THE MATHEMATICS

.. OF ..

NAVIGATION

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. . BY . .

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This book is affectionately dedicated to
My Father, Friend and Teacher,
JOHN PEMBROKE JONES*
U. S. N. 1841-1861†, C. S. N. 1861-1865,
Argentine Service 1872-'78.
Born Feb. 28, 1825, died May 25, 1910.

* The author's mother's family, having become extinct in the male line in 1865, he was legally given his mother's family name.

† Dishonorably dismissed in 1861 along with certain other Southern officers whose resignations were not accepted.

PREFACE

To the beginner, navigation seems a maze of formulas, a mass of tables and a long list of funny names. In fact, the extremely confused and complicated way of presenting the subject reminds the author of the examinations a licensed electrician has to pass, which are apparently gotten up to reduce the number of licensed electricians. Many who navigate ships are poor mathematicians, and though they may be fine men, good sailors and efficient officers they are so in spite of their poor mathematics, not because of it. Navigation is a mathematical subject and should be so taught. No one can understand navigation unless he knows some mathematics—arithmetic, geometry, algebra, analytical geometry and plane trigonometry. The author does not consider spherical geometry and trigonometry necessary, and thinks they should be avoided, as will be seen, but it is absolutely necessary that the general astronomical situation be comprehended. Any one who knows astronomy can skip the chapter headed "Right Ascension," as it was written only to cover this point.

Nearly all books on navigation have a chapter headed "Practice at Sea," "A Navigator's Day at Sea," "A Day's Run," etc. According to these the navigator is on deck by dawn, spends his day, without eating, in a maze of observations, entries, calculations, precalculations, corrections and chart work; takes morning, noon, afternoon, sunset and moon readings, star readings and pole star readings, and if he gets any time for sleep it does not show on the log. He has the author's sympathy, and if this little book really helps him the author will feel fully repaid for his time and trouble.

The kindly encouragement of his friend, Lt. Comndr. George Cole Scott, U. S. N. R. F., and Mr. G. W. Littlehales, of the U. S. Hydrographic Office, is gratefully acknowledged by the author.

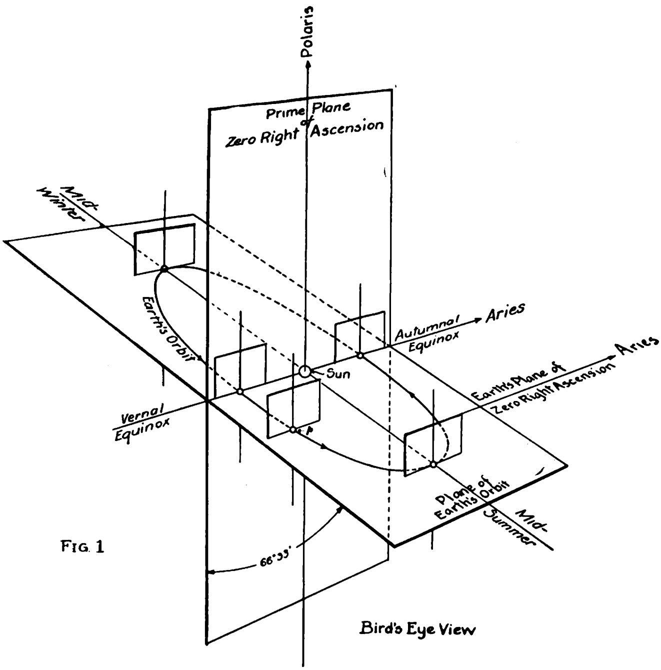

Polaris
Prime Plane of Zero Right Ascension
Mid-Winter
Earth's Orbit
Autumnal Equinox
Aries
Aries
Sun
Earth's Plane of Zero Right Ascension
Vernal Equinox
P
Plane of Earth's Orbit
Mid-Summer
66°33'
Bird's Eye View

FIG. 1

RIGHT ASCENSION

The name is unfortunate; the words do not convey the meaning of the term. In the first place, looking down on our solar system from the North Star, as it is usually shown, right ascension is measured to the left, counter-clockwise, and in the second place, it never measures ascension in the ordinary sense, i. e., how high the object is in the heavens. In practice it has only a very special and technical meaning, which must be understood to be used with confidence and accuracy.

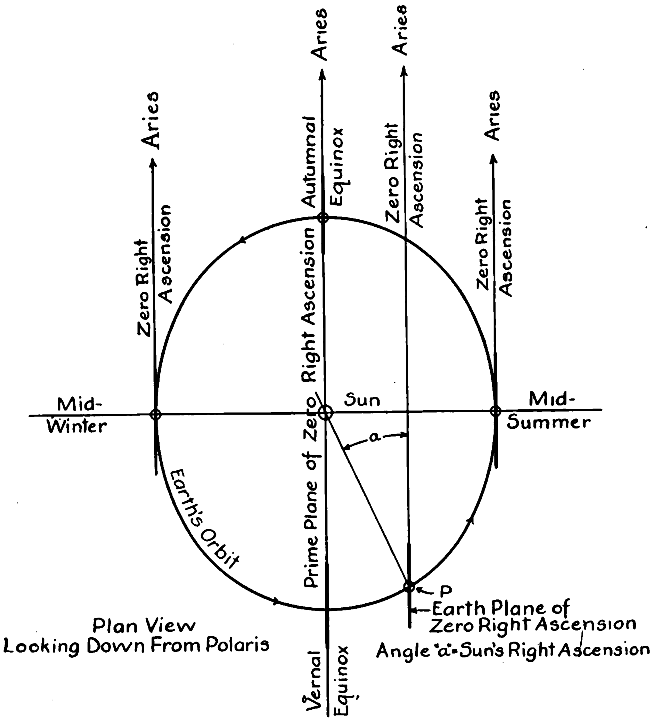
In order to make it easier, we will assume the North Star (Polaris ∝ Ursa Minor) is at the North Pole. It is not by about 1°, but it simplifies matters to have a definite object at this point. At the exact vernal equinox (about March 22nd) pass a plane through the earth's center, sun's center and Polaris. This plane will contain the earth's axis of rotation, and will not be perpendicular to the plane of the earth's orbit around the sun, but will cut it at an angle to the left 23° 26.9′ out of the perpendicular. This plane will almost pass through Alpheratz (∝ Androm) which is then on the other side of the sun from the earth (i. e., sun is in Aries). This plane is the prime zero plane of Right Ascension, and any object having this dirction from the earth has zero Right Ascension. It is to be noted that the zero direction is toward Aries and any object in the opposite direction has 180° or 12 hours Right Ascension.

Right Ascension is measured to the left all around the circle from Aries and is given sometimes in degrees, minutes and seconds—but much more frequently in hours, minutes and seconds, 24 hours making 360°. These hours, minutes and seconds are siderial hours, minutes and seconds, because they are the only hours, minutes and seconds in which 24 hours, 0 minutes and 0 seconds mean exactly 360°. 24 hours, 0 minutes and 0 seconds of solar time is more than 360°, in fact 360°, 59′, 10″, so that if one uses solar time for Right Ascension the angles and figures are about 27/100 of 1% too great. As before stated, this zero plane of Right Ascension does not strike any prominent stars in either direction from the earth. Toward Aries it comes near ∝ Andromeda (Alpheratz) and β Cassiopeia (Calph). In the other direction, 180°, 12 hrs. Right Ascension toward the dipper (Ursa Major) it now comes, almost through the one where the handle joins the bowl (Δ Ursa Major) and Δ Centauri (in the Southern Hemisphere).

Let us consider the earth at any point in its orbit. That is to say, take any day and hour in the year you choose. Pass a plane through the earth's axis parallel to the zero plane. This plane will cut the earth on two meridians. The meridian toward Aries is the meridian of zero Right Ascension and is called the meridian of siderial noon. The meridian the other way (toward the dipper) is the meridian of 180° or 12 hrs. Right Ascension, and is called the meridian of siderial midnight. Strange as it may seem, on the autumnal equinox, siderial midnight is exactly solar midday or noon by the sun (called local apparent noon). It is well to imagine the earth spinning on its axis, traveling slowly in its orbit around the sun, and carrying with it its parallel plane of siderial noon and midnight. Figure 1 shows a birdseye view of this in which the earth with its plane of zero Right Ascension is shown at point "p" in its orbit, also at its four cardinal points, vernal equinox, midsummer, autumnal equinox and midwinter.

If we look down on this from Polaris the view is much simpler, as shown in Figure 2. Unless this view (No. 2) is comprehended, Right Ascension will always be more or less of a puzzle. If, however, it is comprehended, Right Ascension has a physical meaning and can be laid off with a protractor with considerable accuracy, so much so that most problems can easily be worked out graphically within two or three minutes (siderial time minutes) and the fearful mistakes of a parrot-like working of book instructions, formulas and almanac figures entirely prevented.

The orbit of the earth is so small, compared with the distance of the stars, that their Right Ascension does not change with the earth's movement, but such is not the case with the sun, moon and planets. The Right Ascension of the sun increases on an average 3 minutes and 56.555 seconds each day, due to the earth's movement in its orbit around the sun. The moon's Right Ascension increases about one hour each day owing to its movement in its orbit



Aries
Aries
Aries
Aries
Autumnal Equinox
Zero Right Ascension
Zero Right Ascension
Zero Right Ascension
Prime Plane of Zero Right Ascension
Sun
Mid-Winter
Mid-Summer
a
Earth's Orbit
P
Earth Plane of Zero Right Ascension
Angle "a" = Sun's Right Ascension
Plan View
Looking Down From Polaris
Vernal Equinox

FIG. 2

and the earth in its orbit. The increase is variable, running from about 27 minutes to about 65 minutes. The Right Ascensions of the planets vary greatly from day to day, sometimes increasing, at others decreasing, due to their movements in their orbits relative to the earth in its orbit. Indeed they get their name planet (wanderer) from the fact that they apparently move so erratically amongst the fixed stars. The rotation of the earth on its own axis has no effect on the Right Ascension of a celestial object (nor does it affect its declination), and owing to the fact that the earth's diameter is negligible in regard to the distance of all celestial objects, except the moon, the place on the earth from which the view is taken does not affect matters. It is to the fact that a given Right Ascension and declination is applicable all over the world and is not affected by the earth's rotation that these two locative angles are of such value and in such universal use. To show that in actual practice the plane through the earth's center, the sun and Polaris at the exact equinox is absolutely the plane of zero ascension, the following example is given:

On March 21, 1919, the sun had zero declination (i. e., crossed the line) at 4.18 P.M., Greenwich mean time. On March 21, 1919, the Nautical Almanac gives the Right Ascension of the mean sun at Greenwich mean noon as 23 hrs. 51 mins. 48.9 secs. Adding for 4 hrs. and 18 mins. past noon 42.4 secs., gives mean suns right ascension 23 hrs. 52 mins. 31.3 secs. This is the mean sun, i. e. an imaginary sun keeping clock time. The actual sun was at this date and hour 7 mins. and 28.6 secs. behind this, i. e. to the left (the direction in which Right Ascension is measured) and therefore has to be added 23 hrs. 52 mins. 31.3 secs. plus this is 23 hrs. 59 mins. 59.9 secs. Since 24 hrs. is zero ascension and tables only go to 1/10 seconds, this is considered an accurate check.

To show that Right Ascension, in practice, is measured from the plane passing through sun center, polaris and earth's center at the exact vernal equinox, we will consider the earth to have made a quarter turn around the sun, i. e., for it to be midsummer. In making this quarter circle around the sun, the sun will apparently have swung around to the left 90°, i. e., its Right Ascension will have increased from 0 to 6 hrs. At midsummer the sun's declination is highest. The Nautical Almanac for 1919 shows the sun reached its highest declination June 21st, 23 hrs. 54 mins. Greenwich time, and gives its Right Ascension at this time as 5 hrs. 58 mins. 27.5 secs. On this date the equation of time shows the sun behind by 1 min. 32.5 secs., which therefore should be added and gives us 6 hrs. 0 mins. 0 secs. Right Ascension.

Knowing our longitude and the right ascension of any celestial object, we can get the correct time by observing its transit across our meridian. Thus on August 1, 1919, the star Enif (Σ Pegasi) crossed the observer's meridian at 6 hrs. 21 mins. 58 secs. by his chronometer, which was supposed to carry Greenwich mean solar time, the observer being on 100° East Longitude in the island of Sumatra and wished to check his chronometer. In order to show its application and advantages, we will first solve this graphically. In Figure 3, we first lay off the elipse of the earth's orbit as projected on the equitorial plane with its two axis, and the sun in the middle. The mean sun had at time of observation, by the Nautical Almanac, 8 hrs. 37 mins. 13.4 secs. Right Ascension. We lay off angle Aries, Sun, "R." 8 hrs. 37 mins. and carry it back until it strikes the earth's orbit. This gives the position of the earth on August 1st, because it is the only position in the orbit from which the sun has this Right Ascension. According to the Almanac the Right Ascension of Σ Pegasi is 21 hrs. 40 mins. 16.4 secs. We lay off the angle Aries, earth, Σ Pegasi, 21 hrs. 40 mins. Greenwich is 100° west of the point of observation, so we lay off the angle Σ Pegasi, earth, Greenwich 100°. The angle sun, earth, Greenwich, should give Greenwich time at instant of observation. This angle measured by our protractor is 6 hrs. 24 mins. siderial time, equal 6 hrs. 23 mins. mean solar time approximately. As the chronometer reads 6 hrs. 22 mins. approximately, we know our graphical solution is within the error of laying off. We therefore only have to determine this angle, Sun, Earth, Greenwich, accurately. This angle by Fig. No. 3 is angle Aries, Earth, Σ Pegasi, less the sum of the two angles Aries, Earth, Sun and Σ Pegasi, Earth, Greenwich.

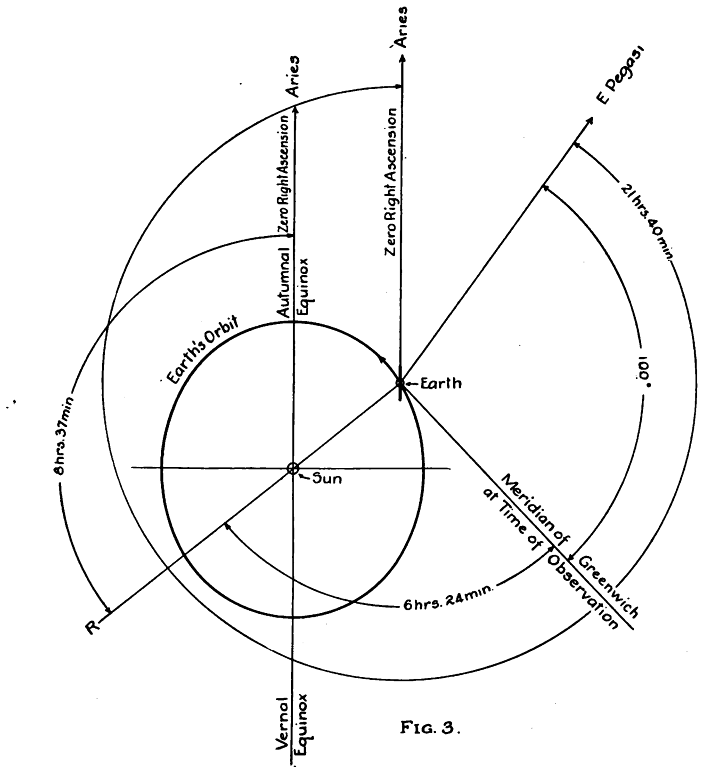

Aries
Aries
E Pegasi
Zero Right Ascension
Zero Right Ascension
Autumnal
Equinox
21hrs. 40 min.
Earth's Orbit
Earth
100°
8hrs. 37min.
Sun
Meridian of Greenwich
at Time of Observation
6hrs. 24min.
R
Vernal
Equinox

Fig. 3.

It is figured thus:

Right Ascension of Σ Pegasi	21 hrs. 40 mins. 16.4 secs.
Sun's Right Ascension— 8 hrs. 27 mins. 13.4 secs.	
100° Siderial time— 6 hrs. 40 mins. 0.0 secs.	
	15 hrs. 17 mins. 13.4 secs.
Angle Sun, Earth, Greenwich in siderial time	6 hrs. 23 mins. 3.0 secs.
Deduction to change to mean solar time	1 min. 2.7 secs.
Greenwich mean solar time of observation	6 hrs. 22 mins. .3 secs.

The observer's chronometer was therefore 2.3 secs. slow.

The preceding example has been given to show how hour angle is obtained. It is the first step in a large part of astronomical work and nearly always the preliminary step in navigation. It is essential, it should be understood, and one should be able to determine with understanding, ease, accuracy and certainty the hour angle which exists at any given instant of time between any given spot on the earth and any given heavenly body. To do this he must have available a nautical or astronomical almanac which covers the instant at which the angle occurs and which gives the right ascension at any time of the heavenly bodies and their declination, also the equation of time, and in addition, times of transit and a few convenient tables such as siderial time into mean solar time and vise-versa, etc. The Superintendent of Documents, Washington, D. C., sells it each year for 35 cts., so there is no excuse for being without it.

In navigation, longitude comes out the sum or difference of two hour angles, one the hour angle between the object sighted and Greenwich, which is determined from the almanac and chronometer time of observation, and the other the hour angle between the object sighted and the observer, which is determined by proper calculation from the angle read on the sextant. This last hour angle expressed in degrees, minutes and seconds is "*t*" in the equations given later. The other hour angle (the first mentioned, i. e., the one between the object and Greenwich) does not appear in the general equation, and to get longitude the two are added or subtracted as their positions in a figure like Fig. 3 call for.

In Fig. 3 we have an example as difficult as any that will be encountered, indeed those usually encountered are a bit simpler. It is well to remember that when the sun itself is not sighted on we can use the mean sun, but when we sight on the sun itself we use the actual sun and we must employ the equation of time and move the mean sun to where the actual sun is, or we will be out an angle equal to the equation of time when the sight is taken.

The ellipse of Figs. 2 and 3 has approximately a major diameter of 180 million miles and a minor diameter of approximately 165 million miles. On the paper it is well to draw 5″ major diameter and 4½″ minor diameter, and if there are any planets in our figuring we must draw this ellipse if we wish to show their location correctly. If, however, we, as is more frequent, have only the sun and a fixed star, we can use a circle instead, for by construction we have the sun's right ascension correct, and since the fixed stars are at infinity even if we put the earth 8 or 10 million miles out of place on our diagram it will not affect their Right Ascension, and therefore the result of our work. It is well, however, to always remember that it is an ellipse, for otherwise we have a very incorrect conception of the situation.

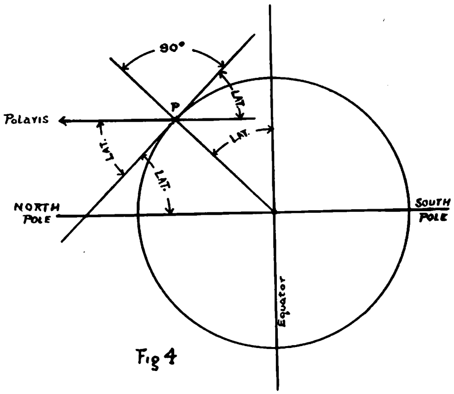


Fig 4

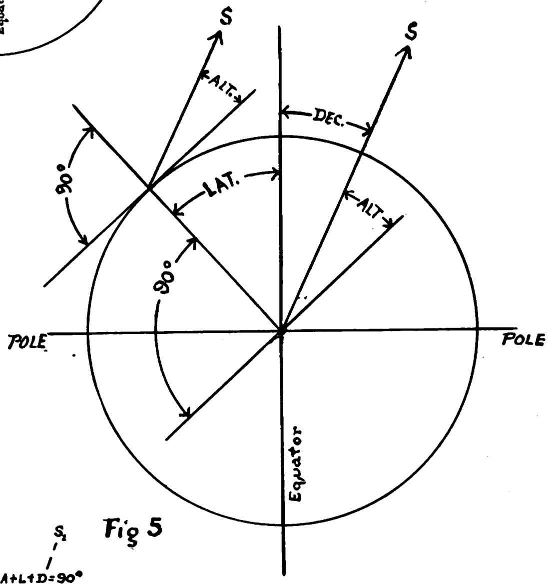


Fig 5

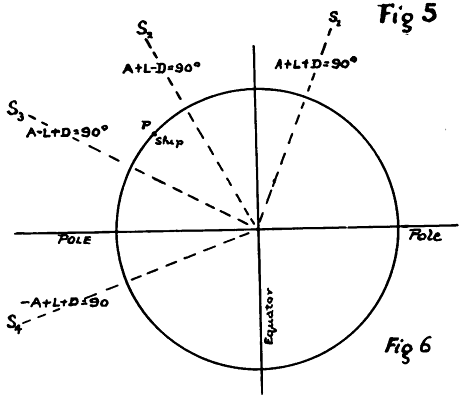


Fig 6

LATITUDE

From diagram 4 it will be seen that the altitude of Polaris is the same as the latitude of the observer, and it would at first seem that all one would have to do to get one's latitude would be to go out after dark and take the altitude of Polaris. There are troubles. In the first place, Polaris is not at the pole, but makes a small circle around it, which will throw us out, and to figure a small circle is about as hard as figuring a large one. In the second, in order to take an altitude at sea, one must see both star and horizon, and since this in general only occurs about dawn and dusk, our opportunities for observation are limited. Furthermore, the North Star cannot be seen in the Southern Hemisphere, and even for the first five or ten degrees of latitude in the Northern is so low to the horizon as not to permit an accurate observation. It is frequently used in the North Atlantic and Pacific, but it cannot be said to be a general solution.

By referring to Figure 5 it will be seen that if we take the altitude of any heavenly body on the meridian,

$$\text{ALTITUDE} + \text{LATITUDE} + \text{DECLINATION} = 90°$$

when the object "S" is on one side of the equator and the observer on the other. When, however, the object and observer are both on the same side of the equator the signs of A, L and D depend on conditions of which there are three. First, upper transit $L > D$ when $A + L - D = 90°$; second, upper transit $L < D$ when $A - L + D = 90°$, and third, lower transit $L > D$ or $L < D$ when $-A + L + D = 90°$. The form $A + L + D = 90°$ is better than one complicated by such things as zenith distance, polar distance and calculating constant, which are entirely unnecessary for either proper understanding or actual working, and the movement of the minus sign to left make the signs easy to remember. (See Fig. 6.)

The ex-meridian determination of latitude involves the use of the fundamental equation connecting altitude, latitude, longitude and declination, which will be established under the head of longitude and is therefore left for mention under that heading.

LONGITUDE

Longitude is the pons-asinorum of the navigation just as wattless currents are the pons-asinorum of electricity. If we could always have some known body visible in the heavens just in the line we wanted, it would be nearly as easy to determine longitude as latitude, but the trouble is the sun, the only thing visible in the day, seldom gets in the line we wish, and even with the large number of stars listed in the almanac, it is rarely possible to find one just in the line we want. The ideal position is one having the same declination as we have latitude, and of course on the same side of the equator. The result is that in order to get our longitude we are forced to a very general solution, that is, we have, at least theoretically, to take the altitude of any known heavenly body whenever and wherever we can see it, and from this meager data calculate our longitude. The following is a perfectly general solution of this proposition, but it is to be remembered that there are practical limitations such as that "S" must be visible from "p," that an accurate altitude cannot be had with "S" too close to the horizon, and that if either OS and Op come too near coinciding with any of the axes, certain angles cannot be accurately determined, etc.

Suppose from any point "p" on the earth the altitude of a heavenly body "S" be taken. Pass a plane through "S" and the axis of the earth and draw a line OS from the earth's center to the object. This plane gives us Fig. 7 Now draw a line through "p" to the earth's center. It will appear as Op and assuming the axis of the earth for our X and the other two (Y and Z), as marked in Figs. 7, 8, and 9, we have in Fig. 7 the angle ZOS, equal to the declination of the object sighted, and in Fig. 8 the angle ZOp, equal to longitude of the observer from the meridian of the object "S," commonly called the hour angle of the observer and is "*t*" in formulae. The line Op is in a cone whose axis is X and whose generatrix makes with X an angle equal to 90° minus the latitude of the observer. We also have the angle SOp (in space between OS and Op) equal to 90° minus the observed altitude.

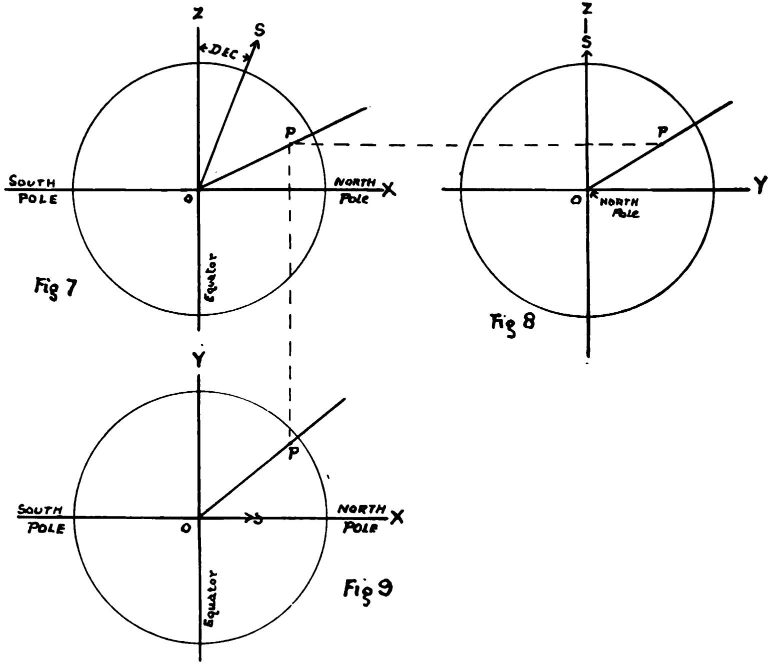


Fig 7

Fig 8

Fig 9

Taking the symmetrical equation to a line in space, i. e. as given by its direction cosines, we have for line OS, cos. α = sin D, cos β = 0, and cos γ = cos D, and the equation of the line OS is

$$\frac{x}{\text{Sin D}} = \frac{y}{0} = \frac{z}{\text{Cos D}}$$

For line Op we have:

$$\rho = x \text{ cosec L}, \quad z^2 + y^2 = x^2 \cot^2 L,$$
$$y = z \text{ tang } t \quad \text{and} \quad z = y \cot t$$

Substituting, we have:

$$\frac{y}{x} = \frac{\cot L}{\text{cosec } t} \quad \text{and} \quad \frac{z}{x} = \frac{\cot L}{\sec t}$$

and its direction cosines are:

$$\cos \alpha^1 = \frac{x}{\rho} = \sin L, \ \cos \beta^1 = \frac{y}{\rho} = \sin t \cos L, \text{ and}$$
$$\cos \gamma^1 = \frac{z}{\rho} = \cos t \cos L$$

and the equation of line Op is:

$$\frac{x}{\sin L} = \frac{y}{\sin t \cos L} = \frac{z}{\cos t \cos L}$$

The angle Θ between any two lines in space is given by the equation:

$$\cos \Theta = \cos \alpha \ \cos \alpha^1 + \cos \beta \cos \beta^1 + \cos \gamma \cos \gamma^1$$

and substituting, we have:

cos (90-Alt) = Sin Alt = Sin D sin L + cos D cos L cos t
which is the usual sine cosine formula.

The signs of the sines and cosines vary with the quadrants in which "S" and "p" occur, and if these sines and cosines are properly inserted the signs in the equation will take care of themselves. This is, however, not easy to do, and it is better to put them in according to the conditions of the case in hand. In the first place, since the earth is opaque, we have to measure altitude upward, and therefore sin A is always positive, though in its usual princely manner mathematics ignore such trivial details, and if the earth were transparent and non-refractive and we could get downward sights the equation would handle them as well as the upward ones. Either of the other two quantities in the equation may be positive or negative. The main question is whether they shall be added or subtracted—that is whether they have the same sign. If both "S" and "p" are on opposite sides of the equator subtract the smaller from the larger. If "S" and "p" are on the same side of the equator add them for upper transit and subtract the smaller from the larger for lower transit.

A very large part of navigation concentrates about this formula or some form of it. In the early days it was chiefly used to determine "*t*" by inserting in it the latitude from noon observation, declination from almanac and altitude from observation. The later method is to assume your chart position correct and figure what altitude you should get. Go on deck and get it if you can. If you can't, correct your chart position to agree with what you get. Either way, in fact, any way you do it, the equation has to be solved to proceed along the generally accepted line. The question here is how to solve it the best way. We will go over the various ways of solving it and see which is best.

The equation is easily solved graphically by methods set forth in any good book on Descriptive Geometry, but even on a large sheet and with careful work 20 or 30 minutes is as close as you can be sure of your result.

The next solution would seem the slide rule, but a moment's consideration will show that this won't do. Even the 24″ rule hardly gives three places down the line. To be sure of minutes we should have five places, i. e., a slide rule 200 ft. long. So it is pretty safe to say no ordinary guessing stick will do the job.

Of course we can solve by going at it straight, i. e., inserting natural functions to five places and multiplying out. This was found very laborious and many arithmetical mistakes occurred. So Logarithims were adopted, which is now the standard method.

In order to "simplify" (?) they usually make use of the fact that haversine X = half versed sine X = ½ (1 — cos x) = $\sin^2 \frac{x}{2}$ and write the equation as follows:

Hav Z = hav (Co. L — P. D.) + [(hav Co. L + P. D.) — hav (Co. L — P. D.)] hav t in which Z = 90 — Alt, and P. D. = 90 — D.

How anyone can call the above simplification the author cannot understand. It almost hopelessly confuses the student, and in fact many who are far beyond this stage. Of course anyone who works like a Chinaman can fill out the printed forms from the tables and get the result, but this is not conducive to any mental grasp of the subject, nor is it any saving of time. In the author's opinion the sooner it is cast aside as obsolete the better.

If one has to use logs, the author would heartily recommend the use of Martell's Tables (W. McGregor & Co., Glasgow). To solve for "*t*" to seconds (time seconds = 15 angular) only the addition of three four-place logs are required and the subtraction and addition of two angles. The chances of error are slight and the results are accurate. In fact where "*t*" in hours, minutes and seconds is sought it is about as good a way as any. It does not, however, lend itself well to the determination of Altitude, which is what is wanted in the new methods of navigation.

Decidedly the easiest and most accurate way to solve the equation

$$\sin \text{Alt} = \sin D \sin L + \cos D \cos L \cos t$$

for altitude is to use Creele's Tables (Vereiningung Wissenchaftlicher Verleger—Berlin). Be not alarmed, it is only the multiplication table up to 999x999, but for ordinary multiplication it is as much ahead of logs as daylight is of darkness.

Take the example given in the U. S. 1919 Bowditch, page 157, in which we have:

$$D = 19^\circ\ 31'\ 18''\ N = \sin^{-1} .33416 = \cos^{-1} .94251$$
$$L = 41^\circ\ 30'\ 0''\ N = \sin^{-1} .66262 = \cos^{-1} .74896$$
$$t = 293^\circ\ 0'\ 0'' = \cos^{-1} .39073$$

Using Creele, we get:

	221108		704616		274950
	207		382		514
"a"	105	"b"	904	"c"	351
	.221420		705902		.275815
	.275815				
	.497235 = sin 29° 49′ 01″				

Bowditch gets Alt = $\sin^{-1}$ .49723 = 29° 49′ 00″, but he only works to five places, and we have used six and in the sixth have found a missing second. "a" is the product of the two sines just as it comes from Crelle. "b," that of the first two cosines, and "c" this product multiplied by the last cosine. If the reader should try Crelle do not follow his instructions for the use of his tables. If you do you will get the product to probably 12 or 18 places when you really need it but to five surely, and use six to make safe. Consider 33416 as 334^{16} and 66262 as 662^{62}, etc. Do not trouble to multiply 16 by 62. It can never amount to as much as one in the sixth place which you dont' need.

It is a good plan first to work out the result to three places on a 20 inch slide rule and find of what angle it is the sine and note if this angle is reasonable or not. By this means a gross error in taking out any of the sines or cosines is detected, a check is obtained on the first three figures and arithmetic is wholly relied on for only the last three places of the result (i. e., for say minutes and seconds).

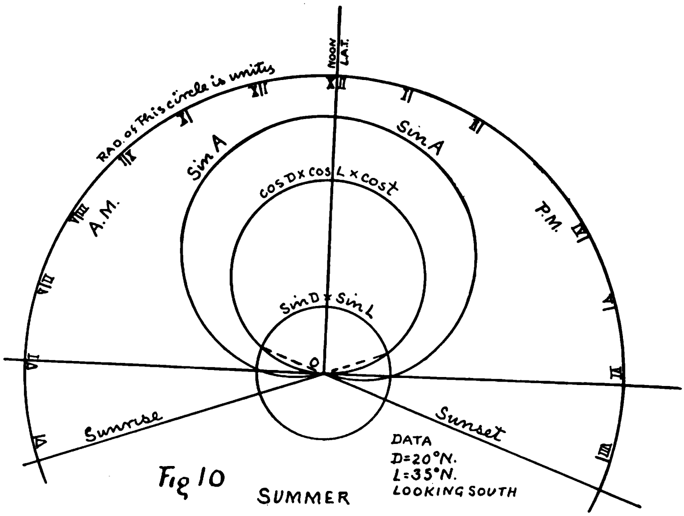
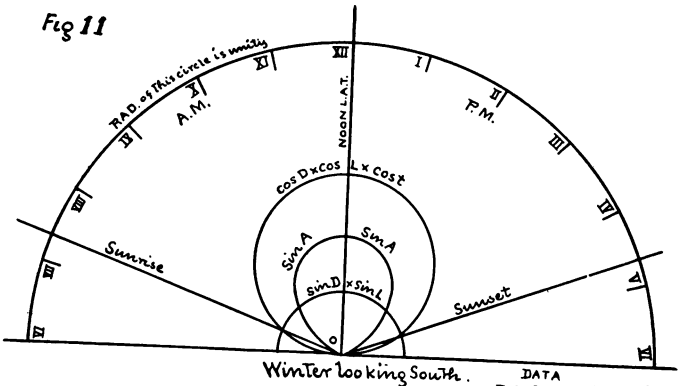


Fig 10

SUMMER

Fig 11



Winter looking South.

Crelle lends itself as well to solving the equation in the form:

$$\cos t = \sin Alt \sec D \sec L \mp \tan D \tan L$$

so that with Crelle available Martelli's tables are not needed. Furthermore Crelle comes in for all sorts of work, and if one once acquires facility in its use logarithims will be seldom employed except for division, powers, roots, integration and laying off curves. The book is much used by actuaries, but strange to say is seldom to be found in an engineer's library.

Logarithims, haversines, zenith distances, polar distances, etc., have for several generations so concealed the real operation of the sine-cosine formula that many good navigators have gone to their graves, or to Davy Jones' locker, without seeing how it really works. It is very much like the equation for the electric circuit. Indeed it is this resemblance that led the writer to take up the subject, particularly the application of calculus. Fig. 10 shows how the general equation operates with time (L. A. T.) with both the position and object on the same side of the equator and upper transit, and Fig. 11 when one is on one side of the equator and one on the other. The small circle with radius sin D $\times$ sin L gives a constant value at all hours. The radius of this circle is zero if either position or object is on the equator and increases as either or both recede from the equator. The larger circle setting on the horizon is the value of cos D $\times$ cos L $\times$ cos t The length of the radius from 0 to it gives its value at the hour to which the radius points. It is maximum at noon and zero at 6 A. M. and 6 P. M. (L. A. T.). If both position and object are on the equator, the diameter of this circle is unity. As either position or object or both recede from the equator its diameter decreases till when either reach the pole it is zero. When both position and object are on the same side of the equator sin A is the sum of these two quantities, as in Fig. 10. When they are on opposite sides sin A is the difference, as in Fig. 11.

LATITUDE AGAIN

The general equation is hard to solve for L, so for ex-meridian determinations of latitude we consider the general vectorially thus:

$(\overline{\sin D}) \sin L \pm \overline{\cos D \cos t}) \cos L = \text{Sin } A$, and the resultant vector is $\sqrt{\sin^2 D + \cos^2 D \cos^2 t}\ \text{Sin } (L \pm \tan^{-1} \sin D \div \cos D \cos t) = \text{Sin } A$, from which we can write

$L = \tan^{-1} \tan D \sec t \pm \cos^{-1} \sin A \operatorname{cosec} D \sin \tan^{-1} \tan D \sec t$ which is the usual $\Phi\text{-}\Phi^1$ formula. The general equation also offers an excellent method of checking dead-reckoning latitude, for in a meridian observation $t = 0$ or $\cos t = 1$ and $\text{Sin } A = \text{Sin } D \sin L \pm \cos D \cos L$, and at exmeridian $\text{Sin } A = \sin D \sin L \pm \cos D \cos L \cos t$, and the difference or change in Sin A is $\cos D \cos L (1 - \cos t)$ which is additive for upper transit and deductive for lower transit. Thus take the example given in U. S. 1919 Bowditch, page 136, in which observed A = 23°, 56′, 01″ sin A = .40567, D = 57°, 39′, 12″, Lat by D. R = 52°, 59′ and t = 33°, 24′, 30″—(2 hrs. 13 mins. 38 secs.). The change in Sin A is .535 x .602 x (1 — .8349) = .0532. Deducting this (since observation was at lower transit) we get at meridian sin A = .35247 or meridian Alt = 20°, 38′, 16″, and therefore latitude = 52°, 59 , 04″. Bowditch gets by $\phi\text{-}\phi'$ method latitude 52°, 59′, 03″. In cases where the observation is nearer noon the correction is smaller, easier calculated and more accurate, for if the observation is taken within a half hour of noon the amount by which the sign is changed won't run over three figures and can be worked out accurately on a 20″ slide rule. If D. R. latitude is out much a second figuring using the obtained latitude as a start will probably give so little change in L as to make one feel safe. Indeed if D. R. latitude is anywhere near right it is easier to feel for L with Δ (sin A) $= \cos L \cos D (1 - \cos t)$ than to go through the $\phi\text{-}\phi'$ method or to use the $a + a t^2$ method on a star.

The right way, however, to deal with an ex-meridian sight for latitude is to use calculus as explained in chapter headed "The Author's Method." It is far superior to any mentioned above.

THE APPLICATION OF DIFFERENTIAL CALCULUS

The rest of the equations used in navigation cannot well be established without the use of either spherical trigonometry or calculus, and the latter is so much the easier method that it surprises the writer to find it is not the one usually employed. In navigation our differentials are minutes, which, in the case of altitude and latitude, are nautical miles on the chart, but in the case of longitude, minutes are not as a rule nautical miles on the chart. At the equator a minute of longitude is a nautical mile on the chart, and at either pole it is nothing on the chart. In between or rather all over the world a minute in longitude on the chart is a nautical mile x cos latitude. Since charts are flat and all directions measured with the same scale we can lay off our dA's and our dL's in nautical miles, but out d*t*'s all have to be laid off in 1 x cos L nautical miles, and in the solution for angles on the chart d*t* alone is not used but is always accompanied with its cos L. Thus

$\frac{dA}{dt \cos L}$ on the chart is the sine of the angle of azimuth but

$\frac{dA}{dt}$ broadly speaking has no physical meaning on the chart.

Its physical meaning is the rate of change of Altitude with time, and since time minutes are 15 times as great as angular minutes, we would have to take altitude readings every four seconds to get $\frac{dA}{dt}$ in keeping with our other differentials.

When one gets a clear conception of the physical meanings of these differentials, calculus opens up a field of short cuts and an insight into the equations of navigation which cannot be acquired as easily any other way. Thus on differentiating the general equation Sin A = sin D sin L +

cos D cos L cos t. Supposing L constant, we get

$$\frac{dA}{dt} = \pm \text{Sec A cos D cos L sin t or } \frac{dA}{dt \cos L} = \text{Sin Z} = \pm \text{Sec A cos D sin t}$$

the usual equation for azimuth. We also have (fig 12)

$$d\, t \cos L = \frac{\cos A}{\cos D \times \sin t}\, dA.$$

Since this equation is readily solved with a 20″ slide rule with all the degree of accuracy used in chart work, and since there are no confusing signs, it seems to the writer it should be used in chart work in place of azimuth when the altitude is known, for if we draw a circle about our chart position with a radius equal our altitude difference, we know our line of position must be a tangent to this circle, and since it must also pass through the point determined by the first equation, we can draw the line readily on the chart without determining azimuth or laying off any angle whatsoever. To make this clear by application, take the example given in U. S. 1919 Bowditch, page 159, in which A = 70°, 25′, 30″, D = 10°, 03′, 00′ N. L = 6°, 00′, 00″ S., t = 11°, 15′ 00″, and Altitude by observation = 70°, 11′, 03″, and altitude difference therefore 14.45 nautical miles. We have dt cos L = 14.45 × .335 ÷ .9847 × .1951 = 14.45 × 1.74 = 25.2 nautical miles. Marking this off on our parallel of latitude (Fig. 12), we get point "t" which is in our line of position, and drawing a circle around our chart position "p" with a radius of 14.45 nautical miles, we get our line of position by drawing a line through "t" tangent to this circle. Bowditch gives aximuth 35°. Since 14.45 ÷ 25.2 = .574 = sin 35°, 2′, we can see that even with so large an altitude difference as 14′, 27″ the differential equation can be relied on for accurate results. Indeed this example was selected for the purpose of showing this. A small dA on a small scale chart gives a circle too small to give direction accurately. Under these conditions move the decimal point of both dA and dt cos L one place to the right, and on this scale get the direction of the line of position. Then with parallel ruler bring it up tangent to the actual dA circle.

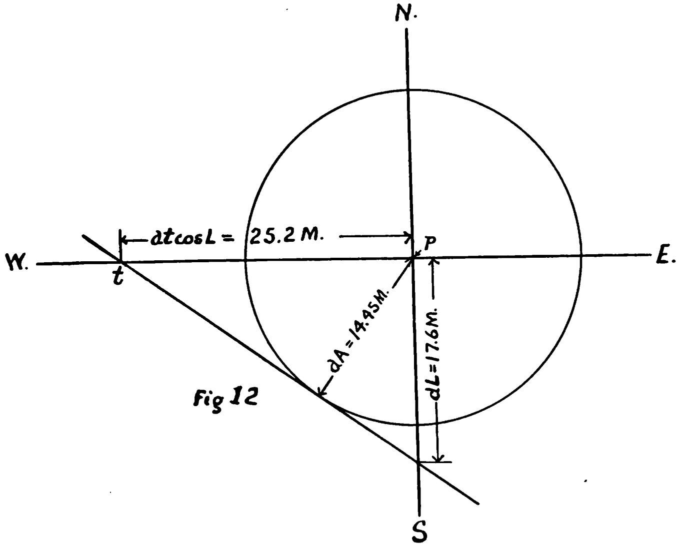


Fig 12

If we differentiate the general equation regarding t as constant, we get:

$$dL = dA \times \cos A \div (\sin D \cos L \mp \cos D \sin L \cos t)$$

and we also have: (Fig. 12).

$$\cos Z = \frac{dA}{dL} = \sec A\ (\sin D \cos L \mp \cos D \sin L \cos t)$$

and since on the horizon $A = 0$, $\sec A = 1$, and $\sin A = \sin D \sin L + \cos D \cos L \cos t = 0$, we get:

$$\sin \text{amplitude} = \sec L \sin D$$

which is the usual formula. It should also be noted that if we differentiate the above equation for azimuth, we get $dZ = dA \cot Z \tan A$, the usual statement.

Captain Sumner's idea of his line was a series of points at which the same altitude applied, at the same instant of time—so that if in the general equation we consider altitude constant we specify a line of position. Differentiating on this basis, we get: (Fig. 12).

$$\frac{dt}{dL} = \frac{\tan D}{\sin t} \mp \frac{\tan L}{\tan t} \text{ and also } \cot \text{azimuth} = \frac{dt \cos L}{dL} =$$

$$\left(\frac{\tan D}{\sin t} \mp \frac{\tan L}{\tan t}\right) \cos L = (\tan L \cos t \mp \tan D) \cos L \operatorname{cosec} t$$

As it is not easy to put the signs of the various trigonometrical functions correctly in this equation, it is best, as in the general equation, to be guided by conditions. In this equation add the two quantities when "S" is on one side of the equator and "P" on the other, but when both "S" and "P" are on the same side of the equator, subtract the smaller from the larger for upper transit and add them for lower transit. With a 20″ slide rule it can easily be solved to within 20 minutes, and in the form $\cot Z = \cot t \sin L \mp \tan D \operatorname{cosec} t \cos L$ it is readily solved by Crelles tables to seconds. Since the result is the tangent of the angle the line of position makes with the meridian it is also the longitude factor of the line of position and its reciprocal the latitude factor, and we get on one figuring a good deal that is desirable. The author has not seen this equation in any book he has read, and, as it may be new, a few examples to show its correctness will be given. Thus referring to U. S. 1919 Bowditch, page 146, in which L = 30°, 25′ N., D = 22°, 07′ S., and t = 39°, 52′, we get cot Z = (.587 × .768 + .406) .862 × 1.56 = .857 × 1.345 = 1.151 = cot 139°, 01′ (Bowditch gives 139°, 03′) and page 157, in which L = 41°, 30′, N., D. = 19°, 31′, 18″ N., and t = 293°, 00′, cot Z= (.885 × .391 — .355) .749 × 1.086 = .009 × .814 = .0073 = cot 89°, 35′ (Bowditch gives 89°, 37′). This equation facilitates an accurate direct precalculation of Z when the D. R. location and time of sight is prearranged and permits prewriting the right side of the equation for a line of position. It is usual practice to precalculate A. The sine cosine formula transformed to

Sin A = (tan D tan L ± cos t) cos D cos L

offers a convenient way to precalculate A. Thus take the example given on page 18 in which D = 19°, 31′, 18″ N = $\tan^{-1}$ .35454, L = 41°, 30′, 00″ N = $\tan^{-1}$ .88472, t = 293°, 00′, 00″ = $\cos^{-1}$ .39073.

$.354^{54}$	354	884
$.884^{72}$	.72	.54
1416	708	3536
2832	2478	4420
2832	254.88	477.36
255		
477		

.313668	= tan D × tan L	
.390730	add cos. t	
.704398	whose log. is	9.84782
	log. cos. 19°, 31′, 18″....	9.97428
	log. cos. 41°, 30′, 00″....	9.87446
	log. sin A	9.69656

or A = 29°, 49′, 01″ (Bodwitch gives 29°, 49′). This method at least presents the advantages of being clear and of requiring only the usual trigonometrical tables and is still shorter if Crelle be at hand and the product of the two tangents be taken therefrom instead of figured in pencil as above. When both A and Z are precloulated and the equation of the line of position prewritten, the only work left to be done after the second sight is tken is to insert the altitude correction dA_2 and solve for the chart corrections, as shown in the next chapter.

THE AUTHOR'S METHOD

Designating the latitude correction as dL and the longitude correction as dt we note, as previously mentioned, that, on the chart, the North or South correction is dL nautical miles and the East or West correction is dt cos L nautical miles. These two chart corrections, dL and dt cos L, are due to different causes and are not in any way co-connected. Either may be zero or any reasonable value and either positive or negative without in any way affecting the other. When, however, we take an observation, ascertain dA (the altitude intercept) calculate Z and draw a line of position, we then tie dL and dt cos L together by the condition that the position must be on this line, or, in the language of calculus, dA, dL and dt cos L must comply with the general equation Sin A = sin D sin L + cos D cos L cos t in the broadest sense, i. e., when A, t and L are all three variable. Differentiating on this basis we get

$$dA = dA_{dt} + dA_{dl} = \text{Sec } A \cos D \cos L \sin t \, dt + \sec A (\sin D \cos L - \cos D \sin L \cos t) \, dL \quad \text{or}$$

$$dA = \text{Sin } Z \, dt \cos L + \cos Z \, dL$$

or, in words, the altitude intercept equals Sin Z times the chart East or West correction plus cos Z times the chart North or South correction, all three measured in nautical miles. For those who do not appreciate calculus it is well to remove the differentials, call the E. & W. correction "x," the N. & S. correction "y," and write the equation "Altitude intercept = x sin Z + y cos Z" in which as before the altitude intercept, the correction "x" and the correction "y" are all three given in nautical miles. While this destroys all trace of the equation's ancestry this form is more easily comprehended and works as well if it be borne in mind that the longitude correction is x ÷ cos L.

The equation "dA = Sin Z dt cos L + cos Z dL" sets forth the sum total of all that can be learned from an observation, which is generally expressed by the statement that it takes two observations to get a location. While this, in a sense, is true, all that is really required besides one observation is some other observed physical fact of sufficient moment and magnitude to be measured accurately by chronometer or sextant or both with which to tie together dL and dt cos L from an independent source. Any condition which will either give us an equation containing both dL and dt or will give either dL or dt alone, will fill our needs. If another sight on the same object is used, time must be allowed for Z to change sufficiently for Sin Z and cos Z not to have values too close to the former ones.

Some have an idea that if an observation gives dA = 0 the position must be correct. Such is not the case. It only shows

$$\cos Z\, dL = \sin Z \cos L\, dt \quad \text{or that}$$

$$\frac{dL}{dt \cos L} = \tan Z$$

which, while true at the D. R. position, is also true at an infinite number of other places on the line of position. Having dA = 0 does not make the D. R. position correct, nor in general when dA is not zero can a really probable position be obtained by drawing a line of position and moving to the nearest point on this line as is frequently done. There is really no more reason to move to one point on the line of position than to move to any other point on the line of position, unless some outside considerations, such as that the error is more likely to be in t than in L or visa versa, have weight and induce the almost arbitrary selection of a special location on the line of position.

In an ex-meridian sight for latitude if M equals the number of G. M. T. minutes the observation is taken before or after L. A. Noon we have t = .25M° and can write

$$dL = \frac{dA}{\cos Z} - \tan Z\, dt \cos L$$

which is easily filled out. If M is accurately known dt = 0 and we have dL = dA sec Z. What can therefore be done with an ex-meridian sight for latitude depends entirely upon how accurately M is known.

From Fig. 13 it can be seen that

$$dA = \sin Z\, dt \cos L + \cos Z\, dL$$

is a line of position and

$$\frac{dt \cos L}{dL} = \cot Z = (\tan L \cos t \pm \tan D) \cos L \operatorname{cosec} t$$

is a line parallel to it through the D. R. position. The expression

$$\sin Z\, dt \cos L + \cos Z\, dL = dA$$

is the equation for a line of position in terms of its direction

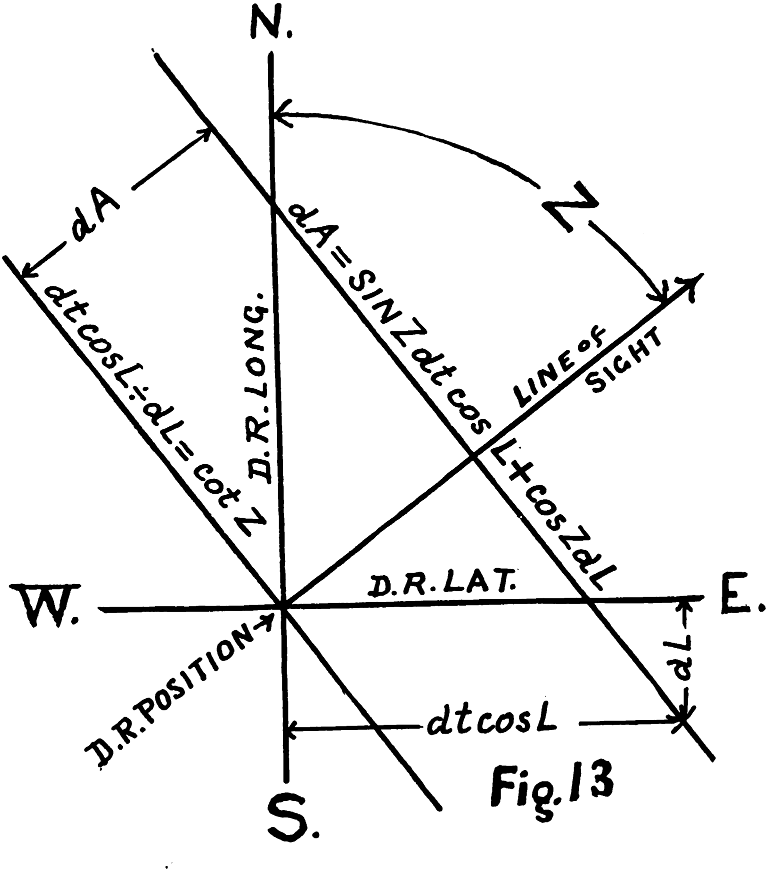

N.
S.
W.
E.
Z
dA
dA = SIN Z dt cos L + cos Z dL
dt cos L ÷ dL = cot Z
D.R. LONG.
LINE of SIGHT
D.R. LAT.
D.R. POSITION
dt cos L
dL

Fig. 13

cosines and perpendicular from the origin. Either of the rectilinear forms

$$dt \cos L = \frac{dA}{\sin Z} - \cot Z dL \quad \text{or} \quad dL = \frac{dA}{\cos Z} - \tan Z\, dt \cos L$$

is the easier to use when solving for dL and dtcos L. In all it is vital to get the signs right. The altitude correction is positive when the observed altitude is higher than the calculated altitude and visa versa and dA should be so entered into the equation. The signs of Sin Z and Cos Z are not easy to get right chiefly due to the fact that longitude does not, as it should, run all the way around to the East to 360°. The following table, if used, will prevent mistakes in these signs:

	E. Longitude		W. Longitude	
North Latitude	Sin Z	Cos Z	Sin Z	Cos Z
First Quadrant (N. E.).....	+	+	—	+
Second Quadrant (S. E.)....	+	—	—	—
Third Quadrant (S. W.)....	—	—	+	—
Fourth Quadrant (N. W.)..	—	+	+	+
South Latitude	Sin Z	Cos Z	Sin Z	Cos Z
First Quadrant (N. E.)....	+	—	—	—
Second Quadrant (S. E.)....	+	+	—	+
Third Quadrant (S. W.)....	—	+	+	+
Fourth Quadrant (N. W.)..	—	—	+	—

When the object is almost due East, or West, if the observed A is less than $\sin^{-1} \sin D \operatorname{cosec} L$ it has not yet passed the prime vertical and Z is, therefore, in one of the quadrants toward the pole. If the observed A is greater than $\sin^{-1} \sin D \operatorname{cosec} L$, then Z is in one of the quadrants next the equator. The best way to tell where Z is on an observation near the meridian is to note at the time of observation or from the series of sights whether A is waxing or waning If the signs of dA, sin Z and cos Z are properly entered in the two equations, on solving between them the two chart corrections dL and dt cos L so obtained will come out with their proper signs and they are each applicable to both D. R. positions, though, of course, it is only to the D. R. position at which the last observation was taken to which the navigator would ordinarily apply them.

The Author's suggestion is to note the miss in time of the rise or set of the object; when or while the object is high enough to get a good sight, take its altitude, from this figure dA and Z, and with these data correct to the true position.

The miss in time of rise or set is due to two causes, one the miss in longitude (dt), and the other the miss in latitude (dL). On the rising or setting of an object $t = \cos^{-1} \tan D \tan L$, from which we have

$$dt_{dl} = -\tan D \text{ cosect } \sec^2 L \, dL$$

and can, therefore, write the miss in time of rise or set,

t_r (observed)—$\cos^{-1} \tan D \tan L = dt \mp \tan D \text{ cosect } \sec^2 L \, dL$

The equation at time of sight is

$$dA = \sin Z \, dt \cos L + \cos Z \, dL$$

and solving between them we get both dt cos L and dL, which are the two corrections necessary for a location. The numerical values for all the known quantities should be inserted in both equations before attempting a solution, and since corrections are only desired to three places a 20″ slide rule will easily do the figuring.

To show the application let us assume D. R. position Lat. 34°, 50′ N. and Long. 150°, 15′ W. May 22nd, 1921. Dec. at sunrise 20°, 21′, 30″, Dec. at time of sight 20°, 22′, 20″. E. T. = 3 min. 33 sec., no chronometer error and sight properly corrected before entry. By observation the sun's center rose at 2 hrs. 56 mins. 13 secs. G. M. T. (about 5 A. M. L. A. T.) and sight was taken 4 hrs. 50 mins. 02 secs. G. M. T. (about 7 A. M. L. A. T.) and Altitude 22°, 11′, 04″ was observed.

At time of rising $t = \cos^{-1} \tan D \tan L = \cos^{-1} .37106 \times .69588 = \cos^{-1} .25822 = 104°, 57' 52'' = 6$ hrs. 59 mins. 51 secs. L. H. A., which from the D. R. longitude 150°, 15′ 00″= 10 hrs. 1 min. 0 secs., gives 3 hrs. 1 min. 9 secs. G. A. T. from which we deduct the E. T. 3 mins. 33 secs. and get the calculated G. M. T. of the rise of the sun's center 2 hrs. 57 mins. 36 secs. Observation gave 2 hrs. 56 mins. 13 secs. G. M. T. The miss was, therefore — 1 min. 23 secs. = — 20.7′ and, therefore $-20.7' = dt - \tan D \text{ cosec } t \sec^2 L \, dL = dt - .371 \times 1.037 \times 1.218 \times 1.218 \times dL = dt - .57 \, dL$. Multiplying both sides by $\cos L = \cos 34°, 50'' = .821$ we have $.821 \times -20.7 = dt \cos L - .821 \times .57 \, dL$, and hence the sunrise equation, $dt \cos L = -17' + .467 \, dL$.

For t at time of sight we have 4 hrs. 50 mins. 2 secs. G. M. T. + 3 mins. 33 secs. = 4 hrs. 53 mins. 35 secs. G. A. T., which from our D. R. longitude 150°, 15′, 00″ = 10 hrs. 1 min. 0 sec. gives t_s = 5 hrs. 7 mins. 25 secs. = 76°, 51′, 10″. According to the sine-cosine formula from D. R. position the observed altitude should have been Sin A = .34810 x .57119 + .93744 x .82082 x cos 76°, 51′ 10″ = .19884 + .76937 x .22745 = .19884 + .17500 = .37384 or A = 21°, 57′, 10″. As the observed altitude is higher than the calculated we have dA = + 13′, 54″ = + 13.9′.

As the sight is nearly East, to calculate Z we utilize the formula Cos Z = Sec. A (sin D cos L — cos D sin L cos t) = 1.0799 (.34811 x .82082 — .93744 x .57119 x .22745) = 1.0799 (.28574 — .12189) = 1.0799 x .16385 = .17696 or Z = 79°, 48′, and since the sight is first quadrant N. Latitude West longitude we have Sin Z = — .984 and Cos Z = + .17696 and we write the equation for the line of position at time of sight

$$13.9' = -.984 \text{ dt} \cos L + .177 \text{ dL}, \quad \text{or}$$

$$\text{dt} \cos L = -14.1' + .18 \text{ dL}$$

We solve between this equation and the one obtained at sunrise thus:

$$\begin{array}{ll} \text{at sunrise} & \text{dt} \cos L = -17.0' + .47 \text{ dL} \\ \text{at sight} & \text{dt} \cos L = -14.1' + .18 \text{ dL} \\ \hline & \qquad\quad 0 = -\ \ 2.9' + .29 \text{ dL} \end{array}$$

or dL = + 10′ and dt cos L = — 14.1′ + 1.8′ = — 12.3 nautical miles and dt = — 12.3′ ÷ .820 = — 15′. We, therefore, correct to Latitude 35° N. and Longitude 150″ West, which is right, as sunrise and sight were calculated from this position as the reader can see by checking the following figures: cos t_r = tan D × tan 35° = .37106 × .70021 = .25982 or t_r = 105°, 03′, 36″ = 7 hrs. 0 min. 14 secs. L. A. T., which from 150°, 00′, 00″ = 10 hrs. 0 min. 0 sec. gives 2 hrs. 59 mins. 46 secs. G. A. T., which minus 3 mins. 33 secs. = 2 hrs. 56 mins. 13 secs. G. M. T. for rise of sun's center and for sight, Sin A = .34811 × .57358 + .93744 × .81915 × cos 76°, 36′, 10″ = .19967 + .76791 × .23170 = .19967 + .17792 = .37759 or A = 22°, 11′, 04″.

If, by reason of a bad horizon or refraction accurate chronometer record cannot be had of time of rise or set, it can be accurately obtained by observing the object just as soon after rise or as late before set as a clear sight can be obtained and correcting the time of observation by

$$\frac{A \cos \frac{1}{2} A}{\text{Cos } D \cos L \times \sin \frac{1}{2} (t_s + \cos^{-1} \tan D \tan L)}$$

As this is an angle it must be changed into M. S. T. to get the chronometer correction. It will usually give accurate results up to 10° but it is better to observe at 5° or 6° if possible—the lower the better, and if the sun be the object use the upper limb where refraction is less.

Should either the time of rising or the meridian sight, or both, be unobtainable, a check with another sight on the same object will do as well, provided Z has changed sufficiently before the second sight be taken. In this case, we have the equations of two lines of position, viz:

$$dA_1 = \text{Sin } Z_1 \, dt \cos L + \cos Z_1 \, dL \quad \text{and}$$
$$dA_2 = \text{Sin } Z_2 \, dt \cos L + \cos Z_2 \, dL$$

and solve for their intersection (i. e. for dt cos L and dL). If conditions are favorable, i. e., Z changing rapidly and the altitude not too high as occurs in winter in the temperate zone when the sun runs low or at all seasons at night if a suitable star be selected in the opposite hemisphere, an accurate location by this method can be obtained in astonishingly short order if the work is properly done. To do so, however, the crude Altitude Azimuth Tables cannot be used and Azimuth must be figured out to minutes by the formula for it best suited to the case. If the object bears North or South the usual equation Sin Z = Sec. A cos D sin t should be used. If it bears East or West Z should be gotten from

$$\text{Cos } Z = \text{Sec. } A \, (\sin D \cos L \mp \cos D \cos t \sin L)$$

in which the minus sign applies when D and L have the same sign. If this be accurately done, dA accurately ascertained in each case, and the sines and cosines of Z_1 and Z_2 properly entered in the equations for the two lines of position a mathematical intersection can be obtained with an accuracy utterly unattainable by any other method the

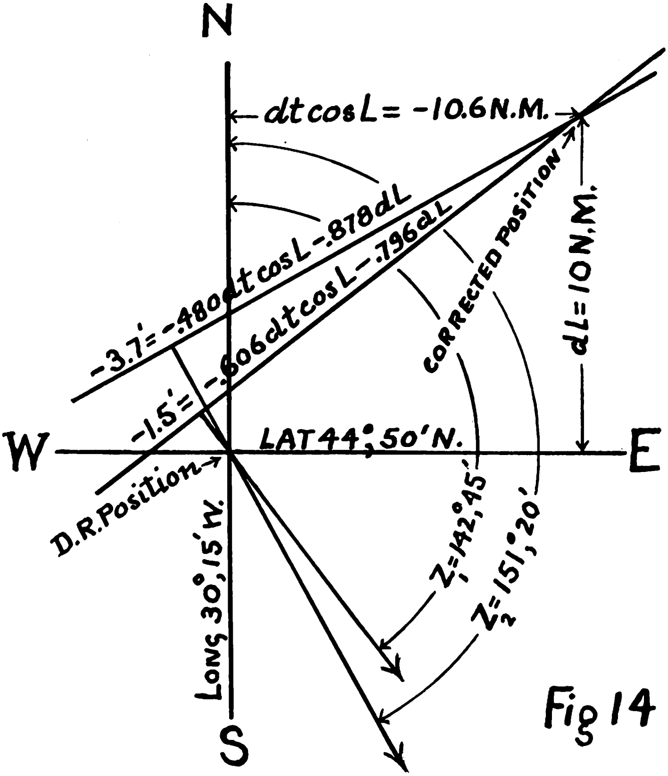

N
S
W
E
dt cos L = -10.6 N.M.
dL = 10 N.M.
-3.7' = -.480 dt cos L -.878 dL
-1.5' = -.606 dt cos L -.796 dL
CORRECTED POSITION
LAT 44°, 50' N.
D.R. Position
LONG 30°, 15' W.
Z1 = 142°, 45'
Z2 = 151°, 20'

Fig 14

the author is aware of. To permit a direct check on the obtained position the ship's run between observations was omitted in the example given on pages 28 and 29. The run does not enter into the calculation but as it always takes place, let us consider the case of two observations with ship's run between taken December 14, 1921, Sun's declination, 23°, 13′, 24″ and sights properly corrected before entry. First sight taken D. R. position L = 44°, 54′ N., and Long. = 29°, 54′ W. when from this position and chronometer reading "t" was 40°, i. e. about 9:20 A.M.L.A.T. and observed altitude 12°, 42′, 07″, Second sight taken 40 mins. later from D. R. position Lat. 44°, 50′ N., and Long. 30°, 15′ W., when "t" figured 30° or about 10 A. M. and observed altitude 16°, 34′, 48″.

From the sine-cosine formula calculated $\sin A_1 = -.39432 \times .70587 + .91898 \times .70834 \times .76604 = -.27834 + .49865 = +.22031$ or calculated $A_1 = 12°, 43', 38''$, therefore $dA_1 = -1', 31'' = -1.52'$ and $\text{Sin } Z_1 = 1.025 \times .919 \times .643 = .606$, and since the position is in N. latitude and West longitude we have $\text{Sin } Z_1 = -.606$ and $\text{Cos } Z_1 = -.796$ and the equation from the first observation was $-1.52' = -.606 \text{ dt} \cos L - .796 \text{ dL}$ or $\text{dt} \cos L = 2.50' - 1.31 \text{ dL}$.

Similiarly calculated $\sin A_2 = -.39432 \times .70505 + .91898 \times .70916 \times .86603 = -.27802 + .56440 = .28638$ or calculated $A_2 = 16°, 38' 29''$ and $dA_2 = -3'.41''$, also $\sin Z_2 = 1.0434 \times .919 \times .50000 = .480$, and we write $\text{Sin } Z_2 = -.480$ and $\cos Z_2 = -.878$ and the equation from the second observation was $-3.7' = -.480 \text{ dt} \cos L - .878 \text{ dL}$ or $\text{dt} \cos L = 7.70' - 1.83 \text{ dL}$.

From second observation,	$\text{dt} \cos L = 7.70' - 1.83 \text{ dL}$
From first observation,	$\text{dt} \cos L = 2.50' - 1.31 \text{ dL}$
Therefore,	$0 = 5.20' - .52 \text{ dL}$

or $\text{dL} = 10'$ and $\text{dt} \cos L = 7.70' - 18.3' = -10.6$ N. M. and since $\cos L = .707$, $\text{dt} = -10.6 \div .707 = -15'$ and we correct to Lat. 45° N. and Long. 30° W. This is correct, as can be seen from $\text{Sin } A_1$ (observed) $= -.39432 \times .70793 + .91898 \times .70628 \times .76884 = -.27915 + .49903 = .21988 = \text{Sin } 12°, 42', 07''$ and $\text{Sin } A_2$ (observed) $= -.39432 \times 70711 + .91898 \times .70711 \times .86820 = -.27883 + .56418 = .28535 = \text{Sin } 16°, 34', 48''$.

As far as the author can ascertain this method of writing the equations for lines of position and solving between them is new. It is, therefore, in order to mention the conditions under which this can be done and the assumptions which have to be made to render it possible. Since, in the equation for a line of position t does not appear (only dt does) we are independent of both the time and the longitude of the observation. On eliminating dt cos L when solving for dL we become independent of any change in L between observations. Any change in Z, due to the change in the ship's position, is cared for when Z is figured. The only assumption necessary is that the chart miss in position be assumed to be the same for both sights, i. e., $dt_1 \cos L_1 = dt_2 \cos L_2$ and $dL_1 = dL_2$. This must be the case if the run between observations is properly entered and is an assumption made in all methods, Sumner's, St. Hilaire's and de Aquino's. Of course the shorter the run between observations the more likely this is to be accurately the case. On the other hand Z must change between observations or no solution for dt cos L or dL can be had and the less Z changes the more uncertain the figuring for dt cos L and dL. The ship's run between observations therefore does not enter into the calculation. The navigator can take his first observation, write the equation therefrom, go where he wishes, wait as long as he pleases, take his second observation, write its equation and then solve for the two corrections dL and dt These corrections, however, as in all methods, are accurate on the basis of the run between observations being correctly charted, a condition probably strictly complied with only when there is no run at all. An absolutely undoubted location cannot therefore be obtained except either from simultaneous observations on two heavenly bodies or by waiting in one position long enough to get two sufficiently different observations on one heavenly body. It should be noted that in moving over to a corrected position the altitude intercept, dA, is thereby consumed and that when a run is charted from a corrected position the equation to be carried forward for use with that from the next observation becomes $0 = \text{Sin } Z \, dt \cos L + \cos Z \, dL$ or $dt \div dL = \tan D \div \text{Sin } t \mp \tan L \div \tan t$ in which neither A nor Z appears. There are several special or critical conditions under which the trigonometric functions of Z have definite relations to D, t, L and A. Thus when $t = 90''$, $\text{Sin } Z = \text{Sec } A \cos D$, $\cos Z = \text{Sec } A \sin D \cos L$ and $\text{Sin } A = \text{Sin } D \sin L$. On the prime vertical $\text{Sin } Z = 1$ and $\text{Cos } Z = 0$, $\text{Sin } A = \text{Sin } D \div \text{Sin } L$, $\cos t = \tan D \div \tan L$ and $dt \cos L = dA$. And on the meridian $\text{Sin } Z = 0$, $\cos Z = 1$ and $dL = dA$. If these situations are taken advantage of the work is easier and more accurate. The simplest and most accurate combination is that of the prime vertical and the meridian, i. e., observe when $t = \cos^{-1} \tan D \times \cot L$ and write $dt \cos L = dA$ and on the meridian write $dL = dA$.

CONCLUSION

The author advocates the abandonment of Spherical Geometry and Trigonometry in the theory, instruction and practice of navigation and advocates the use of plane trigonometry and calculus in their place. He also advises that in many cases where five figures are needed, Crelles Multiplication Table be used instead of logarithims and where three only are necessary a 20″ slide rule be used. When the numerical values of the quantities are given, Crelle is better than logarithims. This is also the case when the quantities themselves have to be added or subtracted in the middle of the work as in the sine cosine formula. Crelle does not lend itself easily to division, so when Crelle is used the equation should be put in the form of a straight multiplication by using suitable trigonometrical functions. Logarithims have for the student the disadvantage of so masking the numerical values of quantities of all kinds and particularly trigonometrical functions that it would seem advisable at least at first to make the student use Crelle and later employ logarithims as a facility only in those cases where logarithims are advantageous. They are really never necessary.

To the navigator an accurate correction is as good as an accurate location—in fact practically the same. An accurate position means work to certainly four and preferably five places. A correction to three places is all that is desired and one accurate through two places is really all that is needed and one within 10% will usually answer fairly well. So it is d t and d L we should seek and not t and L. The author feels absolutely certain that calculus will eventually be the accepted method of handling the mathematical problems of navigation both in teaching and in practice. In most instances the application of calculus involves differentials so minute that many minds fail to conceive of the thing at all. In navigation, however, the differentials are fine, large, visible, measurable, commonly used quantities. In bacteriology it would be like having a germ the size of a cat to work on and study. One who wishes to master navigation should really do so by means of calculus. Indeed, in the author's opinion, navigation can hardly be completely mastered any other way and calculus has, in addition to this, the advantage, like the proper way of doing anything has, of being decidedly the easiest way. When calculus is used the only table needed is one of natural trigonometrical functions. This and the 25c American Nautical Almanac for the year are all that is absolutely necessary, though Crelle's tables and a 20″ slide rule are advisable to save time and mistakes.

The sea and the stars had a strange influence on the old time men of the sea. They hated to change anything from the halliards of the foretopgallantsail to the copper nails on the keel, and even now the author does not ask that the older officers adopt these new methods. It is to younger men and particularly to the professors, teachers and instructors of navigation that the author recommends the methods herein set forth. Those trained and hardened to logarithims, haversines, polar distances, zenith distances, and $\Phi - \Phi'$ calculations had best pursue their way till progress and loving friends attend their funeral.

THE END

THE MERCATOR'S CHART

Any straight line drawn on the surface of the earth is a part of a great circle which cuts the equator at point $L=0$ and longitude t_0 and makes, with the meridian at this point, an angle Z_0 (the azimuth of the line at the equator) see Fig. 1. Its equation on the earth is

$$L=\tan^{-1}[\cot Z_0\times\sin(t-t_0)],\ \text{or}\ t=t_0\pm\sin^{-1}(\tan Z_0\times\tan L)$$

in which L is the latitude of any point and t the longitude of any point, supposing longitude to run all the way around to the East up to 360°.

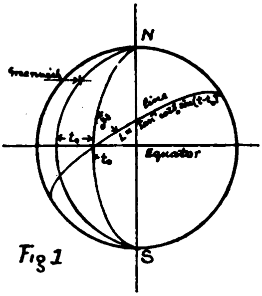


Fig 1

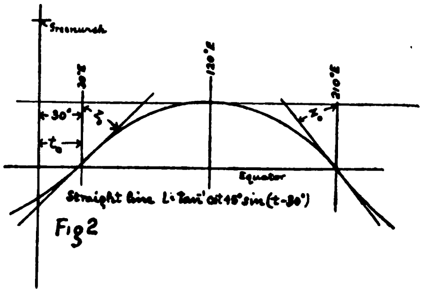


Fig 2

The equation to the line is of course the same on the Mercator's Chart, and in fact on any Chart, but instead of a circle appears as a wave-like curve shown in Fig. 2. All straight lines, such as lines of terrestrial sight, straight courses between two ports (Great Circles) take this form. The usual data is the latitude and longitude of the two ports or positions, i. e. $L_1\ t_1$ and $L_2\ t_2$ from which can write

$$\tan t_0=\frac{\tan L_1\sin t_2\pm\tan L_2\sin t_1}{\tan L_1\cos t_2\pm\tan L_2\cos t_1}$$

and

$$\tan Z_0=\frac{\sin(t_1-t_0)}{\tan L_1}=\frac{\sin(t_2-t_0)}{\tan L_2}$$

and we also have for the compass course Z (in degrees true)

$$\tan Z=\frac{dt\cos L}{dL}=\tan Z_0\times\sec L\times\sec(t-t_0).$$

and the length of the voyage is $\sin^{-1}\left(\frac{\sin L_1}{\cos Z_0}\right)\pm\sin^{-1}\left(\frac{\sin L_2}{\cos Z_0}\right)$ or

$\tan^{-1}\left[\frac{\tan(t_1-t_0)}{\sin Z_0}\right]\pm\tan^{-1}\left[\frac{\tan(t_2-t_0)}{\sin Z_0}\right]$ as best suits, applying the positive sign when L_1 and L_2 have opposite signs.

Thus for a straight course (Great Circle) from Tokio (Harbor entrance $L_1=34°,49'$ N. and $t_1=139°,38'$) to Cape Horn ($L_2=56°,00'$ S. and $t_2=292°,44'$), we have

$$\tan t_0=\frac{.695\times.922-1.48\times648}{.695\times.386-1.48\times762}=\frac{.642-.961}{.268-1.128}=\frac{-.319}{-.860}=.372$$

or $t_0=159°,25'$

$$\tan Z_0=\frac{\sin(139°,38'-159°,25')}{.695}=\frac{\sin 19°,47'}{.695}=\frac{.338}{.695}=.4875$$

or $Z_0=25°,\ 59'$

The voyage equation is, therefore,

$L=\tan^{-1}[2.052\times\sin(t-159°,25')]$ or $t=159°,25'\pm\sin^{-1}[.4875\times\tan L]$

The first form giving the latitude of the course at any meridian and the second form its longitude at any parallel of latitude. The course is $Z\ (\text{true})=\tan^{-1}[.4875\times\sec L\times\sec(t-159°,25')]$

and distance $\sin^{-1}\left(\frac{.571}{.899}\right)+\sin^{-1}\left(\frac{.829}{.899}\right)=\sin^{-1}.635+\sin^{-1}.923=$

$39°.25'+112°.37'=152°.02'$ or 9,122 nautical miles.

This course passes 175 N. M. East of New Zealand, goes within 94 N. M. of the Ant-Arctic circle at longitude 110°,35' W., which it cuts E. and W. It cuts the 30th parallel S.

$$\text{at } t=159°,25+\sin^{-1}(.4875\times\tan 30°)=159°,25'+\sin^{-1}.276=$$
$$159°,25+16°,01'=175°,26' \text{ with compass course}$$

$$Z\ (\text{true})=\tan^{-1}[.4875\times\sec 30\times\sec 16°,01']=\tan^{-1}.4875\times1.155\times1.04$$
$$=\tan^{-1}.586=149°,22'$$

and comes in well from the South at Cape Horn. Thus it cuts the 70th meridian W. at

$L=\tan^{-1}[2.052\times\sin(290°-159°,25')]=\tan^{-1}(2.052\ \sin 130°35')$
$=\tan^{-1},2.052\times.759=\tan^{-1}1,556=57°,20'$ S. with compass course $Z\ (\text{true})=\tan^{-1}(.4875\times1.853\times1.537)=\tan^{-1}1.386=54°,12'$ and continued cuts the Equator at 20°,35' W. terminating on the coast of Portuguese Guinea, North Africa, a straight run of 13,900 N. M.

The following tabe of t_0 and Z_0 will help the reader to acquire confidence in this accurate method.

		t_0	Z_0
1	Sandy Hook and Gibraltar Straits	223°,59'	46°,34'
2	Sandy Hook and Cape Good Hope	338°,23'	43°,14'
3	Sandy Hook and Cape St. Roque	321°,01'	34°,11'
4	Gibraltar Straits and Colon	267°,13'	53°,58'
5	Rio de Janeiro (Harbor entrance) and Cape Good Hope	279°,10'	55°,13'
6	Scilly Isles and Pernambuco, Brazil	328°,08'	19°,54'
7	San Francisco (entrance) and Tokio (entrance)	281°'13'	41°,42'
8	San Francisco (entrance) and Sydney entrance)	190°,58'	43°,08'
9	Panama (Pt. Mala) and Sydney (entrance)	270°,07'	52°,33'

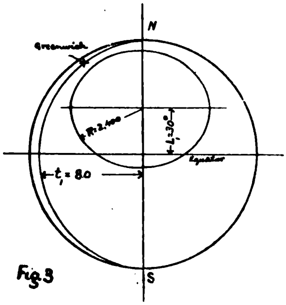


Fig. 3

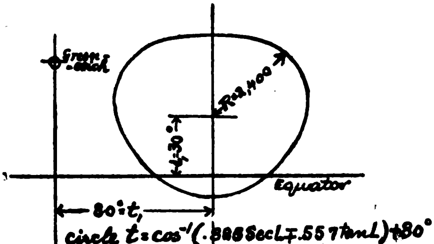


Fig 4 MERCATORS CHART.

On the earth the equation to a circle (see Fig. 3) described about a given point L_1 t_1 is

$$t = \cos^{-1}\left[\cos\left(\frac{R}{60}\right)^0 \sec L \pm \tan L_1 \tan L\right] + t_1$$

in which R is the radius of the circle in nautical miles t the longitude and L the latitude of any point on the circle. The equation on the Mercator's Chart is of course the same, but its form is distorted as can be seen from Fig. 4, which shows the same circle on the Mercator's Chart.

Consider the intersection of two circles on the Mercator's Chart. Let us assume that in winter an observer on January 17th, 1921, at noon G. M. T. obtained 30° corrected altitude simultaneously on both Bellatrix (Gamma Orionis) and Denobla (Beta Leonis). Where was he? At the time of observation Bellatrix was at L=6°,16', 36" and longitude 143°,52' E and Denobla at L=15°,00',36" and longitude 120°,05' W, and the radius of each summer circle was 90°−30°=60°

=3,600 nautical miles. The equation to the circle from Bellatrix was $t = \cos^{-1}$ (.503 sec L—.110 tan L) + 143°,52' and to the one about Denobla $t = \cos^{-1}$ (.518 sec L—.268 tan L) + 239°,54' assuming longitude to go all the way around to the East. Fig. 5, shows

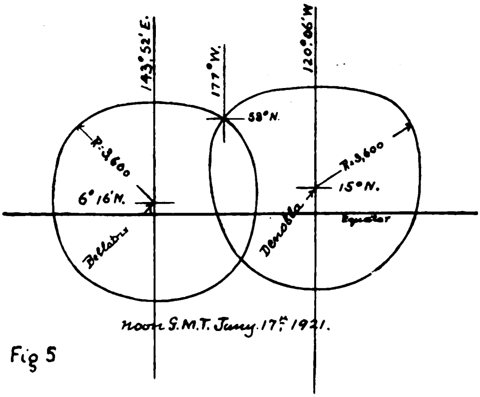


Fig 5

these two circles on the Mercator's Chart, which appear to intersect at L=58° N. and long.=177° W. Let us assume this position which we know is near both Sumner circles. From this position by the sine cosine formula the altitude of Ballatrix should have been 30°,.05' .17'' and of Denobla 29°,56',56''; or dA_1 on Bellatrix=—5.3 and dA_2 on Denobla+3.1'. The Azimuth of Bellatrix was 226°,30', and of Denobla 110°,45'. The equation to the line of position from Bellatrix was therefore by the Author's method ($dA = \sin Z\ dt \cos L + \cos Z\ dL$)

$$dA_1 = -5.3' = +.725\ dt \cos L - .689\ dL, \text{ or } dt \cos L = -7.32' + .95\ dL$$

and the equation of the line of position from Denobla was

$$dA_2 = 3.1' = -.934\ dt \cos L - .358\ dL, \text{ or } dt \cos L = -3.32' - .38\ dL$$

From Bellatrix, $dt \cos L = -7.32' + .95\ dL$

From Denobla, $dt \cos L = -3.32' - .38\ dL$

$$0 = -4.00' + 1.33\ dL \text{ or } dL = +3.00'$$

The latitude correction dL is therefore 3', the East and West Chart correction dt cos L is—7.32'+.95×3'=—7.32'+2.85'=—4.47 nautical miles and since cos L=.529 the longitude correction is —4.47÷.529= —8.45' and the true position at time of sight was Lat. 58°,03' N. and Long. 176°,51',33'' W., as the reader can confirm by the sine cosine formula.

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