence

$$True\ T = \frac{R\,l'}{2t} \quad \ldots\ldots (114)$$

In the present problem

$$R = 137.1;\ l' = 6.709 \text{ and } t = 6.38$$

hence

$$True\ T = \frac{137.1\ (6.709)}{2\ (6.38)} = 72.10$$

If we now take moments about the ordinate of Trial T we obtain;

$$True\ T_1\ (l + l') = True\ T_1\ (2t) = R\,l$$

hence

$$True\ T_1 = \frac{R\,l}{2\ t} \quad \ldots\ldots (115)$$

Since $l = 6.051$

$$True\ T_1 = \frac{137.1\ (6.051)}{2\ (6.38)} = 65.00$$

By equation (111)

$$v_{20}\,m = \frac{True\ T}{Trial\ T} \cdot v_{20}\,n, \text{ since}$$

$$True\ T = 72.10$$

$$Trial\ T = 69.55 \text{ (By Table V.)}$$

$$v_{20}\,n = 6.95$$

hence

$$v_{20}\,m = \frac{72.10}{69.55}\,(6.95) = 7.2$$

Similarly by equation (112)

$$v_1\,m_1 = \frac{True\ T_1}{Trial\ T_1}\,v_1\,n_1$$

hence since

$$True\ T_1 = 65.00$$

$$Trial\ T_1 = 61.70 \text{ (Table V.)}$$

$$v_1\,n_1 = 6.15$$

Therefore,

$$v_1\, m_1 = \frac{65.00}{61.70}\ (6.15) = 6.47$$

Knowing $v_{20}\, m$ and $v_1\, m_1$ we can scale the lengths and determine the position of the true closing line $m\, m_1$

Method II. (Determination of Position of True Closing Line).

If instead of drawing the trial closing line $n\, n_1$ in any position whatever, suppose we locate it so that $v_{20}\, n = v_1\, n_1 = \frac{R}{N}$. Then the quadrilateral $v_{20}\, n\, n_1\, v_1$ becomes a parallelogram, and the triangles $v_{20}\, n\, v_1$ and $n\, v_1\, n_1$ become equal.

If

R equals resultant of the ordinates of parallelogram $v_{20}\, n\, n_1\, v_1$

N equals number of ordinates of parallelogram $v_{20}\, n\, n_1\, v_1$, then

$$\frac{R}{N} = v_{20}\, n = v_1\, n_1 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (116)$$

Hence since $R = 137.1$

$N = 20$

$$\frac{R}{N} = \frac{137.1}{20} = v_{20}\, n = v_1\, n_1 = 6.85$$

Trial T = sum of ordinates triangle $v_{20}\, n\, v_1$

$= \frac{1}{2}$ sum of ordinates parallelogram $v_{20}\, n\, n_1\, v_1$.

Hence

$$Trial\ T = \tfrac{1}{2} R \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (117)$$

Since $R = 137.1$

$$Trial\ T = \frac{137.1}{2} = 68.55$$

Similarly

$$Trial\ T_1 = \tfrac{1}{2} R \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (118)$$

$$Trial\ T_1 = 68.55$$

Since,

$$True\ T = \frac{R\,l'}{2\,t} = 72.10 \text{ and } True\ T_1 = \frac{R\,l}{2\,t} = 65.00$$

and

$$v_1\, m_1 = \frac{True\ T_1}{Trial\ T_1} \cdot v_1\, n_1$$

hence

$$v_1\, m_1 = \frac{65.00}{68.55}\ (6.85) = 6.4$$

Also

$$v_{20}\, m = \frac{True\ T}{Trial\ T}\, v_{20}\, n$$

therefore

$$v_{20}\, m = \frac{72.10}{68.55}\,(6.85) = 7.2$$

In order to avoid the computation of True T and Tiue T_1 we can proceed as follows:

From equation (97), in the quadrilateral $v_{20}\, n\ n_1\ v_1$ we have,

$$R_t = Trial\ T + Trial\ T_1$$

$$\text{But } R_t = R \text{ if } v_{20}\, n = v_1\, n_1 = \frac{R}{N}$$

Also by equations (117) and (118)

$$Trial\ T = Trial\ T_1 = \tfrac{1}{2}\, R$$

hence

$R = 2$ (Trial T) for the parallelogram.
$R = 2$ (Trial T_1) for the parallelogram.

Now, from the above

$$True\ T\ (l + l') = R\, l' \qquad (119)$$

$$True\ T_1\, (l + l') = R\, l \qquad (120)$$

Substituting $R = 2$ (Trial T) in equation (119) and
$R = 2$ (Trial T_1) in equation (120) we obtain

$$True\ T\ (l + l') = 2\, l'\ (Trial\ T)$$

or

$$\frac{True\ T}{Trial\ T} = \frac{2\, l'}{l + l'} \qquad (121)$$

Similarly

$$True\ T_1\, (l + l') = 2\, l\ (Trial\ T_1)$$

and

$$\frac{True\ T_1}{Trial\ T_1} = \frac{2\, l}{l + l'} \qquad (122)$$

But

$$v_{20}\, m = \frac{True\ T}{Trial\ T} \cdot v_{20}\, n$$

$$v_{20}\, n = v_1\, n_1 = \frac{R}{N}$$

If in this expression we substitute the value of $\frac{True\ T}{Trial\ T}$ of equation (121) we get

$$v_{20}\, m = \frac{2\, l'\ .\ R}{(l + l')\, .\ N} \qquad (123)$$

Also since

$$v_1 m_1 = \frac{True\ T_1}{Trial\,T_1} \cdot v_1 n_1$$

hence

$$v_1 m_1 = \frac{2\,l}{l+l'} \cdot \frac{R}{N} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (124)$$

From Fig. 36 and from Tables

$l = 6.05$; $l' = 6.709$; $l + l' = 12.759$

Substituting these values in equations (123) and (124) we obtain

$$v_{20}\, m = \frac{2\,(6.709)}{12.759} \cdot \frac{137.1}{20} = 7.2$$

$$v_1 m_1 = \frac{2\,(6.05)}{12.759} \cdot \frac{137.1}{20} = 6.4$$

In the above we have shown how the true closing line may be located so that the resultant of the quadrilateral $v_{20}\, m\, m_1\, v_1$ will exactly equal the resultant R of the equilibrium polygon $v_{20}\, b_{19}\, b_2\, v_1$ in magnitude and be coincident in location.

The conditional equation $\Sigma\,(v\,m) = \Sigma\,(b\,v)$ referred to at the beginning of the discussion, it is now seen, holds good if point n is located at m and if n_1 is located at m_1; and the condition is satisfied by the expression

$$\Sigma\,(v\,m) = \Sigma\,(b\,v) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (125)$$

Let the coordinates of points a_{20}, a_{19}, ... a_2 and a_1 on the neutral axis of the arch, be x_{20}, y_{20}; x_{19}, y_{19}, ... x_2, y_2 and x_1, y_1, respectively if point O, at the left springing line of the neutral axis of the arch, (Fig. 40) be taken as the origin of coordinates.

The ordinates of the equilibrium polygon $v_{20}\, b_{19}\, b_2\, v_1$ and of the quadrilateral $v_{20}\, m\, m_1\, v_1$ will both have the same abscissas as the respective points a of the arch. Hence if

$\Sigma\,(v\,m) = \Sigma\,(b\,v)$ holds, then

$$\Sigma\,(v\,m\,.\,x) = \Sigma\,(b\,v\,.\,x) \quad \ldots\ldots\ldots\ldots\ldots\ldots (126)$$

It follows that

$$\Sigma\,(v\,m) - \Sigma\,(b\,v) = 0, \text{ hence } \Sigma\,(b\,v - v\,m) = 0$$

But $b\,v - v\,m = m\,b$, therefore

$$\Sigma\,(m\,b) = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (127)$$

Condition (1), equation (29), i. e., $\Sigma\,(M) = 0$ is thus satisfied.

We see by Table VI, column (3) that this condition is satisfied.

TABLE VI.—DETERMINATION OF Σ ($m\,b.y$) OF EQUILIBRIUM POLYGON.

1	2	3	4	5
mb Ordinates of type mb of equilibrium polygon.	mb Ordinates of type mb of equilibrium polygon.	Algebraic sum of Col. 1 + Col. 2 ($m_1 b_1 + m_{20} b_{20}$) etc.	y Length of ordinates type y to neutral axis of arch.	($m\,b.y$) ($m_1 b_1 + m_{20} b_{20}$) y, etc. Col. 3 x Col. 4.
$m_1 b_1 = -6.45$	$m_{20} b_{20} = -7.20$	− 13.65	2.9	− 39.585
$m_2 b_2 = -1.80$	$m_{19} b_{19} = -1.85$	− 3.65	6.15	− 22.44
$m_3 b_3 = -0.45$	$m_{18} b_{18} = -0.25$	− 0.70	7.25	− 5.075
$m_4 b_4 = +0.30$	$m_{17} b_{17} = +0.55$	+ 0.85	7.9	+ 6.715
$m_5 b_5 = +0.80$	$m_{16} b_{16} = +1.05$	+ 1.85	8.3	+ 15.355
$m_6 b_6 = +1.10$	$m_{15} b_{15} = +1.35$	+ 2.45	8.6	+ 21.07
$m_7 b_7 = +1.35$	$m_{14} b_{14} = +1.55$	+ 2.90	8.75	+ 25.37
$m_8 b_8 = +1.50$	$m_{13} b_{13} = +1.70$	+ 3.20	8.8	+ 28.16
$m_9 b_9 = +1.65$	$m_{12} b_{12} = +1.70$	+ 3.35	8.95	+ 29.98
$m_{10} b_{10} = +1.70$	$m_{11} b_{11} = +1.70$	+ 3.40	9.00	+ 30.60
Ordinates mb when above true closing line $m\,m_1$ are regarded as + and when below as −.		+ 18.00 − 18.00 0.00		+ 90.15 = sum = Σ ($m\,b.y$)

Similarly

$$\Sigma\ (b\,v\ .\ x) - \Sigma\ (v\,m\ .\ x) = 0, \text{ hence}$$
$$\Sigma\ (b\,v - v\,m)\ x = 0, \text{ therefore}$$
$$\Sigma\ (m\,b\ .\ x) = 0$$

Condition (3), equation (31), i. e., $\Sigma\ (M\ x) = 0$ is now also satisfied.

(f) Determination of Position of True Closing Line of Neutral Axis Regarded as an Equilibrium Polygon.—(1) If we regard the neutral axis $a_{20}\ a_{10}\ a_1$ of the arch as an equilibrium polygon we must find a closing line $k\,k$ so located that the conditions which held good for the equilibrium polygon $v_{20}\ b_{10}\ v_1$ will also hold good for the neutral axis. Hence the following conditions must be satisfied:

(*a*) $\Sigma\ (M) = 0$

(*b*) $\Sigma\ (M\ x) = 0$

Let the ordinates of the neutral axis, $y_{20}\ .\ .\ .\ .\ .\ y_{10}\ .\ .\ .\ .\ .\ y_1$ from $a_{20}\ .\ .\ .\ .\ .\ a_{10}\ .\ .\ .\ .\ .\ a_1$ to horizontal line OO_1 be designated as of the type y.

Let us, in a way analogous to our procedure with the equilibrium polygon, regard the ordinates y as forces.

The resultant of the ordinates y regarded as forces must coincide with the center line, for these ordinates are symmetrically disposed, and respectively equal, at equal distances from the center line.

The coordinates of the points a about point O as an origin will be, as above, designated as x and y.

Suppose that the closing line $k\,k$ is not parallel to the horizontal line $O\,O_1$. Then the force ordinates y on one side of the crown center line will be greater than on the other side and the resultant will be located on the same side of the center line with the greater force ordinates. Since for $\Sigma\ (M\ x) = 0$ the resultant must pass through the center line, for symmetrically located force ordinates y are equal, hence $k\,k$ must be parallel to $O\,O_1$ and therefore also a horizontal line and figure $O\,k\,k\,O_1$ becomes a rectangle. This becomes clearly evident if we take moments about the center line itself.

For $\Sigma\ (M) = 0$ the ordinates of rectangle $O\,k\,k\,O_1$ must equal the ordinates of neutral axis segment $O\,a\,O_1$.

Hence if the ordinates from line $O\,O_1$ to $k\,k$ be designated as $k\,O$ and from line $O\,O_1$ to the neutral axis as y we must have the following:

$$\Sigma\,(k\,O) = \Sigma\,(y), \text{ which gives}$$
$$\Sigma\,(y) - \Sigma\,(k\,O) = 0 \text{ and}$$
$$\Sigma\,(y - k\,O) = \Sigma\,(k\,a) = 0 \quad \ldots\ldots\ldots\ldots \quad (128)$$

Similarly

$$\Sigma\,(k\,O\,.\,x) = \Sigma\,(y\,.\,x) \text{ and}$$
$$\Sigma\,(y - k\,O)\,x = \Sigma\,(k\,a\,.\,x) = 0 \quad \ldots\ldots\ldots\ldots \quad (129)$$

In order that equations (128) and (129) shall hold good the line $k\,k$ must be placed at such a distance above $O\,O_1$ that the ordinates $k\,a$ measured above $k\,k$, then regarded as positive, shall equal the ordinates $k\,a$ measured below $k\,k$, then regarded as negative.

Let N = number of ordinates

e_1 = constant length of each ordinate of rectangle $O\,k\,k\,O_1$

$\Sigma\,y$ = sum of ordinates of neutral axis segment $O\,a\,O_1$

Then by the preceding,

$$Ne_1 = \Sigma\,(y), \text{ hence}$$
$$e_1 = \frac{\Sigma\,(y)}{N} \quad \ldots\ldots\ldots\ldots \quad (130)$$

The value e_1 measured vertically from O and O_1 determines the location of what may be termed the temperature norm $k\,k$. The function of $k\,k$ in temperature stresses has already been discussed.

From Table IV, we find that $\Sigma\,(y) = 153.20$; since $N = 20$

$$e_1 = \frac{153.20}{20} = 7.66$$

Now since

$$\Sigma\,(e_1) = N\,e_1, \text{ and } \Sigma\,(y) = N\,e_1$$

it follows that

$$\Sigma\,(y) - \Sigma\,(e_1) = 0$$

hence

$$\Sigma\,(y - e_1) = 0 \quad \ldots\ldots\ldots\ldots \quad (131)$$

therefore, since $(y - e_1) = k\,a$

$$\Sigma\,(k\,a) = 0 \quad \ldots\ldots\ldots\ldots \quad (132)$$

Furthermore on account of the symmetrical disposition of the y ordinates,

$$\Sigma\,(k\,a\,.\,x) = 0 \quad \ldots\ldots\ldots\ldots \quad \mathbf{(133)}$$

Hence for the value of $e_1 = \frac{\Sigma\,(y)}{N}$ we fulfill the conditions given above and in equations (67), (68) and (69) of Art. 8, i. e.

$$\Sigma\,(k\,c) = 0$$
$$\Sigma\,(k\,a) = 0$$
$$\Sigma\,(a\,c) = 0$$

This is seen to be true from Table VII.

(g) *Development of Pressure Curve of Arch by Superposition of Equilibrium Polygon on Arch.*—(1) Conditional equation (30), Art. 3, must now be satisfied. $\Sigma\,(M\,y) = 0$. (2) Determination of True Pole-Distance H. (3) Construction of Pressure Curve.

(1) The quotation from Prof. H. T. Eddy's "Researches in Graphical Statics," given at the close of Art. 6, should now be recalled.

We must, in compliance with the procedure there outlined, regard the neutral axis itself as an equilibrium polygon subjected to some system of loading. The closing line $m\,m_1$ of the basic equilibrium polygon must then be made coincident with the closing line $k\,k$ of the neutral axis polygon. Their areas will then partially cover each other, and the real bending moments acting in the arch will be proportional to the ordinates intercepted between these two polygons.

Both polygons, in reference to their own closing lines, satisfy the conditions

$$\Sigma\,(M) = 0 \text{ and } \Sigma\,(M\,x) = 0.$$

There remains, of the three conditions given by equations (29), (30) and (31), but one to be satisfied, i. e., the conditions expressed by equation (30), or

$$\Sigma\,(M\,y) = 0$$

which is expressed by equation (27)

$$\Sigma\,(a\,c\,.\,y) = 0.$$

Since the points along the basic polygon have been represented by the letter b and those along the neutral axis by the letter a, when the basic polygon is superposed on the arch the imaginary intercepted ordinates will be of the type $b\,a$.

Now the bending moments of the sections S of the arch must be proportional to the intercepted ordinates of type $b\,a$, hence

$$\Sigma\,(b\,a\,.\,y) = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (134)$$

TABLE VII.—DETERMINATION OF ORDINATES OF ARCH PRESSURE LINE.

Col. 1.	Col. 2.	Col. 3.	Col. 4.	Col. 5.
$m\,b$ Ordinates of Equilibrium Polygon. $\Sigma\,(mb) = 0$	$\frac{\Sigma(y^2)-e_1\Sigma(y)}{\Sigma(mb.y)}$ $= 0.718$	$k\,c$ Ordinates From Line $k\,k$ to Points c of Pressure Line of Arch. (Col. 1 × Col. 2) $\Sigma\,(k\,c) = 0$	$k\,a$ Ordinates From Line $k\,k$ to Points a on Neutral Axis of Arch. $\Sigma\,(k\,a) = 0$	$a\,c$ Ordinates From Points a to c of Arch. $ac = kc - ka$. (Col. 3—Col. 4) $\Sigma\,(a\,c) = 0$
$m_1\,b_1 = -6.45$	0.718	-4.63	-4.78	$+0.15$
$m_2\,b_2 = -1.80$	"	-1.29	-1.53	$+0.24$
$m_3\,b_3 = -0.45$	"	-0.32	-0.39	$+0.07$
$m_4\,b_4 = +0.30$	"	$+0.22$	$+0.22$	0.00
$m_5\,b_5 = +0.80$	"	$+0.57$	$+0.62$	-0.05
$m_6\,b_6 = +1.10$	"	$+0.79$	$+0.92$	-0.13
$m_7\,b_7 = +1.35$	"	$+0.97$	$+1.07$	-0.10
$m_8\,b_8 = +1.50$	"	$+1.08$	$+1.23$	-0.15
$m_9\,b_9 = +1.65$	"	$+1.18$	$+1.31$	-0.13
$m_{10}b_{10} = +1.70$	"	$+1.22$	$+1.33$	-0.11
$m_{11}b_{11} = +1.70$	"	$+1.22$	$+1.33$	-0.11
$m_{12}b_{12} = +1.70$	"	$+1.22$	$+1.31$	-0.09
$m_{13}b_{13} = +1.70$	"	$+1.22$	$+1.23$	-0.01
$m_{14}b_{14} = +1.55$	"	$+1.11$	$+1.07$	$+0.04$
$m_{15}b_{15} = +1.35$	"	$+0.97$	$+0.92$	$+0.05$
$m_{16}b_{16} = +1.05$	"	$+0.75$	$+0.62$	$+0.13$
$m_{17}b_{17} = +0.55$	"	$+0.39$	$+0.22$	$+0.17$
$m_{18}b_{18} = -0.25$	"	-0.17	-0.39	$+0.22$
$m_{19}b_{19} = -1.85$	"	-1.33	-1.53	$+0.20$
$m_{20}b_{20} = -7.20$	"	-5.17	-4.78	-0.39
$+18.00$		-12.91	$+13.40$	-1.27
-18.00		$+12.91$	-13.40	$+1.27$
0.00		0.00	0.00	0.00

Ordinates above kk are +

Ordinates below kk are −

$$kc = m\,b\,\frac{\Sigma\,(y^2) - e_1\,\Sigma\,(y)}{\Sigma\,(m\,b.y)} = m\,b\,\frac{64.8}{90.15} = m\,b\,(0.718)$$

Ordinates of closing line $k\,k$ of neutral axis of arch

$$= e_1 = \frac{\Sigma(y)}{N} = \frac{153.20}{20} = 7.66$$

Ordinates of basic polygon are of the type $m\,b$.

Ordinates of neutral axis polygon are of the type $k\,a$, hence

$$m\,b - k\,a = b\,a$$

therefore

$$\Sigma\,(m\,b\,.\,y) - \Sigma\,(k\,a\,.\,y) = \Sigma\,(b\,a\,.\,y) = 0 \ldots\ldots\ldots\ldots (135)$$

whence

$$\Sigma\,(m\,b\,.\,y) = \Sigma\,(k\,a\,.\,y) \ldots\ldots\ldots\ldots\ldots\ldots (136)$$

The two members of equation (136) must now be evaluated and if equality does not exist the ordinates of the type $m\,b$ must all be changed in such a constant ratio that equality will ensue. In order to achieve this result, suppose that,

$$\frac{\Sigma\,(k\,a\,.\,y)}{\Sigma\,(m\,b\,.\,y)} = U$$

then

$$U\,\Sigma\,(m\,b\,.\,y) = \Sigma\,(k\,a\,.\,y)$$

But

$$k\,a = y - e_1 \quad \text{(See equation 131 and following.)}$$

hence

$$\Sigma\,(k\,a\,.\,y) = \Sigma\,(y - e_1)\,y$$

therefore

$$\Sigma\,(k\,a\,.\,y) = \Sigma\,(y^2) - e_1\,\Sigma\,(y) \ldots\ldots\ldots\ldots\ldots (137)$$

whence

$$U = \frac{\Sigma\,(k\,a\,.\,y)}{\Sigma\,(m\,b\,.\,y)} = \frac{\Sigma\,(y^2) - e_1\,\Sigma\,(y)}{\Sigma\,(m\,b\,.\,y)} \ldots\ldots\ldots (138)$$

It follows that if the ordinates $m\,b$ be multiplied by the ratio U we will obtain the ordinates $k\,c$ of the pressure curve which will satisfy equation (136), and hence make $\Sigma\,(M\,y) = 0$.

Let the points so determined, on the pressure curve, be designated by the letter c. Equation (134) will then become

$\Sigma\,(a\,c\,.\,y) = 0$, which is the

2d Condition as given by equation (27).

Hence

$$k\,c = m\,b\,.\,\frac{\Sigma\,(y^2) - e_1\,\Sigma\,(y)}{\Sigma\,(m\,b\,.\,y)} \ldots\ldots\ldots\ldots\ldots\ldots (139)$$

In order to solve equation (139) we must evaluate the terms $\Sigma\,(y^2) - e_1\,\Sigma\,(y)$ and $\Sigma\,(m\,b\,.\,y)$. The values $m\,b$ are found from the basic polygon by scaling, using the scale of distance.

From Table IV, we have

$$\Sigma\,(y) = 153.20$$

$$\Sigma\,(y^2) = 1238.30$$

$e_1 = 7.66$ (See below equation 130.)

Hence

$$\Sigma (y^2) - e_1 \Sigma (y) = 1238.30 - 7.66 (153.20) = 64.8.$$

Since symmetrically disposed ordinates y of the neutral axis are equal, it follows that there will be two symmetrical values of $m\,b$ which will be multiplied by the same value of y; in the evaluation of $\Sigma (m\,b\,.\,y)$. Thus

$$(m_1\,b_1 + m_{20}\,b_{20})\,y_1;\ (m_2\,b_2 + m_{19}\,b_{19})\,y_2, \text{ etc.}$$

The evaluation of $\Sigma (m\,b\,.\,y)$ is given in detail in Table VI, from which it is seen that

$$\Sigma (m\,b\,.\,y) = +\ 90.15.$$

Care must be taken to observe that $m\,b$ ordinates above $m\,m_1$ are regarded as positive, and below as negative.

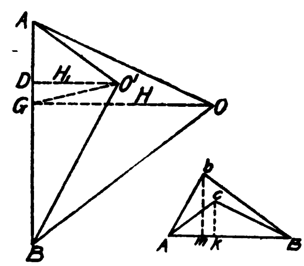


Fig. 38.

It follows that,

$$\frac{\Sigma (y^2) - e_1 \Sigma (y)}{\Sigma (m\,b\,.\,y)} = \frac{64.8}{90.15} = 0.718$$

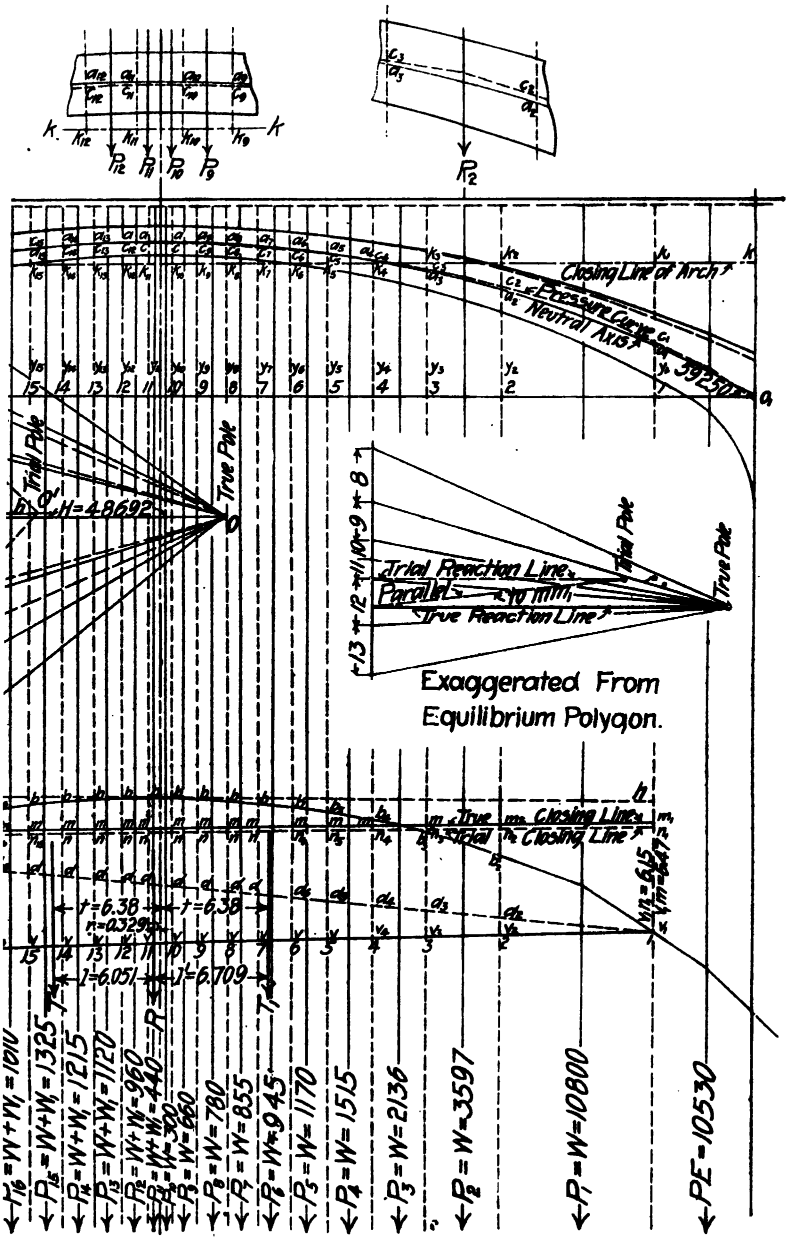
Hence equation (139) becomes

$$k\,c = m\,b\,.\,\frac{\Sigma (y^2) - e_1 \Sigma (y)}{\Sigma (m\,b\,.\,y)} = m\,b\ (0.718)$$

The values of $k\,c$ are found in column (3) of Table VII.

The points $c_1, c_2, c_3 \ldots\ldots c_{19}, c_{20}$ of the pressure curve may now be plotted by scaling the values of $k\,c$ from Table VII, laying them off from the line $k\,k$, along the respective ordinates through points a.

(2) We must now determine the value of the True Pole-distance H. In Fig. 38, let O' be the trial pole and H_1 the trial pole distance. From O' draw the ray $O'\,G$ parallel to the true closing line $m\,m_1$ (not shown in this figure). Then the point of intersection G of the ray $O'\,G$ with the load line $A\,B$ determines absolutely the value of the two end reactions $A\,G$ and $B\,G$. But the true closing line of the arch is the horizontal



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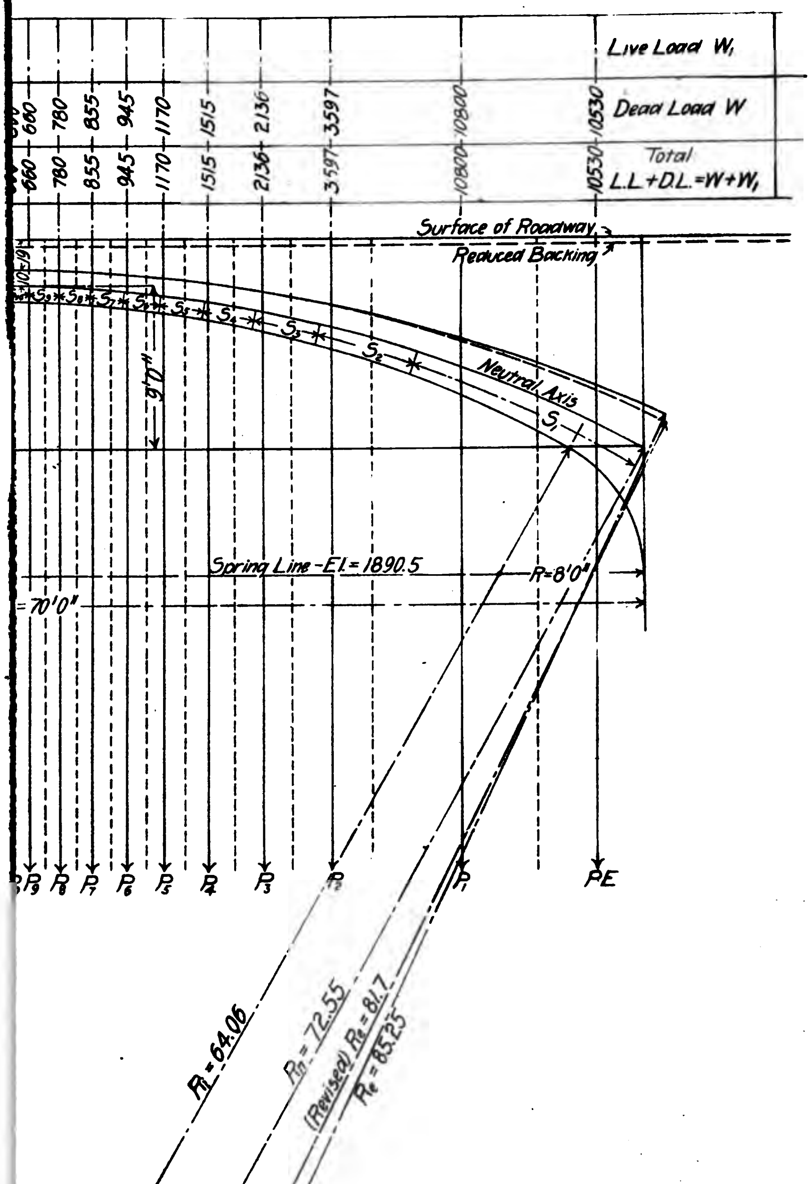

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line $k\,k$, hence the ray for the true pole-distance must be parallel to $k\,k$, hence horizontal, and must start from the point of junction of the two end reactions, that is from point G.

The length of the true pole-distance, $G\,O = H$ may be found from the following considerations:

Let $A\,b\,B$, whose ordinates are of the type $m\,b$, be the polygon corresponding to the trial ray diagram $O'\,A\,B$; and let $A\,c\,B$, whose ordinates are of the type $k\,c$, be the polygon corresponding to the ray diagram $O\,A\,B$; thus assuming that the true pole O has been found. Hence

$A\,b$ is parallel to $B\,O'$
$b\,B$ is parallel to $A\,O'$
$A\,c$ is parallel to $B\,O$
$c\,B$ is parallel to $A\,O$
$A\,B$ is parallel to $G\,O$ and $D\,O'$

Therefore triangle $A\,b\,B$ is similar to triangle $O'\,A\,B$, and triangle $A\,c\,B$ is similar to triangle $O\,A\,B$.

Hence

$$\frac{H}{H_1} = \frac{m\,b}{k\;\;c} = \frac{\Sigma\,(m\,b\,.\,y)}{\Sigma\,(k\;\;c\,.\,y)} \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots(140)$$

Therefore

$$H = H_1 \frac{\Sigma\,(m\,b\,.\,y)}{\Sigma\,(k\;\;c\,.\,y)}$$

or

$$H = H_1 \frac{\Sigma\,(m\,b\,.\,y)}{\Sigma\,(y^2) - e_1\,\Sigma\,(y)} \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots(141)$$

hence

$$H = 35{,}000 \frac{90.15}{64.8} = 48{,}692 \text{ lbs.}$$

If the value 48,692 be laid off from G to O along the horizontal line $G\,O$ (Fig. 40) we locate the true pole O.

(3) We can now proceed to construct the Pressure Curve. Complete the true ray diagram (Fig. 40) by drawing the rays from the true pole O.

Referring to Fig. 34 we have an enlarged section of the arch and a portion of the true ray diagram corresponding to the section.

Through point c_{12} draw line $A\,B$ between load lines P_{12} and P_{13} parallel to ray $O\,B$ which meets the point of juncture B of loads P_{12} and P_{13} in the ray diagram. Similarly, through c_{13} draw $B\,C$ parallel to $O\,C$ of the ray diagram. If the calculations and graphical constructions have been carefully made lines $A B$ and $B C$ so drawn will intersect at B on the load line P_{18}. By continuing this process the entire pressure curve can be drawn as shown in Fig. 40.

Through point c_{20} a line is drawn between load lines P_{20} and $P E$ (left) parallel to ray $O B$ which intersects the load line in the point of juncture of load P_{20} and Left End load.

Where the line through c_{20} intersects the load $P E$ (left) we draw the final and completing line of the pressure curve parallel to the ray $O C$. The ray $O C$ scales 12.55 ins., hence since our load is 5,000 lbs. = 1 in. the value of $O C$ is 62,750 lbs. This value may be resolved by the regular method into thrust and shear components. A similar procedure will determine the last side of the pressure curve at the right. Ray $O A$ is found to equal 59,250 lbs.

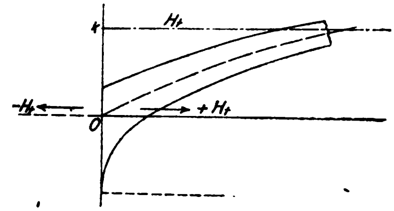

Fig. 41.

18. **Bending Moments, Thrusts and Shears.** — From equation (42), (Art. 6) we see that the bending moment M at any point a of the arch is given by

$$\Sigma (M) = H (a c)$$

Hence we scale the ordinates $k a$ and record them in column 4 (Table VII).

Then since

$k c - k a = a c$ we obtain the values from which the bending moments can be calculated as shown in detail in Table VIII.

To obtain the thrusts and shears at the various points a we construct tangents to the neutral axis. Now as the neutral axis is an arc of a circle we need only draw radii to the various points a and erect at a perpendiculars to the respective radii.

Then from the true pole O (Fig. 40) we draw a line parallel to this tangent corresponding to the ray which refers to the point in question. Thus for point a_{20}, the tangential thrust (tangential component) is found by drawing from point O of the ray diagram a line $O S$ parallel to the tangent through a_{20} and constituting one side of the right triangle $O S B$ whose hypotenuse $O B$ is parallel to the pressure line through c_{20}. The remaining side $B S$ of the right triangle is the normal component and is proportionate to the shear for the particular point considered. By scaling these distances to the scale of loads we determine the amount of the thrust and shear for the various points a of the neutral axis. The relation between these effects and the thrusts due to the temperature is seen from Fig. 30. The results for these quantities are found in Table VIII.

TABLE VIII.—CALCULATION OF MOMENTS, THRUSTS AND SHEARS.

Points.	ac Ordinates from points a to c of arch.	M Bending Moments in foot pounds $M = H\ (a\ c)$ $H = 48,692$	Thrusts $= T$ Tangential Components at points a. Scaled from diagram.	Shears Normal Components at points a. Scaled from diagram.
1	+ 0.15	+ 7,304	53,900	2,050
2	+ 0.24	+11,686	50,200	1,050
3	+ 0.07	+ 3,408	49,500	1,550
4	0.00	0.00	49,000	1,450
5	− 0.05	− 2,434	48,900	1,200
6	− 0.13	− 6,330	48,800	900
7	− 0.10	− 4,869	48,750	600
8	− 0.15	− 7,304	48,700	200
9	− 0.13	− 6,330	48,650	300
10	− 0.11	− 5,356	48,600	550
11	− 0.11	− 5,356	48,500	750
12	− 0.09	− 4,382	48,550	850
13	− 0.01	− 4,869	48,600	750
14	+ 0.04	+ 1,947	48,700	700
15	+ 0.05	+ 2,435	48,850	550
16	+ 0.13	+ 6,330	49,000	350
17	+ 0.17	+ 8,278	49,300	200
18	+ 0.22	+10,712	49,900	350
19	+ 0.20	+ 9,738	51,000	1,500
20	− 0.39	−18,990	55,850	6,000

CHAPTER III.

CALCULATIONS OF FIBER STRESSES.

19. **Calculation of Fiber Stresses.**—We will now give, in detail, the calculation of the stresses for several typical points.

$$S_c = \frac{T}{A_c + n\,A_s} \pm \frac{M\,x_1}{\frac{1}{12}D^3 + n\,A_s\,x_2^2} \text{ (lbs. per square foot)}$$

Let

E_s = modulus of elasticity of steel = 30,000,000 lbs. per sq. in.

E_c = modulus of elasticity of concrete = 1,500,000 lbs. per sq. in.

Then

$$n = \frac{E_s}{E_c} = \frac{30{,}000{,}000}{1{,}500{,}000} = 20.$$

A_c = area section concrete in a slice 1 ft. wide and D ft. deep. (Value of D, Table II) $= D \times 1 = D$ *sq. ft.*

A_s = area steel in square ft. in a slice 1 ft. wide $= \frac{A}{b}$; where

A = cross-sectional area in square ft. of two bars constituting a rib.

and b = distance (in feet) c. to c. of ribs.

Area $\frac{7}{8}$ in. corr. bar = 0.77 sq. in. = 0.00534 sq. ft.

A = 2 (0.00534) = 0.01068.

b = 8 ins. = $\frac{2}{3}$ ft.

$$A_s = \frac{A}{b} = \frac{0.01068}{\frac{2}{3}} = 0.016 \text{ (a constant)}$$

$n\,A_s$ = 20 (0.016) = 0.32 (a constant)
(See Fig. 2.)

D = radial depth in feet of arch ring at point a.

G = distance in feet from c. g. of steel bar to extreme fiber of concrete
= 3 ins. (assumed) = 0.25 ft.

D' = distance in feet between centers of gravity of two steel bars constituting a rib.
= D − 2 G; 2 G = 2 (0.25) = 0.50 (an assumed constant)

$= D - 0.50$ (in ft.)

$$x_1 = \frac{D}{2}$$

$$x_2 = \frac{D'}{2} = \frac{D - 0.50}{2}$$

T = Thrust in lbs. (Tangential component at point *a*.) For value see Table VIII.

M = Bending moment in foot pounds at point *a*.

$= H\,(a\,c)$. For value see Table VIII.

Fiber Stress in Concrete.—

$$S_c = \frac{T}{A_c + n\,A_s} \pm \frac{M\,x_1}{\frac{1}{12}\,D^3 + n\,A_s\,x_2^2}$$

Fiber Stress in Steel.

$$S_s = \left(\frac{T}{A_c + n\,A_s} \pm \frac{M\,x_2}{\frac{1}{12}\,D^3 + n\,A_s\,x_2^2}\right) n$$

When $a\,c$ is + then bending moment M is +

When $a\,c$ is − then bending moment M is −

When M is + then + of ± refers to upper fiber of concrete and steel.

When M is + then − of ± refers to lower fiber of concrete and steel.

When M is − then + of ± refers to upper fiber of concrete and steel.

When M is − then − of ± refers to lower fiber of concrete and steel.

Calculation of Fiber Stresses Due to Variation in Temperature.

H_t = horizontal thrust due to change in temperature.

t° = number of degrees change of temperature

$= \pm 40^\circ$ F.

L = span in feet.

E'_c = modulus of elasticity of concrete in lbs. per sq. ft.

$= 144\,E_c = 144\,(1{,}500{,}000)$

(E_c = modulus per sq. in.)

e = coefficient of expansion of concrete for 1° F. = 0.000006

I_c = moment of inertia of concrete of area A_c

I_s = moment of inertia of steel of area A_s

$$\frac{S}{I_c + n\,I_s} = \text{a constant.}$$

If the subdivisions S of the arch have been obtained under the assumption that $\frac{S}{D^3}$ is a constant, then find the value of the true ratio $\frac{I_c + n\,I_s}{S}$ at, say the three points, a_{20}, a_{15} and a_{11}. for the arch scheme which makes $\frac{S}{D^3}$ constant.

Use the average of these values as the value of $\frac{I_c + n\,I_s}{S}$. This will be accurate enough for all practical purposes.

$$\frac{I_c + 20\,I_s}{S} = \tfrac{1}{12}\,D^3 + 20\,A_s\left(\frac{D}{2} - G\right)^2 = 0.31 \text{ approximately.}$$

$$H_t = \frac{E'_c\,l\,e\,t^\circ}{2\,[\Sigma\,(y^2) - e_1\,\Sigma\,(y)]} \cdot \frac{I_c + n\,I_s}{S}$$

$$\Sigma\,(y^2) - e_1\,\Sigma\,(y) = 64.8$$

$$H_t = \frac{1{,}500{,}000\,(144)\,70\,(0.000006)\,40}{2\,(64.8)}\,(0.31) = 8680$$

Location of temperature norm $k\,k$

Ordinate e_1 of temperature norm $k\,k = \frac{\Sigma\,(y)}{N}$

$$e_1 = \frac{\Sigma\,(y)}{N} = 7{,}66$$

Ordinates of type $k\,a$ above $k\,k$ regarded as +

Ordinates of type $k\,a$ below $k\,k$ regarded as –

M_t = Bending moment due to change in temperature $= H_t\,(k\,a)$.

T_t = Tangential thrust due to change in temperature.

$T_t = H_t \cos \phi = 8680 \cos \phi$ (See Fig. 30)

See Table IX.

Fiber Stress in Concrete Due to Change in Temperature.

$$S_{ct} = \frac{T_t}{A_c + n\,A_s} \pm \frac{M_t\,x_1}{\frac{1}{12}\,D^3 + n\,A_s\,x_2^2}$$

Fiber Stress in Steel Due to Change in Temperature.

$$S_{st} = \left(\frac{T_t}{A_c + n\,A_s} \pm \frac{M_t\,x_2}{\frac{1}{12}\,D^3 + n\,A_s\,x_2^2}\right) n$$

Fiber Stresses Calculated—For Point 1.

Concrete.

$$S_c = \frac{T}{A_c + n\,A_s} \pm \frac{M\,x_1}{\frac{1}{12}\,D_1^3 + n\,A_s\,x_2^2}$$

TABLE IX.—CALCULATION OF TANGENTIAL THRUST, T_t, DUE TO TEMPERATURE.

$\lambda : \frac{1}{2} N = \phi : \frac{1}{2} a$

$\lambda : 36.52 = \phi : 28.84$ N = length of neutral axis = 73.04ft

$\phi = 0.789\ \lambda$ a = subtended central angle=57.68°

Point	λ Distance from Center Line $A\ B$ of Span to points a on Neutral Axis	ϕ $= 0.789\ \lambda$	$\cos \phi$	T_t $= H_t \cos \phi$ $= 8680 \cos \phi$
1	29.77	23°–29′	0.91718	7,961
2	20.42	16°– 7′	0.96070	8,339
3	15.98	12°–36′	0.97592	8,471
4	12.67	10°– 0′	0.98481	8,548
5	10.01	7°–54′	0.99051	8,597
6	7.80	6°– 9′	0.99424	8,630
7	5.86	4°–37′	0.99676	8,652
8	4.06	3°–12′	0.99844	8,666
9	2.38	1°–53′	0.99946	8,675
10	0.78	0°–37′	0.99994	8,679
11	0.78	0°–37′	0.99994	8,679
12	2.38	1°–53′	0.99946	8,675
13	4.06	3°–12′	0.99844	8.666
14	5.86	4°–37′	0.99676	8,652
15	7.80	6°– 9′	0.99424	8,630
16	10.01	7°–54′	0.99051	8,597
17	12.67	10°– 0′	0.98481	8,548
18	15.98	12°–36′	0.97592	8,471
19	20.42	16°– 7′	0.96070	8,339
20	29.77	23°–29′	0.91718	7,961

$T = 53{,}900$ (See Table VIII, point 1.)

$A_c = 3.43$ (Table X.)

$n\,A_s = 20\ (0.016) = 0.32$

$M = +\ 7{,}304$ (Table VIII, point 1.)

$$x_1 = \frac{D}{2} = \frac{3.43}{2} = 1.71 \quad \text{(Table X.)}$$

$$D' = D - 2G = 3.43 - 2\ (0.25) = 2.93$$

$$x_2 = \frac{D'}{2} = \frac{2.93}{2} = 1.46$$

TABLE X.—QUANTITIES NECESSARY FOR FIBER STRESS CALCULATIONS.

Point.	D Radial Depth in feet	D^3	$\frac{1}{12}D^3$	$A_c = D \times 1$	$x_1 = \frac{D}{2}$	$D^1 = D-2G = D-0.5$	$x_2 = \frac{D^1}{2}$	x_2^2	nA_s constant $= 20 \times 0.016$	M Bending Moments	T Thrusts	H_t constant $= 8680$	k_a	$M_t = H_t (ka)$	T_t $H_t \cos \phi$
1	3.43	40.35	3.36	3.43	1.71	2.93	1.46	2.13	0.32	+ 7,304	53,900	8680	−4.78	−41,490	7,961
2	2.49	15.44	1.29	2.49	1.24	1.99	0.99	0.98	0.32	+11,686	50,200	8680	−1.53	−13,280	8,339
3	2.23	11.09	0.92	2.23	1.11	1.73	0.86	0.74	0.32	+ 3,408	49,500	8680	−0.39	− 3,385	8,471
4	2.06	8.74	0.73	2.06	1.03	1.56	0.78	0.61	0.32	0	49,000	8680	+0.22	+ 1,910	8,548
5	1.93	7.19	0.60	1.93	0.96	1.43	0.71	0.50	0.32	− 2,434	48,900	8680	+0.62	+ 5,382	8,597
6	1.82	6.03	0.50	1.82	0.91	1.32	0.66	0.43	0.32	− 6,330	48,800	8680	+0.92	+ 7,986	8,630
7	1.77	5.54	0.46	1.77	0.88	1.27	0.63	0.40	0.32	− 4,869	48,750	8680	+1.07	+ 9,288	8,652
8	1.73	5.18	0.43	1.73	0.86	1.23	0.61	0.37	0.32	− 7,304	48,700	8680	+1.23	+10,676	8,666
9	1.70	4.91	0.41	1.70	0.85	1.20	0.60	0.36	0.32	− 6,330	48,650	8680	+1.31	+11,371	8,675
10	1.67	4.66	0.39	1.67	0.83	1.17	0.58	0.34	0.32	− 5,356	48,600	8680	+1.33	+11,544	8,679
11	1.67	4.66	0.39	1.67	0.83	1.17	0.58	0.34	0.32	− 5,356	48,500	8680	+1.33	+11,544	8,679
12	1.70	4.91	0.41	1.70	0.85	1.20	0.60	0.36	0.32	− 4,382	48,550	8680	+1.31	+11,371	8,675
13	1.73	5.18	0.43	1.73	0.86	1.23	0.61	0.37	0.32	− 4,869	48,600	8680	+1.23	+10,676	8,666
14	1.77	5.54	0.46	1.77	0.88	1.27	0.63	0.40	0.32	+ 1,947	48,700	8680	+1.07	+ 9,288	8,652
15	1.82	6.03	0.50	1.82	0.91	1.32	0.66	0.43	0.32	+ 2,435	48,850	8680	+0.92	+ 7,986	8,630
16	1.93	7.19	0.60	1.93	0.96	1.43	0.71	0.50	0.32	+ 6,330	49,000	8680	+0.62	+ 5,382	8,597
17	2.06	8.74	0.73	2.06	1.03	1.56	0.78	0.61	0.32	+ 8,278	49,300	8680	+0.22	+ 1,910	8,548
18	2.23	11.09	0.92	2.23	1.11	1.73	0.86	0.74	0.32	+10,712	49,900	8680	−0.39	− 3,385	8,471
19	2.49	15.44	1.29	2.49	1.24	1.99	0.99	0.98	0.32	+ 9,738	51,000	8680	−1.53	−13,280	8,339
20	3.43	40.35	3.36	3.43	1.71	2.93	1.46	2.13	0.32	−18,990	55,850	8680	−4.78	−41,490	7,961

$(x_2)^2 = 2.13$

$D^3 = 40.35$

$$S_c = \frac{53{,}900}{3.43 + 0.32} \pm \frac{+7304\,(1.71)}{\frac{1}{12}\,(40.35) + 0.32\,(2.13)}$$

$= (14{,}373 \pm 3{,}092)$

M is +

+ of ± gives + 17,465 lbs. per sq. ft. = + 121 lbs. per sq. in. compression in upper fiber of concrete.

− of ± gives + 11,281 lbs. per sq. ft. = + 78 lbs. per sq. in. compression in lower fiber of concrete.

Steel.

$$S_s = \left(\frac{T}{A_c + n\,A_s} \pm \frac{M\,x_2}{\frac{1}{12}\,D_1^3 + n\,A_s\,x_2^2} \right) n$$

$$= \left(\frac{53{,}900}{3.43 + 0.32} \pm \frac{+7{,}304\,(1.46)}{\frac{1}{12}\,(40.35) + 0.32\,(2.13)} \right) 20$$

$= (14{,}373 \pm 2{,}640)\ 20$

M is +

+ of ± gives + 340,260 lbs. per sq. ft. = + 2,363 lbs. per sq. in. compression in upper fiber of steel.

− of ± gives + 234,660 lbs. per sq. ft. = + 1,629 lbs. per sq. in. compression in lower fiber of steel.

Temperature Concrete.

$$S_{ct} = \frac{T_t}{A_c + n\,A_s} \pm \frac{M_t\,x_1}{\frac{1}{12}\,D_1^3 + n\,A_s\,x_2^2}$$

$M_t = H_t\,(k\,a)$; $k\,a = -4.78$ (See Table VII.)

$M_t = 8{,}680\,(-4.78) = -41{,}490$

$T_t = H_t \cos \phi = 7{,}961.$ (See Table IX.)

$$S_{ct} = \frac{7{,}961}{3.43 + 0.32} \pm \frac{-41{,}490\,(1.71)}{\frac{1}{12}\,(40.35) + 0.32\,(2.13)}$$

+ of ± gives − 107 lbs. per sq. in.

− of ± gives + 137 lbs. per sq. in.

Ordinate $k\,a$ is − at point 1. (See Table XVI.)

TABLE XI.—QUANTITIES NECESSARY FOR FIBER STRESS CALCULATIONS.

CALCULATIONS OF VALUES α AND β.

$\alpha = A_c + n\,A_s$.

$\beta = \frac{1}{12} D^3 + nA_s x_2^2$.

Point	1	2	3	4	5	6	7
	A_c	$n\,A_s$ = constant = 0.32	α = col. 1 + col. 2	$\frac{1}{12} D^3$	x_2^2	$n\,A_s \cdot x_2^2$ col. 2 × col. 5	β = col. 4 + col. 6
1	3.43	0.32	3.75	3.36	2.13	0.68	4.04
2	2.49	0.32	2.81	1.29	0.98	0.31	1.60
3	2.23	0.32	2.55	0.92	0.74	0.24	1.16
4	2.06	0.32	2.38	0.73	0.61	0.20	0.93
5	1.93	0.32	2.25	0.60	0.50	0.16	0.76
6	1.82	0.32	2.14	0.50	0.43	0.14	0.64
7	1.77	0.32	2.09	0.46	0.40	0.13	0.59
8	1.73	0.32	2.05	0.43	0.37	0.12	0.55
9	1.70	0.32	2.02	0.41	0.36	0.12	0.53
10	1.67	0.32	1.99	0.39	0.34	0.11	0.50
11	1.67	0.32	1.99	0.39	0.34	0.11	0.50
12	1.70	0.32	2.02	0.41	0.36	0.12	0.53
13	1.73	0.32	2.05	0.43	0.37	0.12	0.55
14	1.77	0.32	2.09	0.46	0.40	0.13	0.59
15	1.82	0.32	2.14	0.50	0.43	0.14	0.64
16	1.93	0.32	2.25	0.60	0.50	0.16	0.76
17	2.06	0.32	2.38	0.73	0.61	0.20	0.93
18	2.23	0.32	2.55	0.92	0.74	0.24	1.16
19	2.49	0.32	2.81	1.29	0.98	0.31	1.60
20	3.43	0.32	3.75	3.36	2.13	0.68	4.04

	k a is —	
	Upper fiber	Lower fiber
Rise 40° F.	Compression + 137	Tension − 107
Fall 40° F.	Tension − 107	Compression + 137

Temperature Steel.

$$S_{st} = \left(\frac{7{,}961}{3.43 + 0.32} \pm \frac{-41{,}490\ (1.46)}{\frac{1}{12}\ (40.35) + 0.32\ (2.13)} \right) 20$$

+ of ± gives − 1,788 lbs. per sq. in.
− of ± gives + 2,377 lbs. per sq. in.

	k a is —	
	Upper fiber	Lower fiber
Rise 40° F.	Compression + 2,377	Tension − 1,788
Fall 40° F.	Tension − 1,788	Compression + 2,377

Fiber Stresses Calculated.—For Point 4.

Concrete.

M regarded as + [$M = 0$ here]
$S_c = +\ 143 \pm 0$
+ of ± gives + 143 lbs. per sq. in. compression in upper fiber of concrete.
− of ± gives + 143 lbs. per sq. in. compression in lower fiber of concrete.

TABLE XII.—STRESSES IN CONCRETE.

$$S_c = \frac{T}{A_c + n\,A_s} \pm \frac{Mx_1}{\frac{1}{12}D^3 + n\,A_s x_2^2} = \frac{T}{\alpha} \pm \frac{Mx_1}{\beta}.$$

$$\alpha = A_c + nA_s$$

$$\beta = \tfrac{1}{12}D^3 + nA_s \times x_2^2.$$

Points	1	2	3	4	5	6	7	8	9	10	11	12
	T	M	x_1	α	β	Mx_1	$\frac{T}{\alpha}$	$\frac{Mx_1}{\beta}$	Col. 7 + Col. 8 Pounds per Square Foot, Upper fiber	Col. 9 ÷ 144 Pounds per Sq. Inch, Upper fiber	Col. 7 − Col. 8 Pounds per Sq. Foot, Lower fiber	Col. 11 ÷ 144 Pounds per Sq. Inch, Lower fiber
1	53,900	+ 7,304	1.71	3.75	4.04	+12,490	+14,373	+ 3,092	+17,465	+121	+11,281	+ 78
2	50,200	+11,686	1.24	2.81	1.60	+14,491	+17,865	+ 9,057	+26,922	+187	+ 8,808	+ 61
3	49,500	+ 3,408	1.11	2.55	1.16	+ 3,783	+19,412	+ 3,261	+22,673	+157	+16,151	+112
4	49,000	0	1.03	2.38	0.93	0	+20,588	0	+20,588	+143	+20,588	+143
5	48,900	− 2,434	0.96	2.25	0.76	− 2,337	+21,733	− 3,075	+18,658	+129	+24,808	+172
6	48,800	− 6,330	0.91	2.14	0.64	− 5,760	+22,804	− 9,000	+13,804	+ 96	+31,804	+221
7	48,750	− 4,869	0.88	2.09	0.59	− 4,285	+23,325	− 7,262	+16,063	+112	+30,587	+212
8	48,700	− 7,304	0.86	2.05	0.55	− 6,281	+23,756	−11,421	+12,335	+ 86	+35,177	+244
9	48,650	− 6,330	0.85	2.02	0.53	− 5,380	+24,084	−10,152	+13,932	+ 97	+34,236	+238
10	48,600	− 5,356	0.83	1.99	0.50	− 4,445	+24,422	− 8,891	+15,531	+108	+33,313	+231
11	48,500	− 5,356	0.83	1.99	0.50	− 4,445	+24,372	− 8,891	+15,481	+108	+33,263	+231
12	48,550	− 4,382	0.85	2.02	0.53	− 3,725	+24,035	− 7,028	+17,007	+118	+31,063	+216
13	48,600	− 4,869	0.86	2.05	0.55	− 4,187	+23,707	− 7,613	+16,094	+112	+31,320	+218
14	48,700	+ 1,947	0.88	2.09	0.59	+ 1,713	+23,302	+ 2,904	+26,206	+182	+20,398	+142
15	48,850	+ 2,435	0.91	2.14	0.64	+ 2,216	+22,827	+ 3,462	+26,289	+183	+19,365	+134
16	49,000	+ 6,330	0.96	2.25	0.76	+ 6,077	+21,778	+ 7,996	+29,774	+207	+13,782	+ 95
17	49,300	+ 8,278	1.03	2.38	0.93	+ 8,526	+20,715	+ 9,168	+29,883	+208	+11,547	+ 80
18	49,900	+10,712	1.11	2.55	1.16	+11,830	+19,569	+10,250	+29,819	+207	+ 9,319	+ 65
19	51,000	+ 9,738	1.24	2.81	1.60	+12,075	+18,149	+ 7,547	+25,693	+178	+10,599	+ 74
20	55,850	−18,990	1.71	3.75	4.04	−32,473	+14,893	− 8,038	+ 6,855	+ 48	+22,931	+159

+ Compression. − Tension.

$S_s = +\ 2{,}859 \pm 0$

+ of ± gives + 2,859 lbs. per sq. in. compression in upper fiber of steel.

− of ± gives + 2,859 lbs. per sq. in. compression in lower fiber of steel.

Temperature Concrete.

$S_{ct} = [3592 \pm (+\ 2115)] \div 144$ (See Table XIV).

+ of ± gives + 40 lbs. per sq. in. (This value is to be used as it is the maximum.)

− of ± gives + 10 lbs. per sq. in.

No tension.

Ordinate $k\,a$ is + here. (See Table XVI)

	$k\,a$ is +	
	Upper fiber	Lower fiber
	Tension	Compression
Rise 40° F.	0	+ 40
Fall 40° F.	Compression	Tension
	+ 40	0

Temperature Steel.

+ gives + 721 lbs. per sq. in. This value to be used as it is the maximum.

− gives + 276 lbs. per sq. in.

	$k\,a$ is +	
	Upper fiber	Lower fiber
	Tension	Compression
Rise 40° F.	0	+ 721
Fall 40° F.	Compression	Tension
	+ 721	0

TABLE XIII.—STRESSES IN STEEL.

$$S_s = \left\{ \frac{T}{A_c + nA_s} \pm \frac{Mx_2}{\frac{1}{12}D^3 + nA_sx_2^2} \right\} 20 = \left\{ \frac{T}{\alpha} \pm \frac{Mx_2}{\beta} \right\} 20. \qquad \alpha = A_c + nA_s. \quad \beta = \tfrac{1}{12}D^3 + nA_sx_2^2.$$

Points	1	2	3	4	5	6	7	8	9	10	11	12
	T	M	x_2	α	β	$M x_2$	$\frac{T}{\alpha}$	$\frac{M x_2}{\beta}$	(Col. 7+ Col. 8) 20 Lbs. per Sq. Ft. Upper Fiber	Col. 7÷144 Pounds per Sq. Inch. Upper fiber	(Col. 7− Col. 8) 20 Lbs. per Sq. Ft. Lower Fiber	Col. 11÷144 Pounds per Sq. Inch Lower fiber
1	53,900	+ 7,304	1.46	3.75	4.04	+10,664	+14,373	+2,640	+340,260	+2,363	+234,660	+1,629
2	50,200	+11,686	0.99	2.81	1.60	+11,569	+17,865	+7,231	+501,920	+3,485	+212,680	+1,477
3	49,500	+ 3,468	0.86	2.55	1.16	+ 2,931	+19,412	+2,527	+438,780	+3,047	+337,700	+2,345
4	49,000	0	0.78	2.38	0.93	0	+20,588	0	+411,760	+2,859	+411,760	+2,859
5	48,900	− 2,434	0.71	2.25	0.76	− 1,728	+21,733	−2,274	+389,180	+2,703	+480,140	+3,334
6	48,800	− 6,330	0.66	2.14	0.64	− 4,178	+22,804	−6,528	+325,520	+2,261	+586,640	+4,074
7	48,750	− 4,869	0.63	2.09	0.59	− 3,067	+23,325	−5,200	+362,500	+2,517	+570,500	+3,962
8	48,700	− 7,304	0.61	2.05	0.55	− 4,455	+23,756	−8,101	+313,100	+2,174	+637,140	+4,425
9	48,650	− 6,330	0.60	2.02	0.53	− 3,798	+24,084	−7,166	+338,360	+2,350	+625,000	+4,340
10	48,600	− 5,356	0.58	1.99	0.50	− 3,106	+24,422	−6,213	+364,180	+2,529	+612,700	+4,255
11	48,500	− 5,356	0.58	1.99	0.50	− 3,106	+24,372	−6,213	+363,180	+2,522	+611,700	+4,248
12	48,550	− 4,382	0.60	2.02	0.53	− 2,629	+24,035	−4,961	+381,480	+2,649	+579,920	+4,027
13	48,600	− 4,869	0.61	2.05	0.55	− 2,970	+23,707	−5,400	+366,140	+2,543	+582,140	+4,043
14	48,700	+ 1,947	0.63	2.09	0.59	+ 1,227	+23,302	+2,079	+507,620	+3,525	+424,460	+2,948
15	48,850	+ 2,435	0.66	2.14	0.64	+ 1,607	+22,827	+2,511	+506,760	+3,519	+406,320	+2,822
16	49,000	+ 6,330	0.71	2.25	0.76	+ 4,494	+21,778	+5,914	+553,840	+3,846	+317,280	+2,203
17	49,300	+ 8,278	0.78	2.38	0.93	+ 6,457	+20,715	+6,943	+553,160	+3,841	+275,440	+1,913
18	49,900	+10,712	0.86	2.55	1.16	+ 9,212	+19,569	+7,942	+550,220	+3,821	+232,540	+1,615
19	51,000	+ 9,738	0.99	2.81	1.60	+ 9,641	+18,149	+6,025	+483,480	+3,358	+242,480	+1,684
20	55,850	−18,990	1.46	3.75	4.04	−27,725	+14,893	−6,863	+160,600	+1,115	+435,120	+3,021

+ Compression. − Tension.

Fiber Stresses Calculated.—For Point 5.

Concrete.

$M = -$

$S_c = [21733 \pm (-3075)] \div 144$. (See Table XII).

$+$ of $\pm$ gives $+$ 129 lbs. per sq. in. compression in upper fiber.

$-$ of $\pm$ gives $+$ 172 lbs. per sq. in. compression in lower fiber.

$S_s = [21733 \pm (-2274)]\ 20 \div 144$. (See Table XIII).

$+$ of $\pm$ gives $+$ 2703 lbs. per sq. in. compression in upper fiber.

$-$ of $\pm$ gives $+$ 3334 lbs. per sq. in. compression in lower fiber.

Temperature.

$S_{ct} = [3821 \pm (+6798)] \div 144$.

$+$ of $\pm$ gives $+$ 74 lbs. per sq. in.

$-$ of $\pm$ gives $-$ 21 lbs. per sq. in.

Ordinate $k\,a$ is $+$ here. (See Table XVI).

	$k\,a$ is $+$	
	Upper fiber	Lower fiber
Rise 40° F.	Tension $-$ 21	Compression $+$ 74
Fall 40° F.	Compression $+$ 74	Tension $-$ 21

Steel.

$S_{st} = [3821 \pm (+5028)]\ 20 \div 144$. (See Table XV).

$+$ gives $+$ 1,229

$-$ gives $-$ 168

TABLE XIV.—STRESSES DUE TO TEMPERATURE. CONCRETE.

$$S_{ct} = \frac{T_t}{A_c + n A_s} \pm \frac{M_t x_1}{\frac{1}{12} D^3 + n A_s x_2^2} = \frac{T_t}{\alpha} \pm \frac{M_t x_1}{\beta} \qquad \alpha = A_c + n A_s. \quad \beta = \tfrac{1}{12} D^3 + n A_s x_2^2.$$

Points	1	2	3	4	5	6	7	8	9	10	11	12
	T_t	M_t	x_1	x	β	$M_t x_1$	$\frac{T_t}{\alpha}$	$\frac{M_t x_1}{\beta}$	Col. 7+Col. 8 Pounds per Sq. Foot.	Col. 9÷144 Pounds per Sq. Inch.	Col. 7—Col. 8 Pounds per Sq. Foot.	Col. 11÷144 Pounds per Sq. Inch.
1	7,961	−41,490	1.71	3.75	4.04	−70,948	+ 2,123	−17,561	−15,438	−107	+19,684	+137
2	8,339	−13,280	1.24	2.81	1.60	−16,467	+ 2,968	−10,292	− 7,324	− 51	+13,260	+ 92
3	8,471	− 3,385	1.11	2.55	1.16	− 3,757	+ 3,322	− 3,239	+ 83	+ 1	+ 6,561	+ 46
4	8,548	+ 1,910	1.03	2.38	0.93	+ 1,967	+ 3,592	+ 2,115	+ 5,707	+ 40	+ 1,477	+ 10
5	8,597	+ 5,382	0.96	2.25	0.76	+ 5,167	+ 3,821	+ 6,798	+10,619	+ 74	− 2,977	− 21
6	8,630	+ 7,986	0.91	2.14	0.64	+ 7,267	+ 4,033	+11,355	+15,388	+107	− 7,322	− 51
7	8,652	+ 9,288	0.88	2.09	0.59	+ 8,173	+ 4,140	+13,853	+17,993	+125	− 9,713	− 67
8	8,666	+10,676	0.86	2.05	0.55	+ 9,181	+ 4,227	+16,694	+20,921	+145	−12,467	− 87
9	8,675	+11,371	0.85	2.02	0.53	+ 9,665	+ 4,294	+18,236	+22,530	+156	−13,942	− 97
10	8,679	+11,544	0.83	1.99	0.50	+ 9,581	+ 4,361	+19,163	+23,524	+163	−14,802	−103
11	8,679	+11,544	0.83	1.99	0.50	+ 9,581	+ 4,361	+19,163	+23,524	+163	−14,802	−103
12	8,675	+11,371	0.85	2.02	0.53	+ 9,665	+ 4,294	+18,236	+22,530	+156	−13,942	− 97
13	8,666	+10,676	0.86	2.05	0.55	+ 9,181	+ 4,227	+16,694	+20,921	+145	−12,467	− 87
14	8,652	+ 9,288	0.88	2.09	0.59	+ 8,173	+ 4,140	+13,853	+17,993	+125	− 9,713	− 67
15	8,630	+ 7,986	0.91	2.14	0.64	+ 7,267	+ 4,033	+11,355	+15,388	+107	− 7,322	− 51
16	8,597	+ 5,382	0.96	2.25	0.76	+ 5,167	+ 3,821	+ 6,798	+10,619	+ 74	− 2,977	− 21
17	8,548	+ 1,910	1.03	2.38	0.93	+ 1,967	+ 3,592	+ 2,115	+ 5,707	+ 40	+ 1,477	+ 10
18	8,471	− 3,385	1.11	2.55	1.16	− 3,757	+ 3,322	− 3,239	+ 83	+ 1	+ 6,561	+ 46
19	8,339	−13,280	1.24	2.81	1.60	−16,467	+ 2,968	−10,292	− 7,324	− 51	+13,260	+ 92
20	7,961	−41,490	1.71	3.75	4.04	−70,948	+ 2,123	−17,561	−15,438	−107	+19,684	+137

+ Compression. − Tension.

TABLE XV.—STRESSES DUE TO TEMPERATURE. STEEL.

$$S_{st} = \left\{ \frac{T_t}{A_c + n A_s} \pm \frac{M_t x_2}{\frac{1}{12} D^3 + n A_s x_2^2} \right\} 20 = \left\{ \frac{T_t}{\alpha} \pm \frac{M_t x_2}{\beta} \right\} 20 \qquad \alpha = A_c + n A_s. \quad \beta = \frac{1}{12} D^3 + n A_s x_2^2.$$

Points	1	2	3	4	5	6	7	8	9	10	11	12
	T_t	M_t	x_2	α	β	$M_t x_2$	$\frac{T_t}{\alpha}$	$\frac{M_t x_2}{\beta}$	(Col. 7+ Col. 8) 20 Lbs. per Sq. Foot.	Col. 9÷144 Pounds per Sq. Inch	(Col. 7− Col. 8) 20 Lbs. per Sq. Foot	Col. 11÷144 Pounds per Sq. Inch
1	7,961	−41,490	1.46	3.75	4.04	−60,575	+2,123	−14,994	−257,420	−1,788	+342,340	+2,377
2	8,339	−13,280	0.99	2.81	1.60	−13,147	+2,968	− 8,217	−104,980	− 729	+223,700	+1,553
3	8,471	− 3,385	0.86	2.55	1.16	− 2,911	+3,322	− 2,510	+ 16,240	+ 113	+116,640	+ 810
4	8,548	+ 1,910	0.78	2.38	0.93	+ 1,490	+3,592	+ 1,602	+103,880	+ 721	+ 39,800	+ 276
5	8,597	+ 5,382	0.71	2.25	0.76	+ 3,821	+3,821	+ 5,028	+176,980	+1,229	− 24,140	− 168
6	8,630	+ 7,986	0.66	2.14	0.64	+ 5,271	+4,033	+ 8,236	+245,380	+1,704	− 84,060	− 584
7	8,652	+ 9,288	0.63	2.09	0.59	+ 5,851	+4,140	+ 9,918	+281,160	+1,953	−115,560	− 803
8	8,666	+10,676	0.61	2.05	0.55	+ 6,512	+4,227	+11,841	+321,360	+2,232	−152,280	−1,058
9	8,675	+11,371	0.60	2.02	0.53	+ 6,823	+4,294	+12,873	+343,340	+2,384	−171,580	−1,192
10	8,679	+11,544	0.58	1.99	0.50	+ 6,695	+4,361	+13,391	+355,040	+2,466	−180,600	−1,254
11	8,679	+11,544	0.58	1.99	0.50	+ 6,695	+4,361	+13,391	+355,040	+2,466	−180,600	−1,254
12	8,675	+11,371	0.60	2.02	0.53	+ 6,823	+4,294	+12,873	+343,340	+2,384	−171,580	−1,192
13	8,666	+10,676	0.61	2.05	0.55	+ 6,512	+4,227	+11,841	+321,360	+2,232	−152,280	−1,058
14	8,652	+ 9,288	0.63	2.09	0.59	+ 5,851	+4,140	+ 9,918	+281,160	+1,953	−115,560	− 803
15	8,630	+ 7,986	0.66	2.14	0.64	+ 5,271	+4,033	+ 8,236	+245,380	+1,704	− 84,060	− 584
16	8,597	+ 5,382	0.71	2.25	0.76	+ 3,821	+3,821	+ 5,028	+176,980	+1,229	− 24,140	− 168
17	8,548	+ 1,910	0.78	2.38	0.93	+ 1,490	+3,592	+ 1,602	+103,880	+ 721	+ 39,800	+ 276
18	8,471	− 3,385	0.86	2.55	1.16	− 2,911	+3,322	− 2,510	+ 16,240	+ 113	+116,640	+ 810
19	8,339	−13,280	0.99	2.81	1.60	−13,147	+2,968	− 8,217	−104,980	− 729	+223,700	+1,553
20	7,961	−41,490	1.46	3.75	4.04	−60,575	+2,123	−14,994	−257,420	−1,788	+342,340	+2,377

\+ Compression. − Tension.

TABLE XVI.—LOCATION OF FIBER STRESS (UPPER OR LOWER FIBER) DUE TO TEMPERATURE.

	When *k a is* +		When *k a* is −	
	Upper Fiber.	Lower Fiber.	Upper Fiber.	Lower Fiber.
Rise in Temperature..............	Tension.	Compression.	Compression.	Tension.
Fall in Temperature..............	Compression.	Tension.	Tension.	Compression.

Points	*ka* Algebraic Sign	Concrete.				Steel.			
		+ 40° F. (Rise)		− 40° F. (Fall)		+ 40° F (Rise)		− 40° F. (Fall)	
		Upper Fiber	Lower Fiber	Upper Fiber	Lower Fiber	Upper Fiber	Lower Fiber	Upper Fiber	Lower Fiber
1	−	+137	−107	−107	+137	+2,377	−1,788	−1,788	+2,377
2	−	+ 92	− 51	− 51	+ 92	+1,553	− 729	− 729	+1,553
3	−	{ + 46 + 1	0	0	{ + 46 + 1	{ + 810 + 113	0	0	{ + 810 + 113
4	+	0	{ + 40 + 10	{ + 40 + 10	0	0	{ + 721 + 276	{ + 721 + 276	0
5	+	− 21	+ 74	+ 74	− 21	− 168	+1,229	+1,229	− 168
6	+	− 51	+107	+107	− 51	− 584	+1,704	+1,704	− 584
7	+	− 67	+125	+125	− 67	− 803	+1,953	+1,953	− 803
8	+	− 87	+145	+145	− 87	−1,058	+2,232	+2,232	−1,058
9	+	− 97	+156	+156	− 97	−1,192	+2,384	+2,384	−1,192
10	+	−103	+163	+163	−103	−1,254	+2,466	−2,466	−1,254
11	+	−103	+163	+163	−103	−1,254	+2,466	+2,466	−1,254
12	+	− 97	+156	+156	− 97	−1,192	+2,384	+2,384	−1,192
13	+	− 87	+145	+145	− 87	−1,058	+2,232	+2,232	−1,058
14	+	− 67	+125	+125	− 67	− 803	+1,953	+1,953	− 803
15	+	− 51	+107	+107	− 51	− 584	+1,704	+1,704	− 584
16	+	− 21	+ 74	+ 74	− 21	− 168	+1,229	+1,229	− 168
17	+	0	{ + 40 + 10	{ + 40 + 10	0	0	{ + 721 + 276	{ + 721 + 276	0
18	−	{ + 46 + 1	0	0	{ + 46 + 1	{ + 810 + 113	0	0	{ + 810 + 113
19	−	+ 92	− 51	− 51	+ 92	+1,553	− 729	− 729	+1,553
20	−	+137	−107	−107	+137	+2,377	−1,788	−1,788	+2,377

+ Compression. − Tension.

	ka is +	
	Upper fiber	Lower fiber
Rise 40° F.	Tension − 168	Compression + 1,229
Fall 40° F.	Compression + 1,229	Tension − 168

Since the liability of making errors in conducting these calculations is great it will be found imperative that the work be executed by means of a system of tables identical in form with those set forth in this text.

In Table XVII we have combined the stresses in order to obtain the maximum fiber stresses including temperature stresses. From these results it is seen that the depth of the arch can be somewhat reduced without exceeding the allowable fiber stresses for either steel or concrete. It is seen that the maximum stresses for the arch as designed occur at point 9 where the stress for concrete is 394 lbs., and for steel 6,724 lbs. Hence $f_s \div f_c = 17$.

$$f_s \div f_c \text{ at point } 14 = 5478 \div 307 = 17$$
$$f_s \div f_c \text{ at point } 1 = 4740 \div 258 = 18$$

These values are very close and sufficiently near $f_s \div f_c = 20$ to make a very efficient and satisfactory design.

In revising the arch no change whatever should be made in the intrados curve, but the extrados curve alone should be modified. Since the maximum stress occurs at point 9 it is not advisable to lessen the arch depth at the crown, although even here the depth could probably be cut down to 18 inches without exceeding the allowable stresses. It is safer, however, to be well within the allowable limits, hence we will leave the arch depth at the crown at 20 inches. At point 1, we will decrease the depth from 41.2 inches to 38 inches and construct the circular extrados arc accordingly. Furthermore we will modify the disposition of the steel reinforcement, changing the

TABLE XVII.—STRESSES COMBINED TO OBTAIN MAXIMUM FIBER STRESS.

Fiber Stresses in Pounds per Square Inch. Revision of Arch Depth.

Points	Concrete		Steel		Stresses due to Change in Temperature.								Maximum Fiber Stresses Including Temperature Stresses (in pounds per square inch)				Assumed Depth of Arch		Adopted Depth in inches
					Concrete.				Steel.				Concrete		Steel				
					+40° F. (Rise)		−40° F. (Fall)		+40° F. (Rise)		−40° F. (Fall)								
	Upper Fiber Extrados	Lower Fiber Intrados	Upper Fiber Extrados	Lower Fiber Intrados	Upper Fiber Extrados	Lower Fiber Intrados	Upper Fiber Extrados	Lower Fiber Intrados	Upper Fiber Extrados	Lower Fiber Intrados	Upper Fiber Extrados	Lower Fiber Intrados	Upper Fiber Extrados	Lower Fiber Intrados	Upper Fiber Extrados	Lower Fiber Intrados	In Feet	Equivalent in Inches	
1	+121	+78	+2,363	+1,629	+137	−107	−107	+137	+2,377	−1,788	−1,788	+2,377	+258	−29	+4,740	−159	3.43	41.2	38.0
2	+187	+61	+3,485	+1,477	+92	−51	−51	+92	+1,553	−729	−729	+1,553	+279	....	+5,038	....	2.49	29.9	29.0
3	+157	+112	+3,047	+2,345	+46	0	0	+46	+810	0	0	+810	+203	....	+3,657	....	2.23	26.8	26.0
4	+143	+143	+2,859	+2,859	0	+40	+40	0	0	+721	+721	0	+183	+183	+3,580	+3,580	2.06	24.7	24.0
5	+129	+172	+2,703	+3,334	−21	+74	+74	−21	−168	+1,229	+1,229	−168	....	+246	....	+4,563	1.93	23.2	23.0
6	+96	+221	+2,261	+4,074	−51	+107	+107	−51	−584	+1,704	+1,704	−584	....	+328	....	+5,778	1.82	21.8	22.0
7	+112	+212	+2,517	+3,962	−67	+125	+125	−67	−803	+1,953	+1,953	−803	....	+337	....	+5,915	1.77	21.2	21.0
8	+86	+244	+2,174	+4,425	−87	+145	+145	−87	−1,058	+2,232	+2,232	−1,058	....	+389	....	+6,657	1.73	20.8	20.6
9	+97	+238	+2,350	+4,340	−97	+156	+156	−97	−1,192	+2,384	+2,384	−1,192	....	+394	....	+6,724	1.70	20.4	20.3
10	+108	+231	+2,529	+4,255	−103	+163	+163	−103	−1,254	+2,466	+2,466	−1,254	....	+394	....	+6,721	1.67	20.0	20.0
11	+108	+231	+2,522	+4,248	−103	+163	+163	−103	−1,254	+2,466	+2,466	−1,254	....	+394	....	+6,714	1.67	20.0	20.0
12	+118	+216	+2,649	+4,027	−97	+156	+156	−97	−1,192	+2,384	+2,384	−1,192	....	+372	....	+6,411	1.70	20.4	20.3
13	+112	+218	+2,543	+4,043	−87	+145	+145	−87	−1,058	+2,232	+2,232	−1,058	....	+363	....	+6,275	1.73	20.8	20.6
14	+182	+142	+3,525	+2,948	−67	+125	+125	−67	−803	+1,953	+1,953	−803	+307	....	+5,478	....	1.77	21.2	21.0
15	+183	+134	+3,519	+2,822	−51	+107	+107	−51	−584	+1,704	+1,704	−584	+290	....	+5,223	....	1.82	21.8	22.0
16	+207	+95	+3,846	+2,203	−21	+74	+74	−21	−168	+1,229	+1,229	−168	+281	....	+5,075	....	1.93	23.2	23.0
17	+208	+80	+3,841	+1,913	0	+40	+40	0	0	+721	+721	0	+248	....	+4,562	....	2.06	24.7	24.0
18	+207	+65	+3,821	+1,615	+46	0	0	+46	+810	0	0	+810	+253	....	+4,631	....	2.23	26.8	26.0
19	+178	+74	+3,358	+1,684	+92	−51	−51	+92	+1,553	−729	−729	+1,553	+270	....	+4,911	....	2.49	29.9	29.0
20	+48	+159	+1,115	+3,021	+137	−107	−107	+137	+2,377	−1,788	−1,788	+2,377	−59	+296	−673	+5,398	3.43	41.2	38.0

+ Compression. − Tension.

distance between centers of steel ribs from 8 ins. to 12 ins. For the ribs at 12 in. centers,

$$A_s = \frac{0.01068}{1} = 0.01068$$

$$n\,A_s = 20\ (0.01068) = 0.212$$

$$38 \text{ ins.} = 3.17 = D, \text{ hence}$$

$$x_1 = \frac{D}{2} = \frac{3.17}{2} = 1.58$$

$$x_2 = \frac{D - 0.50}{2} = \frac{2.66}{2} = 1.33$$

Hence recalculating the stresses for point 1;

$$S_c = \frac{53{,}900}{3.17 + 0.212} \pm \frac{+\,7{,}304\ (1.58)}{\frac{1}{12}\,(3.17)^3 + 0.212\ (1.33)^2}$$

$$= 15{,}937 \pm 3{,}809$$

Hence for concrete we obtain,

+ 19,746 lbs. per sq. ft. = + 137 lbs. per sq. in., upper fiber, and

+ 12,128 lbs. per sq. ft. = + 84 lbs. per sq. in., lower fiber.

$$S_s = \left(\frac{53{,}900}{3.17 + 0.212} \pm \frac{+\,7{,}304\ (1.33)}{\frac{1}{12}\,(3.17)^3 + 0.212\,(1.33)^2}\right) 20$$

$$= (15{,}937 \pm 3{,}206)\ 20$$

Hence for steel we obtain,

+ 382,860 lbs. per sq. ft. = + 2,659 lbs. per sq. in., upper fiber.

+ 254,620 lbs. per sq. ft. = + 1,768 lbs. per sq. in., lower fiber.

Temperature Stresses. Revised Point 1.

$$S_{ct} = \frac{7{,}961}{3.17 + 0.212} \pm \frac{-\,41{,}490\ (1.58)}{\frac{1}{12}\,(3.17)^3 + 0.212\ (1.33)^2}$$

Hence for concrete we have,

− 19,281 lbs. per sq. ft. = − 134 lbs. per sq. in.

+ 23,989 lbs. per sq. ft. = + 167 lbs. per sq. in.

$$S_{st} = \left(\frac{7{,}961}{3.17 + 0.212} \pm \frac{-\,41{,}490\ (1.33)}{\frac{1}{12}\,(3.17)^3 + 0.212\ (1.33)^2}\right) 20$$

Hence for steel we get,

− 317,160 lbs. per sq. ft. = − 2,203 lbs. per sq. in.

+ 411,320 lbs. per sq. ft. = + 2,857 lbs. per sq. in.

We then obtain the following maximum stress including temperature stress:

Concrete.		Steel.	
Extrados.	Intrados.	Extrados.	Intrados.
+ 304	− 50	+ 5516	− 435

For this value $f_s + f_c = 18$.

It is often well to recalculate a number of points for the revised system. This is not necessary in the present case, except at point 1.

20. **Concluding Remarks.**—It is well to provide the arch with steel reinforcing rods running transversely. Shear bars should also be provided for to take up the normal stresses. In general the transverse bars need not be of as large cross section as the longitudinal bars. The same is true for the shear bars. It is generally sufficient to space both transversals and shear bars 2 ft. center to center. The shear bars should be placed at points of greatest shearing stresses, and preferably along the entire arch.

Where bars must be spliced the following formula will give the length that the bars should be overlapped for the splice:

Let

d = diameter of rod or maximum diagonal of cross-section.

S = length of splice,

then, for plain bars of medium steel,

$$S = 40\,d \qquad (142)$$

For plain bars of high elastic limit steel,

$$S = 70\,d \qquad (143)$$

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APPENDIX A.

GENERAL SPECIFICATIONS FOR CONCRETE-STEEL STRUCTURES.

(Melan, Thacher, and Von Emperger Patents.)

(The Concrete-Steel Engineering Company, New York.)

1. PLANS.—The work will be constructed in accordance with the drawings, herewith submitted, and these specifications.

The specifications and drawings are intended to describe and provide for the complete work. They are intended to be co-operative, and what is called for by either is as binding as if called for by both.

The work herein described is to be completed in every detail, notwithstanding that every item necessarily involved is not particularly mentioned.

The contract price shall be based upon these specifications and drawings, which are hereby made a part of the contract.

2. LOADS.—DEAD LOAD. In estimating dead loads, concrete shall be assumed at 150 lbs. per cu. ft., earth fill at 120 lbs. per cu. ft., ballast, including ties, at 120 lbs. per cu. ft., rails at 60 lbs. per lin. ft. of track. The assumed weight of pavements shall depend on the kind of pavement used, but in the absence of full data the pavement shall be assumed at 150 lbs. per sq. ft. and to occupy a depth of 12 ins.

LIVE LOAD. For bridges and subways carrying railroad or electric car traffic, the live load shall be such as is specified by the company for which the structure is built, and the load on each track shall be assumed as distributed over a width equal to the distance from center to center of tracks. The greatest equivalent load per lineal foot that can come on any span or half span of arch, or span of girder, with all tracks loaded, being used in the calculation.

Bridges and subways carrying highway traffic shall be proportioned to carry a uniformly distributed load of 125 lbs. upon each square foot of roadway and sidewalks, or the following concentrated loads:

(a) For city and suburban bridges, 15 ton steam road roller 11 ft. between axles, 6 tons on forward wheel 4 ft. wide, and 4.5 tons on each of two rear wheels 5 ft. between centers and 20 ins. wide.

(b) For country bridges 5 tons on four wheels 8 ft. between axles and 6 ft. gage.

3. CONDITIONS OF CALCULATIONS.—

Modulus of elasticity of concrete................. 1,500,000 lbs.
Modulus of elasticity of steel......................30,000,000 "

MAXIMUM COMPRESSION ALLOWED ON CONCRETE.

In arches for highway and electric car bridges.

Exclusive of temperature stresses............500 lbs. per sq. in.

Including stresses due to 40° variation in temperature ....600 " " " "

In arches for railway bridges.

Exclusive of temperature stresses............400 " " " "

Including stresses due to 40° variation in temperature ....500 " " " "

Slabs, Girders, Beams, Floors, Walls and Posts.

Subways and girder bridges, carrying highway and electric car traffic, also buildings, roofs, culverts and sewers, shall have a factor of safety of 4 in one month.

Subways and girder bridges for railways shall have a factor of safety of 5 in one month.

MAXIMUM TENSION ALLOWED ON CONCRETE.

In arches, exclusive of temperature stresses... 50 lbs. per sq. in.

In arches, including stresses due to 40° variation in temperature........................ 75 " " " "

In slabs, girders, beams, floors, walls and posts 0 " " " "

MAXIMUM SHEAR ALLOWED ON CONCRETE....... 75 " " " "

MAXIMUM STRESS ALLOWED ON STEEL.

In arches. The steel ribs under a stress not exceeding 18,000 lbs. per sq. in. must be capable of taking the entire bending moment of the arch without aid from the concrete, and have flange areas of not less than the one hundred and fiftieth part of the total area of the arch at crown. The actual stress when imbedded in and acting in combination with concrete shall not exceed twenty times the allowed stress on the concrete.

In slabs, girders, beams, floors, and walls, subjected to transverse stress. The steel shall be assumed to take the entire tensile stress without aid from the concrete, and shall have an area sufficient to equal the compressive strength of concrete composed of 1 part Portland cement, 3 parts sand, and 6 parts of broken stone, of the age of six months.

In walls and posts subjected to compression only. No allowance will be made for the strength of imbedded steel, which will be used only as a precaution against cracks due to shrinkage or changes in temperature.

In tanks. The imbedded steel under a stress not exceeding 15,000 lbs. per sq. in. shall be capable of taking the entire water pressure without aid from the concrete.

4. DISCREPANCIES.—In the event of any discrepancies between the drawings and the figures written on them, the figures are to be taken as correct, and in case of any discrepancy between the drawings and the specifications, the specifications are to be adhered to.

5. FOUNDATIONS.—All foundations shall be as shown on plans, and conform to the dimensions marked thereon.

Foundations on rock shall be prepared by removing all sand, mud, or other soft material, and by excavating the bed rock in such manner as may be described or shown on drawings.

Foundations on hard pan, gravel, gravel and clay, cemented sand, or other material intended to carry the load without piles, shall be excavated to the depths shown on plans.

For deep foundations requiring pneumatic caissons, the caissons shall be of such size and be sunk to such depth as is shown on drawing, and they shall be built in accordance with the detailed plans and specifications specially provided.

Foundations on piles will usually be inclosed by permanent water-tight tongue and grooved sheet piling, or Wakefield piling, and be excavated to the depth shown on plans, and the piles shall be driven after the excavations are made. They shall be sawed off at least 2 ft. below low water. The sheet piling will remain in place and be sawed off at least 1 ft. below low water. The spaces between the piles shall be filled with concrete, and in case it is found necessary to lay the concrete under water, proper appliances must be used to insure its being deposited with as little injury as possible.

If timber cribs with compartments filled with concrete are used on pile foundations, the bed of stream shall be dredged, removing all mud or other soft material, after which piles shall be driven and sawed off at the elevation shown on plans. All piles in the same foundation shall be sawed off at the same level, giving a solid and uniform bearing for the crib, which may extend to within 3 ft. of low water.

The piles shall be oak, yellow pine, or other wood that will stand the blow of the hammer; straight, sound, and cut from live timber; trimmed close, cut off square at the butt, and have all bark taken off. The piles shall not be less than 12 ins. nor more than 16 ins. in diameter at the large end nor less than 8 ins. in diameter at the small end for piles having a length of 30 ft. and under. For greater lengths the diameter of the small end may be reduced 1 in. for each 10 ft. of additional length down to a minimum of 6 ins. The piles shall be driven until they do not penetrate more than ½ in. under the blow of a hammer weighing 2,240 lbs. falling 25 ft.

6. CEMENT.—No cement will be allowed to be used except established brands of high grade Portland cement which have been in successful use under similar conditions to the work proposed for at least three years, and have been seasoned or subjected to aeration for at least 30 days before leaving the factory. All cement shall be dry and free from lumps, and immediately upon receipt shall be stored in a dry, well covered and ventilated place, thoroughly protected from the weather. (If required the contractor shall furnish a certified statement of the chemical composition of the cement, and of the raw material from which it is manufactured.)

The fineness of the cement shall be such that at least 90 per cent will pass through a sieve of No. 40 wire, Stubbs gâge, having 10,000 openings per square inches, and at least 75 per cent will pass through a sieve of No. 45 wire, Stubbs gage, having 40,000 openings per square inch.

Samples for testing may be taken from every bag or barrel, but usually for tests of 100 barrels a sample will be taken from every tenth barrel. The samples will be mixed thoroughly together while dry and the mixture be taken as the sample for test.

Tensile tests will be made on specimens prepared and maintained until tested at a temperature not less than 60° Fahrenheit. Each specimen will have an area of 1 sq. in. at the breaking section, and after being allowed to harden in moist air for 24 hours will be immersed and maintained under water until tested.

The sand used in preparing test specimens shall be clean, sharp, crushed quartz retained on a sieve of 30 meshes per lineal inch, and passing through a sieve of 20 meshes per lineal inch. In test specimens of 1 cement and 3 sand, no more than 12 per cent of water by weight shall be used. Specimens prepared from a mixture of 1 part cement and 3 parts sand, parts by weight, shall after seven days develop a tensile strength of not less than 170 lbs. per sq. in., and not less than 240 lbs. per sq. in. after 28 days. Cement mixed neat with from 20 per cent to 25 per cent of water to form a stiff paste shall after 30 minutes be appreciably indented by the end of a wire 1/12 in. in diameter loaded to weigh ¼ lb. Cement made into thin pats on glass plates shall not crack, scale, nor warp under the following treatment: Three pats will be made and allowed to harden in moist air at from 60° to 70° Fahrenheit; one of these will be placed in fresh water for 28 days, another will be placed in water which will be raised to the boiling point for six hours and then allowed to cool, and the third is to be kept in air of the prevailing outdoor temperature.

7. PORTLAND CEMENT CONCRETE.—The concrete shall be composed of cement, sand, and broken stone or gravel mixed with clean water in the proportions hereafter mentioned.

The sand shall be clean, sharp, and coarse, or coarse and fine mixed, free from sewage, mud, clay, and all foreign matter.

The broken stone shall be clean and hard, broken into approximately cubical pieces, and free from long, thin scales.

The gravel shall be of assorted sizes screened or washed entirely free from clay, loam, or foreign matter, and be free from scale, slime, or humus.

Whenever the amount of work to be done is sufficient to justify it, and for all work exceeding 1,000 cu. yds., approved mixing machines shall be used. The ingredients shall be placed in the machine in a dry state, and in the volumes specified, and be thoroughly mixed, after which clean water shall be added and the mixing continued until the wet mixture is thorough and the mass uniform. The mixture shall be sufficiently wet for the water to come to the surface with moderate ramming. As soon as the batch is mixed it must be deposited in the work without delay. For small bridges, if the mixing is done by hand, the cement and sand shall first be thoroughly mixed dry, in the proportions specified. The stone, previously drenched with water, shall then be deposited in this mixture. Clean water shall be added and the mass be thoroughly mixed and turned over until each stone is covered with mortar, and the batch be deposited without delay.

The concrete shall be deposited in layers of 6 or 8 ins., and be thoroughly rammed until all voids are filled and the water flushes to the surface.

The grades of concrete to be used are as follows:

(a) For the arches, slabs, girders, beams, floors, walls subject to transverse stress, posts and tanks, 1 part Portland cement, 2 parts sand, and 4 parts broken stone that will pass in any direction through a 1½ in. ring, if not otherwise marked on plans.

(b) For spandrel walls, 1 part Portland cement, 3 parts sand, and 6 parts broken stone or gravel that will pass through a 2-in. ring.

(c) For the piers, abutments, foundations, and retaining walls, 1 part Portland cement, 3½ parts sand, and 7 parts broken stone or gravel that will pass through a 3 in. ring.

8. ARTIFICIAL STONE.—(a) All keystones, brackets, consoles, dentiles, pedestals, parapets, hand railings, posts and panels, and other ornamental work when used; also curbs and gutters, shall be of the design shown on plans, and be molded in smooth and suitable molds. For moldings containing curved surfaces, sharp curves, carvings, or other delicate work, the molds shall be plastered with a semi-liquid mortar composed of 1 part cement and 2 parts of fine sharp sand. The mortar coating must be followed up with a backing of only earth damp concrete composed of 1 part cement, 2 parts sand, and 4 parts of fine broken stone, or 1 part cement and 6 parts of gravel that will pass through a ¾ in. ring. The concrete backing must be rammed thoroughly in thin layers.

(b) For plain flat surfaces, the concrete may be rammed directly against the molds, and after the molds have been removed all exposed surfaces shall be floated to a smooth finish with a mortar the same as specified in § 8 (a), care being taken that no body of mortar is left on the face, sufficient only being used to fill the pores and give a smooth finish.

When pedestal posts carry lamp posts, a 4 in. wrought iron pipe shall be built into the concrete from top to bottom, and at bottom it shall be connected with a 3 in. pipe extending under the sidewalk, and connected with gas pipe or electric wire conduit. The pipes shall have no sharp bends, all changes in direction being made by gentle curves.

9. PLASTERING.—No plastering will be allowed on the exposed faces of the work, but the inside faces of the spandrel walls covered by the fill shall be plastered with mortar composed of 1 part cement and 2½ parts sand, the surface being well dampened before plastering.

10. MIXTURES.—The volumes of cement, sand, broken stone, or gravel in all mixtures of mortar or concrete shall be measured loose.

11. CONNECTIONS.—In connecting concrete already set with new concrete, the surface shall be cleaned and roughened, and mopped with a mortar composed of 1 part cement and 1 part sand to cement the parts together.

12. EXPANSION JOINTS.—Expansion joints shall be made in the spandrel walls, cornices, and parapets of each arch above the springing lines, at points one-sixth span from the springing lines, and at such other points, if any, as are shown on plan.

13. SPANDRELS.—The spandrel walls shall have a thickness of not less than 18 ins. at any point, and a thickness at bottom of not less than four-tenths of the height of the wall measured from the top of cornice.

14. ARCHES.—For square arches the concrete shall be laid in transverse sections of the full width of the arch, between timber forms normal to the center line of the arch, the length of sections being such that the center section, or a pair of intermediate or end sections, shall constitute a day's work. Work shall be started at the center section and carried towards the ends, the end sections being laid last.

For skew arches, the concrete shall be started simultaneously from both ends of the arch, and be built in longitudinal sections at least 5½ ft. in width, and wide enough to constitute a day's work. The concrete shall be deposited in layers, each layer being well rammed in place before the previously deposited layer has had time to partially set. The work shall proceed continuously day and night if necessary to complete each longitudinal section. These sections while being built shall be held in place by substantial vertical timber forms, parallel to the face of the arch and to each other, and these forms shall be removed when the section has set sufficiently to admit of it. The sections shall be connected as specified in paragraph 11, and also by steel clamps spaced about 5 ft. apart, connecting the adjacent steel ribs.

15. DRAINAGE.—Provision for drainage shall be made at each pier as follows: A wrought iron pipe of sufficient diameter shall be built into the concrete, extending from the center of each space over piers to the soffit of the arch near the springing line, and project 1 in. below the soffit. The surface of the concrete over piers shall be so formed that any water that may seep through the fill above will be drained to the pipes. The line of drainage will be covered with a layer of broken stone, and the top of pipes will be provided with screens to prevent clogging.

16. STEEL.—Steel ribs shall be imbedded in the concrete of the arches. They shall be spaced at equal distances apart. The design, location, dimensions, and connections of the ribs, also the sections of steel of which they are composed, shall be as shown on plans.

Steel rods shall be imbedded near the tension side of all members subjected to transverse stress. No reliance will be placed on the adhesion between the steel and the concrete, but our patented rods, specially designed for this purpose, shall be used in all cases. The distance of the center of the rods from the outside of the concrete shall not be less than the diameter of the rods. All steel must be free from paint and oil, and all scale and rust must be removed before imbedding in the concrete.

The tensile strength, limit of elasticity and ductility shall be determined from a test piece cut from the finished material and turned and planed parallel. The area of cross section shall not be less than ½ sq. in.; the elongation shall be measured after breaking on an original length of 8 ins. Each melt shall be tested for tension and bending.

Either soft or medium steel may be used in all concrete steel structures. If soft steel is used it shall have an ultimate strength of from 54,000 to 62,000 lbs. per sq. in., an elastic limit of not less than one-half the ultimate strength, shall elongate not less than 25 per cent in 8 ins., and bend cold 180° flat on itself without fracture on outside of bend. If medium steel is used it shall have an ultimate strength of from 60,000 to 68,000 lbs. per sq. in., an elastic limit of not less than one-half the ultimate strength, shall elongate not less than 22 per cent in 8 ins., and bend cold 180° to a diameter equal to the thickness of the piece tested without fracture on outside of bend. In tension tests the fracture must be entirely silky. The workmanship must be first class.

17. CASING.—When concrete facing is used, all piers, abutments, and spandrel walls shall be built in timber forms. These forms shall be substantial and unyielding, of the proper dimensions for the work intended, and all parts in contact with exposed faces of concrete shall be finished to a perfectly smooth surface by plastering or other means, so that no mark or imperfection shall be left on the work.

18. CONCRETE FACING.—If concrete facing is used, the concrete shall be deposited in smooth molds as specified in paragraph 17, and after the molds have been removed the exposed flat surfaces shall be finished in the same manner as specified in paragraph 8 (b).

If the arch faces, quoins, or other exposed surfaces are marked to represent masonry or other division lines, either straight or curved, are shown in the faces of the arch or spandrels, such division lines shall be made by triangular moldings of wood 2 ins. wide and 1 in. deep, fastened to the casing in true lines as shown on plans. The face of the arch at intrados shall be beveled to correspond, and all angles or intersections of the moldings shall be neatly beveled and fitted in a workmanlike manner to give a smooth finish. Before depositing the concrete the moldings shall be coated in the same manner as specified in paragraph 8 (a).

The soffits of the arches shall be floated and finished in the same manner as specified in paragraph 8 (b).

19. OTHER FACING.—If ashlar masonry, boulder, brick, terra cotta, or other facing is used on the work, it will be shown or noted on the drawings, and a specification therefor will be attached.

20. CENTERING.—The contractor shall build an unyielding falsework or centering. The lagging shall be dressed to a uniform thickness so that when laid it shall present a smooth surface, or it shall be made smooth by plastering or other efficient means.

In framing the centers allowance shall be made for settlement of centerings, deflection of arch after the removal of centerings and for permanent cambre. The centers shall be framed for a rise of arch greater than the rise marked on drawings by an amount equal to one-eight hundredth part of the span, and shall not be struck until at least 28 days after the completion of the arch, and not until the fill has been put on. Great care shall be used in lowering the centers evenly and uniformly, preferably by means of sand boxes, so as not to throw undue strains upon the arches. The tendency of the centers to rise at the crown as they are loaded at the haunches must be provided for in the design, or, if not, the centers must be temporarily loaded at the crown and the load so regulated as to prevent distortion of the arch as the work progresses.

21. WATER-PROOFING.—After the completion of the arches and spandrels, and before any fill is put in, the top surface of the arches, piers and abutments, and the lower 6 ins. of the inner surface of the spandrel walls shall be coated with a heavy coat of semi-liquid mortar consisting of 1 part cement, ½ part thoroughly slaked lime, and 3 parts sand, spread to leave a smooth finish, and after this has set hard it shall be given a heavy coat of pure cement grout.

22. FILL.—The space between the spandrel walls shall be filled with sand, earth, cinders, or other suitable material, thoroughly compacted by ramming or rolling, and be finished to the proper grade to receive the curbing and pavement.

The fill over any arch shall not be put in until at least two weeks after the arch concrete has been completed.

23. ROADWAY PAVEMENT.—The pavement shall be of the kind shown on plans, or mentioned in the proposal, and shall be built according to the specifications adopted in the locality where used unless otherwise mentioned.

24. CONCRETE SIDEWALKS.—The ground on which the concrete sidewalk is to be laid shall be rammed or rolled to a hard bearing surface, 10 ins. below the finished grade. After the curbing has been set true to line and grade, a foundation of gravel or cinders shall be laid and thoroughly rammed or rolled to a thickness of 6 ins. On this shall be deposited a layer of dry concrete, 3 ins. thick after ramming, consisting of 1 part Portland cement, 2 parts sand, and 4 parts broken stone or gravel that will pass through a 1½ in. ring. On the concrete shall then be laid a wearing surface 1 in. thick, composed of 1 part Portland cement and 1½ parts of coarse sharp sand or of broken granite or other acceptable stone in size from ⅜ in. downward. The mortar for the wearing surface shall not be too wet; it must be spread before the concrete base has had time to partially set, shall be pressed down hard into the latter, and shall be troweled to a smooth and even surface. All concrete sidewalks shall be divided into blocks of not more than 36 sq. ft. All blocks shall be separated from those adjoining by pieces of heavy tar paper or other effective means to prevent adhesion of the blocks. The divisions between blocks shall reach entirely through the concrete and the wearing surface, and

shall be neatly finished on top with a jointing tool. As soon as the wearing surface has well set a 2 in. layer of sand shall be carefully spread over it and kept moist for one week by frequent sprinkling.

Concrete curbs shall be divided into blocks corresponding with those forming the sidewalk, and shall be neatly finished in the same manner.

25. HAND RAILING AND PARAPETS.—The hand railing or parapets shall be of the material and of the form and dimensions shown on plans, and shall be brought true to line, and be firmly fastened in the position shown. If an iron hand railing is used, it shall receive, after erection, two coats of paint of a color and quality approved by the engineer. Concrete parapets shall be made as specified in paragraph 8 (a). They shall be provided with expansion joints at intervals not greater than 10 ft.

26. LAMP POSTS AND TROLLEY POLES.—If required, and furnished by the contractor, they shall be of a design approved by the engineer. The number of pieces and the minimum cost of each, delivered on the work, shall be specially mentioned.

27. NAME PLATES.—Two name plates shall be furnished by the contractor. They shall be of a design approved by the engineer, and built into the roadway side of the abutment pedestals or such other places as may be directed by the engineer, one plate being inscribed with the names of the city or county officials and year of completion, the other being inscribed with the names of the designers and contractors, and date of patents. If the contract price exceeds $20,000, the plates shall be made of bronze; if it is less than $20,000, the plates may be made of cast iron, in which case they shall receive two coats of bronze paint and two coats of best varnish.

28. ERECTION.—The contractor shall employ suitable and competent labor for every kind of work. The contractor shall furnish all staging, piling, cribbing, centering, casing, and material of every description required in the erection of the work; also all plant, including dredges, engines, pumps, barges, pile drivers, derricks, mixing machines, conveyors, or other appliances necessary for carrying on all parts of work. The contractor shall make all the provisions necessary to maintain and protect buildings, fences, trees, conduits, sewers. and other structures, and shall repair all damage occasioned; shall provide watchmen, red lights, fences, and other precautionary measures necessary to the protection of persons and property. The contractor shall assume all risks for loss or damage incurred by ice, floods, fire, or other causes during the construction of the work, and until the same is accepted.

29. WORK EMBRACED BY CONTRACT.—The contractor shall do all the work prescribed in these specifications and as shown on the plans, for the structure complete from out to out of abutments or retaining walls, including fill, pavement, curbs, sidewalks, and hand railing or parapets for this length, unless otherwise mentioned.

30. APPROACHES.—The approaches will commence where the work mentioned in paragraph 29 ends, and they are not included in the contract, except when specially mentioned.

31. CLEANING UP.—After the completion of the work, and before final acceptance thereof, the contractor shall remove all temporary structures and rubbish, and leave the work and surrounding grounds in a neat and satisfactory condition.

32. MAINTAINING PUBLIC TRAVEL.—If public travel is to be maintained during the construction of the new bridge, by the construction of a temporary bridge, or otherwise, it shall be specially mentioned.

33. REMOVAL OF OLD BRIDGE.—If the site of the proposed structure is occupied by an old bridge, the same shall be removed by the contractor. The iron work shall be piled on the bank, and the timber and stone shall become the property of the contractor.

34. ENGINEER.—Whenever the word "Engineer" is used in these specifications, the same shall mean the Engineer or other public official who may be properly authorized to act for the party who makes this contract with the contractor.

35. LINES AND GRADES.—Lines and grades will be established by the Engineer, and no work shall be commenced until these are given.

36. EXTRA WORK.—The contractor must be prepared to do any extra work that may be ordered in writing by the Engineer, and for this he shall be paid at current contract rates for work of a similar character, or if the extra work should be of a class for which no rate is fixed by current contracts, the actual reasonable cost to the contractor, as determined by the Engineer, plus 15 per cent of said cost. The contractor shall have no claim for compensation for extra work unless the same is ordered in writing by the Engineer.

37. INSPECTION.—All material furnished by the contractor shall be subject to the inspection and approval of the Engineer, and the Engineer shall have power to condemn all work which in his opinion is not done in accordance with this contract and specifications.

38. MODIFICATIONS.—Any modifications of the prescribed lines, grades, positions, methods, or materials of construction which in the judgment of the Engineer may be expedient, shall be made by the contractor.

39. INTERPRETATION OF PLANS AND SPECIFICATIONS. —The decision of the Engineer shall control as to the interpretation of the plans and specifications during the execution of the work thereunder, but this shall not deprive the contractor of his lawful rights to redress after the completion of the work for any improper orders or decisions which may have been received during the execution of the work.

40. ESTIMATES.—Approximate estimates of the work done and material furnished shall be made on or about the last day of every month, and a valuation of the same in proportion to contract prices for the completed work will be made by the Engineer, which sum will be paid to the contractor in cash on or about the 10th day of the following month, less a deduction of 10 per cent upon said valuation, which shall be retained until the final completion of the work.

41. ROYALTY, PLANS, AND SPECIFICATIONS.—These specifications and the accompanying plans contemplate a concrete-steel bridge to be built under the patents owned by the Concrete-Steel Engineering Company of New York. As a compensation for royalty, plans, and specifications the contractor shall include in his bid and pay to the said Concrete-Steel Engineering Company a sum equal to ten per cent (10%) of the total contract price, and to insure this payment the party representing the city, county, or corporation, as the case may be, and who enters into this contract with the contractor, is authorized, and hereby agrees to deduct, retain, and pay for the contractor to the said Concrete-Steel Engineering Company from each monthly estimate, such proportion of the above sum as the monthly estimate bears to the contract price, until the whole amount shall have been so deducted, retained, and paid.

42. FINAL PAYMENT.—Upon the completion of the work, the contractor shall be promptly paid the balance of the contract price which shall then remain due and unpaid.

APPENDIX B.

SPECIFICATIONS FOR REINFORCED CONCRETE STRUCTURES EMBODIED IN THE BUILDING ORDINANCES OF THE CITY OF ST. LOUIS.

(Report of the Special Committee on Reinforced Concrete of the Engineers' Club of St. Louis.)

DEFINITIONS.

1. Reinforced concrete is a concrete in which steel is embodied in such manner that the two act in unison in resisting stresses due to external loading.

2. Concrete is an artificial stone resulting from a mixture of Portland cement, water, and an aggregate.

3. Portland cement shall be as defined in the Standard Specifications, adopted on June 14, 1904, by the American Society for Testing Materials.

4. An aggregate, as herein used, means one or more of the following materials: Sand, broken stone, gravel, hard burned clay. Aggregates will be divided into two classes, Fine Aggregates and Coarse Aggregates. A fine aggregate will include all aggregate passing a No. 8 sieve. A coarse aggregate will include all aggregate passing a 1 in. ring and retained on a No. 8 sieve.

QUALITY OF MATERIALS.

5. Portland cement shall conform to the requirements of the specifications of the American Society for Testing Materials, as adopted June 14, 1904, with all subsequent amendments thereto.

6. Aggregates.—Fine Aggregates shall be well graded in size from the finest to at least the size retained on a No. 10 sieve. Coarse Aggregates shall also be well graded in size from the finest to at least the size retained by a 9/16 in. ring. Fine Aggregates may contain not more than 5 per cent, by weight, of clay, but no other impurities. Coarse Aggregates shall contain no impurities.

7. Sand shall be equal in quality to the Mississippi River sand.

8. Broken stone shall be either limestone, chatts, or granite, or some other stone equal to one of these in the opinion of the Commissioner of Public Buildings.

9. Hard burned clay shall be made from suitable clay free from sand or silt, burned hard and thoroughly. Absorption of water should not exceed 15 per cent.

10. Concrete.—The solid ingredients of the concrete shall be mixed by volume in one of the following proportions:

(a) Not more than three parts Fine Aggregate to one of cement.

(b) Not more than two parts of Fine Aggregate and four parts of

Coarse Aggregate to one of cement; but in all cases the Fine Aggregate shall be 50 per cent of the Coarse Aggregate.

11. Concrete shall have an ultimate strength in compression in 28 days of not less than the following:

Burned clay concrete—1,000 lbs. per sq. in.

All other concrete—2,000 lbs. per sq. in.

12. Steel shall be Medium Steel or High Elastic Limit Steel. The physical properties shall conform to the following limits:

	MEDIUM STEEL.	HIGH ELASTIC LIMIT STEEL.
Elastic Limit,	Not less than 30,000	Not less than 50,000
Percentage of Elongation, min. in 8 ins.,	$E = \frac{1,800,000}{f-10,000} - 10$	$E = \frac{1,800,000}{f-10,000} - 10$
Cold bend without fracture on outer circumference	180° flat.	90° to radius = 5 times thickness
Character of fracture.	silky	silky or fine granular

f = unit stress in steel at rupture.

13. Tests shall be made on specimens taken from the finished bar, and certified copies of test reports shall be furnished the Commissioner of Public Buildings at his request.

14. Bending tests shall be made by pressure.

15. Finished material shall be free from seams, flaws, cracks, defective edges or other defects, and have a smooth, uniform and workmanlike finish, and shall be free from irregularities of all kinds.

16. The net area of cross-section of finished steel members shall not be less than ninety-five per cent (95%) of the area shown in the approved design.

EXECUTION.

17. All reinforced concrete work shall be built in accordance with approved detailed working drawings. These drawings shall be submitted to the Commissioner of Public Buildings for approval and no work shall be commenced until the drawings shall have been approved by him.

18. The steel used for reinforcing concrete shall have no paint upon it, but shall present only a clean or slightly rusted surface to the concrete. All dirt, mud and other foreign matter shall be removed.

19. If the steel has more than a thin film of rust upon its surface it shall be cleaned before placing in the work.

20. In proportioning materials for concrete, one bag containing not less than 93 pounds of cement shall be considered one cubic foot.

21. The ingredients of the concrete shall be so thoroughly mixed that the cement shall be uniformly distributed throughout the mass and that the resulting concrete will be homogeneous.

22. The concrete shall be mixed as wet as possible without causing a separation of the cement from the mixture, and shall be deposited in the work in such manner as not to cause the separation of mortar from coarse aggregate.

23. Concrete shall be placed in the forms as soon as practicable after mixing, and in no case shall concrete be used if more than one hour has

elapsed since the addition of its water. It shall be deposited in horizontal layers not exceeding eight inches in thickness and thoroughly tamped with tampers of such form and material as the circumstances require.

24. The steel shall be accurately placed in the forms and secured against disturbances while the concrete is being placed and tamped, and every precaution shall be taken to insure that the steel occupies exactly the position in the finished work as shown on the drawing.

25. Before the placing of concrete is suspended the joint to be formed shall be in such place and shall be made in such manner as will not injure the strength of the completed structure.

26. Whenever fresh concrete joins concrete that has set, the surface of the old concrete shall be roughened, cleaned and thoroughly slushed with a grout of neat cement and water.

27. No work shall be done in freezing weather, except when the influence of frost is entirely excluded.

28. Until sufficient hardening of the concrete has occurred, the structural parts shall be protected against the effects of freezing, as well as against vibrations and loads.

29. When the concrete is exposed to a hot or dry atmosphere special precautions shall be taken to prevent premature drying by keeping it moist for a period of at least twenty-four hours after it has taken its initial set. This shall be done by a covering of wet sand, cinders, burlap, or by continuous sprinkling, or by some other method equally effective in the opinion of the Commissioner of Public Buildings.

30. If during the hardening period the temperature is continually above 70° F., the side forms of concrete beams and the forms of floor slabs up to spans of eight feet shall not be removed before four days, the remaining forms and supports not before ten days from the completion of tamping.

31. If during the hardening period the temperature falls below 70° F., the side forms of concrete beams and the forms of floor slabs up to spans of eight feet shall not be removed before seven days; the remaining forms and the supports not before fourteen days from the completion of the tamping. But if, during the hardening period, the temperature falls below 35° F., the time for hardening shall be extended by the time during which the temperature was below 35° F.

32. Forms for concrete shall be sufficiently substantial to preserve their accurate shape until the concrete has set, and shall be sufficiently tight so as not to permit any part of the concrete to leak out through cracks or holes.

33. Before placing the concrete, the inside of the forms shall be thoroughly cleaned of all dirt and rubbish, the forms of all beams, girders and columns being constructed with a temporary opening in the bottom for this purpose.

34. If loading tests are considered necessary by the Commissioner of Public Buildings, they shall be made in accordance with his instructions,

but the stresses induced in all parts of a structural member by its test load shall be the same as if the member were subjected to twice the dead load plus twice the assumed load.

35. All tests of material herein required shall be made by testing laboratories of recognized standing, and certified copies of such test reports shall be filed with the Commissioner of Public Buildings.

DESIGN.

36. The weight of burned clay concrete, including the steel reinforcement, shall be taken at 120 lbs. per cu. ft.

37. The weight of all other concrete, including the reinforcement, shall be taken at 150 lbs. per cu. ft.

38. Besides the above, in calculating the dead loads, the weights of the different materials shall be assumed as given in Table I:

TABLE I.—WEIGHTS OF BUILDING MATERIALS, ETC., IN POUNDS PER CUBIC FOOT.

Material.	Weight.	Material.	Weight.
Paving brick	150	Plaster	140
Building brick	120	Glass	160
Granite	170	Snow	40
Marble	170	Spruce	25
Limestone	160	Hemlock	25
Sandstone	145	White Pine	25
Slag	140	Oregon Fir	30
Gravel	120	Yellow Pine	40
Slate	175	Oak	50
Sand, clay and earth	110	Cast iron	450
Mortar	100	Wrought iron	480
Stone concrete	150	Steel	490
Cinder concrete	90	Paving asphaltum	100

39. The following table gives the uniformly distributed live loads for which structural members shall be designed when their dead loads are as given in the first column A:

TABLE II.—ASSUMED CORRESPONDING LIVE LOADS FOR VARIOUS DEAD LOADS

DEAD LOAD. Pounds per Square Foot. (Column A)	CORRESPONDING LIVE LOAD. Pounds per Square Foot. (1)	(2)	(3)	(4)
40 or under	72	103	155	194
50	63	93	140	175
60	59	84	126	158
70	53	76	114	143
80	48	69	104	130
90	46	64	96	120
100	41	58	87	109
110	37	53	80	100
120	34	49	74	93
130	31	44	66	81
140	29	41	62	78
150 or over	27	39	59	74

40. The live loads on floors for dwellings, apartment houses, dormitories, hospitals and hotels shall be as given in column (1) of Table II.

41. For school rooms, churches, offices, theatre galleries, use column (2), Table II.

42. For ground floors of office buildings, corridors and stairs in public buildings, ordinary stores, light manufacturing establishments, stables and garages, use column (3), Table II.

43. For assembly rooms, main floors of theatres, ball rooms, gymnasiums or any room likely to be used for dancing or drilling, use column (4), Table II.

44. For sidewalks, 300 pounds per square foot.

45. For warehouses, factories, special according to service, but not less than column (4) of Table II.

46. For columns the specified uniform live loads per square foot shall be used with a minimum of 20,000 pounds per column.

47. For columns carrying more than five floors the live loads may be reduced as follows:

For columns supporting the roof and top floor, no reduction.

For columns supporting each succeeding floor, a reduction of 5 per cent of the total live load may be made until 50 per cent is reached, which reduced load shall be used for the columns supporting all remaining floors.

48. This reduction is not to apply to live load on columns of warehouses and similar buildings which are likely to be fully loaded on all floors at the same time.

49. The method used in computing the stresses shall be such that the resultant unit stresses shall not exceed the prescribed unit stresses as computed on the following assumptions:

(1) That a plane section normal to the neutral axis remains such during flexure, from which it follows that the deformation in any fibre is directly proportional to the distance of that fibre from the neutral axis.

(2) That the modulus of elasticity remains constant within the limits of the working stresses fixed in these regulations and is as follows:

Steel, 30,000,000 lbs. per square inch.
Burnt clay concrete, 1,500,000 lbs. per square inch.
All other concrete, 2,000,000 lbs. per square inch.

(3) That concrete does not take tension, except that in floor slabs, secondary tension induced by internal shearing stresses may be assumed to exist.

UNIT STRESSES.

50. The allowable unit stresses under a working load shall not exceed the following:

Burned clay concrete—

Direct compression, 300 lbs. per sq. in.
Cross bending, 400 lbs. per sq. in.

Direct shearing, 150 lbs. per sq. in.

Shearing where secondary tension is allowed, 15 lbs. per sq. in.

All other concretes—

Direct compression, 500 lbs. per sq. in.

Cross bending, 800 lbs. per sq. in

Direct shearing, 300 lbs. per sq. in.

Shearing where secondary tension is allowed, 25 lbs. per sq. in.

STEEL.

	Medium Steel.	High Elastic Limit Steel.
Tension,	14,000	20,000

51. The compression in the steel shall be computed from the corresponding compression in the concrete, except for hooped columns.

52. The bonding stress between steel and concrete under working load shall not exceed the following for plain steel:

For medium steel, 50 lbs. per superficial sq. in. of contact.

For High El. Lim. Steel, 30 lbs. per superficial sq. in. of contact.

53. For bars of such shape throughout their length that their efficiency of bond does not depend upon the adhesion of concrete to steel, the allowable bonding stress under working load shall be determined as follows:

The bars shall be imbedded not less than six inches in concrete as herein defined and the force required to pull out the bar shall be ascertained. At least five such tests shall be made for each size of bar and an affidavit report of the test shall be submitted to the Commissioner of Public Buildings, who shall then fix one-fourth of the average stress thus ascertained at failure as the allowable working stress.

54. The unsupported length of a column shall not exceed fifteen times its least lateral dimension.

55. In a column subjected to combined direct compression and flexure, the extreme fiber stress resulting from the combined actions shall not exceed the unit stress prescribed for direct compression.

56. All columns shall have longitudinal steel members so arranged as to make the column capable of resisting flexure. These longitudinal members shall be stayed against buckling at points whose distance apart does not exceed twenty times the least lateral dimension of the longitudinal member. In no case shall the combined area of cross-section of these longitudinal members be less than one per cent of the area of the concrete used in proportioning the column, and the stays shall have a minimum cross section of three one-hundredths of a square inch (0.03 sq. in.).

57. If a concrete column is hooped with steel near its outer surface either in the shape of circular hoops or of a helical cylinder, and if the minimum distance apart of the hoops or the pitch of the helix does not exceed one-tenth the diameter of the column, then the strength of such a column may be assumed to be the sum of the following three elements:

(1) The compressive resistance of the concrete when stressed not to exceed five hundred pounds per square inch for the concrete enclosed by the hooping, the remainder being neglected.

(2) The compressive resistance of the longitudinal steel reinforcement when stress does not exceed allowable working stress for steel in tension.

(3) The compressive resistance which would have been produced by imaginary longitudinals stressed the same as the actual longitudinals, the volume of the imaginary longitudinals being taken at two and four-tenths (2.4) times the volume of the hooping. In computing the volume of the hooping it shall be assumed that the section of the hooping throughout is the same as its least section. If the hooping is spliced the splice shall develop the full length of the least section of the hooping.

58. The minimum covering of concrete over any portion of the reinforcing steel shall be as follows:

For flat slabs not less than 1 in.

For beams, girders, ribs, etc., not less than 1½ ins.

For columns not less than 2 in. In computing the strength of columns, other than hooped columns, the outside 1 in. around the entire column shall be neglected.

59. For flat slabs continuous over two or more supports and uniformly loaded, the bending moment may be taken as $\frac{WL}{12}$, in which W equals total load on the span and L the center to center distance between supports.

60. Beams continuous over supports shall be reinforced to take the full negative bending moment over the supports, but shall be computed as non-continuous beams.

61. The minimum distance center to center of reinforcing steel members shall not be less than the maximum diameter or diagonal dimensions of cross section plus 2 ins.

62. In designing T beams, the width of floor slab, which may be assumed to act as compression flange of the beam, shall not exceed one-fourth (¼) of the span of the beam, but in no case shall it exceed the distance, center to center, of beams.

63. If it is necessary to splice steel reinforcing members, either in compression or tension, the splice shall be either a steel splice that in tension will develop the full strength of the member, or else the members shall be lapped in the concrete for a length equal to at least the following: For plain bars of medium steel, forty times the diameter or maximum diagonal of cross section. For plain bars of high elastic limit steel, seventy times the diameter or maximum diagonal of cross section. For other than plain bars, the length of lap shall be in inverse ratio to the ratio of the allowed bonding stresses as herein required. In no case, however, shall the steel reinforcement in a beam or girder be lap spliced.

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