THE BALANCING OF ENGINES

BY

W. E. DALBY, F.R.S.,

M.INST.C.E., M.I.M.E., A.M.I.N.A., M.A., B.SC.

UNIVERSITY PROFESSOR OF CIVIL AND MECHANICAL ENGINEERING, CITY AND GUILDS OF LONDON INSTITUTE, CENTRAL TECHNICAL COLLEGE

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PREFACE.

During the last ten years the subject of Engine Balancing has gradually forced itself upon the attention of Marine Engineers, chiefly because the unbalanced periodic forces of the engine and the natural periods of vibration of the hull have mutually approached the sensitive region of synchronism. Electrical Engineers have had vibration troubles at Central Stations and on Electric Railways, and many cases of undue wear and tear and hot bearings in Mills and Factories undoubtedly arise from unbalanced machinery, though the actual vibration produced may not be great.

In general, the running of an unbalanced engine or machine provokes its supports to elastic oscillations, and adds a grinding pressure on the bearings, and the obvious way to prevent these undesirable effects from happening is to remove the cause of them, that is to say, balance the moving parts from which the unbalanced forces arise.

The balancing of the marine engine and the peculiar problems connected therewith have been investigated by many engineers, and most of the original papers on the subject are to be found in the *Transactions of the Institution of Naval Architects.* The gradual introduction of the Yarrow-Schlick-Tweedy system of balancing the reciprocating parts of an engine amongst themselves, is familiar to all who are in touch with modern marine engine design. The Balancing of Locomotives is carried out in a traditional way, and the compromise which makes a hammer blow on the rails a necessary accompaniment to approximate uniformity of tractive force is accepted by Railway Engineers as the best practical solution of the problem. The advent of the four-cylinder locomotive, however, brings with it practical possibilities of balancing the inertia forces as great as in a four-cylinder marine engine.

The main object of this book is to develop a semi-graphical method which may be consistently used to attack problems connected with the balancing of the inertia forces arising from the relative motion of the parts of an engine or machine. In the case of a system of revolving masses, or a system of reciprocating masses where the motion may be assumed simple harmonic without serious error, the application of the method is simple in the extreme, as it requires nothing but a knowledge of the four rules of arithmetic and good draughtsmanship. Moreover, the work can be easily checked, and in the case of symmetrically arranged engines, like locomotives, for example, the method is self-checking. The application of the method to the case of a reciprocating system, in which the motion of the several masses is constrained by connecting-rods which are short relatively to the cranks they turn, is considered in Chapter V. The use of the method to compute the unbalanced forces arising from the running of an engine or machine of given dimensions in which the mass of each moving part is known, is illustrated in Chapter VI.

The precise effect of an engine on its supports cannot be predicted from a knowledge of the magnitudes of the unbalanced forces alone. The effect depends upon the elastic peculiarities of the support in relation to the periodic times and places of action of the unbalanced external forces acting upon it. A brief discussion of the principles governing the behaviour of elastic supports under the action of external forces is given in Chapter VII. I am indebted to Lord Rayleigh's "Sound," Vol. I., for the fundamental ideas of the first seven articles of the chapter. The motion of the connecting-rod and its action upon the frame is considered in Chapter VIII.

Those who are approaching the subject for the first time are recommended to work up to Art. 30, and then to check the balanced system given there by drawing out the force and couple polygons for several different positions of the reference plane. Having done this, proceed to Art. 33, and then go straight to Chapter III. Those interested in locomotive work should begin Chapter IV. after working the examples of Arts. 48 and 49, leaving Example 50 for subsequent consideration. Progress should be tested by working the exercises at the end of the book. Exercises 1 to 42 are based upon Chapters I. to IV.

A knowledge of the principles explained and illustrated through the book, will enable an engineer to apply the method to the many problems of balancing which he will find on every hand, not only with regard to engines, but in connection with machinery of all kinds. In fact, there is a balancing problem proper to every machine which has a moving part, and the consideration of this should form an essential part of the drawing-office work connected with the design of the machine.

I must thank the Council of the Institution of Naval Architects for permission to make free use of the two papers I have had the honour to communicate to the Institution, entitled respectively, "The Balancing of Engines, with Special Reference to Marine Work" (March, 1899); "On the Balancing of the Reciprocating Parts of an Engine, including the Effect of the Connecting-rod" (March, 1901). I must also thank the Council of the Institution of Mechanical Engineers for permission to use the substance of a

paper I had the honour to communicate to the Institution, entitled, "The Balancing of Locomotives" (November, 1901).

My acknowledgments are due to Dr. W. E. Sumpner for help in connection with Chapter V.; and to Mr. C. G. Lamb for kindly reading the proofs. I am also indebted to Mr. J. A. F. Aspinall for the data of the Lancashire and Yorkshire Engines; to Mr. F. W. Webb for suggestions regarding the balancing of four-cylinder locomotives; to Mr. J. Holden for details of locomotive connecting-rods; to Mr. C. A. Park for details of the balancing of Carriage Wheels; and to Mr. Yarrow for data supplied for the application of the method to the balancing of a Torpedo Boat Destroyer.

It is too much to hope that a book involving so much numerical computation will be free from error, and I shall be grateful for any corrections.

W. E. DALBY.

December, 1901.

PREFACE TO THE SECOND EDITION.

It has not been found necessary to make any essential changes in the second edition. Numerical errors and a few verbal ambiguities have been corrected, and five Appendices have been added. Appendices I. and IV. contain simple geometrical constructions for finding the acceleration of the piston, and for fixing a point on the line of action of the force producing the instantaneous acceleration of a link having plane motion. I am indebted to Mr. G. T. Bennett, of Emmanuel College, Cambridge, for these simple and useful constructions. In Appendix V. an investigation is made of the balancing of a few cases where the problem reduces to balancing in a plane.

W. E. DALBY.

March, 1906.

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THE

BALANCING OF ENGINES.

CHAPTER I.

THE ADDITION AND SUBTRACTION OF VECTOR QUANTITIES.

SIR WILLIAM HAMILTON divided quantities into two kinds. The one kind called Scalar quantities, the other Vector quantities.

1. A Scalar Quantity does not involve the idea of direction. Sums of money; the capacity of a tank; a quantity of matter, say a ton of coal; the energy stored in a moving body; all these are scalar quantities, and they are defined completely by the simple statement of their magnitudes. Quantities of this kind are added and subtracted by the ordinary rules of arithmetic.

2. A Vector or Directed Quantity involves the idea of direction as well as magnitude. The simple statement of the magnitude of a quantity of this kind is not enough to define it. The direction in which the quantity is active must be given in such a way that there can be no ambiguity. In general, a vector quantity is said to have—

Magnitude.
Direction.
Sense or Way of action.

It will be shown immediately that direction may be defined in a way which will include the last two properties of a vector in a single statement.

Force, acceleration, velocity, displacement, momentum, couples, an electric current, are all examples of vector quantities.

A vector quantity may be represented by a line, whose length is proportional to the magnitude of the quantity, and whose direction is parallel to the direction of the quantity, the sense or way of action being indicated by an arrow-head placed on the line.

The upper line AB (Fig. 1) represents a vector quantity whose magnitude is the length of AB to scale, whose direction is parallel to AB, and whose way of action is *from* A *towards* B. The line is referred to as "vector AB."

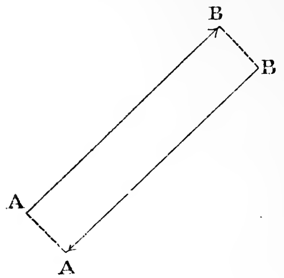


Fig. 1.

If +AB represent a given vector quantity, a reversal of its sign, denoted by a reversal of the arrow-head, shown on the lower line of Fig. 1, is another vector quantity specified by −AB or +BA, so that a change of sign, or the reversal of the order of the letters specifying a line, is equivalent to reversing the way of action of the vector.

3. Addition of Vector Quantities.—The extension of the idea of addition to quantities of this kind is already familiar to every draughtsman through the use of the polygon of forces, to find the resultant of a number of forces acting at a point. The sum of the separate effects of the forces is equivalent to the effect of the single force, "the resultant." Or, the resultant is the vector sum of the several vector quantities, which in this case happen to be forces. The term "vector sum" is therefore synonymous with the term "resultant." Both terms may be used with reference to vector quantities of all kinds, though in each particular addition the vectors must represent quantities of the same kind.

The rule for addition may be stated as follows :—

Starting from any point, set out the lines representing the vector quantities as if to form a polygon, the arrow heads all pointing round in the same direction: the line drawn *from* the starting-point, closing the polygon, represents their sum or resultant.

The **Order** of setting out the lines is immaterial.

Let A, B, C, D, E (Fig. 2) be a series of lines set out in order representing a set of vector quantities, as forces acting on a point. Notice that the letter at a corner denotes the end of one line and the beginning of the next. The operation of setting them out may be conveniently indicated by the expression—

Vector sum (AB + BC + CD + DE)

the direction of each being specified, either numerically or by a drawing. The sum is given by the line AE, so that the first and last letter of the series are the two letters in order specifying the line representing the sum. In what follows an expression of the kind—

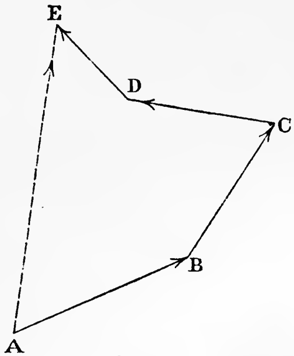


FIG. 2.

$$\text{Vector sum } (AB + BC + CD + DE) = AE \quad . \quad . \quad (1)$$

must be understood to mean that the lines indicated by the letters in the brackets are to be set out in order, their several directions being otherwise specified, generally by a drawing, and that the length, direction, and sense of AE, the closing line, are to be found graphically.

4. Condition that the Vector Sum may be Zero.—Bring the right-hand side of equation (1) to the left; then—

$$\text{Vector sum } (AB + BC + CD + DE - AE) = 0$$
$$\text{But } -AE = +EA \text{ (Art. 2)}$$
$$\text{therefore, Vector sum } (AB + BC + CD + DE + EA) = 0 \quad . \quad (2)$$

The first and last letters of the series in the brackets denote the same point. This expression represents the fact that when the lines are set out in order, they will form a closed polygon, in which case the vector sum of the quantities is zero, since no line has to be drawn to close the polygon. If the vectors represent forces, this expresses the fact that the forces are in equilibrium.

It must be carefully remembered that the expressions (1) and (2) are not equations in the ordinary sense. The first is merely a convenient way of indicating an operation and its result, the second only a convenient way of stating that the vector polygon closes, and in both the sign "=" should be interpreted to mean "is equivalent to." The greatest care must be taken when drawing a polygon to get the arrow-heads rightly placed. The accidental reversal of an arrow-head on a line means that a quantity, presumably specified by the line, has been left out, and one exactly equal and opposite included. When the lines are set out, the arrow-heads must all point in the same circuit, with the exception of the one on the line representing the sum. This must point against the rest.

5. Displacement Vectors.—The properties of vectors may be illustrated by the displacement of a point from a position A to a position B. Let the vector AB (Fig. 3) represent a displacement from A to B. A further displacement from B to C is represented by the line BC. The sum of the two operations, that is—

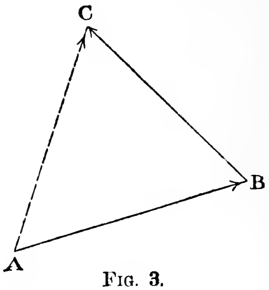


Fig. 3.

$$\text{Vector sum } (AB + BC) = AC$$

has resulted in a change of position from A to C, which might have been attained by the single displacement AC. If the two transferences are simultaneous, as when the point is carried in the direction AB, across the deck of a ship steaming the distance BC in the same time, the result of the two transferences is a displacement from the position A to the position C. The idea of the transference of a point, kept in mind, is of great assistance in thinking about vectors. Every individual unconsciously illustrates the principles of vector addition by every movement and by every walk he takes. A return to the same spot means that so far as transference is concerned the vector sum of all his displacements, reckoned from the starting-point, is zero; and if the lines representing the successive straight parts of the walk are plotted, they will form a closed polygon. They need not even be plotted; the series of places passed through, joined up on a map, is a vector polygon, and is obviously closed by a return to the starting-point. Or wherever he gets to, the position attained might have been attained by walking straight to it from the start. The sense of the vector being traced out at any instant is given by the direction of walking.

6. The Subtraction of two vectors is performed by setting them out from the same point; the line joining their ends represents their difference.

The meaning of

$$\text{Vector sum } (AB + BC) = AC$$

has been defined in Art. 3, also the meaning of the minus sign in Art. 2. Remembering these, the following expressions, obtained by transposing the terms of the above expression, evidently indicate the operation of finding the vector difference:—

$$\text{Vector difference } (AB - AC) = -BC = CB \quad . \quad . \quad (3)$$
$$\text{Vector difference } (AC - AB) = +BC \quad . \quad . \quad . \quad . \quad (4)$$

The corresponding triangles are shown in Figs. 4 and 5. The only difficulty in performing this graphical subtraction is, that

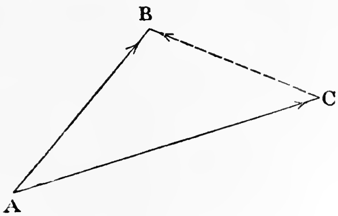


FIG. 4.

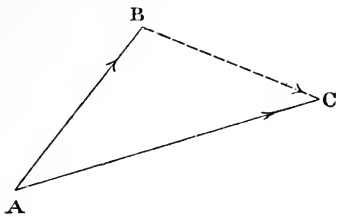


FIG. 5.

having drawn the triangle, it is not at once obvious how to place the arrow-head on the line representing the difference. The rule is, that it must always be placed in circuit with the quantity being subtracted.

This method is of great use in finding the value of one quantity relative to a second, both being originally expressed relatively to a third quantity. For instance, if AC (Fig. 4) represent the velocity of a train relative to the earth and AB represent the velocity of a second train relative to the earth, CB represents the velocity of the second train relative to the first and BC (Fig. 5) represents the velocity of the first train relative to the second. In the well-known proposition of the parallelogram of forces, which is merely a method of adding two vectors, equivalent to the rule already given, one diagonal of the parallelogram represents the vector sum, the other the vector difference.

The principle of taking a vector difference is the basis of the geometrical constructions used in the design of turbines and centrifugal pumps, for finding the angles of the vanes; it is the key to the geometrical methods given to find relative velocities and accelerations in kinematics and in mechanisms. Illustrations of its utility in this respect will be found in Art. 105.

7. Definition of Direction to include Sense.—Let OX (Fig. 6) be a line from which direction is to be measured. Suppose an arm to be hinged at O, a point called the origin, forming with the line OX a kind of open compass. Opening the arm out, counter-clockwise, to an angle θ, the direction it indicates is always to be measured *from* O outwards along the arm, or radius vector, as it is called, for a positive vector quantity. No ambiguity in sense can arise, because the sense from O along the dotted line would be specified by the angular direction being given 180° greater. Thus OA might be inclined 60°; OA_1 or AO, the direction of the opposite sense, would be inclined 240°. In the case where the vector is negative, it would be measured from O in the direction opposite to the direction in which the arm points. For instance, $-OA$ when θ is 30°, is equivalent to $+OA$, where $\theta = 180 + 30°$.

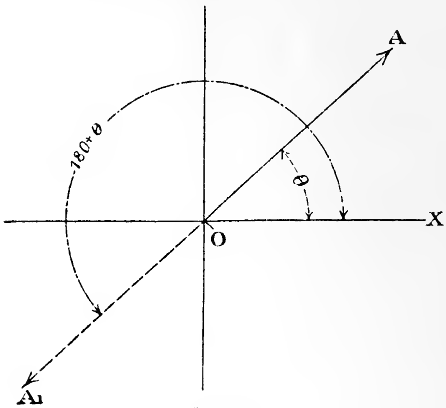


Fig. 6.

The initial line OX, and the lines measured from it, may be all moved about together in any way, without altering the directions of lines relatively to it, or to one another. If the hub of a cart-wheel, for instance, be selected for an origin, and one of the spokes be fixed upon as the line from which to measure the angular position of all the other spokes, any motion whatever may be given to the wheel without in the least affecting the inclination of the other spokes either to the initial spoke or to one another. A rotating initial line, or line of reference, is one of the essential features of the method explained in the next chapter.

8. On the Two Quantities determined by closing a Polygon.—Measuring direction by the method of Art. 7, a vector quantity is defined by two quantities, a **Magnitude** and a **Direction.**

In a closed polygon of n sides there are n magnitudes and n directions, $2n$ quantities in all. These $2n$ quantities cannot all be chosen at will, since they are subject to the condition that they form a closed polygon. This condition requires that two, and two only, of the quantities shall remain unspecified, their values being found, graphically or by calculation, by closing the polygon. If less than two of the quantities are left undetermined, the data will in general be inconsistent; if more, the closing of the polygon is indeterminate. The two unknown quantities may be—

A, A magnitude and its direction.
B, Two magnitudes.
C, Two directions.
D, A magnitude and the direction of another known magnitude.

Case A.—Consider the case of a five-sided polygon. Ten quantities in all are concerned in the specification of its sides; of these eight must be completely specified. Let the eight quantities be specified by the following schedule, leaving a magnitude and its direction unknown:—

SCHEDULE 1.

	Magnitude.	Direction.
AB	2·0	0°
BC	2·5	30°
CD	3·0	150°
DE	2·5	210°
EA	1·61	291°

Setting out the four vectors, AB, BC, CD, DE, they might by some fortuitous chance form a closed polygon; in general, however, an unknown term, EA, involving a magnitude and a direction, is required to close the polygon. Setting them out as shown in Fig. 7, it will be found that the closing side EA measures 1·61, inclined 291° to the initial direction AB.

Case B.—Let AB, BC, CD be completely given, and the directions, θ_1 and θ_2, of DE and EA. Set out the first three quantities (Fig. 8). At D set out the direction θ_1, and at A the

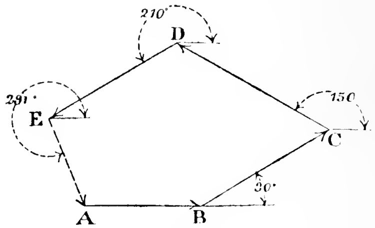


Fig. 7.

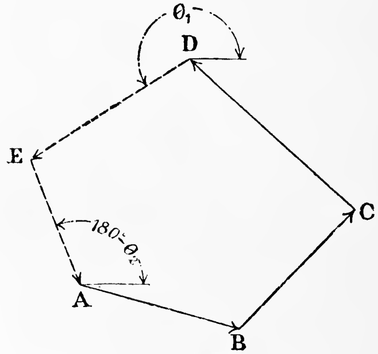


Fig. 8.

direction 180° – θ_2. The magnitudes of DE and EA are determined by the intersection at E.

Case C.—Let AB, BC, CD be given completely, and the magnitudes only of DE and EA. Set out the completely given vectors as in Fig. 9, arriving at the point D. From centre D describe an arc with radius DE, and from centre A describe an arc with radius AE. These two arcs will—

(1) cut one another in two points, in which case there are two solutions to the problem;

(2) they will touch, in which case DE and EA are in the same direction;

(3) they will not intersect, showing that the data are inconsistent, and that there is no solution possible.

Case D.—Let AB, BC, CD be completely given, together with the direction of DE and the magnitude of EA. Set out the completely given vectors (Fig. 10), arriving at the point D. From D set out a line in the direction θ_1, and from A draw an arc of radius AE. This arc will—

(1) cut the direction of DE in two points, giving two solutions;
(2) touch DE, in which case EA is at 270° with DE;
(3) not touch the line, showing that there is no solution to the problem.

In general, therefore, two, and only two, quantities can be

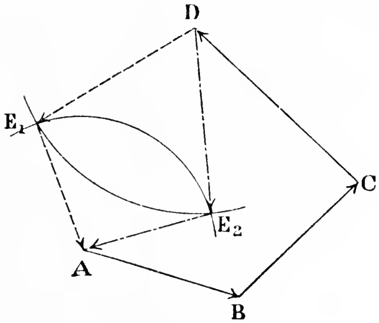


FIG. 9.

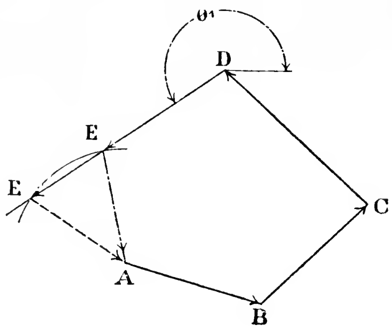


FIG. 10.

found from a vector polygon, and if the polygon have n sides, of the $2n$ quantities concerned, $(2n - 2)$ must be given, but no more, to make the problem determinate. Even then the data may possibly be inconsistent, as exemplified in Cases C and D above.

CHAPTER II.

THE BALANCING OF REVOLVING MASSES.

9. The Force required to constrain Motion in a Circle.—The natural mode of motion of a mass of matter, unacted upon by any external force, is in a straight line with uniform speed. The action of an external force is required to change either the direction of motion, or the speed. The force, in lbs. weight, required to constrain a mass of M pounds to move in a circular path, r feet radius, the mass centre moving at a uniform speed of v feet per second, is given by the expression—

$$\frac{Mv^2}{gr} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

and its direction of action is in a line through the mass centre of the body, towards the centre of the path.

If ω is the angular velocity of the mass in radians per second, $v = \omega r$, and the above expression may be written—

$$\frac{M\omega^2 r}{g} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

or simply, $M\omega^2 r$, when the force is measured in absolute units.

The radius of the path, r, means the distance measured from the centre of the path to the mass centre of the circularly constrained body. For all practical purposes the mass centre is coincident with the centre of gravity of the body, so that the usual methods for finding the centre of gravity may be employed to find the mass centre. The above expressions may be adjusted for revolutions per minute, N, or revolutions per second, n, by the relations—

$$\omega = 2\pi n = \frac{2\pi N}{60}$$

The different forms in which the magnitude of the constraining force may be expressed are collected together in the following schedule for convenience of reference :—

SCHEDULE 2.

M in pounds ; r in feet ; $g = 32 \cdot 2$.

Constraining force F.	Angular velocity in			Speed along the path = v feet per second.
	radians per second = ω.	revolutions per second = n.	revolutions per minute = N.	
F in poundals ...	$M\omega^2 r$	$M 4\pi^2 n^2 r$	$\frac{M 4\pi^2 N^2 r}{3600}$	$\frac{Mv^2}{r}$
F in lbs. weight ...	$\frac{M\omega^2 r}{g}$	$\frac{M 4\pi^2 n^2 r}{g}$	$\frac{M 4\pi^2 N^2 r}{g \times 3600}$	$\frac{Mv^2}{gr}$
F in lbs. weight ...	$0 \cdot 031 M\omega^2 r$	$1 \cdot 224 M n^2 r$	$0 \cdot 00034 M N^2 r$	$0 \cdot 031 \frac{Mv^2}{r}$

The most convenient form to use in balancing problems is generally—

$$M\omega^2 r$$

10. Methods of applying the Constraining Force: the Reaction on the Axis.—The constraining force F may be applied either as a push or a pull towards the centre. In either case the force has two aspects, the action on the body it is constraining in a circular path, and an equal and opposite reaction on some other body. A railway train is constrained to pass round a curve by the forces exerted between the outer rail and the flanges. The rails push the train continually away from the straight line, the train tries to push the curved constraining rail straight. These two aspects of the push constitute a pressure between the rails and the wheels. For a given speed the opposite aspect of the constraining force F may be supplied by tilting the sleepers, so that the resolved component of the weight of the train is equal to F, thus removing the pressure from the outer rail.

The force F is frequently of necessity applied to a body by means of a radial connector. A stone whirled in a circle by means of a sling is constrained in its path by the pull exerted by the string, which is necessarily accompanied by an equal and opposite pull at the centre. The two aspects of the constraining force exerted by a radial connector, which together always form a tension, have been named respectively the centripetal and centrifugal forces. By centripetal force is to be understood the action of the radial connector with reference to the body it is constraining, and by centrifugal force the action of the radial connector on the axis.

Example.—A mass of 10 pounds is constrained to move in a circle 4 feet radius, at a speed of 5 feet per second, by means of a radial connector. Find the reaction on the axis, *i.e.* the centrifugal force.

The tension F in the connector is—

$$\frac{Mv^2}{gr} = \frac{10 \times 5^2}{32{\cdot}2 \times 4} = 1{\cdot}94 \text{ lbs. weight.}$$

Hence, if 1·94 lbs. weight is the pull on the axis, 1·94 is the pull on the mass necessary to constrain the motion.

11. Dynamical Load on a Shaft.—A shaft supporting a rotating body whose mass centre is not on the axis of rotation becomes loaded therefore with what may be called a dynamical load—a load varying directly as the radius of the mass centre of the body, as the square of the angular velocity, and changing continuously in direction. At a high speed, such a load tends to set up vibrations of the framework carrying the shaft, and of the floor or foundation to which the framework is attached. In some cases even moderate speeds cause trouble, and if the period of vibration of any part of the supporting framework or foundation should happen to coincide with the period of revolution of the mass, the disturbances set up may become dangerous.

A mass of 1 pound at 1 foot radius attached to a shaft and turned at 12,000 revolutions per minute, imposes on the shaft a dynamical load of nearly 50,000 lbs. weight. Or a mass weighing only 1 pound requires a force nearly 50,000 times as great as its own weight to constrain the motion. Heavy foundations, strong holding-down bolts, and stiffened frames are of little avail against these disturbances; and moreover, even if such means reduce the extent of the vibrations to a practical minimum, the dynamical load has still to be taken by the bearings, causing unnecessary wear and tear, often heating, and always a loss of energy, thereby decreasing the efficiency of the machine.

The Balancing of a Rotating System consists in the arranging of the masses forming the system, so that the centrifugal forces acting on the shaft, in consequence of the rotation, form a system in equilibrium.

In developing the method of effecting such an arrangement, it will be convenient to consider first those problems where the masses are all in the same plane of revolution.

Definition.—The plane in which a mass is said to revolve is the plane in which its mass centre revolves.

Definition.—A plane of revolution is a plane at right angles to the axis of revolution.

An exact knowledge of the next three paragraphs is necessary for the proper comprehension of the method developed in the rest of the book.

12. Balancing a Single Mass by Means of a Mass in the Same Plane of Revolution.—Let a mass, M_1, be attached to a truly turned disc (Fig. 11), at radius r_1. Let M_0, at radius r_0, be the mass which will balance the effect of M_1. The condition that there may be no unbalanced centrifugal force acting on the axis, is, that the resultant or vector sum (see Arts. 3 and 4) of the forces due to M_1, the disturbing mass, and M_0, the balancing mass, is zero for all values of ω, the angular velocity of the system. This condition is expressed by—

Fig. 12.

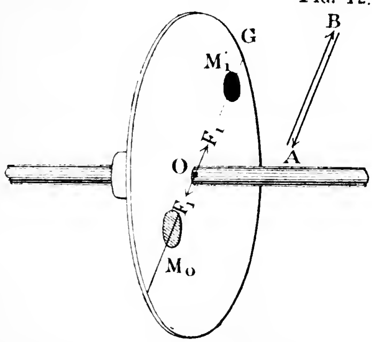


Fig. 11.

$$(M_1r_1 + M_0r_0)\omega^2 = 0$$

ω being put outside the bracket because it is the same for every point in the system. Now ω is by the terms of the problem not zero; therefore, to satisfy this condition, the sum of the terms in the brackets must be zero. Each term is a vector quantity, and is called a **mass moment.** The magnitude of such a term is specified by the numerical value of a product of the form Mr, and its direction by the radius r, specified by a drawing, the sense being determined by the rule that the way of drawing the vector is always *from* the axis of the shaft *towards* the mass. The condition that the vector sum of the terms in the brackets is zero, means simply that when the vectors representing them are set out in order, they must form a closed polygon. The angular velocity ω may have any value without affecting the result. Let it equal unity, then a term of the form Mr is the centrifugal force when ω = unity. In this way the use of the term "mass moment" may be avoided. The solution is to be carried out as follows:—

Set out AB (Fig. 12) parallel to OG to represent M_1r_1; then BA is the vector required to close the polygon, in this case a line returning on itself, so that the sum—

$$AB + BA = 0$$

BA, therefore, represents in magnitude and direction the quantity M_0r_0. Draw OM_0 (Fig. 11) parallel to BA, remembering that it is to be drawn in the direction *from* B *to* A. The balancing mass M_0 can be found directly the radius at which it is to be placed is given.

The analytical condition is in this case expressed by the equation—

$$M_1r_1 = -M_0r_0$$

M is always to be considered positive; the negative sign therefore refers to the radius, and since the radii r_1 and r_0 are in the same straight line, the negative sign indicates that, measuring r_1 from the axis outward along a diameter, r_0 must be measured from the axis along the diameter in the opposite direction.

Example.—If the given mass M_1 is 5 pounds at a radius of 2 feet—

$$M_0r_0 = -10$$

Fig. 13 shows graphically the way in which M_0 varies with r_0, so that their product may remain constant and equal to 10. The choice of the mass to be used is evidently a very wide one if the radius is not specified. Thus a mass of 5 pounds at 2 feet radius, equally with a mass of 1000 pounds at $\frac{1}{100}$ foot, or 1 pound at 10 feet, will effect balance.

The centrifugal forces at the axis are not eliminated; they are merely balanced. The connector is in tension, along the line connecting the masses under the action of the constraining force F_1, acting on M_1, and an equal and opposite force F_1, acting on M_0. The one force is the reaction to the other at any speed of rotation.

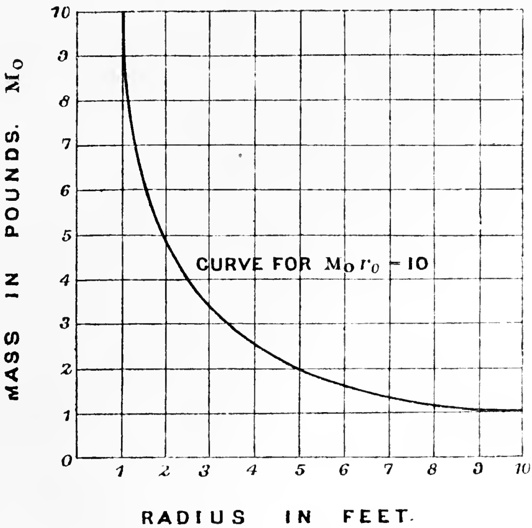


FIG. 13.

As the speed of rotation increases, F_1 may become sufficiently great to rupture the connector. For example, a pair of 100-pound balls, attached to an iron rod 1 inch in diameter, at 10 feet centre to centre, and rotated ten times per second about a central axis at right angles to the rod, would require a tension in the rod of 61,250 lbs. weight to constrain their motion—quite enough to break the rod.

13. Balancing Two rigidly connected Masses by Means of a Third Mass, all being in the Same Plane of Revolution.—Let M_1 and M_2 (Fig. 14) be the two given masses at radii r_1 and r_2 respectively, and M_0 the balancing mass at a radius r_0. When the angular velocity is ω, the masses give rise to centrifugal forces $M_1\omega^2r_1$, $M_2\omega^2r_2$, $M_0\omega^2r_0$, ω being the same for all.

In order that there may be no dynamical load upon the shaft—

$$\text{Vector sum } (M_1r_1 + M_2r_2 + M_0r_0)\omega^2 = 0$$
$$\text{that is, } (AB + BC + CA) = 0$$

where M_1r_1, M_2r_2, M_0r_0 are represented in magnitude, direction, and sense by the lines AB, BC, CA, respectively. To find M_0r_0, therefore, set out AB (Fig. 15) to scale equal to M_1r_1, drawing

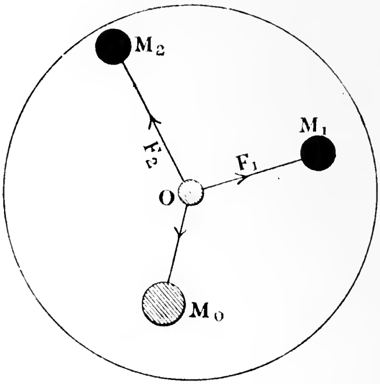


FIG. 14.

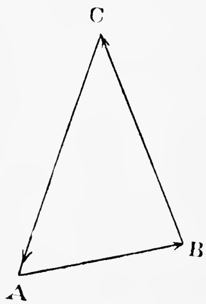


FIG. 15.

in a direction from the axis to the mass M_1, and BC equal to M_2r_2, drawing from the axis to M_2. Then CA, the closing side of the triangle, represents M_0r_0, the balancing product. Transfer the direction CA to Fig. 14, remembering to draw from the shaft. The magnitude of M_0 can be fixed as before, when the radius at which it is to be placed is given.

The method of this article may be extended to any system of co-planar masses, and the next article is a general statement of the proposition.

14. Balancing any Number of Masses, rigidly connected to an Axis, by Means of a Single Mass, all being in the Same Plane of Revolution.—Let M_1, M_2, M_3, . . . M_n, be the given masses, at radii r_1, r_2, r_3, . . . r_n, feet respectively. The angular positions of the radii are to be specified by a drawing. Let M_0 be the balancing mass at radius r_0, and ω the common angular velocity of the system.

When the angular velocity is ω, the masses give rise to centrifugal forces $M_1\omega^2r_1$, $M_2\omega^2r_2$ . . . $M_n\omega^2r_n$, $M_0\omega^2r_0$.

The condition that there may be no dynamical load on the shaft is—

$$\text{Vector sum } (M_1r_1 + M_2r_2 + \ldots + M_nr_n + M_0r_0)\omega^2 = 0$$

In order that this may be true for all values of ω, the vector sum of the terms in the brackets must be zero. Representing them by the lines AB, BC, . . . , these lines set out in order must form a closed polygon. The necessary closure is effected by the side corresponding to M_0r_0, which, being measured to scale, gives the value of the product. Its direction, transferred to the drawing specifying the angles between the radii, fixes the direction of the radius r_0 relative to the given radii. The operation of finding the balancing product M_0r_0 may therefore be stated thus—

Set out the magnitudes of the products of the given masses and their radii as if to form a polygon. The closing side, taken in order with the rest, represents the product M_0r_0.

Example.—Masses of 3 pounds, 4 pounds, and 3 pounds are attached to a disc rotated by the shaft O, at the respective radii

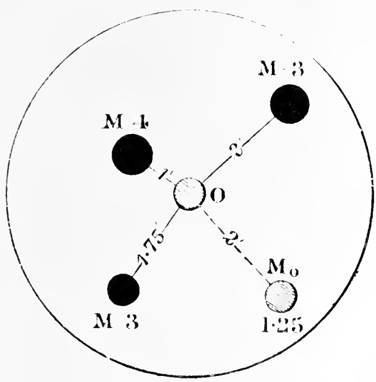


Fig. 16.

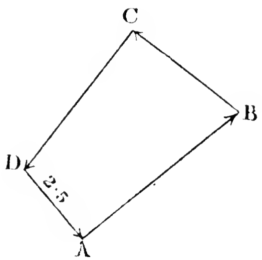


Fig. 17.

2 feet, 1 foot, and 1·75 foot, at angles specified by Fig. 16. Find the balancing product M_0r_0.

The centrifugal forces, when $\omega = 1$, or products of the given masses and their respective radii, are 6, 4, and 5·25. Starting from A (Fig. 17), AB, BC, CD, represent these products set out in order. The closing side measures 2·5; hence—

$$M_0 r_0 = 2{\cdot}5$$

Assuming M_0 to be conveniently placed at 2 feet radius, its magnitude is 1·25 pounds, and its direction is completely specified by DA.

15. Magnitude of the Unbalanced Force due to a Given System of Masses in the Same Plane of Revolution.—If $M_0 r_0$ is the balancing product for the system, the centrifugal force due to the rotation of M_0 is—

$$M_0 r_0 \omega^2$$

Since this balances the centrifugal forces due to the given system of masses, it is equal and opposite to their resultant. Considering the previous paragraphs, the length of AD to scale, Fig. 17, multiplied by ω^2, gives the magnitude of the unbalanced force for the system of masses shown in black by Fig. 16; similarly, $AC\omega^2$ is the unbalanced force due to the two masses M_1, M_2 of Figs. 14 and 15.

In general, the unbalanced force is found by taking the vector sum of the centrifugal forces, assuming $\omega = 1$, and multiplying this sum by ω^2.

16. Mass Centre.—The examples considered will have shown how necessary it is to make as exact a determination as possible of the position of the mass centre of each individual mass before proceeding to find the balancing mass. Where the unbalanced masses have been machined, their form is generally simple, and there is little difficulty in finding this point near enough for ordinary work. Any of the methods generally used for finding the position of the centre of gravity may be used, since the mass centre is a point which may be looked upon as coincident with the centre of gravity. Locating it for irregular masses is more difficult and sometimes impossible, and then recourse must be made to experiment. For instance, a pulley running at a high speed will sometimes cause trouble through being out of balance, even though the rim has been turned inside and out, and obviously the only possible way of balancing it is by experiment.

It may be of interest to show that the general method explained for finding the balancing mass may be extended to find the mass centre of a system of masses whose individual mass centres are known.

The resultant centrifugal force acting on the axis due to the rotation of a given system of co-planar masses, is equal to the centrifugal force due to a single mass equal in magnitude to the arithmetical sum of the masses forming the given system, concentrated at the mass centre of the system. The combination of this principle with the methods of Art. 14 gives a simple rule for finding the mass centre.

Evidently the mass balancing the concentrated mass at the mass centre is the balancing mass for the system. From Art. 12 it follows that the mass centre of the given system and of the balancing mass are on a diameter, and are on opposite sides of the axis. Hence, if x be the distance of the mass centre of the given system from the axis of rotation—

$$(M_1+M_2+M_3+\ldots M_n)x\omega^2=M_0r_0\omega^2=(M_1r_1+M_2r_2+\ldots M_nr_n)\omega^2$$

from which—

$$x=\frac{\text{vector sum }(M_1r_1+M_2r_2+\ldots M_nr_n)}{\text{scalar sum }(M_1+M_2+\ldots M_n)}$$

If the numerator of the fraction is zero, $x=0$; hence, if the vector sum of the centrifugal forces about a given axis is zero, the mass centre of the system is on the axis.

Referring to Arts. 12, 13, 14, it will be seen that the result of the balancing operations is in each case to move the mass centre of the given system on to the axis of rotation by means of the balancing mass.

Example.—Find the mass centre of the system of masses specified in the example of Art. 14 and shown in black in Fig. 16.

The vector sum of the centrifugal forces is represented by AD, the magnitude of which is by measurement 2·5. The scalar sum of the masses is 10. Therefore—

$$x=\frac{2\cdot5}{10}=0\cdot25\text{ foot}$$

measured from O in the direction from A to D.

17. Experimentally Testing the Balance.—The positions of the mass centres of the several masses forming a system, and consequently the position of the mass centre of the system, cannot be found with mathematical accuracy, even when the parts are machined, by any method of calculation, because of the small variations of density throughout the material and the slight deviations from the form assumed in the calculations. There will always be a small error in a presumably balanced system on this account. The error may be quite negligible at a relatively low speed of rotation, but may become important at a higher speed. After a system has been balanced as nearly as possible by the methods given, the most delicate test which can be applied is to run the system at a high speed, mounted, if possible, on springs. The mass which must be added to make the system run quietly may then be found by trial.

The carriage-wheels on the London and North-Western Railway are all balanced experimentally. The system formed by a pair of wheels and their axle, each part of which is of regular form and placed so that the mass centre of the whole is presumably on the axis of rotation, is placed in bearings mounted on springs, as shown in Fig. 18. The system is driven by gear seen at the left hand of the figure, so that the peripheral speed of the wheels is one mile per minute. This corresponds to about 465 revolutions per minute for the standard $43\frac{1}{2}$-inch wheel. Any want of balance is at once shown by the vibration of the bearings on their spring supports. Plates are attached to the inside of the wood centres of the wheels until the system runs steadily.

When the mass and position of a balancing plate is known, the original deviation x of the mass centre of the wheel may be readily calculated. Suppose, for instance, that a mass of 1 pound attached at 1·2 foot radius effectively balanced a wheel weighing 969 pounds—

$$969x = 1 \cdot 2$$
$$\text{or } x = 0 \cdot 00124 \text{ foot}$$

measured from the axis on the line joining the mass centre of the balancing plate to the axis, produced.

This seems a small amount to trouble about, yet at 80 miles per hour the centrifugal force due to this slight deviation of the mass centre from the axis of rotation is 157 lbs. weight approximately. This force changes its direction of action at the horns 618 times per minute, and tends to set up unpleasant tremors in

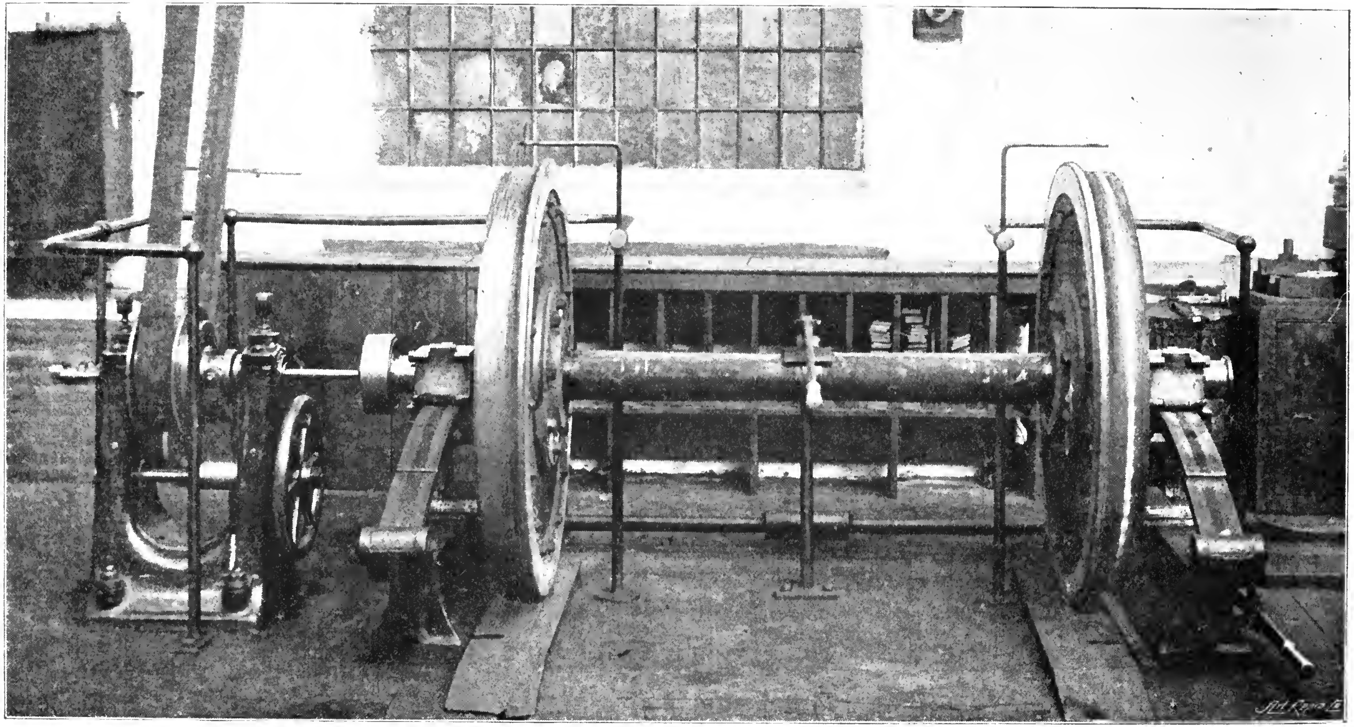


FIG. 18.

the carriage body. However excellent the permanent way may be, for smooth running at high speeds it is essential that every carriage-wheel in the train should be experimentally balanced.

18. Centrifugal Couple.—If the two masses of Art. 12 are placed in different planes of revolution (Fig. 19), their centrifugal forces, though always exactly equal in magnitude and of opposite sign, do not free the shaft from dynamical loading. They form a couple tending to turn the shaft in a plane containing the axis

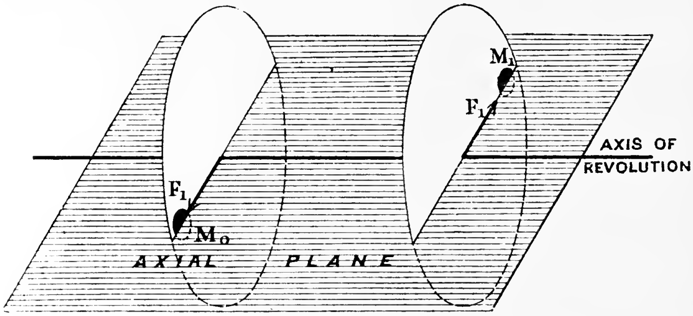


FIG. 19.

of revolution and the centrifugal forces, and therefore the mass centres of the two masses. This may be conveniently referred to as an **axial plane,** because it always contains the axis of revolution. An axial plane is indicated by shading in Fig. 19. It will be noticed that the radii of the masses lie on its intersections with the planes of revolution.

DIGRESSION ON THE PROPERTIES OF COUPLES.

19. A Couple.—A **couple** is the name given to a pair of equal and opposite forces acting in parallel lines.

The perpendicular distance between the lines of action of the forces is called the **arm** of the couple.

In Fig. 20 the pair of equal and opposite forces F, acting in parallel lines a feet apart, form a couple whose arm is a feet long.

Proposition 1.—The turning effort of a couple with respect to any axis at right angles to its plane is the same, and is measured by the product of one of the forces and the arm of the couple.

Let AB, CD (Fig. 21) be the directions of action of two equal, opposite, and parallel forces, acting upon a rigid body free to turn about the axis, O, at right angles to the plane of the couple.

From O draw a common perpendicular to the forces, cutting them respectively at C and A. Let AB = CD represent the

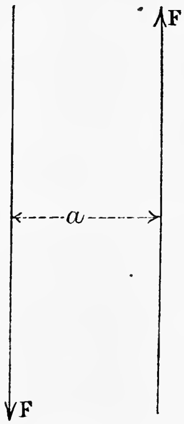


Fig. 20.

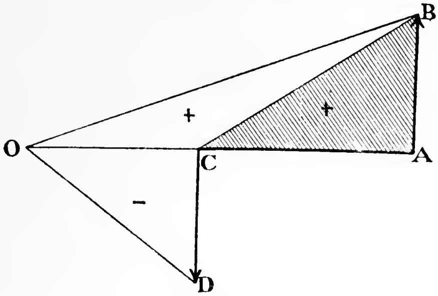


Fig. 21.

respective magnitudes of the forces. Join BC. The turning effort of the couple is equal to the sum of the moments of the two forces with respect to O.

The moment of AB about O is positive, and is represented by twice the area of the triangle OAB.

Similarly, the moment of CD is negative, and is represented by twice the area of the triangle OCD. The resultant moment is represented by twice the difference of these areas—that is, by twice the triangle ABC—since the triangle OCB is equal to the triangle OCD, both being on the same base and of equal altitude.

The **turning effort** or **moment of the couple** is therefore equal to the product of one force and the arm of the couple, and this is evidently constant for a given couple and independent of the position of the axis O. The product is usually expressed in "foot-lbs."

Corollary.—Since the moment of the couple is the same with respect to all axes at right angles to its plane, it follows that the couple may be moved to any new position in its plane, without affecting its moment with respect to a given axis. The several couples shown in Fig. 22 exert equal turning moments on the disc, though they are applied in such different positions.

20. Equivalent Couples.—The moment of a couple is represented by the product of two factors, the arm and a force. Provided that this product remains constant the turning effort of the couple remains constant, however the individual factors are varied. If F, the magnitude of the forces of a couple, be changed to F_1, the arm a must be changed to a_1, so that—

$$Fa = F_1a_1$$

Hence the arm varies inversely as the force for constant turning effort. This is exhibited graphically in Fig. 23. The arm a, of a couple whose moment is 10 foot-lbs., is plotted horizon-

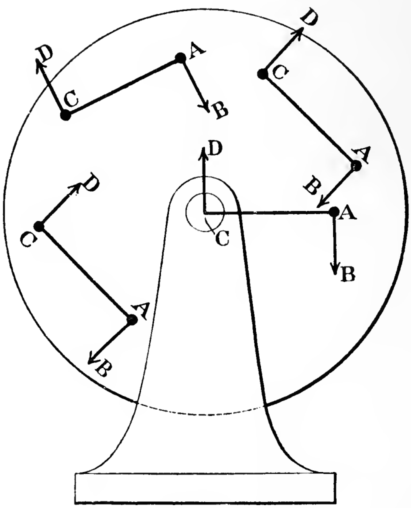


Fig. 22.

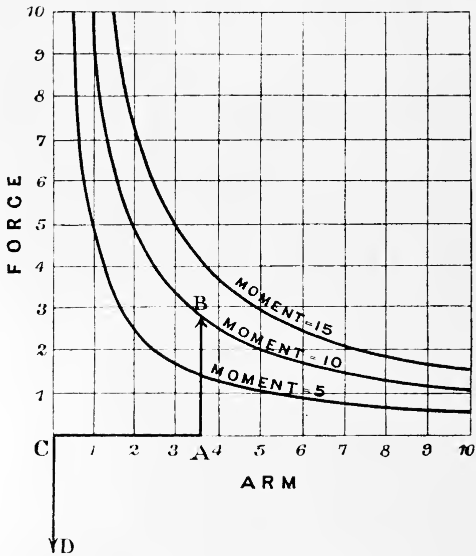


Fig. 23.

tally against the force vertically. Taking an arm of length CA, the corresponding force is represented by the ordinate AB, to the curve marked 10. Notice how rapidly the magnitude of the force must increase to keep the turning effort constant, as the arm is shortened. Curves are also added for moments of 5 and 15 foot-lbs. These curves and the curve of Fig. 13 are the same in form, and follow precisely the same law, being in fact rectangular hyperbolas;

but they represent different quantities, though the quantities themselves have apparently the same name, viz. foot-pound. The pound in the case of the mass moment refers to the quantity of material in the mass, and in the case of the couple to the magnitude of the forces. To save confusion the moment of a couple might be written "foot-lbs. weight," but this is an awkward combination. Generally the context is enough to indicate the meaning of the term "foot-pound." There is no ambiguity if forces are measured in absolute units, because then whilst a mass moment is measured in "foot-pounds," the moment of a couple is measured in "foot-poundals." Again, neither the terms foot-pound nor foot-poundal must be confused with the corresponding work units. To avoid this, it has been suggested to invert the order of the words when the combination refers to the moment of a couple—that is, to write "a moment of so many lbs.-feet," or poundals-feet. The usual way of writing the moment of a couple, viz. "foot-lb.," will be followed, the abbreviation lb. distinguishing it from a mass moment.

21. Axis of a Couple.—A line drawn at right angles to the plane of a couple, whose length, measured *from* the plane, is proportional to the moment of the couple, and on the side of the plane such that, looking along the axis towards the plane, the couple appears to be exerting a counter-clockwise turning effort, is called the **axis** of the couple.

The axis of the couple shown in Fig. 20 is a line Fa units long, projecting at right angles above the surface of the paper. The axis of each couple in Fig. 22 is a line AB × AC units long, projecting below the paper.

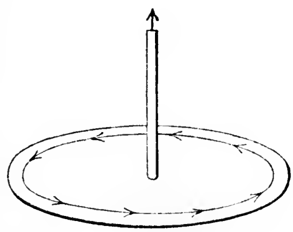


FIG 24.

To find which side of the plane of a given couple the axis is to be drawn, think of an instrument consisting of a handle attached to a cardboard disc (Fig. 24). Suppose a circle to be drawn on the disc indicating the direction of positive rotation as shown. To find the axis, imagine the disc placed in the plane of the couple so that its director circle indicates the direction in which the couple tends to turn. The handle then shows the side of the plane on which the axis is to stand, or, as it is called, the sense of the axis. The moment of the couple is then to be set out in a direction *from* the plane along this axis.

22. Addition of Couples.—A couple is a directed quantity, and it has been shown in the preceding paragraph how to represent it by a line, the axis. Couples may be added by taking the vector sum of their axes.

Three classes of problems present themselves. The couples may act—

(1) In the same or parallel planes;

(2) In planes mutually inclined to one another, but all at right angles to a given plane;

(3) In planes inclined anyhow.

In the first case the axes of the couples are parallel, and their vector sum is taken by adding their lengths algebraically, or, what is the same thing, adding the moments of the couples algebraically. This is the case of a large number of familiar problems; for instance, questions on the equilibrium of levers, of beams and girders, roofs, spur-wheel gearing, all afford examples in which the couples involved have parallel axes.

The system of planes in the second case is represented by an open Japanese screen. The leaves are all inclined to one another, but they are all at right angles to a given plane, the floor. Suppose each leaf of the screen to be the plane of a couple. Each couple will be represented by its axis, standing out at right angles to the one or other side of its leaf. All the axes will be parallel to the floor. To find their vector sum, suppose each to be moved parallel to itself so that the whole group may be laid out in order on the floor. The single line representing their sum is the axis of the resultant couple—that is, it represents the united turning effort of the several couples on the screen, considered as a rigid system. Examples illustrating this case are to be found in questions relative to the equilibrium of three-legged derrick cranes, tripods, the gyrostat; and the application of this principle to find the vector sum of a system of centrifugal couples is one of the leading features of the sequel.

In the third case, the axes of the couples form a system of lines inclined to one another in all directions. Setting them out in order from a selected origin, they form with the closing side a gauche polygon. The actual setting out of the lines can be done by the principles of solid geometry.

23. The Condition that there shall be no turning moment is, in each case, that the axes form a closed polygon. The closed polygon in the first case is a line returning on itself to the origin.

With this slight digression on the properties of couples the course of the main argument may be resumed.

24. Effect of a Force acting on a Rigid Body fixed at one Point.—Let F (Fig. 25) be a given force acting at a perpendicular distance, a, from the fixed point O. The force causes a pressure on the point, equal and parallel to itself. The two aspects of this pressure, together with the original force, form a system of three forces, which split up into—

(1) A force equal and parallel to F acting on the point O;

(2) A couple, whose moment is Fa, tending to cause rotation about an axis through O, at right angles to its plane.

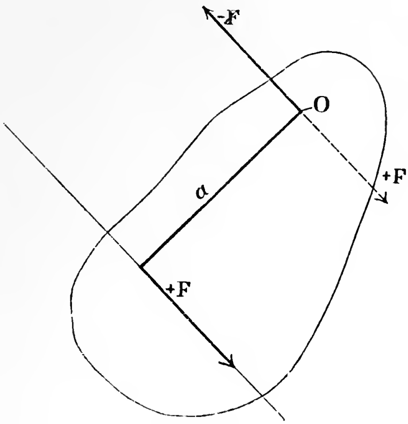


FIG 25.

Fig. 26 illustrates this in a more general way. A sphere is supported at its centre. A force, F, acts upon it at the point p. The ellipse shows the plane of the couple. OC, at right angles to this, is the line about which the sphere will tend to turn, urged by a clockwise or negative couple of magnitude Fa. The dotted force F is the action on the fixed point of support. The line AB, representing the axis of the couple, must be set out parallel to OL, drawing from O towards L.

25. Effect of any Number of Forces acting simultaneously on a Rigid Body fixed at one Point.—Each force is equivalent to an equal and parallel force at the fixed point, and a couple. The resultant force on the fixed point is the vector sum of the "equal and parallel forces" there. The resultant turning effort is specified by the axis, which is the vector sum of the axes of the different couples.

The condition that there may be no force acting on the fixed point is, that the vector sum of the forces be equal to zero; and the condition that there may be no turning effort is, that the

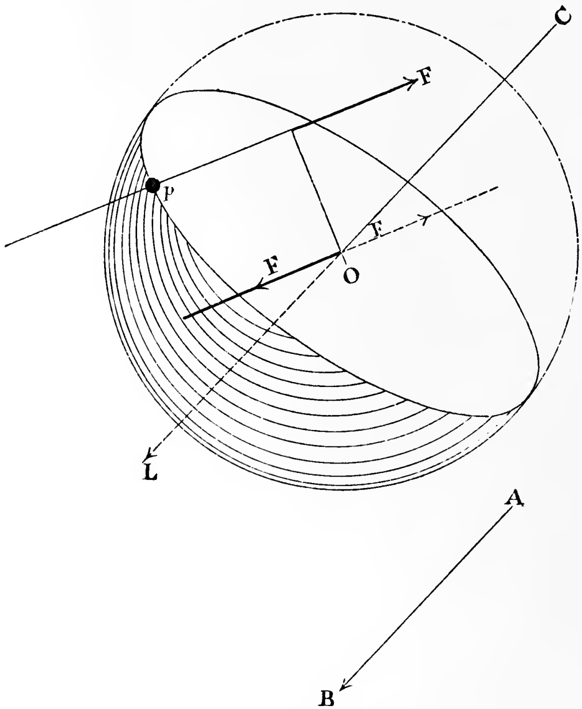


Fig. 26.

vector sum of the axes of the couples be equal to zero. These are two independent conditions, and must be separately satisfied.

The extension of this way of considering the effect of a force to the centrifugal forces acting at the axis of a rotating system is the key to the solution of many balancing problems, and is the feature of the paper communicated by the author to the Institution of Naval Architects, March 24th, 1899. The rotating system is supposed to be free from the constraint of bearings, and to be held only at a fixed point selected at any convenient place on the axis of rotation. The effect of the weight of the system is to be entirely neglected; it acts constantly in one direction, and only loads the bearings with a constant load. In fact, the system to be balanced may be imagined at the centre of the earth, where it would be weightless, but every other condition of the problem would remain the same.

26. Effect of a Centrifugal Force with Reference to a Fixed Point on the Axis of Rotation.—Consider the effect of the mass M (Fig. 27) attached to a truly turned disc D, when rotated by the

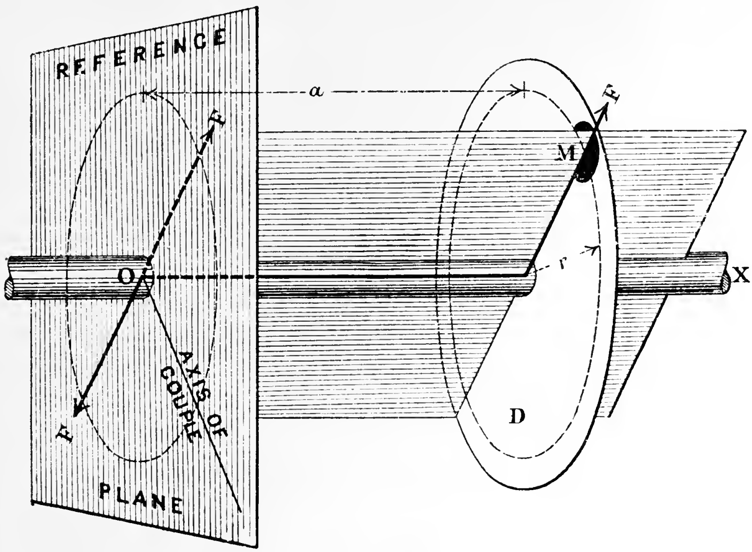


Fig. 27.

shaft OX, which is held only at the fixed point O, distant a feet from M's plane of revolution.

A force, $M\omega^2r$, is exerted on the shaft in the plane of the disc D. This is equivalent to—

(1) an equal and parallel force, $M\omega^2r = F$, acting at the fixed point O, and shown by a dotted line;

(2) a couple whose moment is $M\omega^2ra$, tending to cause

rotation about an axis through O, at right angles to the plane of the couple, in the positive direction.

A plane through the fixed point O, at right angles to the axis of rotation, and revolving with it, will be called the **reference plane.** It contains both the force at the fixed point O, and the axis about which the system is assumed to be free to turn under the action of the centrifugal couple. The **reference plane** may be thought of as a sheet keyed to the shaft, or as the drawing-board on which all the vector summation which is required in the problem may be imagined carried out.

Example.—The effect of a mass of 10 pounds, revolving 4 times per second at 5 feet radius, in a plane distant 5 feet from the reference, is—

(1) A force $\frac{M4\pi^2n^2r}{g} = 980{\cdot}47$ lbs. weight, acting at the fixed point O, in the plane of reference;

(2) A couple of moment $980{\cdot}47 \times 5 = 4902$ foot-lbs., tending to turn the system about the axis shown in Fig. 27.

If a balancing mass or masses be applied to the system, giving rise to an equal and opposite centrifugal couple, there will be no tendency to turn about the fixed point. If at the same time the balancing masses have a resultant centrifugal force, equal and opposite to the resultant centrifugal force at the fixed point, there will be no pressure acting on it. Under these circumstances, the constraint applied to the fixed point may be removed, and the system will continue to rotate without trying to change the direction of the main shaft. It would be a balanced rotating system, and held in bearings, would put no dynamical load upon them.

27. To balance a Single Mass by Means of Masses in Given, Separate, Planes of Revolution.—Many cases arise in practice in which it is inconvenient or impossible to apply the balancing mass in the same plane of revolution as the disturbing mass, in the way illustrated in Arts. 12, 13, and 14.

Under these circumstances at least two balancing masses are required, placed in separate planes of revolution, which may be selected either with the disturbing mass between them, or outside both of them, whichever may be the most convenient arrangement. Having selected the position of these two planes, choose one of them for a reference plane. Then the conditions to be fulfilled by the three masses—that is, the disturbing mass and the two masses balancing it—are—

(1) The sum of the forces at O, the supposed fixed point, must be zero;

(2) The sum of the centrifugal couples must be zero.

Let M (Fig. 28) be the mass to be balanced, by masses placed

FIG. 29.

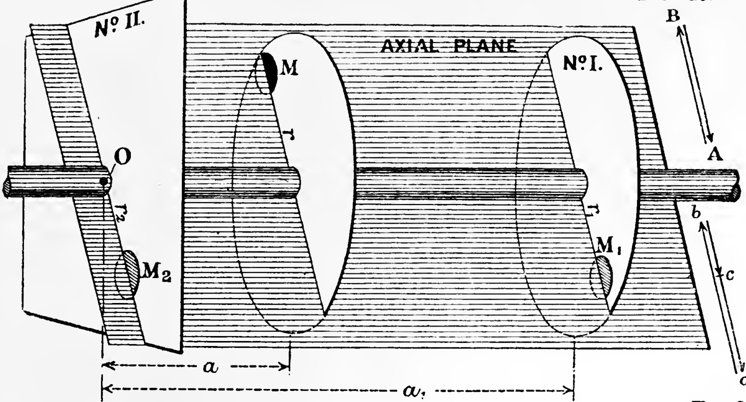


FIG 30.

FIG 28.

in planes Nos. I. and II. respectively. Choose No. II. for the reference plane, O being therefore the fixed point. Let plane No. I. and the given plane be distant respectively a_1 and a feet from the reference plane. The mass in the reference plane, whatever be its magnitude or position, will have no moment about O. Let M_1 be the mass in plane No. I., which will balance the couple due to M about O, acting at radius r_1. The condition that the sum of the moments of the couples about O vanish is expressed by—

$$(M_1 r_1 a_1 + Mra)\omega^2 = 0$$

That is, the vector sum of the centrifugal couples in the brackets must be zero. The direction of the vector representing the couple Mra is at right angles to the axial plane, and it may be drawn by the rule of Art. 21. It is, however, more convenient to imagine that its axis is turned through 90°, so that the direction of the axis corresponds with the direction of the crank, measuring *from* the shaft outwards. Then the side closing the couple polygon is parallel to the crank to be added with the balancing mass, and shows the sense in which the crank radius is to be drawn, viz. *from* the axis of the shaft, *outwards,* in the direction indicated by the arrow-head on the closing side, when, as in the present case, the cranks are all on one side of the reference plane. Hence set out AB (Fig. 29) to represent Mra to scale. The line closing the polygon is BA, equal in length to AB, but points in the opposite direction. This at once fixes the direction of No. 1 crank. The magnitude of the mass it carries is found from—

$$M_1r_1 = \frac{Mar}{a_1} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

when r_1 is fixed.

The separate centrifugal forces due to M and M_1 are each accompanied by an equal and parallel force at O in the reference plane. If M_2 at radius r_2 is the mass in the reference plane, whose centrifugal force will balance the resultant of the transferred forces, the expression—

$$\text{Vector sum } (Mr + M_1r_1 + M_2r_2) = 0$$

states the condition of equilibrium. Setting out ab, bc (Fig. 30) to respectively represent Mr and M_1r_1, the line ca, closing the polygon, represents the force at O in magnitude and direction required to balance the forces there. ca is the crank direction, and the magnitude of the mass it is to carry is evidently given by—

$$Mr - M_1r_1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

for the case shown. When the radius at which M_2 can be conveniently placed is given, the magnitude of M_2 can be found at once.

If the reference plane is between plane No. I. and the given plane, a and a_1 must be considered of opposite sign.

Example.—Suppose M = 10 pounds at 2 feet radius, and a and a_1 are 2 feet and 5 feet respectively.

$$M_1r_1 = 8, \text{ from equation (1)}$$
$$\text{and } M_2r_2 = 12, \text{ from equation (2)}$$

M_1 and M_2 in this case are to be placed in opposition to M, as shown in Fig. 28.

The effect on M_1r_1 and M_2r_2 of varying the position of the plane No. I. relative to the others, is shown by the curves A and B respectively (Fig. 31). In any position of the plane—for instance, when it is a_1 feet from the reference plane—the ordinate

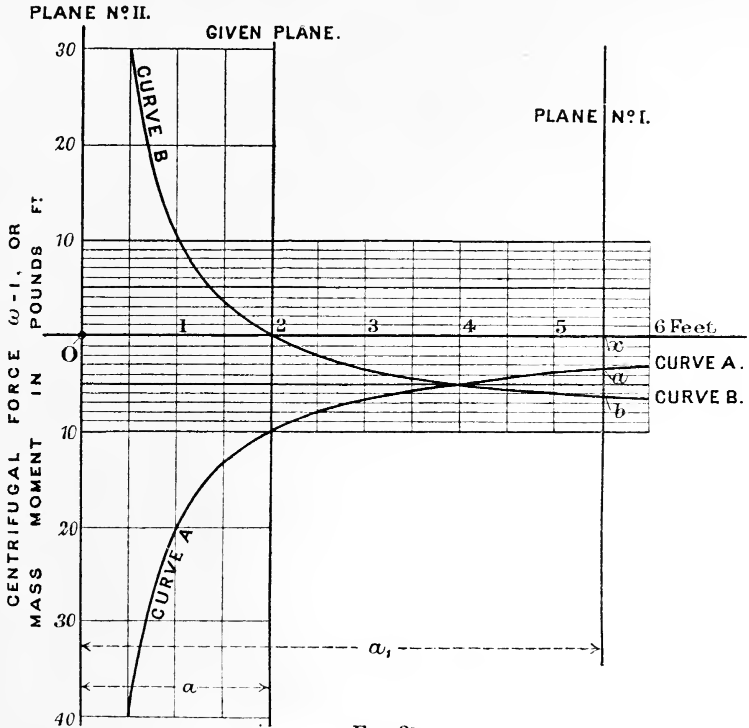


Fig. 31.

of curve A, xa, gives the value of M_1r_1, and the ordinate of curve B, xb, the value of M_2r_2. The figures refer to the case where $Mr = 10$ and $a = 2$. The curves show clearly that M_1r_1 is always negative relative to Mr, and M_2r_2 is negative so long as plane No. I. is to the right of the given plane, and positive when it is on the same side as plane No. II., the reference plane.

28. Balancing any Number of Given Masses by Means of Masses

placed in Two Given Planes.—This and the next three articles state the general method of which the problems and examples already given are particular cases. The earlier and simpler propositions have, however, many practical applications, and should be carefully studied.

Let a_4 be the distance between the two given planes in which the balancing masses are to be placed.

Choose one of these planes for the reference plane, thus fixing the point O. Let M_1, M_2, M_3, etc., be the given masses, at radii r_1, r_2, r_3, etc., respectively, revolving in planes a_1, a_2, a_3, etc., feet from the reference plane.

Let M_5 be the balancing mass in the plane of reference at radius r_5, and M_4 the balancing mass at radius r_4 in the plane, which is by the terms of the problem a_4 feet from the reference plane.

When the system rotates, the centrifugal force corresponding to each mass acts upon the axis, which in turn causes an equal and parallel force to act at the fixed point O, and a couple. The condition that there may be no couple is expressed by—

$$\text{Vector sum } (M_1r_1a_1 + M_2r_2a_2 + \ldots + M_4r_4a_4)\omega^2 = 0 \quad . \quad (1)$$

and the condition for no force on O by—

$$\text{Vector sum } (M_1r_1 + M_2r_2 + \ldots + M_4r_4 + M_5r_5)\omega^2 = 0 \quad . \quad (2)$$

The artifice used to obtain a solution of the problem consists in taking the reference plane coincident with the plane of revolution of one of the balancing masses, so that it has no moment about O, and consequently the balance for couples may be adjusted without it. The reference plane being at No. 5 plane, in the case under discussion, M_5r_5 does not appear in equation (1).

Whatever be the value of ω, the two conditions are separately fulfilled if the vector sums of the terms in the brackets are in each case zero—that is, if when set out to scale they form a pair of closed polygons. Consider equation (1). All the terms are given but $M_4r_4a_4$, of which, however, the factor a_4 is given. Calculate their arithmetical values and set them out to scale, their relative directions being specified by a drawing. The axes of the couples the terms represent are of course at right angles to the axial planes in which they respectively act. A little consideration will show, however, that the directions of the cranks themselves may be used in actually drawing the couple polygon, if the following rules are observed:—

Rules for the Way of drawing Couple Vectors.—If the masses are all on the same side of the reference plane, the direction of drawing is *from* the axis, *outwards*, to the mass, in a direction parallel to the respective crank directions. If the masses are some on one side of the reference plane and some on the other, the direction of drawing is *from* the axis, *outwards,* towards the mass, for all masses on one side; and *from* the mass, *inwards,* towards the axis for all masses on the opposite side of the reference plane, drawing always parallel to the respective crank directions.

The line closing the polygon represents $M_4r_4a_4$. Scaling this off and dividing by a_4, M_4r_4 is known.

Again, calculate the arithmetical values of the terms in equation (2), and set them out to scale, the relative directions being given by the drawing, and include of course the value of M_4r_4 just found from the couple polygon, observing the following rule:—

Rule for the Way of drawing Force Vectors.—Draw always *from* the axis *outwards* towards the mass parallel to the respective crank directions.

The line closing the polygon represents M_5r_5.

Choosing the radii, r_4 and r_5, the magnitude of the balancing masses may be calculated at once. These added to the given system, so that their radii are placed in the relative positions specified by the closing sides of the two polygons respectively, completely balance it for all speeds of rotation.

Checking the Accuracy of the Work.—Having found the balancing masses, add them to the drawing in their proper positions relatively to the given masses; choose a new reference plane anywhere, and draw a new couple polygon relatively to it. If it close, it is safe to infer that no mistake has been made in the work. The force polygon is the same for all positions of the reference plane.

29. Nomenclature.—It will be noticed that each term in the brackets of equation (2), Art. 28, is a mass moment, and that each term in the brackets of equation (1) is the moment of a mass moment with reference to a, the reference plane. The term, "product of inertia," is also used to denote terms of this form. The conditions of balance may evidently be concisely stated as follows :—

(1) The sum of the products of inertia about the axis of rotation must vanish ;

(2) The sum of the mass moments about the axis of rotation must vanish.

When these conditions are fulfilled, the axis is called a principal axis. To avoid using these somewhat unfamiliar terms, the angular velocity may be supposed equal to unity, since it may have any value, in the problem under discussion, without affecting the balance in any way. Then, as already indicated in Art. 12, a term of the form Mr may be referred to as a centrifugal force, and a term of the form Mra as a centrifugal couple.

30. Typical Example.—Three masses (shown black, Fig. 32), rigidly connected to a shaft, are specified in the following list, the distances, a, being measured from a given plane of reference :—

$M_1 = 1{\cdot}0$ pound	$r_1 = 1{\cdot}5$ foot	$a_1 = 7{\cdot}0$ feet
$M_2 = 2{\cdot}0$,,	$r_2 = 1{\cdot}0$,,	$a_2 = 3{\cdot}5$,,
$M_3 = 1{\cdot}8$,,	$r_3 = 1{\cdot}25$,,	$a_3 = 1{\cdot}8$,,

The angles between the mass radii are specified by the dotted lines (Fig. 33).

Find the magnitude and position of two balancing masses, which are to be placed, one in the plane of reference at unity radius, the other in the plane of M_1, also at unity radius.

It will be found convenient to arrange the data in a schedule of the following kind, in order to calculate the arithmetical values of the different terms.

SCHEDULE 3.

Number of plane of revolution.	Magnitude of the mass in pounds (M).	Mass radius measured in feet (r).	Distance from the reference plane in feet (a).	$\omega = 1$.	
				The products Mr, the centrifugal forces.	The products Mra, the centrifugal couples.
No. 1	1	1·5	7 0	1·5	10·5
No. 2	2	1 0	3·5	2·0	7·0
No. 3	1·8	1 25	1·8	2·25	4·05
No. 4	unknown	1·0	7·0	0·63	4·4 (closure)
No. 5	unknown	1·0	0·0	1·7 (closure)	

Draw the couple polygon first (Fig. 34), setting out AB, BC, CD parallel respectively to the crank directions given by Fig. 33, and representing 10·5, 7·0, and 4·05 to scale. The closure DA measures 4·4, and this is the value of $M_4r_4a_4$, in which a_4 is given equal to 7 feet; therefore—

$$M_4r_4 = \cdot 63$$

The angular position of the radius in the plane of M_4, which in this case is given coincident with the plane of M_1, is determined by drawing a line parallel to DA, *from* the axis. Further, if $r_4 = 1$ foot, ·63 pound placed at this radius, as shown in Fig. 32, will balance the centrifugal couples.

$M_4r_4 = \cdot 63$ may now be written in the schedule.

Again, set out the force polygon (Fig. 35), *abcde*, the sides being parallel to the radii given by Fig. 33, and proportional to 1·5, 2·0, 2·25, and ·63 respectively, the forces acting at the point O. The closure *ea* measures 1·7, and this is the value of M_5r_5. Taking $r_5 = 1$, M_5 is equal to 1·7 pound. The angular position of the radius in the reference plane is determined by drawing a line parallel to *ea*, *from* the axis, as indicated in Fig. 33.

These two masses completely balance the given system; in mathematical language, they convert the axis of rotation into a principal axis.

31. To find the Unbalanced Force and the Unbalanced Couple,

with respect to a Given Reference Plane due to a System of Masses rotating at a Given Speed.—The unbalanced force is the vector sum

Fig. 32.

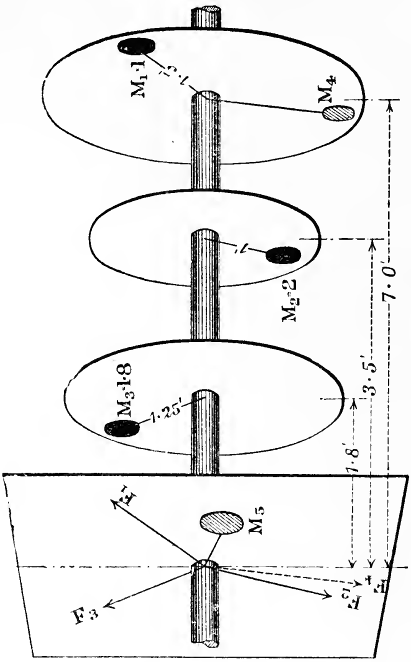


Fig. 33.

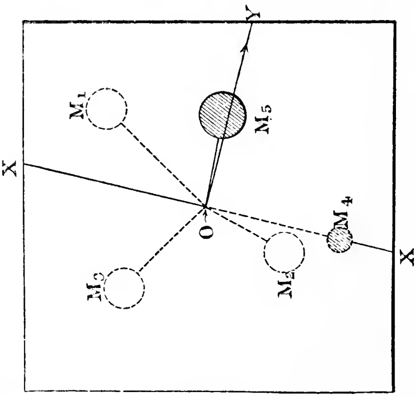


Fig. 35.

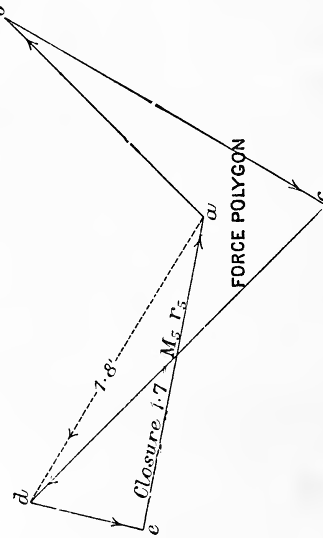
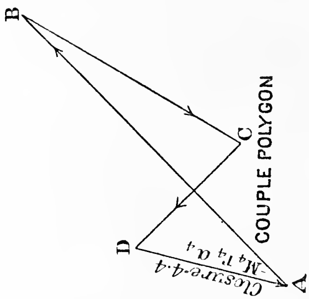


Fig. 34.



of the forces at O, equal and parallel to the centrifugal forces. The unbalanced couple is the vector sum of the centrifugal couples.

Considering the example of Art. 30, the unbalanced couple is represented by AD (Fig. 34), the magnitude of which is $4{\cdot}4\frac{\omega^2}{g}$ foot-lbs. The axis of the couple is at right angles to AD, and is shown by OY (Fig. 33). The magnitude of the resultant force is represented by *ad* (Fig. 35), which measures 1·8 feet to scale; the magnitude of the force is therefore $1{\cdot}8\frac{\omega^2}{g}$ lbs. weight acting at O, parallel to AD.

If the shaft rotates at 10 revolutions per second, these values are 543 foot-lbs., and 221 lbs. weight respectively.

The force and the couple may now be balanced in a more general way than that given in Art. 30. To balance the force, a mass, M, must be placed in the reference plane in a direction, *da*, at such a radius, r, that—

$$Mr = 1{\cdot}8$$

To balance the couple, masses M_a, M_b, at radii r_a, r_b respectively, may be placed anywhere in the axial plane, of which XX (Fig. 33) is the trace, so that if a_c be the axial distance between their radii—

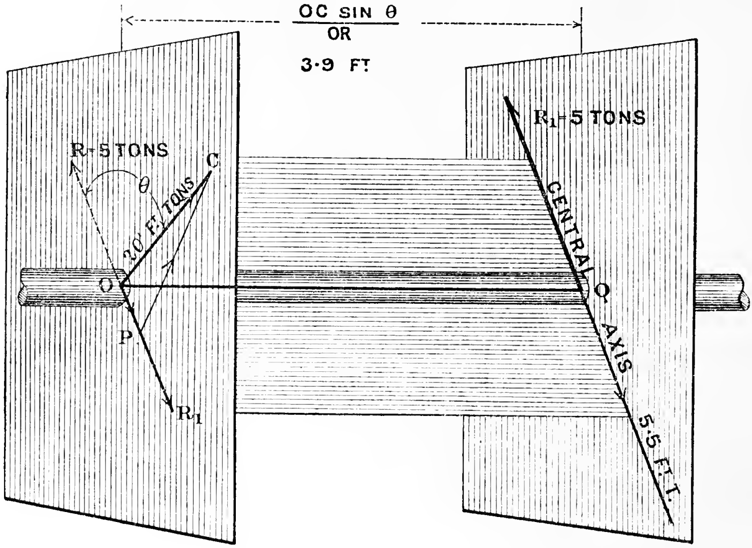
$$M_a r_a a_c = 4{\cdot}4 = M_b r_b a_c$$

their disposition being such that they give rise to a couple opposite in sign to the unbalanced couple. This method gives in general three balancing masses, which of course may be combined into the two of Art. 30. The first method is by far the most convenient for practical use, because it gives the balancing masses without any necessity of thinking of their way of action, these being determined automatically by the closure of the polygons.

32. Reduction of the Unbalanced Force and Couple to a Central Axis.—It will readily be perceived that the magnitude of the unbalanced force is independent of the position of the reference plane, but the magnitude of the couple is different for every new position the reference plane is moved into. No fair comparison of the unbalanced couples belonging to two different systems can be made unless the respective reference planes be moved into the positions for which the magnitudes of the couples are respectively a minimum. But this is a property of Poinsot's central axis, so that the problem resolves itself into finding the central axis for the unbalanced force and couple found with respect to any reference plane. This may be done in the following way:—

Let OR (Fig. 36) be the resultant force for a revolving system; OC the axis of the resultant couple. Resolve the axis in and at

Fig. 36.



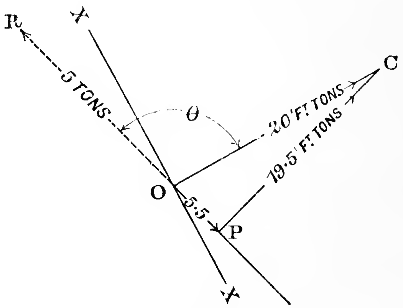


Fig. 37.

right angles to the direction of R. Thus OC is equivalent to a couple, CP, acting in a plane containing OR and the axis of revolution, and OP in a plane at right angles to this—the axis of rotation is the common intersection of these two planes. This resolution is shown in Fig. 37 without distortion. Adjust the couple CP so that its forces are each equal to R, and so that one of them, R_1, acts in opposition to R. R, R_1, being equal, annul one another, leaving a single force, R_1, at a distance $\frac{CP}{R_1} = \frac{OC \sin \theta}{R_1}$ from the plane of reference, and a couple, OP, whose axis coincides in direction with the remaining force. The disturbing forces are then reduced to a single force R_1, and a couple whose axis is along R_1.

The line of action of R_1 is the central axis for the system. It has the property that the disturbing couple about it is a minimum. The central axis may be looked upon as fixed to, and revolving with, the system. Its position relative to the cranks never alters, being determined solely by the disposition of the masses in motion. The magnitudes of the force and couple set off along it vary with the square of the speed.

In Figs. 36 and 37, if OR = 5 tons and the resultant couple OC = 20 foot-tons, the resolved couples acting in and at right angles to the plane containing the resultant force and the axis of rotation are 19·5 and 5·5 foot-tons respectively. The plane containing the central axis is therefore $\frac{19 \cdot 5}{5} = 3 \cdot 9$ feet from the reference plane.

Q is the new origin, and marks the point in the shaft through which the central axis passes.

A fair comparison of the want of balance of different systems can now be made by comparing their central axes. The reduction to the central axis is not of practical importance in a general way.

33. Reduction of the Masses to a Common Radius.—It is often convenient to make a preliminary reduction of the masses to a common radius, the crank radius in an engine problem, or unity in a general problem. The centrifugal force $M\omega^2 r$ is proportional to the product Mr, and the individual factors, M and r, may have any value, providing that their product remains constant. This point has already been exemplified in Art. 12, Fig. 13. If R represents the common radius, the reduced masses $\mathsf{M}_1, \mathsf{M}_2, \mathsf{M}_3$, etc., are obtained from the several equations—

$$\mathsf{M}_1 R = M_1 r_1,\ \mathsf{M}_2 R = M_2 r_2,\ \mathsf{M}_3 R = M_3 r_3, \text{ etc.}$$

Substituting these equivalent products in equations (1) and (2) of Art. 28, they become—

$$\text{Vector sum } (\mathbf{M}_1 a_1 + \mathbf{M}_2 a_2 + \ldots + \mathbf{M}_4 a_4)\omega^2 R = 0 \quad . \quad (3)$$
$$\text{Vector sum } (\mathbf{M}_1 + \mathbf{M}_2 + \ldots + \mathbf{M}_5)\omega^2 R = 0 \quad . \quad (4$$

If the vector sums in the brackets separately vanish, the two conditions of balance are fulfilled for all speeds. **M** is really a mass moment, and **M**a a moment of a mass moment, but, to avoid using unfamiliar terms, the factor ω^2R may be supposed equal to unity; then **M** may be referred to as a centrifugal force, and **M**a as a centrifugal couple.

To calculate the value of the centrifugal force and couple in general, if the vector sum in the brackets be not zero, the quantities **M** and **M**a representing the respective sums must be multiplied by $\frac{\omega^2}{g}$R, the g being introduced to obtain the result in lbs. weight units of force. In what follows the ordinary capital M denotes mass at crank radius unless otherwise stated.

34. Conditions which must be satisfied by a Given System of Masses so that they may be in Balance amongst themselves.—Suppose all the masses to be first reduced to equivalent masses at a common radius so that the terms "equivalent mass" and "equivalent mass moment" may be used instead of "mass moment" and "moment of mass moment" respectively. Then—

(1) It must be possible to draw a closed polygon whose sides are proportional to the equivalent masses, and parallel in direction to the corresponding mass radii;

(2) It must be possible to draw a closed polygon whose sides are proportional to the equivalent mass moments taken with respect to any reference plane.

If condition (1) is satisfied and not (2), there is no unbalanced force, but there is an unbalanced couple.

Condition (2) cannot be satisfied unless (1) is satisfied, for although the couple polygon may be closed for any reference plane, yet if the plane is moved into a new position, the couple polygon for the new position will close only if there is no force in the old reference plane—that is, only if condition (1) is satisfied.

Balancing problems are conditioned, therefore, by the geometrical properties of two polygons, whose sides are parallel, but of different lengths, the sides of the one being obtained from the sides of the other by multiplication, the multiplier in each case being the distance of the equivalent mass from the reference plane.

35. On the Selection of Data.—The balancing of a given system in a specified way can only be done if there are four independent variables left to be determined by the pair of closed polygons which condition the balancing, since the closure of each polygon requires that two, and two only, quantities be left unknown (Art. 8). In fixing the preliminary data, therefore, it is necessary to know exactly how many variables there are concerned in the proposed problem. To completely specify any one mass in the system there must be given—

(1) The equivalent magnitude of the mass at unity radius;

(2) The direction of the radius of the mass measured from a line of reference in the reference plane;

(3) The distance of the plane of revolution of the mass from the reference plane;

so that, if there are n masses there will be $3n$ quantities altogether in the complete system. But these quantities are not all independent variables.

Consider the number of independent variables as regards—

(1) *The Magnitudes of the Masses.*—The closed force polygon only determines the ratios of the several equivalent masses represented by its sides, and not their absolute magnitudes. An infinite number of similar polygons could be drawn satisfying the same conditions. Consequently, any one side may be considered unity, and the magnitudes of the rest expressed in terms of it. The number of independent variables is, therefore, one less than the number of sides forming the closed polygon—that is, one less than the number of masses concerned in the problem—so that if there are n masses, the number of independent variables of magnitude is—

$$n - 1$$

(2) *With regard to the Directions of the Mass Radii or Cranks.*—The specification of $(n - 1)$ variables of direction is sufficient to determine the n angles of a closed polygon, so that the number of independent variables of direction corresponding to n masses is—

$$n - 1$$

(3) *With regard to the Distances apart of the Planes of Revolution.*—Measuring from any arbitrarily chosen reference

plane, there are n quantities concerned in fixing the position of n planes of revolution relative to it: dividing each distance by any one of the distances, there will result—

$$n - 1$$

variables of distance.

The total number of independent variables corresponding to n masses is therefore in general—

$$3(n - 1)$$

The first step in the process of balancing a system is to ascertain how many masses there are concerned in it, including, of course, any masses it may be proposed to add as balancing masses. Call this number n. The number of independent variables is then $3(n - 1)$. The number of independent variables which must be left to effect closure of the two polygons is four; therefore $3(n - 1) - 4 = 3n - 7$ independent quantities must be fixed, and no more.

In choosing these quantities, it must not be forgotten that the number specifying the magnitude of a mass is only to be considered an independent variable if one of the masses is called unity; when this is not done, it is the ratio of a pair of masses which is an independent variable, so that fixing the magnitude of two masses is equal to fixing one variable quantity, fixing three masses equivalent to two, and generally fixing n masses is equivalent to fixing $n - 1$ independent variables. Also, if the reference plane does not coincide with the plane of revolution of one of the masses, fixing n distances from it is equal to fixing $n - 1$ independent variables. Coincidence between the reference plane and a plane of revolution determines that one of the $n - 1$ variables of distance $= 0$.

By way of example, suppose a balanced system is to consist of five masses; in general, $3 \times 5 - 7 = 8$ of the independent quantities must be settled to start with, but no more. Suppose now that the distances of the five planes of revolution are given from any arbitrarily chosen reference plane; this is equivalent to fixing four of the eight quantities. Next, suppose the magnitudes of three masses to be fixed; this is equal to fixing two of the variables, leaving two more to be fixed, which may be two of the crank angles. If anything else is fixed, the data becomes inconsistent, and the problem cannot be solved.

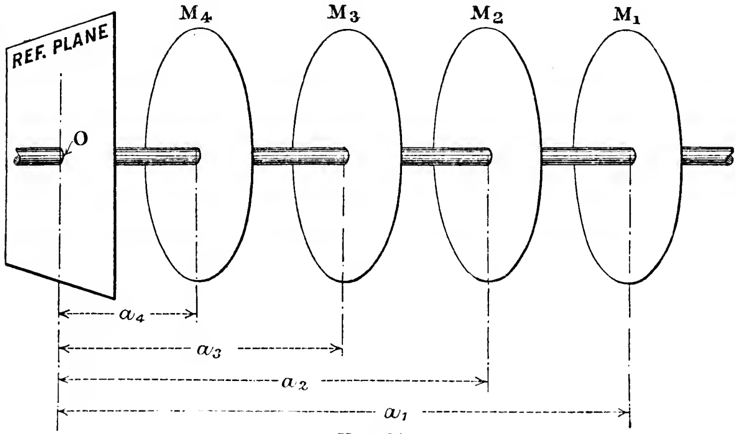


FIG. 38.

FIG. 40.

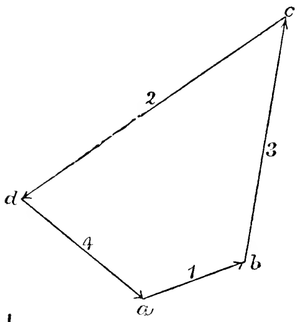
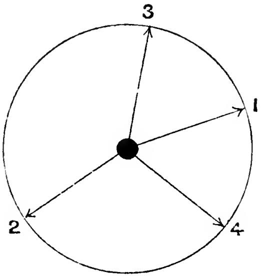


FIG 41.



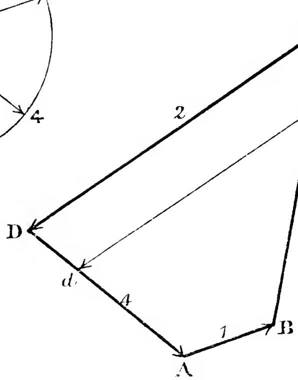


FIG. 39.

36. Relation between the Polygons.—Let equivalent masses, M_1, M_2, M_3, M_4, be distant a_1, a_2, a_3, a_4, feet respectively from the reference plane at O (Fig. 38), and suppose the system to satisfy the conditions of balance, and that ABCD (Fig. 39) and *abcd* (Fig. 40) are the force and couple polygons, the order of drawing the sides being 1, 3, 2, 4, to avoid re-entrant angles. Then—

$$\begin{aligned} AB : ab &= M_1 : M_1a_1 = 1 : a_1 \\ BC : bc &= M_3 : M_3a_3 = 1 : a_3 \\ CD : cd &= M_2 : M_2a_2 = 1 : a_2 \\ DA : da &= M_4 : M_4a_4 = 1 : a_4 \end{aligned}$$

If the polygons are similar, the ratio between corresponding sides will be constant, in which case—

$$a_1 = a_3 = a_2 = a_4$$

that is, all the masses are in the same plane of revolution. If a pair of ratios are equal, the corresponding masses are in the same plane of revolution.

This principle shows that **two masses** cannot balance one another unless they are in the same plane of revolution, for the only pair of polygons which can be drawn are a pair of lines returning on themselves, ABA and *aba*, in which the ratio—

$$AB : ab = BA : ba$$

Assuming **three masses** to be in balance, let ABC (Fig. 42)

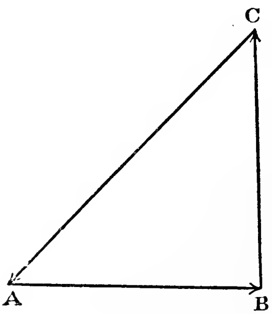


FIG. 42.

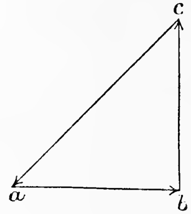


FIG. 43.

be the force polygon, in this case a triangle. No other triangle, as *abc* (Fig. 43), can be drawn with parallel sides unless it be similar; therefore the three masses must be in the same plane of revolution.

If, however, the force polygon closes up into a line returning on itself, as AB + BC + CA (Fig. 44), a second line may be drawn, as *ab* + *bc* + *ca* (Fig. 45), so that the ratios of corresponding sides, or segments, are different. It follows that the masses may be placed in different planes of revolution, but now they must all lie in an axial plane.

A pair of quadrilaterals for **four masses** may be drawn in which the four ratios of corresponding sides are all different,

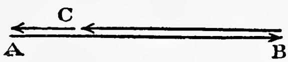


FIG. 44.

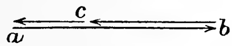


FIG. 45.

providing opposite pairs of sides in one of the polygons are not parallel. Let Figs. 38, 39, and 40 illustrate this case. Alter the scale of the couple polygon until *ab* is the same length as AB. This is equivalent to making a_1 equal to unity. Then—

$$\text{AB} : ab = 1 : 1 \qquad \text{CD} : cd = 1 : \frac{a_2}{a_1}$$

$$\text{BC} : bc = 1 : \frac{a_3}{a_1} \qquad \text{DA} : da = 1 : \frac{a_4}{a_1}$$

Then the couple polygon may be superposed on the force polygon so that *ab* coincides with AB, D*a* with *da*, and B*c* with *bc*; *cd* remaining parallel to CD (Fig. 39). From this it is evident that if any quadrilateral, as ABCD, is drawn, and a line *dc* is drawn anywhere parallel to one side, cutting the other sides, produced if necessary in *d* and *c*, then—

(1) The common side as AB represents unity mass at unity distance from the reference plane;

(2) The sides of the quadrilateral ABCD are proportional to the equivalent masses, that is, to the centrifugal forces;

(3) The directions of the sides transferred from the force polygon to an end view of the shaft give the crank angles;

(4) The ratios of a pair of corresponding sides of the two quadrilaterals, ABCD, AB*cd*, is the distance of the corresponding mass from the reference plane, *a* being unity.

These things are true for a pair of polygons of any number of sides, superposed so that they have a common side, the remaining sides being parallel; if the ratios between two pairs of sides are equal the corresponding masses will be in the same plane of revolution.

Example.—Suppose the quadrilateral ABCD and the parallel *cd* of Fig. 39 to have been drawn at random. Measure off the different lengths concerned to any convenient scale and arrange them as in the following schedule:—

SCHEDULE 4.

Plane of revolution.	Proportional numbers for the masses at unit radius.	Proportional numbers for mass moments.	Proportional numbers for the distances of the several planes of revolution from the reference plane.
No. 1	AB = 1·5	AB = 1·5	a = 1·0
No. 2	CD = 5·85	cd = 4·55	a = 0·778
No. 3	BC = 4·9	Bc = 3·5	a = 0 715
No. 4	DA = 3·3	dA = 2·2	a = 0·667

Divide the numbers representing the mass moments by the corresponding numbers representing the masses, to obtain the numbers proportional to the distances of the planes of revolution from the reference plane. These numbers are given in the column to the extreme right of the schedule.

Then if a system be arranged in which the magnitudes of the masses are in the proportion of the figures of column 2 of the schedule, the crank directions being given by the direction of the sides of the force polygon, and the spacing of the planes of revolution in the proportion of the figures of column 4, the system will be perfectly balanced, and will run at any speed without putting any dynamical load on the shaft.

Mr. Macfarlane Gray * has shown how the related polygons for a four-crank system may be built up of wooden laths, leaving them freedom of distortion. Then pulling the frame into any position, the crank angles, etc., for balance can be derived from it. Obviously polygons of any number of sides may be built up to form a flexible frame, the balancing conditions being determined

* *Trans. I.N.A.*, vol. xlii., 1900.

for any configuration in the way already illustrated by Schedule 4.

The parallel *cd* (Fig. 39) may be drawn anywhere; it merely fixes the position of the reference plane (Fig. 38) relatively to the planes of revolution. Consider Fig. 46. Suppose *cd* to coincide with CD.

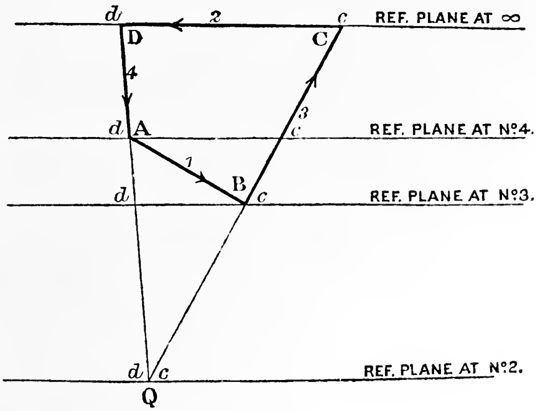


FIG. 46.

Then—

$$Ad : AD = Bc : BC = 1$$

this cannot be true unless the reference plane be at an infinite distance from the plane of revolution. If *cd* passes through the point A—

$$Ad : AD = 0$$

therefore $a_4 = 0$, and the reference plane coincides with the plane of M_4.

Similarly, if it passes through B, the reference plane coincides with the plane of M_3.

At Q—

$$cd : CD = 0$$

and the reference plane coincides with the plane of M_2.

Notice that the ratios $a_4 : a_1$ and $a_3 : a_1$ become negative when *cd* cuts either DA or CB produced. All the ratios approach infinity as the reference plane approaches the plane of M_1, simply because its distance from that plane is reckoned unity.

37. Geometrical Solutions of Particular Problems. Four-Crank Systems.—The problem of finding the positions of the planes of revolution for any force polygon drawn at random may be solved geometrically. Let ABCD (Fig. 47) be drawn at random. Number the sides in the way shown. Select the point of intersection of the two shorter sides; B in the figure. Draw the diagonal which this point subtends, AC. Draw B*c* and B*d* parallel to AD and CD respectively, intersecting AC in *e* and *f*. Then if AC represents the distance between the extreme planes of revolution, *e* and *f* give

Fig. 48.

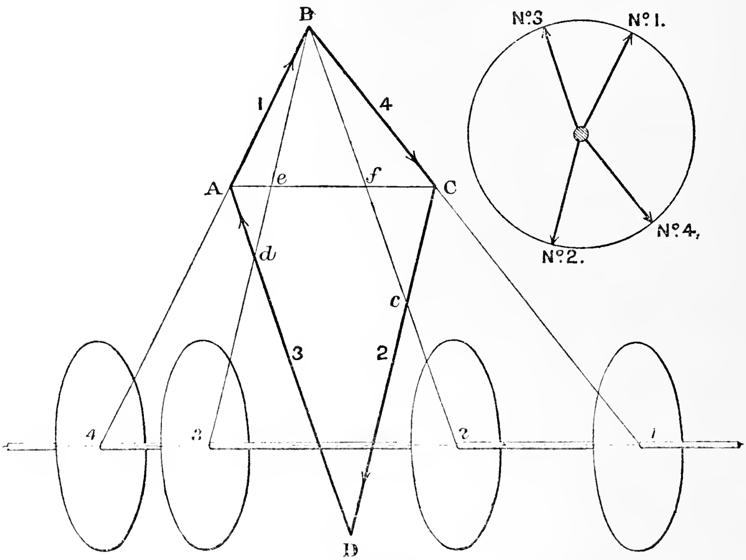


Fig. 47.

the positions of the two inner planes. If the lines meeting at B are produced indefinitely, forming a pencil of four rays, any line drawn across the pencil parallel to AC is divided in the same ratio as AC, and may therefore be taken to represent to some scale the relative positions of the planes of revolution. Notice that the position of No. 2 plane is fixed by a parallel to No. 3 side, that No. 3 plane is fixed by a parallel to No. 2 side, and that Nos. 1 and 4 planes are determined respectively by Nos. 4 and 1 rays of the pencil. There is thus a reciprocal connection between the positions of the inner planes and the sides of the corresponding force polygon, and a similar connection between the outer planes and the force polygon. The proof of this construction depends upon the fact that if the plane of reference is at No. 4 plane, AB*d* is the couple triangle (see Fig. 46); if at No. 1 plane, BC*c* is the couple triangle. Under these circumstances—

$$Cc : CD = a_2 : a_4$$

and—

$$Ad : AD = a_3 : a_1$$

But in the triangle ACD, *cf* is parallel to AD, therefore—

$$Cf : CA = Cc : CD = a_2 : a_4$$

Similarly—

$$Ae : AC = Ad : AD = a_3 : a_1$$

Therefore if CA represents a_4, C*f* represents a_2, measuring from C. The reference plane is at No. 1, and therefore point C corresponds with No. 1 plane. Also if AC represents a_1, A*e* represents a_3, measuring from A. The reference plane is at No. 4 plane, therefore A corresponds with No. 4 plane. But $a_4 = a_1$, therefore AC is divided by *e* and *f*, so that the four points A, *e*, *f*, C represent to scale the relative positions of the planes of revolution along the axis corresponding to the force polygon ABCD.

If the positions of the four planes of revolution are given, an inverse construction will disclose the force polygon, the first step in the process discovering the crank angles for which balance is possible. Stating the method categorically, suppose the shaft and the position of the planes 1, 2, 3, 4 (Fig. 47) to be given. Take any point B. There is no restriction in the selection of this point; it may be taken anywhere. Join the points on the axis to B, forming a pencil of four rays. The directions of these rays fix the relative positions of the crank angles, though in drawing the cranks from them the reciprocal relation already stated must not be forgotten. That is, the ray B4 (the 4 referring to the figure 4 on the shaft) gives the direction of No. 1 crank (Fig. 48), B1 No. 4 crank, B3 No. 2 crank, and B2 No. 3 crank. The way the cranks radiate from the axis is best fixed by drawing parallels to the side of the force polygon; ambiguity of sense is thereby avoided.

To determine the force polygon, draw any line across the pencil, as AC in the figure, parallel to the axis of revolution. This determines the two sides AB and BC. Draw CD and AD parallel to B3 and B2 respectively. Then the sides of the polygon so formed represent the masses at crank radius which will be in balance for the crank angles determined by the position of the point B and the given planes of revolution.

The method of fixing the crank angles for which balance is possible, by drawing a pencil of rays to a point B, was indicated by Dr. Schubert, in Mr. Schlick's paper on Balancing Engines, Institute of Naval Architects, 1900, and earlier in a paper contributed by Dr. Schubert to the Hamburg Mathematical Society, 1898.

38. Experimental Apparatus.—The principles of this chapter may easily be verified experimentally by means of the apparatus shown in Fig. 49. A wood frame carries an accurately turned and well-mounted steel shaft, on which are arranged four carefully turned and balanced discs. One disc, shown to the front in the figure is fixed to the shaft and carries a protractor, the others are capable of angular and longitudinal adjustment relatively to the shaft. The radius defined by a disc is fixed by a small hole, drilled near the periphery. The system is driven by a motor also carried on the frame, so that the discs may be driven at any speed by the motor, free from the action of any external driving force.

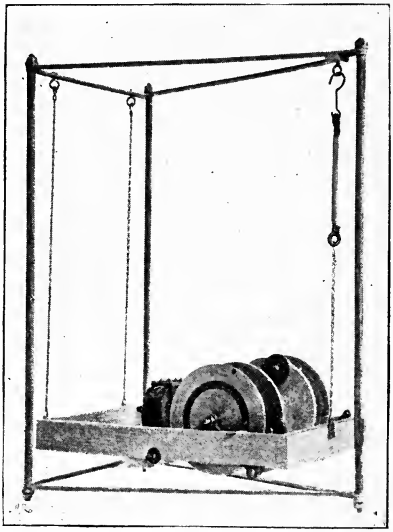


Fig. 49.

The only unbalanced forces acting on the system are therefore those due to the rotation of the discs, which should be nothing. This is tested by slinging the frame on chains in the way shown in the figure. Any want of balance is at once apparent, when the system is driven, by the vibration of the apparatus. Assuming the system to be balanced, the four discs now serve to carry any assigned set of crank-pin masses. These are bolted to the discs at the crank-pin holes (two such masses are shown in the figure), the discs are set to the proper crank angles by means of the protractor on the front one, and the proper distance apart by means of the longitudinal adjustment. Any want of balance is at once shown by the oscillations set up when the system is driven. The first apparatus of this kind was designed by Professor Ewing for the Engineering Laboratories at Cambridge.

CHAPTER III.

THE BALANCING OF RECIPROCATING MASSES.—LONG CONNECTING-RODS.

39. The Force required to change the Speed of a Mass of Matter moving in a Straight Line.—The last chapter was devoted to the consideration of the effect of forcing a system of masses to move in circular paths at uniform speeds. Turn now to the case where the natural straight path of a mass is not interfered with, but the speed in that path is changed from instant to instant by the action of a force. In general, if M is the mass in pounds of the moving body, and A the acceleration produced by the force F, acting at the mass centre—

$$F = MA \text{ poundals}$$

or—

$$F = \frac{MA}{g} \text{ lbs. weight}$$

Since in balancing problems the magnitudes of the forces are generally not concerned, it is more convenient to use the first expression, avoiding thereby the introduction of g into the work. In the steam-engine mechanism, when the motion of the crank-pin is given, the corresponding acceleration of the piston can be found for any given position of the gear, though the expression giving it is a complicated one; its consideration will be deferred for the present. On the other hand, if the connecting-rod be imagined infinitely long, the expression is a simple one. In many cases in practice the difference between the true accelerating force acting on the piston, and the force calculated on the assumption that the connecting-rod is infinitely long, is small enough to be negligible. The present chapter is concerned in showing how the effects of the accelerating forces, acting on the reciprocating masses, may be balanced, supposing these masses to be operated by infinitely long connecting-rods, that is to say, supposing their motion to be simple harmonic.

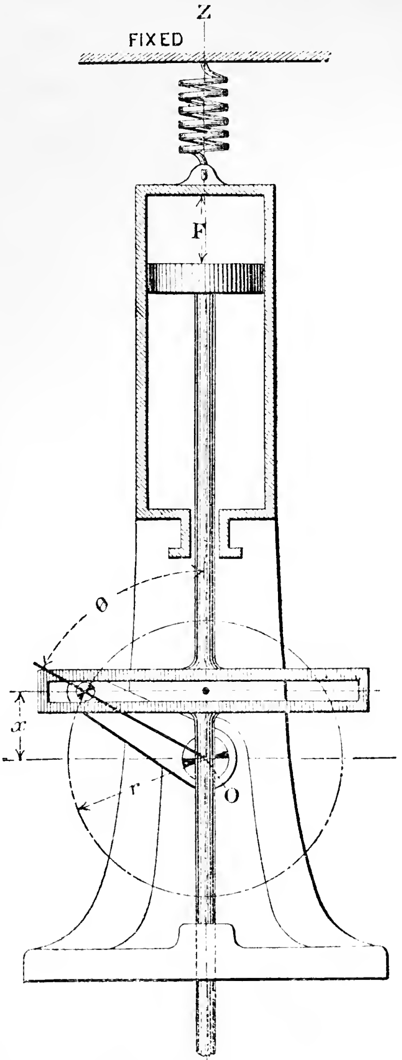


Fig. 50.

40. Value of the Acceleration of One Set of Reciprocating Masses, when the Connecting-rod is infinitely Long.—The motion given by the crank (Fig. 50) to the slotted bar, which here represents a set of reciprocating parts, is precisely the same as if the bar were connected to the crank by an infinitely long connecting-rod. The mechanism is, in fact, a practical way of expressing the conditions of the problem. Let θ be the variable angle between a fixed line of reference OZ, the vertical centre line of the gear, say, and the crank whose radius is r. The movement of the slotted bar from its central position, in terms of the angle, is given by the expression—

$$x = r \cos \theta$$

Differentiating this twice with respect to the time, and considering the angular velocity, $\frac{d\theta}{dt} = \omega$, of the crank to be sensibly constant, the acceleration of the slotted bar is—

$$\frac{d^2x}{dt^2} = -\omega^2 r \cos\theta = \mathrm{A} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

and consequently the instantaneous value of the accelerating force, F, is given by the expression—

$$-\mathrm{M}\omega^2 r \cos\theta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Thus, if the mass of the slotted bar and all its attachments were 500 pounds, and the crank radius were 1·5 feet, the revolutions per second being 3, the magnitude of the accelerating force when θ is 60°, would be 4140 lbs. weight approximately.

41. The Accelerating Force and its Action upon the Frame.— However the reciprocating mass of Fig. 50 be moved, whether by the effect of fluid pressure acting within the cylinder, or by the action of a torque applied to the crank-shaft, given that the crank revolves uniformly, the instantaneous value of the accelerating force can always be calculated for a given crank angle from equation (2) of the previous article. In the case of steam-engines or gas-engines this force must be taken from or added to the total fluid pressure acting on the piston, to find the force transmitted along the piston-rod to the crank, according as the acceleration is positive or negative. Whatever be the instantaneous value of the accelerating force, its action on the reciprocating masses necessarily involves the action of an equal and opposite force on the frame of the engine. The force is indicated by F (Fig. 50), the one aspect accelerating the motion of the reciprocating parts, the other the motion of the framework. The acceleration of the motion of a shot from a gun is always accompanied by the recoil, the movement of the shot and the gun being necessary consequences of the action of the pressure due to the explosion. In explosion motors, as gas-engines, the same thing happens, only that the shot, now a piston, must move in a way prescribed by its connection to the crank-shaft, and if the engine frame were free to move in the direction of the line of stroke, the recoil would take place just as if the cylinder were a gun. In the steam-engine the phenomenon is precisely similar. In considering these examples, it should be distinctly borne in mind that the recoil of the framework depends only upon the acceleration of the shot, the piston or the reciprocating masses, whatever be the name given to the bodies accelerated ; and has nothing whatever to do with the agent causing the acceleration. For instance, given that an engine crank-shaft revolves uniformly three times a second, and that the reciprocating masses weigh 500 pounds, the magnitude of the accelerating force is 4140 lbs. weight when the crank angle is 60°. The recoil of the framework is at that instant progressing under the action of this force, whether the motion of the reciprocating parts is due to the action of steam pressure, or to the explosion of a gas, or to the action of a turning effort on the crank-shaft as when the gear is working as an air compressor.

If such an arrangement as Fig. 50 were suspended from a spring and driven with fluid pressure, the framework would be seen to oscillate up and down, oppositely to the piston, but in time with it. The mass of the framework being so much greater than that of the piston, the oscillations would be in general of smaller amplitude. If held from oscillation by holding-down bolts, they have continually to sustain the effect of the accelerating force. They transmit the effect to the foundations, and troublesome tremors are the usual consequences. In engines with more than one cylinder, the frame forces corresponding to F form a system which can, at any instant, be reduced to a force and a couple. Then, in addition to the effect of the force, the couple tends to tilt or rock the frame in the plane containing the reciprocating masses.

42. The Balancing of a Reciprocating System consists in the arranging of the masses forming the system, so that the accelerating forces acting on the framework form a system in equilibrium.

A **system** of reciprocating masses is formed of any number of masses moving in parallel lines in the same plane, and whose periodic times are equal. In the steam-engine the masses forming the system move in parallel paths, and with few exceptions there is only one system of reciprocating masses to be considered. The exceptions are those cases where the cylinders are inclined to one another. In cases of this kind the number of systems is equal to the numbers of different planes in which reciprocation takes place. The principle to be observed in balancing engines of this class is that each system must be balanced independently of the others.

43. Simple Harmonic Motion and its Relation to Circular Motion.

—If a mass M, equal to the reciprocating mass of Fig. 50, is concentrated at the crank-pin, the centripetal force required to constrain its motion in the circular path is, by Art. 9—

$$-M\omega^2 r$$

The resolved component of this in the line of stroke is—

$$-M\omega^2 r \cos \theta$$

This latter expression is the same in form as expression (2) (Art. 40). Similarly, the resolved component of the centrifugal force in the line of stroke is the same in form as the force acting on the frame in consequence of the action of the accelerating force to produce reciprocation. Hence—

The disturbing force on the frame due to the reciprocation of a mass M *is equal to the disturbance which would be produced by the component of the centrifugal force in the line of stroke, due to an equal mass* M *supposed concentrated at the crank-pin.*

Thus, instead of fixing the mind on the reciprocating mass and the force accelerating it, picture this mass transferred to and revolving with the crank-pin; the force actually disturbing the frame for any position of the gear is then disclosed by the projection of the centrifugal force due to the transferred mass, on the line of stroke.

If there are several reciprocating masses connected to the same shaft by infinitely long connecting-rods, the whole action on the frame may be considered due to the combined effect of the components of the centrifugal forces due to the rotation of the several reciprocating masses at their respective crank-pins.

44. Method of investigating the Balancing Conditions of a System of Reciprocating Masses whose Motion is Simple Harmonic or may be considered so without Serious Error.—Let ABCD (Fig. 51) be a closed force polygon in the reference plane, which is supposed to be keyed to, and to revolve with the shaft, whose end is shown at O. The shaft carries four masses whose respective centrifugal forces are represented by the sides of the polygon. Let ZZ be any fixed line. It is clear that in the position shown in Fig. 51, the sum of the projections of the sides of the polygon on the line ZZ is zero. But these projections are the components of the centrifugal forces represented by the sides of the force polygon, and therefore, if the masses concerned in drawing the polygon are reciprocated in the plane of which *ZZ* is the trace, the sum of the disturbing forces due to their reciprocation is, for the position shown, zero, since these forces are instantaneously represented by the several projections (drawn to the right of *ZZ* for clearness) shown in Fig. 51. Again, consider the system when the shaft, and therefore the reference plane and the polygon on it, has turned into the position shown in Fig. 52, the line *ZZ*, and the plane of which it is the trace, remaining fixed. The sum of the projections of the centrifugal forces, viz. *ab*, *bc*, *cd*, *da*, is still zero, and therefore there is still balance amongst the

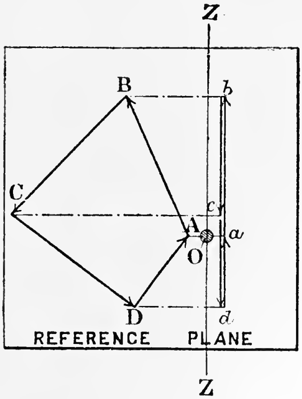


Fig. 51.

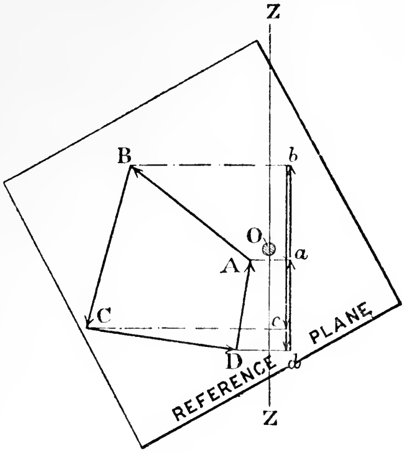
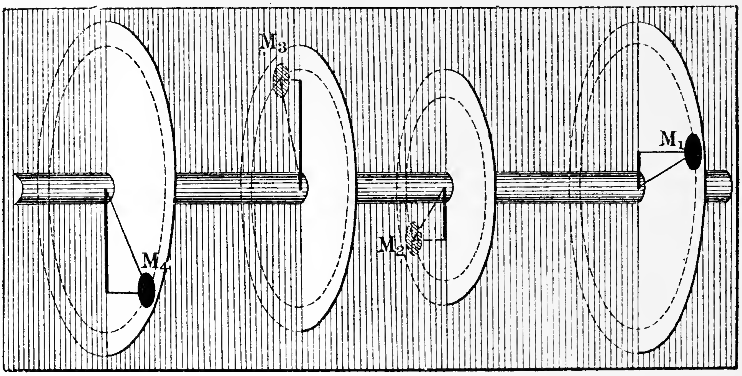


Fig. 52.

reciprocating forces. In fact, always providing that the force polygon is closed, the sum of the projections of its sides on *ZZ* is continuously zero during the rotation of the shaft, though the individual magnitudes of the projections are continually varying. Similar reasoning applies to a closed couple polygon. During rotation the sum of the projections of its several sides on *ZZ* is always zero, and therefore the reciprocation of a corresponding set of masses in the plane, of which *ZZ* is a trace, gives rise to no tilting action on the engine frame. From this it follows that to investigate the balancing conditions amongst a given system of reciprocating masses, it is only necessary to imagine them transferred to their respective crank-pins, and then to proceed by the rules of Chapter II. In fact, every example on revolving masses in the previous chapter may be looked upon as

Fig. 53.



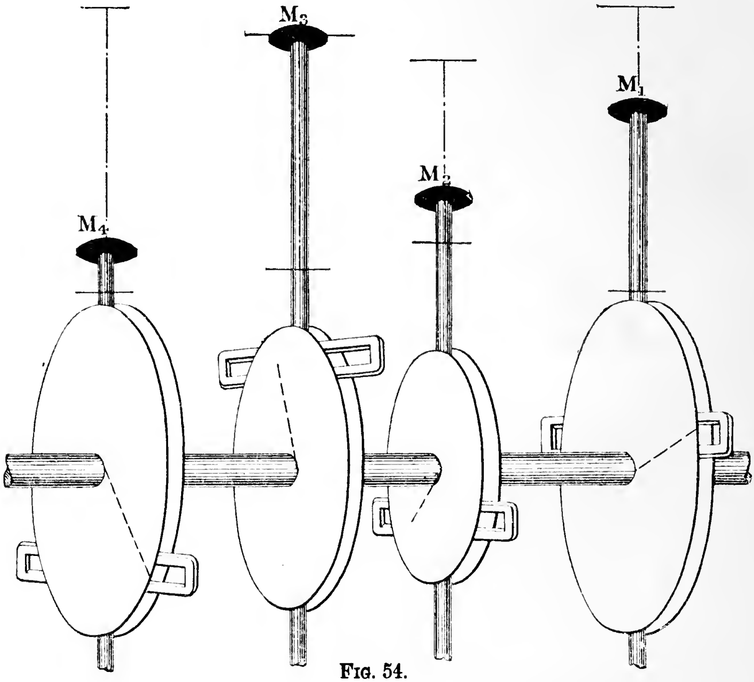


Fig. 54.

an example in reciprocating masses, assuming, of course, that the reciprocation is simple harmonic.

To fix these principles in the mind, consider the system of revolving masses M_1, M_2, M_3, M_4, shown in Fig. 53, to be in balance amongst themselves. Taking a reference plane anywhere, the corresponding force and couple polygons will be closed. The sum of the projections of these two polygons on the plane indicated by shading will, therefore, be continuously zero during the rotation of the system. If, therefore, the revolving system be changed into the reciprocating system shown in Fig. 54, where the masses have been taken from their respective crank-pins and placed as pistons on the slotted bars (whose mass is here neglected), the system of reciprocating masses so formed will be balanced.

45. Estimation of the Unbalanced Force and Couple due to a Given System of Reciprocating Masses assuming Simple Harmonic Motion.*—Let ABCD (Fig. 55) be the unclosed force polygon corresponding to a system of three revolving masses carried by the shaft, whose end is indicated by O. Let ZZ be a fixed line, and let OX be a line drawn in the reference plane by means of which the angular position of the plane may be specified relatively to the fixed line ZZ. When the angle between OX and ZZ is known, the angles between all the crank directions and ZZ are known. The unbalanced centrifugal force is represented by the vector AD. The projection of this on ZZ is the instantaneous value of the unbalanced force due to a corresponding system of reciprocating masses. If a is the angle between the vector AD and OX, and θ is the instantaneous value of the crank angle, the magnitude of this projection is—

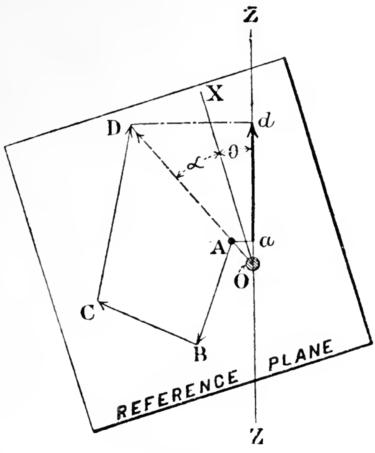


FIG. 55.

$$AD \cos(\theta + a)$$

* The method of finding the unbalanced force and couple by means of drawing polygons was given in a paper, "The Causes of the Vibrations of Screw Steamers," by Mr. D. W. Taylor, and published in the *Journal of the American Society of Naval Engineers*, vol. iii., 1891.

Similarly, if AD represent the unbalanced couple, the expression will give the value of the tilting couple acting on the engine frame for the corresponding system of reciprocating masses in terms of the crank angle.

It will be noticed that the value of the unbalanced force is a maximum at the instant AD is parallel to ZZ. And remembering the rules for drawing a couple polygon, the tilting couple is a maximum also when AD, taken now to represent the unbalanced couple, is parallel to ZZ. Referring to Art. 32, it will be evident that the unbalanced force and the minimum values of the unbalanced couple are the projections on the plane in which reciprocation takes place of the central axis belonging to the corresponding system of revolving masses.

Example.—Find the unbalanced force and couple due to the reciprocating masses of a three-crank engine arranged with cranks mutually at 120°, and running at 88 revolutions per minute, having given—

Mass of each set of reciprocating parts ...	5 tons.
Crank radius	2 feet.
Cylinders	16 feet pitch.

Take a reference plane at the central crank. Imagine the reciprocating parts concentrated and moving with their respective crank-pins, and apply the method of Art. 31.

The force polygon is a closed equilateral triangle, so that there is no unbalanced force.

The couple polygon is open, requiring a side 138·5 units long, inclined 90° to the direction of the central crank to close it. The couple this represents at 88 revolutions per minute is $138{\cdot}5\frac{\omega^2 r}{g} = 728$ foot-tons. This is the maximum value of the tilting couple for the reciprocating masses, and the value in terms of the crank angle is given by—

$$728 \cos (\theta + 90°)$$

46. Elimination of the Connecting-rod.—Not only does the connecting-rod disturb the simple harmonic motion of the reciprocating masses, but the motion of the rod itself, partly reciprocating, partly turning, and in each case changing from instant to instant, requires the action of varying accelerating forces to constrain its motion, and the equal and opposite aspects of these forces disturb the frame. Therefore, in addition to the assumption that the connecting-rod is so long that the differences between the accelerations of the mass it reciprocates, and the accelerations it would give if it were infinitely long, are negligible, the forces due to its own acceleration must be reckoned with before it can be finally discarded from the problem. A full discussion of this subject is given in Chapter VIII. For the present purpose it will be sufficient to state that its effect on the frame in the line of stroke is imitated by two separate masses, one being concentrated at the crank-pin, the other at the crosshead. These two masses are then included, the one with the reciprocating masses of the engine, the other with the revolving masses. The magnitudes of these masses are inversely as the mass centre of the rod divides the line joining the crank-pin centre to the centre of the crosshead-pin, and their sum is equal to the mass of the rod.

To find the mass centre of the rod, take it in its finished state, complete in every detail, and balance it on a knife-edge. Let c be the distance from the crank-pin centre to the knife-edge, and l the length of the rod centre to centre. Weigh the rod, and let M be its mass in pounds or tons as the case may be. Then—

$$\left.\begin{array}{l}\text{Mass supposed attached to, and moving}\\ \quad\text{with the crosshead}\end{array}\right\} = \frac{\mathrm{M}c}{l} = m$$

$$\left.\begin{array}{l}\text{Mass supposed attached to, and revolving}\\ \quad\text{with the crank-pin}\end{array}\right\} = \mathrm{M} - m$$

Another way of arriving at the proper division of the mass is to place the rod with its centres on knife-edges, supported on the platforms of two separate weighing-machines. The reading given by the scale supporting the crank-pin end gives the mass to be included with the revolving masses. The reading given by the other scale, gives the mass which must be included with the masses at the crosshead. It is obviously only really necessary to support one knife-edge on a scale, the pressure on the other edge being found by difference when the mass of the rod is known.

Example.—A connecting-rod, 6 feet centre to centre, weighs 500 pounds. It is found to balance about a knife edge 1 foot 6 inches from the crank-pin centre. The mass to be included with the reciprocating masses is 125 pounds, the rest, 375 pounds, being reckoned with the revolving masses.

47. General Method of Procedure for Balancing an Engine, when the Motion of the Reciprocating Parts may be considered Simple Harmonic.—(1) Reduce each mass to an equivalent mass at a common crank radius by Art. 33, distinguishing between revolving and reciprocating masses.

(2) Distribute the mass of each connecting-rod between the revolving and reciprocating parts which it connects by the method of Art. 46.

(3) Fill in a schedule of the following type for the reciprocating masses, choosing the reference plane to coincide with a plane of revolution for which the reciprocating mass is unknown.

SCHEDULE 5.

RECIPROCATING MASSES. Plane of Reference at ....

Number of crank.	Distance of centre line of cylinder from plane of reference.	Column I. Equivalent mass at crank-pin or centrifugal force, when $\omega^2 R = 1$.	Column II. Equivalent mass moment or centrifugal couple, when $\omega^2 R = 1$.

(4) Treat the quantities in the schedule exactly as though they formed a revolving system, and find the balancing masses by Art. 30. These masses when reciprocated will be the balancing masses for the system of reciprocating masses under consideration.

(5) Choose a new reference plane. Fill up a new schedule, including in it the balancing masses found in (4). Draw the couple polygon corresponding to it. If the work has been correctly done this polygon will close. This should always be done to check the accuracy of the work. Notice, however, that this only checks (3) and (4). (1) and (2) must be checked independently.

The result of these five operations is to fix the crank angles and the reciprocating masses so that the reciprocating system is in balance. There now remains the revolving masses connected with the crank-shaft to deal with. Since in this revolving system the crank angles are fixed, masses must be added to the system to balance it. An example is worked out in Art. 51.

It will be noticed that the problem of balancing reciprocating masses presents itself in a slightly different form from problems on revolving masses. In the latter case, the problem is usually—

Given a system of revolving masses to find the masses which added to the given masses, will produce a balanced system.

The least number of masses in the general case which must be added is 2, so that if there are n masses given, there results a system of $(n + 2)$ masses in balance. There is little difficulty in adding revolving balancing masses to a system. With reciprocating masses it is more difficult. A balancing mass in this case means a new crank, connecting-rod, guides, etc., to operate a mass which in every other respect but the enclosing cylinder may be looked upon as a piston. Such a mass has been called a "bob-weight." Mr. Yarrow, in a paper * at the Institution of Naval Architects in 1892, described how the engines of a torpedo boat had been balanced by the addition of two bob-weights. This paper of Mr. Yarrow's is full of interest, as in it are given the details of the calculations used to determine the bob-weights for balancing the reciprocating masses of a three-cylinder engine. It will be perceived that, providing there are a sufficient number of cranks, the reciprocating parts operated by any two cranks may be considered as the bob-weights balancing the rest of the reciprocating parts.

The general problem of balancing a reciprocating system, therefore, presents itself in this way—

Given a system of n reciprocating masses to find how the masses must be arranged so that they mutually balance.

It has been shown, in Art. 35, that the number of independent variables concerned in a revolving system of n cranks is in general $3(n - 1)$. This applies equally to a system of reciprocating masses. Consequently, if an engine is to be built with n cylinders, there will be $3(n - 1)$ variables to be considered in the balancing of the reciprocating masses, which have to satisfy four conditions; consequently, $3n - 7$ of these must be fixed, but no more. The

* On "Balancing Marine Engines and the Vibration of Vessels." By Mr. A. F. Yarrow. *Trans Inst. Naval Architects.* London, 1892.

remaining four variables can then be found by the foregoing methods to balance the system.

Thus, in a four-crank engine, there are nine independent variables; these are, or, rather, may be (for any one of the M's or a's may be used for the common divisor)—

$$\text{Variables of magnitude } \frac{M_2}{M_1}, \frac{M_3}{M_1}, \frac{M_4}{M_1} = 3$$

$$\text{Variables in } a \quad \frac{a_2}{a_1}, \frac{a_3}{a_1}, \frac{a_4}{a_1} = 3$$

$$\text{Variables of direction } \theta_{12}, \theta_{13}, \theta_{14} = 3$$

$$\text{Total} = 9$$

The double subscript to the θ's indicate the particular numbers of the cranks between which θ is measured. A good way is to put the magnitude of one of the masses equal to unity, and one of the distances from the reference plane equal to unity. The letters left in then represent independent variables; they are, of course, now proportional numbers, and in the solution must be interpreted in terms of the M and the a which were put equal to unity.

If, for instance, M_1 and a_1 be considered each equal to unity, the variables are—

$$M_2, M_3, M_4 = 3$$

$$a_2, a_3, a_4 = 3$$

$$\theta_{12}, \theta_{13}, \theta_{14} = 3$$

$$\text{Total} = 9$$

Of these $3 \times 4 - 7 = 5$ must be fixed. Fixing the pitch of the cylinders is equivalent to fixing all the variables in a, that is, three. The remaining two may be chosen at will from the above set. For instance, any two of the M's may be fixed, or any two of the angles, or one M and one angle. It is evident that if all the masses are fixed as well as the centre lines of the cylinders, there are too many data chosen, and no solution of the problem is possible. It is also clear that there are as many solutions possible as there are ways of choosing the two quantities, supposing the centre lines to be fixed.

48. Example.—Given the stroke, the cylinder centre lines, and the masses corresponding to three cylinders in a four-cylinder engine, find the crank angles, and the mass of the reciprocating parts belonging to the fourth cylinder so that the reciprocating masses may be in balance amongst themselves.

The first step is to examine the data. There are nine variables concerned in the problem (see Art. 47), and of these five must be fixed. The fixing of the cylinder centre lines accounts for three, and the fixing of three masses for the remaining two, because, although three masses are given, this only corresponds to two ratios. It is the same as putting one of the given masses equal to unity.

Take the reference plane so that it contains the centre line of the gear whose mass is to be determined as shown in Fig. 56, the dimensions there shown being given. From it fill up Schedule 6.

SCHEDULE 6.

RECIPROCATING MASSES. Plane of reference at No. 4 cylinder.

Number of crank.	Distance of centre line of gear from plane of reference.	Column I. Equivalent mass at crank-pin or centrifugal force, when $\omega^2 R = 1$.	Column II. Equivalent mass moment or centrifugal couple, when $\omega^2 R = 1$.
	Feet.	Tons.	
No. 4	0·0	Unknown (4·95)	0·0
No. 3	3·3	7·025	23·2
No. 2	8·9	6 49	57·8
No. 1	11·5	5·1	58·6

By supposition there is to be balance, therefore the couple polygon must close. In this case the polygon becomes a triangle.

Choosing a convenient scale, draw a triangle, as in Fig. 57, in which—

$$AB = 58{\cdot}6; \quad BC = 57{\cdot}8; \quad CA = 23{\cdot}2$$

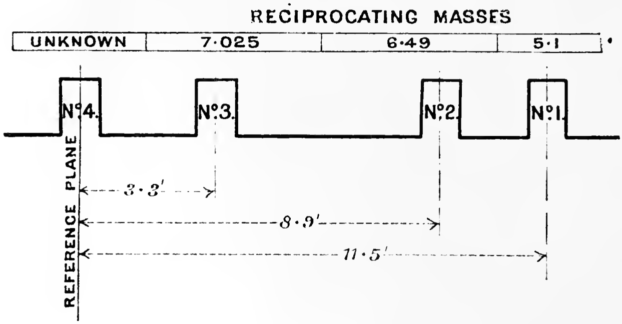
Then—

AB is the direction of No. 1 crank
BC „ „ of No. 2 „
CA „ „ of No. 3 „

These directions are transferred to the end view of the crank-shaft centre lines (Fig. 58).

One condition of balance is fulfilled, viz. that the couple polygon close. The second condition, viz. that the force polygon close, is easily satisfied by taking advantage of the adjustment

FIG. 56.



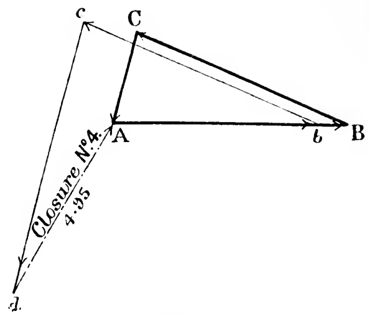


FIG. 57.

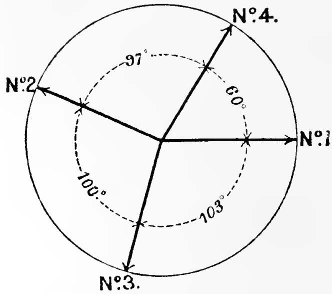


FIG. 58.

which may be made by crank No. 4, since, in whatever direction it is placed, the mass it operates has no moment about the reference plane, and consequently it may be fixed in any desired position without disturbing the balance amongst the couples.

Choosing a convenient scale, make A*b* (Fig. 57) = to 5·1, *bc* = 6·49 and parallel to crank No. 2, *cd* = 7·025 and parallel to crank No. 3. The polygon fails to close by the side *d*A.

Close it by means of the fourth crank. Thus, *d*A is the direction of crank No. 4 relatively to the others, and its length represents the equivalent mass (of the reciprocating parts) attached to the crank—dA scales 4·95 tons.

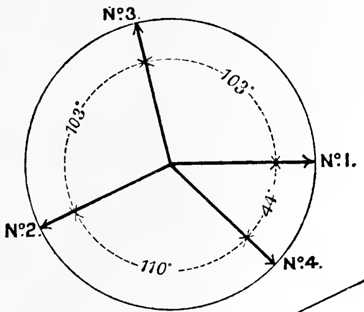
Check the work in this way—

Suppose the reference plane to be at No. 1 crank. Make a new schedule for the masses with reference to this plane, including, of course, No. 4 crank. Draw the couple polygon. If it closes, the work is correct.

A consideration of the couple triangle (Fig. 57) will show that the lightest mass should be placed in plane No. 1; that in plane No. 2 the crank should be arranged oppositely to crank No. 1, otherwise the mass at No. 3 would have to be relatively very great to effect balance.

49. Example.—Given the cylinder centre lines of a four-cylinder

FIG. 59.



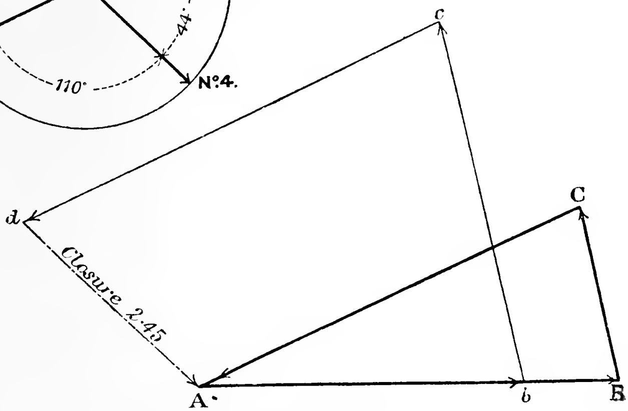


FIG. 60.

engine, find the crank angles and the masses so that the reciprocating parts may be in balance amongst themselves.

It will be noticed at once that not enough data are given to solve the problem. There are nine variables concerned in the balancing, and of these five must be given to get a solution. Since only three of the variables are fixed by the cylinder centre lines, two more must be fixed. Let these be two angles. Assume the angles, which are shown in Fig. 59, between cranks 1, 2, and 3. Take a reference plane at No. 4 cylinder. Set out AB (Fig. 60) in the direction of No. 1 crank, and make it = 100, to some convenient scale. Draw BC, CA parallel respectively to No. 3 and No. 2 crank; measure them off, and enter them in Column II. of Schedule 7.

SCHEDULE 7.

RECIPROCATING MASSES. Reference plane at No. 4 crank.

Number of crank.	Distance from reference plane.	Column I. Proportional equivalent mass or centrifugal force, when $\omega^2 R = 1$.	Column II. Proportional equivalent mass moment or centrifugal couple, when $\omega^2 R = 1$.
	Feet.		
No. 4	0·0	2·45	—
No. 3	12·16	3·72	45·2
No. 2	22·87	4·375	100·0
No. 1	33·04	3·025	100·0

The actual couples about No. 4 plane must be in the ratio—

45·2 : 100 : 100 for balance

Divide each of these by the corresponding distance from the reference plane given in the schedule. The quotients are the ratios of the masses at crank-pin radius. These are entered in Column I. To find the proportional number for the mass at No. 4 crank, set out (Fig. 60)—

Ab = 3·025 parallel to No. 1 crank,
bc = 3·72 parallel to No. 3 crank,
cd = 4·375 parallel to No. 2 crank

dA, the closure, gives the direction of crank No. 4, and its length, 2·45, is the proportional equivalent mass number.

The masses must be adjusted so that they are in the ratio of—

$$2·45 : 3·72 : 4·37 : 3·02$$

50. Example.—Both the preceding examples ignore the effect of the valve-gear. The next example includes it, and is worked out in somewhat greater detail to serve as a typical illustration of a way of dealing with torpedo-boat engines. The peculiarity of the problem is that the eccentric sheave angles are functions of the corresponding main crank angles. The method of dealing with the problem will be apparent in the working out.

Given the centre lines of four cranks and the corresponding ahead and astern eccentric sheaves; the mass of the different parts of the valve-gears and the mass of one piston; to fix the crank angles and the masses of the pistons so that the reciprocating masses may be in balance amongst themselves, and to find the balancing masses for the crank-shaft.

Fig. 61 shows the crank-shaft and centre lines. Above each crank and sheave is written the reciprocating and revolving masses at the crank radius, it being understood that the connecting-rods and eccentric rods are included by the method of Art. 46.

In calculating the reciprocating masses, it may be noted that the engine is supposed to be in forward gear, and that the equivalent reciprocating mass for the ahead eccentric in each case includes the mass of the valve, valve spindle, etc., an appropriate part of the eccentric rod, and one-half the link.

The valve motion of No. 4 crank cannot be taken into consideration in the general method, because its crank angles are functions of crank No. 4, the last angle to be determined.

Assume the mass of the reciprocating parts of No. 1 crank to be 1000 pounds.

Fill in Schedule **8**.

SCHEDULE 8.

RECIPROCATING MASSES. Reference plane at No. 4.

Number of crank.	Distance from reference plane.	Column I. Equivalent mass at crank radius or centrifugal force, when $\omega^2 R = 1$.	Column II. Equivalent mass moment or centrifugal couple, when $\omega^2 R = 1$.
	Feet.	Pounds.	
No. 4 astern ecc sheave	−3·3	60	} Not included
No. 4 ahead ecc. sheave	−3·0	140	
No. 4 crank	0·0	Unknown (1040)	——
No. 3 crank	4·0	Unknown (1275)	Unknown (5,100)
No. 3 astern ecc. sheave	7·0	60	420
No. 3 ahead ecc. sheave	7·3	150	1,095
No. 2 crank	10·0	Unknown (1500)	Unknown (15,000)
No. 2 astern ecc. sheave	13·0	70	910
No. 2 ahead ecc. sheave	13·3	150	1,995
No. 1 crank	16·0	1000	16,000
No. 1 astern ecc. sheave	20·0	70	1,400
No. 1 ahead ecc. sheave	20·3	145	2,943

Crank angles between Nos. 1 and 2, 200°; between 1 and 3, 100° (Fig. 70).

Set out each crank and the centre lines of its eccentric sheaves (Figs. 62, 64, 66, and 68). The sheaves are all drawn for an angular advance of 30°.

Draw (Fig. 63) AB, BC, CD parallel to OK, OF, OE (Fig. 62) respectively, and representing on some convenient scale the corresponding couples given in the schedule. Mark these vectors carefully with arrows to indicate their way of action.

Set out (Fig. 65) A_1B_1, B_1C_1, C_1D_1 parallel to OK, OF, OE (Fig. 64) respectively, B_1C_1, C_1D_1 representing the corresponding couples. A_1B_1 is of indefinite length, since the couple due to crank No. 2 is not known. Draw similarly A_0B_0, B_0C_0, C_0D_0 (Fig. 67) for crank No. 3, leaving C_0D_0 indefinite as to length.

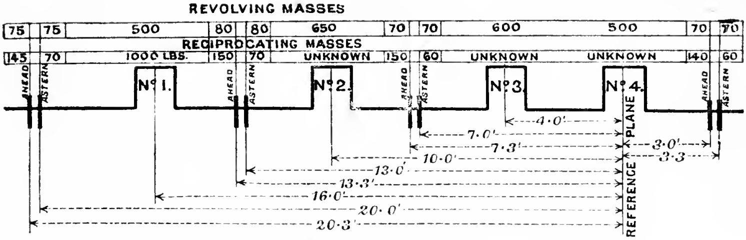
There is now a choice of two ways of continuing the work.

(1) Assume values for the unknown couples of Figs. 65 and 67.

(2) Assume the angles between cranks Nos. 1, 2, and 3.

Proceed by the latter method, assuming the angles between cranks Nos. 1 and 2 to be 200°, and between 1 and 3, 100°.

Fig. 61.



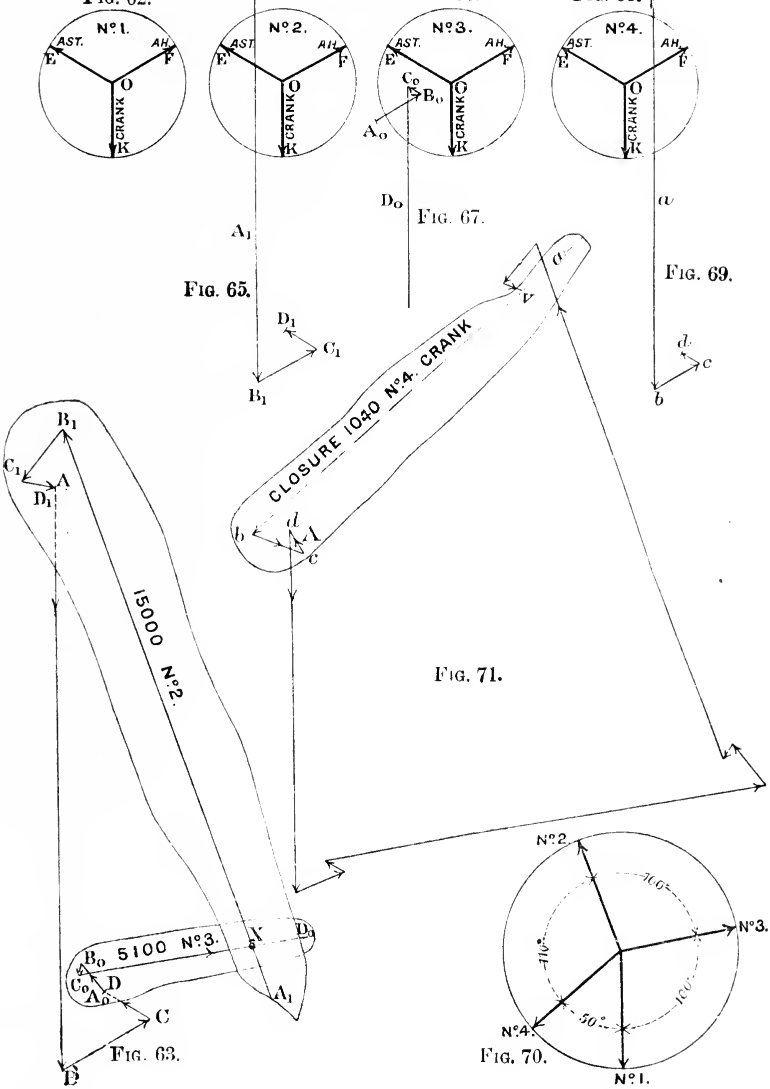


Fig. 62. Fig. 64. Fig. 66. Fig. 68.

Fig. 65. Fig. 67. Fig. 69.

Fig. 71.

Fig. 63. Fig. 70.

Set out (Fig. 70) the angle between No. 1 and No. 2 cranks to be 200°, and between Nos. 1 and 3 100°.

Draw the couple polygon thus—

Trace the vectors of Figs. 65 and 67 on separate pieces of tracing-paper. Pin the tracing from Fig. 67 over the vector drawing of Fig. 63 by a single pin through A_0 on the tracing and D on the drawing. Similarly, pin the tracing of Fig. 65 through D_1 on the tracing and A on the drawing. Turn the tracings round their pins until A_1B_1 is parallel to crank No. 2, and C_0D_0 to crank No. 3. X, the intersection of the indefinite lines A_1B_1 and C_0D_0, defines their length, and therefore the couples for cranks Nos. 2 and 3. Measure these off and enter them in the schedule. Divide each by its appropriate distance from the reference plane, and enter the force so obtained in Column I. It is at once settled that the reciprocating crank-pin mass for No. 2 crank is 1500 pounds, and for No. 3 crank 1275 pounds.

Draw a force polygon whose sides are parallel to the couple polygon (Fig. 63), and proportional to the numbers in Column I. (Fig. 71). It fails to close by the vector VA. The valve-gear of No. 4 crank may now be included. To do this, draw *ab, bc, cd* (Fig. 69) parallel to OK, OF, OE (Fig. 68). Make *bc, cd* proportional to the corresponding equivalent masses. The magnitude of *ab* is as yet unknown. Pin the tracing of these vectors over Fig. 71, *d* on the tracing being over A on the drawing. Turn it until the indefinite line passes through V.

V*b* represents the mass of the reciprocating parts of No. 4 crank, and gives the direction of the crank. This crank is added to Fig. 70.

Care must be taken to see that in these vector polygons the arrows are all pointing in the same way.

Notice that the valve-gear of No. 4 crank is balanced as regards forces, but not as regards couples.

These couples are of small magnitude, and may be neglected. They can be balanced, however—

(1) By repeating the whole process and including the couples in the couple polygon, assuming the radii of the two sheaves in question to have the directions found by the first application of the method.

(2) By two reciprocating masses arranged in two planes. Pumps worked from the shaft may sometimes be arranged to do this.

(3) By considering them as couples to be balanced with the revolving masses of the crank-shaft.

Assume that it can be done by No. 2 method.

51. Balancing the Crank-shaft.—The crank-shaft may be balanced in either of two ways—

(1) by the addition of two balancing masses in two separate planes of revolution;

(2) by the extension of the arms of each crank to form balancing masses for the revolving masses of that crank.

The first case is treated by the general method; the second case by Art. 12.

In the first case a crank-shaft in which there are any number of cranks may be balanced by the addition of two masses only. In the second case there are as many balancing masses as there are crank-arms, but this method has the advantage that the intermediate parts of the crank-shaft have not to transmit force from one crank to another, since each mass is balanced in its own plane of revolution. The shaft is thus freed from bending moment due to this cause, so far as the revolving masses are concerned.

The addition of balancing masses to the crank-shaft may be avoided altogether if it is balanced in the vertical plane only, leaving the component force and couple in the horizontal plane unbalanced. This may be done in marine engines in those cases in which the ship's hull is stiff enough horizontally to render the disturbances due to a crank-shaft unbalanced horizontally negligibly small. To balance an engine in this way, add the revolving masses and the reciprocating masses in each plane together, and proceed by the method of the previous article. Many examples of combining reciprocating and revolving masses into one schedule will be found in the next chapter.

Returning to the example of the previous article, balance the crank-shaft by the first method. The crank angles (Fig. 70) are all fixed by the conditions of balance found for the reciprocating masses. The equivalent revolving masses are all given in Fig. 61. Fill in Schedule 9.

SCHEDULE 9.

REVOLVING MASSES. Reference plane at No. 4.

Number of crank.	Distance from reference plane.	Equivalent mass at crank radius or centrifugal force, when $\omega^2R = 1$.	Equivalent mass moment or centrifugal couple, when $\omega^2R = 1$.
	Feet.	Pounds.	
No. 4 astern ecc. sheave	−3·3	70	−231*
No. 4 ahead ecc. sheave	−3·0	70	−210*
No. 4 crank	0·0	500	——
No. 3 crank	4·0	600	2400
No. 3 astern ecc.	7·0	70	490
No. 3 ahead ecc.	7·3	70	511
No. 2 crank	10·0	650	6500
No. 2 astern ecc.	13·0	80	1040
No. 2 ahead ecc.	13·3	80	1064
No. 1 crank	16·0	500	8000
No. 1 astern ecc.	20·0	75	1500
No. 1 ahead ecc.	20·3	75	1522

Fig. 72 shows the couple polygon for Schedule 9. It fails to close by the vector VO, which measures 1200 foot-pounds. This couple may be balanced in any convenient way. Obviously, from an inspection of the direction, it would be most convenient to attach a balancing mass opposite crank No. 1. This is 16 feet from the reference plane. The mass required at the crank radius is therefore 75 pounds.

Fig. 73 shows the corresponding force polygon. This must be drawn to include the 75 pounds added to balance the couples. The polygon requires the vector V_1O_1 to close it, viz. a mass of 12 pounds at the crank radius and in the reference plane in the direction indicated. This could easily be arranged for on crank No. 4.

Thus the engine is completely balanced both for reciprocating and revolving masses by the addition of the small masses of 75 and 12 lbs.

* Note the way of drawing these vectors, viz. TOWARDS the centre of the crank-shaft. The force vectors remain positive. See Art. 28.

An end view of the crank-shaft centre lines is shown in Fig. 74.

Fig. 72.

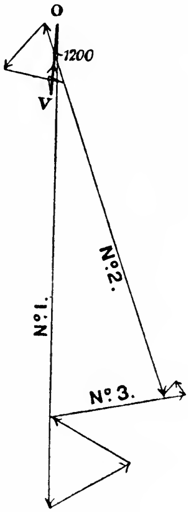
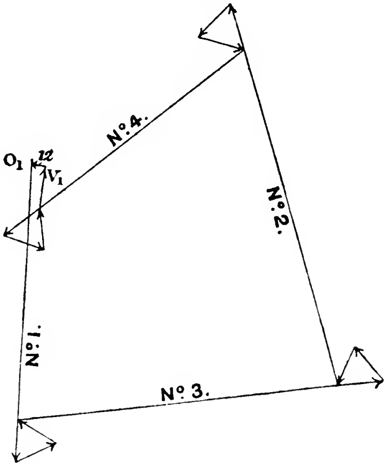


Fig. 73.



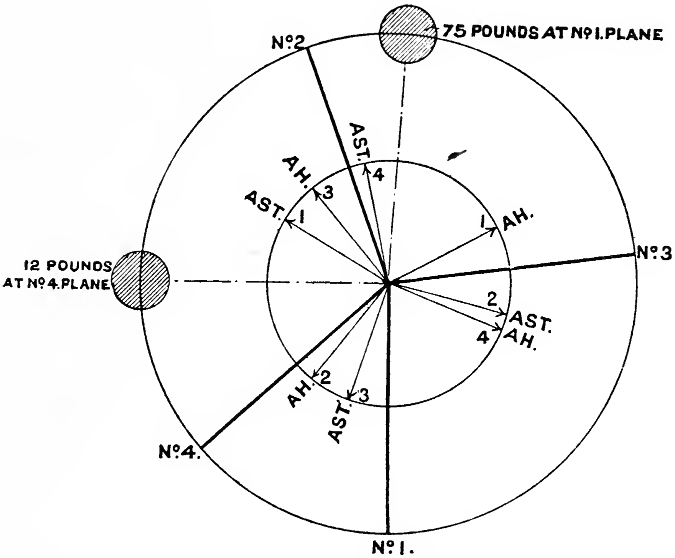


Fig. 74.

52. Conditions that an Engine may be balanced without the Addition of Balancing Masses either to the Reciprocating Parts or to the Crank-shaft.—If an engine has four cranks or more, the reciprocating parts may be balanced amongst themselves without the addition of balancing masses in the way already exemplified in Arts. 48, 49, and 50; and this necessarily fixes the crank angles. The distances between the planes of revolution of the revolving masses are practically the same as the distances between the cylinder centre lines, since the crank-arms, etc., are symmetrical with respect to their respective cylinder centre lines, and therefore the mass centre of each revolving mass is in the same plane in which the corresponding reciprocating mass moves; hence, for the revolving masses to be in balance, the force and couple polygon corresponding to them must be similar to the force and couple polygon belonging to the reciprocating masses, and therefore the revolving masses must be in the same proportion to one another as the reciprocating masses. Or, briefly, no balancing masses will be required if—

(1) The mass centres of the revolving and reciprocating masses of each line of parts are in the same plane;

(2) The masses of the reciprocating parts are in the same proportions as the masses of the revolving parts;

(3) The reciprocating system is balanced by the method of Art. 47.

For instance, in the example of Art. 49, the masses of the reciprocating parts must be in the following proportions for balance—

$$2{\cdot}45 : 3{\cdot}72 : 4{\cdot}37 : 3{\cdot}02$$

and these, therefore, represent the ratios of the revolving masses if no balancing masses are to be added to the crank-shaft.

This condition rarely obtains in practice, and it is, therefore, necessary to add masses to the crank-shaft in order to balance it.

53. To find the Resultant Unbalanced Force and Couple due to the Revolving and Reciprocating Parts together.—The way to find the unbalanced force and couple for a given system of revolving masses has been given in Art. 31, and the method of estimating the unbalanced force and couple due to a given system of reciprocating masses, moving with simple harmonic motion, has been given in Art. 45. The resultant force or couple at any instant is the resultant of the two vectors representing the force belonging to each system, or of the vectors representing the couples.

Suppose the vector OA (Fig. 75) represents the unbalanced force belonging to the revolving system attached to the shaft, the end of which is shown at O, and OR the unbalanced force of the imaginary revolving system replacing the reciprocating system operated by the same shaft. The projection of OR on ZZ, that is, O*r*, is the instantaneous value of the unbalanced reciprocating force (Art. 45). The resultant of OA and O*r*, that is, OB, is the instantaneous value of the whole disturbing force. It may be shown that the locus of the point B is an ellipse. Notice that OR and OA revolve with the shaft, and that the angle a between them is constant.

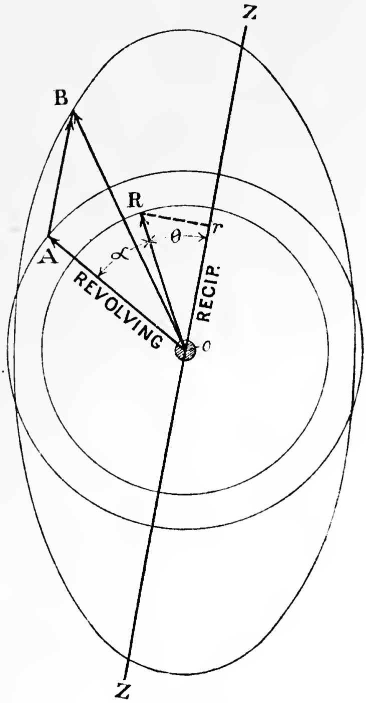


FIG. 75.

The couples may be dealt with in a similar way.

This need not be further pursued, because a method will be given in Chapter VI. by means of which both the unbalanced force and couple may be found exactly for short rods by a simple geometrical process.

54. Experimental Apparatus.—Fig. 76 shows an apparatus which has been designed by the author to illustrate the principles of balancing reciprocating parts. There are four cranks, all mutually adjustable, three of the flanges of the crank-shaft being divided into degrees for this purpose. Adjustable masses may be fixed to the tails of the respective piston-rods. Of the nine variables concerned in the balancing of a four-crank engine six are susceptible of variation in this apparatus. Suspended from a frame as shown, its motions when running unbalanced exhibit the way a marine engine tries to wobble when running under similar conditions. Properly balanced, it hangs motionless at all speeds, showing only a little uneasiness when the speed is passing through the natural period of oscillation of the supporting springs. Placed on rollers in the way shown in Fig. 101, it shows the effect on the tractive force due to the unbalanced parts of a four-cylinder locomotive. In this model the revolving parts are so arranged that their balance is not disturbed by any adjustment that may be made in the reciprocating parts or crank angles. If provided with short connecting-rods, it serves also to illustrate the principles of Chapter V.

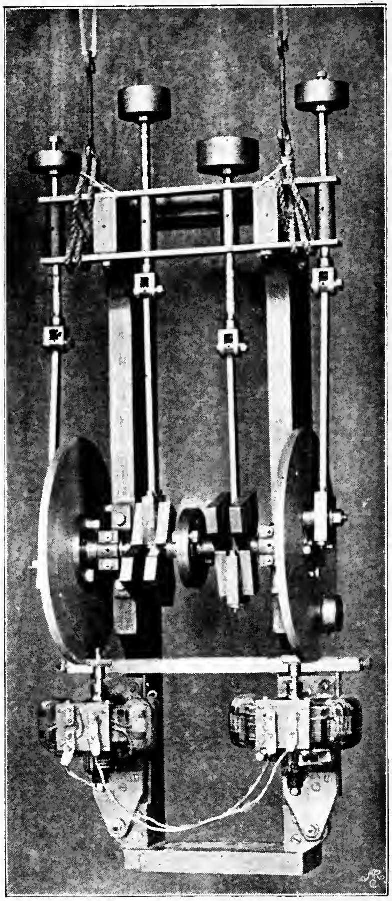


Fig. 76.

55. Balancing Reciprocating Masses by the Addition of Revolving Masses.—The balancing masses found by Art. 30 must, of course, be reciprocated, forming, with the given system of reciprocating masses, a system in a balance. If they are added to the system as revolving masses, being attached to the crank-shaft, although they do produce balance amongst the reciprocating masses, they introduce at the same time forces at right angles to the plane of reciprocation exactly equal to the forces they are balancing in the reciprocating system, though differing in phase by 90°. It may be that the unbalanced forces in this direction are less serious than in the original direction, though in general balancing in this way cures one trouble only to introduce another. This case is similar, though opposite, to the method mentioned in Art. 51 of avoiding the addition of balancing masses to the crank-shaft by arranging the reciprocating parts to balance it vertically, leaving it unbalanced horizontally. The method of procedure is alike in both cases, and, in the case of locomotives, is fully illustrated in the next chapter.

CHAPTER IV.

THE BALANCING OF LOCOMOTIVES.

56. General Consideration of the Effects produced by the Unbalanced Reciprocating Parts of a Locomotive.—The disturbances caused by the machinery of a locomotive may be divided into those due to the revolving and reciprocating masses respectively. There is no need to discuss the effects due to the unbalanced revolving masses, crank-arms, crank-pins, coupling-rods, etc. These always can and always should be balanced forming a revolving system in equilibrium at all speeds, and affecting, therefore, neither the tractive force nor the rail pressure.

The effect due to the unbalanced reciprocating masses may be investigated by reducing them to a reference plane taken centrally, at right angles to the axis of the driving-axle. The unbalanced forces then reduce to a single force and a couple.

In Fig. 77 the left-hand crank of a driving-axle belonging to either an inside or outside 2-cylinder engine is shown on the trailing dead centre, and it stands to the front of the reference plane. The right-hand crank is, of course, behind the plane. The reciprocating system consists of two masses, each equal to M pounds say, connected to cranks at right angles by rods which are relatively long with respect to the crank radius. The unbalanced force and couple are found by the method of Art. 45. Thus Oab is the force triangle in which Oa and ab are equal, each representing M: the vector Ob, therefore, represents the unbalanced force of the system of revolving masses corresponding to the reciprocating masses. Its projection on the line of stroke, Oa in Fig. 77, Oc in Fig. 78, where the crank axle has turned through the angle θ, θ being the angle between the centre line and the left-hand crank, represents the instantaneous value of the unbalanced force in the centre line of the engine.

Since for a given speed $Oa = ab = \frac{M\omega^2 r}{g}$ lbs. weight, the numerical value of this force in terms of the angle θ is—

$$\frac{\sqrt{2}\,M\omega^2 r \cos(\theta + 45^\circ)}{g} \text{ lbs. weight}$$

or, in terms of the revolutions per second of the crank-axle n, M being in pounds and r in feet, it becomes—

Unbalanced force in lbs. weight acting at the centre line $= 1{\cdot}7Mn^2r \cos(\theta + 45^\circ)$ approximately . . . (1)

Similarly, OAB (Fig. 77) is the couple triangle in which OB

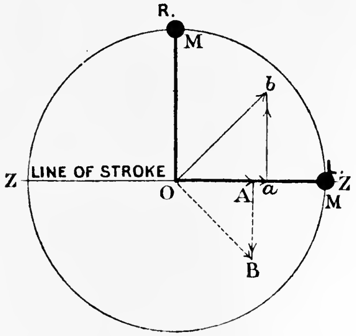


FIG. 77.

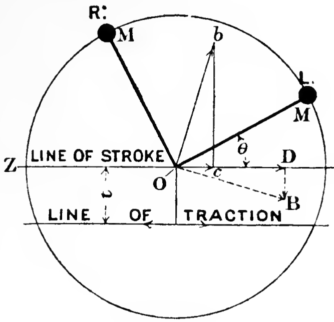


FIG. 78.

represents the value of the unbalanced couple for the corresponding system of revolving masses; its projection, OA in Fig. 77, OD in Fig. 78, therefore, represents the instantaneous value of the unbalanced couple acting in the plane containing the centre lines of the cylinders. This couple oscillates the engine about a vertical axis, or one very nearly vertical when the cylinders are inclined. This axis is always at right angles to the plane of reciprocation.

Since for a given speed $OA = AB = \frac{M\omega^2 ra}{g}$ foot-lbs., where a is equal to half the distance between the cylinders, the numerical value of the couple in terms of the angle θ is—

$$\frac{\sqrt{2}M\omega^2 ra \cos(\theta - 45^\circ)}{g}$$

or, in terms of the revolutions per second n, and the distance between the centre lines of the cylinders, d—

$$\cdot 85 M n^2 r d \cos(\theta - 45^\circ) \text{ foot-lbs. approximately} \quad . \quad . \quad . \quad (2)$$

The maximum values of the force occur when Ob turns into the line ZZ, that is, when $\theta = -45^\circ$, or 135°; and the maximum values of the couple when OB turns into the line ZZ, that is, when $\theta = +45^\circ$, or 225°.

There is still another disturbance due to the fact that the centre line of the crank-axle is above or below the line of traction. If the distance between the line of action of the force and the line of traction is t (Fig. 78), the transference of the force to the line of traction is equivalent to (see Art. 24)—

(1) An equal and parallel force in the line of traction;

(2) A couple whose arm is t.

This couple, which is continually varying in magnitude and sign, tends to oscillate the engine in a vertical plane. This would be perceived most with comparatively short tank engines with small wheels. Assuming the distance of the line of traction above the rails to be about 3 feet $4\frac{1}{2}$ inches, with a 6 feet 9 inch driving-wheel, this cause of disturbance would be entirely absent, since $t = 0$.

It may be noticed incidentally that the tractive pull of the engine must be transferred from the plane of reciprocation to the line of traction, which transference gives rise to a couple varying in magnitude, but acting to turn the engine about a horizontal axis always in the same direction so long as the engine is pulling. The effect of this couple is to cause a redistribution of the weights on the springs. Also the turning couples on the crank-axle are necessarily accompanied by equal and opposite turning couples on the engine as a whole which modify the distribution of weight on the springs to a much greater extent than the couple just mentioned. The effect of the turning couple is to decrease the load on the springs at the leading end for forward running, and to increase it for backward running. This point is again discussed on page 103. These turning couples are variable, and the variation for each crank and of the total are shown for a particular case in Fig. 91. Since the load is brought on to the main bearings by means of springs, the variations of the turning couples set up oscillations which are superposed upon the oscillations due to the inertia forces of the unbalanced machinery. The disturbances due to the variation of turning moment cannot be eliminated by the addition of balancing masses; they can be minimized, however, by placing the cylinders as close together as possible. A discussion of this question in connection with marine engines is given in Arts. 130–132.

Summarising, if—

M is the mass in pounds of the reciprocating parts, equal in each cylinder, belonging to one cylinder;
r the crank radius in feet;
n the revolutions of the crank-axle per second;
d the distance in feet between the cylinder centre lines;
t the distance between the centre line of the driving-axle and the line of traction;

the unbalanced reciprocating parts cause—

(1) An unbalanced force, the maximum value of which is given by $1{\cdot}7\text{M}n^2r$ lbs. weight. This force accelerates the whole mass of the train positively and negatively in the direction of travelling.

(2) A couple whose maximum value is $\cdot 85\text{M}n^2rd$ foot-lbs. This couple produces an oscillatory motion about a vertical axis, which, superposed upon the general forward motion of the engine, causes a swaying from side to side, which, acting on a short engine, may become dangerous at high speeds. The effect of this couple must be judged with reference to the moment of inertia of the engine about a vertical axis through its mass centre. The couple is much less in magnitude for inside cylinders than for outside cylinders, since it varies directly as d.

(3) A couple whose maximum value is $1{\cdot}7\text{M}n^2rt$ foot-lbs. This couple tends to cause oscillation in a vertical plane about a horizontal axis. Its magnitude is usually small, and it disappears altogether if the driving-wheel radius is equal to the height of the line of traction above the rails.

57. Example.—The mass of one set of reciprocating parts of an unbalanced locomotive is 600 pounds, and the cylinders are 2-feet pitch. Find the maximum values of the unbalanced force, and of the couples about a vertical and a horizontal axis, assuming the diameter of the driving-wheel to be 4 feet 6 inches, and the line of traction to be 3 feet 4½ inches above the rail, the speed being 38·5 miles per hour, and the stroke 26 inches. Find also the values when $\theta = 30°$ (see Fig. 78).

The number of revolutions of the driving-wheel per second at the given speed = 4 approximately.

M = 600 pounds, and r = 1·08 feet; therefore—

(1) Maximum value of the unbalanced force } = 1·7 × 600 × 16 × 1·08
= + and − 17,625 lbs. weight
= + and − 7·87 tons weight

This is greater than the average tractive force exerted by the engine.

(2) Maximum value of couple about a vertical axis = + and − 17,625 foot-lbs.

For outside cylinders d is about 6 feet, so that the value of this couple would in this case be increased to 52,875 foot-lbs.

(3) Maximum value of the couple about a horizontal axis is, t being 1·125 feet—

17,625 × 1·125 = + and − 19,828 foot-lbs.

When $\theta = 30°$,

Instantaneous value of the unbalanced force is—

17,625 cos (30° + 45°) = 17,625 × ·26 = 4582 lbs. weight

Instantaneous value of the couple about a vertical axis—

17,625 cos (30° − 45°) = 17,625 × ·96 = 16,920 foot-lbs.

Instantaneous value of couple about horizontal axis—

4582 × 1·125 = 5155 foot-lbs.

This example sufficiently illustrates the effects of the unbalanced reciprocating masses. If the revolving masses are left unbalanced as well, the maximum values are very much increased, and in addition they introduce a couple acting about a longitudinal axis, which causes a variation of the driving-wheel rail-load.

It is instructive to set out the maximum values found above to scale on a piece of cardboard, dotting in the position of the cranks, in the way shown in Fig. 78. Pin the cardboard to the drawing-board through O, and bring the edge of the **T**-square up to the centre. Then turn the cardboard disc, which really represents the reference plane, into various positions: the changing values of the force and swaying couple, represented by the projections of O*b* and OB respectively, on to the edge of the **T**-square, may then be easily studied.

58. Method of Balancing the Reciprocating Masses of a Locomotive.—To properly balance the reciprocating masses requires the addition of two more sets of parts reciprocated in the same plane, forming either a two-cylinder engine with two sets of "bob-weights," as suggested by Mr. Yarrow, or a four-cylinder engine in which the crank angles are found by the principles of Chapter III. At the present time no locomotives in this country are balanced with bob-weights, and no four-crank locomotive has been built in which the crank angles and masses are arranged so that the forces and couples balance, the usual arrangement being to place the four cranks at right angles, in which case the forces are, or rather may be, completely balanced, but the couple cannot be balanced without the addition of balance weights. Four-cylinder engines will be discussed more fully later on. It is almost the universal custom to partially balance the force and couple due to the reciprocating parts in a two-cylinder engine, by masses placed between the spokes of the driving-wheels, these being combined with the masses balancing the revolving parts to form a single pair of balancing masses, or, to use the more familiar term, balance weights.

The addition of revolving balancing masses to the reciprocating system introduces forces exactly equal to the forces they balance, acting at right angles to the plane of reciprocation, which, in the present case, causes a variation of the pressure between the wheels and the rail, a variation which in extreme cases is sufficient to double the rail-pressure per wheel at one instant, and lift the wheel clear of the rail in the next, the interval of time corresponding to half a revolution of the wheel. This variation of rail-pressure is injurious to the permanent way, to the bridges, and to the engine tyres, and should be kept as small as possible. The designer has therefore to keep in mind two contradictory conditions. If the reciprocating parts are fully balanced by revolving masses, there is no unbalanced force, and no swaying couple, but there is a large variation in the rail-pressure. If, on the other hand, the reciprocating parts are left entirely unbalanced (the revolving parts are assumed to be completely balanced), there is no variation of rail-pressure, but there is an unbalanced force and a swaying couple which make it dangerous to run at high speeds.

A compromise is usually made, a common practice in this country being to balance about two-thirds of the reciprocating parts. The effect of the unbalanced part will be examined in detail in Arts. 75 and 76. The next three examples represent typical cases of locomotive balancing. The method followed is to take a set of reciprocating parts and consider them common to different classes of engines. The set of parts taken, dimensions and revolving parts where possible, are those common to a large number of engines on the Lancashire and Yorkshire Railway, the data of which has kindly been supplied by Mr. Aspinall.

When revolving masses are used to balance the reciprocating parts of an engine, it is unnecessary to divide the work into two stages, as directed in Art. 47. The proportion of the reciprocating masses to be balanced is to be included with the revolving masses; the revolving balance weight for the two systems is then found at one operation, that is, by one schedule.

Following the usual custom, two-thirds of the reciprocating parts are balanced in the typical examples given in Arts. 62, 63, and 64.

59. A Set of Reciprocating Parts (Lancashire and Yorkshire Railway. Cylinder, 18 inches diameter × 26 inches stroke)—

1 piston, 18 inches diameter	146	pounds
2 piston-rings	13	"
1 piston-rod and 1 crosshead	151	"
1 nut	6	"
1 crosshead pin	17½	"
2 slide-blocks	66	"
Total ...	399½	"

The connecting-rod weighs 444 pounds, and this mass is to be divided between the reciprocating masses and the revolving masses by the method of Art. 46. The position of mass centre is ·659 the length, measured from the small end, therefore—

$$\cdot 659 \times 444 = 292\tfrac{1}{2} \text{ pounds}$$

is to be included with the revolving masses, the rest, 151½ pounds, being included with the reciprocating masses.

This gives finally—

Mass reciprocated by the connecting-rod	...	399½ pounds
Proportion of mass of connecting-rod	...	151½ „
Total reciprocating mass per cylinder	...	551 „

60. The corresponding **Revolving Parts of the Crank Axle** are—

1 pair of crank-arms ...	...	296 pounds	at 13 inches
1 crank-journal	...	56 „	„
Proportion of connecting rod	...	292½ „	„
Total revolving mass per crank-pin		644½ „	„

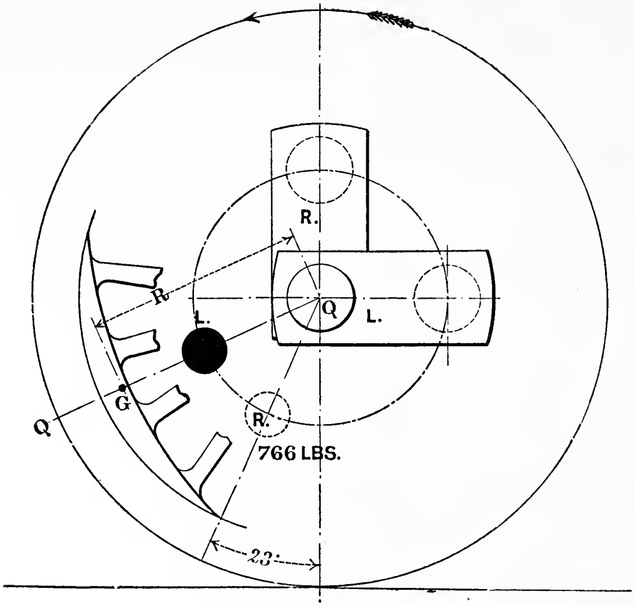
61. Scales.—In the following examples the distances from the reference plane are expressed in inches. As a consequence, the mass moments, or couples, are given by numbers involving sometimes five figures. It will be sufficiently exact for all practical purposes if the scale to which the couple polygons are drawn is chosen so that three significant figures can be read, a fourth being estimated, the fifth being considered zero. The scale of the force polygon should allow three significant figures to be read.

62. Balancing an Inside Cylinder Single Engine.—

DATA.

Stroke	26 inches
Distance centre to centre of cylinders	1 foot 11 inches
Distance between the planes containing the mass centres of the balance weights	4 feet 11 „
Mass of unbalanced revolving parts per crank-pin reduced to 13 inches radius	644 pounds
Mass of reciprocating parts per cylinder at crank radius	551 „
Proportion of reciprocating parts to be balanced.	two-thirds

Fig. 79.


R.
R
L.
Q
L.
G
Q
R.
766 LBS.
23"

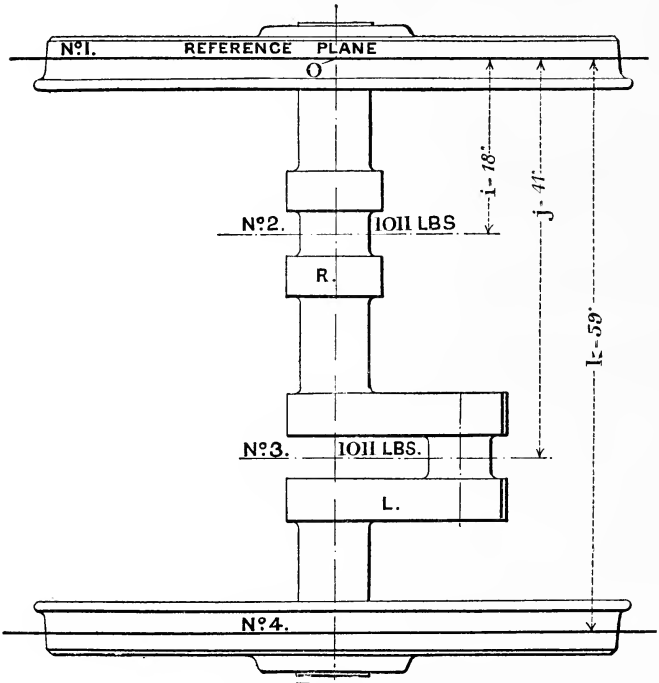

Nº1.
REFERENCE PLANE
O
i - 18"
j - 41"
k - 59"
Nº2.
1011 LBS
R.
Nº3.
1011 LBS.
L.
Nº4.

Fig. 80.

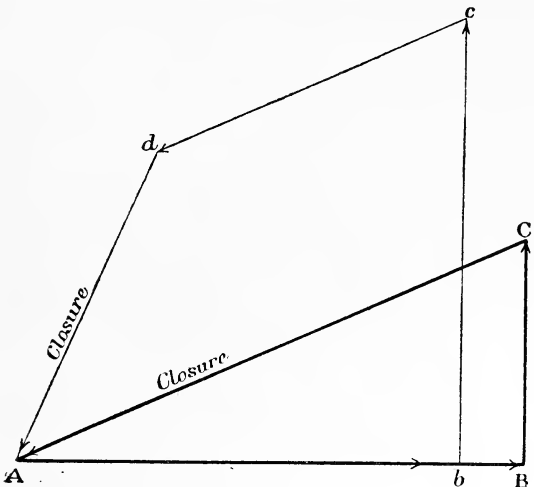


Fig. 81.

Masses to be balanced are therefore—

Revolving	644 pounds
$\frac{2}{3}$ reciprocating	367 "
Total at each crank-pin	1011 "

Draw the plan and elevation of the crank-axle as shown in Figs. 79 and 80. Notice that the left-hand driving-wheel shows to the front in elevation. Choose a reference plane to coincide with the plane containing the mass centre of the right-hand balance weight, and mark on the drawing the three dimensions i, j, and k. The balance weights are found by the general method. The quantities concerned are shown in Schedule 10.

Schedule 10.

Inside cylinder single engine. Reference plane at No. 1 (Fig. 80).

Number of crank.	Distance from reference plane.	Equivalent mass at crank radius, or centrifugal force, when ω^2R=unity.	Equivalent mass moment, or centrifugal couple, when ω^2R=unity.
	Inches.	Pounds.	
No. 1. R.H. balance weight ...	0	766	0
No. 2. R.H. crank	18	1011	18198
No. 3. L.H. crank	41	1011	41451
No. 4. L.H. balance weight ...	59	766	45220

ABC (Fig. 81) is the couple polygon, the closure CA measuring 45,220.

This represents the product of the mass of the left-hand balance weight and its distance from the reference plane, which distance is 59 inches. The mass of the balance weight is therefore 766 pounds. Its angular position in relation to the cranks is at once given by drawing QQ (Fig. 79) parallel to CA (Fig. 81), remembering to draw from the centre of the axle in the direction from C to A. The balance weight is shown in black.

The force polygon is *Abcd* (Fig. 81). Its closure measures 766 pounds, and this is the mass of the balance weight in the right-hand wheel, its angular position being defined by the direction *d*A (Fig. 81). The balance weight is shown dotted.

It is unnecessary to take a new reference plane to check the work, since the polygons check one another when the masses at each crank-pin are equal, and their planes of revolution and the planes in which the balance weights are placed are symmetrically disposed with reference to the central plane of the engine. Under these conditions the balance weights are equal in magnitude, and their angular positions are symmetrical with respect to the cranks. One balance weight is found from the couple triangle ABC (Fig. 81), the other is therefore known, and the drawing of the force polygon *Abcd* is therefore only necessary to check the accuracy of the work.

The actual mass M_0 of the balance weight depends upon the distance R of its mass centre G from the axis. If r is the crank radius, M_0 is found from—

$$M_0R = M_4r = 766r \text{ for the present example.}$$

Taking $r = 13$ inches and $R = 36$ inches, which would be about the practicable distance for a 7 feet 3 inches wheel—

$$M_0 = 376 \text{ lbs.}$$

This should be arranged in crescent form between the spokes, as shown in Fig. 79.

63. Balancing an Outside Cylinder Single Engine.—The revolving parts in an engine of this class consist of the crank-arm formed by the protrusion of the wheel-boss between the spokes of the wheel, the crank-pin, and a proportion of the connecting-rod. The planes in which the mass centre of the crank-arms revolve are not, as in the previous example, coincident with the planes in which the centres of the crank-pins revolve. There are therefore six masses to consider, revolving in six planes, as shown in Fig 83.

DATA.

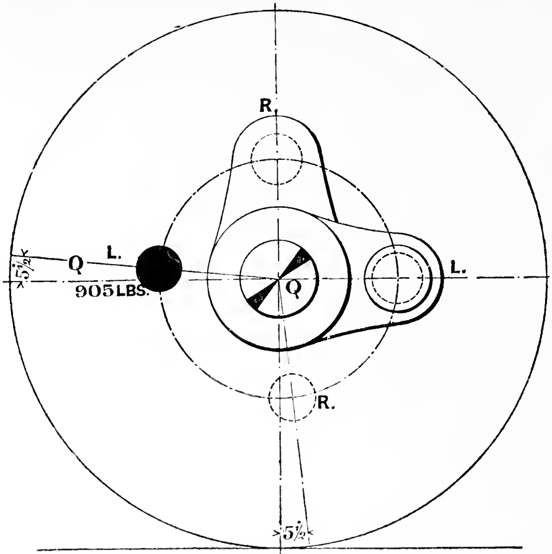
Stroke	26 inches
Distance centre to centre of cylinders	6 feet $1\frac{3}{8}$ inches
Distance between planes containing the mass centres of the balance weights	4 „ 11 „
Distance between the planes containing the mass centres of the wheel-cranks	5 „ $1\frac{3}{4}$ „
Reciprocating mass per cylinder	551 pounds
Unbalanced mass of one crank-arm and the part of the crank-pin therein reduced, to 13 inches radius	130 „
Unbalanced mass of the part of the crank-pin and washer outside the crank, 25 pounds; together with 292 pounds of the connecting-rod, both reduced to 13 inches radius ...	317 „

Mass at each crank-pin to be considered in the balancing—

Revolving	317 pounds
$\frac{2}{3}$ reciprocating	367 „
Total at each crank-pin	684 „

The plan and elevation of the crank-axle are shown in Figs. 82 and 83. The cranks are arranged as in the previous case; that is, the left-hand crank is to the front and horizontal. Take the reference plane at No. 3 to contain the mass centre of the right-hand balance weight. Fill in Schedule 11.

Fig. 82.


R.
L.
Q
905 LBS.
5½
Q
L.
R.
5½

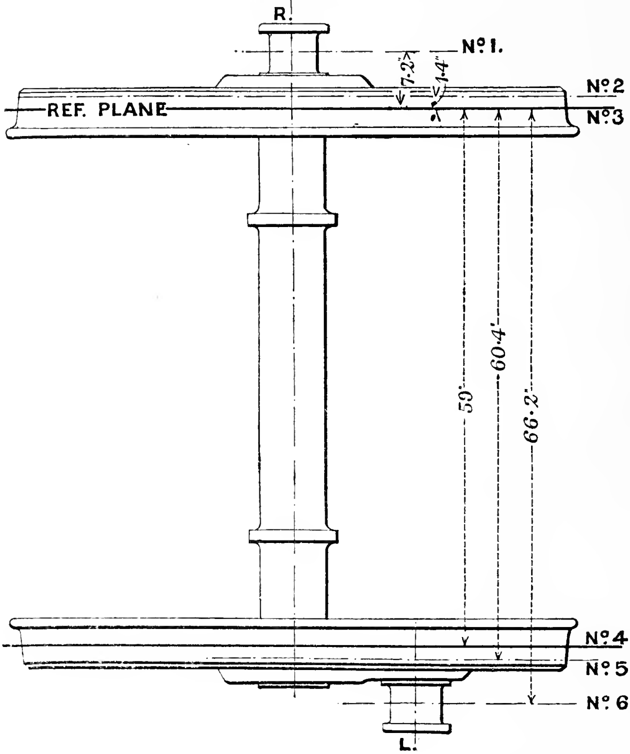

R.
No. 1.
7·2"
1·4"
No. 2
REF. PLANE
No. 3
59'
60·4'
66·2'
No. 4
No. 5
No. 6
L.

Fig. 83.

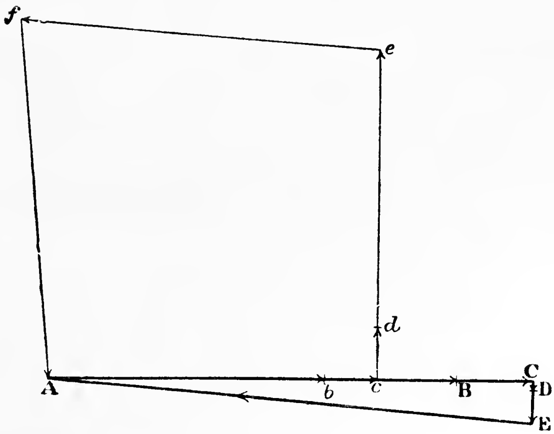


FIG. 84.

SCHEDULE 11.

Outside cylinder single engine. Reference plane at No. 3 (Fig. 83).

Number of crank.	Distance from reference plane.	Equivalent mass at crank radius, or centrifugal force, when ω^2R=unity.	Equivalent mass moment, or centrifugal couple, when ω^2R=unity.
	Inches.	Pounds.	
No. 1. R.H. crank-pin	−7 2	684.	−4925
No. 2. R.H. wheel-crank ...	−1·4	130	−182
No. 3. R.H. balance weight ...	0	905	0
No. 4. L.H. balance weight ...	59·0	905	53395
No. 5. L.H. wheel-crank ...	60·4	130	7852
No. 6. L.H. crank-pin	66·2	684	45280

ABCDE (Fig. 84) is the couple polygon. Observe that the vectors CD and DE are drawn oppositely to the direction of the right-hand crank, because these cranks are on the opposite side of the reference plane to the left-hand crank (Art. 28). The closure EA measures 53,395. The mass of the balance weight is therefore 905 pounds, and its angular position is found by drawing QQ (Fig. 82) parallel to EA (Fig. 84).

Abcdef is the force polygon (Fig. 84), the closure *f*A measuring

905, thereby checking the work. Remember, in drawing the force polygon, that the direction of drawing is always from the axis outwards to the mass, so that the force vectors *cd* and *de* are in the direction of the right-hand crank, and therefore in the opposite direction of the couple vectors CD and DE.

The balancing masses at crank radius are shown in Fig. 82, the left-hand black, the right-hand dotted.

64. Balancing an Inside Cylinder Six-coupled Engine.—(The data for this example correspond with an 18 inches by 26 inches six-coupled goods engine of the Lancashire and Yorkshire Railway. Wheels, 5 feet $0\frac{7}{8}$ inch diameter.) The new feature in this example is the coupling-rod. Each coupling-rod is to be divided between the three outside crank-pins in the proportion that they respectively support its weight since the rod is made with a joint near the centre pin. The proportion in which the division is to be made may be arrived at expeditiously by placing the rod on three knife-edges at its pin centres, each knife-edge being suitably supported on the platform of an independent weighing-machine; or by separately weighing each part of the coupling-rod in the way indicated in Art. 46. In the present example the leading and trailing wheels each take 143 pounds per crank-pin, and the driving-wheel 257 pounds. The total mass of each rod is 543 pounds.

Very often the radius of the outside cranks is less than the radius of the inner cranks, so that care must be taken that the masses at the outside crank-pins are reduced to the inside radius before including them in the schedule.

DATA FOR THE DRIVING-WHEEL (Figs. 85 and 86).

Stroke	26 inches
Radius of outside cranks	10 „
Distance centre to centre of cylinders	1 foot 11 inches
Distance centre to centre of coupling-rods ...	6 feet $1\frac{3}{8}$ „
Distance between planes 2 and 3 containing the mass centres of the wheel-cranks	5 „ $1\frac{3}{4}$ „
Distance between the planes containing the mass centres of the balance weights	4 „ 11 „
Unbalanced mass at each outside crank-pin in planes 1 and 8, made up of 257 pounds of	

the coupling-rod, and 25 pounds for the crank-pin and washer, in all 282 pounds at 10 inches radius, equivalent at 13 inches radius to ... ... 217 pounds
Unbalanced mass of each wheel-crank and part of pin in it reduced to 13 inches radius, in planes 2 and 7 ... ... 96 „
Unbalanced mass of revolving parts at each inside crank-journal ... ... 644 „
Mass of reciprocating parts per cylinder ... 551 „

Mass to be considered at each inside crank-journal in the balancing is—

Revolving	644 pounds
$\frac{2}{3}$ reciprocating	367 „
Total per crank	1011 „

Fill in Schedule 12.

Schedule 12.

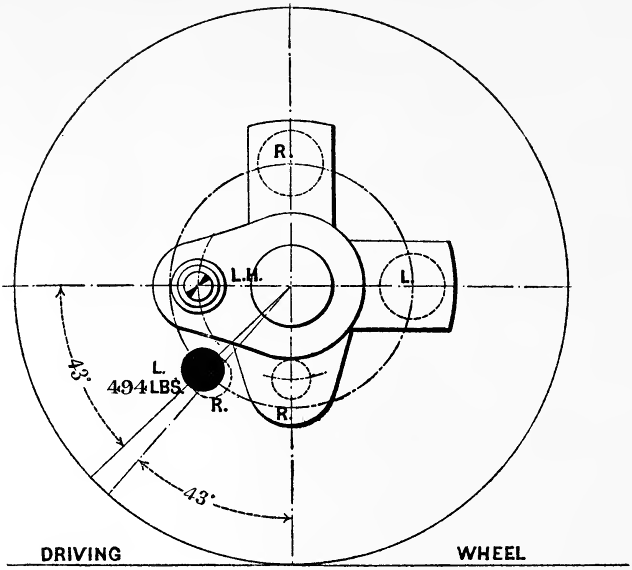
Six-coupled inside cylinder engine. DRIVING-WHEEL. Crank radius = 13 inches. Reference plane at No. 3 (Fig. 86).

Number of crank.	Distance from reference plane.	Equivalent mass at crank radius, or centrifugal force, when ω^2R=unity.	Equivalent mass moment, or centrifugal couple, when ω^2R=unity.
	Inches.	Pounds.	
No. 1	−7·2	217	−1,562
No. 2	−1·4	96	−134
No. 3	0·0	494	0
No. 4	18·0	1011	18,198
No. 5	41·0	1011	41,451
No. 6	59·0	494	29,140
No. 7	60·4	96	5,798
No. 8	66·2	217	14,365

ABCDEFG (Fig 87) is the couple polygon, in which the closure GA measures 29,140; dividing this by 59, the quotient 494 gives the balance weight for the left-hand wheel.

Abcdefgh is the force polygon. The closure *h*A measures 494, thereby checking the work.

Fig. 85.


R.
L.H.
L.
L.
494 LBS.
R.
R.
43°
43°
DRIVING
WHEEL

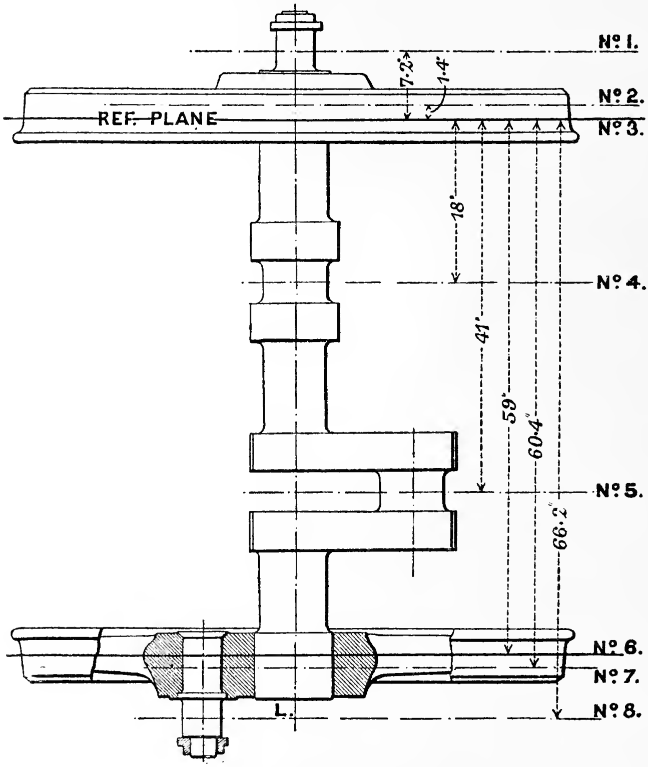

No. 1.
7·2"
1·4"
No. 2.
REF. PLANE
No. 3.
18"
No. 4.
41"
59"
60·4"
No. 5.
66·2"
No. 6.
No. 7.
L.
No. 8.

Fig. 86.

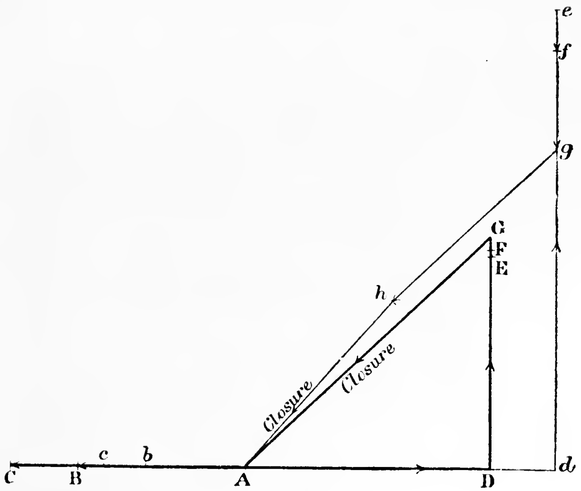


Fig. 87.

Notice that the couple vectors EF and FG are drawn oppositely to the corresponding crank directions, because they are on the opposite side of the reference plane to the rest of the cranks. (See Art. 28 for the rules for drawing couple vectors.)

Leading-wheel.—The unbalanced masses are entirely revolving, and are therefore to be completely balanced. They consist of the crank-arm, crank-pin, and a proportion of a coupling-rod, in the present example 143 pounds. The masses form a system similar to the system considered in Art. 63. Fig. 83 may be taken to represent the plan of the leading-wheel of the present example. The left-hand outside crank is there shown to the right; it should, of course, be to the left to correspond with the driving-wheel of Fig. 85, but no confusion is possible, since in whatever position the wheel is placed, the position of the balance weights is found relatively to its cranks.

Additional Data.

Mass due to coupling-rod	143	pounds
Part of crank-pin outside the crank, and washer ...	25	"
Total in planes Nos. 1 and 6 at 10 inches radius	168	"

Mass of wheel-crank and the part of the crank-pin in it, revolving in planes Nos. 2 and 4 at 10 inches radius ... ... 125 pounds

Fill in Schedule 13.

SCHEDULE 13.

Six-coupled inside cylinder engine. LEADING-WHEEL. Crank radius=10 inches. Reference plane at No. 3 (Fig. 83).

Number of crank.	Distance from reference plane.	Equivalent mass at crank radius, or centrifugal force, when $\omega^2 R$=unity.	Equivalent mass moment, or centrifugal couple, when $\omega^2 R$=unity.
	Inches.	Pounds.	
No. 1	−7·2	168	−1,209
No. 2	−1·4	125	−175
No. 3	0·0	317	0
No. 4	59·0	317	18,720
No. 5	60·4	125	7,550
No. 6	66·2	168	11,122

The couple and force polygons corresponding to the schedule are shown in Fig. 89. The balance weights they determine are shown on the elevation re-drawn in Fig. 88.

The balancing of the **trailing-wheel** is the same as the balancing of the leading-wheel in every respect.

The radius of the mass centre of each of the actual balance weights may be taken at 1 foot 10 inches.

Therefore the actual balancing masses are—

$$\text{Driving-wheel} = \frac{494 \times 13}{22} = 292 \text{ pounds.}$$

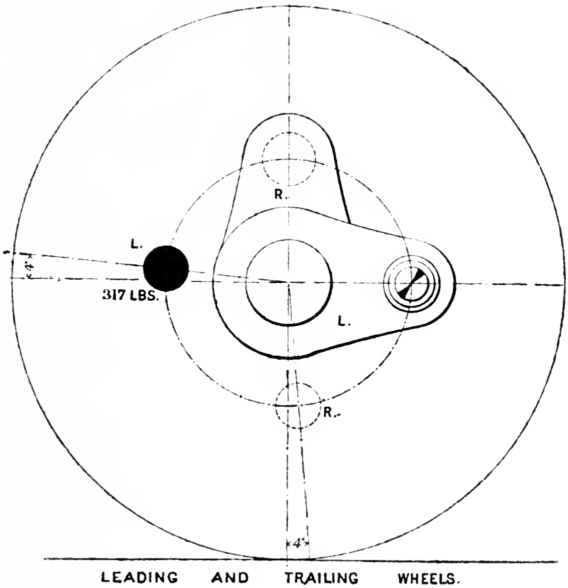
$$\text{Leading and trailing-wheel} = \frac{317 \times 10}{22} = 138 \quad ,,$$

The angles (measured from the drawings of the polygons), which the respective radii of the left-hand masses make with the horizontal, are—

Driving-wheel, 43° below the centre line:
Leading and trailing-wheels, 4° below the centre line.

The left-hand side of the engine is shown in Fig. 96, the cranks being shown in their proper relation to one another, and the balancing masses in black.

FIG. 88.



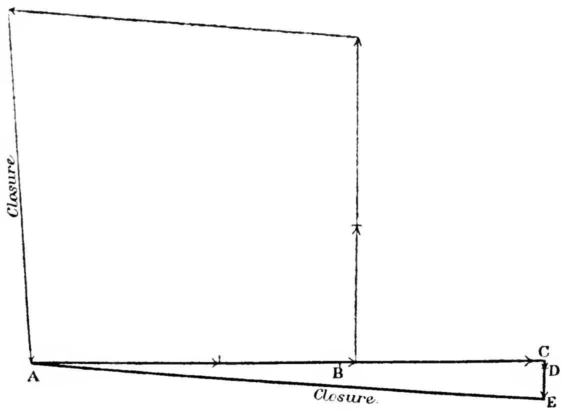


FIG. 89

65. Variation of Rail-load: "Hammer Blow."—The variation of load on the rail caused by the vertical component of the centrifugal force due to the part of the balance weight concerned in balancing the reciprocating parts is called the "hammer blow." This description of the effect does not describe what takes place very well, because the variation of load is not sudden, but continuous, except in the extreme case where the maximum value of the variation is greater than the weight on the wheel; in this case the wheel lifts for an instant, and gives the rail a true blow in coming down.

To estimate the variation of load on one rail in any given case, the balance weight concerned in balancing the reciprocating parts must be separated from the main balance weight. The quickest way to do this is to find the balance weight for the proportion of the reciprocating masses balanced, neglecting altogether the revolving masses. The schedule for this problem would be similar to Schedule 10. It is merely necessary to write in for the mass at each crank-pin the proportion of the reciprocating parts to be balanced. A more convenient way is to consider the crank-pin mass unity. Then, in the couple polygon (Fig. 81), AB would represent the dimension j, BC the dimension i. The closure is therefore given by—

$$CA = \sqrt{i^2 + j^2}$$

and the magnitude of the balance weight for unity mass by—

$$\frac{CA}{k} = \frac{1}{k}\sqrt{i^2 + j^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Then, if M is the mass in pounds of the reciprocating parts per crank-pin and q the fraction of this quantity which is to be balanced, the magnitude, m, of the balance weight required is—

$$m = \frac{qM}{k}\sqrt{i^2 + j^2} \text{ pounds} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

The value of the angle of direction is given by—

$$\tan \theta = -\frac{i}{j} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

considering AB to be the initial direction.

Let w be the variation of rail-load, that is, the vertical component of the centrifugal force due to m,

α the instantaneous value of the angle, measured in the positive direction, that is, counter-clockwise between the line of stroke and the radius of the balance weight m,

r the crank radius in feet,

ω the angular velocity of the wheel in radians per second.

Then—

$$w = \frac{m\omega^2 r}{g} \sin \alpha \text{ lbs. weight} \quad . \quad . \quad . \quad . \quad . \quad (4)$$

The sign of w is determined by the sign of sine α; a positive value indicates a diminution of rail-load, a negative sign an increase.

If V is the speed of the train in miles per hour, and D the diameter in feet of the driving-wheel, containing the balance weight—

$$\omega = \frac{2 \times 5280V}{3600D}$$

Substituting this in 4, and dividing by 2240 to obtain w in tons weight (m is in pounds)—

$$w = \frac{\cdot 00012 mrV^2}{D^2} \sin \alpha \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

A further variation of the rail-load is brought about by the obliquity of the connecting-rods. Considering the R.H. rod, the transmission of force along it is accompanied by a force acting on the slide-bar, and an equal and opposite force acting at the centre of the driving-axle. These two forces, in fact, form a couple instantaneously equal to the turning couple acting on the crank. In the ordinary stationary engine this couple tends to turn the engine-frame as a whole. In a locomotive, the main bearings being spring-connected to the frame, and free to move relatively to the frame in the direction in which these forces act, the effect is somewhat different. Supposing the engine to be running forward, the force acting on the slide-bar is upwards for the greater part of the stroke. The equal and opposite force acting at the driving-axle causes an increase of the rail-load. The effect of the force on the slide-bar is to lift the leading end of the engine slightly, thereby changing the loads on all the springs and causing a slight diminution of the load on the driving-springs, the amount of which it would be impracticable to calculate. It is certain, however, that this diminution is some fraction of the force acting at the bars because the other springs share the effect, and the total change in load on all the springs must be just equal to the slide-bar force supposing no oscillations are going on. The variation of rail-load due to the obliquity of the connecting-rod is therefore made up of two parts—the one, a force equal and opposite to the force at the slide-bars acting directly on the driving-axle; the other, a change in the load transmitted to the axle by the driving-springs in consequence of the action of the force at the slide-bars on the frame. These two parts are always of opposite sign, and the second part is always less than the first, so that if $+f$ denotes a force acting upwards at the slide-bars, $-f$ is the force acting at the driving-axle, and if i and k have the same meaning as in Fig. 80 (k now being taken equal to the distance between the wheel treads), the instantaneous value of the variation of the rail-load at the left-hand wheel is given by—

$$f \times \frac{i}{k} \textit{ minus } \text{change of load on L.H. spring.}$$

Similarly for the R.H. wheel. The changes from the left-hand gear added to these give the total results.

The magnitude of these quantities depends upon the cut-off in the cylinders, upon the speed, upon the method of springing the engine, upon the strength of the springs, and upon the ratio of the length of the connecting-rod to the crank. The later the cut-off the greater the average value of f, the higher the speed the more uniform the instantaneous values, and the longer the connecting-rod the less the values of f. To find the value of f for a given crank position, deduce the resultant driving pressure from the indicator diagrams, and take from this the force required for the acceleration of the reciprocating parts. This nett driving pressure multiplied by the tangent of the angle the connecting-rod makes with the line of stroke gives the value of f. Multiply f by the distance between the crosshead centre and the centre of the driving-axle, and the product is the turning couple (Art. 130). The nett value of the driving pressure in a particular case is shown in Fig. 95, and the corresponding turning couple by the curve marked L.H., Fig. 91.

Compared with the effect of the balance weights, this cause of variation of rail-load is practically negligible at high speeds. The effects of balance weights increase as the square of the speed, the value of f decreases as the speed increases, in engines of the ordinary proportions.

66. Example.—Consider that the example of Art. 62 represents a 7-foot inside single.

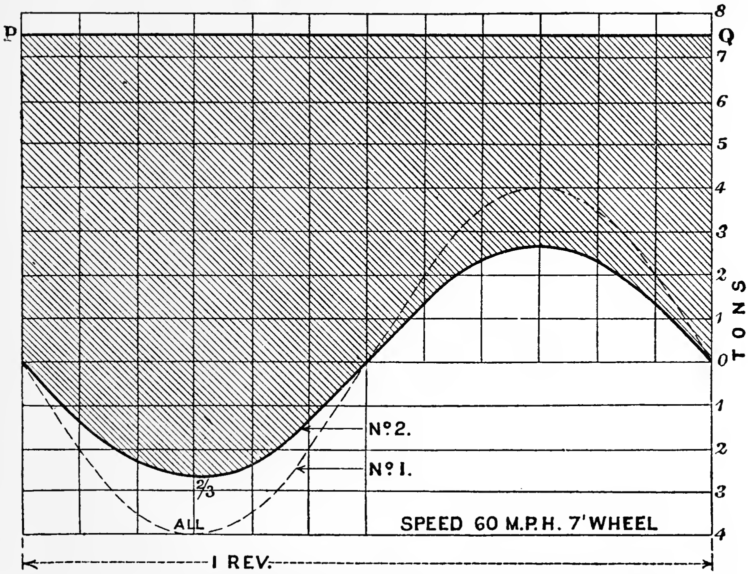
The value given by equation (1) of the previous article is approximately 0·76; therefore the magnitude of the balance weight required to balance the reciprocating masses is—

$$0{\cdot}76qM \text{ pounds}$$

If the whole of the reciprocating parts are balanced, $q = 1$, and $M = 551$ pounds; therefore—

$$m = 0{\cdot}76 \times 551 = 419 \text{ pounds}$$

Let V be 60 miles per hour, and D, the diameter of the wheel,



MASS OF RECIPROCATING PARTS 551 LBS. STROKE 1·03'.

FIG. 90.

7 feet. Then, from equation (5) of the previous article, the crank radius being 1·08 foot—

$$w = 4 \sin a \ldots \text{ tons weight nearly.}$$

When this balance weight is passing through its highest and lowest positions respectively, sin a is $+1$ and -1. The load on the rail is decreased in the first case by 4 tons, increased in the second by 4 tons. Supposing the load on the axle to be 15 tons, that is, 7½ tons per wheel, at every revolution the load per wheel is alternately decreased and increased by about 54 per cent. If two-thirds of M be balanced, the percentage variation is reduced to 36 per cent. Taking a horizontal base line through 0 (Fig 90) to represent the circumference of the driving-wheel, curves Nos. 1 and 2 respectively represent the variation of the rail-load when the whole and when two-thirds of the reciprocating parts are balanced. The static load on the wheel is represented by the ordinate to the horizontal line PQ. The part of a vertical line cut off by the shaded figure therefore represents the load available for adhesion, at 60 miles per hour, when the point at which it cuts the circumference is in contact with the rail, assuming two-thirds of the reciprocating parts to be balanced.

67. Speed at which a Wheel lifts.—When small wheels are used, the piston speed increases for a given speed of travelling, and the rail-load variation must be carefully considered in the balancing, or the wheels may leave the rail altogether at every revolution, a mistake in design not entirely unknown in practice. The formula, equation (5), Art. 65, may easily be adjusted to find the speed at which lifting takes place.

Let W be the static load on the wheel.

The rail-load at any instant is given by—

$$W - w$$

If w is numerically equal to W, this becomes 0 when the balance weight is passing through its highest position, and 2W when passing through the lowest. Hence, putting W for w in equation (5), Art. 65, and solving for V, sin a being 1—

$$V_0 = \sqrt{\frac{D^2W}{0 \cdot 00012mr}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

V_0 is now the speed in miles per hour at which the rail-load vanishes when the balance weight passes through its highest position. Taking the data of the example of Art. 66, where $W = 7 \cdot 5$ tons, and $m = 419$ pounds for full balance—

$$V_0 = 82 \text{ miles per hour approximately}$$

If $m = 280$ lbs., thus balancing two-thirds of the reciprocating masses—

$$V = 100 \text{ miles per hour approximately.}$$

These two calculations show that two-thirds is about the greatest proportion of the reciprocating masses which should be balanced in a single engine, and in a coupled engine also, if the balancing mass m is all put in the driving-wheel.

Although the rail-load vanishes, slipping may not occur, because the other wheel on the same axle may be able to provide sufficient adhesion at the instant. To detect if slipping is about to take place, the turning effort on the crank must be compared with the couple resisting slipping, this couple depending upon the instantaneous sum of the rail-loads.

68. Slipping.—The driving-wheels tend to slip when the turning effort on the driving-axle is equal to the couple resisting slipping. The forces of this latter couple are, the frictional resistance at the rail and the equal, parallel, and opposite tractive force at the driving-horns; the arm of the couple is the radius of the driving-wheel.

The force due to the frictional resistance varies directly as the pressure between the wheel and the rail. If W_1 is the static load on the two driving-wheels, w_1 the resultant variation of rail-load for the two wheels, the greatest value of the frictional resistance is about—

$$\frac{W_1 - w_1}{5}$$

Therefore, if the turning couple on the driving-axle is greater than the couple—

$$\left(\frac{W_1 - w_1}{5}\right)R \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

R being the radius of the driving-wheel, slipping will tend to take place.

The resultant of the right-hand and left-hand balance weights concerned in balancing the reciprocating parts is equal and opposite to the resultant of the proportion of the reciprocating masses balanced, the latter being considered concentrated at the respective crank-pins. This latter resultant (Art. 56, Fig. 77) is equal to $Mq\sqrt{2}$ pounds at 45° to each crank. The resultant balance weight is, therefore, a mass $Mq\sqrt{2}$ pounds, attached to an imaginary crank at the centre of the driving-axle, placed at 135° to each crank, as shown in Fig. 92. The instantaneous value of the whole variation of the rail-load for the two wheels is given by the projection of the force vector belonging to this imaginary central crank on a line at right angles to the plane of the rails. If θ is the angle between the left-hand crank (Fig. 78) and the line of stroke, measured counterclockwise, the magnitude of this projection is—

$$\frac{1{\cdot}41qM\omega^2 r}{g} \sin(\theta + 225^\circ) \text{ lbs. weight.}$$

or in terms of the revolutions per second, n, of the crank-axle, M being in pounds, r in feet, and dividing the constant by 2240—

$$w_1 = {\cdot}00077qMn^2r \sin(\theta + 225^\circ) \text{ tons weight} \quad . \quad . \quad . \quad (2)$$

The instantaneous value of the couple resisting slipping is therefore—

$$R\left\{\frac{W_1 - {\cdot}00077qMn^2r \sin(\theta + 225^\circ)}{5}\right\} \text{ foot-tons} \quad . \quad (3)$$

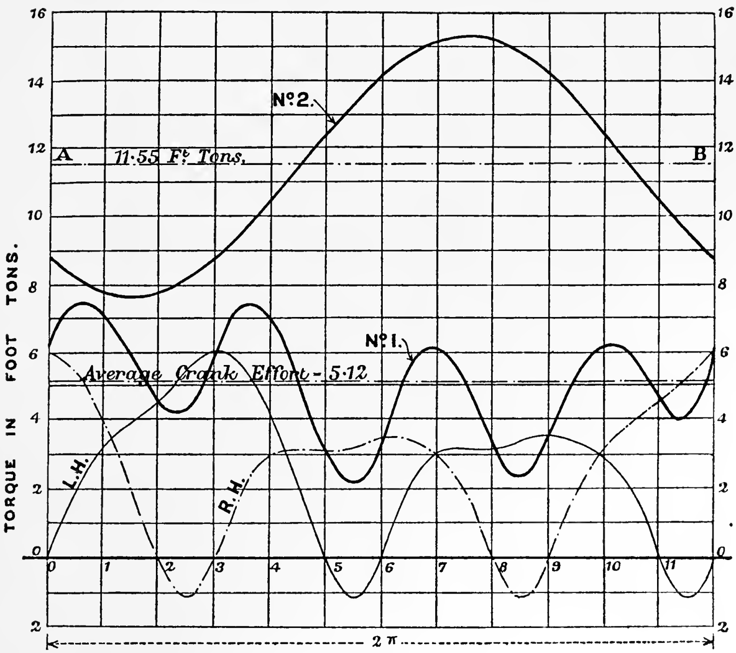
This is a maximum when the imaginary central crank is passing through its lowest position, that is, when $\theta = 45^\circ$; and a minimum when passing through the highest position, that is, when $\theta = 225^\circ$.

For instance, if $W = 16$ tons, on 7-feet driving-wheels, the mass of the reciprocating parts per cylinder being 551 lbs., of which $q = \frac{2}{3}$ are balanced, stroke 26 inches, the minimum value of the resisting couple when n is 4 per second, corresponding to 60 miles per hour, is—

$$3{\cdot}5\left\{\frac{16 - {\cdot}00077 \times \frac{2}{3} \times 551 \times 16 \times 1{\cdot}08}{5}\right\} = 7{\cdot}77 \text{ foot-tons}$$

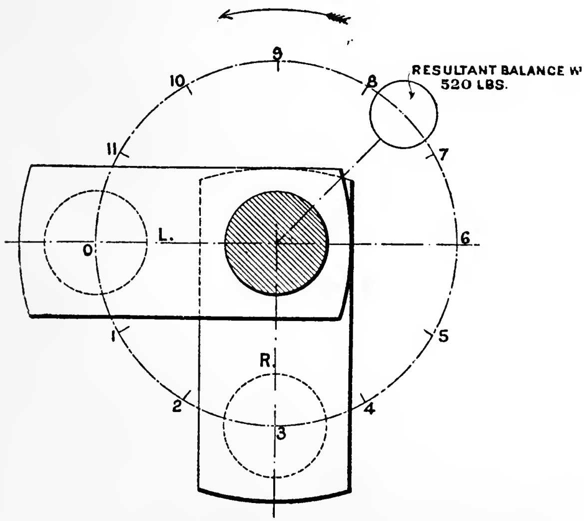
The value corresponding to the static load of 16 tons is 11·2 foot-tons.

69. Example.—To further illustrate this point, the actual driving effort or torque is compared with the couple resisting slipping for a complete revolution in Fig. 91, in the case of a Lancashire and Yorkshire 4-coupled bogie express passenger engine running at 65 miles per hour, taking two-thirds of the reciprocating parts to



TORQUE IN FOOT TONS.
A
11·55 Ft Tons.
B
No 2.
No 1.
Average Crank Effort - 5·12
L.H.
R.H.
2 π

FIG. 91.



RESULTANT BALANCE W' 520 LBS.
L.
R.

FIG. 92.

be balanced in the driving-wheels, which are 7 feet diameter; cylinder, 18 inches diameter; 26 inches stroke.

The ordinates of curve No. 1 (Fig. 91) represent the torque or driving-couple acting on the driving-axle, those of curve No. 2 the couple resisting slipping. It will be noticed that the two ordinates are nearly equal for crank position 1. A little more steam and curve No. 1 would have cut curve No. 2, and if this had been a single engine, slight slipping would be the result. In the case in question, the coupled wheels would come into play and prevent it. Between crank positions 7 and 8 there is a large difference in the ordinates, and slipping is not to be feared; the instantaneous load on the rails is increased from $16\frac{1}{2}$, however, to 21·8 tons. The rail-load corresponding to the minimum value of the resisting couple is 11·2 tons, so that the rail-load under the driving-wheels at 65 miles per hour is continually varying between 11·2 and 21·8 tons once per revolution, that is, 4·2 times per second.

The method of drawing the curves is as follows:—

Fig. 92 shows the cranks and the resultant balancing mass, which is 520 pounds. Notice the way the crank positions are numbered round from the initial position of the left-hand crank. The instantaneous angular distance of the radius of the resultant mass from the initial position, 0, of the left-hand crank is given by $\theta + 225°$, θ being measured downwards from the horizontal centre line. The left-hand crank shows to the front and the engine is running forward.

(*a*) Divide the crank circle into twelve or more equal parts, and find the corresponding positions of the crosshead graphically.

(*b*) Find the nett driving pressure from the indicator cards which are shown in Fig. 93, by taking the intercepts between the steam-line of one diagram and the exhaust-line of its fellow. The shaded parts of the diagrams show the width to be taken for the left-hand end. These are plotted in Fig. 94, curve No. 1, for both ends, and calibrated to give the total pressure acting on the piston in tons. The numbers on the horizontal axis are those corresponding to the numbers on the crank circle in Fig. 92.

(*c*) The pressures of Fig. 94 are modified by the forces required to accelerate the motion of the reciprocating masses. These are quickly found by Klein's construction (Art. 104). The curve representing them is No. 2 (Fig. 94). The driving pressures on the piston are decreased by the accelerating forces during the first part of the stroke, and increased during the second part. The vertical width of the shaded figure gives the instantaneous value of the force operating at the crosshead to turn the crank for any given crank angle. These widths have been re-plotted in Fig. 95.

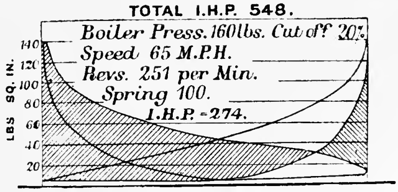


FIG. 93.

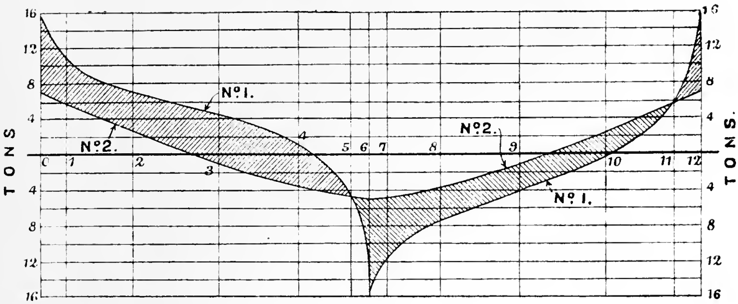


FIG. 94.

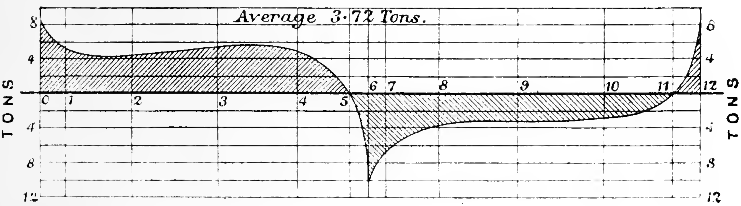


FIG. 95.

Notice how much more uniform this driving force is made by the action of the accelerating force. In the neighbourhood of the points 5 and 11 the driving force vanishes and becomes negative, that is, the crank is now driving the piston instead of the piston the crank. These are the points at which a change over takes place all through the driving system. The pressure between the connecting-rod brasses and the crank-pin changes from one side to the other at these points. The slide-blocks leave the top bar for the bottom bar in forward running, returning to the top bar at positions 6 and 12. These changes are accompanied by a knock if there is any slack at the places where they occur.

(*d*) The crank-effort diagram may be constructed by any of the usual methods from the curve of pressures in Fig. 95. A full discussion of this, and a simple construction for the purpose, are given in Art. 130. The curve marked L.H. in Fig. 91 is the crank effort or torque curve corresponding to the driving pressures of Fig. 95. The curve corresponding to the right-hand crank is assumed to be the same in form; the left-hand curve is, therefore, simply redrawn with an angular difference of 90° to get the crank-effort curve for the right-hand crank. The two are then added to get curve No. 1 (Fig. 91), giving the total crank effort in terms of the position of the left-hand crank.

(*e*) The data for drawing the curve resisting slipping are $W = 16\frac{1}{2}$ tons; $R = 3{\cdot}5$; $M = 551$ lbs.; $q = \frac{2}{3}$; $n = 4{\cdot}2$; and $r = 1{\cdot}08$ feet.

Expression 3, Art. 68, reduces to—

$$\text{Resisting couple} = 11{\cdot}55 + 3{\cdot}78 \sin (\theta + 225°)$$

the positive sign being used because the angle θ is measured counter-clockwise and downwards from 0 to the left-hand crank radius.

This is represented by curve No 2, Fig. 91. The average crank effort is 5·12 foot-tons. The average resisting couple is 11·55 foot-tons, more than double the average torque. Judging from these figures alone, there would appear to be ample margin to prevent slipping, and yet, as the diagram shows, the engine would be just on the point of slipping if it were not prevented by the coupled wheels.

70. Distribution of the Balance Weights for the Reciprocating Parts amongst the Coupled Wheels.—A way of decreasing the variation of rail-load in coupled engines is to divide the balance weight necessary to balance the reciprocating parts, amongst the coupled wheels. The effects of these separate weights on the engine-frame add up to the same horizontal effect as that due to the single balance weight m in the driving-wheel. The variation of rail-load is reduced at the driving-wheel, a proportional variation being introduced at the coupled wheels to which part of the balance weight is transferred. There is also a redistribution of pressures at the horns.

To illustrate this method of distribution, consider the example of Art. 64 again. Fig. 96 shows the crank circles drawn out with the balancing masses, shown in black, already found for the com-

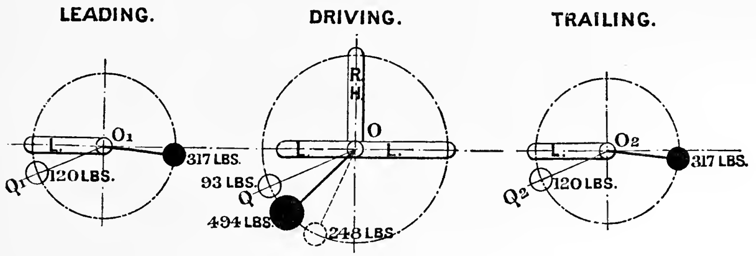


FIG. 96.

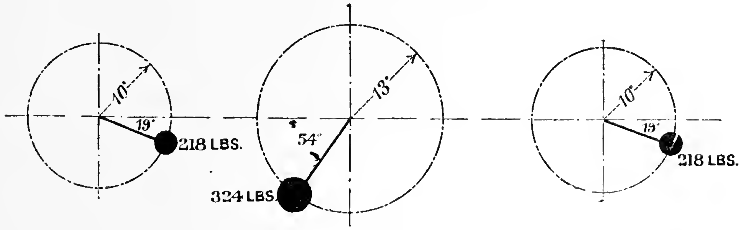


FIG. 97.

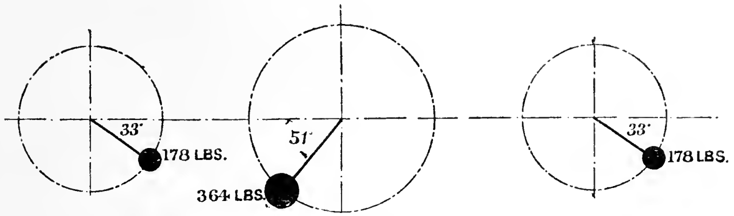


FIG. 98.

plete balance of the revolving parts and two-thirds of the reciprocating parts, the latter being balanced in the driving-wheel. The balance weight required in the driving-wheel to balance the revolving masses alone is 248 pounds, placed as shown by the dotted circle. The balance weight (m of Art. 65, equation (2)) required to balance the reciprocating parts alone is 279 pounds, placed as shown by the full open circle. The black mass of 494 pounds is the resultant of these two.

Draw lines O_1Q_1 and O_2Q_2, in the leading and trailing wheels respectively, parallel to the radius OQ in the driving-wheel, and place one-third of the 279 pounds, that is, 93 pounds at 13 inches, in each wheel. This is equivalent to 120 pounds at 10 inches, the radius of the cranks for the leading and trailing wheels.

Considering the leading or trailing wheel, the 120 pounds due to transferred mass combines with the 317 pounds already found for the revolving masses to form a resultant mass of 218 pounds at 10 inches radius, placed as shown in Fig. 97. Recombining the 248 pounds balancing the revolving masses of the driving-wheel, with the 93 pounds left of m, a resultant mass of 324 pounds is obtained. The new balance weights are shown in Fig. 97. Balancing is effected to precisely the same extent by them as by the heavier set of Fig. 96; the variation of the rail-load is reduced by two-thirds, though there is now a load variation under each of the coupled wheels. Fig. 98 shows the set of balance weights which will balance the whole of the reciprocating parts (551 lbs.). In this case m, the mass corresponding to the open circle in the driving-wheel of Fig. 96, is 419 pounds, Art. 66 giving 140 pounds at 13 inches radius per wheel to be combined with the masses balancing the revolving parts. The result of this combination is to give 178 pounds in the trailing and leading wheel, and 364 pounds in the driving-wheel, a set of masses weighing less than the sets of Figs. 96 and 97, though balancing the whole of the reciprocating parts, and causing less variation of rail-load than the first set. The three sets of balance weights are drawn one under the other for the sake of comparison.

This method of distribution is unquestionably the best way of dealing with whatever proportion of the reciprocating parts may be balanced so far as the permanent way is concerned. The variation of the tractive effort may be completely balanced since the whole of the reciprocating masses may be balanced without introducing too great a variation of the rail-load.

The division between the coupled wheels may be made in any proportion. Having decided what proportion is to be balanced in the different wheels, the proper balance weights are found directly by the use of a schedule of the style of No. 12. Each wheel, in fact, to which a part of the reciprocating mass is to be transferred is to be looked upon as a crank-axle coupled by an imaginary connecting-rod at the imaginary inside crank-journals to the real inside crank-journals belonging to the crank-axle, and carrying the part of the reciprocating mass assigned to it at the crank-journals.

If an engine without a bogie or small leading-wheel is to be balanced in this way, it would be advisable to assign a less proportion than one-third to the leading-wheel if the engine is to run very fast.

71. American Practice.—Mr. Henszey, of the Baldwin Locomotive Works, has kindly furnished the following details of their practice :—

All the revolving parts are balanced and two-thirds of the reciprocating parts on single-expansion engines, three-quarters on the Vauclain Compound. The weights balancing the reciprocating parts are distributed equally between the coupled wheels. One-third of the connecting-rod is included with the reciprocating parts, two-thirds with the revolving parts. This is the distribution for a rod whose mass centre is $\cdot 66 \times l$, measured from the small end. The coupling-rods are "weighed" (see Art. 46), to find the mass to be assigned to each crank-pin. The parts are balanced as though their respective mass centres revolved in the same plane, that is, the balance weights are put exactly opposite the cranks.

72. Example—Eight-coupled Engine, Class "E," Baldwin Company.—Fig. 99 shows the arrangement of the wheels.

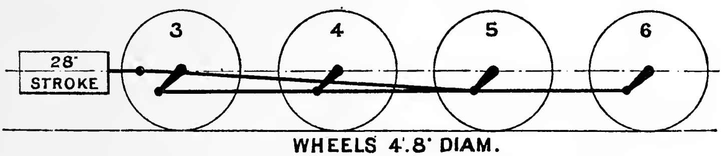


Fig. 99.

Mass of reciprocating parts, including piston, crosshead, one-third of connecting-rod = 1170 lbs. Of this two-thirds is balanced, which, equally distributed between the four wheels, gives 195 lbs. per wheel.

The mass to be balanced in each wheel is made up as follows:—

	Wheel numbers.				
	No. 3.	No. 4.	No. 5.	No. 6.	
Reciprocating parts equally distributed	195	195	195	195	pounds
Revolving parts—					
Two-thirds connecting-rod ...	—	—	464	—	„
Coupling-rod	169	214	265	106	„
Wrist-pins	73	90	275	86	„
Crank-hubs	184	204	272	204	„
At 14 inches radius ...	621	703	1471	591	„
At $16\frac{1}{4}$ inches the radius of the mass centres of the balance weights	531	605	1267	508	„

73. Four-cylinder Locomotives.—The reciprocating masses in a four-crank locomotive may be arranged to balance amongst themselves without the use of balance weights at all. Under these circumstances, always supposing the revolving masses to be balanced, there will be no variation of rail-load, no unbalanced force, and no horizontal swaying couple. The engine will, in fact, be perfectly balanced, neglecting the error due to the obliquity of the connecting-rod.

The crank angles involved in balancing four reciprocating masses amongst themselves require the employment of a separate set of valve-gear per cylinder. Considerable mechanical simplicity may be obtained by arranging the cranks in two pairs, the two cranks in each pair being at 180° with one other, the pairs themselves being at 90°. If the reciprocating masses be equal, the force polygon would close, forming a square or a right angle returning on itself; there would be, therefore, no unbalanced force. The couple polygon, however, would not close; there would be, therefore, a horizontal swaying couple left whose maximum magnitude is by the principles of Art. 56—

$$1{\cdot}7Mn^2br\cos(\theta + 45^\circ)$$

where b is the distance between the cranks forming a 180° pair. The disturbing effect of this couple will depend upon the moment of inertia of the engine about a vertical axis through its mass centre. The longer and heavier the engine, and the more the mass is grouped at the leading and trailing ends, the less the disturbance. If this couple is left in there will be no variation of rail-load because there are no revolving masses in the system applied to balance reciprocating masses; if, however, revolving masses are added to balance this couple, they introduce a variation of rail-load. If the engine will run steadily at high speeds, there is no doubt that the best thing to do under the circumstances is to leave the swaying couple in, so that the engine may run without variation of rail-load. If this couple is left unbalanced, the whole static load on the wheel is always available for adhesion. If, however, there is much swaying, masses may be put in the driving-wheels to reduce it.

An example of a successful four-cylinder engine in which advantage is taken of the four sets of reciprocating parts to avoid the use of balance weights is furnished by the four-cylinder compounds introduced on the London and North Western Railway in 1897, by Mr. F. W. Webb, for running between Euston and Crewe without a stop with the heavy trains for the North. A description and drawings are published in *Engineering*, December 3, 1897. The cranks are arranged in two 180° pairs at right angles, and only two sets of valve-gear are employed, one set being arranged to work the two valves of a 180° pair.

There is another point in connection with the balancing of this engine which should be noticed. Each inside crank is balanced by prolonging the crank-arms on the opposite side of the axle to form a balance weight. In this way the loading of the axle with centrifugal force between the wheels is avoided. In the usual arrangement, where the balance weights are placed in the wheels, although the axle, as a whole, is thereby freed from dynamical load due to the rotation of the masses belonging to each crank, yet the axle has to transmit the centrifugal force from each revolving mass to the planes in which the balance weights are placed, thereby causing a bending moment on the axle.

To emphasise this point, consider the case of a crank-axle of the usual type for an inside cylinder engine where each crank and the part of the connecting-rod included with revolving masses is equivalent to 700 pounds at a radius of 1 foot. At 60 miles per hour, with a 7-foot driving-wheel, the axle is making 4 revolutions per second. The centrifugal force corresponding to this is 6·12 tons weight. At 80 miles per hour the centrifugal force increases to 10·9 tons per crank. So that at this latter speed acting at each inside crank is a force of 10·9 tons due to the motion alone. This force loads the axle almost as severely as the full steam pressure at starting, so far as the bending moment is concerned.

74. Crank Angles for the Elimination of the Horizontal Swaying Couple.—If four sets of valve-gear be employed, the crank angles may be arranged for complete balance in a large number of ways for a given set of cylinder centre-lines. The only solution which is practicable for coupled engines is that in which the outside cranks are at right angles. Consider the case of a symmetrical engine in which the pitch of the outside cylinders is 6·7 feet and the pitch of the inside cylinders 2·2 feet. Proceeding by the method of Art. 49 or Art. 37, it will be found that the crank angles must be those shown in Fig. 100, and that each set of inside reciprocating parts must be 2·27 times a set of outside reciprocating parts. Under these circumstances there would be no unbalanced force, no variation of rail-pressure, and no swaying couple, except to the extent introduced by the obliquity of the connecting-rods, which in the case of locomotives where the ratio of the rod to the crank is always relatively great, is negligible. Up to the present, no locomotive in this country has been balanced so perfectly as this, and whether the gain in smoothness of running and absence of vibration is worth the extra mechanical complication, or whether an engine with such crank angles would work well on the road, are points which can only properly be decided by a practical trial. It should be noticed that an engine with the crank angles of Fig. 100 has only one dead centre at a time. To obtain a good crank-effort curve the work may be distributed amongst the cylinders by the principles of Art. 131. The revolving parts must be balanced independently as a separate system. See Art. 51.

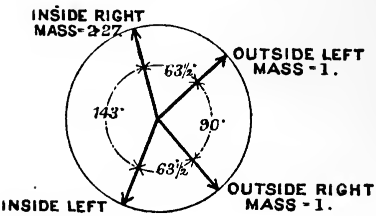


Fig. 100.

75. Estimation of the Unbalanced Force and Couple.—If the proportion q of the reciprocating masses is balanced, there remains $(1 - q)M$ pounds unbalanced. This causes disturbances like those enumerated in Art. 56. The maximum value of the unbalanced force is—

$$1{\cdot}7M(1 - q)n^2r \text{ lbs. weight, from equation 1, Art. 56}$$

the maximum value of the swaying couple is—

$$\cdot 85M(1 - q)n^2rd \text{ foot-lbs., from equation 2, Art. 56}$$

similarly—

$$1{\cdot}7M(1 - q)n^2rt$$

is the maximum value of the couple acting in a vertical plane. These values are, of course, only true for engines of the ordinary type, that is, with equal reciprocating masses symmetrically arranged. For any other type, two-cylinder compounds with unequal reciprocating masses, four-cylinder engines with arbitrarily determined reciprocating masses, etc., the general method of Art. 45 must be used to find the magnitudes of the closures to the force and couple polygons.

76. Comparative Tables.—The following schedules have been calculated for the purpose of comparing the different types of engines with regard to their possibilities of balance. Schedule 14 gives the general formulæ for the different cases. The expressions in column A. are reduced from the formula—

$$1{\cdot}22qMn^2r\frac{\sqrt{i^2 + j^2}}{k} \text{ lbs. weight} \quad . \quad . \quad . \quad . \quad \textbf{(1)}$$

formed by the combination of formulæ 4 and 2, Art. 65. The values of $i, j, k,$ assumed in calculating the quantities stated, are those given in Fig. 80, viz. 18, 41, and 59 inches respectively, for the inside cylinder engines, and those given in Fig. 83, viz. 7·2, 66·2, and 59 inches respectively, for the outside cylinder engines.

Column B. is calculated from—

$$1{\cdot}7M(1 - q)n^2r \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

and column C. from—

$$\cdot 85M(1 - q)n^2rd \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

The errors of the four-cylinder engines are estimated in the way indicated in Art. 73.

SCHEDULE 14.

M = the mass of the reciprocating parts per cylinder in pounds.
r = the crank radius in feet.
n = the number of revolutions of the crank-axle per second.
d = the distance between the cylinder centre lines in feet.

Type of engine.	Proportion of reciprocating masses balanced per cylinder (q).	A. Maximum value of the variation of rail-load per wheel in lbs. weight (hammer-blow).	B. Maximum value of the unbalanced force in lbs. weight.	C. Maximum value of the swaying couple in foot-lbs.
Inside single	All	$0{\cdot}93Mn^2r$	Nil	Nil
Outside single	All	$1{\cdot}38Mn^2r$	Nil	Nil
Inside single	$q = \frac{2}{3}$	$0{\cdot}62Mn^2r$	$0{\cdot}57Mn^2r$	$0{\cdot}28Mn^2rd$
Outside single	$q = \frac{2}{3}$	$0{\cdot}92Mn^2r$	"	"
Inside 4-coupled ...	$\left\{ \begin{array}{c} q = \frac{2}{3} \\ \frac{1}{2} \text{ per wheel} \end{array} \right\}$	$0{\cdot}31Mn^2r$	"	"
Outside 4-coupled ...	$\left\{ \begin{array}{c} q = \frac{2}{3} \\ \frac{1}{2} \text{ per wheel} \end{array} \right\}$	$0{\cdot}46Mn^2r$	"	"
Inside 6-coupled ...	$\left\{ \begin{array}{c} q = \frac{2}{3} \\ \frac{1}{3} \text{ per wheel} \end{array} \right\}$	$0{\cdot}21Mn^2r$	"	"
Outside 6-coupled ...	$\left\{ \begin{array}{c} q = \frac{2}{3} \\ \frac{1}{3} \text{ per wheel} \end{array} \right\}$	$0{\cdot}31Mn^2r$	"	"
4-cylinder engine, two 180° pairs of cranks at right angles	All	Nil	Nil	$1{\cdot}73Mn^2rd$
4-cylinder engine (Fig. 100)	All	Nil	Nil	Nil

The actual values of the different quantities in Schedule 14 are given in Schedule 15, on the assumption that the mass of the reciprocating parts per cylinder is in each case 551 lbs. weight, that the stroke is 26 inches, and that the number of revolutions of the crank-axle is 4 per second. The piston speed corresponding to this assumption is 1036 feet per minute. For the inside cylinder engines the distance between the cylinders, d, is taken, 1·92 feet (Fig. 80); for the outside cylinders, 6·117 feet (Fig. 83). For the four-cylinder engine the distance, d, between the two cranks forming a 180° pair is assumed to be 2 feet. When the reciprocating parts are balanced by masses in the driving-wheel alone the quantities for the coupled engines are the same as for the corresponding type of single engine.

SCHEDULE 15.

$Mn^2r = 9521.$
$M = 551$ lbs.
$n = 4$ revolutions per second.
$r = 1 \cdot 08$ foot (26-inch stroke).
Piston speed, 1036 feet per minute.

Type of engine.	Reciprocating mass balanced per cylinder.	Maximum value of the variation of rail-road per wheel in lbs. weight (hammer-blow).	Maximum value of the unbalanced force in lbs. weight.	Maximum value of the swaying couple in foot-lbs.
	lbs.			
Inside single	551	8,854	Nil	Nil
Outside single	551	13,138	Nil	Nil
Inside single	367	5,902	5426	5,117
Outside single	367	8,759	"	16,302
Inside 4-coupled	183 per wheel	2,951	"	5,117
Outside 4-coupled ...	183 per wheel	4,379	"	16,302
Inside 6-coupled	122 per wheel	1,999	"	5,117
Outside 6-coupled ...	122 per wheel	2,951	"	16,302
4-cylinder engine, two 180° pairs of cranks at right angles		Nil	Nil	32,942
4-cylinder engine (Fig. 100)		Nil	Nil	Nil

The results of Schedule 15 should be considered, together with Schedule 16, to form an idea of the approximate magnitude of the forces and couples corresponding to a given speed.

SCHEDULE 16.

Speed corresponding to different diameters of driving-wheel for a piston speed of 1036 feet per minute, and a 26-inch stroke. Revolutions per second = 4.

Diameter of driving-wheel in feet.		Speed in miles per hour.
ft.	in.	
4	0	34·0
4	6	38·5
5	0	42·6
5	6	48·0
6	0	51·2
6	6	55·5
7	0	59·7
7	6	64·0
8	0	68·24

A peculiarity belonging to locomotives, depending upon the principles of the next chapter for its investigation, may be mentioned here. In this chapter the effect of the obliquity of the connecting-rod has been neglected. As a matter of fact there is no secondary error in the balancing of the forces, nor in the estimation of the magnitude of the unbalanced forces, arising from this cause, because in the case of an engine with two cranks at right angles connected to equal reciprocating masses, the secondary forces arising from the obliquity of the connecting-rod mutually balance, the secondary force polygon being a line returning upon itself. There is, however, secondary couple error. An engine with four cranks at right angles severally connected to equal reciprocating masses has no secondary force error, the secondary forces mutually balancing one another. In this case the primary forces mutually balance as well, but there is a primary couple and secondary couple left unbalanced (see Schedule 21, p. 196). The primary couple can be balanced, still maintaining the balance amongst primary and secondary forces, either by the addition of revolving masses, thereby introducing a hammer-blow, or by altering the crank angles and masses in the way explained in Art. 96, though this method carries with it the disadvantage that it is impracticable to arrange the outside cranks at right angles to one another. The secondary couple cannot be balanced as well in a four-crank engine by any practicable arrangement of the cranks. Balanced in the way explained in Art. 74, there is no primary force nor couple error, but both secondary force and couple errors. With the long connecting-rods usual in locomotive practice these secondary errors are negligible.

77. Experimental Apparatus.—Fig. 101 shows a model of an inside four-coupled engine, by means of which the various problems of locomotive balancing may be studied. It is shown resting on rollers. Supported in this way, the effect of the unbalanced masses on the tractive force is separated from the other effects. Unbalanced, the model rolls backwards and forwards when the

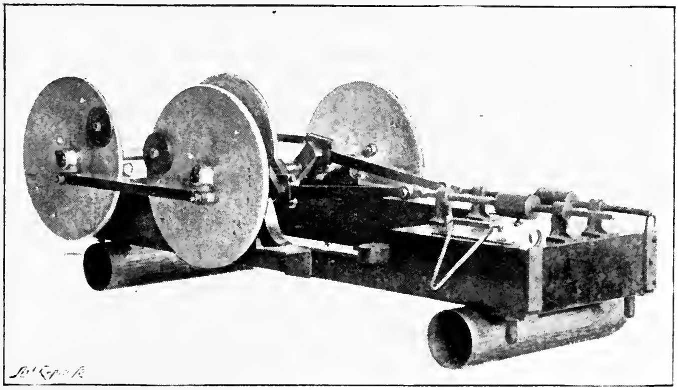


Fig. 101.

gear is driven. When the proper masses are added to balance the whole of the reciprocating and the revolving parts, the model stands quite still on the rollers at all speeds of rotation. If the model be suspended by three chains after the manner of the apparatus shown in Fig. 49, the effect of the swaying couple would be seen; if one of these chains be replaced by an elastic link, or a spring, the vertical oscillations would indicate the hammer-blow.

CHAPTER V.

SECONDARY BALANCING.

When the ratio between the length of the connecting-rod and the crank is small, the difference between the true motion of the piston and the motion it would have if the rod were infinitely long, is often sufficiently great to introduce considerable error in the balancing made on the assumption of an infinite rod. The error is called a secondary one, because its chief part is of half the periodic time of the engine, and the maximum value of the force due to it is smaller than the maximum value of the force due to the simple harmonic motion of the reciprocating mass. The two forces are often referred to as the primary and secondary forces due to the reciprocation, and similarly, the terms primary and secondary balancing refer to the balancing of these respective forces. The object of this chapter is to show how to arrange an engine so that the primary and secondary effects of the reciprocating masses may be balanced without the addition of balancing masses. The geometrical ideas of the previous chapters are merged into an analytical method, by means of which a general method is obtained of treating both primary and secondary balancing. The first step in the investigation is to substitute the expression (2), Art. 78, for the simpler expression for the acceleration, equation (2), Art. 40. Notice that the new expression is the old one with the new term $\frac{M\omega^2 r^2}{l}\cos 2(\theta + \alpha)$ added. This new term represents the secondary effect. This expression has been used by Mr. Mallock, by Mr. Mark Robinson, and Captain Sankey; it is the basis of Mr. Macfarlane Gray's Accelerity Diagram; it has been used by

M. Normand, and, more recently, by Herr Schlick. The degree to which it approximates to the real acceleration is examined in Art. 79. The error is small, and its effect is, probably, very much less than the effects of the unbalanced auxiliary engines, or of a propeller, even slightly out of balance. After establishing the general method, it is applied to several actual examples, including the case of partial balancing, given in Herr Schlick's recent paper (*Trans. I.N.A.*, vol. xlii., 1900). The solution for the balancing of a four-crank engine completely is given in Art. 94, and, though it cannot be applied practically, it is used, in Arts. 99 and 100, to obtain new solutions, with respect to the balancing of five- and six-crank engines, which solutions are shown to satisfy four more conditions in Art. 101.

78. Analytical Expression for the Acceleration of the Reciprocating Masses, including the Secondary Effect.—Let θ be the variable angle between a fixed line of reference OZ (Fig. 102), the centre line of the engine, say, and a line of reference OX_1, drawn in the revolving reference plane, containing the crank OP. The circle may be looked upon, in fact, as a crank disc, on which the lines OP and OX_1 are scribed.

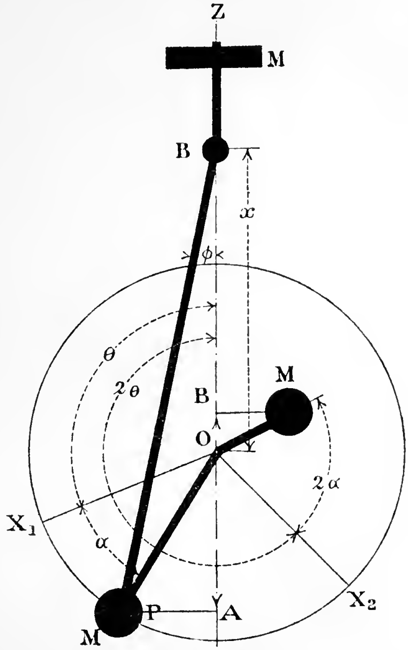


Fig. 102.

Let α be the constant angle between the direction of the crank OP and the line of reference OX_1; ϕ, the angle between the connecting-rod and OZ; r, the crank radius; l, the length of the connecting-rod.

The distance, x, of the crosshead, B, from the origin O, is given by the expression—

$$r \cos (\theta + \alpha) + l \cos \phi$$

But since—

$$r \sin(\theta + a) = l \sin \phi$$

$$\cos \phi = \sqrt{1 - \frac{r^2}{l^2} \sin^2(\theta + a)} \quad . \quad . \quad . \quad . \quad . \quad . \quad (a)$$

$$\therefore \cos \phi = 1 - \frac{r^2}{2l^2} \sin^2(\theta + a), \text{ approximately}$$

$$\therefore \cos \phi = 1 + \frac{r^2}{4l^2}\{\cos 2(\theta + a) - 1\}$$

$$\therefore x = r \cos(\theta + a) + \frac{r^2}{4l} \cos 2(\theta + a) + \left(l - \frac{r^2}{4l}\right). \quad . \quad . \quad (1)$$

Differentiating twice, with respect to the time, and considering the angular velocity, $\frac{d\theta}{dt} = \dot{\theta} = \omega$, of the crank to be sensibly constant, the acceleration of B is—

$$\frac{d^2x}{dt^2} = -r\dot{\theta}^2 \cos(\theta + a) - \frac{r^2\dot{\theta}^2}{l} \cos 2(\theta + a)$$

Let M be the mass reciprocated by the point B; then, writing ω for $\dot{\theta}$, the instantaneous value of the unbalanced force, acting on the engine-frame in the line of stroke, which is equal and opposite to the force required for M's acceleration, is given by—

$$M\omega^2 r\{\cos(\theta + a) + \frac{r}{l} \cos 2(\theta + a)\} \quad . \quad . \quad . \quad . \quad (2)$$

79. On the Error involved by the Approximation.—It will be observed that the approximate formulæ (1) and (2), of Art. 78, depend upon the extraction of the square root of the expression giving $\cos \phi$, by the Binomial Theorem, to two terms only, it being tacitly assumed that the remaining terms in the expansion may be neglected, without involving serious error. How surprisingly small the error is may be seen by comparing the real acceleration for a few crank positions, with the acceleration calculated by the approximate formula (2). Differentiating the true expression for the displacement x, twice, with respect to the time, the true value of the acceleration is given by—

$$-\omega^2 r\left\{\cos(\theta + a) + \frac{rl^2 \cos 2(\theta + a) + r^3 \sin^4(\theta + a)}{\{l^2 - r^2 \sin^2(\theta + a)\}^{\frac{3}{2}}}\right\} \quad . \quad (3)$$

The values calculated by this formula are compared with the approximate values calculated by formula (2), for 30° intervals, in the following schedule, for a rod $3\frac{1}{2}$ times the length of the crank. It will be noticed that when $(\theta + a) = 0°$, or $180°$, the values given by the formula (2) are exact, because $\sin(\theta + a) = 0$, and therefore the true expression reduces to the same form as the approximate one.

The greatest percentage error in the table is at 90°, and is of the order 4 per cent.

SCHEDULE 17.

Crank angle.	True acceleration.	Acceleration from approximate formula.
Degrees.		
0	+1·286	+1·286
30	+1·0148	+1·009
60	+0·3571	+0·357
90	−0·2981	−0·286
120	−0·6429	−0·643
150	−0·7172	−0·723
180	−0·714	−0·714
210	−0·7172	−0·723
240	−0·6429	−0·643
270	−0·2981	−0·286
300	+0·3571	+0·357
330	+1·0148	+1·009
360	+1·286	+1·286

80. Graphical Interpretation of Expression (2), Art. 78.—The first term of the expression, namely—

$$M\omega^2 r \cos(\theta + a)$$

is equal to the projection OA (Fig. 102), on the line of stroke, of the centrifugal force, due to a mass M, concentrated at the crank radius.

Multiply, and divide the second term by 4, giving—

$$M(2\omega)^2\left(\frac{r^2}{4l}\right)\cos 2(\theta + a)$$

This is equal to the projection OB, on the line of stroke, of the centrifugal force due to M, concentrated at a crank radius $\frac{r^2}{4l}$ rotating twice as fast as the main crank.

In this way, the unbalanced force caused by the reciprocation of M may be separated into a primary and a secondary part.

The primary part is simply the projection, on the line of stroke, of the centrifugal force, due to the reciprocating mass, supposed transferred to the crank-pin.

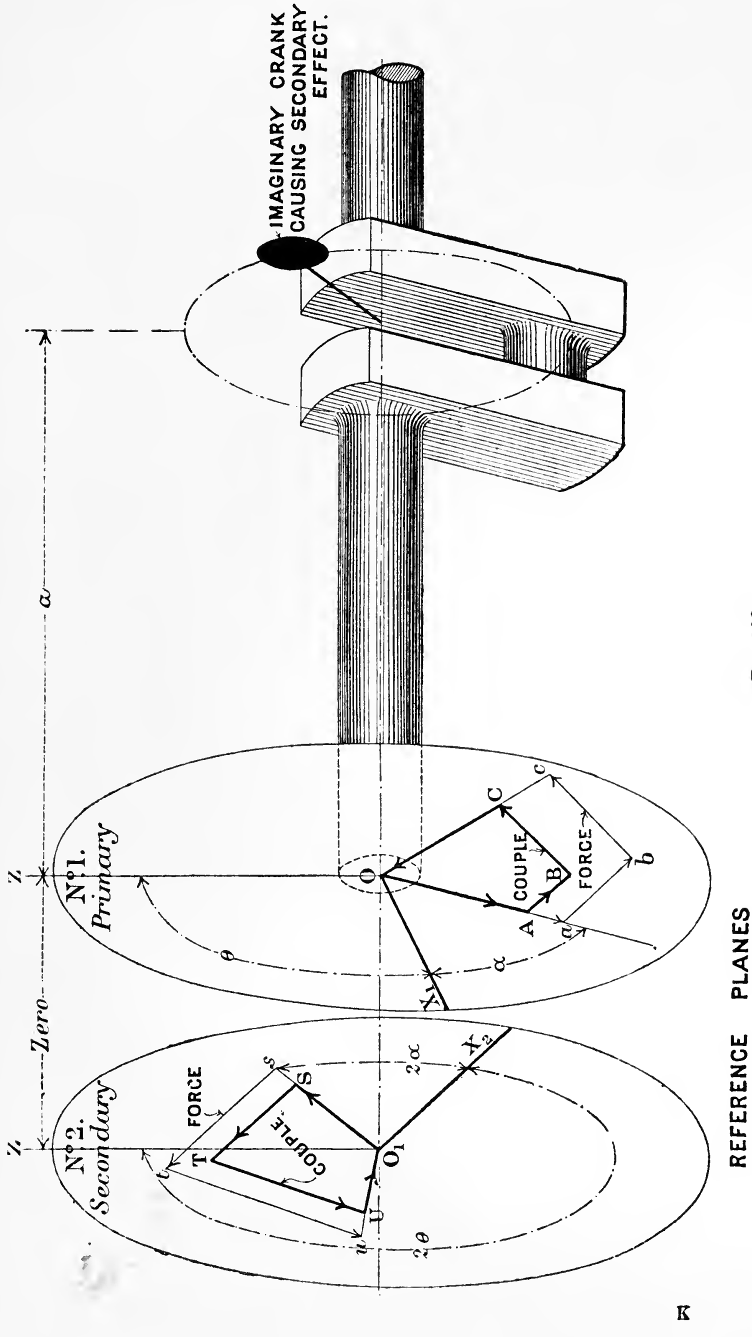
The secondary part is the projection, on the line of stroke, of the centrifugal force, due to the reciprocating mass, supposed transferred to the crank-pin of an imaginary crank, $\frac{r}{4l}$ times the radius of the main crank, revolving in the same plane twice as fast as the main crank.

The line of reference OX_1 (Fig. 102) revolves at the same speed as the crank; the angle a, therefore, remains constant. Similarly, the line of reference OX_2 revolves twice as fast as the main crank, and the angle $2a$ remains constant.

81. The Effect of the Primary and Secondary Unbalanced Forces with respect to a Reference Plane *a* Feet from the Plane of Revolution of the Crank.—Fig. 103 shows a crank, and the instantaneous relative position of the imaginary crank causing the secondary forces. The effect of the two cranks, with reference to any chosen reference plane, is conveniently treated by supposing that there are two coincident reference planes: one belonging to the main crank, and revolving as though keyed to the shaft; and one belonging to the secondary crank, and revolving twice as fast as the shaft. The two planes may be thought of as a pair of infinitely narrow fast-and-loose pulleys. These planes are shown separated in Fig. 103, for the sake of clearness. The principles of the method explained in Chapter II., Arts. 24–28, may now be applied to the consideration of these cranks. Due to the mass M, at the main crank radius r, whose angular velocity is ω, there is—

(1) A force $M\omega^2r$, equal and parallel to the centrifugal force due to M, shown by Oa, to scale, in No. 1 reference plane;

(2) A couple, whose moment is $M\omega^2ra$, represented by OA in plane No. 1.


IMAGINARY CRANK CAUSING SECONDARY EFFECT.
α
Zero
Z
Z
Nº1. Primary
Nº2. Secondary
O
A
B
C
a
b
c
COUPLE
FORCE
θ
α
X_1
X_2
2α
O_1
S
T
U
s
t
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COUPLE
FORCE
2θ
REFERENCE PLANES

Fig. 103.

Due to the mass M, at the imaginary crank radius $\frac{r^2}{4l}$, whose angular velocity is 2ω, there is—

(1) A force $\frac{M\omega^2r^2}{l}$, equal and parallel to the centrifugal force due to M, and shown by O_1s in plane No. 2;

(2) A couple, whose moment is $\frac{M\omega^2r^2a}{l}$, and shown by O_1S in plane No. 2.

82. Effect of More than One Crank on the Same Shaft.—If there are several cranks on the same shaft, each will be accompanied by its imaginary fellow, and each crank will, therefore, give rise to a set of forces and couples, in a given pair of coincident reference planes, similar to the set stated above. The angles between the set of imaginary cranks will remain constant, although the imaginary shaft is revolving twice as fast as the main shaft; in fact, they form an imaginary crank-shaft, in which the angles between any pair of the imaginary cranks is always double the angle between the corresponding pair of real cranks. If the effect of each crank is referred to one pair of reference planes, as in Fig. 103, the whole effect will be represented by the vector sums of the several forces and couples. The sum of the projections of the resultant force vectors on the line of stroke will give, at any instant, the value of the disturbing force; and the sum of the projections of the resultant couple vectors, the value of the disturbing couple. Evidently the conditions that there shall be no force and couple are that the four polygons shall separately close. For instance, if there were four cranks on the shaft (Fig. 103), and if it were possible to draw the closed polygons—

$$OABC, Oabc; O_1STU, O_1stu$$

the engine would be balanced both for primary and secondary forces and couples; because, obviously, their several projections on the line of stroke would be zero for all positions of the polygons, that is, for all the values of θ.

83. The Conditions of Balance.—The conditions of balance, to include the effect of the connecting-rod, are, therefore, completely stated thus. Choosing a pair of coincident reference planes anywhere along the shaft—

(1) The primary force polygon must close.
(2) The primary couple polygon must close.
(3) The secondary force polygon must close.
(4) The secondary couple polygon must close.
} **B**

84. Analytical Representation of a Vector Quantity.—Take a pair of axes OX, OY (Fig. 104). Let μ represent a vector whose magnitude is OV. The direction of the vector is determined by the two quantities x and y, measured along OX, and parallel to OY, respectively, having regard to the usual convention respecting signs. The end of y, remote from the X axis, fixes a point, q. The line joining O to q defines the direction of the vector completely.

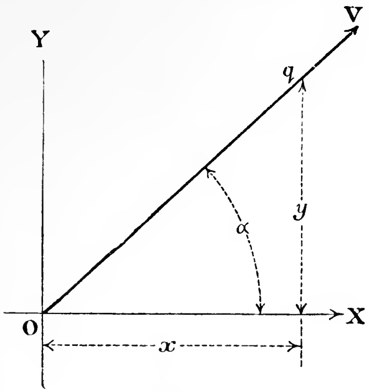


FIG. 104.

There are an infinite number of pairs of values of x and y, $\frac{y}{x}$ being constant, which will define the same direction, and the choice of a pair depends upon the particular work in hand. If the direction is to be set out on a drawing, x and y should be chosen to bring q as far from O as possible, to ensure accuracy. For purposes of analytical investigation, it is usually more convenient to choose that pair of values which make—

$$x^2 + y^2 = 1$$

in which case the vector is represented by—

$$OV(x + iy)$$

where i may be looked upon as a symbol of operation, directing that y is to be set out parallel to the axis of Y, *from* the axis OX *towards* q. Considering the figure, it is evident that, assuming—

$$x^2 + y^2 = 1$$

it follows that $x = \cos a$, and that $y = \sin a$; also, if $x = 0$, then

$y = 1$; and, if $y = 0$, $x = 1$, since $x^2 + y^2$ is, by hypothesis, always equal to unity.

The magnitude of OV is always considered as a positive quantity. Further, if, in the two vectors, $OV(x + iy)$ and $O_1V_1(x_1 + iy_1)$, $x = x_1$, then $\pm y = \pm y_1$; the sign of the y in each case being determined by the conditions of the problem.

85. Relation between the Quantities defining the Directions a and $2a$.—Let x and y (Fig. 105) define the direction Oq, and x_1, y_1

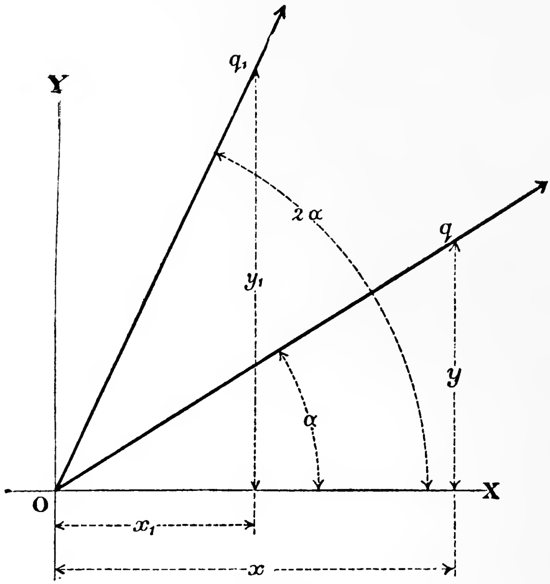


FIG. 105.

the direction Oq_1. The four quantities are connected by the condition that the angle XOq_1 is to be double the angle XOq. Since—

$$XOq_1 = 2a$$

and—

$$x_1 = \cos 2a = \cos^2 a - \sin^2 a$$

$$\text{or} \quad x_1 = x^2 - y^2$$

since x and y are respectively, $\cos a$ and $\sin a$; similarly—

$$y_1 = \sin 2a = 2 \sin a \cos a$$

$$\text{or} \quad y_1 = 2xy.$$

Therefore, if x and y define a direction a, $(x^2 - y^2)$ and $2xy$ are the pair of values of the quantities defining the direction $2a$. These are, of course, to be measured along and parallel to OX and OY respectively.

86. Application to the Balancing Problem.—Taking $OV(x + iy)$ to represent a centrifugal force, the magnitude OV is given by $M\omega^2 r$. If it represents a centrifugal couple $OV = Ma\omega^2 r$. Hence the side of a primary force polygon is represented by a vector of the form—

$$M\omega^2 r(x + iy)$$

the side of a primary couple polygon by—

$$M\omega^2 ra(x + iy)$$

the side of a secondary force polygon by—

$$\frac{M\omega^2 r^2}{l}\{(x^2 - y^2) + i2xy\}$$

and the side of a secondary couple polygon by—

$$\frac{M\omega^2 r^2 a}{l}\{(x^2 - y^2) + i2xy\}$$

Adding subscripts to distinguish the different cranks, the statement of the conditions B, Art. 83, may now be transformed into—

$$\left.\begin{aligned}
&\omega^2 r\{M_1(x_1 + iy_1) + M_2(x_2 + iy_2) + M_3(x_3 + iy_3) + \ldots\} &= 0\\
&\omega^2 r\{M_1 a_1(x_1 + iy_1) + M_2 a_2(x_2 + iy_2) + \ldots\} &= 0\\
&\frac{\omega^2 r^2}{l}\{M_1(x_1^2 - y_1^2 + i2x_1y_1) + \ldots\} &= 0\\
&\frac{\omega^2 r^2}{l}\{M_1 a_1(x_1^2 - y_1^2 + i2x_1y_1) + \ldots\} &= 0
\end{aligned}\right\} \text{C}$$

It is a property of the quantities of the kind considered that, in any one equation of the above type, all the quantities associated with the symbol i must of themselves form an expression equal to nothing; the remaining quantities forming a second expression, also equal to nothing.

87. The Eight Fundamental Equations.—Using the principle of the previous article, the equations of Group C become as below, ω, r, and l cancelling out—

$$\left.\begin{array}{ll}
\left\{\begin{array}{l}\text{Primary}\\ \text{forces vanish}\end{array}\right. & \begin{array}{ll}\{M_1x_1 + M_2x_2 + \ldots\} = 0 & (1)\\ \{M_1y_1 + M_2y_2 + \ldots\} = 0 & (2)\end{array}\\
\left\{\begin{array}{l}\text{Primary}\\ \text{couples vanish}\end{array}\right. & \begin{array}{ll}\{M_1x_1a_1 + M_2x_2a_2 + \ldots\} = 0 & (3)\\ \{M_1y_1a_1 + M_2y_2a_2 + \ldots\} = 0 & (4)\end{array}\\
\left\{\begin{array}{l}\text{Secondary}\\ \text{forces vanish}\end{array}\right. & \begin{array}{ll}\{M_1(x_1^2 - y_1^2) + M_2(x_2^2 - y_2^2) + \ldots\} = 0 & (5)\\ \{M_1(x_1y_1) + M_2(x_2y_2) + \ldots\} = 0 & (6)\end{array}\\
\left\{\begin{array}{l}\text{Secondary}\\ \text{couples vanish}\end{array}\right. & \begin{array}{ll}\{M_1(x_1^2 - y_1^2)a_1 + M_2(x_2^2 - y_2^2)a_2 + \ldots\} = 0 & (7)\\ \{M_1(x_1y_1)a_1 + M_2(x_2y_2)a_2 + \ldots\} = 0 & (8)\end{array}
\end{array}\right\} \text{D}$$

It should be carefully remembered that every x is connected to the y, with the same subscript, by the relation $(x^2 + y^2) = 1$; so that, when any x is known, the corresponding y is known also. A direction is, in fact, completely specified by the value of an x, or, of course, the value of a y. Although the eight equations appear to involve the four sets of unknown quantities in M, a, x, and y, respectively, they are really in the three sets M, a, and x, since the y's can everywhere be expressed in terms of the corresponding x.

The above eight equations express completely the analytical conditions of balance amongst the reciprocating parts, for an engine with any number of cranks, the cylinders being arranged in the way usual in marine work, *i.e.* all on one side of the crank-shaft, their centre lines being all in the vertical plane, which contains the axis of the crank-shaft. The connecting-rods are supposed to be equal in length, and the masses are the equivalent masses, reduced to the crank radius.

83. On the Relation between the Number of Conditional Equations and the Number of Variables.—In the application of these equations to any particular example, the possibilities of a solution are indicated by the following propositions, quoted from "Chrystal's Algebra," pp. 286 and 288, Part I. :—

(1) "The solution of a system of equations is in general determinate when the number of equations is equal to the number of variables."

(2) "If the number of equations be less than the number of

variables, the solution is in general indeterminate" (that is, several solutions are possible).

(3) "If the number of independent equations be greater than the number of variables, there is in general no solution, and the system of equations is said to be inconsistent."

89. On the Number of Variables.—It has already been shown (Art. 35) that there are in general $3(n - 1)$ variables in balancing problems where n is the number of cranks. This formula applies equally to a set of revolving or reciprocating masses.

It should be carefully remembered that the variables in M and a are ratios. The value of any one magnitude may always be put equal to unity, if found convenient; for this does not fix the value of a variable in M—it only means that the M's remaining in the equations represent, not their values in pounds or tons, but their respective ratios to the absolute value of the particular M fixed to unity. Similarly, any one of the a's may be considered unity.

90. On the Selection of the Conditional Equations.—If the eight conditional equations of Art. 87 are satisfied, the solution is independent of the position of the reference plane. The possibilities of balancing the reciprocating parts of an engine amongst themselves in which the number of variables is less than eight, are to be investigated by solving a set of equations selected from the fundamental Group D, Art. 87, equal in number to the number of variables concerned in the problem. For instance, the number of variables in a three-crank engine, whose cylinder centre lines are fixed, is $2(n - 1) = 4$. This indicates that four of the eight conditions may be satisfied.

The selection cannot be made at will. In the first place, the equations must be taken in pairs, that is, (1) and (2), (3) and (4), etc., must be taken together; and, in the second place, the solution of the chosen equations must be independent of the position of the reference plane; for there is no axis which can be specified as the particular one about which the engine will tend to turn, consequently any solution which is dependent upon the position of the reference plane is illusory, and is of no practical value. The following theorem shows how this second condition operates to restrict the selection.

Theorem.—In order that the balancing of the secondary couples may be independent of the position of the reference plane, the conditions for the balancing of the secondary forces must be satisfied as well.

Let the functions—

$f(M,x,y) = 0$,	represent equations (1) and (2) of Group D, Art. 87		(*a*)
$F(M,a,x,y) = 0$,	,, ,, (3) ,, (4) ,, ,,		(*b*)
$f'(M,x,y) = 0$,	,, ,, (5) ,, (6) ,, ,,		(*c*)
$F'(M,a,x,y) = 0$,	,, ,, (7) ,, (8) ,, ,,		(*d*)

Suppose the reference plane moved a distance z from its original position, then the values of all the a's change by this amount, becoming $a_1 + z$, $a_2 + z$, etc. The functions (*b*) and (*d*) become—

$$F\{M(a+z),x,y\} = F(M,a,x,y) + zf(M,x,y) \quad . \quad . \quad . \quad (e)$$
$$F'\{M(a+z),x,y\} = F'(M,a,x,y) + zf'(M,x,y) \quad . \quad . \quad . \quad (f)$$

The condition, that the balancing may be independent of the position of the reference plane, is, that the functions (*e*) and (*f*) vanish, when the functions (*b*) and (*d*) vanish. The first term of each of the functions (*e*) and (*f*) is similar in form to the respective functions (*b*) and (*d*), which, by supposition, vanish, leaving—

$$zf(M,x,y) = 0$$

and—

$$zf'(M,x,y) = 0$$

The z's divide out, leaving functions of the same form as (*a*) and (*c*), which must vanish to make (*e*) and (*f*) vanish. Hence the theorem.

Corollary 1.—It is evident, from equation (*e*), that the primary couples cannot be balanced independently of the reference plane, unless the primary forces are balanced as well; for, in order that the expression may vanish, each term must separately vanish.

Corollary 2.—It is also clear, from equation (*e*), that if the primary force and couple polygons close for any one position of the reference plane, they will close for all positions of it.

Corollary 3.—The unbalanced primary couple is constant for all positions of the reference plane, if the primary force polygon close. For, suppose $F(M,a,x,y) = A$, then, from equation (*e*), if $f(M,x,y) = 0$, $F\{M(a+z),(x,y)\} = A$, for all values of z.

Corollary 4.—The unbalanced secondary couple is constant for all positions of the reference plane, only if the secondary force polygon close.

This is proved in a similar manner to Corollary 3, by using equation (f).

It follows, from the above theorem, that, to obtain any useful practical result, equations (7) and (8) cannot be taken, without at the same time taking equations (5) and (6), and that equations (3) and (4) must be accompanied by equations (1) and (2).

Returning now to the selection of the four equations which contain the possibilities of balancing a three-crank engine, whose cylinder centre lines are given, it is evident that, under the operation of the two conditions stated above, the selection can only be made in two ways. These are—equations (1), (2), (3), and (4), or (1), (2), (5), and (6).

Any other set of four would be inconsistent with the condition that the equations be taken in pairs, or with the above theorem. This at once shows that a three-crank engine may presumably be balanced—

(1) For primary forces and primary couples, leaving the secondary forces and couples unbalanced;

(2) For primary forces and secondary forces, leaving the primary and secondary couples unbalanced.

The possibilities of balancing the reciprocating parts of an engine of this type amongst themselves, are confined to these two cases, and no arrangement of three cranks is possible whereby anything further can be done, though there remains the practical problem, in all such cases of partial balancing, how to select the data so that the unavoidable errors left are as small as possible.

91. Application of the Method to One- and Two-crank Engines.— Obviously, nothing can be done to balance the reciprocating parts of a one-crank engine without adding another set of reciprocating parts. This evident fact may be used to illustrate the use of the expression giving the number of conditions which may be satisfied; for, substituting the value 1 for n in expression (1), Art. 89, it becomes—

$$3(1 - 1) = 0$$

which means that none of the conditions of balance can be fulfilled.

Considering a two-crank engine, the expression becomes—

$$3(2 - 1) = 3$$

showing that three of the balancing conditions may be satisfied. The conditions of Art. 90 limit the selection to equations (1) and (2), from the fundamental Group D, Art. 87, because the selection must be made in pairs; therefore, two only can be taken, and (1) and (2) must be taken before (3) and (4), by Corollary 1, Art. 90, and it is evidently no use taking (5) and (6) until (1) and (2) have been satisfied. Hence equations (1) and (2) express all that can be done in the way of balancing the reciprocating parts of a two-crank engine amongst themselves.

Taking these two equations to two terms each, they will at once reduce to values of x and y, showing that the cranks must lie in the same plane of rotation, exactly opposite to one another, the masses at crank radius being equal.

These two cases are given merely to illustrate the way in which the expression giving the number of variables may be used at the beginning of a problem, to define the limits between which a solution is possible.

92. Application to Three-crank Engines.—The number of conditions which may presumably be satisfied is equal to the number of variables, and this is, by equation (1), Art. 89—

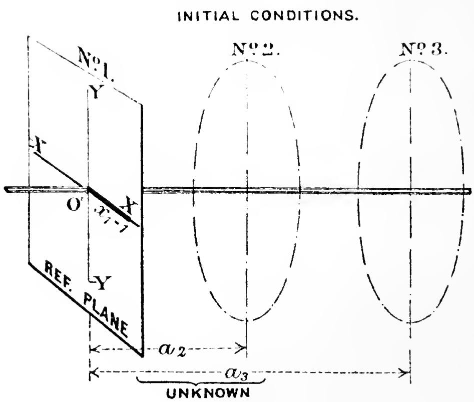


FIG. 106.

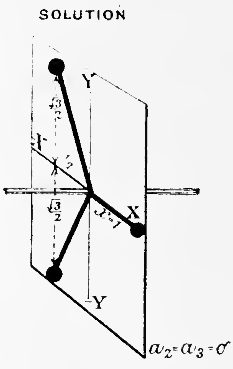


FIG. 107.

$$3(3 - 1) = 6$$

The conditions of Art. 90 limit the selection to the first six equations of the fundamental Group D, Art. 87.

Assume No. 1 crank to be in coincidence with the line of reference. Then $x_1 = 1$ and $y_1 = 0$. Also take the reference plane at No. 1 crank so that $a_1 = 0$. This latter assumption reduces the number of variables by 1, leaving 5. These initial conditions are shown in Fig. 106.

To investigate the general conditions, however, take the first six equations and substitute the above values. The equations become—

$$\begin{aligned}
M_1 + M_2x_2 + M_3x_3 &= 0 \quad . \quad . \quad . \quad (1)\\
M_2y_2 + M_3y_3 &= 0 \quad . \quad . \quad . \quad (2)\\
M_2a_2x_2 + M_3a_3x_3 &= 0 \quad . \quad . \quad . \quad (3)\\
M_2a_2y_2 + M_3a_3y_3 &= 0 \quad . \quad . \quad . \quad (4)\\
M_1 + M_2(x_2^2 - y_2^2) + M_3(x_3^2 - y_3^2) &= 0 \quad . \quad . \quad . \quad (5)\\
M_2x_2y_2 + M_3x_3y_3 &= 0 \quad . \quad . \quad . \quad (6)
\end{aligned}$$

Eliminating the M's from equations (2) and (6)—

$$y_2x_3y_3 = y_2x_2y_3$$

From this, either $x_3 = x_2$, in which case $y_3 = y_2$ (Art. 84), or x_3 is not equal to x_2, in which case the above equation requires that $y_2 = y_3 = 0$, an untenable solution, since this requires that $x_2 = x_3 = 1$, which is contrary to the second supposition.

Taking $x_2 = x_3$, and therefore $y_2 = y_3$, it follows, from equation (2), that—

$$M_2 = M_3$$

Equations (1) and (5), therefore, reduce to—

$$\left.\begin{aligned}
M_1 + 2M_2x_2 &= 0\\
M_1 + 2M_2(x_2^2 - y_2^2) &= 0
\end{aligned}\right\}$$

Eliminating the M's—

$$x_2^2 - y_2^2 = x_2$$

that is, substituting $(1 - x_2^2)$ for y_2^2—

$$(2x_2 + 1)(x_2 - 1) = 0$$

From this—

$$x_2 = -\tfrac{1}{2} = x_3$$

or—

$$x_2 = 1 = x_3$$

this is untenable, since it involves—

$$y_2 = y_3 = 0$$

and this has been shown above to require that x_2 is not equal to x_3. The first value of x_2 is, therefore, the only tenable one.

The values of y_2 and y_3 corresponding to this are—

$$y_2 = y_3 = \pm\sqrt{1 - \tfrac{1}{4}} = \pm\frac{\sqrt{3}}{2}$$

Substituting the values of x_2 and x_3 in equation (1), remembering that $M_2 = M_3$—

$$M_1 = M_2 = M_3$$

From equation (2) it is clear that y_2 is of opposite sign to y_3, since the M's must be positive. The crank directions are, therefore, completely specified by Schedule 18, from which it appears that the cranks are mutually at 120°.

SCHEDULE 18.

Crank.	x.	y.
No. 1	1	0
No. 2	$-\frac{1}{2}$	$\frac{\sqrt{3}}{2}$
No. 3	$-\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$

Substituting these values, in equations (3) and (4), they become, remembering that the M's are all equal—

$$-\tfrac{1}{2}a_2 - \tfrac{1}{2}a_3 = 0$$

$$\frac{\sqrt{3}}{2}a_2 - \frac{\sqrt{3}}{2}a_3 = 0$$

From which—

$$a_2 = 0$$
$$a_3 = 0$$

indicating that the planes of revolution must be coincident, as shown in Fig. 107. Three equal masses, therefore, disposed in a plane of revolution, so that their respective radii are mutually at 120°, are balanced for primary and secondary forces. The force triangle corresponding to this is, of course, equilateral.

If the planes of revolution are not coincident, couple errors, both primary and secondary, will be introduced to an extent depending upon the distances apart of the three planes. This is, in fact, the case with the three-crank engine, which has been used so much in marine work.

Given that the masses are equal, and that the crank angles are mutually at 120°, such an engine would be completely balanced for primary and secondary forces; but there remains a large couple error. Mr. Mark Robinson and Captain Sankey * investigated the force errors very carefully in this arrangement, and communicated their results to the Institution of Naval Architects in 1895. By the graphic method they used, no errors could be detected; calculation from an exact formula, however, disclosed a force error of about one lb. weight in a 300 H.P. engine, running at 350 revolutions per minute, the ratio of connecting-rod to crank being 4·77 to 1.

93. Three-crank Engine Cylinder, Centre Lines fixed.—There is another way of looking at the three-crank engine problem which may be noticed. Suppose it given that the cylinder centre lines are fixed. Then the number of independent variables in the problem is—

$$2(3 - 1) = 4$$

This shows that only four conditions can be satisfied, that is, that only four of the eight equations of Art. 87 can be chosen. Now, consistent with the conditions of Art. 90, the selection can only be made in two ways, viz.—

Equations Nos. (1), (2), (3), and (4)

* "On a Method of preventing Vibrations in Marine Engines" By Mr. Mark Robinson and Captain H. Riall-Sankey. *Trans. Inst. Naval Architects.* London. 1895.

or—

Equations Nos. (1), (2), (5), and (6)

The latter set have been fully considered in the first part of the preceding article, and they are completely satisfied by an engine with equal masses, and cranks mutually at 120°.

The first set merely lead to a solution for complete balancing, supposing the motion to be simple harmonic. The way to obtain the solution of (1), (2), (3), and (4) is as follows, assuming the initial conditions to be the same as before, and as shown by Fig. 106.

Eliminating the M's from equations (2) and (4) of the preceding article—

$$y_2 a_3 y_3 = y_2 a_2 y_3$$

a_2 is, by supposition, not equal to a_3, therefore—

$$y_2 = y_3 = 0$$

and therefore—

$$x_2 = x_3 = \pm 1$$

since—

$$x^2 + y^2 = 1$$

Substituting these values in equation (3)—

$$\pm M_2 a_2 \pm M_3 a_3 = 0$$

If a_2 and a_3 are both positive, x_2 must be of opposite sign to x_3, in order to leave the above two quantities connected with a minus sign, since the M's are always positive.

Suppose x_2 negative and x_3 positive, then the conditions of balance are stated by the two equations—

$$M_1 + M_3 = M_2, \text{ from equation (1)}$$

and—

$$M_2 a_2 - M_3 a_3 = 0$$

The a's are both of the same sign here, and therefore the corresponding planes of revolution lie on the same side of the reference plane which is at crank No. 1. Also, since all the y's are zero, the cranks all lie in the same axial plane. The solution is shown in Fig. 108.

If the two x's were of the same sign, the corresponding a's would, of necessity, be of opposite signs, and the two planes of revolution would be on opposite sides of the reference plane.

Summarizing, the general conditions of balance for a three-crank engine with fixed centre lines are—

(1) The three cranks must lie in the same axial plane.

(2) The masses in the two outer planes of rotation must be such that their moments, with respect to the plane of rotation between them, must be equal and opposite, and the sum of the two outer masses must be equal to the central mass.

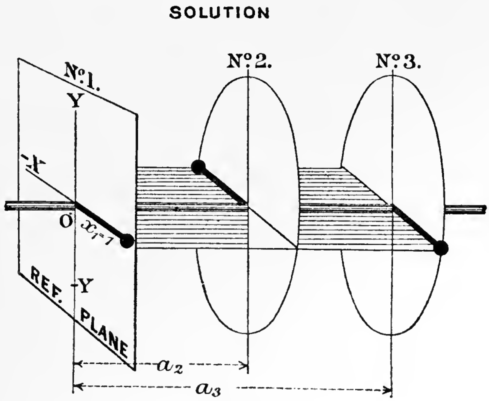


Fig. 108.

94. Application to a Four-crank Engine.—The number of variables is in general—

$$3(4 - 1) = 9$$

In this case, all the conditions expressed by the fundamental group of equations D, Art. 87, may, presumably, be satisfied.

Take the reference plane at No. 4 crank so that $a_4 = 0$, thus reducing the number of variables by one, and put No. 1 crank in coincidence with the reference line OX, so that $x_1 = 1$, and $y_1 = 0$. These initial conditions are similar to those shown by Fig. 106, except that the reference plane is now at No. 4 crank, which is not shown there. Substituting these values, and taking four terms, in each of the equations of Group D, Art. 87, they become—

$$M_1 + M_2x_2 + M_3x_3 + M_4x_4 = 0 \quad . \quad (1)$$
$$0 + M_2y_2 + M_3y_3 + M_4y_4 = 0 \quad . \quad (2)$$
$$M_1a_1 + M_2a_2x_2 + M_3a_3x_3 + 0 = 0 \quad . \quad (3)$$
$$0 + M_2a_2y_2 + M_3a_3y_3 + 0 = 0 \quad . \quad (4)$$
$$M_1 + M_2(x_2^2 - y_2^2) + M_3(x_3^2 - y_3^2) + M_4(x_4^2 - y_4^2) = 0 \quad . \quad (5)$$
$$0 + M_2x_2y_2 + M_3x_3y_3 + M_4x_4y_4 = 0 \quad . \quad (6)$$
$$M_1a_1 + M_2a_2(x_2^2 - y_2^2) + M_3a_3(x_3^2 - y_3^2) + 0 = 0 \quad . \quad (7)$$
$$0 + M_2a_2x_2y_2 + M_3a_3x_3y_3 + 0 = 0 \quad . \quad (8)$$

Equations (3), (4), (7), (8), in Ma above, are precisely similar in form to equations (1), (2), (5), (6) of Art. 92, in M. The solution of (3), (4), (7), (8), in Ma, is, therefore, the same as the solution of (1), (2), (5), (6) of Art. 92, in M. Hence—

$$M_1a_1 = M_2a_2 = M_3a_3 = 1, \text{ say} \quad . \quad . \quad . \quad . \quad (9)$$

And the cranks are mutually at 120°. Substituting the values of the M's in terms of the a's, and the values of the x's and y's from Schedule 18, Art. 92, in equations (1), (2), (5), and (6), they reduce to—

$$M_4x_4 = -\frac{1}{a_1} + \frac{1}{2a_2} + \frac{1}{2a_3} = M_4(x_4^2 - y_4^2) \quad . \quad . \quad (10)$$

$$M_4y_4 = -\frac{\sqrt{3}}{2a_2} + \frac{\sqrt{3}}{2a_3} = -2M_4x_4y_4 \quad . \quad . \quad . \quad . \quad . \quad (11)$$

From (10) –

$$x_4^2 - y_4^2 = x_4$$

that is—

$$(2x_4 + 1)(x_4 - 1) = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

From (11) –

$$-x_4y_4 = \frac{y_4}{2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

From (12)–

$$x_4 = 1$$

and therefore—

$$y_4 = 0$$

or —

$$x_4 = -\tfrac{1}{2}$$

and therefore—

$$y = \pm\frac{\sqrt{3}}{2}$$

Both these solutions also satisfy equation (13).

Considering the first solution, it shows that crank No. 4 is parallel to crank No. 1, and since the right and left hand expressions of equation (11) vanish, y_4 being zero, a_2 must be equal to a_3. The second solution leads to the same result, in terms of different letters. Hence, from (9)—

$$M_2 = M_3 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

Substituting the values—

$$x_4 = 1, \ a_2 = a_3$$

in equation (10)—

$$M_4 = \frac{1}{a_2} - \frac{1}{a_1} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

From this it is clear that a_1 must always be greater than a_2, in order to make M_4 positive.

The relation between the masses is deduced from this equation with the aid of (9), above, from which—

$$M_2a_2 = M_1a_1 = 1$$

Substitute these values in the numerators of the terms on the right of equation (15), then—

$$M_4 = \frac{M_2a_2}{a_2} - \frac{M_1a_1}{a_1}$$

From which—

$$M_4 + M_1 = M_2 = M_3 \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

Summarizing, a completely balanced four-crank engine must satisfy the following conditions:—

(1) The four cranks must be arranged in three planes of revolution, two of the cranks being in the central plane.

(2) The two cranks in the central plane must be at 120° with one another: the two outside cranks must point in the same direction in the same axial plane, which plane is at 120° with each of the cranks in the central plane of rotation.

(3) The masses in the central plane must be equal.

(4) The masses in the outer planes must be such that their moments, with respect to the central plane of rotation, are equal and opposite, and their sum must be equal to one of the equal masses in the central plane.

The arrangement is shown in Fig. 109, and, though impracticable to realize, is useful in the solution of other problems.

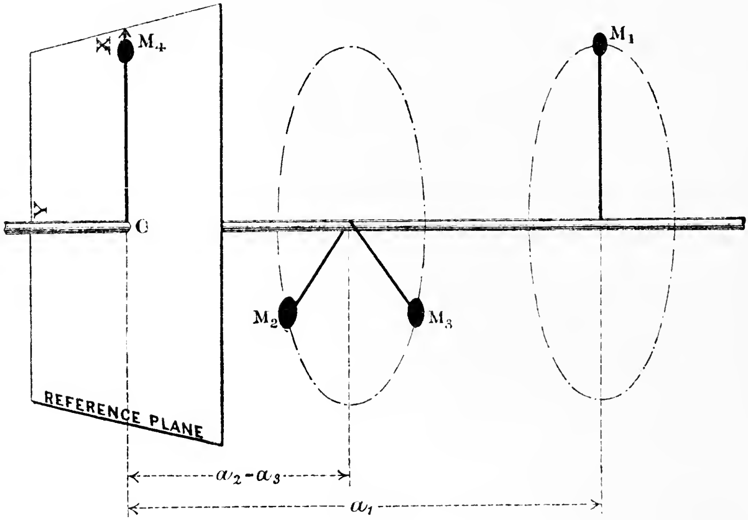


FIG. 109.

95. Example.—Let the ratio $\frac{a_1}{a_2}$ be 3. Measuring from No. 4 crank—

$$a_1 = 3$$
$$a_2 = a_3 = 1$$

If $M_1 a_1$ is unity, from (9)—

$$M_1 = \tfrac{1}{3}$$

and—

$$M_2 = M_3 = 1$$

from (15) or (16)—

$$M_4 = \frac{1}{a_2} - \frac{1}{a_1} = \frac{2}{3}$$

96. Four-crank Engine satisfying Six Conditions. *Case* I. *Schlick Symmetrical Engine.*—Conditions 1, 2, 3, 4, 5, and 6 may, presumably, be satisfied, leaving in general three of the nine variables concerned in a four-crank engine to be fixed arbitrarily. The solution of this problem was the main feature of Herr Schlick's paper * at the 1900 Spring Meeting of the Institute of Naval Architects.

Choose the line of reference so that it bisects the angle between No. 1 and No. 4 cranks, as OX (Fig. 111), then, $x_1 = x_4$, and therefore $y_1 = - y_4$.

The three conditions which may conveniently be fixed are—

(1) That the line of reference shall also bisect the angle between cranks No. 2 and No. 3. This involves that—

$$x_2 = x_3$$

and therefore y_2 is equal in magnitude and opposite in sign to y_3.

(2) That the ratio $M_1 : M_4 = 1$.

(3) That the ratio $a_1 : a_2$ is given.

The first two equations from Group D, Art. 87, become, under these conditions—

$$2x_1M_1 + x_2(M_2 + M_3) = 0 \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$0 + y_2(M_2 - M_3) = 0 \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Considering equation (2), M_2 must evidently equal M_3, since y_2 is not zero; therefore (1) becomes—

$$M_1x_1 + M_2x_2 = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Taking a reference plane to bisect the distance between planes Nos. 1 and 4, so that $a_1 = - a_4$, the third equation from Group D becomes—

$$M_2x_2(a_2 + a_3) = 0$$

whence—

$$a_2 = - a_3$$

The consequences of the three assumptions detailed above are, therefore, that $M_2 = M_3$, and $a_2 = a_3$, the two a's being of opposite sign.

The cylinder lines are, therefore, symmetrical, with reference

* "On Balancing Steam Engines." By Herr Otto Schlick. *Trans. Inst. Naval Architects.* London. 1900.

to the central plane of the engine. Substituting the foregoing equalities in the first six equations of Group D, Art. 87, they become—

$$M_1x_1 + M_2x_2 = 0 \quad . \quad . \quad . \quad . \quad (4)$$
$$M_1a_1y_1 + M_2a_2y_2 = 0 \quad . \quad . \quad . \quad . \quad (5)$$
$$M_1(x_1^2 - y_1^2) + M_2(x_2^2 - y_2^2) = 0 \quad . \quad . \quad . \quad . \quad (6)$$

Eliminating the M's from (4) and (6)—

$$x_1(x_2^2 - y_2^2) = x_2(x_1^2 - y_1^2)$$

Introducing $1 - x^2$ for y^2, this at once reduces to—

$$x_1x_2 = -\tfrac{1}{2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

Eliminating the M's from (4) and (5)—

$$x_1y_2a_2 = x_2y_1a_1$$

Squaring each side, and substituting $-\dfrac{1}{2x_1}$ for x_2, from (7), this reduces to—

$$x_1^4(4a_2^2) + x_1^2(a_1^2 - a_2^2) - a_1^2 = 0 \quad . \quad . \quad . \quad . \quad (8)$$

If—

$$P = \frac{a_1^2 - a_2^2}{8a_2^2}, \text{ and } Q^2 = \frac{a_1^2}{4a_2^2}$$

$$x_1^2 = -P \pm \sqrt{P^2 + Q^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

Also—

$$x_1^2 = \frac{M_2}{2M_1} \text{ from (7) and (4)} \quad . \quad . \quad . \quad . \quad (10)$$

x_1 can be calculated from (9), when the ratio $a_1 : a_2$ is given. Then x_2 can be found from equation (7). The corresponding values of the y's are found from the relation—

$$y^2 = 1 - x^2$$

their signs being determined from equations (4), (5), (6), (7). The ratio $M_1 : M_3$ is found from equation (10).

The relation given in equation (7), above, is that stated in the form $\cos\frac{\alpha}{2}\cos\frac{\gamma}{2} = \frac{1}{2}$, in Herr Schlick's paper.

Case II. *Unsymmetrical Engine.*—The general solution of the equations (1)–(6), Art. 94, has been published by Dr. Lorenz in a book, entitled *Dynamik der Kurbelgetriebe* (Leipzig, 1901), from which equations (11)–(14) have been quoted, the symbols having been altered where necessary, to bring them in conformity with those already used in the previous articles.

The elimination of the M's from equations (1), (2), (5), and (6), gives a relation between the angles, which, stated in its trigonometrical form, is—

$$\cos\frac{\gamma_2 + \gamma_1}{2} + \cos\frac{\gamma_2 - \gamma_1}{2} = \cos\frac{\beta - \delta}{2} \text{ *} \quad . \quad . \quad . \quad (11)$$

where γ_2, γ_1, β, δ, are pairs of opposite angles arranged in the way shown in Fig. 111. This equation may be reduced to the form—

$$2\cos\frac{\gamma_1}{2}\cos\frac{\gamma_2}{2} = \cos\frac{\beta - \delta}{2}$$

which, if $\beta = \delta$, as in the case of the symmetrical engine, becomes—

$$\cos\frac{\gamma_1}{2}\cos\frac{\gamma_2}{2} = \frac{1}{2}$$

a relation already expressed by equation (7) of this article.

Returning to the consideration of equation (11), it will be seen that by assuming values for β and δ, and thereby fixing the value of $(\gamma_2 + \gamma_1)$ since—

$$\gamma_2 + \gamma_1 = 360° - (\beta + \delta)$$

the value of $(\gamma_2 - \gamma_1)$ can be calculated, from which and the previous data γ_2 and γ_1 are at once determined.

The corresponding masses are to be found by the elimination of the factors of the M's in equations (1), (2), (5), and (6). If M_4 is unity, then measuring the angles a_1, a_2, and a_3 from crank No. 4, Fig. 111—

* This condition and similar conditions to those given by equation (12), are given in a paper by Dr. Schubert to the Hamburg Mathematical Society, dated February, 1898. (Mittheilungen der Mathematischen Gesellshaft in Hamburg Band 2, Heft 8.) The geometrical condition between the angles and centre lines, which is the basis of Art. 37, and a solution of the general equations for a five-crank engine, are also given.

$$M_1^2 = \frac{\sin\frac{a_3}{2}\sin\frac{3}{2}a_3\sin\frac{a_2}{2}\sin\frac{3}{2}a_2}{\sin\frac{a_3 - a_1}{2}\sin\frac{3}{2}(a_3 - a_1)\sin\frac{a_2 - a_1}{2}\sin\frac{3}{2}(a_2 - a_1)} \quad . \quad (12)$$

The expression for M_2^2 is obtained from this by writing the subscripts 1 for 3, 3 for 2, and 2 for 1.

For M_3^2 write 2 for 3, 1 for 2, and 3 for 1.

Having found the masses, and knowing the angles, the cylinder pitches are to be found from equations (3) and (4) of Art. 94. They reduce to, expressed in the trigonometrical form—

$$\frac{a_3}{a_1} = -\frac{M_1 \sin(a_2 - a_1)}{M_3 \sin(a_2 - a_3)} = -\frac{M_1 \sin(\beta + \gamma_2)}{M_3 \sin \gamma_2} \quad . \quad . \quad (13)$$

$$\frac{a_2}{a_1} = -\frac{M_1 \sin(a_3 - a_2)}{M_2 \sin(a_3 - a_2)} = -\frac{M_1 \sin \beta}{M_2 \sin(-\gamma_1)} \quad . \quad . \quad (14)$$

In these expressions, the reference plane is taken at No. 4 crank, so that $a_4 = 0$. Thus, assuming the values of the angles β and δ, which assumption, of course, carries with it the value of $(\gamma_1 + \gamma_2)$, the masses and cylinder centre lines can be calculated from the above equations.

97. Examples.—*Case* I. *Symmetrical Engine.*—Given that $M_1 = M_4$, that the crank angles are symmetrical, and that the ratio $a_1 : a_2 = 6{\cdot}5 : 2$. Find the crank angles, and the ratio of the masses $M_2 : M_1$, so that the engine may be balanced for primary and secondary forces and primary couples.

The data necessitate a symmetrical engine, as shown in Fig. 110, *i.e.* $M_2 = M_3$, and, taking the reference plane at the centre so that $a_1 = a_4$, $a_2 = a_3$, of necessity.

Calculate the values of the quantities P, P^2, and Q^2, and solve equation (9) of Art. 96.

$$P = \frac{6{\cdot}5^2 - 2^2}{8 \times 2^2} = 1{\cdot}195$$

therefore—

$$P^2 = 1{\cdot}428$$

$$Q^2 = \frac{6{\cdot}5^2}{4 \times 2^2} = 2{\cdot}643$$

Equation (9) becomes—

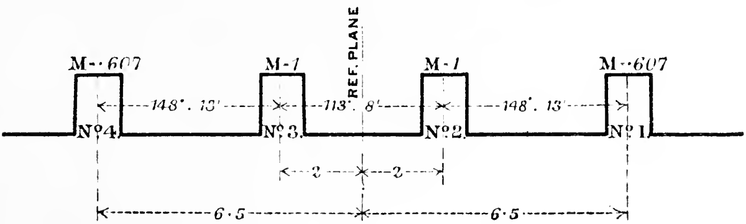
$$x^2_1 = -1{\cdot}195 \pm \sqrt{1{\cdot}428 + 2{\cdot}643}$$

therefore—

$$x^2_1 = + {\cdot}823$$

the negative value being untenable, since x_1 must be real. Therefore, retaining the positive sign, $x_1 = + {\cdot}908$.

Fig. 110.



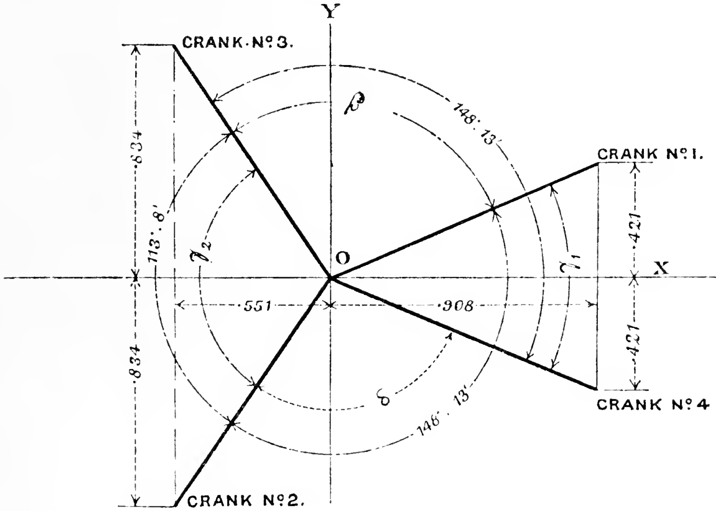


Fig. 111.

From equation (7), Art. 96—

$$x_2 = -\frac{1}{2 \times {\cdot}908} = -{\cdot}551$$

Then—

$$y_1 = \pm\sqrt{1 - {\cdot}908^2} = \pm{\cdot}421$$
$$y_2 = \pm\sqrt{1 - {\cdot}551^2} = \pm{\cdot}834$$

The three assumptions made include—

y_1 is equal in magnitude and opposite in sign to y_4
y_1 " " " y_3

The individual signs of y_1 and y_2 are to be determined from the general equations.

From equation (5)—

$$M_1 a_1 y_1 = -M_2 a_2 y_2$$

M_1 and M_2 are positive, a_1 and a_2 are of the same sign; therefore y_1 must be opposite in sign to y_2.

Arranging the results—

$x_1 = +{\cdot}908$	$y_1 = +{\cdot}421$, giving the direction of	No. 1 crank
$x_2 = -{\cdot}551$	$y_2 = -{\cdot}834$, " " "	No. 2 "
$x_3 = -{\cdot}551$	$y_3 = +{\cdot}834$, " " "	No. 3 "
$x_4 = +{\cdot}908$	$y_4 = -{\cdot}421$, " " "	No. 4 "

From equation (4), Art. 96—

$$M_1 : M_2 = x_2 : x_1 = {\cdot}551 : {\cdot}908 = {\cdot}607 : 1$$

This is also the ratio $M_4 : M_3$

And—

$$M_1 = M_4 = 1 \text{ by hypothesis}$$

Therefore—

$$M_2 = M_3 = {\cdot}607$$

Figs. 110 and 111 show the centre lines and crank angles, set out in their proper relative positions. The crank angles, in degrees, are added between the successive cranks and between the cranks Nos. 1 and 4, and 3 and 2.

Case II. *Unsymmetrical Engine.*—Given that $\beta = 100°$ and that $\delta = 90°$, find the angles γ_1, γ_2, the ratio of the reciprocating masses and the cylinder centre lines so that the engine may be in balance for primary and secondary forces and primary couples.

The following is a convenient semi-graphical way of solving this problem. Calculate the values of γ_2 and γ_1 from equation (11) of the previous article. Next calculate the value of M_1 from equation (12). Now draw an end view of the crank angles, numbering them as in Fig. 111, and set out the two sides of the corresponding force polygon which are known, namely, $M_4 = 1$, and M_1 from the value just computed from equation (12). Complete the polygon by lines parallel to cranks 2 and 3, thus determining the magnitudes of M_2 and M_3. Now apply the construction of Art. 37 and Fig. 47 to determine the pitches of the cylinders.

From the given data—

$$\frac{\beta - \delta}{2} = 5°$$

and therefore—

$$\frac{\gamma_2 + \gamma_1}{2} = 85°$$

Hence from equation (11)—

$$\cos\frac{\gamma_2 - \gamma_1}{2} = \cos 5° - \cos 85° = 0·909039$$

The angle corresponding to this is 24° 37′.

Hence—

$$\gamma_2 - \gamma_1 = 49° \ 14'$$
$$\gamma_2 + \gamma_1 = 170° \ 0'$$

Therefore—

$$\gamma_2 = 109° \ 37'$$
$$\gamma_1 = 60° \ 23'$$

Again, substituting the values of the angles in equation (12)—

$$M^2_1 = \frac{\sin 80° \ 11' \cdot \sin 240° \ 34' \cdot \sin 135° \cdot \sin 405°}{\sin 50° \cdot \sin 150° \cdot \sin 104° \ 48' \cdot \sin 314° \ 24'}$$

From this—

$$M_1 = 1·2735$$

Drawing the force polygon, and measuring the sides corresponding to cranks 2 and 3—

$$M_2 = 1·685$$
$$M_3 = 1·727$$

Applying the construction of Art. 37, Fig. 47—

$$\frac{a_3}{a_1} = 0·386$$

$$\frac{a_2}{a_1} = 0·790$$

98. Symmetrical Engine. On the Variation of the Different Quantities in Terms of Pitch of the Cylinders. Two Graphical Methods.—Let b be the ratio of $\frac{a_1}{a_2}$, or, since the engine is symmetrical, the ratio of the pitch of the pair of outside cylinders to the pitch of the pair of inner cylinders. Then—

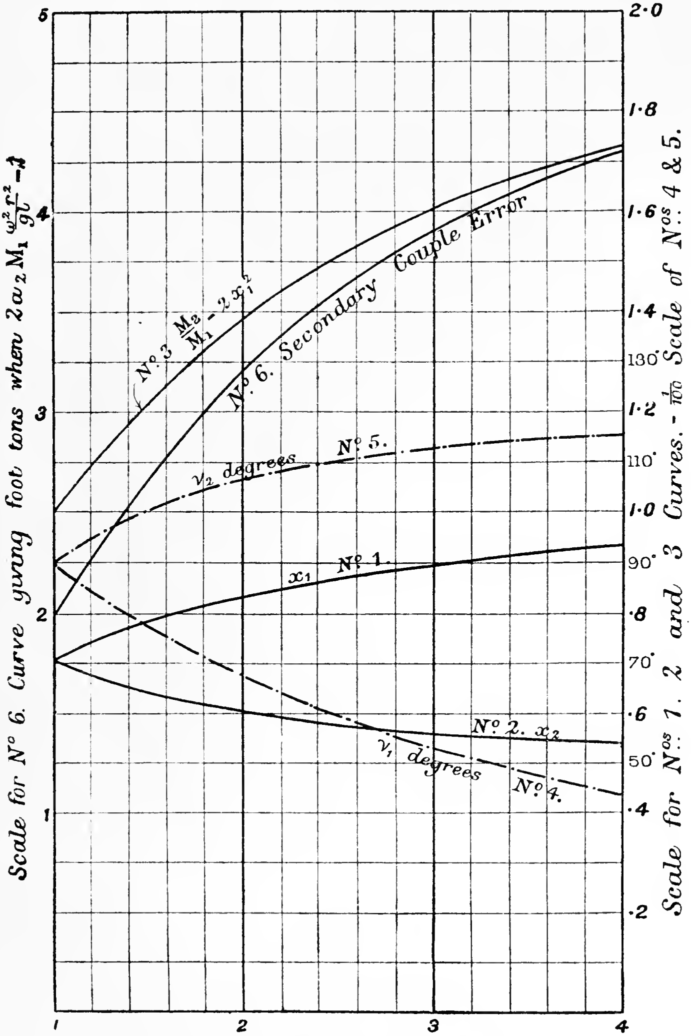
$$P = \frac{b^2 - 1}{8}, \text{ and } Q^2 = \frac{b^2}{4}$$

The way the different quantities vary for different values of b is shown by the curves in Fig. 112. The values of x_1 and x_2 are shown by the ordinates to curves Nos. 1 and 2 respectively. Curve No. 3 gives the ratio $\frac{M_2}{M_1}$. If the pitch of the middle pair of cylinders is considered unity, the value of b is the pitch of the outer pair. Also, if M_1 is considered unity, the ordinates to curve No. 3 give the value of M_2. This curve shows how rapidly the ratio $\frac{M_2}{M_1}$ increases as b increases. The outer cylinders, therefore, should be kept as near the middle cylinders as possible, *i.e.* b should be as near unity as possible, in order to avoid the necessity of increasing the weight of the reciprocating parts of the inner pair of cylinders too much beyond what would be necessary if designed simply for strength.

The angles corresponding to γ_1 and γ_2 between cranks Nos. 1 and 4, and 2 and 3, respectively (Fig. 111), are given by curves Nos. 4 and 5. The scale for these two curves is that on the right-hand side multiplied by 100. For example, suppose the ratio b to be 2. Then, taking the lengths of the ordinates corresponding with 2 to the different curves, $x_1 = {\cdot}83$, $x_2 = {\cdot}6$, the angle $\gamma_1 = 67\frac{1}{4}°$ and the angle $\gamma_2 = 106\frac{1}{8}°$, $\frac{M_2}{M_1} = 1{\cdot}38$.

The crank angles may easily be set out from the curves when the ratio b is given. With the distance unity on the right-hand vertical scale as radius, draw a circle (Fig. 113). Set out along a horizontal diameter the distances x_1 and x_2 taken from the curves, measuring x_2 to the left of a vertical diameter. The intersection of the verticals through p and q with the circumference of the circle determine points r, s, b, u, which fix the crank positions.

Anticipating Chapter VI., Equation (8), Art. 118 gives the maximum value of the unbalanced secondary couple. Considering



$b = \frac{a_1}{a_2} = \frac{\text{Pitch of outer Cyl}^{rs}}{\text{Pitch of middle Cyl}^{rs}}$

FIG. 112.

the factor $2Ba_2M_1$ = unity, the value of the couple for different values of b is shown by curve No. 6 (Fig. 112). The scale for this curve is on the left of the diagram. This couple increases rapidly as b increases, furnishing an additional reason for keeping the ratio b as near unity as possible. To find the actual value of the couple in any given case, measure off the ordinate to the curve, and multiply by $2M_1a_2B$, where $B = \frac{\omega^2r^2}{gl}$.

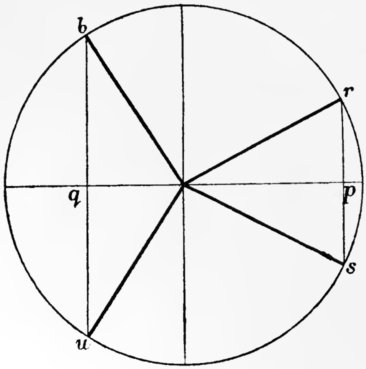


FIG. 113.

When $b = 2$, the ordinate to the curve is 3·2. If $2a_2 = 16$, and $M_1 = 5$, and $B = \frac{\omega^2r^2}{gl} = 1{\cdot}5$, which is the case when $r = 2$ feet, $l = 7$ feet, and the revolutions per minute are 88, the maximum value of the unbalanced secondary couple is—

$$1{\cdot}5 \times 5 \times 16 \times 3{\cdot}2 = 384 \text{ foot tons}$$

When $b = 1$, cranks Nos. 1 and 2 are opposite and in the same plane of revolution, and cranks Nos. 3 and 4 are similarly circumstanced, the two being at right angles to Nos. 1 and 2. M_1 is then equal to M_2 and the secondary error is a minimum.

The following construction, which is slightly modified from one given by Mr. Macfarlane Grey in the discussion of Mr. Schlick's paper to the Institute of Naval Architects, 1900, gives the relations between the masses, crank angles, and cylinder centre lines for a symmetrical engine satisfying the conditions of Case I., Art. 96.

Draw a circle of any diameter and draw a pair of diameters at right angles AB, CD (Fig. 114). From C with any radius cut the circle in G and the diameter AB in E. Join EC and ED. Join CG, and produce it to cut the diameter AB produced in F. Join FD. Then EDFC is a force polygon such that the angles of the corresponding cranks satisfy the relation of equation (7), Art. 96. To prove this, number the cranks as shown. Consider CE = ED = DH = CG each equal to unity, and draw perpendiculars Gg

and Hh, to CD. Then CO represents x_1, and Dh = gC represents x_2. Also Gg = Hh = y_2. Therefore—

$$\text{since } Ch \times Dh = Hh^2$$
$$(2x_1 - x_2)x_2 = y_2^2$$
$$2x_1x_2 - x_2^2 = 1 - x_2^2$$
$$i.e.\ x_1x_2 = \tfrac{1}{2}$$

But the direction of x_2 is negative; therefore—

$$x_1x_2 = -\tfrac{1}{2}$$

which is the condition of equation (7), Art. 96. The centre lines may be found from the force polygon by the method of Art. 37.

In Art. 37 and Fig. 47 it has been shown that angles for which balance is possible may be determined by drawing a pencil

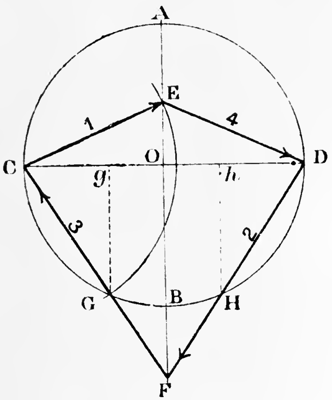


Fig. 114.

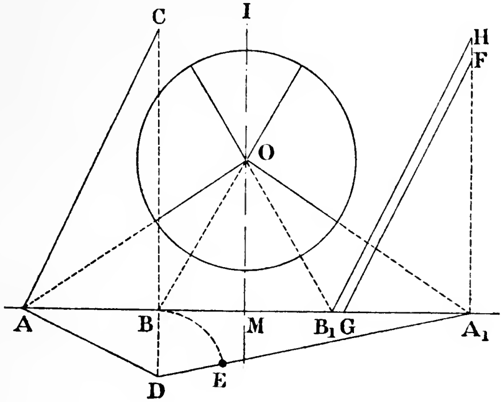


Fig. 115.

of rays through the traces of the cylinder centre lines, to meet in a point B. In the paper quoted there Mr. Schlick has shown how to find the point B to satisfy the condition of equation (7), Art. 96, for a symmetrical engine. The construction is as follows (Fig. 115):—

Set out the centres of the cylinders to scale, ABB_1A_1. At B erect a perpendicular BC, and make AC = AB_1. Produce CB to cut a line drawn at A at right angles to AC in D. Join D and A_1, and make DE = DB. Erect another perpendicular at A_1 and make $A_1F = A_1E$. With radius BC and centre F describe an arc cutting the line AA_1 at G. Produce A_1F to cut a parallel to FG through B^1 in H. Erect a perpendicular at M and make $MI = B^1H$. Bisect MI in O. O is the required point. The angles corresponding to this are shown in the figure.

99. Five-crank Engine.—The number of variables in this case is in general—

$$3(5 - 1) = 12$$

All the conditions expressed in the eight fundamental equations of Art. 87 may presumably be satisfied, therefore, leaving four of the variables concerned in the problem to be fixed arbitrarily. The solution of the problem may be derived from the general solution of the four-crank problem given in Art. 94, without reference to the eight equations of Art. 87.

Fig. 109 shows the disposition of four cranks and the corresponding masses for complete balance. All the eight equations of Art. 87 are satisfied. A combination of two such systems would also satisfy all the conditions of balance, for each is in equilibrium; and the combination of two systems, each in equilibrium, must form a single system in equilibrium. Combine two four-crank systems, of the type shown in Fig. 109, to form a single system, in the way shown in Fig. 116, one system being distinguished by the radii of its masses being shown in full lines, the masses forming the system being M_1, M_2, m_1, and m_2, the other system being formed of the masses M_3, M_4, M_5, and m_6, their radii being shown dotted. It will be observed that the two systems are placed coaxially, with their central double cranks in the same plane of revolution, which plane may conveniently be chosen for the reference plane.

The angular disposition of the two systems, relative to one another, is to be such that the four cranks in the reference plane are mutually at 120°. This arrangement is only possible, if one crank of the central pair, belonging to one system, coincides with one crank of the central pair belonging to the other system, as shown in the reference plane (Fig. 116). The set of four masses in the reference plane may now be divided into two groups—

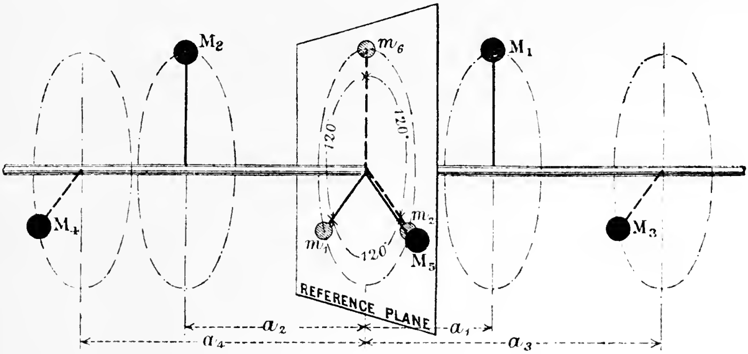
(1) A group of three masses, m_1, m_2, m_6, whose radii are mutually inclined at 120°; these are indicated by cross-hatching in Fig. 116.

(2) A single mass lettered M_5, and shown black; this is drawn

at a slightly greater radius, for the sake of clearness. It is, of course, really coincident in position with m_2.

If the shaded masses were equal in magnitude, they would form a system in equilibrium amongst themselves, both for the primary and the secondary forces they give rise to. Art. 92 is a proof of this, or it may be proved by substituting these values of

Fig. 116.



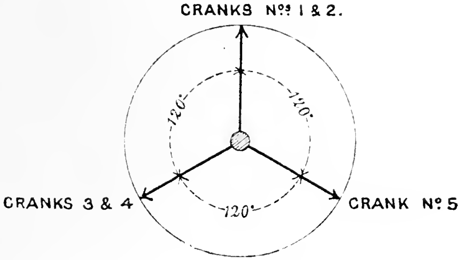


Fig. 117.

the masses and angles in equations (1), (2), (5), (6), of Group D, Art. 87. They might, under these circumstances, be subtracted from the combined system, without interfering with its equilibrium. Let them be equal, therefore, and let them be subtracted from the combined system; there will be left, in the reference plane, the single mass M_5.

Now, the masses forming each of the original systems must satisfy the relation expressed in equation (16), Art. 94, which is, considering one system, that the magnitude of each of the two central masses is equal to the sum of the magnitudes of the two outer masses; the two outer masses are always, of course, in the same axial plane as shown in Fig. 109. Therefore, for the combined system under discussion, the following two equations must hold, the first line referring to the system whose mass radii are shown by thick lines, the second to the system whose mass radii are shown dotted in Fig. 116:—

$M_1 + M_2 = m_1 = m_2$ from equation (16), Art. 94
$M_3 + M_4 = m_2 = m_6$ „ „ „ „

But $m_1 = m_2 = m_6$, by supposition; therefore—

$$M_1 + M_2 = M_3 + M_4 = M_5 \quad \ldots \ldots \quad (1)$$

And from the conditions of balance in the original four-crank systems—

$$M_1 a_1 = M_2 a_2 \quad \ldots \ldots \quad (2)$$
$$M_3 a_3 = M_4 a_4 \quad \ldots \ldots \quad (3)$$

Equations (2) and (3) each express the relation given by expression (9) in Art. 94.

From these three equations the values of the masses for given values of the a's may be found in terms of M_5. Notice that, in this case, fixing the pitch of the cylinders is equivalent to fixing four of the twelve variables concerned in the problem, and that the remaining eight quantities must be found from the equations.

Assuming the pitch of the cylinders to be settled, the magnitudes of the several masses may be more explicitly stated as follows:—

From equations (1) and (2) above—

$$M_1 + M_2 = M_5 = 1, \text{ say} \quad \ldots \ldots \quad (4)$$
$$M_1 a_1 = M_2 a_2 \quad \ldots \ldots \quad (5)$$

Eliminating M_2 from (4) and (5)—

$$\left.\begin{array}{l} \left.\begin{array}{l} M_1 = \dfrac{a_2}{a_1 + a_2} \\ M_2 = 1 - M_1 \end{array}\right\} \text{in the same axial plane} \\ \text{Similarly—} \\ \left.\begin{array}{l} M_3 = \dfrac{a_4}{a_3 + a_4} \\ M_4 = 1 - M_3 \end{array}\right\} \text{in the same axial plane} \end{array}\right\} \text{at } 120°$$

Example.—Suppose the pitch of the cylinders to be such that $a_1 = 2$ feet, $a_2 = 3$ feet, $a_3 = 4$ feet, $a_4 = 5$ feet, each of these distances being measured from the plane of the central crank of the five. Substituting, in the above expressions, and taking M_5 = unity, the masses must be proportional to the numbers placed below them in the following rows :—

$$M_1 : M_2 : M_3 : M_4 : M_5$$
$$\tfrac{3}{8} : \tfrac{2}{5} : \tfrac{5}{9} : \tfrac{4}{9} : 1$$

The crank angles have the same sequence as those shown in Fig. 117.

106. Six-crank Engine.—The number of variables in this case is in general—

$$3(6 - 1) = 15$$

A solution of this problem may be derived from the general solution of the four-crank problem given in Art. 94, by the method explained in the previous article, for a five-crank engine, although there are presumably other solutions possible. Combine three four-crank systems of the type shown in Fig. 109, to form a single system, in the way shown in Fig. 118, the masses forming the systems being respectively—

$$\begin{array}{llll} M_1 & M_2 & m_1 & m_2 \\ M_3 & M_4 & m_3 & m_4 \\ M_5 & M_6 & m_5 & m_6 \end{array}$$

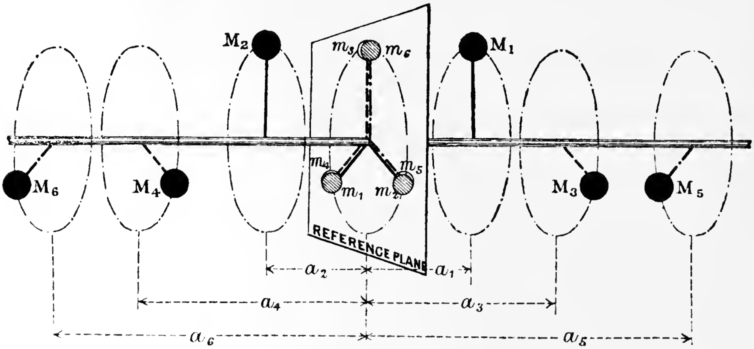
It will be observed that the three systems are placed coaxially, with their central double cranks in the same plane of revolution, which plane may conveniently be chosen for the reference plane. The angular disposition of the three systems, relatively to one another, is to be such that the six cranks in the reference plane are mutually at 120°. This necessitates the arrangement shown in Fig. 118, in which the cranks form coincident pairs. The set of six masses in the reference plane may now be divided into two groups—

(1) A group of three masses, m_1, m_2, m_6, shown cross-hatched, whose radii are mutually at 120°.

(2) A group of three masses, m_3, m_4, m_5, shown slightly displaced behind the former three, whose radii are mutually inclined at 120°.

If the masses in each group were equal, they would form two systems, each in equilibrium, both for the primary and the secondary forces they give rise to when reciprocated. (Art. 92 contains the proof of this.) The two systems might, under these circumstances, be subtracted from the combined system, without interfering with its equilibrium. Let them be equal, therefore, and let them be subtracted from the combined system; there will be left six masses, viz. M_1, M_2, M_3, M_4, M_5, M_6, three on each

Fig. 118.



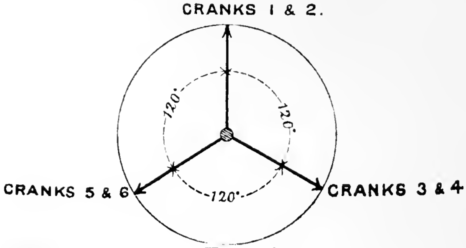


Fig. 119.

side of the reference plane, forming a six-crank engine, whose cranks are shown in Fig. 119. The following equations are true of each system, respectively, from the conditions of balance of the original systems :—

$$M_1 + M_2 = m_1 = m_2 \text{ from equation (16), Art. 94}$$
$$M_3 + M_4 = m_3 = m_4 \quad \text{,, \quad ,, \quad ,, \quad ,,}$$
$$M_5 + M_6 = m_5 = m_6 \quad \text{,, \quad ,, \quad ,, \quad ,,}$$

But $m_1 = m_2 = m_6$, and $m_3 = m_{m_4} = m_{m_5}$, by supposition therefore—

$$M_1 + M_2 = M_3 + M_4 = M_5 + M_6 \quad . \quad . \quad . \quad . \quad (1)$$

and, from the conditions of balance of the three original four-crank systems used in the combination—

$$M_1a_1 = M_2a_2 \quad \ldots \ldots \quad (2)$$
$$M_3a_3 = M_4a_4 \quad \ldots \ldots \quad (3)$$
$$M_5a_5 = M_6a_6 \quad \ldots \ldots \quad (4)$$

Equations (2), (3), and (4), severally, express the relations (9) in Art. 94.

This solution adjusts itself most happily to the general conditions of design. The rules for designing a six-crank engine of this type may be thus stated—

Take a reference plane, and group three pairs of cranks about it, each pair to be in an axial plane and to satisfy the conditions—

$$M_1a_1 = M_2a_2 \quad \ldots \ldots \quad (5)$$
$$M_1 + M_2 = \text{a constant} = 1, \text{ say} \quad \ldots \quad (6)$$

Assuming the spacing of the cylinders to be settled, all the a's are known, and the magnitudes of the several masses may be more explicitly stated as follows :—

$$\left.\begin{array}{l}\left.\begin{array}{l} M_1 = \dfrac{a_2}{a_1 + a_2} \\ M_2 = 1 - M_1 \end{array}\right\} \text{in the same axial plane} \\ \left.\begin{array}{l} M_3 = \dfrac{a_4}{a_3 + a_4} \\ M_4 = 1 - M_3 \end{array}\right\} \text{in the same axial plane} \\ \left.\begin{array}{l} M_5 = \dfrac{a_6}{a_5 + a_6} \\ M_6 = 1 - M_5 \end{array}\right\} \text{in the same axial plane} \end{array}\right\} \text{at } 120^\circ$$

101. Extension of the General Principles of the Foregoing Articles to the balancing of Engines when the Fundamental Expression (2), Art. 78, includes Terms of Higher Order than the Second.—If the extraction of the square root of expression (a), Art. 78, be continued to more terms, the final expression for the displacement of B (Fig. 102) will contain terms in $\cos 4(\theta + \alpha)$, $\cos 6(\theta + \alpha)$, etc. This converging series of cosines, differentiated twice, gives a converging series for the acceleration of B (Fig. 102), which, when multiplied by the mass, gives the values of the unbalanced force acting on the frame. Expression (2), Art. 78, then becomes—

$$M\omega^2 r\{\cos(\theta + a) + A\cos 2(\theta + a) - B\cos 4(\theta + a) + C\cos 6(\theta + a) \ldots\} \quad \ldots \quad (1)$$

where the coefficients A, B, C have the values, c being the ratio $\frac{r}{l}$—

$$A = c + \frac{c^3}{4} + \frac{15c^5}{128} + \ldots$$

$$B = \frac{c^3}{4} + \frac{3c^5}{16} \ldots$$

$$C = \frac{9c^5}{128} + \ldots$$

The details of the calculation of the above expression are given in full in an interesting paper by Mr. John H. Macalpine, *Engineering*, October 22, 1897.

Each term in the series may be interpreted in the way explained in Art. 80 for $\cos 2(\theta + a)$. Thus, the force corresponding to the third term may be considered as the result of the rotation of an imaginary crank, revolving four times as fast as the main crank. The fourth term represents an imaginary crank, revolving six times as fast, and so on.

For balancing purposes, therefore, the main crank-shaft of an engine may be looked upon as associated with a series of imaginary crank-shafts, rotating with speeds twice, four times, six times, etc., the speed of the main shaft, about the same axis of rotation; the angles between the cranks of the imaginary shafts being respectively twice, four times, six times, etc., the actual angles between the cranks of the main shaft, the masses carried by each imaginary shaft being in the same proportion as those of the main shaft, the planes of rotation of the series of imaginary cranks, corresponding to any one of the actual cranks, being, of course, coincident. Keyed to each shaft is an appropriate reference plane. The conditions that each imaginary shaft may be in balance are, simply, that force and the couple polygons belonging to it shall close. Consider a force vector belonging to one of the main cranks, and let its direction, with the initial line in the reference plane, be $\boldsymbol{a}$. The series of imaginary cranks belonging to this crank will have directions $2a$, $4a$, $6a$, etc., with the initial lines in their respective reference planes.

These directions may be severally represented analytically in terms of the quantities specifying the direction of the force vector in the reference plane corresponding to the main crank. The way of doing this has been explained in Art. 85 for the $2a$ direction. By an obvious extension of the method, a vector in the reference plane corresponding to the $4a$ crank is denoted by—

$$R\{(8y^4 - 8y^2 + 1) + i4xy(1 - 2y^2)\}$$

where R is a function of the mass carried by the main crank the length of the rod and the crank radius, but which, for balancing purposes, may be written equal to M, since everything else cancels out when the final conditional equations are formed.

Similarly, a vector in the next plane of the series is represented by—

$$M[\{(x^2 - y^2)^3 - 3(x^2 - y^2)\} + i(6xy - 8x^3y^3)]$$

The eight conditional equations of Art. 87 will be increased by four for every additional imaginary crank-shaft taken in the series. Thus, corresponding to the $4a$ crank-shaft will be the four additional conditions—

$$\begin{aligned} \Sigma M(8y^4 - 8y^2 + 1) &= 0 \\ \Sigma Mxy(1 - 2y^2) &= 0 \\ \Sigma Ma(8y^4 - 8y^2 + 1) &= 0 \\ \Sigma Maxy(1 - 2y^2) &= 0 \end{aligned}$$

Similar sets of four more must be added to the conditions, if the $6a$ crank-shaft is taken in, and so on.

The solution of the set of sixteen simultaneous equations, corresponding to the main shaft and the first three imaginary crank-shafts, presents formidable analytical difficulties. They may be used, however, without much trouble, to test the balancing of a proposed arrangement; and, by the introduction of proper coefficients, the magnitude of the unbalanced force and couple corresponding to any crank-shaft in the series may be found.

Reverting to the polygons in the series of reference planes, it will be evident that, if a given arrangement of engine is in perfect balance, corresponding to the closed force polygon in the first and second reference planes (see Fig. 103), there will be a closed polygon in the third plane, whose sides are proportional in magnitude to the sides of the force polygon in the first plane, but make four times the angle with their initial line that the sides of the first polygon makes with its initial line. There will be a closed polygon in the next reference plane, the directions of whose sides with their initial line is six times the angle of the sides of the first polygon with its initial line. And so on through the whole series. There will, of course, be a similar series of couple polygons.

The application of this test to the arrangement shown in Fig. 109 discloses that, not only is the primary and secondary shaft in balance, but that the imaginary shaft, corresponding to $4a$, is also in balance for forces. The $6a$ shaft is out of balance. The $4a$ shaft is also balanced for couples. Since the $6a$ forces are not balanced, the $6a$ couples are not balanced independently of the position of the reference planes. In general, the infinite series of imaginary shafts are in balance, with the exception of the $6a$ shaft, and all multiples of it. If the reference plane is taken at the centre, the whole infinite series of couples are in balance, but those belonging to the $n(6a)$ series appear again, if the reference plane is taken in any other position; this follows from an extension of the theorem of Art. 90. To prove these statements, take the reference planes at the centre (Fig. 109), and the lines of reference in them, to correspond with the direction of the two outer cranks. The force polygon, in the first plane, is the equilateral triangle (Fig. 120). In the second plane, corresponding to the $2a$ shaft, it is also an equilateral triangle (Fig. 121). Fig. 122 shows the triangle for the $4a$ shaft, still closed. In the next plane, for the $6a$ shaft, it becomes the line of Fig. 123. Continuing in this way, the $8a$, $10a$ polygons close, opening into a line for $12a$. The couple polygon for all planes is a line returning on itself, since the angle of the two outer cranks with the initial line is in each case 0, and therefore all the multiples are nothing.

The actual magnitude of the maximum unbalanced force due to the $6a$ shaft is represented by OC (Fig. 123), and this is evidently = 3OA.

OA represents to scale the maximum force due to the mass M. This is given by the value of—

$$\frac{M\omega^2 r}{g} \times C \text{ lbs. weight}$$

the fourth term in expression (1) of this article, so that it is only necessary to calculate the value of C. As a matter of interest,

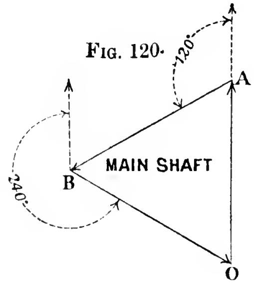


FIG. 120.

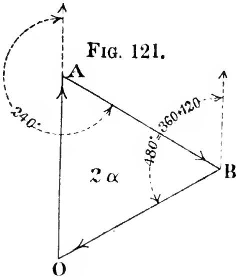


FIG. 121.

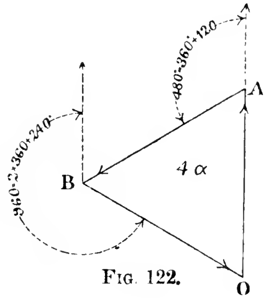


FIG. 122.

FIG. 123.

however, the whole series is given for the case of a rod equal to three and a half times the length of the crank. It is—

$$\frac{M\omega^2 r}{g}(\cos\theta + \cdot 291\cos 2\theta - \cdot 006\cos 4\theta + \cdot 0002\cos 6\theta + \ldots)$$

The maximum unbalanced force is then—

$$0\cdot 0006 \times \frac{M\omega^2 r}{g} \text{ lbs. weight}$$

If $M = 11,200$ pounds, $r = 2$ feet, revolutions per minute = 88, this force is equal to 33·6 lbs. weight, a quantity negligible compared with the unbalanced forces, even from the auxiliary engines about the ship.

From these properties of the arrangement of Fig. 109, it follows that the five- and six-crank engines derived from it are similarly in balance. Thus, an engine balanced by the rules of Art. 99, or a six-crank engine designed from Art. 100, would be in balance for forces and couples up to those of the $6a$ class. There can be no doubt but that these are the kinds of engines to use to avoid vibration. The four-crank engine cannot approach them in this respect, since, even in the best arrangement, that of Art. 96, couples of the $2a$ class of considerable magnitude are left unbalanced. Moreover, the crank angles of the five- and six-crank engines fit in so well with the other conditions of design. A uniform crank-effort diagram can be obtained with ease, and there are no awkward starting angles.

102. General Summary.—(*a*) One or any number, n, of rigidly connected masses revolving in the same plane may be balanced, by the addition of one mass in that plane, whose magnitude and position are found by Art. 14, forming with the given masses a system of $n + 1$ masses in balance; or, by the addition of two masses, each in a given separate plane of revolution, forming a system of $n + 2$ masses in balance. These two masses may be found by the general method of Art. 28, or by using first Art. 14 to find the mass in the plane of the given masses and then dividing it between the given planes, inversely as their distances from the plane of revolution of the given masses.

(*b*) One or any number, n, of rigidly connected masses, each revolving in a different plane, may be balanced by the addition of two balancing masses in any two given planes of revolution, forming with the given masses a system of $n + 2$ masses in balance. These masses are to be found by the general method of Art. 28.

(*c*) It is generally better to reduce the masses to a common radius, the crank radius in engine problems, before filling in the schedules. This is done by the method of Art. 33. Any balancing mass "at crank radius" may be placed at any radius, provided it is changed so that the product Mr remains constant. (See Art. 12 and Fig. 12.)

(*d*) Assuming that the masses forming a reciprocating system, as in a multi-cylinder engine, move with simple harmonic motion, any number, n, may be balanced by the general method of Art. 28 (see Art. 44), the balancing masses found being reciprocated in the common plane of reciprocation. The system then consists of $n + 2$ reciprocating masses. The problem of balancing the reciprocating parts of an engine generally presents itself in this form—given n cylinders to find the masses of the corresponding reciprocating parts and the crank angles so that the system may be in balance without the addition of balancing masses of any kind. Typical solutions for four masses, and for four masses and their valve-gear, are given in examples Arts. 48 to 50.

(*e*) In some cases, locomotives in particular, after finding the balancing masses for the reciprocating parts, in order to avoid the practical objection to reciprocating them, they are added to the system as revolving masses. Under these circumstances, the balancing masses introduce an unbalanced force and couple equal to those they are balancing, in a plane at right angles to the plane of reciprocation. (See Art. 65 for an illustration.)

(*f*) Secondary balancing can be partially effected in four-crank engines and completely effected in five- and six-crank engines. The following general directions show how to set about balancing an engine, and the possibilities of balancing each type are briefly indicated :—

(*g*) **Two-cylinder Engine.**—Put the cranks at 180°, and make the reciprocating masses equal. This ensures balance for primary forces. The primary couples and the secondary forces and couples cannot be balanced. Keep the two planes of revolution as close as possible to minimize the couple error.

Three-cylinder Engine.—The cranks must all be in the same axial plane as shown in Fig. 28, and the masses proportioned so that the middle one is equal to the sum of the outer two, and the moments of the outer two about the plane of the middle one are equal. In this arrangement the primary forces and couples are balanced. The secondary forces and couples are not balanced. If equal masses at crank radius are operated by cranks at 120°, the primary and secondary forces are balanced (Art. 93), but the primary and secondary couples are unbalanced. To minimize the couple error, keep the planes of revolution as close together as possible.

Four-cylinder Engine.—Set out the centre lines of the cylinders, and choose any three of the remaining seven variables (see Art. 35), and apply the method of the example of Arts. 48 or 49. Do not fail to check the work as directed in Art. 47. To include the valve-gear, proceed as in the example of Art. 50. A four-cylinder engine balanced in this way is completely balanced for primary forces and couples, but is unbalanced for secondary forces and couples. If the cylinder centre lines are set out symmetrically as shown in Fig. 110, lay down the centre lines, and then calculate the ratio of the pitch of the extreme cylinders to the pitch of the inner cylinders, which has been called b. Find this number on the horizontal scale of Fig. 112, and read off the quantities x_1, x_2 from curves Nos. 1 and 2 respectively, and set out the crank angles in the way illustrated in Fig. 113. Then read off the mass ratio $\frac{M_2}{M_1}$ from curve No. 3. An engine constructed from these angles and masses will be in balance for primary and secondary forces and primary couples. There will be a secondary couple error, the variation of which is shown by curve No. 6. The amount of the error in foot-tons is found by multiplying the ordinate from the diagram by the product of the pitch of the inner cylinders in feet and the mass in tons of one of the outer reciprocating masses, and by $1{\cdot}2\frac{n^2r^2}{l}$, where n = the revolutions per second, and l and r are the length of the connecting-rod and the crank radius respectively. This error is smaller the smaller the value of b; therefore keep the distance between the outer pairs of cylinders as small as possible relatively to the pitch of the middle pair. The balancing of a symmetrical engine in this way was described by Mr. Schlick in his paper to the Institute of Naval Architects, 1900. For an example, see the s.s. *Deutschland*, *Engineering*, November 23, 1900. To include the valve-gear, find the angle between the two inner cranks either from equation (9), Art. 96, or from Fig. 112; assume two equal masses for the inner cranks, and then apply the method of Art. 50, taking a reference plane at one of the outer cranks. The final angles will be slightly different from those required for the balancing of the secondary forces. See exercises 43 and 44 at the end.

If the angles β and δ (Fig. 111) are not equal, the conditions for balancing the primary and secondary force and the primary couple necessitate an unsymmetrical engine, the details of which are to be found by the method illustrated in Art. 97, Case II.

Five- and Six-cylinder Engines.—Set out the cylinder centre lines. Then the crank angles must be placed in 120° pairs, two pairs and a central crank for the five cylinders (Figs. 116 and 117), and three pairs for six cylinders, as in Figs. 118 and 119. The masses are then calculated by the formulæ at the ends of Arts. 99 and 100. In these engines the primary and secondary forces and couples are completely balanced, and further, as shown in Art. 101, the fourth period forces are balanced also. In fact, five- and six-crank engines arranged in this way are the most perfectly balanced engines of the usual multi-cylinder type that it is possible to construct.

The Crank-shaft.—The crank-shaft is in general an unbalanced revolving system in which the planes of revolution and the crank angles are fixed by the reciprocating system which it operates. It can be balanced by either of the methods of Art. 51. If the ratio of the revolving to the reciprocating masses is the same for every cylinder in the engine, no balancing masses will be required for the crank-shaft.

CHAPTER VI.

ESTIMATION OF THE PRIMARY AND SECONDARY UNBALANCED FORCES AND COUPLES.

103.—In this chapter it is shown how the unbalanced force and couple may be estimated for an engine whether any attempt has been made to balance it or not. Methods have been given in Art. 31 for dealing with a set of revolving masses, and in Art. 53 for finding the resultant force and couple due to a revolving and reciprocating system together, the reciprocation, however, being simple harmonic. The method about to be explained combines that of Art. 31 with, in the first place, a construction which gives the accelerating force acting on the piston accurately for any length of connecting-rod, and, in the second place, with an approximate graphical method, requiring rather less work, based upon the formula (2), Art. 78. Finally, it is shown how the equations of Art. 87 may be used to calculate the maximum values of the primary and secondary components of the resultant force and couple.

104. Klein's Construction for finding the Acceleration of the Piston.*—This construction is theoretically accurate. It is, in fact, the graphical solution of expression (3), Art. 79. It is simple, quickly applied, and gives the result in a convenient way. The construction is given in the *Journal of the Franklin Institute*, Vol. CXXXII., September, 1891.†

Let OK (Fig. 124) be the crank, KB the connecting-rod, BO the line of stroke. The acceleration of the piston masses is the same as the acceleration of the point B, representing the cross-head. It is required to find the acceleration of the point B in terms of the crank angle θ, assuming the crank to rotate uniformly.

* See also Bennett's Construction, Appendix I.

† See reference on page 234.

Produce the connecting-rod BK to meet a perpendicular to BO, the line of stroke, in V. On BK as diameter draw a circle. From K as centre, with radius KV, draw an arc cutting this circle in QQ_1. Join QQ_1, producing the line if necessary to cut the line of stroke in A. Then AO represents the acceleration of the cross-head B, to the same scale that KO, the crank radius, represents

FIG. 124.

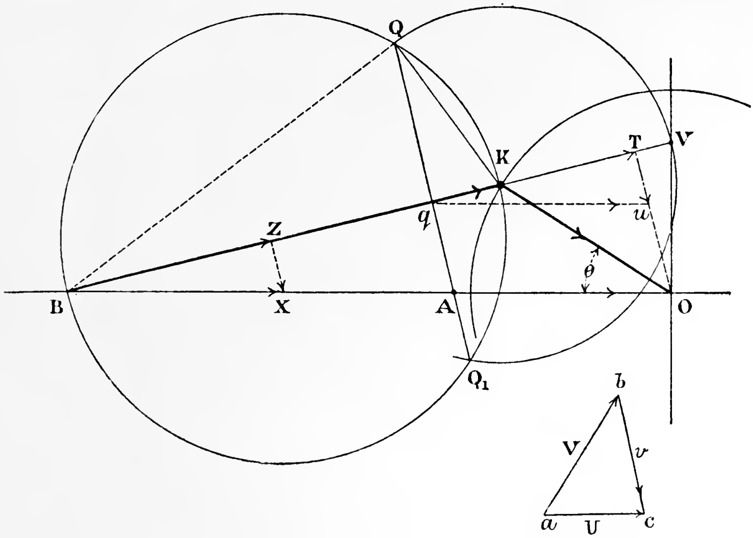


FIG. 125.

the acceleration of the crank-pin K. The latter is uniform and is equal to ω^2KO. For instance, suppose the drawing to be made to a scale of $1\frac{1}{2}$ inches = 1 foot, and that AO scales 2·2 feet. Then the acceleration of B, from B towards A, is $2{\cdot}2 \times \omega^2$. If the crank is making 88 revolutions per minute, $\omega^2 = 84{\cdot}5$, and therefore the acceleration of B is 185·9 feet per second per second.

105. Proof.—Let BX be the acceleration which is to be found. It may be resolved into two components, BZ along the rod, ZX at right angles to the rod. If the magnitude of either of these components is found, the scale of the acceleration triangle BZX is fixed, and the values of BX, and the component ZX, may be scaled off. Now, BZ is equal to the radial acceleration of B relatively to K, plus the component acceleration of the crank-pin K along the rod.

Draw OT at right angles to BV and *qu* parallel to BO. If KO represents the radial acceleration of the crank-pin, KT is clearly its component along the rod. It will be proved immediately that *q*K is the radial acceleration of B relatively to K. Therefore *q*T represents the whole acceleration of B along the rod, and *qu* is evidently the acceleration of B in the line of stroke, which is again equal to AO.

It remains to show that *q*K is the radial acceleration of B relatively to K.

Let V be the velocity of the crank-pin K. This is known in magnitude and direction.

U the velocity of the cross-head B, known in direction only.

v the velocity of B relatively to K, known in direction, since it must always be at right angles to the rod BK.

$$v = \text{the vector difference } (\mathrm{U} - \mathrm{V}) \text{ (Art. 6)}$$

This is a case of B (Art. 8). Set out *ab* (Fig. 125) to represent V, and *ac* in the direction of the velocity U. Draw *bc* at right angles to the rod BK, thereby fixing the point *c* and the magnitudes of the velocities *v* and U. Comparing this triangle with the triangle OVK (Fig. 124), it will be perceived that the two are similar, OK corresponding to *ab* and OV with *ac*. Therefore KV represents *v* to the scale on which OK, the crank radius, represents V. Thus the first operation in Klein's construction gives the velocity triangle KOV in which the radius represents the velocity of the crank-pin. Under these circumstances KO also represents $\frac{\mathrm{V}^2}{r} = \omega^2 r$, the radial acceleration of the crank-pin.

Again, the triangles BKQ and QK*q* are similar, BK corresponding with QK. Therefore—

$$q\mathrm{K} : \mathrm{KQ} = \mathrm{KQ} : \mathrm{BK}$$

But $\mathrm{KQ} = \mathrm{KV} = v$. Therefore $q\mathrm{K} = \frac{v^2}{\mathrm{BK}}$, the radial acceleration of B relatively to K. Hence the construction.

The way this construction may be used to investigate the state of balance of an engine is most easily explained by applying it directly to an example.

106. Data of a Typical Engine.—The following data may be taken to apply to a modern engine of about 12,000 I.H.P. They are given in schedule form at once to avoid repetition, and for convenience of reference. The engine is supposed to be in full gear ahead. The H.P. and L.P. forward reciprocating masses include an allowance for the air-pump gear operated by their respective cross-heads. The ahead gear includes the mass of the valves, valve spindle, motion block, half the link, a proportion of the eccentric rod. In the two L.P. gears the mass of the balancing pistons is included. The astern gear includes a proportion of the eccentric rod and half the link, and reciprocation of the rod is supposed to take place in the main plane of reciprocation, an assumption which involves a negligibly small error. The rest of the valve-gear is included with the revolving masses. The angular advance of all the eccentric sheaves is assumed to be 45°. In an actual engine they would differ for the H.P., Int., and L.P. cylinders.

SCHEDULE 19.

RECIPROCATING MASSES.
100 revolutions per minute, Crank radius, 2 feet.
Ratio crank to rod, 1 : 3·9.

Crank.	Distance from reference plane.	Mass at crank radius or centrifugal force, when $\omega^2R=1$.	Mass ratios.	Mass moment or centrifugal couple, when $\omega^2R=1$.	Moment ratios.
	Feet.	Tons.			
4. L.P. Aft	0	6·6	1·1	0	—
Ast. sheave	4·7	0·1	—	0·47	—
Ah. sheave	5·4	0·6	—	3·24	—
Ah. sheave	10 6	0·6	—	6·36	—
Ast. sheave	11·3	0·1	—	1·13	—
3. L.P. Forward	16·0	7·0	1·166	112·0	1
Ah. sheave	23·6	0 6	—	14·16	—
Ast. sheave	24·3	0·1	—	2·43	—
2. Int.	29·0	6·3	1·05	182·7	1·631
Ah. sheave	36·6	0·6	—	21·96	—
Ast. sheave	37·3	0·1	—	3·73	—
1. H.P.	42·0	6·0	1·0	252·0	2·25

SCHEDULE 20.

REVOLVING MASSES.

Crank.	Distance from reference plane.	Mass at crank radius or centrifugal force, when $\omega^2 R=1$.	Mass ratios.	Mass moment or centrifugal couple, when $\omega^2 R=1$.	Moment ratios.
	Feet.	Tons.			
4. L.P.A.	0	4·41	1	—	—
Ast. sheave	4·7	0·2	—	0·94	—
Ah. sheave	5·4	0·2	—	1·08	—
Ah. sheave	10·6	0·2	—	2·12	—
Ast. sheave	11·3	0·2	—	2·26	—
3. L.P.F.	16·0	4·41	1	70·56	1
Ah. sheave	23·6	0·2	—	4·72	—
Ast. sheave	24·3	0·2	—	4·86	—
2. Int.	29·0	4·41	1	127·89	1·81
Ah. sheave	36·6	0·2	—	7·32	—
Ast. sheave	37·3	0·2	—	7·46	—
1. H.P.	42·0	4·41	1	185·22	2·62

107. Calibration of a Klein Curve to give the Accelerating Force.—The changing values of the force acting on the frame due to any one of the set of reciprocating masses may be conveniently exhibited by a curve plotted on a crank base. Consider the H.P. cylinder of the example. The ratio of the crank to the rod is 1 : 3·9. Choose a suitable scale, and apply Klein's construction at, say, twelve equidistant angular positions of the crank. Let XX (Fig. 126), taken any length, be divided into the same number of equal divisions. Set out the accelerations found by the constructions, vertically above or below the corresponding angles on XX, as the case may be. AA, for instance, represents the acceleration of the reciprocating parts when the crank angle is 30°. Mark off XR to represent the crank radius. Draw a curve through the ends of the ordinates. All the intermediate values of the acceleration may be scaled off this curve. Since the mass accelerated is constant, the ordinates are also proportional to the accelerating force. The scale is at once fixed for a given speed from the fact that XR represents $\omega^2 r$, the acceleration of the crank-pin, the corresponding force in tons is therefore $\frac{M\omega^2 r}{g}$. At 100 revolutions per minute ω^2 is very nearly equal to 110. From the schedule, M, for the H.P. cylinder, is 6 tons, and r is 2 feet, so that XR represents a force of 41 tons. Draw Xr at any angle, and set off 41 to a suitable scale, marking off the points 10, 20, etc., at the same time, to correspond to 10-ton intervals; join rR, and draw parallels to it from 10, 20, etc., thereby reducing the 10-ton intervals to the

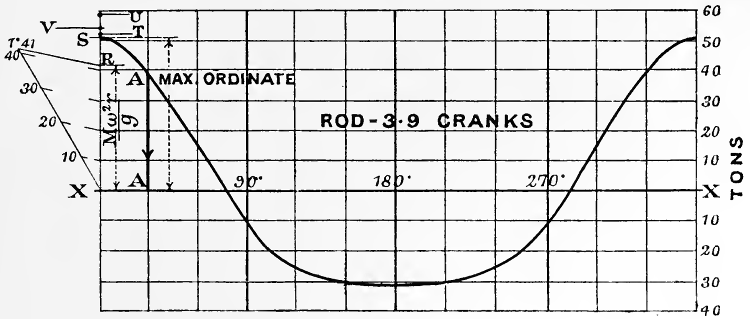


FIG. 126.

scale of the drawing at the points where these parallels cut the line XR. The curve is properly calibrated by drawing horizontals through these points, repeating them and subdividing the intervals as much as may be required. The curve shows that the force at the top, dead centre, is about 51 tons, and on the bottom centre 31 tons, that when the crank is at right angles to the line of stroke it is 10 tons; in fact, the disturbing force for this particular cylinder is known for any assigned position of the crank belonging to it.

108. Derivation of Curves to represent the Forces due to the Other Reciprocating Masses in the Engine, the Ratio of Crank to Rod being Constant.—When the ratio of the crank to rod is a constant for all the cylinders, the Klein curves are all similar. Further, if the strokes are the same, the only variable in the expression for the value of the force is M. If, therefore, a series of similar curves be drawn, the maximum ordinates being in the ratio of the several masses, the curves will severally represent the magnitudes of the disturbing forces belonging to each cylinder. Applying this to the example, Schedule 19, if lengths XT, XU, XV are set out so that—

$$XS : XT : XU : XV = 6 : 6 \cdot 3 : 7 : 6 \cdot 6 = 1 : 1 \cdot 05 : 1 \cdot 166 : 1 \cdot 1$$

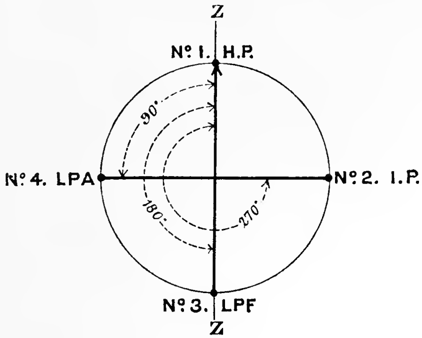
and curves be drawn through the points T, U, V similar to the curve through S, from the appropriate curve of the set so obtained the force corresponding to the motion of the parts belonging to any one of the cylinders may be read off for any given position of the crank belonging to that cylinder. The proportional increase in the ordinates of the curves are quickly found by the use of a proportional compass.

109. Combination of the Curves for their Phase Differences.— In order that the simultaneous values of the forces corresponding to an assigned angular position of *one* of the cranks may be read off, the curves must be combined for the phase differences of their cranks relatively to it. In Fig. 128, the four curves corresponding to the cranks 1, 2, 3, 4 of Schedule 19 are combined in their phase relation to the H.P. crank. Thus when the crank is at 30° with the line of stroke, the ordinates ab, ac, ad, ae respectively represent the instantaneous value of the forces acting on the frame due to the reciprocation of the parts belonging to cylinders Nos. 1, 2, 3, and 4. Their algebraic sum—

$$ab + ac - ad - ae = -af$$

is the instantaneous value of the unbalanced force acting on the engine-frame. The ordinates for a number of positions are quickly added by means of a pair of dividers, and through the series of points so obtained, of which f is one, the error curve may be sketched in. It will be observed that the maximum error is about 10 tons. The combination for difference of phase may be made in the following way. Draw an end elevation of the cranks (Fig. 127), marking on the drawing their angular distances from the H.P. crank. Consider the intermediate crank No. 2. Assuming the four curves, through the points S, T, U, V (Fig. 126), to have been drawn according to the instructions of Art. 108, trace the curve belonging to the intermediate cylinder, that is, the one through T, and mark on the axis XX the point corresponding to the angular distance of its crank from the H.P. crank, in this case 270°. Place the tracing over a drawing of the H.P. curve so that the angle marked on the XX of the tracing coincides with the zero of the H.P. axis XX, and so that the XX of the tracing coincides with the XX of the drawing. Then prick or rub the curve through on to the drawing. Do this for each curve in turn, obtaining finally a set like those shown in Fig. 128. The calibration of the diagram is obtained, of course, from the calibration of any one of the curves, since they are all drawn to the

Fig. 127.



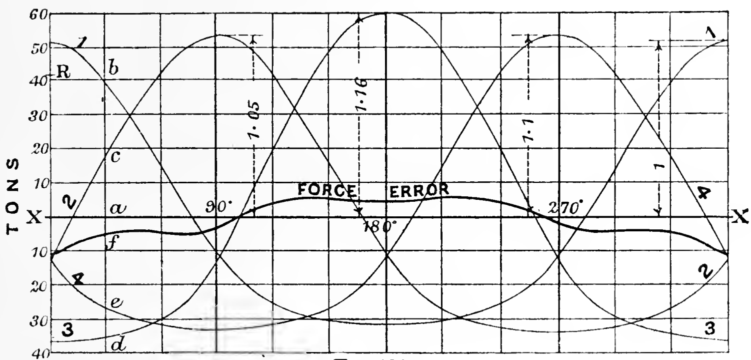


Fig. 128.

same force scale. For instance, XR on Fig. 128 is the XR of Fig. 126, and this has been shown to represent 41 tons for the example under discussion. The force-error curve shows at a glance the variation of the unbalanced force acting through a complete revolution of the crank-shaft. The component of the unbalanced force due to the revolving parts in the plane of reciprocation must be added to this to get the whole force in that plane. In the present example the force polygon for the revolving parts is a closed square, so that there is nothing to add to the curve shown. The work entailed by strictly following the directions of this and the preceding two articles may be considerably curtailed in ways which will be obvious after one trial of the method.

110. Calibration and Combination of Klein Curves to obtain the Unbalanced Couple belonging to the Reciprocating Masses of an Engine.—If a is the distance of the centre line of a cylinder from the reference plane, the magnitude of the couple in foot-tons is given by the

$$\frac{M}{g}a \times \text{acceleration}$$

The only variable in this expression is the acceleration. But this is given by a Klein curve. Therefore the ordinates of the curve represent to some scale the changing value of the couple in terms of the crank angle. To fix the scale it is only necessary to observe that the length XR, representing the radius of the crank, now stands for the couple $\frac{Ma\omega^2r}{g}$. This is equal to 1722 foot-tons for the quantities corresponding to the H.P. cylinder, Schedule 19. The curve marked 1 (Fig. 130), which is merely the curve of Fig. 126 recalibrated, thus represents the couples at any instant due to the H.P. parts taken with reference to a plane at the L.P.A. cylinder centre line. Curves Nos. 2 and 3 are drawn so that their maximum ordinates have the ratios to No. 1 curve given in the last column of Schedule 19, and so that their phase differences are those given by Fig. 127. The thick curve marked "reciprocating masses" is the resultant curve of the three found by taking the sum of their ordinates at points along the base in the way already explained. The drawing of this curve finishes the problem so far as the reciprocating masses are concerned, for corresponding to any assigned crank angle of the H.P. crank, the instantaneous value of the unbalanced couple can be read off. Thus when the crank angle is about 45°, the couple acting due to the motion of the piston masses is about 1600 foot-tons, changing to about 1700 foot-tons, whilst the H.P. crank turns through about half a revolution, the corresponding time interval being 0·3 second. Serious as this couple is, it is not by any means the whole unbalanced couple, for there must be added to it the component of the unbalanced couple due to the revolving masses.

Fig. 130.

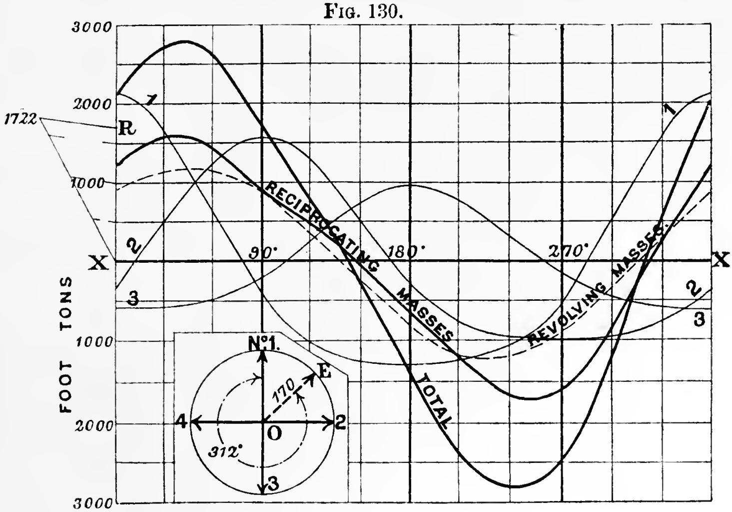


Fig. 129.

111. Addition of Couple due to Revolving Parts.—To find the effect of the revolving parts, set out the couple polygon corresponding to the data for the main cranks given in Schedule 20. It will be found that the vector sum of the couples, that is, the line joining the origin to the end of the last vector, or the "closure" reversed, scales 170, and makes an angle of 312° with the H.P. crank (Fig. 129). The projection of this line, as the system revolves, on the plane of reciprocation gives the instantaneous value of the component couple. These projections, set out on the crank base (Fig. 130), form a cosine curve whose maximum ordinate represents the actual value of the couple. If the diagram is calibrated in foot-tons for a given value of ω and r, then the maximum ordinate of the cosine curve must be taken so that it represents—

$$\frac{170\omega^2 r}{g} \text{ foot-tons}$$

This curve is shown dotted in the figure. The thick curve marked "total" is obtained by adding the ordinates of the dotted curve and the curve marked "reciprocating masses." It will be noticed now that the unbalanced couple changes from a positive value of about 3000 foot-tons to a negative value of the same amount in about half a revolution. A ship with a similar set of engines to those specified in the schedules vibrated so much under the action of the unbalanced couple that they had to be altered.

112. Acceleration Curve corresponding to the Approximate Formula (2), Art. 78.—It has been shown (Art. 79) that the difference between the true acceleration and that given by the approximate formula is small. It follows that the difference between the acceleration curve constructed from this latter formula and the Klein curve, which realizes the exact formula, will be negligibly small also. The method of building up the approximate acceleration curve from a primary and a secondary curve is exhibited in Fig. 131. XX and XR are taken equal in length to the correspond-

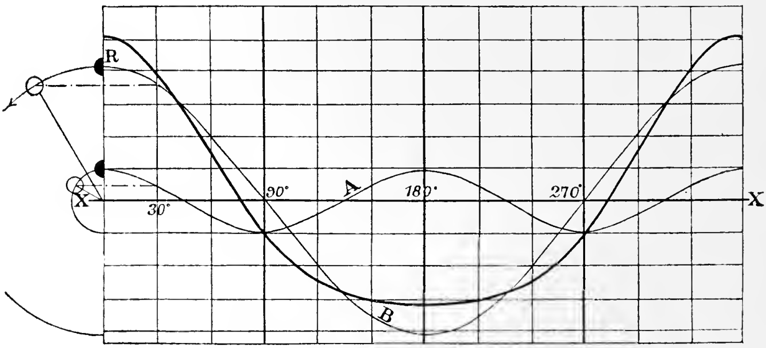


Fig. 131.

ing lines in Fig. 126, so that the two curves may be compared. Referring to formula (2), Art. 78, the form of the acceleration curve is given by the quantities in the brackets, since the factor $M\omega^2 r$ is constant. Measuring angles from the H.P. crank, a becomes zero. The first term in the brackets is represented graphically by a cosine curve whose maximum ordinate is XR (Fig. 131), the second term by a cosine curve whose maximum ordinate is the fraction $\frac{r}{l}$ of XR, in phase with the other curve at O, but of half the periodic time. In fact, the values of the ordinate of the two component curves are given respectively by the projections of the main crank and its imaginary fellow, revolving twice as fast, on the line XR. The positions of the two cranks corresponding to 30° of the main crank are shown to the left of the figure. Adding these two curves together, the thick curve is obtained, which will be found to differ only slightly from the true curve of Fig. 126. The approximate curve may, therefore, be used as the basis for estimating the unbalanced force and couple for most practical purposes. The calibration of the curve is fixed as before from XR $= \frac{M\omega^2 r}{g}$, or $\frac{Ma\omega^2 r}{g}$, as the case may be.

A well-known and convenient way of finding the ordinate for this approximate curve is illustrated in Fig. 132. Let θ be any

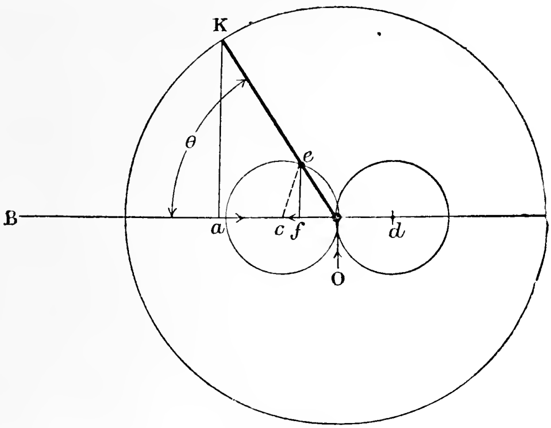


FIG. 132.

crank angle, BO the line of stroke, OK the crank. Draw two circles, each with radius $\frac{r}{l}$, so disposed that they touch one another externally at the centre O, and that the line joining their centres coincides with the line of stroke. From the points K and e drop perpendiculars to the line of stroke. Then—

$$aO + fc$$

represents the acceleration of the piston to the same scale that KO represents $\omega^2 r$, the acceleration of K.

The intercept fc is a vector quantity (c is the centre of the circle nearest the cross-head) always directed from f towards c; but when the crank cuts the other circle, the direction of the intercept is from the centre d towards f, the foot of the perpendicular. Also aO is a vector always directed towards O. In the position shown in the figure the quantities happen to be oppositely directed. When the crank is at 90°, the point f coincides with O, and the intercept fc is then equal to the radius of the circle, that is, $\frac{r}{l}$. The proof of this construction is simple. If KO represents $\omega^2 r$, then fc represents $\omega^2 r\left(\frac{r}{l}\cos 2\theta\right)$, since the exterior angle ace of the triangle cOe, is equal to the two interior and opposite angles, each of which is equal to θ. The projection aO represents $\omega^2 r\cos\theta$; hence the vector (aO $+ fc$) represents $\omega^2 r(\cos\theta + \frac{r}{l}\cos 2\theta)$, a form identical with that of the expression giving the acceleration. The advantage of this method is that it can be applied without having to draw the connecting-rod in at all.

113. Process of finding the Primary and Secondary Components of the Resultant Force and Couple Curves.—The resultant of curves like Fig. 131, might be found in the way that has just been described when exact curves like Fig. 126 are used. A much simpler method is available, however, since the component vectors of all the curves are known. The total unbalanced force may be represented by—

$$\omega^2 r\Sigma\{\mathrm{M}\cos(\theta + a) + \mathrm{M}\frac{r}{l}\cos 2(\theta + a)\}$$

To find the effect of the first terms alone, set out the M's in order parallel to the crank directions in the usual way, to find their vector sum. Set them out again, only this time draw parallel to the cranks of the imaginary shaft, that is, a shaft in which all the crank angles are doubled. This second sum, multiplied by $\frac{r}{l}$, is a vector representing the effect of the second terms in the expression. These resultant vectors may now be conceived attached, the one to the crank-shaft, the other to the imaginary shaft, which is revolving twice as fast. The respective projections of these two vectors on the plane of reciprocation will then represent the instantaneous values of the ordinates of the two curves which are the components of the resultant curve, in the same way that the two thin curves of Fig. 131 are the components of the thick curve, only now there will in general be a phase difference. This method avoids the necessity of drawing any but the two component curves to obtain the resultant curve. A similar process may be used to find the unbalanced couple curve. The method is illustrated by applying it to find the unbalanced couples of the example previously considered.

114. Application to the Couples of the Example.—The vector OC (Fig. 133) represents the sum of the couples in Schedule 19, found by drawing a polygon relatively to the main cranks of the engine.

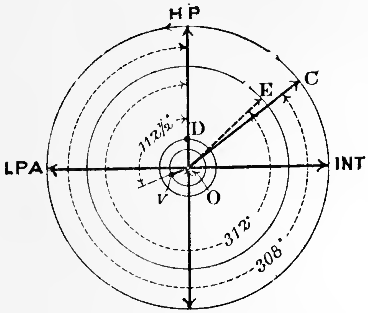
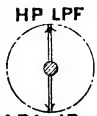


Fig. 133.

Fig. 134.



An end view of the imaginary crank-shaft is shown in Fig. 134. The angle between the H.P. and L.P.A. crank is now 180°, between the L.P.A. and the L.P.F. 180°, and between the L.P.F. and Int. 180°.

OD (Fig. 133) is the vector sum of the couples, multiplied by $\frac{r}{l}$, found from a polygon whose sides are drawn parallel to the cranks on the imaginary shaft (Fig. 134). The curves corresponding to the rotation of these two vectors, the latter twice as fast as the former, are shown in Fig. 135, the one marked *b* corresponding to the vector OC, the one marked *a*, to the vector OD. Their sum is shown by the thick line. Comparing this line with the corresponding one in Fig. 130, marked "reciprocating masses," very little difference will be observable to the scale of the illustrations. The sum of the couples belonging to the crank-shaft system is shown by OE (Fig. 133). The curve corresponding to this may be drawn on the diagram, as in Fig. 130, and then added to the

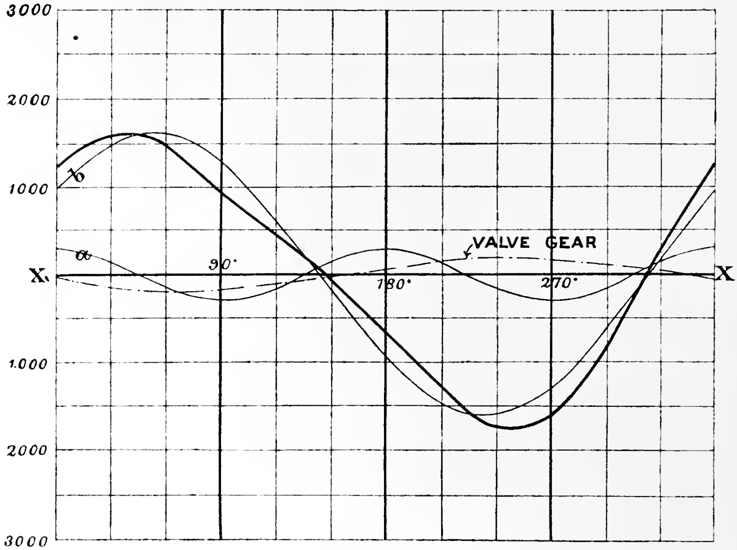


Fig. 135.

resultant curve for the reciprocating masses as before. Or the sum of the vectors OC and OE may be taken, and a curve drawn for their resultant. This curve added to curve *a* will give the curve representing the total unbalanced couple acting in the plane of reciprocation. The principle of the method explained in the last two articles is the principle of Mr. Macfarlane Gray's * Accelerity Diagram.

115. Valve-gear and Summary.—The ratio of the eccentricity to the length of the eccentric rod is usually so small that the effect of the rod's obliquity may be neglected, and the motion of the valves, therefore, treated as simple harmonic. Also considering the engine in full gear ahead, the effect of the idle eccentric rod

* "On the Accelerity Diagram of the Steam Engine." By Mr. J. Macfarlan Gray. *Trans. Inst. Naval Architects.* London, 1897.

may, without sensible error, be supposed to take place in the main plane of reciprocation. On this understanding the unbalanced forces and couples due to the valve-gear are to be found by the principles of the preceding articles, and added to the total curves given for the main cranks. The several masses belonging to the reciprocating and revolving parts of the valve-gear, all reduced to the crank radius of 2 feet, are given in Schedules 19 and 20 for the example under consideration. Instead of drawing two polygons, one for the reciprocating and one for the revolving masses, one only need be drawn for the purpose in hand, using for the equivalent mass the sum of the reciprocating masses and the revolving masses given in the schedules. The angular advance of the eccentrics for both ahead and astern sheaves is 45°. An end view of the eight eccentric cranks concerned can therefore be drawn. It will be found that the force polygon closes, and that the vector sum of the couples is a line 29(Ma) units long, making an angle of $112\frac{1}{2}$° with the H.P. crank. It is shown in its proper phase relation and scale to the other vectors in Fig. 133 by OV, its length being given by the radius of the small inner circle. The cosine curve corresponding to it is shown by chain-dotted lines marked valve-gear in Fig. 135.

It will be observed that in this particular case the unbalanced couple due to the valve-gear is not so great as the unbalanced secondary couple due to the piston masses. In other cases it might be greater. This shows that if an engine is balanced for secondary forces or couples the balancing can hardly be considered satisfactory unless the valve-gear is balanced also. This can be done conveniently by the addition of balancing masses to the crank-shaft, it being unnecessary to distinguish between the reciprocating and revolving parts of the valve-gear, since the projection of the centrifugal force due to the proportion of the masses balancing the reciprocating parts on a horizontal plane will cause little effect about a vertical axis.

The processes of this and the preceding two articles may be summarized thus—

(1) Make a schedule in which the equivalent mass M represents the sum of the revolving and reciprocating masses appropriate to each crank in the system, including the eccentrics.

(2) Find the vector sum of the forces and the corresponding sets of couples.

(3) Make a schedule of the reciprocating parts alone belonging to the main cranks of the engine, and draw an end view of the imaginary crank-shaft in which all the main crank angles are doubled. Using the same force and couple scale as in (2), find the vector sum of the forces and couples for these angles, remembering that the lengths scaled, as the sum or closure reversed, is to be multiplied by $\frac{r}{l}$ in each case, to reduce them to the scale of the vectors found under (2).

(4) Combine these vectors, two for the forces and two for the couples, in the way shown in Fig. 135, by the curves *a* and *b* for the couples.

116. Calculation of the Maximum Ordinates of the Components of the Resultant Force and Couple Curves. Extension to any Number of Terms in the General Series.—The "vector sums" of the forces and couples may be calculated directly from the general equations D, Art. 87, the coefficients being introduced which divide out when those equations are equal to 0. Consider the primary force polygon. Equation (1), of the set D, gives the value of the sum of the x components of all the disturbing forces, equation (2), the sum of the y components, $\frac{\omega^2 r}{g}$ being equal to unity. Calling the first sum X, and the second Y, the magnitude of the sum is—

$$\sqrt{X^2 + Y^2}$$

This is, in fact, the length of the closure to the polygon. The direction of this is given by—

$$\frac{\tan^{-1}Y}{X} = a$$

Care must be taken to give the proper signs to the numerator and denominator of this fraction in order to fix the quadrant in which the vector lies. The magnitudes of the four sums are given in the following general form for the sake of reference. The summation sign indicates that a number of terms equal to the number of cranks in the problem is to be taken of the form to which the sign is prefixed.

Magnitude of the maximum unbalanced primary force in lbs. weight—

$$\frac{\omega^2 r}{g}\{(\Sigma Mx)^2 + (\Sigma My)^2\}^{\frac{1}{2}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Direction—

$$\tan \alpha = \frac{\Sigma(My)}{\Sigma(Mx)} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Magnitude of the maximum unbalanced primary couple in foot-lbs.—

$$\frac{\omega^2 r}{g}\{(\Sigma Max)^2 + (\Sigma May)^2\}^{\frac{1}{2}} \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Direction—

$$\tan \beta = \frac{\Sigma(May)}{\Sigma(Max)} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Magnitude of the maximum unbalanced secondary force in lbs. weight—

$$\frac{\omega^2 r^2}{gl}[\{\Sigma M(2x^2 - 1)\}^2 + \{\Sigma(M2xy)\}^2]^{\frac{1}{2}} \quad . \quad . \quad . \quad . \quad (5)$$

Direction—

$$\tan \gamma = \frac{\Sigma(2Mxy)}{\Sigma\{M(2x^2-1)\}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

Magnitude of the maximum unbalanced secondary couple in foot-lbs.—

$$\frac{\omega^2 r^2}{gl}[\{\Sigma Ma(2x^2 - 1)\}^2 + \{\Sigma(2Maxy)\}^2]^{\frac{1}{2}} \quad . \quad . \quad . \quad (7)$$

Direction—

$$\tan \delta = \frac{\Sigma(2Maxy)}{\Sigma\{Ma(2x^2 - 1)\}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

117. Application to the Example of Art. 106.—The values of the x's and y's to be substituted, taking the H.P. crank for the axis of X, are (Fig. 127)—

$$\begin{array}{ll} x_1 = 1 & y_1 = 0 \\ x_2 = 0 & y_2 = -1 \\ x_3 = -1 & y_3 = 0 \\ x_4 = 0 & y_4 = 1 \end{array}$$

The values of the M's are given in Schedule 19.

Substituting these values in equations (1) and (2) of D, Art. 87—

$$\Sigma Mx = -1$$
$$\Sigma My = +0{\cdot}3$$

Substituting these sums in (1) of the previous article—

$$\text{Maximum primary force} = \frac{1{\cdot}04 \times \omega^2 r}{g} \text{ lbs. weight app.}$$

Substituting them in (2)—

$$\text{Direction relatively to the H.P. crank is } \tan^{-1}\frac{0{\cdot}3}{-1} = -0{\cdot}3$$

Similarly, the secondary force error, by substitution in (5) and (6), reduces to—

$$\frac{{\cdot}026 \times \omega^2 r}{g} \text{ lbs. weight}$$

and the direction is coincident with the H.P. crank, since $\Sigma(2Mxy)$ is zero.

The set of expressions for primary effects may be used to find the maximum values of unbalanced forces and couples due to the revolving masses. It will be found, using the numbers from Schedule 20, that the sum of the forces is zero.

Again, substituting the proper values from Schedule 19 in equations (3) and (4) of D, Art. 87—

$$\Sigma Max = +140$$
$$\Sigma May = -182{\cdot}7$$

and these sums substituted in (3) of the previous article give—

$$\text{Maximum primary couple } \frac{230 \times \omega^2 r}{g} \text{ foot-lbs.}$$

Substituting them in (4)—

$$\text{Direction relatively to H.P. crank is } \tan^{-1}\frac{-182{\cdot}7}{+140{\cdot}0}$$

the corresponding angle being 308°. This vector is shown by OC (Fig. 133).

For the secondary couples, substituting the proper values in equations (7) and (8) of D, Art. 87—

$$\Sigma Ma(2x^2 - 1) = 181{\cdot}3$$
$$\Sigma Ma\, 2xy = 0$$

Substituting these values in (7) of the previous article—

$$\text{Maximum secondary couple is } \frac{46{\cdot}5 \times \omega^2 r}{g} \text{ foot-lbs.}$$

The direction of the vector is coincident with the H.P. crank. It is shown by OD (Fig. 133).

Similarly, the couple due to the revolving masses may be found by using the proper quantities from Schedule 20 in expressions (3) and (4) of D, Art. 87, and (3) and (4) of the preceding article. The valve-gear may, of course, be treated in the same way.

Thus the maximum values of the component curves can be calculated directly for any given arrangement of cranks and magnitudes of parts. From these values a very good idea can be formed of the engine as to the balance. Proceeding further and calculating the directions, all the data are obtained for Fig. 133, and the resultant curves may be built up from these vectors in the way already explained.

118. General Formulæ for Typical Cases.—Let A represent $\frac{\omega^2 r}{g}$, and let B represent $\frac{\omega^2 r^2}{gl}$.

Type 1.—Three-crank engine, the cranks all being in the same plane, the masses being proportioned by Art. 93, so that the primary force and couple polygons are closed, but the secondary force and couple polygons are open to an unknown extent. The values to be used in the general equations of Art. 116 are, taking the reference plane at the central crank—

$$x_1 = 1 \qquad y_1 = 0$$
$$x_2 = -1 \qquad y_2 = 0$$
$$x_3 = 1 \qquad y_3 = 0$$
$$M_1 a_1 = M_3 a_3$$
$$M_1 + M_3 = M_2$$

The closure for the secondary force polygon reduces to—

$$2BM_1\left(\frac{a_1 + a_3}{a_3}\right) \text{ lbs. weight} \quad . \quad . \quad . \quad . \quad . \quad (1)$$

in terms of the pitch of the cylinders; or—

$$2B(M_1 + M_3), \text{ or } 2BM_2 \text{ lbs. weight} \quad . \quad . \quad . \quad . \quad (2)$$

in terms of the masses.

The length of the closure to the secondary couple polygon depends upon the position of the reference plane, since there is secondary force error (Theorem 1, Art. 90).

If the reference plane is taken at the central crank, the secondary couple polygon closes, and there is, therefore, no error; if it be taken at an outer crank, the closure reduces to $2BMa$ foot-lbs., the M and a being respectively the mass and the distance, corresponding to the crank farthest from the reference plane.

Type 2.—Three-crank engine, arranged as in Art. 92.

Taking the reference plane at the central crank, the quantities to be substituted in the general equations, Art. 116, are—

$$x_1 = 1 \qquad y_1 = 0$$

$$x_2 = -\tfrac{1}{2} \qquad y_2 = \frac{\sqrt{3}}{2}$$

$$x_3 = -\tfrac{1}{2} \qquad y_3 = -\frac{\sqrt{3}}{2}$$

$$M_1 = M_2 = M_3$$

In this case the primary and secondary force polygons are both closed, the remaining two polygons being open.

The closure for the primary couple polygon reduces to—

$$AM\sqrt{a^2_1 + a^2_3 + a_1a_3} \text{ foot-lbs.} \quad . \quad . \quad . \quad . \quad (3)$$

and for the secondary couple polygon to—

$$BM\sqrt{a^2_1 + a^2_3 + a_1a_3} \text{ foot-lbs.} \quad . \quad . \quad . \quad . \quad . \quad (4)$$

If the cylinders are equally spaced, so that $a_1 = a_3$, these two closures reduce respectively to—

$$AMa\sqrt{3}, \text{ and } BMa\sqrt{3} \text{ foot-lbs.} \quad . \quad . \quad . \quad . \quad (5)$$

The magnitudes of both the primary and secondary couple errors are constant, because both the primary and secondary force polygons close.

Type 3.—Symmetrical four-crank engine, arranged as in Art. 96, illustrated by Figs. 110 and 111.

The general data in this case, the reference plane being at the centre, are—

$$x_1 = x_4 \qquad y_1 = -y_4$$
$$-x_2 = -x_3 \qquad y_3 = -y_2$$
$$a_1 = -a_4$$
$$a_2 = -a_3$$

The only open polygon is that representing the secondary couples; the other three are closed, by the conditions of the problem. The closure reduces to—

$$4B(M_1a_1x_1y_1 + M_2a_2x_2y_2) \text{ foot-lbs.} \quad . \quad . \quad . \quad . \quad (6)$$

Referring to equation (10), Art. 96, it will be seen that—

$$x_1^2 = \frac{M_2}{2M_1}$$

By means of this relation, and—

$$a_2 = -\frac{M_1a_1y_1}{M_2y_2}, \text{ from equation (5), Art. 96}$$

and—

$$x_2 = -\frac{1}{2x_1}, \text{ from equation (7), Art. 96}$$

the x's and y's may be eliminated from the above expression for the error; for, substituting these values for x_2, a_2, and putting $\sqrt{1 - x_1^2}$ for y_1, and multiplying and dividing by x_1, where necessary, to obtain the x_1^2 form, the expression becomes—

$$2Ba_1\left(M_1\sqrt{\frac{1}{x_1^2} - 1} + M_2\sqrt{\frac{1}{x_1^2} - 1}\right) \text{ foot-lbs.}$$

and this, substituting $\frac{M_2}{2M_1}$ for x_1^2, gives—

$$2Ba_1\left\{(M_1 + M_2)\sqrt{\frac{2M_1}{M_2} - 1}\right\} \text{ foot-lbs.} \quad . \quad . \quad . \quad (7)$$

an expression of the same form as that given by Herr Schlick in the paper already quoted.

If $b = \frac{a_1}{a_2}$, this expression may be written—

$$2Ba_2M_1b\left(1 + \frac{M_2}{M_1}\right)\left(\sqrt{\frac{2M_1}{M_2}} - 1\right) \text{ foot-lbs.} \quad . \quad . \quad . \quad (8)$$

a convenient form for use when $2a_2$, M_1, and B are each put equal to unity (see Art. 98, and Fig. 112).

Type 4.—Four-crank engine with cranks at 90°, and equal reciprocating masses.

Take the reference plane at the centre, so that a_1 and a_2 are positive; and a_3 and a_4 are negative. The data are—

$$\begin{array}{ll} x_1 = +1 & y_1 = 0 \\ x_2 = -1 & y_2 = 0 \\ x_3 = 0 & y_3 = +1 \\ x_4 = 0 & y_4 = -1 \end{array}$$
$$M_1 = M_2 = M_3 = M_4$$

Substitute these values in the proper equations, and the following results will be obtained:—

Primary force polygon—

$$\text{Length of closure} = 0$$

Primary couple polygon: the closure reduces to—

$$AM\sqrt{(a_1 - a_2)^2 + (a_4 - a_3)^2} \text{ foot-lbs.}$$

which is equal to—

$$AM(a_1 - a_2)\sqrt{2} \text{ foot-lbs.} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

if the distances between each pair of cranks, which are at 180°, are equal.

The secondary force polygon closes, and the secondary couple polygon reduces to $BM(a_1 + a_2 + a_3 + a_4)$.

This is equal to—

$$2BM(a_1 + a_2)$$

if the distances between each pair of cranks which are at 180°, are equal.

119. Comparative Examples.—It is interesting and instructive to compare the disturbing effect, due to different arrangements of an engine, on the assumption that the magnitudes of the lightest set of reciprocating parts is the same in each case, and that the engine cylinders are spaced a given distance apart. For this purpose, assume the following data:—

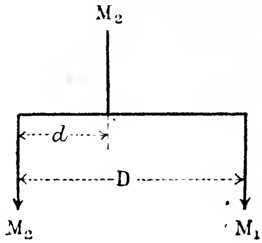
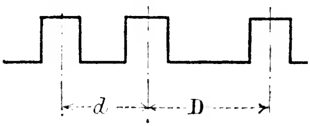
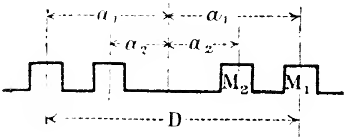
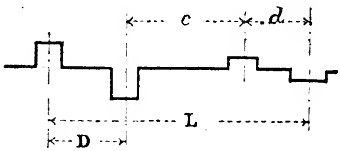
Mass of the lightest set of reciprocating parts = 5 tons
Crank radius = 2 feet. Cylinders, 16 feet pitch
Ratio of crank to rod = 1 : 3·5
Revolutions per minute = 88

Then—

$$\omega = 9{\cdot}2 \text{ approximately}$$
$$\omega^2 = 84{\cdot}5$$
$$\frac{\omega^2 r}{g} = 5{\cdot}25 = \mathbf{A}$$
$$\frac{\omega^2 r^2}{gl} = 1{\cdot}5 = \mathbf{B}$$

The force errors may be compared with the maximum disturbing force, due to the reciprocation of the lightest mass, with simple harmonic motion, which, in the case under discussion, is $\frac{5\omega^2 r}{g} = 26{\cdot}25$ tons weight.

The different results are set forth in Schedule 21. The general formulæ for the errors will be found in the first horizontal row, corresponding to each type. The second row gives the forms the formulæ assume when the several pitches of the cylinders are equal. The figures in the third row are the numerical results corresponding to the above data.

Type.	Primary force error.	Primary couple error.
	Tons. 0	Foot-tons. 0
CRANKS AT 120°. 	0	$AM\sqrt{D^2 + d^2 + Dd}$ $AMD\sqrt{3}$ 728
SYMMETRICAL ARRANGEMENT. 	0	0
CRANKS IN 180° PAIRS AT 90°. 	0	$AM\sqrt{D^2 + d^2}$ $AMD\sqrt{2}$ 594
5- and 6-crank engines arranged as in Figs. 105, 106, 107, and 108.	0	0

Revolutions per minute, 88. A = 5·25. B = 1·5. Disturbing

Secondary force error.	Secondary couple error.
Tons.	Foot-tons.
$2BM_1\frac{D}{d}$, or $2B(M_1 + M_3)$ $4BM_1$ 30	Variable. Reference plane at centre, error = 0, at No. 1 = 480.
0	$BM\sqrt{D^2 + d^2 + Dd}$ $BMD\sqrt{3}$ 208
0	$B(M_1a_1r_1y_1 + M_2a_2r_2y_2)$ $BD\left\{(M_1 + M_2)\sqrt{\frac{2M_1}{M_2} - 1}\right\}$ $1{\cdot}28M_1DB$ 465
0	$BM(L + c)$ $BM(L + c)$ 480
0	0

force due to the lightest mass of 5 tons is 26·25 tons weight.

Although the magnitudes of the unbalanced force and couple of a given engine of any one of the types considered may readily be inferred from Schedule 21, yet nothing can be predicted about the behaviour of the support or foundation of the engine from this knowledge alone. On some supports the engine may run without causing vibration enough to be troublesome, even though there is a large unbalanced couple. Place the same engine on another support, and the vibration it causes may be exceedingly great. Again, a support may be sensitive to a small unbalanced force, and at the same time undisturbed by a large unbalanced couple, and *vice versâ*. It is always necessary, therefore, when considering the effect likely to be produced by an unbalanced engine, to have in mind the general principles governing the vibration of supports under the action of an external force or couple. A short account of the main features of this subject is given in the next chapter.

CHAPTER VII.

THE VIBRATION OF THE SUPPORTS.

120. Preliminary.—The foundation or support of an engine or machine is in general an elastic system susceptible of vibrating in a variety of ways. If any part of the system is displaced from its position of equilibrium by the action of an external force which suddenly ceases to act, the system, in returning to its position of rest, overshoots the mark, and begins to vibrate in one of the ways peculiar to it. The energy of the vibration is gradually dissipated in heat, and by communication to the surrounding medium. If the application of the disturbing force is repeated at regular intervals, that is to say, if a periodic force acts on the system, it is compelled to vibrate in a way foreign to it, the period of this forced vibration being equal to the period of the force producing it. This period is in general different to the period of any of the natural modes of vibration of which the system is susceptible if disturbed and then left to itself. If the periodic time of the disturbing force should happen to be equal to the periodic time of any one of the many natural modes of vibration peculiar to the system, a large disturbance is produced, even though the magnitude of the force producing it is extremely small. The work done by the force in displacing the system from its position of rest appears as the energy of the vibration; if the next application of the force is so timed that it begins to act just at the time that the vibration caused by the first application is about to repeat itself, it is able to communicate another small amount of energy to the system without interfering with the amount communicated by its first action. If this timed action of the force be continued indefinitely, energy is gradually accumulated in the system to such an extent that the consequent vibration may be sufficient to break it down.

An unbalanced engine or machine applies a periodic force and couple to its supports, compelling them to execute forced vibrations, which may be of small amplitude and little consequence, though the force and couple acting may be large. If, however, the support possesses amongst its many natural modes of vibration, one whose period is equal or nearly equal to the time of revolution of the engine or machine bolted to it, then the amplitude of the force vibration is large, and out of all proportion to the force producing it. In this way the engines of an electric light station, though bolted to large concrete blocks, may set up vibrations in the ground in which the block is embedded, which may be transmitted all round the neighbourhood. The hull of a steamer may be thrown into violent vibration to the discomfort of the passengers and the injury of the ship. An unbalanced machine bolted to a shop floor may shake the whole building, though the actual value of the force and couple may be insignificant. A locomotive may be thrown into violent and dangerous oscillation if the swaying couple coincides in period with the natural period of oscillation of the engine on its springs. A carriage will ride roughly if the speed of the train is such that the interval of time between the blows from the fish-joints is equal to a periodic time of one or more of the loaded carriage springs. The precise investigation of even the simpler problems of this kind is difficult. A great deal of instruction, however, may be obtained from the detailed consideration of the natural mode of vibration of the simplest kind of support and its behaviour under the action of a periodic force.

121. Natural Period of Vibration of a Simple Elastic System.—Let a mass M be supported by a steel bar whose mass is negligibly small compared with M; and let the bar rest in two closed V's (Fig. 136). The bar is the support whose behaviour is to be examined. One of the objects in the investigation of the vibration of such a system is to find an expression which will give the displacement of the point *c*, relatively to its position of rest, at the end of any time after the commencement of the vibration. The assumption is made that the system is only free to vibrate in a vertical plane, and that the displacements are not so great that the bar or spring is strained beyond the elastic limit. The magnitude of the displacement will be fixed by one variable, which will be denoted by y. This symbol will, therefore, always stand for the distance of the centre of the mass M from its position

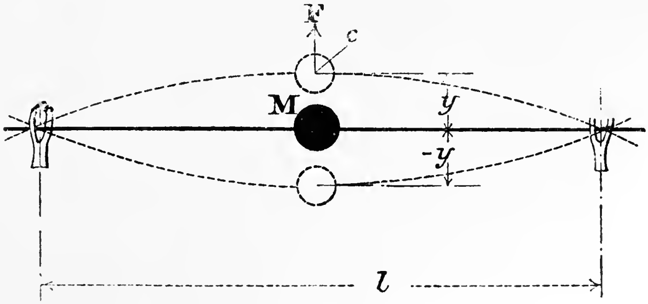


FIG. 136.

of rest. The equations are formed from the familiar fundamental form—

Force = mass × acceleration

In the problems under discussion the force and the acceleration are continuously variable and functions of the time. Newton's method of indicating differentiation with respect to the time is used, so that—

$$y = \text{the displacement of } c \text{ from its position of rest}$$

$$\dot{y} = \frac{dy}{dt} = \text{the instantaneous value of } c\text{'s velocity}$$

$$\ddot{y} = \frac{d^2y}{dt^2} = \text{the instantaneous value of } c\text{'s acceleration}$$

It may be noticed that the problem is just the converse of that in Art. 78. There the displacement x is fixed for a given crank angle, which, of course, is a function of the time, by the mechanism itself, and it is desired to find an expression for the acceleration. This is accomplished by two differentiations. In the problems about to be discussed the acceleration is stated in terms of the force, and the displacement is required in terms of the time. This involves two integrations to get to the displacement equation. In what follows the equations will be stated and their solutions given, because the interest lies in their solution and not in the method of obtaining it. The equations all belong to the type known as linear differential equations with constant coefficients, and rules for their solution are to be found in any work on differential equations.* The solutions may always be verified by substituting the value of y in the original equation and differentiating.

Returning to the problem, let F be the force in absolute units, acting at c, which produces the deflection y. This force varies as the deflection, and therefore $\frac{F}{y}$ is the force which must be applied to produce unit displacement of one foot. This may be found experimentally or by calculation from the formula—

$$\frac{F}{y} = \frac{48EI}{l^3}$$

for the case under consideration, E being Young's modulus, and I the moment of inertia of the section of the bar about the neutral axis, l the distance between the V's.

Let $\frac{F}{y}$ be represented by μ. Then when the deflection is y, the accelerating force is given by $M\ddot{y}$. This force is also equal to $-\mu y$, the minus sign being introduced because the force acts to oppose the motion. Hence the equation of motion is—

$$M\ddot{y} + \mu y = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

of which the solution is—

$$y = a \cos (qt - \alpha) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

The two constants, a and α, are determined by the initial conditions of the motion, and $q = \sqrt{\frac{\mu}{M}}$.

Thus, if the mass of M be 50 pounds, and it requires a force of

* "Calculus for Engineers," Perry. London, 1897. "Integral Calculus," Edwards. London, 1894.

10 lbs. weight = $10g$ poundals, to produce a deflection of 0·0071 feet, μ, the force which will produce unit deflection, is—

$$\frac{10}{0\cdot0071} \times 32\cdot2 = 45{,}352 \text{ poundals}$$

q therefore is 30·14.

To find the constants a and α, the initial conditions of the motion must be specified in some way. Suppose that the system be held at 0·1 foot displacement, and that the time be counted from the instant it is let go, then y, the displacement, is 0·1 foot when t is nothing, and the constant α is 0, since the angle is nothing when t is nothing; substituting these values of t, y, and α in equation (2), $a = 0\cdot1$. Substituting the values of a and q in equation (2), the displacement for any given time t is—

$$y = 0\cdot1 \cos 30\cdot14t \text{ feet}$$

The displacement curve for this free vibration is drawn and calibrated in Fig. 137.

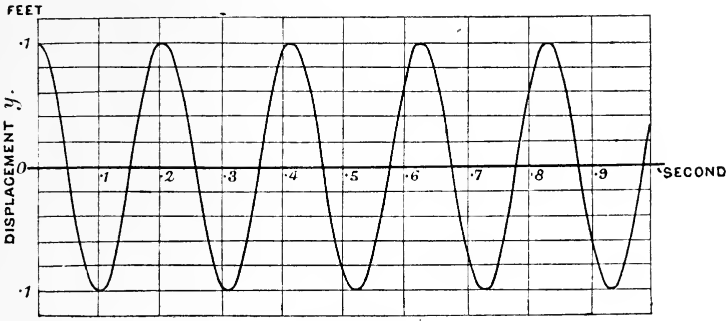


FIG. 137.

The cosine of an angle repeats itself exactly if the angle be increased by 2π. Reverting to equation (2), all the circumstances of the motion will be the same, therefore, when—

$$(qt_2 - \alpha) - (qt_1 - \alpha) = 2\pi$$

from which—

$$t_2 - t_1 = \tau = 2\pi\frac{1}{q}$$

giving on substitution of the value of q, the well-known formula—

$$\tau = 2\pi\sqrt{\frac{M}{\mu}}$$

τ being called the periodic time of the vibration.

The system under consideration will, therefore, execute vibrations in periodic time—

$$\frac{2\pi}{30 \cdot 14} = 0 \cdot 208 \text{ seconds}$$

or its frequency will be 4·8 vibrations per second.

This vibration is the principal one natural to the system, and whatever be the magnitude of the force producing the initial displacement, providing always that it does not strain the system beyond the limit of its elasticity, the vibrations following its removal will always have the same periodic time, though the amplitude will depend upon the magnitude of the force starting the motion.

122. Damping.—The successive ordinates denoting the maximum amplitude of the vibration in Fig. 137 are shown equal. This could only be true if none of the energy of the vibration were lost. In any actual system, part of the energy is gradually frittered away in heat, partly through the imperfect elasticity of the bar, partly through the frictional resistances between the surface of the system and the medium in which it is vibrating, another part being communicated to the environment, so that eventually the system is brought to rest. In many cases the loss through frictional resistance is very great relatively to the other causes of loss, and its effect in modifying or damping the amplitude of the vibration is considerable. In some cases it is artificially increased by means of dash-pots and such-like apparatus. Frictional resistance is equivalent to a force acting to oppose the motion, and its magnitude may be assumed proportional to the velocity of the mass M in the simple case under discussion. In the consideration of the motion of the system (Fig. 136), the effect of the resistance at the surface of the bar may be neglected. The frictional force at any instant may therefore be written $-\delta\dot{y}$, δ being a constant determined by experiment. The force acting to retard the motion is now—

$$-\delta\dot{y} - \mu y$$

and the equation of motion is—

$$M\ddot{y} + \delta\dot{y} + \mu y = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

the solution of which is—

$$y = Ae^{-\frac{1}{2}bt}\cos\{\sqrt{(q^2 - \tfrac{1}{4}b^2)}t - a\} \quad . \quad . \quad . \quad . \quad (4)$$

where $b = \dfrac{\delta}{M}$, and $q = \sqrt{\dfrac{\mu}{M}}$, A and a being constants determined by the initial circumstances of the motion. If $\frac{1}{4}b^2$ is less than q^2, so that the quantity under the root is positive, the motion is oscillatory, otherwise the displaced mass M returns gradually towards its position of equilibrium, but does not overshoot it.

Fig. 138 is a copy to scale of a displacement curve which

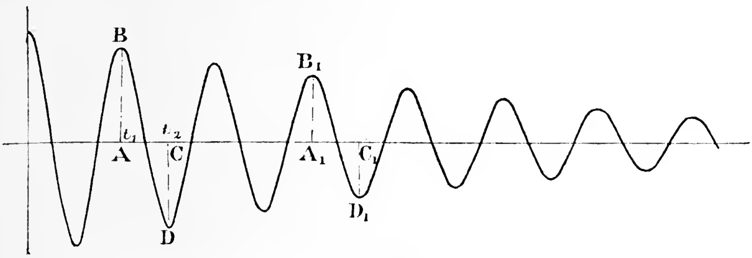


FIG. 138.

was automatically drawn by a damped system of the simple kind under consideration, and in which the mass was 6·83 pounds and the support was a spiral spring. Equation (4), with proper constants, may be taken to represent this curve. It is an interesting exercise to determine the values of these constants from the curve. This incidentally indicates a method of experimentally finding δ and μ. The time of a complete vibration is represented by twice the distance AC or A_1C_1. This was observed to be 1·46 seconds. The isochronous character of the motion can be verified by measuring the series of intercepts made by the curve along the axis. At the point A, midway between the points where the curve crosses the axis, the angle is such that its cosine is unity, whatever may be the value of the individual quantities in the brackets of equation (4). Similarly, after 0·73 seconds, the angle has changed by π, and its cosine is -1. Therefore the ratios—

$$\frac{C_1D_1}{A_1B_1} = \frac{CD}{AB}, \text{ etc.}, = \frac{Ae^{-\frac{1}{2}bt_2}}{Ae^{-\frac{1}{2}b}} = e^{-\frac{1}{2}b(t_2-t_1)}$$

Measuring several pairs of successive ordinates the ratio was found to be 0·91. And $t_2 - t_1 = 0{\cdot}73$, therefore—

$$-0{\cdot}36b = \log_\epsilon 0{\cdot}91$$

from which—

$$b = 0{\cdot}262$$

The mass being 6·83 pounds—

$$\delta = 0{\cdot}262 \times 6{\cdot}38 = 1{\cdot}67$$

This means that the frictional resistance to motion is 1·67 poundals when the velocity is unity.

Again, choosing the origin of co-ordinates at a point midway between two successive zero values of y, so that $\cos\{(\sqrt{q^2 - \frac{1}{4}b^2})t - a\} = 1$; when $t = 0$, $a = 0$.

The coefficient of t is found from the consideration that for a change in the time 0·73 seconds the change in the angle is π. Hence—

$$0{\cdot}73\sqrt{q^2 - \tfrac{1}{4}b^2} = \pi$$

In this $b = 0{\cdot}262$. Solving for q^2—

$$q^2 = 18{\cdot}54$$

and since this $= \frac{\mu}{M}$, and $M = 6{\cdot}38$ pounds—

$$\mu = 118 \text{ poundals}$$

meaning that a force of 118 poundals will displace the system 1 foot from its position of rest.

To find A, measure a pair of simultaneous values of y and the angle. One pair of values is $y = {\cdot}094$ feet, when $t = 0$. Substituting these in equation (4), it at once reduces to—

$$A = {\cdot}094$$

Hence—

$$y = 0{\cdot}094e^{-0{\cdot}13t}\cos(4{\cdot}3t) \text{ feet}$$

123. Vibration of the System under the Action of a Periodic Force.—Suppose now that the mass M (Fig. 136) is an engine, self-contained, whose crank-shaft makes n revolutions per second. Let there be an unbalanced force in it whose maximum value is E poundals. Whether this be from revolving or reciprocating parts, the unbalanced force in the vertical plane, which is all that is under consideration at present, will have the value—

$$\text{E}\cos pt$$

where $p = 2\pi n$. As in the previous case, suppose the frictional forces resisting motion to be expressed by $\delta\dot{y}$. Then the sum of the forces acting to cause motion is—

$$-\delta\dot{y} - \mu y + \text{E}\cos pt$$

and the equation of motion is—

$$\text{M}\ddot{y} + \delta\dot{y} + \mu y = \text{E}\cos pt \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

The solution of this is—

$$y = \frac{\text{P}\sin\varepsilon}{pb}\cos(pt - \varepsilon)\ ^{*} \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

where the value of ε, which fixes the phase of the motion, is given by—

$$\tan\varepsilon = \frac{pb}{q^2 - p^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

In these expressions $b = \frac{\delta}{\text{M}}$, and $q = \sqrt{\frac{\mu}{\text{M}}} = 2\pi n_1$, $\text{P} = \frac{\text{E}}{\text{M}}$.

When $q = p$, the revolutions of the engine per second are equal the number of oscillations per second natural to the system, since $p = 2\pi n$ and $q = 2\pi n_1$. Under these circumstances, $\tan\varepsilon$ = infinity, and therefore $\varepsilon = 90°$, so that $\sin\varepsilon = 1$ and $\cos\varepsilon = 0$. Equation (6) then becomes—

$$y = \frac{\text{P}\sin pt}{pb} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

* Equation (4), page 205, should strictly be added to this equation, but this damped oscillation soon disappears if it is present initially.

If b is small, y becomes large; in fact, y tends to infinity as b tends to zero. This shows that when b is small, and when $q = p$, the engine could set up vibrations of sufficient amplitude to break down its supporting rod altogether. This equation is one of great importance, and it should be carefully studied, trying the effect on y of different values of b, and gradually approaching values of p and q.

The effect of the frictional resistance in modifying the maximum amplitude of the forced vibration is illustrated by the diagram (Fig. 139). Considering a system similar to Fig. 136, whose natural

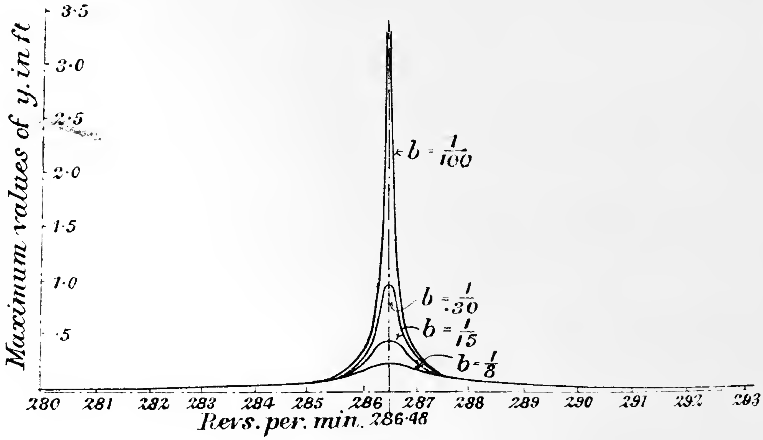


Fig. 139.

number of vibrations per minute is 286·48, suppose the speed of the engine, assumed contained in the mass M, to be gradually increased from 283 to 291 revolutions per minute. Further, suppose P = 1, and that b has the different values shown in the figure against the corresponding curves (Fig. 139). The maximum amplitude of the forced vibration, calculated from equations (7) and (6) of the present article, are shown by the ordinates of the curves, for the different speeds and for the different values of b. When the engine is running at 283 revolutions per minute, the maximum values of the amplitudes are practically the same for all values of b taken, and they are relatively insignificant in amount. At 286 revolutions per minute the effect of the different values of b is distinctly shown. At the synchronizing speed,* b exerts its maximum influence. It

* See Appendix II.

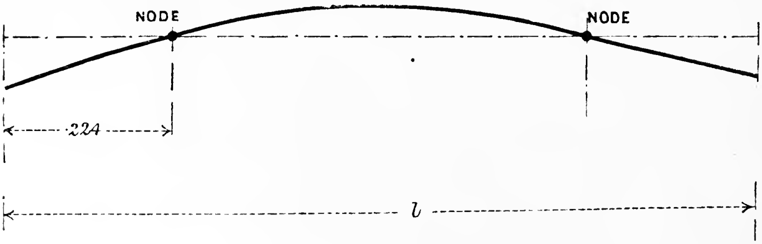
should be remembered in considering this diagram, that if $b = 0$, the maximum amplitude of the forced vibration at the synchronizing speed is infinite. Beyond the synchronizing speed the curves rapidly drop again to insignificant values of y. This indicates how it is that a dangerous speed may be run through to higher speeds at which the disturbance becomes negligible. These curves, though only illustrating an example of the simplest kind, show how necessary it is to include the frictional resistance into any actual problem to get even an approximate result. At the synchronizing speed the phase difference is 90°. Below 285 revolutions per minute the phase difference is small; above 288 revolutions per minute it is nearly 180°. When $b = \frac{1}{30}$, the phase differences corresponding to different speeds are shown below.

Revs. per min.	Phase difference.	
284	3° 48′	
285	5° 3′	
286	16° 51′	
286·3	41° 14′	
286·48	90°	Synchronism.
286·6	126° 51′	
286·8	153° 44′	
287	162° 26′	
288	172° 51′	
292	178° 21′	

Thus within a few revolutions of synchronism the phase difference is practically nothing if p is less than q, and about 180° if p is greater than q.

124. Natural Vibrations of an Elastic Rod of Uniform Section.— The vibrating system of the last three articles consists of a single mass whose motion is controlled by a steel bar. The mass of the bar itself has been neglected. Consider now that the system consists simply of a steel rod, so that the forces called into play when any point of it is displaced from the position of equilibrium act on the mass of the bar only. This is a much more complex system to deal with than the first one. The bar possesses several natural modes of vibration. The first three, corresponding to the gravest periodic time, and the next two higher, are shown in Figs. 140, 141, and 142. The points marked with the dots are the nodes, or places of rest for that particular vibration, though they need not be points of absolute rest, since all these separate modes of vibration may exist simultaneously. The respective numbers of vibrations per second corresponding to successive modes of division are very nearly in the ratios of the squares of the odd

FIG. 140.



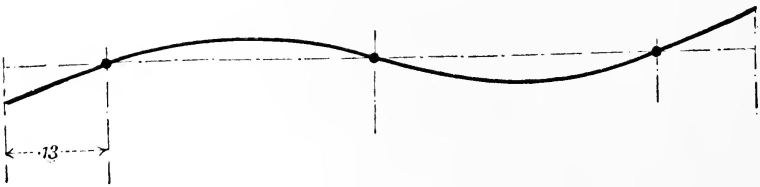


FIG. 141.

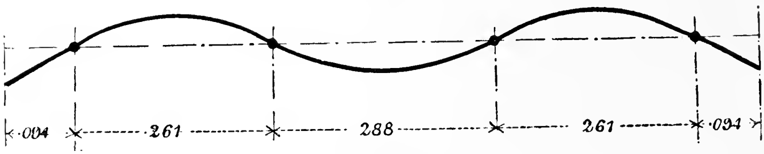


FIG. 142.

numbers. The next three lines show the characteristics of these natural modes of vibration.

Number of nodes ...	2	3	4	5	6	7
Frequency ratio	3^2	5^2	7^2	9^2	11^2	13^2
Ratios of periodic times	1	0·36	0·183	0·111	0·074	0·053

The gravest frequency may be calculated from the formula—

$$n = \frac{22{\cdot}4\kappa}{2\pi l^2}\sqrt{\frac{E}{\rho}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where ρ is the density of the material, E Young's modulus, l the length in feet, κ the radius of gyration of the section about an axis perpendicular to the plane of bending.

125. On the Point of Application of a Force and the Vibrations produced.—If the rod considered in the last article be supported in such a way that it is free to vibrate in any of its natural modes, the application of an external force, whose periodic time is equal to or is a multiple of any of the natural periods of the bar, is capable of forcing vibrations of considerable magnitude of that period, corresponding with the case of Art. 123. It is a fundamental principle that the point displaced by the action of the force cannot be a node in the subsequent vibration, so that all modes of vibration requiring the point of application for a node disappear. Advantage is taken of this principle in fixing the point at which the hammer shall strike a piano string. A point is chosen which would form a node in a mode of vibration inharmonious with the principal modes of vibration of the string. The act of striking the string at this point eliminates those particular modes from the note. Suppose now that the period of the applied force acting on the bar is equal to the gravest mode of vibration natural to the bar. If the force be applied at either of the corresponding nodes, it cannot induce the vibration, though applied at any other part of the bar large forced vibrations will result.

If a couple be applied at the bar, it can force vibrations of its period if applied at a node of that period, but not if applied midway between a pair of nodes. These two principles will perhaps be more clearly understood if the matter is considered in another way. When the bar is vibrating in any one of the modes peculiar to it, every point is constrained to move in a certain way. A point midway between a pair of nodes moves vertically in a straight line, and the element of length surrounding this point has no freedom to turn. The point forming a node is not free to move, but the element of length in its neighbourhood is free to turn about an axis through the node. Motion in a straight line is produced by the action of a force, turning about an axis by the action of a couple, and neither can produce the effect of the other. Hence a periodic force in agreement with a natural period of the bar is unable to force the corresponding mode of vibration if it is applied at a node belonging to that mode. Neither can a periodic couple in agreement force the corresponding vibration, if it is applied midway between a pair of nodes corresponding to the system. If the engine of Art. 123 have a period of revolution in agreement with one of the natural mode of vibration of the bar under consideration, it would still be possible to prevent it forcing the corresponding vibration on the bar by properly choosing its position of attachment to the bar, supposing it had either an unbalanced force or an unbalanced couple. These principles may be studied practically by suspending the model already described in Art. 54, from a flexible bar, itself supported in hooks hanging freely from a cross-bar, so that their distance apart may be varied quickly. At a synchronizing speed of the model the part of the bar projecting beyond the frame may be thrown into vibration with nodes which can be made to disappear by an increase or decrease of the speed. The model can be adjusted to give either a force or a couple.

126. Longitudinal and Torsional Vibrations.—In addition to the lateral vibrations already considered, the bar is susceptible of vibrating in the direction of its axis and about its axis, these being called longitudinal and torsional modes of vibration respectively. Corresponding to each kind, there is a fundamental vibration with one node at the centre and a series of higher vibrations with two, three, etc., nodes. The nodes and the corresponding ratios of the frequencies and periodic times are given in the following lines:—

Nodes	1	2	3	4	5
Ratios of frequencies	1	2	3	4	5
Ratios of periodic times ...	1	·5	·333	·25	·2

The frequency of the gravest period is given in the case of longitudinal vibrations by—

$$n = \frac{1}{2l}\sqrt{\frac{E}{\rho}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

and for torsional vibrations by—

$$n = \frac{1}{2l}\sqrt{\frac{C}{\rho}} . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

where ρ is the density, E Young's modulus, C the modulus of rigidity, l the length in feet.

Example.—Let l be 10 feet; E, 30,000,000 × 144 × 32 poundals per square foot; ρ, 490 pounds per cubic foot. Introducing these values in equation (2), n, the frequency of the gravest longitudinal mode of vibration, is 842 per second. The gravest frequency of the torsional vibrations would be less than this in the ratio of $\sqrt{E}$ to $\sqrt{C}$. For hard steel, E : C about in the ratio 1 : ·4, consequently the ratio of the frequencies is 1 : ·63, or the gravest torsional vibration corresponding to one node at the centre is 530 per second.

The gravest vertical vibration may be calculated from equation (1), Art. 124, when the diameter of the bar is given. Suppose it to be ·2 feet diameter, then κ, the radius of gyration, will be ·05. Substituting this and the data of the example in the equation, the gravest frequency is found to be 30 vibrations per second. The higher frequencies follow the ratios given in Art. 124. Summarizing the results—

	Vertical.	Longitudinal.	Torsional.	
Gravest frequency ...	30	842	530	Vibrations per second
Frequency of next mode	83	1684	1060	
Frequency of third mode	163	2526	1590	
Frequency of fourth mode	270	3368	2120	

A periodic force or couple agreeing with the periodic time of any of these modes of vibration, and applied in the proper manner to the bar, is capable of exciting large vibrations of that period.

127. Simultaneous Action of Several Forces and Couples of Different Periods.—The effect of the simultaneous action of a set of forces is indicated by the following quotation from Lord Rayleigh's "Sound," Vol. I. p. 49 : "From the linearity of the equations it follows that the motion resulting from the simultaneous action of any number of forces is the simple sum of the motions due to the forces taken separately. Each force causes the vibration proper to itself, without regard to the presence or absence of any others. The peculiarities of a force are thus in a manner transmitted into the system. For example, if the force be periodic in time τ, so will the resulting vibration. Each harmonic element of the force will call forth a corresponding harmonic vibration in the system. But since the retardation of phase ϵ and the ratio of the amplitudes are not the same for the different components, the resulting vibration, though periodic in time, is different in character from the force. It may happen, for instance, that one of the components is isochronous, or nearly so, with the free vibration, in which case it will manifest itself in the motion out of all proportion to its original importance." From this it appears that the vibration for each periodic force is to be found as though it alone acted. If the displacement of any point from its position of rest be examined, its magnitude and position at any instant will be the vector sum of the several displacements which would be caused by each force acting separately.

A knowledge of the principles of the preceding articles of this chapter will often indicate a way in which troublesome vibrations of foundations or supports may be minimized.

128. Possible Modes of Vibration of a Ship's Hull and the Forces present to produce them.—A ship's hull is an elastic structure susceptible of vibrating in all the different modes which have been considered for the solid bar, although the positions of the nodes and the corresponding periodic times of vibration are very different. Exact mathematical treatment is impossible. The different parts of the hull are loaded differently at different times, and since the loads have to share the vibrations, they must be included even in an approximate treatment of the problem. Any resulting state of vibration may be analyzed into the following components:—

(1) Vibrations in a vertical plane after the manner of the rods in Figs. 140 to 142, though the nodes will not be in the same relative positions as there shown.

(2) Vibrations in a horizontal plane.

(3) Longitudinal vibrations.

(4) Torsional vibrations.

The two latter types approximate much more closely to the modes of the rigid bar than the former two. Their frequencies are, however, so great relatively to the frequency of the engine that synchronism is a remote contingency.

The periodic forces acting to throw the hull into vibration are—

(1) The unbalanced force and couple due to the motion of the

reciprocating parts of the engine. These force oscillations in a vertical plane.

(2) The unbalanced force and couple from the revolving parts of the engine. The horizontal and vertical components of these force vibrations in a horizontal and vertical plane respectively.

(3) The variation of turning moment on the propeller shaft. This tends to force torsional oscillations.

(4) Corresponding to the variation of turning moment, there is a variation of the thrust on the propeller.

(5) The variation of thrust due to a partially immersed propeller.

(6) Want of symmetry in the propeller, and slight variations of the pitch of the blades.

With all these exciting forces in simultaneous activity, each producing a forced vibration proper to itself, the hull is thoroughly searched for any of its natural modes of vibration of corresponding periodic time. Any that are found are immediately exalted in amplitude above all the rest of the co-existing forced vibrations, and as a result the ship is thoroughly uncomfortable to voyage in. This agreement in periodic time may take place at relatively slow speeds—for instance, a ship may be in violent vibration at half speed, and quite comfortable at full speed, or the oscillations may be unbearable at 80 revolutions per minute, and insignificant at 90. It has been shown by experiment, and may be predicted from theoretical considerations, that the unbalanced forces from the engines are the most important agents in producing forcing oscillations. The magnitude of the force or the couple may be large, and in addition the engine may be placed in just that part of the ship, relatively to the nodes, most favourable for forcing the oscillations.

There is a peculiarity in the vibration of twin-screw steamers which may be mentioned here. The two engines cannot be run at exactly the same speed. One is continually gaining slightly on the other. Suppose each engine to force a vibration. Then two vibrations of very nearly equal period will be impressed on the hull. The combination of two such vibrations gives a resultant vibration of varying amplitude. One instant the resultant is equal to the sum of the components, at another instant equal to the difference of the component amplitudes, the number of maxima or minima per minute being given by the difference between the number of revolutions of the engines per minute. The phenomenon is similar to that of beats in music. If, for example, the speeds of the two engines are 80 and 81 revolutions per minute, and the maximum amplitudes of the vibration due to each separately are equal, each being a inches, there will be once per minute an amplitude of $2a$ inches, decreasing gradually to 0, and then increasing again to the maximum $2a$. This peculiarity may be illustrated if two curves like the one in Fig. 137 are added to get a resultant curve, the base of one curve, however, being taken slightly longer than the base of the other.

129. Experimental Results.—Mr. Yarrow * has given direct experimental confirmation of the foregoing principles by means of a series of costly and beautiful experiments, in which the actual vibration of the hull of a torpedo-boat was measured under different circumstances by a "Vibrometer." To separate the effect of the propeller from the effect of the engines, vibration diagrams were taken, firstly with the boat under weigh, secondly with the boat at rest and the propeller removed, thereby entirely eliminating whatever vibration it caused. The recorded vibrations were practically alike when the engines ran at the same speed in the two experiments, and this agreement continued in over a hundred similar experiments at different speeds, showing that the effect of the propeller was small, and that the real cause of vibration was to be looked for in the engines. Experiments were then made on a first-class, 23-knot torpedo-boat, 130 feet long, 13 feet 6 inches beam, carrying 20 tons. The reciprocating parts of the three-crank engines were balanced by means of bob-weights (Art. 47), so that the reciprocating system (neglecting the valve-gear) really consisted of five cranks. It was found that 248 revolutions per minute corresponded with a natural mode of vibration of the hull. The experiments were made under three conditions of balancing, in all of which the speed was kept at 248. The amplitude of the vertical vibration at the stem was $\frac{27}{64}$ inch, when no balancing of any kind was used, the cranks being at 120°, and the engines being of the usual design. This was reduced to $\frac{20}{64}$ when balance-weights were attached to the crank-shaft, properly placed to balance the revolving masses only, and to $\frac{7}{64}$ when bob-weights

* "On Balancing Marine Engines, and the Vibration of Vessels." By Mr. A. F Yarrow. *Trans. Inst. Naval Architects.* London, 1892.

were added to balance the reciprocating parts. Thus at a critical speed, the proper balancing of the engine, neglecting the effect of the obliquity of the connecting-rod, reduced the maximum amplitude of the vibrations in the ratio of 27 to 7. The hull being moored and the propeller removed, instantaneous photographs were taken of the ripples on the surface of the water caused by the vibrating hull for each of the three conditions above stated. The ripples indicated the nodes in the hull and the places of maximum vibration, and their decreasing amplitude showed the decreasing amplitude of the hull's vibrations as the balancing of the engines was improved. The photographs are published in *Engineering*, April 5, 1892.

In some observations made by Mr. Schlick on the twin-screw despatch vessel, *Meteor*, belonging to the Imperial German Navy,* the maximum amplitude of the vertical vibration just near the stern post was $\frac{6}{16}$ inch, and the maximum horizontal vibration $\frac{2}{16}$, at a speed of 186 revolutions per minute. At 220 revolutions per minute the amplitudes of both directions were comparatively small, the maximum amplitude being observed at 175 revolutions per minute. At 120 revolutions the vibrations practically disappeared. Mr. Schlick attributed the lateral vibrations recorded by his instrument (described in *Trans. I.N.A.*, 1893) to the effect of torsional oscillations, the stiffness of the ship horizontally being too great to have a natural period corresponding to the speed of the engine.

Mr. Schlick † has devised the following simple formula from his experimental observations, designed to give the number of vibrations per minute of the gravest natural mode of vibration of the hull:—

N = vibrations per minute.
D = the displacement in tons.
L = length of the hull in feet.
I = the moment of inertia of the midship section, in the calculation of which the several areas constituting the section are to be expressed in square inches, and the respective distances of their centres of gravity from the neutral axis in feet.

* "On an Apparatus for Measuring and Registering the Vibrations of Steamers." By Herr Otto Schlick. *Trans. Inst. Naval Architects.* London, 1893.

† "Further Investigations of the Vibrations of Steamers." By Herr Otto Schlick *Trans. Inst. Naval Architects.* London, 1894.

Then—

$$N = \phi\sqrt{\frac{I}{DL^3}}$$

where for—

Vessels with very fine lines, such as torpedo-boat destroyers	$\phi = 156,850$
Large transatlantic passenger steamers with fine lines	$\phi = 143,500$
Cargo-boats with full lines	$\phi = 127,900$

Mr. Schlick * also states that for ships with very sharp lines, like cruisers and despatch-boats, the after node is from 0·23L to 0·25L from the after perpendicular, and the forward node from 0·31L to 0·36L measured from the fore perpendicular, these corresponding to the gravest period of vibration of the hull. The number of vibrations corresponding to the next mode of vibration is sometimes only twice the number of the gravest kind. Referring to Art. 124, it will be seen that for the solid rod the ratio is much higher.

Mr. Mallock † has suggested a method by means of which the natural period of vibration and the position of the nodes in a proposed ship may be approximately found from the behaviour of a plank shaped so that its width is everywhere proportional to the moment of inertia of the corresponding section of a wood model of the hull under consideration, and loaded so that the weight at any cross-section is proportional to the weight at the corresponding cross-section of the model.

In a recent paper to the Institution of Naval Architects, Mr. Schlick ‡ describes some interesting experiments carried out on the s.s. *Deutschland* to ascertain the cause producing the vibrations of the hull at the synchronizing speed. The instrument recording the vibrations was placed at the extreme after end, and at the synchronizing speed of 67 revolutions

* "On Vibrations of Higher Order in Steamers and on Torsional Vibrations." By Herr Otto Schlick. *Trans. Inst. Naval Architects.* London, 1895.

† "On the Vibration of Ships and Engines." By Mr. A. Mallock. *Trans. Inst. Naval Architects.* London, 1895.

‡ "On Some Experiments made on Board the Atlantic Liner *Deutschland* during her Trial Trip, June, 1900." By Herr Otto Schlick. *Trans. Inst. Naval Architects.* 1901.

per minute the maximum amplitude of the vertical vibrations recorded was about $\frac{3}{16}$ inch, a small amount for a ship 662 feet long and 37,000 horse-power. The curve indicated a simple vibration of the same periodic time as the engine. The engines were balanced by Mr. Schlick's method, so that the primary forces and couples in both engines are presumably completely balanced, and there is only a secondary couple unbalanced. Electrical apparatus connected each crank-shaft to the drum of the recording instrument, so that an indication was made when the respective forward cranks of the engines were vertical. A time-line was also drawn, so that from the lines on the drum the revolutions of the engines could be exactly computed, and the position of the cranks relatively to the vibration curve fixed at any instant. It has been shown, in Art. 123, that in the immediate neighbourhood of synchronism, the phase difference between the force and the vibration it causes is about 90°, being exactly 90° at the critical speed. Using this principle, Mr. Schlick inferred that the vibrations produced in the vertical plane were caused by a difference of resistance amongst the blades of the propellers, which difference is attributed to slight differences in their pitch.

130. Turning Moment on the Crank-shaft.—When the forces and couples due to the motion of the parts of an engine have been balanced, there still remains a couple acting on the frame equal and opposite to the turning couple on the crank-shaft. This couple may have a periodic variation sufficiently great to cause vibration. In the case of a ship, whatever be the turning moment or couple exerted by the engine on the propeller shaft, there is of necessity an equal and opposite couple acting on the hull of the ship, which, if the couple is uniform, holds the hull steadily in a position imperceptibly inclined to the vertical. There is a proper position of equilibrium corresponding to every value of the turning couple. A periodic variation of the couple causes an oscillation of the hull about the position of equilibrium corresponding to its average value, that is, to the average value of the turning moment on the shaft, and is thus able to force the hull into torsional vibrations, which may be insignificant or important according as the periodic time of the variation approaches the period belonging to one of the hull's natural modes of torsional vibration. The cause of the oscillations is removed if the turning effort on the shaft is made uniform.

Fig. 143 shows an engine-frame in diagrammatic form; the forces in thick lines are those acting on the frame in consequence of the driving pressures exerted by the fluid pressure P. The couple acting on the frame is represented by R × OB for the crank position shown. Considering the question in more detail, let M be the mass of the reciprocating parts, and a their acceleration. Then, if p is the steam-pressure or gas-pressure per square inch in the cylinder, p_1 the back pressure, and A the area of the cylinder in square inches, the total resultant pressure acting on the piston and equally on the cylinder cover is $(p - p_1)A$. Of this, the part $\frac{Ma}{g}$, in which the acceleration a can be found by Klein's construction (Art. 104), is required for the acceleration of the reciprocating masses; the remainder—

$$(p - p_1)A - \frac{Ma}{g} = P\ *$$

is the pressure acting to produce the turning of the crank. (The continuous variation of P is shown in Fig. 95 for the case of a locomotive running at 65 miles per hour. The indicator cards are shown in Fig. 93. The ordinates of curve No. 1, in Fig. 94, represent the values of $(p - p_1)A$, and those of curve No. 2, $\frac{Ma}{g}$, for one revolution of the crank.) The forces and couples due to the motion of the parts can be balanced by the methods already given; the finding of the effect of P may therefore be considered as a statical problem. Assuming, therefore, that all the inertia effects are balanced, P is kept in equilibrium at the cross-head B (Fig. 143), by the force Q, along the connecting-rod, and the slide-bar reaction R (acting to the right as shown by the dotted force R). The values of these three forces are given by the force triangle *abc* (Fig. 144), where *ab* represents P, and *ca* and *bc* represent respectively the force Q and the reaction R. The equal and opposite value of R, viz. *cb*, is the force acting on the frame from the cross-head. The connecting-rod applies the force Q to the crank-pin K in the direction shown, and its effect with respect to the axis O of the crank-shaft is equal to (Art. 24)—

* See Appendix III.

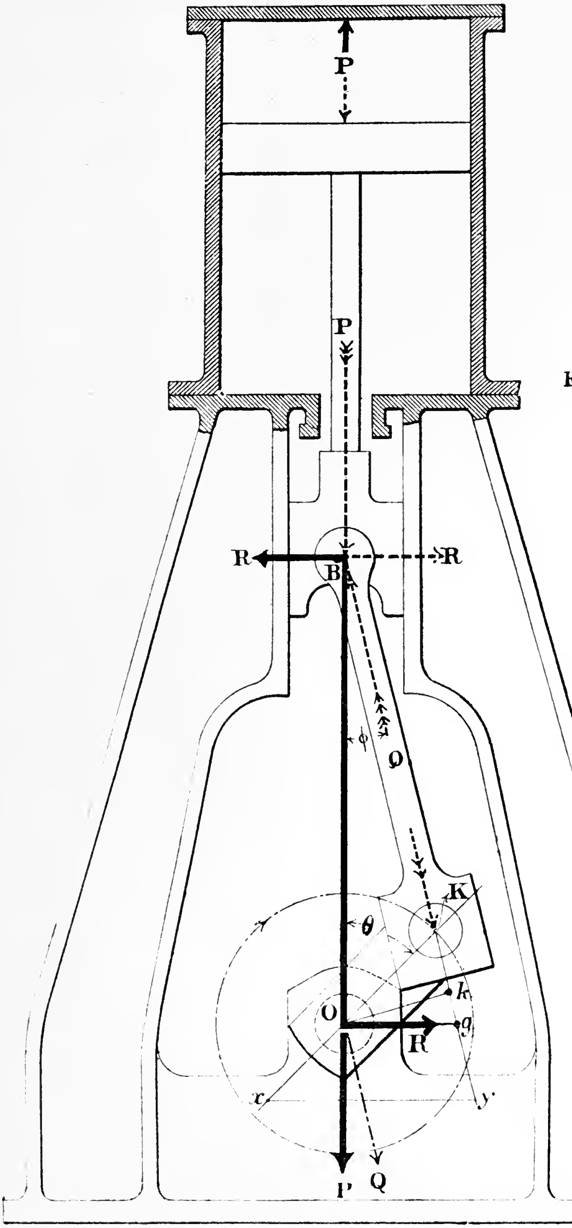


FIG. 143.

FIG. 144.

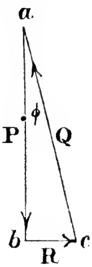
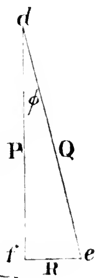


FIG. 145.



(1) An equal and parallel force acting at O.

(2) A couple, $Q \times Ok$, acting to turn the crank, Ok being perpendicular to the rod BK.

The force Q, represented by *de* in the force triangle *def* (Fig. 145), causes pressure on the main bearing, and its horizontal and vertical components are evidently equal to R and P of Fig. 144. The forces acting on the frame at the cylinder end are *cb*, that is *bc* reversed, at the bars; and *ba*, or *ab* reversed, at the cylinder cover. Thus R at the main bearing and R at the slide-bars form a couple acting on the frame from the gear. The vertical downward component, $df = P$ at the main bearing, is balanced by the vertical upward force $ba = P$ acting on the top cylinder cover.

To show that the couple $R \times OB$ is equal to the turning couple on the crank, it is only necessary to express Q in terms of R, and OB in terms of the angles θ and ϕ. The angle OKg is equal to $(\theta + \phi)$, therefore $Ok = OK \sin(\theta + \phi) = BO \sin \phi$. And $Q = R \operatorname{cosec} \phi$, and hence the couple $Q \times Ok = R \operatorname{cosec} \phi \times OK \sin(\theta + \phi)$.

But $BO = OK \sin(\theta + \phi) \operatorname{cosec} \phi$, therefore $R \times OB$ represents the magnitude of the turning couple on the crank-shaft.

Again, draw Og at right angles to the line of stroke; then, since $P = R \cot \phi = R \times \frac{OB}{Og}$, the product $P \times Og = R \times OB$, also represents the moment of the couple. Also, if Y represents the resolved component of Q at right angles to the crank, that is, the tangential force, $Y \times OK$ is another product representing the couple. Collecting these results, any one of the four products—

$$Q \times Ok = R \times OB = P \times Og = Y \times OK \quad . \quad . \quad . \quad (1)$$

may be taken to represent the moment of the turning couple, and therefore the equal and opposite couple acting on the frame, as may be most convenient for the problem in hand. It will be observed that the factors of the first three are all variable, but that Y only is variable in the fourth. Therefore, the variation of Y represents the variation of the couple. A convenient construction for finding this is as follows:—

Set out Kx (Fig. 143) to represent the value of P, measuring from K along the crank radius; draw xy parallel to Og, that is, horizontal; then xy represents the value of Y. To prove this—

$$Kx : xy = KO : Og$$

that is—

$$P : xy = KO : Og$$

therefore—

$$xy \times KO = P \times Og$$

But $P \times Og$ is equal to the moment of the couple from (1), therefore $xy = Y$. This method may be very quickly applied to find the value of the turning couple for any position of the crank, and the variations of the couple may be exhibited by plotting Y vertically over a straight line drawn to represent the circumference of the crank circle. Such a curve is usually called a crank-effort diagram. The curve may obviously be drawn so that the vertical ordinate represents the couple $Y \times OK$, in which case the length of the base-line represents 2π. It is a matter of indifference whether the constant multiplier OK, the crank radius, be introduced vertically or horizontally, the form of the curve and its area remain the same.

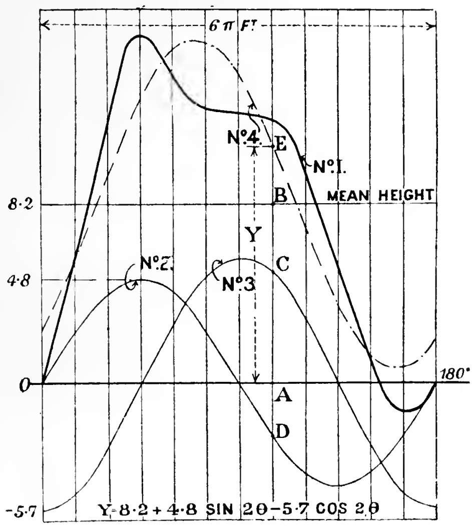


FIG. 146

The curve marked L.H. (Fig. 91) represents the turning couple acting on the left-hand crank of the locomotive under consideration in that article, resulting from the varying values of P shown in Fig. 95. There the base represents 2π, and consequently the ordinates represent the couple $Y \times OK$. Curve No. 1 (Fig. 146) shows the crank-effort curve for an engine, 6 feet stroke, running at 60 revolutions per minute, cutting off at about 20 per cent. of the stroke, and in which the reciprocating parts weigh $8\frac{1}{8}$ tons. In this case the base represents half the circumference of the crank-pin circle, so that the ordinates represent the tangential force. The vertical scale must be multiplied by 2·18 to find the tangential force in tons, the figures on the diagram being proportional only. Fig. 147 shows the crank-effort curve of a gas-engine, and may be taken as typical for many explosion motors. The cycle of operation in the cylinder requires four strokes to complete it; Y has very great values in the first stroke, and in the following three becomes relatively insignificant.

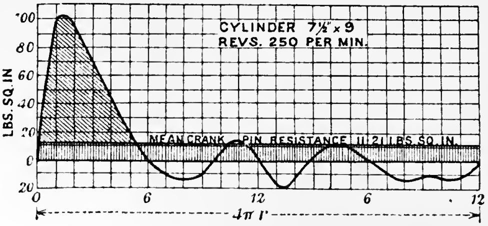


Fig. 147.

Each of the crank-effort curves mentioned represents also the couple acting to turn the frame. It will be noticed what a large variation there is between the maximum and minimum values. The actual values may be read off the curve (Fig. 91). The maximum value in Fig. 146 is over 100 foot-tons and the minimum just under −10 foot-tons. In Fig. 147 the variation is from 1700 foot-lbs. to −332 foot-lbs.

Expressing this in another way, the maximum value in the first case is 2·3 times the mean, which is 2·56 foot-tons, and the minimum is negative; in the second case the maximum is nearly twice the mean, falling to a negative quantity; and in the third case, where the mean is 186 foot-lbs., the maximum is 9·2 times the mean. These variations take place in a fraction of a second in each case.

There is therefore, in each case acting in the plane containing the cylinder centre line, and at right angles to the axis of the crank-shaft, a couple, whose variations are approximately periodic, and which is therefore capable of forcing vibration in the frame of the same period as itself about any axis parallel to the axis of the crank-shaft. Notice that the plane in which this couple acts is at right angles to the plane of the couples of the unbalanced moving parts of the engine.

131. Uniformity of Turning Moment.—In a single-cylinder engine there will always be a large variation of turning moment of the order shown in the cases just discussed. Where there are more cylinders than one, the cranks may be so arranged that the combination of the crank-effort curves corresponding to each results in a more uniform curve. In motor-cars, where the cylinders are often put in the same plane with cranks at 180° or parallel, the variation is of the type shown in Fig. 147. Consequently, even if these engines were properly balanced for the moving parts, there will always be acting on the frame a variable couple tending to cause oscillations. Speaking generally, synchronizing oscillations of large amplitude are not to be feared, because the speed of the engine is so high that the supporting springs have no grave periods to correspond to it. There are, however, forced oscillations of small amplitude, and to get rid of these the engines must be arranged, not only for balance amongst the moving parts, but so that the turning effort is much more uniform than is usually the case.

The result of combining two crank-effort curves when the cranks are at right angles is shown in Fig. 91, by curve No. 1. The variation is now much less, but takes place twice as fast. The ratio of the maximum to the mean is now reduced to 1·46, and, instead of the minimum being negative, it is positive and 0·45 of the mean.

The crank-effort curves and their resultant curve for the s.s. *Kaiser Wilhelm der Grosse* are given in *Engineering* for April 8, 1898, p. 434. There are four cylinders, and the ratio of the maximum to the mean is 1·19, and the minimum to the mean ·66.

In three- or four-cylinder engines with cranks at 120° and 90° respectively, the usual equal division of work amongst the cylinders is the best to get a uniform turning moment on the shaft. The same rule would give equally good results applied to the five- and six-crank engines of Arts. 99 and 100. Applied to four-crank engines in which the cranks are arranged at angles specially found for balancing the reciprocating masses amongst themselves, the equal division of work amongst the cylinders results in a turning effort in which there may be considerable variation during a stroke. Dr. Lorenz* has shown how the

* "On the Uniformity of Turning Moments in Marine Engines." By Dr. Lorenz *Trans. Inst. Naval Architects.* London, 1900. Also "Dynamik der Kurbelgetriebe." By Dr. Lorenz. Leipzig, 1901.

division of work may be made under these circumstances, to secure a better approach to uniformity of effort. The rule applies to engines of any number of cranks with any angles between them. The process of finding the rule is given in full, to show exactly what assumptions are made to obtain the results.

The form of the crank-effort curve corresponding to one cylinder, depends upon the cut-off, back pressure, mass of the reciprocating parts, the length of the connecting-rod relatively to the crank, and the speed of the engine, yet, when the average speed and the rate of working are constant, the curve repeats itself at every revolution, or neglecting the effect of the obliquity of the connecting-rod, at every stroke. In other words, the turning effort is continuous and periodic in the time occupied by half a revolution, and it may therefore be represented by a Fourier series. That is to say, if Y is the moment of the turning couple at any instant corresponding to a crank angle θ, which is dependent upon the time, the value of Y is expressed by—

$$Y = A_0 + A_2 \cos 2\theta + A_4 \cos 4\theta \ldots$$
$$+ B_2 \sin 2\theta + B_4 \sin 4\theta \ldots$$

where A_0 is the average height of the crank-effort curve, and A_2, B_2, etc., are numerical coefficients. The odd values of the angles do not appear in the series because the curve is periodic in half a revolution. There are a variety of ways of finding the values of the coefficients in the series corresponding to a given curve, the quickest and most convenient being to use some form of harmonic analyzer.

It is an essential feature of Dr. Lorenz's method that terms above 2θ must be discarded, consequently the value of Y is assumed to be given by three terms only of the series, or—

$$Y = A_0 + A_2 \cos 2\theta + B_2 \sin 2\theta \quad . \quad . \quad . \quad . \quad (1)$$

To show to what extent this value of Y differs from the true value in a typical case, the author analyzed the tangential force shown by the crank-effort curve marked No. 1, in Fig. 146, into the above harmonic constituents, finding the coefficients by means of a Henrici analyzer. The value of A_0 is found by measuring the area of the curve with a planimeter and calculating the average height in the usual way. The result is—

$$Y = 8{\cdot}2 + 4{\cdot}8 \sin 2\theta - 5{\cdot}7 \cos 2\theta$$

The component curves are shown in Fig. 146, and are numbered Nos. 2 and 3. Their sum, including the mean height, is shown by the chain-dotted curve No. 4. A comparison between this curve and the true curve, No. 1, with which it is assumed to coincide, shows to what extent equation No. 1 is able to realize the actual conditions of any given case. It does so nearly enough to use as a basis for obtaining a working rule.

To find the turning moment on the propeller-shaft for a multi-cylinder engine, the ordinates of the crank-effort curves corresponding to each cylinder are added together after the curves have been adjusted relatively to one another for their phase differences, in precisely the same way that has been explained for the force curves in Art. 109. Or stating the process more generally, if there are n cylinders, the turning effort on the propeller-shaft is represented in terms of the crank angle of any one assigned crank by the proper combination of n curves of the type No. 1 (Fig. 146); or, assuming No. 1 curve to be represented nearly enough by No. 4 curve, by the proper combination of n curves of No. 4 type. But the resultant curve in this latter case may be found by combining its components separately, thus finding the components of the resultant curve. Now, the sum of the components of the mean heights will be a straight line parallel to the axis under all circumstances, assuming uniform speed. The resultant components of the curves of the types Nos. 2 and 3 give two resultant components which are variable. If, however, it were possible to arrange that these two resultant components were separately zero, then the resultant turning effort would be constant, since under those circumstances it would be represented by the sum of the average heights of the n crank-effort curves. Dr. Lorenz shows how the separate sums of the n component curves of the types 2 and 3 (Fig. 146) may be made nothing if the n crank-effort curves are similar. This assumption of similarity cannot be exactly realized in four-cylinder balanced engines, because the inertia correction for each set of reciprocating masses is different, and therefore, even if cut off, etc., and all the other circumstances of a stroke could be kept constant through all the n cylinders of the engine, there would always remain the different inertia corrections to destroy the assumed similarity of the crank-effort diagrams.

Dr. Lorenz finds the rule analytically as follows:—

Let Y_1, Y_2, Y_3, etc., be the turning moments on cranks Nos. 1, 2, 3, etc. Also let θ be the variable angle between an initial line of reference revolving with the crank-shaft and a fixed line, the fixed vertical centre line of the engine, and let a_1, a_2, a_3, etc., be the respective crank angles measured from this revolving line of reference. (Fig. 102 illustrates this: OX_1 is the revolving line of reference and OM is one crank, OZ being the fixed line from which the variable angle is measured.) Then for crank No. 1—

$$Y = A_0 + A_2 \cos (2\theta + 2a_1) + B_2 \sin (2\theta + 2a_1)$$

or expanding the cosine and sine and rearranging the terms,

$$\begin{aligned} Y = A_0 &+ \cos 2\theta(A_2 \cos 2a_1 + B_2 \sin 2a_1) \\ &- \sin 2\theta(A_2 \sin 2a_1 - B_2 \cos 2a_1) \end{aligned}$$

The total turning effort on the shaft is then—

$$\Sigma Y = \Sigma A_0 + \cos 2\theta\Sigma(A_2 \cos 2a + B_2 \sin 2a) - \sin 2\theta\Sigma(A_2 \sin 2a - B_2 \cos 2a)$$

If this total effort is not to be influenced by the variations originating from the double angle 2θ, then both equations—

$$\Sigma(A_2 \cos 2a + B_2 \sin 2a) = 0$$
$$\Sigma(A_2 \sin 2a - B_2 \cos 2a) = 0$$

must be separately fulfilled.

Assuming a similar form of indicator diagram for all cylinders, the coefficients A and B (dropping the subscript 2) are proportional to the average turning effort Y_m of each crank, or if a and b denote constant quantities—

$$A = aY_m \text{ and } B = bY_m$$

Then the conditions are—

$$a\Sigma Y_m \cos 2a + b\Sigma Y_m \sin 2a = 0$$
$$a\Sigma Y_m \sin 2a - b\Sigma Y_m \cos 2a = 0$$

which for finite values of a and b require that—

$$\Sigma Y_m \cos 2a = 0$$
$$\Sigma Y_m \sin 2a = 0$$

Interpreted graphically, this means that it must be possible to draw a closed polygon whose sides are respectively proportional to the average turning effort exerted by the cranks, and the directions of whose sides are parallel to directions which are double the actual crank angles.

132. Example.—To illustrate this method, consider the example in Chapter III., Art. 48. Find what must be the distribution of work amongst the cylinders in order to obtain the most uniform turning moment.

Draw an end view of the crank-shaft showing the crank angles (Fig. 148). Measuring angles from crank No. 1, draw another end view in which the angles are all doubled (Fig. 149). Draw AB (Fig. 150) parallel to No. 1, and BC parallel to No. 4 in Fig. 149, taking them any lengths. Close the quadrilateral by two

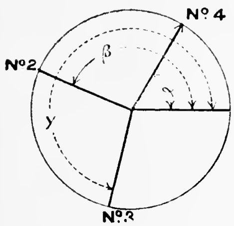


FIG. 148.

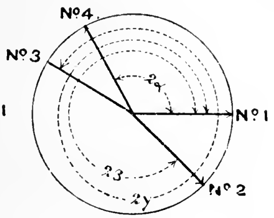


FIG. 149.

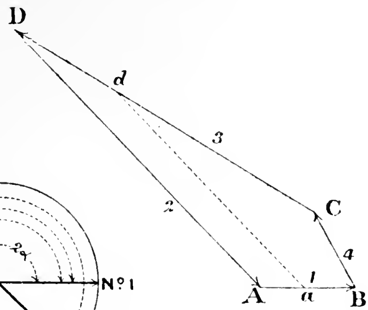


FIG. 150.

lines, CD and AD, drawn parallel respectively to cranks Nos. 3 and 2 (Fig. 149). Then the distribution of work should be in the proportion—

$$AB : BC : CD : DA$$

for the cylinders Nos. 1, 4, 3, 2 respectively.

A parallel *da* gives another set of lengths, *a*B, BC, C*d*, *da*, which satisfy the necessary conditions, and there are obviously a great many other ways in which they may be satisfied.

Fig. 151, taken from Dr. Lorenz's paper already quoted, is the crank-effort diagram of the s.s. *Medjerda*. The line AB represents the circumference of the crank circle, the angles between the cranks being indicated by the figures below it. The work done in each cylinder is written against the corresponding vertical for the respective cylinders. It will be found that the horse-powers are very nearly in the ratio of the sides of the double-angled polygon corresponding to the angles given. In this case the maximum is

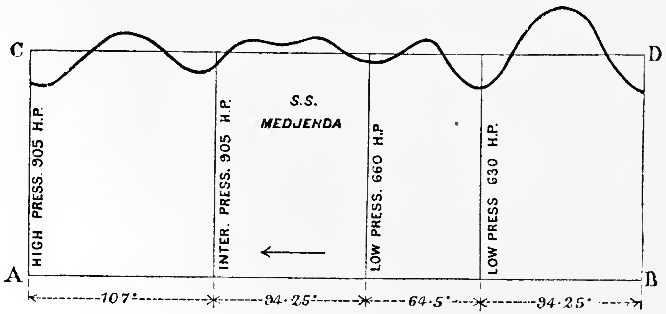


Fig. 151.

1·21, and the minimum ·85 of the mean. Of course, if the conditions stated in obtaining the rule were exactly fulfilled, the curve would be a straight line corresponding with CD, the mean value.

133. Short-framed Engines.—Although the forces acting on the frame due to a set of reciprocating and revolving masses in balance, form a system of forces in equilibrium, the individual forces of the system cause elastic deformation of the frame at the places where they act. If the frame is flexible these local deformations, being of course periodic, may conceivably be of sufficient amplitude to cause vibration of the supports. Generally, engine-frames are stiff enough to limit the elastic deformations to negligibly small amplitude, and a frame may be indefinitely stiffened if these elastic deformations become troublesome. In balanced engines of the usual type there are forces belonging to the system acting on the frame, at the intermediate bearings of the crank-shaft. Mr. Macalpine has proposed a form of engine * in which none of the forces forming the system acting on the frame act at any of the main bearings. In this engine, which is balanced in the longitudinal vertical plane and in a horizontal plane, the

* *Trans. Inst. Naval Architects*, 1901, and *Engineering*, July 12, 1901.

cylinders are placed in pairs across the shaft. The cross-heads of a pair are connected by a rocking beam, and one of the pair of cross-heads is connected to a crank by the usual form of connecting-rod. The two sets of reciprocating parts forming a pair are made equal in mass, and being moved in opposite directions by the rocking beam, the forces in the arrangement proposed are very nearly perfectly balanced, but there remains a couple in the plane of the two cylinders tending to force torsional oscillations. Mr. Macalpine's contention is, that it is preferable to have couples in a plane across the ship of relatively great magnitude, to having forces and couples of exceedingly small magnitude in the longitudinal vertical plane, since in the former case the periodic time of the gravest torsional oscillation of the hull is so small compared with the periodic time of the engine that trouble from torsional vibrations is not to be feared.

In the Wigzell engine the cylinders are all placed athwart the frame in one plane, and balance amongst the reciprocating masses effected so that there is no torsional couple left. The frame is very short, and the only forces acting on it are those due to the angle of the connecting-rods. A full description of this engine will be found in *Engineering*, September 7, 1900. There are three cylinders, each containing two pistons, which move oppositely to one another in each case. The three piston-rods coming through the lower cylinder-covers are connected to one central crank by a triangular connecting-rod. The corresponding three piston rods taken through the upper cylinder-covers are similarly connected to two outer cranks by triangular connecting-rods connected to the upper set of cross-heads by coupling-rods. An incidental advantage of this arrangement is that there is practically no pressure between the crank-shaft and the main bearings. What pressure there is, is chiefly due to the weight of the crank-shaft system alone.

CHAPTER VIII.

THE MOTION OF THE CONNECTING-ROD.

134. THE motion of the connecting-rod is one of periodic acceleration, and therefore the forces required to produce the motion have also a periodic variation. These forces ultimately appear as reactions on the frame, and, being periodic, may cause vibration. The effect of the rod may be divided into two distinct parts: first, it disturbs the simple harmonic motion of the reciprocating parts; secondly, the forces required for its acceleration appear as forces on the frame tending to cause oscillation. Chapter V. shows how the first effect is dealt with in the balancing of the engine, and a rule has been given in Art. 46 for dividing the mass of the rod between the revolving and reciprocating parts of the gear to eliminate the second effect, it being tacitly assumed that the accelerations of these two masses require reactions on the frame equal to those required by the motion of the actual rod. The object of this chapter is to investigate the motion of the rod, and show to what extent the rule given is a valid one.

135. Dynamical Principles on which the Investigation is based.— Let C (Fig. 152) be the mass centre of a body, free to move in the plane of the paper and acted upon by a system of co-planar forces whose resultant is R. This force is equivalent to—

(1) An equal and parallel force acting at the mass centre.

(2) A couple whose moment is $R \times CZ = L$, say.

The effect of the force is to accelerate the motion of the mass centre in the direction of its line of action. The effect of the couple is to accelerate the angular motion of the body about an axis through the mass centre, at right angles to the plane containing the force R and the point C. It is a fundamental dynamical principle that the acceleration of the mass centre C is the same as if the whole mass of the body were concentrated at the one point C, and that the angular acceleration of the body about C is the same as if the axis through C were fixed. Hence, if R be given in position and magnitude, the instantaneous acceleration of the mass centre, and the angular acceleration about the mass centre, can be found when the mass of the body and its moment of inertia about the perpendicular axis through C are respectively given. The acceleration of any point in the body is then determined, being the vector sum of the acceleration of C, and the acceleration of the point about C.

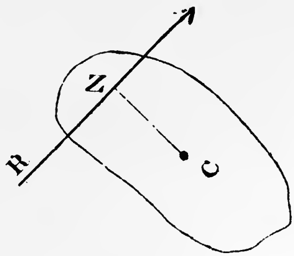


FIG. 152.

The connecting-rod problem is precisely the converse of this. Two points in the rod are compelled to move in a definite manner, the one in a circle guided by the crank-pin, the other in a straight line guided by slide-bars, and if the acceleration of the first point be given, the acceleration of the second can be readily found. But if the accelerations of two points in a body moving in a plane be given, the acceleration of every point can be immediately deduced. From the acceleration of the mass centre and the mass of the rod, the force R producing this acceleration can be at once calculated. Similarly, the angular acceleration of the rod about the mass centre can be found when the accelerations of two points are given, and hence the couple L may be inferred when the moment of inertia about the mass centre is known.

The problem naturally divides itself into two parts: first, the determination of the acceleration of the mass centre and the magnitude and direction of the force R; secondly, the determination of the position of R relative to the mass centre so that it causes a couple equal to L. In the geometrical method of finding R, which will be explained first, the first construction gives the acceleration of the mass centre, after which there are several ways of finding the position of R relative to the mass centre, depending upon the artifice of concentrating the mass of the rod at two points, so that the two masses thus concentrated form an equivalent dynamical system to the actual rod. Then the two forces which must act to produce the instantaneous acceleration of this two-mass system, which has, of course, the same acceleration as the actual rod, must have R for a resultant. The magnitude and direction of R is found by the first construction. The aim of the second is to find a point on the line of action of R. This point is discovered by the intersection of the directions of the two forces producing the instantaneous motion of the equivalent two-mass system.

136. Graphical Method for finding the Acceleration of the Mass Centre of the Rod.*—Assume the angular velocity of the crank to be sensibly constant. Let OK (Fig. 153) be the crank, KB the connecting-rod, and BO the line of stroke. Let C be the mass

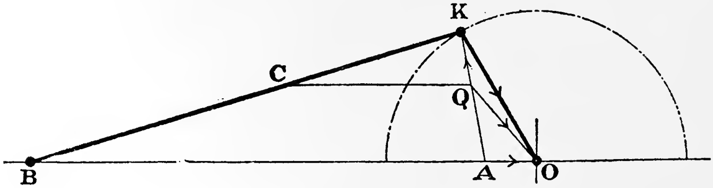


Fig. 153.

centre of the rod. Draw CQ parallel to BA. The acceleration of the point K is ω^2KO. The acceleration of the point B may be found by Klein's construction (Art. 104). This acceleration is shown by AO in the figure. From a reference to Art. 6, it will be understood that AK is the acceleration of B relatively to K. QK is then the acceleration of C relatively to K, since QK : AK = CK : BK. The whole acceleration of C is the vector sum of its acceleration relatively to K and the acceleration of K, that is, the vector sum of QK and KO = QO. It is evident that there is nothing to restrict this reasoning to the point C. Therefore, the acceleration of any point on the rod may be found by projecting the point on to KA by a line parallel to BO, and joining the point so found to O. Any point thus projected divides KA in the

* Many of the following graphical methods are given in "New Constructions of the Force of Inertia of Connecting-rods and Couplers and Constructions of the Pressures on their Pins." By Professor J. F. Klein. *Journal of the Franklin Institute*, Vol. CXXXII., September and October, 1891.

same ratio that it divides KB. In fact, KA is the rod drawn to a smaller scale, and in such a position that the line joining any point on KA to O, represents the acceleration of the corresponding point in the actual rod. On account of this property, KA has been called the acceleration image of the rod. QO is to be measured to the scale on which KO represents the crank radius; then the actual value of the acceleration is ω^2QO.

If M is the mass of the rod, the magnitude of the force R, is—

$$\frac{M}{g}\omega^2 QO \text{ lbs. weight}$$

acting at C in a direction parallel to QO from Q towards O. This force applied at the mass centre, would produce the instantaneous acceleration of the mass centre which it actually undergoes.

137. Equivalent Dynamical System.—The conditions to be satisfied by the concentration of the mass of the rod at two points are—

(1) The sum of the two masses must be equal to the mass of the rod.

(2) Their mass centre must coincide with the mass centre of the rod.

(3) Their moment of inertia about an axis through the mass centre at right angles to the plane of motion of the rod, must be equal to the moment of inertia of the rod about the same axis.

Let m_1, m_2 be the two masses into which M is divided, distant respectively d_1 and d_2 feet from the mass centre of the rod, and let k be the radius of gyration of the rod about the axis at the mass centre. The three conditions stated above are expressed by the equations—

$$m_1 + m_2 = M \quad \ldots \ldots \quad (1)$$
$$m_1 d_1 - m_2 d_2 = 0 \quad \ldots \ldots \quad (2)$$
$$m_1 d_1^2 + m_2 d_2^2 = Mk^2 \quad \ldots \ldots \quad (3)$$

From (1) and (2)—

$$m_1 = \frac{Md_2}{d_1 + d_2} \quad \ldots \ldots \quad (4)$$

and—

$$m_2 = \frac{Md_1}{d_1 + d_2} \quad \ldots \ldots \quad (5)$$

Combining these with (3)—

$$d_1 d_2 = k^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

This is the relation governing the position of the masses. Their magnitudes are, as shown by equation (2), inversely as the mass centre divides the distance $d_1 + d_2$. Observe that k is a mean proportional between d_1 and d_2. Hence, if the position of one mass and k be given, the position of the second mass can be found by the following construction:—

Let C be the mass centre of the body, which is symmetrical about the line sH, shown in Fig. 154. Let s be the given position

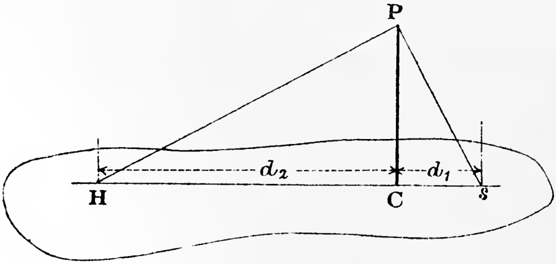


Fig. 154.

of one mass. Draw CP at right angles to Hs, its length representing to scale the radius of gyration k. Join s and P, and draw PH at right angles to sP. H then marks the position of the second mass.

It is evident that there is nothing to restrict the position of the given point s, consequently an infinite number of pairs of points, s and H, can be found. It may be noticed that if the body be suspended from an axis through the point s, sH would be the length of the simple equivalent pendulum, and that H would be the centre of percussion relatively to s, and s the centre of percussion relatively to H.

Returning to the connecting-rod, the value of k must be known, or the position of the two points s and H, before the division of the mass can be made.

The best way to arrive at the position of a pair of points of a finished rod is to let it oscillate about some selected point s, the axis through the centre of the small end usually, and adjust the length of a plumb-line until it swings in unison with the rod. The length of the plumb-line, measured from the point about which it swings to the centre of the bob, gives the distance sH, that is, $d_1 + d_2$. The position of the mass centre C can be found by balancing the rod on a knife-edge; thus the individual values of d_1 and d_2 are known, and k can at once be calculated.

Example.—The length of a plumb-line adjusted to swing in unison with the connecting-rod belonging to an inside-cylinder, four-coupled passenger engine, the rod swinging about an axis at the small end-centre, was found by experiment to be 5·9 feet. The distance of the mass centre, the position of which was found by balancing the rod on a knife-edge, was 4·92 feet from the axis of suspension. Therefore—

$$d_1 = 4{\cdot}92 \text{ feet}, \ d_2 = 0{\cdot}98 \text{ feet}$$

Hence—

$$k^2 = 4{\cdot}82, \text{ and } k = 2{\cdot}19 \text{ feet}$$

138. Constructions for fixing a Point in the Line of Action of R.*—Having found a pair of points, s and **H**, on the assumption that k is known, the object of each of the following constructions is to find the respective lines of action of the forces acting on the concentrated pair of masses. The intersection of these two lines of action fix a point X on the resultant, defining thereby its line of action, since its direction is known from the construction of Art. 136.

Construction 1 (Fig. 155).—Choose the point s to correspond

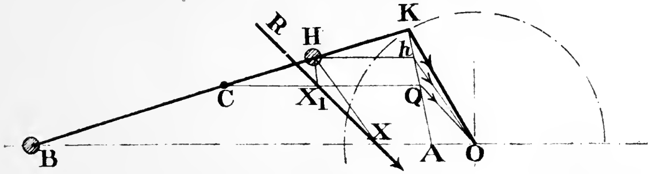


FIG. 155.

with B, and find H by the method of Fig. 154. Repeat the construction of Fig. 153 to find KA, the acceleration image of the rod. The direction of the acceleration of B is along the line of stroke. To find the direction of acceleration of H, refer it to the line KA, then hO gives the required direction. Therefore, the

* See Appendix IV.

intersection X, of a line parallel to hO drawn through H, with BO fixes a point in the line of action of R. The line through X parallel to QO, R's direction, is the line of action of the resultant R. Its perpendicular distance from C is such that it causes the couple L, producing the angular acceleration of the rod.

Construction 2 (Fig. 155).—The motion of the rod may be analyzed into a translation of the rod parallel to itself, every point moving with the acceleration of B, and an angular motion about B. The force to produce the first acceleration must be applied to the system at the mass centre C in a direction parallel to the line of stroke. The force to produce the angular acceleration about B must be applied at H in the direction of acceleration of H relatively to B, that is, in the direction hA. The intersection of a line through H, parallel to hA, with CQ fixes a point, X_1, in the line of action of R. A line through X_1 parallel to QO is therefore the line of action of R.

Construction 3 (Fig. 156).—Choose s to correspond with the big end-centre K, and apply the construction of Fig. 154 to find H. The direction of acceleration of K is along the crank radius. Project H on to the image of the rod KA, to find hO, the direction of acceleration of H. A line through H, parallel to hO, intersects the crank produced in a point, X_2, on the line of action of the resultant R.

Construction 4 (Fig. 156).—The motion of the rod may be analyzed into a motion parallel to itself, where every point moves in a circle of radius equal to the crank radius, and an angular motion about K, the crank-pin. To produce the first, a force must be applied at the mass centre C in a direction parallel to the crank KO. To produce the angular motion about K, a force must be applied at H in the direction of the acceleration of H relatively to K, that is, in the direction hK. Therefore, the intersection X_3 of a line through H parallel to hK, with a line through C parallel to the crank radius KO, fixes a point in the line of action of R.

Construction 5.—The point s may be taken in any position on the centre line of the rod, and a point, H, to correspond with it found by the construction of Fig. 154. The directions of their accelerations are found at once from the acceleration image, from which a point, X, can immediately be fixed.

Any one of these constructions may be used in combination with the construction of Art. 136 to find R. The combination with Construction 1, and, in fact, the whole question of the inertia loading of links, is discussed in Professor Klein's paper already quoted. The combination with Construction 2 is due to Professor Dunkerley and Mr. J. B. Peace. The use that may be made of

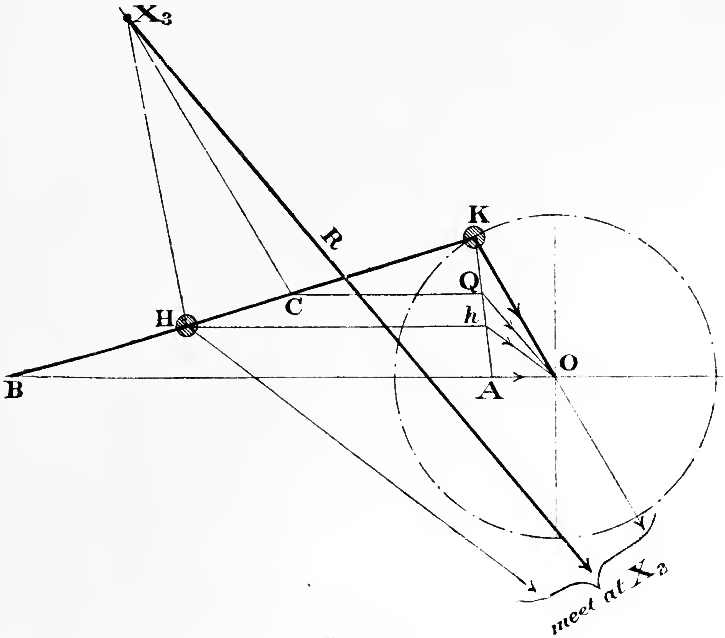


Fig. 156.

it to find the inertia loading of a connecting-rod is shown in an article by Professor Dunkerley in *Engineering*, June 2, 1899. The whole question of acceleration and velocity images is treated in a general way in Professor Smith's "Graphics."

139. Combined Construction.—For convenience of reference, the steps in the combination of the construction of Art. 136, with construction No. 1 of the previous article, are restated.

To find the force R (Fig. 155) for any given crank angle: mark the points C and H on the rod; find AO, the acceleration of B, by Art. 104; join KA; draw CQ and H*h*, respectively parallel to BO. Then the intersection of a line through H parallel to *h*O with the line of stroke fixes a point, X, in the line of action of R.

A line through X parallel to QO is the line of action of R, and R's magnitude is—

$$\frac{M}{g}\omega^2 QO \text{ lbs. weight}$$

140. Effect on the Frame and on the Turning Moment exerted by the Crank.—Since the force R, under whose action the instantaneous acceleration might be produced, can be found for any crank angle by the construction of the previous article, the effect of the rod's motion on the frame is reduced to the statical problem of finding the effect of R on the frame.

Let R be the resultant force (Fig. 157) for the crank angle shown. Referring R to the crank-pin, it is equivalent to—

(1) An equal and parallel force R acting on the crank-pin.

(2) A couple whose moment is $R \times gi$ acting on the rod.

The couple is actually applied to the rod by a pair of parallel forces acting respectively at the cross-head and crank-pin in a direction at right angles to the line of stroke BO, since this is the only direction in which a force can act from the frame at the slide-bars, neglecting friction. Draw Kp at right angles to BO, then the magnitude of each force of the couple is—

$$\frac{R \times gi}{Bp} = S, \text{ say}$$

So that acting at the crank-pin there are two forces, S, acting always at right angles to the line of stroke, and a force equal and parallel to R. Referring these forces to the main bearing, O, each is equivalent to an equal and parallel force acting at the bearing and a couple acting on the crank. The whole couple acting on the crank is thus—

$$R \times Oj + S \times Op$$

This is the extent to which the turning moment exerted by the crank is modified by the motion of the connecting-rod.

The resultant force R, therefore, requires that—

(1) A force, S, acts at the slide-bars from the bars to the rod.

(2) Forces S and R act at the main bearing O.

These are the forces which must act from the frame on the gear to produce the acceleration of the rod. The forces acting from the gear on the frame are therefore equal and opposite to these, and are shown below in Fig. 158. Notice that S at the bars and $-S$ at the main bearing form a couple. The effect of R on the frame is thus equal to—

(1) A force, equal, opposite, and parallel to R, acting on the main bearing.

(2) A couple $S \times OB$.

Fig. 157.

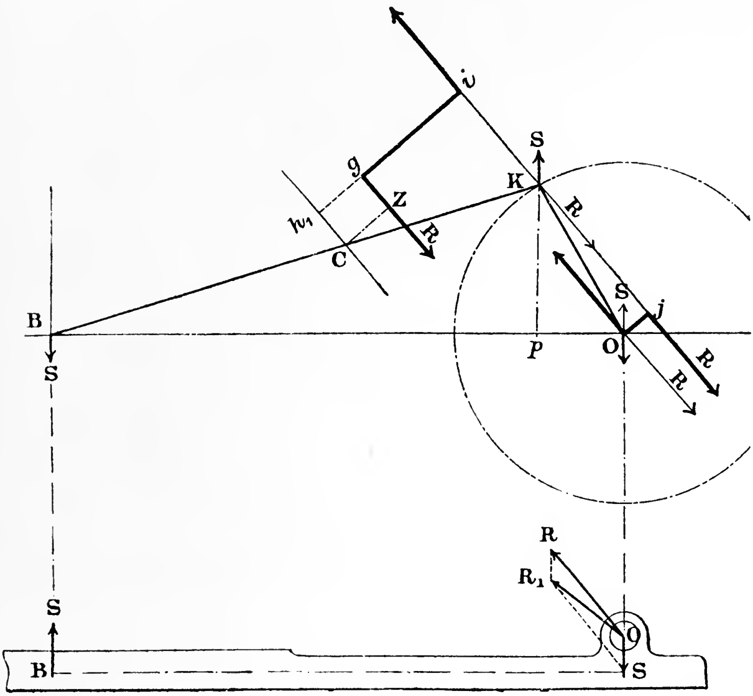


Fig. 158.

Another way of stating the effect is—

(1) The resultant OR_1 of R and S at the main bearing. Call this F.

(2) A single force S at the slide-bars.

The forces F and S may be found directly by a simple construction (Fig. 159). The rod is at any instant in equilibrium under the action of three forces, viz. the forces acting through its ends and R reversed. These three forces must therefore meet in a point. The force acting at the slide-bars must be at right angles to the line of stroke, and the line of action of R is known; hence, draw a line at right angles to the line of stroke through B, to meet R produced in the point V. Join VK. Set off VV_1 equal to R. Draw V_1V_2 parallel to VB. Then V_2V is the whole force acting at the crank-pin, from the rod to the pin, and V_1V_2 is the force from the rod to the slide-bars. The force V_2V at the crank-pin is equivalent to an equal and parallel force at the main bearing and a couple $VV_2 \times Oa$. This couple is equal and opposite to that

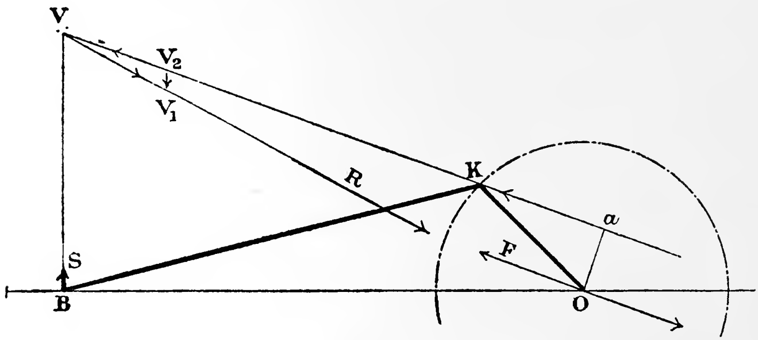


Fig. 159.

previously given, which modifies the turning moment of the crank, and the force F is the same as the resultant of R and S (Fig. 158).

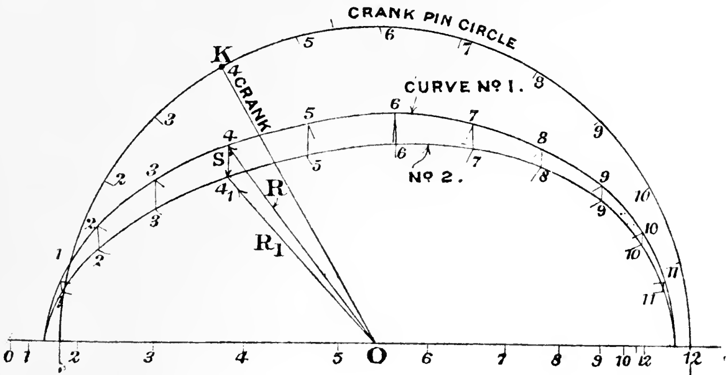
When the construction is carried out on the figure used to find the value of the resultant R, that is, QO in Fig. 153, the point V is only required to fix the direction VV_2, since, if a perpendicular to the line of stroke be drawn from Q to cut a line through O parallel to VV_2, the point of intersection fixes the value of F and S. The triangle formed in this way is shown in Fig. 158, where OR is the resultant, OR_1 the direction V_2V, and RR_1, perpendicular to the line of stroke which fixes the point R_1, defining thereby the lengths of OR_1 and RR_1, that is, F and S.

141. Examples.—Fig. 160 shows the frame reactions at the main bearing and the slide-bars respectively for the rod of an inside cylinder locomotive. The specification of the rod is as follows:—

Length, centre to centre, 6·81 feet
Distance of mass centre from the small end, 4·93 feet
Distance of the centre of percussion H from the small end, 6·0 feet
Mass of rod, 454 pounds

The method of finding points on the curves was first to find R by the construction of Art. 139, and then to apply the construction

Fig. 160.



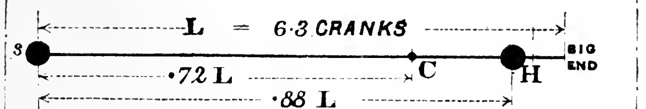


Fig. 162.

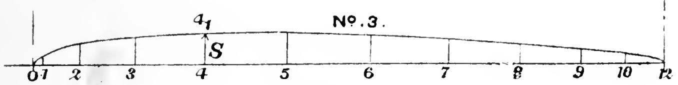


Fig. 161.

of Fig. 159 to find S and F, for twelve different angular positions of the crank.

Curve No. 1 is the locus of the point corresponding to R (Fig. 158), and curve No. 2 the locus corresponding to point R_1 (Fig. 158). The vertical distance between the curves represents RR_1 (Fig. 158), that is, the force S, whose equal and opposite acts at the bars. Curve No. 3 (Fig. 161) represents the changing values of S, set out at the slide-bars. Thus, when the crank is at OK (Fig. 160), the length of O4 measured to curve No. 1 represents the magnitude and direction of R, the length of $O4_1$ the magnitude and direction of $OR_1 = F$, the whole force on the main bearing, and the intercept, 44_1, equal to 44_1 on curve No. 3 (Fig. 161), represents the force at the slide-bars. The actual magnitudes of these forces are to be found by measuring the lengths of the lines respectively

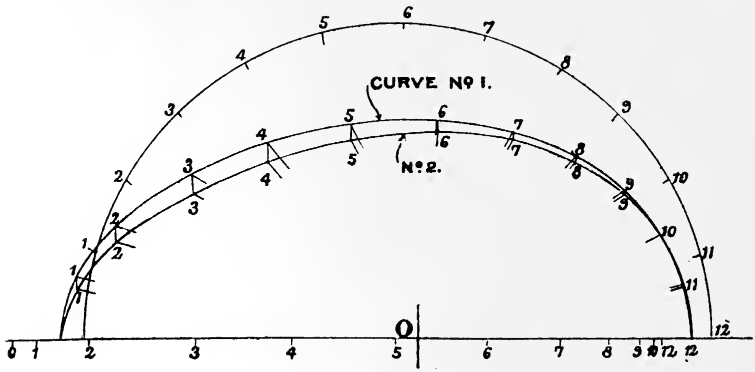


Fig. 163.

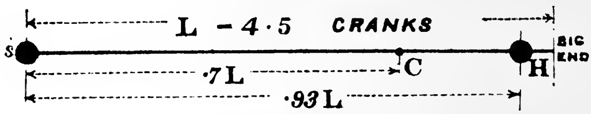


Fig. 165.

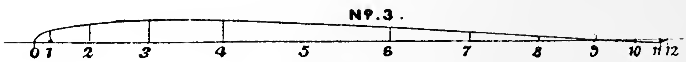


Fig. 164.

representing them on the diagram, to the scale on which OK represents the crank radius, and multiplying the lengths so found, in feet, by the mass of the rod and the square of the angular velocity of the crank, dividing by g to get the result in lbs. weight. For example, if the crank radius OK represents 1·08 feet, O4 measures 0·846 feet, $O4_1$ measures 0·76 feet, and 44_1 measures 0·114 feet. At 240 revolutions per minute ω^2 is 632.

Therefore—

$$04 = \frac{M}{g}\omega^2 \times 0{\cdot}846 = 7538 \text{ lbs. weight} = R$$

$$04_1 = \frac{M}{g}\omega^2 \times 0{\cdot}76 \quad = 6770 \text{ lbs. weight} = F$$

$$44_1 = \frac{M}{g}\omega^2 \times 0{\cdot}114 = 1016 \text{ lbs. weight} = S$$

Figs. 163 and 164 show sets of curves for the rod of a torpedo-

Fig. 166.

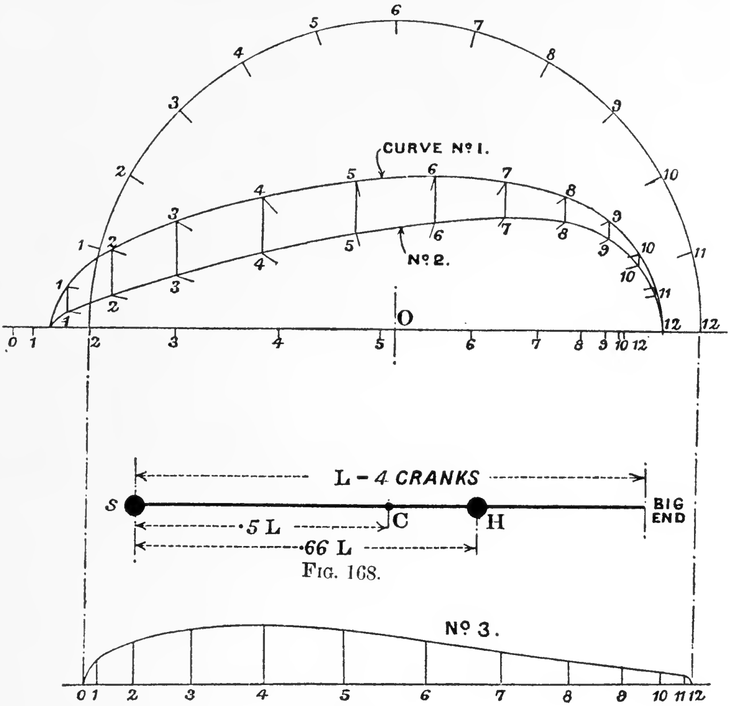


Fig. 168.

Fig. 167.

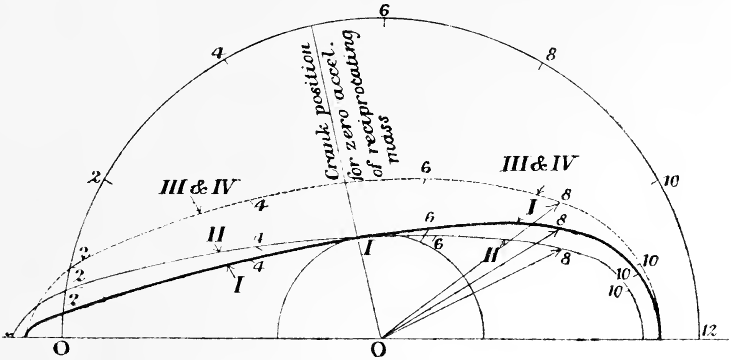
boat destroyer, and Figs. 166 and 167 curves for a rod of uniform section. These latter curves are merely of theoretical interest, since the big and small ends of the rod must always be of considerably greater section than the body connecting them. In each case the dynamical peculiarities of the rods are indicated by the centre-line drawings (Figs. 162, 165, and 168). Comparing Figs.

160, 163, and 166, it will be noticed that the nearer the point H, the centre of percussion relatively to the small end, is to the big end, the smaller is the maximum vertical distance between the curves Nos. 1 and 2, which indicates that the couple, of which the vertical distance represents one force, is small also. If H coincide with the big end, the line of action of the resultant R passes through the point O in every position of the crank; and if in any position of the crank R pass through K, the couple vanishes. This is the case in Fig. 163 at No. 10 crank position.

142. Balancing the Rod.—Suppose the form of the rod to be such that, when s is taken at the small end, H falls on the crank-pin centre. The two masses forming the equivalent system are then concentrated at these two points, their magnitudes being inversely as the mass centre divides the rod; this follows from expressions (4) and (5) (Art. 137). If the masses are treated independently, the one considered to be attached to and moving with the cross-head reciprocating masses, the other attached to and moving with the crank-pin, the forces required for their acceleration must have R for their resultant, since the masses are equivalent to a dynamical system of which R is the resultant accelerating force. If these forces are exactly balanced, it is clear that R at the main bearing is balanced, since in all positions of the gear the line of action of R passes through the centre of the main bearing. There still remains the couple $S \times OB$ acting on the frames, consequent upon the existence of the couple $R \times gi$ (Figs. 157, 158). The frame-couple only vanishes when this couple vanishes, that is, when the line of action of R passes through the crank-pin centre.

The rod is seldom of such a form that s and H fall at its centres. But it will be noticed that their position has no effect on either the magnitude or the direction of the force R. These points only determine the line of action of R relative to the mass centre, and therefore affect only the magnitude of the couple $R \times ig$, and ultimately the frame-couple $S \times OB$; and since S is always at right angles to the line of stroke, it has no component in the line of stroke. Hence, so far as the unbalanced forces in the line of stroke are concerned, it is only necessary to consider R at the main bearing for any form of rod. If the mass of the rod is distributed between the crank-pin and cross-head inversely as the mass centre divides the rod, the assumption is tacitly made that these two separate and independent masses form an equivalent dynamical system; so far as the forces in the line of stroke are concerned, there is no error in the assumption; at right angles to the line of stroke, however, this assumption involves an error in S depending upon the position of the point H. No attempt is usually made to balance this couple on the frame.

The author suggested a way of dividing the mass of the rod between the crank-pin and cross-head whereby the resultant of the



BH = ·66L
BC = ·5L
L = ·4 CRANKS

FIG. 169.

forces required to accelerate the two masses is approximately, but very nearly, coincident with the force OR_1, the whole force on the main bearing, during the whole revolution. The mass at the crank-pin is found from the expression—

$$\frac{M \times BC \times BH}{KB^2}$$

M being the mass of the rod, BC and BH the respective distances of the mass centre and the centre of percussion from the small end centre, KB the length of the rod. The remainder of M is placed at the cross-head.

The curves of Fig. 169 show to what extent the resultant force at the main bearing, due to the acceleration of the two masses found from this formula, differs from the true value of

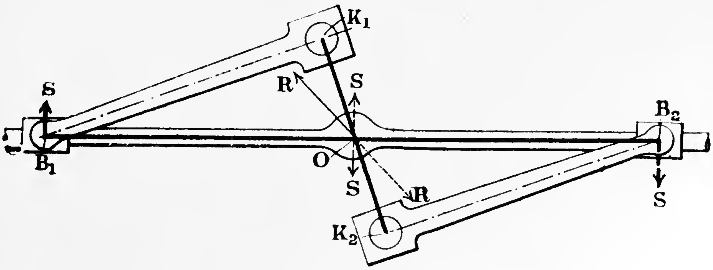
OR_1. The thick curve (No. I.) shows the locus of the point corresponding to R_1 of Fig. 158, and is, in fact, curve No. 2 of Fig. 166. The thin curve (No. II.) shows the locus of the end of the force acting on the main bearing due to the acceleration of the masses divided in the way suggested, and the dotted curve (No. III.) shows the locus of the end of the force due to the acceleration of the cross-head and crank-pin masses divided inversely as the mass centre divides the rod. In the position 8, for example, the length of O8 measured to curve No. I. is the actual force on the main bearing, the length of O8 measured to curve No. II. is the force consequent upon the mass division suggested, the length of O8 measured to curve No. III. is the force consequent upon the inverse mass division. This shows that a better agreement at the main bearing can be obtained if desired, leaving the force S at the slide-bars unbalanced. There is an error introduced in the line of stroke by this method, but it should not be forgotten that in many cases the mass added to balance the reciprocating mass can only be arranged to do so in a very approximate manner, and that therefore a small error in the line of stroke in the mass assumed concentrated at the cross-head is of no consequence. The method of division suggested may sometimes be found useful when it is desired to relieve the main bearing of the whole force OR_1 due to the acceleration of the rod, and when the point H is some distance from the big end.

143. Particular Form of Balanced Engine.—The rod is sometimes balanced by opposing to it a similarly formed rod of equal mass, moving similarly but in opposition. Fig. 170 shows the arrangement. The crank is prolonged so that there are two cranks at 180°, and the mass centres of each rod move in the same plane. The forces acting on the frame are two sets like those of Fig. 158. They are indicated in Fig. 170, one set being shown in dotted lines. It will be seen that the forces at the main bearing mutually balance, leaving a force S at each slide-bar, which together form an unbalanced couple of moment $S \times B_1B_2$.

To realize this arrangement practically, one connecting-rod must be divided into two, each, half the mass of the original rod, and formed similarly to it, and each being placed at equal distances on opposite sides of the central plane of motion. It should be noticed also that in this arrangement equal reciprocating masses of any magnitude, placed at B_2 and B_1 respectively, balance each other exactly. The effect of the angle of the connecting-rod is entirely eliminated in the line of stroke, since B_2 and B_1 move with exactly equal and opposite acceleration.

Suppose that instead of splitting one rod, the two are displaced relatively to one another along the axis of the shaft so that their

FIG. 170.



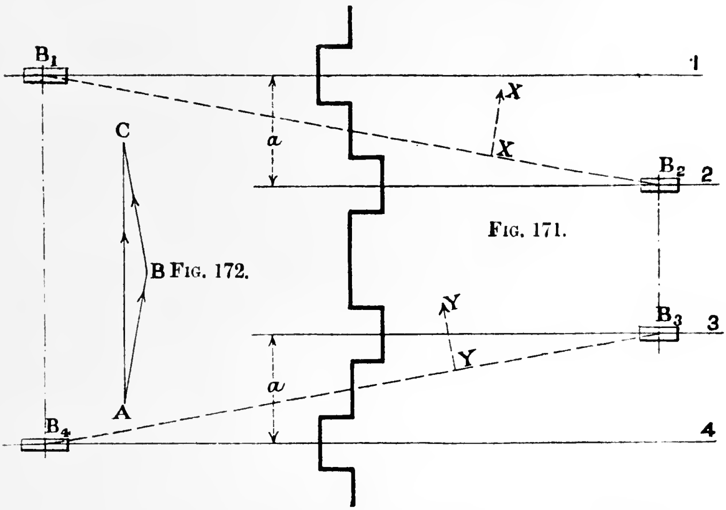


FIG. 171.

FIG. 172.

respective planes of motion, 1 and 2 (Fig. 171), are *a* feet apart. It will be apparent that there will be now couples $R \times a$ and $S \times a$, acting on the frame derived from the forces at the main bearing, and a couple, $S \times B_2B_1$, the diagonal distance between the cross-heads, derived from the pair of equal and opposite forces acting at the bars. If a similar system of two cranks is arranged anywhere along the shaft in the manner shown in planes 3 and 4 (Fig. 171), the couples arising from the forces at the main bearing mutually balance. The forces at the four cross-heads form two couples acting respectively in the planes of which B_1B_2 and B_3B_4 are the horizontal traces. Their axes are in the respective directions XX and YY, and their magnitudes are equal. The vector sum of these couples, found by the triangle (Fig. 172), is represented by the line AC. This couple is always equal in magnitude to twice the projection of the distance $B_1B_2 = B_3B_4$, on the line B1 (Fig. 171), which equals B_1B_2 (Fig. 170) multiplied by S, and it tends to rock the frame about the axis of the crank-shaft. There is always, in addition to this, the couple equal and opposite to the driving couple (see Art. 130), and the couple due to the acceleration of the reciprocating masses by the rod, tending to rock the frame about the crank-shaft. These are usually great in comparison with the couple due to the connecting-rod.

Mr. W. G. Wilson has designed a motor-car engine on these principles, and its smoothness of running at all speeds, the maximum being 1500 revolutions per minute, is remarkable.

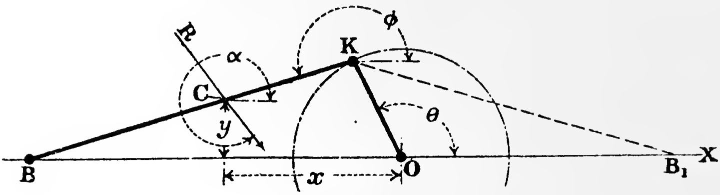


FIG. 173.

144. Analytical Method of finding R and L.

Let M be the mass of the rod (Fig. 173);

θ, the crank angle measured from the initial direction OX;

ω, the constant angular velocity of the crank;

ϕ, the angle made by the rod with the initial direction OX;

α, the angle made by the resultant force R with OX;

$OK = a$, the crank radius in feet;

$KC = b$, the distance of the mass centre C from the crank-pin K;

$KB = l$, the length of the rod;

k, the radius of gyration about the mass centre;

x, y, the co-ordinates of the mass centre C.

Use the Newtonian notation for representing differentiations with regard to the time, viz.—

$$\dot{x} \text{ for } \frac{dx}{dt}, \quad \ddot{x} \text{ for } \frac{d^2x}{dt^2}, \quad \ddot{\phi} \text{ for } \frac{d^2\phi}{dt^2}, \text{ etc.}$$

In all that follows, $\dot{\theta} = \omega$, the angular velocity of the crank is assumed to be sensibly constant, so that $\theta = 0$.

Suppose the force R transferred to the mass centre C, the transference giving rise to the couple L (see Art. 135).

The component of R parallel to the X axis is $R \cos a$. The acceleration of the mass centre parallel to the X axis is $\ddot{x}$.

Therefore—

$$R \cos a = M\ddot{x} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Similarly—

$$R \sin a = M\ddot{y} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

and—

$$L = Mk^2 \ddot{\phi} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

The direction of R is evidently given by—

$$\tan a = \frac{\ddot{y}}{\ddot{x}}$$

Its magnitude can be calculated from either (1) or (2).

The position of R relatively to the mass centre is given by—

$$\frac{L}{R} = CZ \text{ (Fig. 157)}$$

The constraint applied to the rod by the crank-pin and slide-bars is such that, for all positions of the gear, whether the rod is arranged to work to the left or to the right of the crank as shown in Fig. 173, by full and dotted lines respectively—

$$\sin \phi = -\frac{a}{l} \sin \theta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Therefore—

$$l \cos \phi = \pm \sqrt{l^2 - a^2 \sin^2 \theta} \quad . \quad . \quad . \quad . \quad . \quad (5)$$

the positive sign to be taken if the rod is to the right of the crank, the negative sign if to the left, because, in the first case, $\cos \phi$ is positive during the whole motion of the crank; in the second case, it is always negative.

In what follows the minus sign will be retained in the radical corresponding with the arrangement of gear shown in full lines in Fig. 173.

From equation (4)—

$$\phi = \sin^{-1}\left(-\frac{a}{l}\sin\theta\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

The angular velocity of the rod—

$$\dot{\phi} = \frac{\dot{\theta}a\cos\theta}{\sqrt{l^2 - a^2\sin^2\theta}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

The angular acceleration—

$$\ddot{\phi} = -\frac{\dot{\theta}^2 a(l^2 - a^2)\sin\theta}{(l^2 - a^2\sin^2\theta)^{\frac{3}{2}}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

If the rod is arranged to the right of the crank $\dot{\phi}$ will be negative and $\ddot{\phi}$ positive.

Using the value of ϕ given by equation (8) in (3), and writing ω, the constant angular velocity of the crank, for $\dot{\theta}$, the couple L is given by—

$$\mathrm{L} = -\mathrm{M}k^2\left\{\frac{\omega^2 a(l^2 - a^2)\sin\theta}{(l^2 - a^2\sin^2\theta)^{\frac{3}{2}}}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

from which its magnitude can be computed for any given value of θ, the crank angle. It will be noticed that at the dead centres, where $\sin\theta$ is zero, the couple vanishes.

Again, the position of the mass centre relatively to O, the origin, is found by taking the sum of the two vectors, a, θ, and b, ϕ; or by taking the sum of their horizontal and vertical components. Thus, x is always the vector sum of $a\cos\theta$ and $b\cos\phi$, and y the vector sum of $a\sin\theta$ and $b\sin\phi$. Then—

$$x = a\cos\theta + b\cos\phi \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

Velocity of C parallel to X—

$$\dot{x} = -a\dot{\theta}\sin\theta - b\dot{\phi}\sin\phi \quad . \quad . \quad . \quad . \quad . \quad (11)$$

Acceleration of C parallel to X, remembering that $\ddot{\theta} = 0$—

$$\ddot{x} = -a\dot{\theta}^2\cos\theta - b\dot{\phi}^2\cos\phi - b\ddot{\phi}\sin\phi \quad . \quad . \quad . \quad (12)$$

Similarly—

$$y = a \sin \theta + b \sin \phi \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

Velocity of C parallel to Y—

$$\dot{y} = a\dot{\theta} \cos \theta + b\dot{\phi} \cos \phi \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

Acceleration of C parallel to Y—

$$\ddot{y} = -a\dot{\theta}^2 \sin \theta - b\dot{\phi}^2 \sin \phi + b\ddot{\phi} \cos \phi \quad . \quad . \quad . \quad (15)$$

Therefore—

$$\tan \alpha = \frac{\ddot{y}}{\ddot{x}} = \frac{-a\dot{\theta}^2 \sin \theta - b\dot{\phi}^2 \sin \phi + b\ddot{\phi} \cos \phi}{-a\dot{\theta}^2 \cos \theta - b\dot{\phi}^2 \cos \phi - b\ddot{\phi} \sin \phi} \quad . \quad (16)$$

and—

$$R = \frac{M\ddot{x}}{\cos \alpha} = \frac{M\ddot{y}}{\sin \alpha} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

145. Values of $\dot{\phi}$, $\ddot{\phi}$, and $\ddot{x}$, $\ddot{y}$, at the Dead Centres.—When $\theta = 0$, or 180°, $\cos \theta = \pm 1$ and $\sin \theta = 0$. Substituting these values in equation (7)—

$$\dot{\phi} = \frac{a\dot{\theta}}{l} = \frac{a\omega}{l} \text{ when } \theta = 0$$

and—

$$\dot{\phi} = -\frac{a\dot{\theta}}{l} = -\frac{a\omega}{l} \text{ when } \theta = 180°$$

A substitution of the value of the sine in equation (8) shows that—

$$\ddot{\phi} = 0$$

Again, substituting these values of $\dot{\phi}$ and $\ddot{\phi}$ in equation (12), writing ω for $\dot{\theta}$, and noting that $\phi = 180°$ when $\theta = 0$ or 180°, so that $\cos \phi$ is negative—

$$\ddot{x} = -a\omega^2\left(1 - \frac{ab}{l^2}\right) \text{ when } \theta = 0 \quad . \quad . \quad . \quad (18)$$

and—

$$\ddot{x} = a\omega^2\left(1 + \frac{ab}{l^2}\right) \text{ when } \theta = 180° \quad . \quad . \quad . \quad (19)$$

$\ddot{y}$ vanishes for both values of θ.

146. The Acceleration of the Cross-head in the Line of Stroke may be found by putting l for b in equation (12).

Let $\ddot{x}_1$ represent the acceleration in the line of stroke; then—

$$\ddot{x}_1 = -a\dot{\theta}^2 \cos \theta - l\dot{\phi}^2 \cos \phi - l\ddot{\phi} \sin \phi \quad . \quad . \quad . \quad (20)$$

If the values of $\dot{\phi}$, $\ddot{\phi}$, from equations (7) and (8), and the values of $\sin \phi = -\frac{a}{l} \sin \theta$ and $\cos \phi = \pm\frac{1}{l}\sqrt{l^2 - a^2 \sin^2 \theta}$, be substituted in this equation, it reduces to—

$$\ddot{x}_1 = -a\omega^2\left\{\cos \theta \mp \frac{al^2 \cos 2\theta + a^3 \sin^4 \theta}{(l^2 - a^2 \sin^2 \theta)^{\frac{3}{2}}}\right\} \quad . \quad . \quad . \quad (21)$$

in which the upper sign is to be taken if the rod is to the left of the crank, and the lower sign if to the right.

The acceleration of the cross-heads at the dead centres may be found by writing l for b in equations (18) and (19), giving,

when $\theta = 0$ . . . . (22)

$$\ddot{x}_1 = -a\omega^2\left(1 - \frac{a}{l}\right),$$

and when $\theta = 180°$ . . . . (23)

$$x_1 = a\omega^2\left(1 + \frac{a}{l}\right)$$

147. Example.—Given that—

$$a = 1 \text{ foot}$$
$$b = 1{\cdot}2 \text{ feet}$$
$$l = 3 \text{ feet}$$
$$k^2 = 0{\cdot}81$$

and that the rod is arranged to the left of the crank in the way shown in Fig. 173, find the magnitude, direction, and position of R, and the value of the couple L, when $\theta = 120°$ and at the dead centres, in terms of the mass of the rod M and the angular velocity ω.

If the values of $\dot{\phi}$ and $\ddot{\phi}$ in terms of θ, given by equations (7) and (8), be substituted in equations (12) and (15), giving the value of $\ddot{x}$ and $\ddot{y}$, $\dot{\theta}^2$ will appear as a factor of every term. Hence, in computing the values of these expressions, $\dot{\theta}$ may be taken equal to unity; $\dot{\theta}^2 = \omega^2$ must then be introduced as a factor in the final result. Hence, wherever $\dot{\theta}$ appears, take it equal to unity.

From the given data—

$\phi =$ 196° 46′ from equation (6)
$\dot{\phi} = -0{\cdot}174\omega$ radians per second from equation (7)
$\ddot{\phi} = -0{\cdot}292\omega^2$ radians per second per second from equation (8)
$\ddot{x} =$ $0{\cdot}434\omega^2$ feet per second per second from equation (12)
$\ddot{y} = -0{\cdot}520\omega^2$ feet per second per second from equation (15)

Then—

$$\tan a = \frac{-0{\cdot}520\omega^2}{0{\cdot}434\omega^2}$$

from which—

$$a = 309° \ 55'$$

Also—

$$R = \frac{M\ddot{x}}{\cos a} = \frac{M\ddot{y}}{\sin a}, \text{ from equations (1) and (2)}$$

Therefore—

$$R = 0{\cdot}68M\omega^2 \text{ in absolute units of force}$$

The value of the couple L, found by substituting the value of $\ddot{\phi}$ in equation (3), is—

$$-0{\cdot}292Mk^2\omega^2 = -0{\cdot}236M\omega^2$$

The perpendicular distance, h_1g (Fig. 157), of R from the mass centre, to cause this couple, is found from—

$$h_1g = \frac{L}{R} = \frac{0{\cdot}236M\omega^2}{0{\cdot}68M\omega^2} = 0{\cdot}35 \text{ feet approximately}$$

R must be placed so that the couple is negative, *i.e.* clockwise.

At the dead centres, using equations (18) and (19)—

$$R = -0{\cdot}96M\omega^2 \text{ when } \theta = 0$$
$$R = 1{\cdot}96M\omega^2 \text{ when } \theta = 180°$$

L vanishes for both angles.

The acceleration of the cross-head, found from equations (20) or (21), is—

$$0{\cdot}33\omega^2$$

At the dead centres, using equations (22) and (23), the acceleration is—

$$-0{\cdot}66\omega^2 \text{ when } \theta = 0°$$

and—

$$1{\cdot}33\omega^2 \text{ when } \theta = 180°$$

APPENDICES.

APPENDIX I.

(*With reference to Article* 104.)

The Acceleration of the Piston.—Professor Schröter has pointed out to the author that Klein's construction (Art. 104) is to be found in a paper by the late Professor Kirsch, entitled, "Ueber die graphische Bestimmung der Kolbenbeschleunigung," which was published in the *Zeitschrift Verein Deutsche Ingenieure*, 1890, page 1320.

Mr. G. T. Bennett, of Emmanuel College, Cambridge, sent the author the following simple construction in August, 1902:—

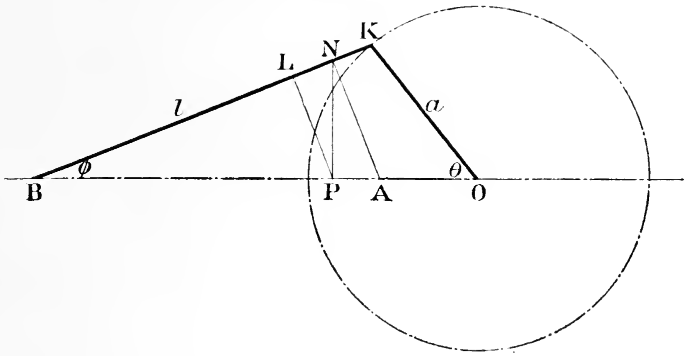


FIG. 174.

Let OK (Fig. 174) be the crank and KB the connecting-rod. On the connecting-rod take a point L, such that—

$$KL \times KB = KO^2$$

That is, KO is a mean proportional between KL and KB, so that the point L may be fixed by dropping a perpendicular from O, on to the rod, when the crank is at right angles to the line of stroke.

Then, the crank standing at any angle with the line of stroke, draw LP at right angles to the connecting-rod, PN at right angles to the line of stroke, and, finally, NA at right angles to the rod. AO is the acceleration of the point B to the scale on which KO represents the acceleration of the crank-pin K.

At the dead points, when the connecting-rod and therefore the point L is in the line of stroke, the distance between the points L and O gives the acceleration. Thus, in Fig. 175, LO is the acceleration at the 0° dead point, and L_1O the acceleration at the 180° dead point.

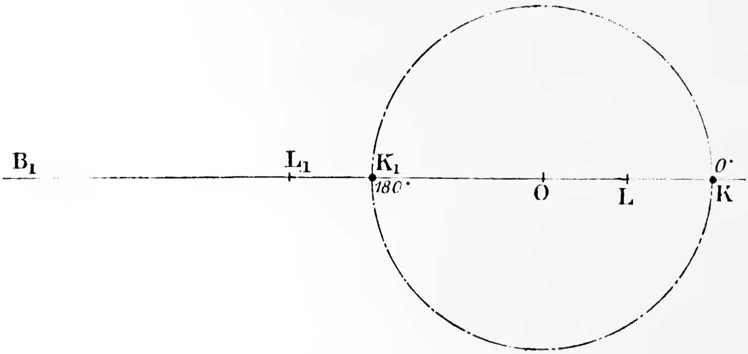


FIG. 175.

Proof:—

Let θ be any crank angle (Fig. 174), and let ϕ be the corresponding angle between the connecting-rod and the line of stroke. Also let a be the crank radius, and l the length of the connecting-rod. Then—

$$l \sin \phi = a \sin \theta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Differentiating this with regard to the time t, and writing $\dot{\phi}$ for $\frac{d\phi}{dt}$, $\dot{\theta}$ for $\frac{d\theta}{dt}$, and assuming θ constant and equal to ω—

$$l\dot{\phi} \cos \phi = a\omega \cos \theta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Eliminating θ from equations (1) and (2)—

$$\dot{\phi}^2 = \omega^2 - \frac{l^2 - a^2}{l^2} \cdot \omega^2 \cdot \sec.^2 \phi \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Differentiating this again with regard to the time—

$$\ddot{\phi} = -\frac{l^2 - a^2}{l^2} \cdot \omega^2 \cdot \frac{\sin \phi}{\cos^3 \phi} \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Again, the position of the cross-head B is given by—

$$x = a \cos \theta + l \cos \phi \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Differentiating this with regard to the time, the velocity of the cross-head towards O is—

$$-\dot{x} = a\omega \sin \theta + l\dot{\phi} \sin \phi \quad . \quad . \quad . \quad . \quad . \quad (6)$$

Differentiating this velocity with regard to the time, the acceleration is—

$$-\ddot{x} = a\omega^2 \cos \theta + l\ddot{\phi} \sin \phi + l\dot{\phi}^2 \cos \phi \quad . \quad . \quad . \quad (7)$$

Substituting the values of $\dot{\phi}^2$ and $\ddot{\phi}$ from (3) and (4)—

$$-\ddot{x} = \omega^2 \left\{ a \cos \theta + l \cos \phi - \frac{l^2 - a^2}{l} \sec^3 \phi \right\} \quad . \quad . \quad (8)$$

In the construction given above—

$$\text{OB} = a \cos \theta + l \cos \phi$$

and from the relation KB . KL $= a^2$, that is, $l(l - \text{BL}) = a^2$

$$\text{BL} = \frac{l^2 - a^2}{l}$$

Therefore—

$$\text{BP} = \frac{l^2 - a^2}{l} \sec \phi$$

$$\text{BN} = \frac{l^2 - a^2}{l} \sec^2 \phi$$

and, finally—

$$\text{BA} = \frac{l^2 - a^2}{l} \sec^3 \phi$$

Therefore OB − BA = AO is seen by expression (8) to represent the acceleration of the point B to the scale on which KO represents the acceleration of the crank-pin K.

APPENDIX II.

(*With reference to Article* 123.)

On the Maximum Amplitude of the Forced Vibration.—As the value of p increases towards the value of q, the amplitude y of the forced vibration gradually increases, until, just before synchronism, y attains a maximum value. The value of p corresponding to the maximum value of y is so nearly that of q, that for all practical purposes it may be taken equal to q, unless the damping is very great. In order to find the value of p, which makes y a maximum, it is convenient to write equation (6), page 207, in a different form.

Since—

$$\tan \varepsilon = \frac{pb}{q^2 - p^2}$$

$$\sin \varepsilon = \frac{pb}{\sqrt{(q^2 - p^2)^2 + p^2b^2}}$$

and, substituting this value of $\sin \varepsilon$ in equation (6), page 207—

$$y = \frac{\mathrm{P}}{\sqrt{(q^2 - p^2)^2 + p^2b^2}} \cos (pt - \varepsilon)$$

For given values of q, p, P, and b, y is a maximum when $\cos (pt - \varepsilon) =$ unity.

For given values of q, P, and b, the value of p which makes y a maximum is found by putting $\cos (pt - \varepsilon)$ equal to unity, and then equating the differential coefficient of y with regard to p to zero. Thus—

$$2(p^2 - q^2) + b^2 = 0 \text{ for a maximum}$$

Hence—

$$p = \sqrt{q^2 - \frac{b^2}{2}}$$

The form of this equation shows to what a small extent p differs from q when y is a maximum. Taking a numerical case corresponding to the curve marked $b = \frac{1}{8}$ in Fig. 139, $q = 30$, therefore—

$$p = \sqrt{900 - 0{\cdot}0078}$$
$$= 29{\cdot}999$$

APPENDIX III.

(*With reference to Article* 130).

Weight of the Parts, and the Turning Moment on the Crank.—The effect of the weight of the parts on the turning moment is not included in the expression for P given on p. 220. Strictly a term should be added to allow for this, but usually the effect is negligible in comparison with the effect of the steam pressure and the accelerating forces. In a horizontal engine the weight of the reciprocating parts has no effect on the turning moment; and the weight of the unbalanced revolving parts has a small effect, which is practically reduced to zero if the revolving parts are balanced by masses placed on the prolongation of the crank-arms. This is also true for the revolving masses of a vertical engine, but in this case the weight of the reciprocating masses exerts its maximum influence on the turning moment. The effect of the weight of the reciprocating parts on the turning moment in the case of a vertical engine may be allowed for by adding a term to the formula for P given on p. 220, so that the value of P is increased for the downstroke, and is diminished for the upstroke by an amount equal to the weight of the reciprocating parts.

Thus, if W is the weight of the reciprocating parts in pounds—

$$P = (p - p_1)A - \frac{Ma}{g} + W \text{ for the downstroke}$$

$$P = (p - p_1)A - \frac{Ma}{g} - W \text{ for the upstroke}$$

APPENDIX IV.

(*With reference to Article* 138.)

Bennett's Constructions for finding a Point in the Line of Action of the Force producing the Instantaneous Acceleration of a Rigid Link moving in a Plane.—Several methods of finding a point in the line of action of the resultant force producing the instantaneous acceleration of the connecting-rod are given in Art. 138. All these methods require that the directions of the acceleration of two points in the rod be given, and, in addition, the magnitude of one of them must be given also. Mr. G. T. Bennett has given the author the following general proposition and corollaries relating to the motion of a lamina in a plane, by means of which a point may be found in the line of action of the resultant accelerating force when only the directions of acceleration of two points in the lamina are given. Stating the proposition formally :—

> Given the instantaneous directions of the acceleration of any two points of a lamina moving in a plane, find a point in the line of action of the resultant accelerating force.

Let B and K (Fig. 176) be any two points in the lamina, and let the respective directions of acceleration be BO and KO. Let C be the mass centre of the lamina, and k the radius of gyration about an axis perpendicular to the lamina through its mass centre.

Construction.—Produce the given directions of acceleration to meet in O.

Draw a circle through the three points, B, K and O.

Produce the line joining O and C to cut the circle in Y.

Take a point Z on the line OY so that $YC \cdot CZ = k^2$.

Then Z is a point in the line of action of the resultant accelerating force.

Proof.—The proof depends upon the fact that the angle between the direction of acceleration of any selected point in the lamina, and the line joining that point to the centre of acceleration in the lamina is constant. Thus, referring to Fig. 177, let P be any point in the lamina and X, the centre of zero acceleration. If ω is the angular velocity of the lamina, and ω' the angular acceleration, the radial acceleration of P is $\omega^2 \,.\, PX$: and the tangential acceleration of P is $\omega' \,.\, PX$. Hence the angle ϕ, made by the resultant acceleration of P with the line PX, is such that—

$$\tan \phi = \frac{\omega' PX}{\omega^2 PX} = \frac{\omega'}{\omega^2}$$

and thus, wherever P is taken in the lamina, the angle ϕ remains constant. Thus the angles made by the respective accelerations of P_1 and P_2 with the lines P_1X and P_2X are each equal to ϕ.

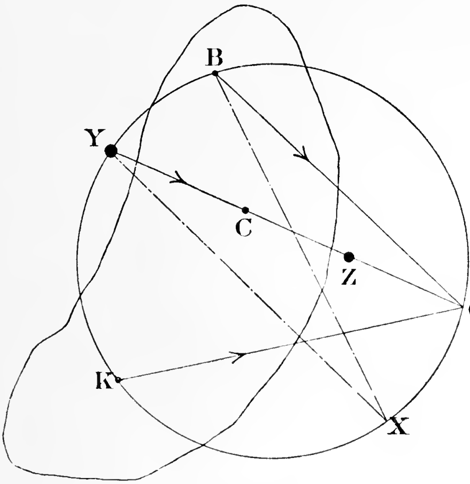


FIG. 176.

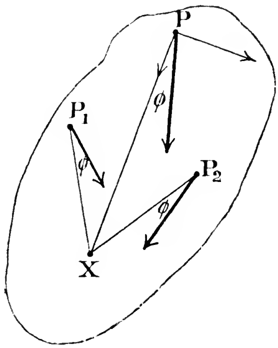


FIG. 177.

The proof of the preceding construction is now simple. Take X to be the centre of zero acceleration in the lamina. Then the angles XKO and XBO are equal. Hence X is a point on the circle OBX; and YO, making an angle with XY equal to the other two angles, is the direction of acceleration of Y.

Again, since Z is taken so that $YC \,.\, CZ = k^2$, the lamina may be represented by two particles placed respectively at Y and Z, these two particles forming an instantaneous equivalent dynamical system.

The force causing the acceleration of the particle at Y acts along the line YZO, and the force causing the acceleration of the particle at Z acts through Z. Therefore the resultant of these two forces acts through Z. Q.E.D.

Corollary 1.—The point Z may be found also by the following construction (Fig. 178):—

Produce the directions of acceleration to meet in O, and draw the circle as before, and then produce the line joining K and C to cut the circle in V.

Take a point A so that KC . CA = k^2. Join VO, and through A draw AZ parallel to VO to cut the line joining O and C in Z.

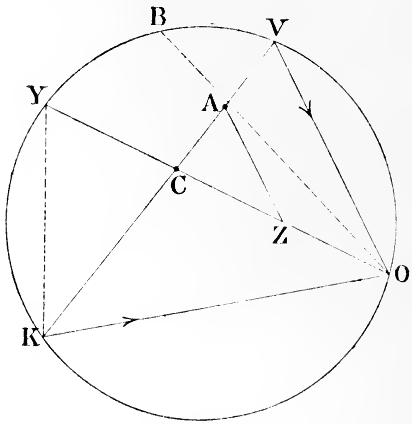


Fig. 178.

The proof of this follows from the preceding proposition. Assuming the point Z to be known, by the preceding proposition YC . CZ = k^2, and this is also equal to KC . CA. Therefore—

$$KC : CY = CZ : CA$$

Therefore the triangles KCY and ZCA are similar, and hence the angle at A is equal to the angle at Y. But by the property of the circle the angle at V is equal to the angle at Y, therefore the angles at A and at V are equal. Therefore the line AZ is parallel to VO, hence the construction.

Corollary 2.—If the points K, C, and B are in a line, the construction reduces to a simple one, which may be applied to the connecting-rod problem in order to find Z. Thus, in Fig. 179, suppose BO and KO are the given directions of the acceleration of the points B and K, and that C is the mass centre of the rod BK. Produce the given directions of acceleration to meet in O, and join OC. Take the point A such that KC . CA = k^2. Draw AZ parallel to

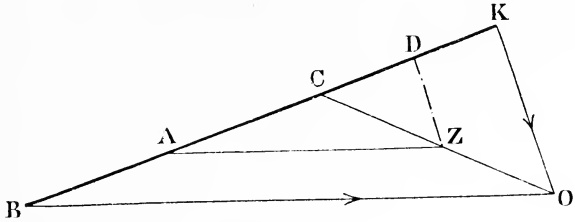


FIG. 179.

BO. Z is then a point in the line of action of the resultant accelerating force. The point may equally well be found by taking D such that BC . CD = k^2, and then by drawing DZ parallel to KO.

Another proof of this construction is given in "Valves and Valve Gear Mechanisms," by the author, and the construction is applied to problems connected with the accelerating forces acting on the links of a Joy valve gear.

Corollary 3.—If the points KBO (Figs. 176 and 178) lie on a

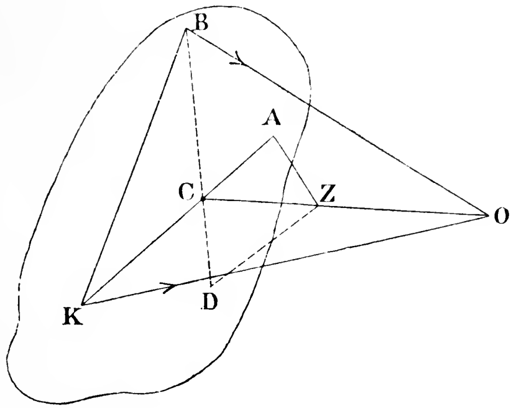


FIG. 180.

circle of large diameter, it may be inconvenient to draw the circle. Then the following construction may be used:—

Produce the directions of acceleration to meet in O (Fig. 180) and join O to C, the mass centre of the lamina.

Produce the line joining the points K and C to A, A being taken so that KC . CA $= k^2$. Then make the angle at A equal to the angle KBO, and complete the triangle CAZ. Z is the point required.

A reference to Fig. 178 will make the proof of this clear. In Fig. 178, the angle at A is equal to the angle at V, and this is also equal to the angle KBO, since KBO and KVO are angles subtended by the common chord KO.

The construction may equally well be carried out by producing the line joining B and C to D, D being taken so that BC . CD $= k^2$, and then by making the angle at D equal to the angle BKO.

APPENDIX V.

The Balancing of Engines of the Type where the Axis of the Crank-Shaft is at Right Angles to the Plane containing the Centre Lines of the Cylinders.—The conditions of balance have been investigated in the preceding pages for multi-cylinder engines of the usual marine type, that is to say, the axis of the crank-shaft and the centre lines of the cylinders are in the same plane; the centre lines of the cylinders are parallel, and the cylinders themselves are all placed on the same side of the crank-shaft, with the single exception of the case considered on page 249.

There are some interesting problems, however, in connection with types of engines which differ from this standard form, but which possess the common property that the axis of the crank-shaft is at right angles to the plane containing the centre lines of the cylinders. The respective mass centres of all the moving parts of properly designed engines of this class are in one plane, the plane of the cylinder centre lines, and therefore there is no centrifugal couple to consider of the kind discussed on page 22. The balancing problem, in fact, reduces to problems concerned with the equilibrium of forces acting in a plane.

Fig. 181 shows a diagrammatic view of an engine of this type. It will be seen that there are two cylinders at right angles driving on to one crank.

The unbalanced force along the line of stroke of either cylinder is given by Expression (2), page 126. In applying this to the type of engine under discussion, care must be taken to measure the respective crank-angles for the several lines of parts from initial lines similarly situated with regard to each line of stroke. It is convenient to define the initial line for any one line of parts as the line drawn from the centre of the crank-shaft along the line of stroke towards the cylinder. Also it is to be remembered that all angles are to be measured from these initial lines in the positive or counter-clockwise direction. Thus, referring to Fig. 181, the angle from which the instantaneous value of the unbalanced force along the line AB is to be calculated is measured from the initial line OA, in the positive direction, and is represented by θ: and the angle from which the corresponding instantaneous value of the unbalanced forces along the line CD is to be calculated is measured from the initial line OC, in the positive direction, and is $270 + \theta$.

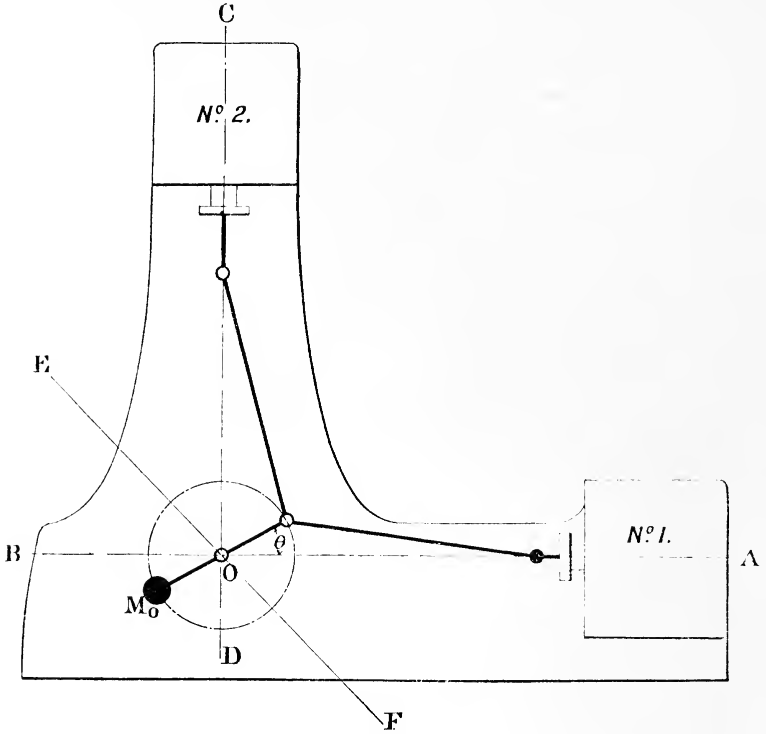


Fig. 181.

Consider the primary and secondary forces separately by means of the principle explained on page 127.

The instantaneous value of the primary unbalanced force along the line AB due to the acceleration of the reciprocating masses M_1 of No. 1 cylinder is, r being the crank radius,—

$$M_1\omega^2 r \cos \theta$$

At the same instant, the corresponding instantaneous value of the primary unbalanced force along the line CD due to the reciprocating masses M_2 of No. 2 cylinder is—

$$M_2\omega^2 r \cos(270 + \theta) = M_2\omega^2 r \sin\theta$$

The primary unbalanced force due to the reciprocating masses is therefore the resultant of these two forces, that is—

$$\omega^2 r\sqrt{(M_1^2 \cos^2\theta + M_2^2 \sin^2\theta)}$$

If $M_1 = M_2$, this reduces to—

$$M_1\omega^2 r$$

acting along the crank-arm.

Thus the reciprocating masses cause a primary unbalanced force of constant magnitude to act along the crank-arm, and this force may therefore be balanced by placing a balance-weight M_0 at radius r_0 opposite to the crank, so that—

$$M_0 r_0 = M_1 r$$

The revolving masses may also be balanced at the same time by suitably increasing M_0. The unbalanced revolving mass is the sum of the unbalanced parts of the crank itself, and the unbalanced revolving masses belonging to each line of parts. Let this sum be M at crank radius. Then to include the balance-weight required for the revolving parts with that required above for the reciprocating parts M_0 must be found from—

$$(M_1 + M)\,r = M_0 r_0$$

Referring to Expression (2), page 126, and to Fig. 181, it will be seen that the secondary unbalanced force along AB is to be computed from the angle 2θ and along CD from the angle $2(270 + \theta) = (180 + 2\theta)$.

The secondary unbalanced force along AB is therefore—

$$\frac{M_1\omega^2 r^2}{l}\cos 2\theta$$

and along CD it is—

$$\frac{M_1\omega^2 r^2}{l}\cos(180 + 2\theta) = -\frac{M_1\omega^2 r^2}{l}\cos 2\theta$$

The resultant secondary unbalanced force is therefore—

$$\frac{M_1\omega^2 r^2}{l}\sqrt{\cos^2 2\theta + \cos^2 2\theta} = \sqrt{2}\,\frac{M_1\omega^2 r^2}{l}\cos 2\theta$$

inclined at an angle ϕ to AB such that—

$$\tan \phi = \frac{-\cos 2\theta}{\cos 2\theta} = -1$$

therefore—

$$\phi = 315^\circ$$

That is to say, the resultant unbalanced secondary force is variable in magnitude and sense, but acts along the constant direction EF (Fig. 181). Since the magnitude of the force is a function of the crank-angle, and its direction is always along EF it cannot be balanced by a revolving mass. To balance it a mass of M_1 pounds would have to be reciprocated with simple harmonic motions along the line EF by a crank whose length is $\frac{r}{4l}$ times the radius of the main crank, and whose speed is twice the speed of the main crank, as explained on page 128.

Let the turning moment on the crank due to the line of parts AB be represented by the crank-effort curve, Fig. 146: and let the corresponding turning moment due to the line of parts CD be also represented by the same curve. The actual turning moment on the crank at any instant will then be represented by the ordinates of a curve found by taking the sum of the ordinates of two curves like Fig. 146 after they have been displaced horizontally relative to one another through a distance representing 270°, since when No. 1 piston is at a dead point, and the crank angle is zero, the crank angle corresponding to No. 2 piston is 270°. The engine is therefore equivalent as regards turning moment to an engine of the ordinary type with two cranks at right angles.

Fig. 182 shows a diagrammatic arrangement of an engine in which three cylinders are arranged in one plane, their centre lines meeting at O, the intersection of the axis of the crank-shaft with the plane of the cylinders. The centre lines are mutually inclined to one another at 120°, and the pistons are coupled to one crank.

Adhering to the convention regarding the measurement of the crank angles specified above, it will be understood from the diagram that the angle to be used in Expression (2), page 126, for the calculation of the unbalanced force along the line OX is θ, along the line OA it is $240^\circ + \theta$, and along the line OB, $120^\circ + \theta$.

Let P, Q, R be the simultaneous instantaneous values of the unbalanced forces along the directions OX, OA, and OB respectively.

Then resolving these along and at right angles to the direction OX, the instantaneous force along OX is—

$$P + Q \cos 120 + R \cos 240 = S \qquad (1)$$

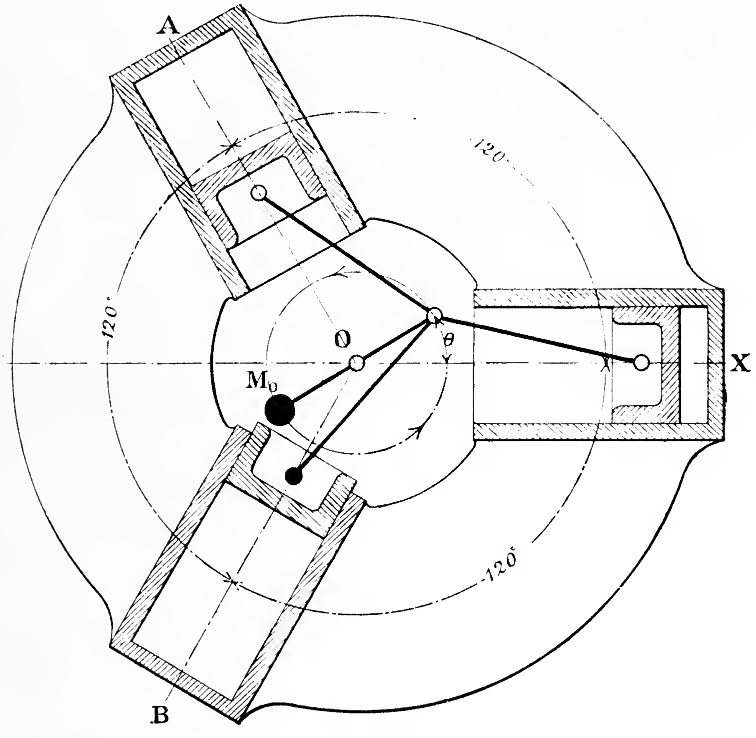


FIG. 182.

and at right angles to OX

$$Q \sin 120 + R \sin 240 = T \qquad (2)$$

The instantaneous value of the resultant unbalanced force is

$$\sqrt{S^2 + T^2} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (3)$$

acting in the direction ϕ such that—

$$\tan \phi = \frac{T}{S} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (4)$$

It only remains to insert the proper values of P, Q, and R in these equations. Considering first the primary forces—

$$P = M_1\omega^2 r \cos\theta$$
$$Q = M_2\omega^2 r \cos(240 + \theta)$$
$$R = M_3\omega^2 r \cos(120 + \theta)$$

If $M_1 = M_2 = M_3$ the equations reduce to a simple form. Thus, introducing these values in equations (1) and (2), with M_1 for the mass of the reciprocating parts for each cylinder, they reduce to

$$S = \tfrac{3}{2}M_1\omega^2 r \cos\theta$$

$$T = \tfrac{3}{2}M_1\omega^2 r \sin\theta$$

and the resultant force becomes $\frac{3}{2}M_1\omega^2 r$ acting along the crank from the centre of the crank-shaft towards the crank-pin. Hence the unbalanced primary force due to the reciprocation of the three sets of parts may be balanced by a single balance-weight M_0 placed in the prolongation of the crank-arm at a radius r_0, such that

$$\tfrac{3}{2}M_1 r = M_0 r_0$$

As before, the revolving masses may be balanced also by suitably increasing the mass of M_0. The unbalanced revolving mass is the sum of the unbalanced parts of the crank itself, and the unbalanced revolving masses from each line of parts. Let this sum be M at crank radius. Then to include the balance-weight required for the revolving parts with that required above for the reciprocating parts, M_0 must be found from

$$(\tfrac{3}{2}M_1 + M)r = M_0 r_0$$

The angles from which the secondary unbalanced forces are to be calculated are—

along OX, 2θ
along OA, $2(240 + \theta) = (120 + 2\theta)$
along OB, $2(120 + \theta) = (240 + 2\theta)$

Equations (1), (2), (3), and (4) may now be used to calculate the resultant unbalanced secondary force, providing that the values of P, Q, and R are calculated from the expression—

$$P = \frac{M_1\omega^2 r^2}{l}\cos 2\theta$$

$$Q = \frac{M_1\omega^2 r^2}{l}\cos(120 + 2\theta)$$

$$R = \frac{M_1\omega^2 r^2}{l}\cos(240 + 2\theta)$$

Inserting these values in expressions (1) and (2) they reduce to—

$$S = \frac{\frac{3}{2}M_1\omega^2 r^2}{l}\cos 2\theta$$

$$T = -\frac{\frac{3}{2}M_1\omega^2 r^2}{l}\sin 2\theta$$

and the resultant force is therefore equal to—

$$\frac{\frac{3}{2}M_1\omega^2 r^2}{l}$$

acting in the direction ϕ given by—

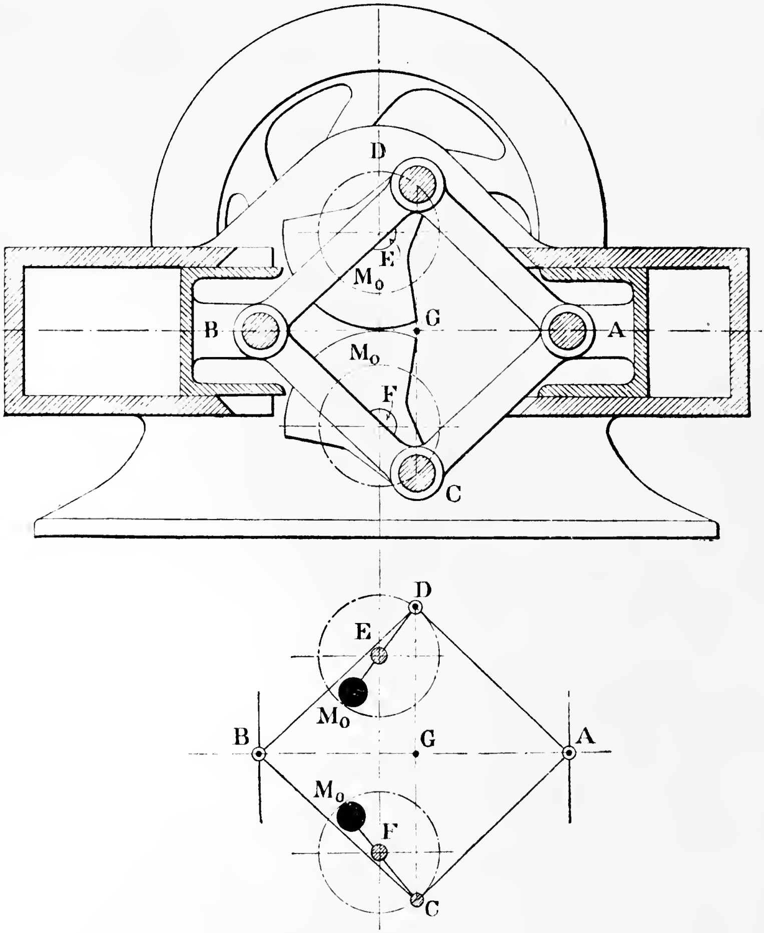
$$\tan\phi = \frac{-\sin 2\theta}{\cos 2\theta}$$

Interpreted geometrically, this means that the secondary unbalanced force due to the three reciprocating masses is equivalent to that due to a mass of M_1 pounds at radius $\frac{r}{4l}$ revolving with twice the speed of the main crank and in the opposite direction, the imaginary radius being in coincidence with the main crank when $\theta = 0$. This could be balanced by a mass properly placed and revolving twice as fast as the main crank.

The turning moment in this type of engine is similar to that of an engine of the usual type arranged with three cylinders and with cranks at 120°.

Another interesting example of an engine where the mass centres of the moving parts are in a plane is afforded by the well-known Lanchester Motor Car Engine, shown in diagrammatic form in Fig. 183, the centre lines of the parts being sketched below in Fig. 184.

A frame of links, or four-bar chain, is connected at the joints D and C with the crank-pins of the cranks which are centred at E and F, and the chain carries a piston at each of the joints A and B as shown. The crank-shafts revolve in opposite directions, with equal angular velocities being connected by gearing (not shown in the sketch) to insure this. The pistons are constrained by the cylinders to move along a common centre line. Supposing for the moment that the joints D and C are disconnected from the crank-pins, the motion possible to the chain as a whole is one of trans-



FIGS. 183, 184.

lation along the common centre line of the cylinders, and the pistons are free at the same time to move relatively to one another since the bars of the chain are free to close up or open out, and whilst doing so the joints D and C move vertically. When the joints C and D are connected to their respective crank-pins the common motion of translation of the pistons and the chain, together with the superposed relative motion of the pistons, becomes constrained because the joints C and D are now compelled to move in circles. It will be observed that, assuming the bars of the chain to be equal amongst themselves and the pistons also to be equal, the mass centre of the combination of the pistons and bars falls at the intersection of the diagonals of the bars G in every configuration of the gear. It will be seen therefore that, assuming the cranks to revolve uniformly, the point G moves with simple harmonic motion, notwithstanding the complex motion of the pistons.

The unbalanced force along the line of stroke is that due to the mass of the two pistons and the four-bar chain concentrated at the point G, moving with simple harmonic motion. This force can therefore be balanced by means of equal revolving masses M_0, M_0 placed opposite to the cranks, because the sum of the projections of the forces due to them parallel to the line of stroke is a true simple harmonic motion, and the projections on a line at right angles to the stroke mutually balance since the cranks revolve in opposite directions. If M_0 is the mass of one balance-weight at a radius r_0, and if M is the mass of the unbalanced parts of one crank at crank radius r, and if M_1 is the total mass of the two pistons and the four-bar chain, the relation between these quantities is

$$M_0 r_0 = (\tfrac{1}{2}M_1 + M)r$$

Another feature in the design is the manner in which the impulsive reaction on the frame, due to the explosions in the cylinder, is balanced. A flywheel is keyed to each crank-shaft (one only is shown in the figure), and the moments of inertia of the two wheels are equal. It will be understood from the sketch that, when the speed varies, the angular accelerations of the wheels are equal and opposite, and therefore the frame of the engine is entirely relieved of the couple equal and opposite to that required for the acceleration of one wheel.

EXERCISES.

GRAPHICAL work is connected with most of the following exercises. The student is recommended to get a set of scales in which the foot is divided decimally instead of into inches. The glass scales made by Zeiss, divided into centimetres and tenths, are very useful, and are more accurate than the ordinary type of boxwood or ivory scale. The scale divisions are engraved on the side of the glass, which is put in contact with the paper; parallax in reading the scale is thereby avoided.

To set out an angle, as α (Fig. 104), measure out the distance x equal to unity, using as large a scale as possible. Then set out y at right angles to OX, equal in length to the trigonometrical tangent of the given angle, the value of which is to be found from a table of tangents.

Conversely to measure any angle, set out x equal to unity, to as large a scale as possible, and draw y at right angles. Measure y, and look out the corresponding value of the angle from a table of tangents.

In the following examples many of the results are given approximately, angles being given to the nearest degree, and magnitudes to the nearest whole number.

Unless otherwise stated, an angle is always to be measured in the way shown in Fig. 6, from a horizontal initial line. Direction will be indicated in some cases by subscript figures or signs; thus—

$$12_{30°}, \quad 603_{340°}, \quad AB_{\theta°}$$

mean, respectively, a vector quantity whose magnitude is 12 and whose direction is 30° with the initial direction; a quantity of magnitude 603 whose direction is 340° with the initial line, measured of course counter-clockwise; a vector of magnitude AB inclined θ degrees to the initial line.

1. Find the sum of the vectors given in Schedule 1, setting them out in any four of the possible twenty-four different orders.

2. Show that, if the magnitudes of the three vectors are equal—

$$AB_{0^\circ} + BC_{120^\circ} + CD_{240^\circ} = 0$$

and that—

$$AB_{90^\circ} + BC_{180^\circ} + CD_{270^\circ} = AB_{180^\circ}$$

and that—

$$AB_{0^\circ} + BC_{60^\circ} + CD_{60^\circ} = 2{\cdot}65_{41^\circ}$$

3. Assuming the following set of vectors to represent forces acting at a point, find the force required to maintain equilibrium—

$$12_{0^\circ},\quad 24_{30^\circ},\quad 36_{120^\circ},\quad 10_{190^\circ}$$

Answer.—$41{\cdot}7_{263^\circ}$.

4. Find the velocity of a point in the rim of a driving-wheel turning in the positive direction, belonging to a locomotive running at 60 miles per hour—

(1) When the point is in its highest position.

(2) When in its lowest position.

(3) When the radius of the point is at 120° with the initial line.

Diameter of the wheel, 7 feet.

(Take the vector sum of the velocity of the wheel-centre and the velocity of the point in the several cases, supposing the wheel-centre to be fixed.)

Answers.—

(1) 176_{180° feet per second. (2) 0. (3) 170_{195° feet per second.

5. Find the difference of the vectors—

$$(1)\ 12_{30^\circ} - 30_{180^\circ}.$$
$$(2)\ 30_{180^\circ} - 12_{30^\circ}.$$

Answers.—(1) $40{\cdot}8_{9^\circ}$. (2) $40{\cdot}8_{189^\circ}$.

6. A bicyclist rides at 12 miles per hour due north. Find the direction from which the wind appears to him to be blowing, and find the component velocity of the wind he must actually ride against when the actual velocity of the wind is—

(1) 12 miles per hour blowing from the E.
(2) 12 " " " S.
(3) 12 " " " N.
(4) 12 " " " N.E.

Answers.—

(1) Direction N.E. Northerly component, 12 miles per hour.
(2) No wind.
(3) N. Northerly component, 24 miles per hour.
(4) 23° to E. of North. Northerly component, 20·5 miles per hour.

7. The crank of an engine makes 5 revolutions per second in the clockwise direction, and the crank-pin is 9 inches radius, the connecting-rod is 3 feet long. Find—
(1) the velocity of the cross-head pin relative to the frame,
(2) the velocity of the cross-head pin relative to the crank pin,
(3) the velocity of the crank-pin relative to the cross-head pin,
when the crank angles are respectively 30, 90, and 120 degrees, measured from the initial direction, the cross-head being arranged to the left of the crank-shaft (as shown in Fig. 153).

(The velocity of the cross-head relatively to the crank-pin is given by—

Vector difference (velocity of cross-head *minus* velocity of crank-pin)

The direction of relative velocity is at right angles to the connecting-rod; hence the directions of the three sides of the vector triangle are given and the magnitude of one of them from which the various problems above can be solved.)

Answers.—

Angles	30°	90°	120°
	Feet per second.		
Velocity of cross-head pin relative to frame	$9{\cdot}2_{180^\circ}$	$23{\cdot}55_{180^\circ}$	23_{180°
Velocity of cross-head pin relative to crank-pin	$20{\cdot}6_{277^\circ}$	0°	12_{102°
Velocity of crank-pin relative to cross-head	$20{\cdot}6_{97^\circ}$	0°	12_{282°

8. Defining acceleration as the change of velocity per second, prove formula (2), Art. 9.

(The change of velocity in time dt is the vector difference of the velocity at the end and at the beginning of the interval. When a body is moving uniformly in a circle, the magnitude of the velocity at the beginning and at the end of the interval is the same, but the direction is different. Drawing a triangle to find this difference for a small angular displacement of the radius $d\theta$, which takes place in time dt, it will be observed that in the limit, the vector difference may be expressed by $Vd\theta = \omega r d\theta$, in a direction towards the centre of the circle. The change per second is therefore $\omega r \frac{d\theta}{dt} = \omega^2 r$, and the force corresponding to this acceleration is—

$$M\omega^2 r \text{ poundals}$$
$$\frac{M\omega^2 r}{g} \text{ lbs. weight.}$$

9. Find the two vectors, α, β, required to make—

$$\alpha + \beta + 14_{30} + 20_{90} + 10_0 = 0$$

having given that the magnitudes of the two vectors are respectively—

(1) 20 and 14·9.
(2) 12·5 and 32·7.

Answers.—(1) $20_{230°}$ and $14{\cdot}9_{230°}$.
(2) $12{\cdot}5_{300°}$ and $32{\cdot}7_{210°}$.
or—
$12{\cdot}5_{161°}$ and $32{\cdot}7_{252°}$.

10. A clack-box is bolted to an angle-plate on the face-plate of a lathe for boring out the valve seating. The mass of the box and the angle-plate and attaching bolts is equivalent to 50 pounds at 3 inches radius. What mass must be added to the face-plate at 1 foot 6 inches radius to effect balance?

(The seating cannot be bored truly unless the work is balanced.)

Answer.—8·33 pounds.

11. A rope-wheel, weighing 1 ton, driving the main shaft of a mill at 150 revolutions per minute, caused the bearing nearest to it to heat. The distances of the right-hand and the left-hand bearings from the centre of the wheel are respectively 1 foot 9 inches and 5 feet. The shaft was disconnected, and it was found by experiment that the wheel was out of balance (the rim was not turned inside) to the extent of 34 pounds at 2·8 feet radius. Find the distance of the mass centre of the wheel from the centre of the shaft, and the dynamical load on the bearings. Assuming a coefficient of friction of 0·1, find the horse-power required to overcome the friction of the bearings due to the dynamical load alone. The diameter of the shaft is 5 inches. What fraction is the dynamical load of the static load?

(Horse-power required is given by—

$$\frac{\text{Dynaml. load on bearing} \times \text{coefficient of friction} \times \text{rad. of jour. in ft.} \times \omega}{550}$$

Answers.—

(1) $\frac{1}{2}$ inch nearly. (2) Dynamical load on right-hand bearing, 540 lbs. weight; on the left-hand, 189 lbs. weight. (3) 0·44 H.P. (4) $32\frac{1}{2}$ per cent.

12. The crank-arms and crank-pin of a crank-shaft are equivalent to a mass of 700 pounds at 1 foot radius. The shaft is supported in two bearings, 5 feet centre to centre, and the centre of the crank is 1·5 feet from the left-hand bearing. The diameter of the shaft at the journals is 8 inches. Find the dynamical load on the shaft and on the bearings for a speed of 240 revolutions per minute, and calculate the rate, in horse-power, at which work is dissipated in heat at each bearing, assuming a coefficient of friction of 0·05.

Answers.—

Dynamical load on shaft, 13,708 lbs. weight, load on bearings, left hand, 9595·6 lbs. weight; right hand, 4112·4 lbs. weight.

H.P. loss, left-hand bearing, 7·30; right hand, 3·12; total, 10·42.

13. Draw the bending moment diagram and the shearing force diagram for the shaft of the previous question due to the revolution of the unbalanced mass, assuming the shaft to be a straight one. State the numerical value of the maximum bending moment.

Answer.—Maximum bending moment, 172,720 inch lbs.

14. Find the single mass at 3 feet radius which will balance the mass of question 12. Find also the magnitudes of two masses which will effect balance when they are placed in the same plane of revolution as the disturbing mass, at radii of 4 feet and 5 feet respectively, these radii being inclined to the radius of the disturbing mass at 160 degrees and 220 degrees respectively.

Answers.—(1) $233\frac{1}{3}$ pounds.

(2) $\begin{cases} \text{130 pounds at 4 feet radius.} \\ \text{55 pounds at 5 feet radius.} \end{cases}$

15. What is the direction of the axis of the couple exerted on a double-ended wrench, to tap a nut with a right-hand thread?

Answer.—

A line drawn parallel to the axis of the tap with the arrow-head on it pointing from the wrench towards the point of the tap.

16. Add the couples—

$$14_{0^\circ},\ 20_{30^\circ},\ 10_{90^\circ}$$

Answer.—

$37{\cdot}2_{33^\circ}$, meaning that the axis is inclined 33 degrees to the initial line, and that the moment of the couple is 37·2.

17. A three-legged table touches the ground at points which joined, form an equilateral triangle, ABC, of 2-feet side. A force of 7 lbs. weight acts vertically on the table at a point D, 1·2 feet from A and 1·4 feet from B. Find the pressure at the points of support.

(Take A for the origin. Consider the lines AB, AC, AD, to be the horizontal traces of three vertical planes. Referring the given force to the origin A, there will be an equal and parallel force at A, and a couple 7 × AD foot-lbs. acting in the plane of which AD is the trace. This couple must be balanced by couples in the planes of which AB and AC are the horizontal traces. Therefore set out the axis of the couple in plane AD, 7 × AD units long, and complete the triangle by lines drawn at right angles to the remaining traces, the intersection of these lines will determine the length and sense of the two unknown axes. Measuring these off and dividing respectively by the arms AC and AB, the forces of the two couples at the supporting points are known.)

Answer.

Force at A = 2·23 lbs. weight
" at B = 1·4 " "
" at C = 3·37 " "

18. Find the two masses which will balance the mass of question 12, when they are placed at 4 feet and 5 feet radii respectively—

(1) In planes of revolution, the first 1 foot to the left of the plane of the given mass, the second 2 feet to the right.

(2) In planes distant 1 foot and 3 feet respectively to the right of the given mass.

Answers.—

(1) Mass in plane to the left 116·6 pounds at 4 feet radius, the radius being at 180° with the radius of the given mass.

Mass in plane to the right 46·6 pounds at 5 feet radius, the radius being at 180° with the radius of the given mass.

(2) Mass in plane nearer the given mass, 262·5 pounds at

4 feet radius, the radius being at 180° with the radius of the given mass.

Mass in further plane 70 pounds at 5 feet radius, the radius being at 0° with the radius of the given mass.

19. Draw the bending moment and shearing force diagrams for the shafts of the two balanced systems of question 18, due to the dynamical loading alone, when the speed is 250 revolutions per minute.

20. Five pullies, equally spaced at 2 feet apart, are keyed to a shaft which is supported on bearings 12 feet apart. The pullies are out of balance to the following extent:—

No. 1, 5	pounds at	1 foot	radius.
No. 2, 6	„	2 feet	„
No. 3, 7	„	1 foot	„
No. 4, 2	„	2 feet	„
No. 5, 6	„	1 foot	„

The angles between the several mass radii and the mass radius of No. 1 pulley are respectively 45, 90, 120, and 240 degrees. Find the two masses which will balance the system—

(1) When placed in Nos. 1 and 5 pullies at 1 foot radius.
(2) „ „ „ 2 „ 4 „ „ „

Answers.—

(1) $\begin{cases} 3{\cdot}84_{308^\circ} \text{ in No. 5.} \\ 15{\cdot}25_{225^\circ} \text{ in No. 1.} \end{cases}$ (2) $\begin{cases} 9{\cdot}1_{5^\circ} \text{ in No. 4.} \\ 22{\cdot}7_{220^\circ} \text{ in No. 2.} \end{cases}$

The angles are measured from the direction of No. 1 radius.

21. Find at what speed the maximum value of the unbalanced force of a locomotive is 4 tons, assuming that the revolving masses are balanced and that the reciprocating masses, which weigh 600 pounds per cylinder, are unbalanced. Diameter of wheels, 4 feet 6 inches. Stroke, 2 feet. (Use formula in (1), page 85.)

Answer.—28·6 miles per hour.

22. What is the speed in question 21, if the revolving parts, which weigh 700 pounds per cylinder, and are at 1 foot radius, are unbalanced as well?

Answer.—19·4 miles per hour.

23. Calculate the values of the swaying couples in questions 21 and 22, assuming the cylinders to be inside and 2 feet centre to centre. (Use formula in (2), page 85.)

Answer.—8950 foot-lbs. in each case.

24. What is the speed in question 21, if two-thirds of the reciprocating masses are balanced and all the revolving masses ?

Answer.—49·5 miles per hour.

25. Calculate the speed in the previous question when the diameter of the driving-wheel is 7 feet.

Answer.—77 miles per hour.

26. Calculate the values of the respective swaying couples for a speed of 60 miles per hour and their respective periodic times, for the following engines, in each of which the revolving masses are balanced and the mass of the reciprocating parts per cylinder is 600 pounds, two-thirds of which is balanced. Stroke, 2 feet.

(1) A 7-foot inside single, cylinders 2 feet pitch.
(2) A 7-foot outside single, cylinders 6 feet pitch.
(3) An 8-foot outside single, cylinders 6 feet pitch.
(4) A 5-feet inside cylinder tank, cylinders 2 feet pitch.

Answers.—(1)	5440 foot-lbs.	0·25 seconds.
(2)	16320 „	0·25 „
(3)	12495 „	0·286 „
(4)	10662 „	0·179 „

27. Two engines are built with similar sets of reciprocating parts, one as a 7-foot outside single in which the cylinders are 6 feet pitch, the other as an inside cylinder tank engine in which the cylinders are 2 feet pitch. The revolving masses and two-thirds of the reciprocating masses are balanced in each case. For what diameter of the driving-wheels would the swaying couple acting on the tank engine be equal to that acting on the single engine, when both engines are running at the same speed ?

Answer.—4 feet diameter approx.

28. Assuming that the tractive force exerted by an engine varies inversely as the speed, and that the tractive force is 2 tons when the speed is 30 miles per hour, and also that there are 200 pounds reciprocating mass unbalanced per cylinder, find the speed at which the maximum value of the unbalanced force becomes equal to the average tractive force. Wheels, 7 feet diameter. Stroke, 2 feet.

Answer.—44·6 miles per hour.

29. Find the balance weights for the inside single engine specified by the following data:—

Stroke, 26 inches. Cranks at right angles, left-hand crank leading. (In Fig. 79 the right-hand crank is leading.)
All the revolving and two-thirds of the reciprocating masses are to be balanced.

Distance centre to centre of the cylinders	...	2 feet 4 inches
Distance between the planes containing the mass centres of the balance weights	... ...	4 „ $11\frac{3}{4}$ „
Mass of the reciprocating parts per cylinder	...	612 pounds
Mass of the revolving parts per cylinder	...	720 „

Answer.—

Left-hand wheel. 880 pounds at 13 inches radius at an angle of 160 degrees measured from the left-hand crank, counter-clockwise, when facing the left-hand wheel.

30. Find the balance weights for the inside cylinder four-coupled engine specified by the following data:—

Stroke, 24 inches. Inside cranks at right angles, right-hand crank leading. Outside cranks, 11 inches radius, placed oppositely to the corresponding inside cranks. All the revolving and two-thirds of the reciprocating masses to be balanced. The mass of each coupling-rod to be divided equally between the driving and trailing wheels. The balancing mass for the reciprocating parts to be divided equally between the driving and trailing wheels.

Distance centre to centre of cylinders	2 feet
Distance between planes of motion of wheel-cranks	5·166 feet
Distance between planes of motion of coupling-rods	6·27 „

Distance between the planes containing the mass centres of the balance weights	4·94 feet
Mass of reciprocating parts per cylinder	642 pounds at 12 inches
Inside revolving masses per cylinder	723 „ „ 12 „
Mass of each coupling-rod	224·5 „ „ 11 „
Mass of each wheel-crank in driving and trailing wheels	117·6 „ „ 11 „
Mass of part of crank-pin outside crank, together with the pin and washer for each outside crank	38·2 „ „ 11 „

Answer.—

Left-hand driving-wheel. 493 pounds at 12 inches radius at an angle of 37 degrees, measured from the outside crank, counter-clockwise, when facing the left-hand wheel.

Trailing-wheel. 145 pounds at 12 inches radius at an angle of 144 degrees, measured from the outside crank, counter-clockwise, when facing the left-hand wheel.

31. Find the balance weights for the engine of the previous question when the two-thirds of the reciprocating masses are balanced entirely in the driving-wheels.

Answer.—

Left-hand driving-wheel. 651 pounds at 12 inches radius at an angle of 34 degrees, measured from the outside crank, counter-clockwise, when facing the left-hand wheel.

Trailing-wheel. 268 pounds at 12 inches radius at an angle of $175\frac{1}{2}$ degrees, measured from the outside crank, counter-clockwise, when facing the left-hand wheel.

32. Calculate the maximum value of the hammer-blow for the driving-wheel in the two preceding examples when the crank-shaft is making four turns per second (corresponding to 60 miles per hour with a 7-foot wheel), and hence find the maximum and minimum load on the rail, supposing the static load to be $7\frac{1}{2}$ tons. (Use expression (1), Art. 76.)

Answers.—

For example 30, where one-third is balanced in the driving-wheel, hammer-blow = 3185 lbs. weight.

Maximum load on the rail, 8·92 tons weight; minimum, 6·08 tons weight.

For example 31, where two-thirds is balanced in the driving-wheel, hammer blow = 6370 lbs. weight.

Maximum load on the rail, 10·34 tons weight; minimum, 4·66 tons weight.

33. Assuming the mass of the reciprocating parts shown in Fig. 50 to be 1000 pounds, and the crank radius to be 1 foot, find the accelerating force when the crank makes 480 revolutions per minute, for the crank angles, 0, 60, 90, 120, and 180 degrees.

Answer.—

-78440, -39220, 0, $+39220$, $+78440$ lbs. weight.

34. Find the maximum values of the unbalanced force and the unbalanced couple in terms of the revolutions per second, due to the reciprocating masses of a four-crank engine in which the cylinder pitches, reckoning from the left, are 10 feet, 12 feet, and 10 feet respectively, the corresponding masses, reckoning from the left, being 2, 3, 4, and 2 tons, and in which the crank angles are, reckoning from the left, between cranks 1 and 2, 90 degrees; between cranks 2 and 3, 90 degrees; between cranks 3 and 4, 90 degrees. Stroke, 4 feet.

Answer.—

Unbalanced force, $176{\cdot}5\frac{n^2}{g}$ lbs. weight.

Unbalanced couple in plane of reciprocation about an axis at the centre of the engine, $5920\frac{n^2}{g}$ foot-lbs.

35. Find the unbalanced force and couple using the data of the previous question when the sequence of cranks is changed to the following, reckoning from the left: angle between cranks Nos. 1 and 2, 180 degrees; between cranks 2 and 3, 90 degrees; between cranks 3 and 4, 180 degrees.

Answer.—Unbalanced force, $176{\cdot}5\frac{n^2}{g}$ lbs. weight.

Unbalanced couple at the centre, $1275\frac{n^2}{g}$ foot-lbs.

36. The mass centre of a connecting-rod, l feet long, is $0{\cdot}8l$ from the small end. Its mass is 850 pounds. Find the masses which must be included with the revolving and reciprocating parts which it connects, in order that the effect of the rod may be balanced when these masses are balanced.

Answer.—680 pounds with the revolving mass.
170 „ „ reciprocating mass.

37. Reckoning from the left in order, let the letters A, B, C, D denote the cylinders of a four-crank engine. The distances between them are, 5 feet between A and B, 8 feet between B and C, 6 feet between C and D. The revolving masses corresponding to A, B, C, D are respectively 1, $1\frac{1}{2}$, $1\frac{1}{4}$, and 1 ton at crank radius. Given that the angle between the cranks of cylinders B and C is 105 degrees, and that the reciprocating masses of cylinders B and C are respectively $2\frac{1}{4}$ and 2 tons, find the remaining crank angles and masses so that the reciprocating parts may be in balance amongst themselves, neglecting the obliquity of the connecting-rod. Find also the masses which must be added to the crank-shaft at cranks A and B to balance it.

Answer.—

Angle between cranks C and A, $97\frac{1}{2}$ degrees ⎫
„ „ „ A „ D, 58 „ ⎬ measured in order.
„ „ „ D „ B, $99\frac{1}{2}$ „ ⎭
Reciprocating mass at A, 1·615 tons.
„ „ at D, 1·343 „
Revolving mass attached to crank-shaft at A, $0{\cdot}07_{184^\circ}$ tons.
„ „ „ at D, $0{\cdot}158_{92^\circ}$ „
The subscript directions being measured from crank B towards crank C.

38. Taking the data of the previous example, find the remaining crank angles and reciprocating masses, including the revolving masses with the reciprocating, so that the reciprocating and revolving masses are together in balance in the plane of reciprocation, but the revolving masses are unbalanced in the plane at right angles to it.

Answer.—

Angle between cranks C and A, $96\frac{1}{3}$ degrees
" " " A " D, $57\frac{1}{2}$ "
" " " D " B, $101\frac{1}{6}$ "
} measuring in order.

Reciprocating mass at A, 1·69 tons.
" " at D, 1·19 tons.

39. Taking the data of question 37, balance the reciprocating masses amongst themselves, and find what the corresponding revolving masses should be so that they may be in balance without the addition of balance weights, having given that the revolving mass at A is 1 ton.

Answer.—

Revolving masses must be in the same ratio as the reciprocating masses.

Revolving mass at B, 1·394 tons.
" " at C, 1·238 "
" " at D, 0·833 "

40. The reciprocating masses of a four-crank engine are respectively $5\frac{1}{2}$, 7, 6, and 5 tons, taken in order. Find the cylinder centre lines having given that the pitch of the extreme cylinders is 39 feet, and that the remaining two are to be arranged symmetrically with respect to them, neglecting the obliquity of the connecting-rod. Find also the corresponding set of crank angles.

(This problem may be solved by taking advantage of the geometrical properties of the combined force and couple polygons, Art. 37, Fig. 47, to calculate the ratio $\frac{a_2}{a_1}$, the distances a_1 and a_2 being measured from a central reference plane, as in Fig. 110, and then drawing the couple triangle ABd or CBc, from either of which the force polygon may be completed. Referring to Fig. 47, page 50, the condition of symmetry requires that—

$$\frac{\mathrm{A}d}{\mathrm{AD}} = \frac{\mathrm{C}c}{\mathrm{CD}}$$

and therefore the line joining d to c is parallel to AC.

Hence—

$$Be = \frac{ef}{dc} \times Bd = \frac{2a_2}{a_1 + a_2}Bd \quad . \quad . \quad . \quad (1)$$

Since if AC represents $2a_1$, dc represents $a_1 + a_2$, and ef represents $2a_2$.

Again—

$$Bd = cD = CD\left(1 - \frac{a_1 - a_2}{2a_1}\right)$$

since—

$$cC = \frac{Cf}{CA}DC = \left(\frac{a_1 - a_2}{2a_1}\right)DC$$

Substituting the value of Bd, in (1)—

$$Be = \frac{a_2}{a_1}CD$$

Similarly—

$$Bf = \frac{a_2}{a_1}AD$$

Again considering the triangle ABC, let O be the central point of AC, and let the angle BOC be θ. Then—

$$AB^2 - BC^2 = 2a_1 OB \cos \theta$$
$$Be^2 - Bf^2 = 2a_2 OB \cos \theta$$

Dividing and substituting the values of Be and Bf—

$$\frac{AB^2 - Bc^2}{CD^2 - AD^2} = \frac{a_2}{a_1} = \frac{M_1^2 - M_4^2}{M_2^2 - M_3^2} \; *$$

After the value of this ratio has been calculated from the four given masses, either of the couple triangles ABd, or BcC, may be drawn. Completing the force polygon all the crank angles are determined. The lengths to be used in drawing the triangle ABd are—

$$AB = 2a_1 M_1$$
$$Bd = (a_1 + a_2)M_2$$
$$Ad = (a_1 - a_2)M_3$$

* This relation is given in Herr Schlick's paper "On Balancing Steam Engines," *Trans. Inst. Naval Architects*, 1900.

Answers—

Ratio $\frac{a_2}{a_1} = \frac{21}{52}$

Pitch of inner cylinders 15·75 feet

Angle between cranks 1 and 3, 118 degrees (see Fig. 48)

"	"	"	3 and 2, $81\frac{1}{2}$	"
"	"	"	2 and 4, 123	"
"	"	"	4 and 1, $37\frac{1}{2}$	"

41.—Referring to the engine of which the data are given in Art. 50, find the change which must be made in the masses belonging to Nos. 2, 3, and 4 cranks, and the change in the direction of crank No. 4, relatively to crank No. 1, in order that the reciprocating parts may be in balance amongst themselves, when the valve-gear is neglected.

Answer.—

No. 2 mass, increased 100 pounds, 6·66 per cent. of the original mass.

No. 3 mass, increased 125 pounds, 9·8 per cent. of the original mass.

No. 4 mass, increased 75 pounds, 7·2 per cent. of the original mass.

Angle between cranks 1 and 4, $47\frac{1}{2}$ degrees, being a change of $2\frac{1}{2}$ degrees.

42.—Referring to the engine of which the data are given in Art. 50, assume that the masses of the reciprocating parts corresponding to cranks Nos. 1, 2, and 3 are given, being respectively 1000, 1500, and 1275 pounds, and find the two solutions which are possible when the valve-gear is included.*

(This is case C of Art. 8. Neglecting the valve-gear, the solutions differ only in the respect that the sequence of angles in the one is opposite to that of the other. Or, holding the drawing of the angles from one solution in front of a looking-glass, the reflection is the other solution. Including the valve-gear, there are two different solutions, one of which in the case in question is of course the solution of Art. 50.)

* I am indebted to Professor Dunkerley for this extension of Art. 50.

Answer.—

2nd solution. Angles between cranks 1 and 4, 4 and 2, 2 and 3, are respectively $44\frac{1}{2}$ degrees, 118 degrees, and $93\frac{1}{2}$ degrees measured counter-clockwise.

Mass at No. 4 = 1095 pounds.

43. The cylinders of a four-crank engine are arranged symmetrically. The pitch of the outer pair is 35 feet, and of the inner pair 15 feet. The mass of each set of reciprocating parts belonging to the outer cylinders is 6 tons. Find the crank angles and the inner masses, so that the reciprocating masses may be in balance for primary and secondary forces, and primary couples.

(Use equation (9), Art. 96, and check the work by Fig. 112.)

Answer.—

Let A, B, C, D indicate the cylinders taken in order.

Angle between cranks A and D, 61 degrees 42 minutes.

" " " B " C, 108 " 48 "

Reciprocating mass corresponding to cylinders B and C, 8·844 tons per cylinder.

44. Details of the valve-gear belonging to the engine of the previous question are given below.

	Distance from Crank A.	Mass at crank radius.
	Feet.	Tons.
Ahead sheave	–5·0	0·6
Astern sheave	–4·3	0·1
Crank A	0·0	
Crank B	10·0	8·844
Ahead sheave	14·3	0·6
Astern sheave	15·0	0·1
Astern sheave	20·0	0·1
Ahead sheave	20·7	0·6
Crank C	25·0	8·844
Crank D	35·0	
Astern sheave	39·3	0·1
Ahead sheave	40·0	0·6

Assuming that the angle between cranks B and C is 108 degrees 48 minutes, and that the corresponding masses are 8·844 tons per cylinder, as found in the previous question, find the remaining angles and the masses corresponding to cylinders A and D, including the effect of the valve-gear.

(Take a reference plane at crank A and include the couples belonging to the valve-gear of crank A, by assuming that the direction of crank A is that found in the previous question.)

Answer.—

Angle between cranks A and D, 63 degrees 53 minutes.
" " " A " C, 95 " 56 "
" " " B " D, 91 " 23 "
Mass at A, 6·3 tons.
" at D, 6·0 "

45. An opposite pair of crank angles, β and δ, in a four-crank engine, have the values $\beta = 110$ degrees and $\delta = 90$ degrees. Find the remaining angles, the ratio of the reciprocating masses, and the pitch of the cylinders so that the engine may be in balance for primary and secondary forces and primary couples.

(Find γ_1 and γ_2 from equation (11), Art. 96, and M_1 from equation (12), Art. 96; then find M_2, M_3, a_2, a_3, either by calculation or by the graphical method of Art. 37.)

Answer.—

$\gamma_1 = 44$ degrees 12 minutes
$\gamma_2 = 115$ " 48 "

If $M_4 = 1$, then—

$$M_1 = 2{\cdot}3155$$
$$M_2 = 2{\cdot}9026$$
$$M_3 = 2{\cdot}9563$$

If the reference plane is taken at No. 4 crank, so that $a_4 = 0$, and a_1 be put equal to unity—

$$a_2 = 0{\cdot}8326$$
$$a_3 = 0{\cdot}6236$$

46. Given that the crank radius of the symmetrical engine detailed in question 43 is 2 feet, that the connecting-rod is 7 feet long, and that the engine runs at 80 revolutions per minute, find the maximum value of the unbalanced secondary couple.

(See Art. 118, type 3, or the formulæ in Schedule 21.)

Answer.—385 foot-tons.

47. Given that the crank radius of the unsymmetrical engine detailed in question 45 is 2 feet, that the connecting-rod is 7 feet long, that the pitch of the extreme cylinders is 35 feet, that M_4 is 6 tons, and that the speed is 80 revolutions per minute, find the maximum value of the unbalanced secondary couple.

(The most expeditious way of doing this is to find the closure of the secondary couple polygon graphically. This measures 312 units. Multiply this by $\frac{\omega^2 r^2}{gl}$.)

Answer.—385 foot-tons.

48. A steel tube, 6 feet long say, is firmly fixed at one end in a vertical position, and loaded at the free end with a mass of 10 pounds. It is found by experiment that a force of 10 lbs. weight, applied at the centre of the mass at right angles to the length of the tube, produces a displacement from the position of equilibrium of a tenth of a foot. Find the time of vibration of the system, and the displacement of the centre of the mass from the position of equilibrium at the end of 10 seconds, having given that the displacement is 0·3 feet when the time is nothing, and that a is nothing when the time is nothing. Neglect the mass of the tube.

Answer.—Time of vibration = 0·35 seconds.
Displacement at the end of 10 seconds is −0·28 feet.

49. Find the displacement at the end of 1 second for the system of question 48 when there is a frictional resistance acting, which is proportional to the velocity, having given that the resistance to motion is 3 lbs. weight when the velocity is 1 foot per second.

Answer.—0·00004 feet.

50. Suppose a periodic force whose periodic time is 0·4 seconds, and whose maximum value is $\frac{10}{g}$ lbs. weight, to act on the mass of the system of question 48. Find the maximum amplitudes when the frictional resistance to motion at unit velocity is respectively 3 lbs. weight and $\frac{1}{g}$ lbs. weight. Find the maximum amplitudes in the two cases when the periodic time of the force is 0·35 seconds.

Answers.—

0·0059 feet and 0·0066 feet when the periodic time is 0·4 seconds.

0·1306 feet and 0·637 feet when synchronism takes place.

51. The reciprocating masses of a single-cylinder engine weigh 6 tons. The cylinder is 70 inches diameter and 4 feet stroke, and the connecting-rod is 7 feet long. When the connecting-rod subtends a crank angle of 60 degrees, the difference between the forward and the back pressure in the cylinder is 30 lbs. per square inch. Find the turning moment on the crank when the speed is 80 revolutions per minute.

(Calculate the acceleration of the reciprocating masses by equation (2), Art. 78 (a is zero in this case), or use Klein's construction, Art. 104. Calculate the length of Og (Fig. 143), and use the appropriate product from equation (1), Art. 130.)

Answer.—83·6 foot-tons.

52. Find the ratio in which the total horse-power should be distributed between the four cylinders of the example, case 2, Art. 97, in order to obtain a good crank-effort curve, having given that the horse-power of No. 2 cylinder is to be equal to the horse-power of No. 3 cylinder.

Answer.—

Calling the horse-power of cylinders Nos. 2 and 3 each equal to unity—

Horse-power of No. 1 cylinder = 0·74
" " " 4 " = 0·6

53. Find the ratios in the previous example, if the horse-powers in cylinders 1 and 4 are each equal to unity.

Answer.—Horse-power in cylinder No. 2 = 1·53.
" " " No. 3 = 1·34.

54. The mass centre of a connecting-rod 7 feet long from centre to centre, is 4·9 feet from the small end. The rod, suspended so that it is free to swing about an axis through the small-end centre, swings in unison with a plumb-line 6·3 feet long. Find k, the radius of gyration about a parallel axis through the mass centre.

Answer.—2·61 feet.

55. The rod of the previous question is suspended so that it is free to swing about an axis through the big-end centre. Find the length of a plumb-line which will swing in unison with it.

Answer.—5·36 feet.

56. The connecting-rod specified in question 54 drives a crank 2 feet radius. The mass of the rod is 1610 pounds. Find the force required to produce the instantaneous motion of the rod when the rod subtends a crank angle of 60 degrees, and the crank revolves uniformly at 80 revolutions per minute.

Answer.—

5305 lbs. weight in a direction inclined 307 degrees with the line of stroke (the crank radius in the line of stroke, representing the initial direction, points away from the cross-head), and cutting the line of stroke at 0·71 feet measured from the centre of the crank-shaft towards the cross-head.

57. Taking the data of the previous question, find the instantaneous values of the reactions on the frame.

Answer.—

(1) A force of 5305 lbs. weight inclined 127 degrees with the line of stroke (the initial direction pointing away from the cross-head) acting at the axis of the crank-shaft.

(2) A couple whose moment is 4930 foot-lbs., the equal forces of which are each 634 lbs. weight.

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