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SOLUTION OF TRIANGLES

PART I

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CHAPTER I

THE USE OF FORMULAS*

In mathematical and mechanical books and treatises, as well as in articles containing calculations published in the engineering journals, formulas are used to a great extent instead of rules. In these formulas, signs and symbols are used in order to condense into a small space the essentials of what would otherwise be long and cumbersome rules. The symbols used are generally the letters in the alphabet, and the signs are simply the ordinary signs for arithmetical calculations, with some additional ones necessary for special purposes. Letters from the Greek alphabet are commonly used to designate angles, and the Greek letter π (pi) is always used to indicate the proportion of the circumference of a circle to its diameter; π, therefore, is always, in formulas, equal to 3.1416. The most commonly used Greek letters, besides π, are α (alpha), β (beta), and γ (gamma).

Knowledge of algebra is not necessary in order to make possible the successful use of formulas for the solving of problems such as occur in the solution of triangles; but a thorough understanding of the rules and processes of arithmetic is very essential. The symbols or letters used in the formulas simply stand in place of the actual figures or numerical values which are inserted in the formula in each specific case, according to the requirements of the problem to be solved. When these values are inserted, the result required may be obtained by simple arithmetical processes.

There are two main reasons why a formula is preferable to a rule expressed in words. Firstly, the formula is more concise, it occupies less space, and it is possible for the eye to catch at a glance the whole meaning of the rule laid down; secondly, it is easier to remember a short formula than a long rule, and it is, therefore, of greater value and convenience, as it is not always possible to carry a handbook or reference book about, but the memory must be relied upon to store up a number of the most frequently occurring mathematical and mechanical rules.

The use of formulas can be explained most readily by actual examples. In the following, therefore, a number of simple formulas will be given, and the values will be inserted so as to show, in detail, the principles involved.

Example 1.—When the diameter of a circle is known, the circumference may be found by multiplying the diameter by 3.1416. This rule, expressed as a formula, is:

$$C = D \times 3.1416$$

in which C = circumference of circle,
D = diameter of circle.

* This chapter has been practically reproduced from MACHINERY'S Reference Series No. 52, "Advanced Shop Arithmetic for the Machinist," in order to make the present treatise complete in itself.

This formula shows at a glance, that no matter what the diameter of the circle be, the circumference is always equal to the diameter times 3.1416. Let it be required to find, for example, the circumference of a circle 22 inches in diameter. If then we insert 22 in the place of D in the formula, we have:

$$C = 22 \times 3.1416 = 69.1152 \text{ inches.}$$

Hence, our formula gives, by means of a simple multiplication, the result required.

Assume that the diameter of a circle is 3.72 inches. The circumference of this circle is found by inserting this value instead of D in the formula:

$$C = 3.72 \times 3.1416 = 11.6867 \text{ inches.}$$

Example 2.—In spur gears, the outside diameter of the gear can be found by adding 2 to the number of teeth, and dividing the sum obtained by the diametral pitch of the gear. This rule can be expressed very simply by a formula. Assume that we write D for the outside diameter of the gear, N for the number of teeth, and P for the pitch. Then the formula would be

$$D = \frac{N + 2}{P}$$

This formula reads exactly as the rule given above. It says that the outside diameter (D) of the gear equals 2 added to the number of teeth (N), and this sum divided by the pitch (P).

If the number of teeth in a gear is 16 and the pitch 6, then simply put these figures in the place of N and P in the formula, and find the outside diameter as in ordinary arithmetic.

$$D = \frac{16 + 2}{6} = \frac{18}{6} = 3.$$

D, or the outside diameter, then, is 3 inches.

In another gear the number of teeth is 96 and the pitch 7; find the outside diameter of the gear.

$$D = \frac{96 + 2}{7} = \frac{98}{7} = 14 \text{ inches.}$$

From the examples given it will be seen that in formulas, each letter stands for a certain dimension or quantity. When using a formula for solving a problem, replace the letters in the formula by the figures given in a certain problem, and find the result as in a regular arithmetical calculation.

Example 3.—The formula for the horse-power of a steam engine is as follows:

$$\text{H. P.} = \frac{P \times L \times A \times N}{33{,}000}$$

in which H. P. = indicated horse-power of engine,

P = mean effective pressure on piston in pounds per square inch,

L = length of piston stroke in feet,
A = area of piston in square inches,
N = number of strokes of piston per minute.

Assume that $P = 90$, $L = 2$, $A = 320$, and $N = 110$; what would be the horse-power?

If we insert the given values in the formula we have:

$$\text{H. P.} = \frac{90 \times 2 \times 320 \times 110}{33{,}000} = 192.$$

In formulas, the sign for multiplication ($\times$) is often left out between letters the values of which are to be multiplied. Thus AB means $A \times B$, and the formula

$$\frac{P \times L \times A \times N}{33{,}000} \text{ can also be written } \frac{PLAN}{33{,}000}$$

Thus, if $A = 3$, and $B = 5$, then:

$$AB = A \times B = 3 \times 5 = 15.$$

If $A = 12$, $B = 2$, and $C = 3$, then:

$$ABC = A \times B \times C = 12 \times 2 \times 3 = 72.$$

It is only the multiplication sign ($\times$) that can be thus left out between the symbols or letters in a formula. All other signs must be indicated the same as in arithmetic.

A parenthesis ( ) or bracket [ ] in a formula means that the expression inside the parenthesis or bracket should be considered as one single symbol, or in other words, that the calculation inside the parenthesis or bracket should be carried out by itself, before other calculations are carried out.

Examples:

$6 \times (8 + 3) = 6 \times 11 = 66.$

$5 \times (16 - 14) + 3\ (2.25 - 1.75) = 5 \times 2 + 3 \times 0.5 = 10 + 1.5 = 11.5.$

In the last example above it will be seen that 5 is multiplied by 2 and 3 by 0.5, and then the products of these two multipiications are added. From the order of the numbers $5 \times 2 + 3 \times 0.5$, one might have assumed that the calculation should have been carried out as follows: 5 times 2 = 10, plus 3 = 13, times 0.5 = 6.5. This latter procedure, however, is not correct.

When several numbers or expressions are connected by the signs $+$, $-$, $\times$ and $\div$, the operations are carried out in the order written, except that *all multiplications should be carried out before the other operations*. The reason for this is that numbers connected by a multiplication sign are only factors of the product thus indicated, which product should be considered by itself as one number. Divisions should be carried out before additions and subtractions, if the division is indicated in the same line with these other processes.

Examples:

$5 \times 6 + 4 - 6 \times 4 = 30 + 4 - 24 = 34 - 24 = 10.$

$5 + 3 \times 2 = 5 + 6 = 11.$

$100 \div 2 \times 5 = 100 \div 10 = 10.$

$3.5 + 16.5 \div 3 - 1.75 = 3.5 + 5.5 - 1.75 = 7.25.$

But $5 \times (6 + 4) - 6 \times 4 = 5 \times 10 - 24 = 50 - 24 = 26.$

$(5 + 3) \times 2 = 8 \times 2 = 16.$

$(100 \div 2) \times 5 = 50 \times 5 = 250.$

$(3.5 + 16.5) \div (3 - 1.75) = 20 \div 1.25 = 16.$

Formulas Containing Square and Cube Roots

The square of a number is the product of that number multiplied by itself. The square of 2 is $2 \times 2 = 4$, and the square of 10 is $10 \times 10 = 100$; similarly the square of 177 is $177 \times 177 = 31,329$. Instead of writing 4×4 for the quare of 4, it is often written 4^2 which is read *four square*, and means that 4 is multiplied by 4. In the same way 128^2 means 128×128. The small figure (^{2}) in these expressions is called *exponent.*

The square root of a number is that number which, when multiplied by itself, will give a product equal to the given number. Thus, the square root of 4 is 2, because 2 multiplied by itself gives 4. The square root of 25 is 5; of 36, 6, etc. We may say that the square root is the reverse of the square, so that if the square of 24 is 576, then the square root of 576 is 24. The mathematical sign for the square root is $\sqrt[2]{\ }$, but the *index figure* (^{2}) is generally left out, making the square-root sign simply $\sqrt{\ }$, thus:

$\sqrt{4} = 2$ (the square root of four equals two),

$\sqrt{100} = 10$ (the square root of one hundred equals ten).

The operation of finding the square root of a given number is called *extracting* the square root.* Squares and square roots as well as cubes and cube roots of all numbers up to 1,000 (sometimes up to 1,600) are generally given in all standard handbooks.

The cube of a number is the product obtained if the number itself is repeated as a factor three times. The cube of 2 is $2 \times 2 \times 2 = 8$, and the cube of 12 is $12 \times 12 \times 12 = 1,728$. Instead of writing $2 \times 2 \times 2$ for the cube of 2, it is often written 2^3, which is read "two cube." In the same way 128^3 means $128 \times 128 \times 128$. The small figure (^{3}) in these expressions is called *exponent*, the same as in the case of the figure (^{2}) indicating the square of a number. An expression of the form 18^3 may also be read the "third power of 18."

In the same way as square root means the reverse of square, so cube root means the reverse of cube; that is, the cube root of a given number is the number which, if repeated as factor three times, would give the number given. Thus the cube root of 27 is 3, because $3 \times 3 \times 3 = 27$. If the cube of 15 is 3,375, then the cube root of 3,375 is, of course, 15. The mathematical sign for the cube root is $\sqrt[3]{\ }$, thus:

$\sqrt[3]{64} = 4$ (the cube root of sixty-four equals four),

$\sqrt[3]{4096} = 16$ (the cube root of four thousand ninety-six equals sixteen).

* See MACHINERY'S Reference Series No. 52, "Advanced Shop Arithmetic for the Machinist", Chapter I.

Assume, for an example, that a formula is given as follows:

$$A = \frac{\sqrt{B} \times C}{D}$$

Let $B = 36$, $C = 3.5$, and $D = 10.5$. Find the value of A.
If we insert these values in the formula, we have:

$$A = \frac{\sqrt{36} \times 3.5}{10.5} = \frac{6 \times 3.5}{10.5} = \frac{21}{10.5} = 2.$$

As another example, find the value of A in the formula

$$A = \frac{B^2 + C^2}{D^2}, \text{ if } B = 5,\ C = 7, \text{ and } D = 2.$$

If we insert these values in the formula, and carry out the calculation, remembering that $5^2 = 5 \times 5$, $7^2 = 7 \times 7$, etc., we have:

$$A = \frac{5^2 + 7^2}{2^2} = \frac{25 + 49}{4} = \frac{74}{4} = 18.5.$$

Find the value of A in the formula

$$A = \sqrt{B^2 + C^2}, \text{ if } B = 8 \text{ and } C = 6.$$

If we insert the given values in the formula, we have:

$$A = \sqrt{8^2 + 6^2} = \sqrt{8 \times 8 + 6 \times 6} = \sqrt{64 + 36} = \sqrt{100} = 10.$$

The examples given indicate the principles involved in the use of formulas, and show, as well, how easily formulas may be employed by anyone who has a general understanding of arithmetic.

CHAPTER II

ANGLES AND ANGULAR MEASUREMENTS

When two lines meet as shown in Fig. 1, they form an angle with each other. The point where the two lines meet or intersect is called the *vertex* of the angle. The two lines forming the angle are called the sides of the angle.

Angles are measured in degrees and subdivisions of a degree. If the circumference (periphery) of a circle is divided into 360 parts, each part is called one degree, and the angle between two lines from the center to the ends of this small part of the circle is a one-degree angle, as shown in Fig. 2. As the whole circle contains 360 degrees, one-half of a circle contains 180 degrees, and one-quarter of a circle, 90 degrees, as shown in Fig. 9.

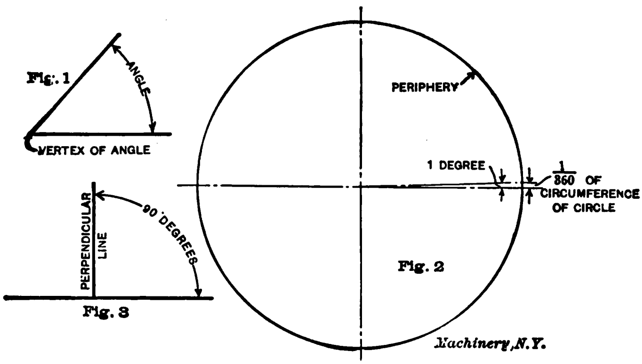
A 90-degree angle is called a *right* angle. An angle larger than 90 degrees is called an *obtuse* angle, and an angle less than 90 degrees is called an *acute* angle. (See Fig. 10.) Any angle which is not a right angle is called an *oblique* angle.

When two lines form a right or 90-degree angle with each other, as shown in Fig. 3, one line is said to be *perpendicular* to the other.

Angles are said to be equal when they contain the same number of degrees. The angle in Fig. 4 and the angle in Fig. 5 are equal, because they are both 60 degrees; that the sides of the angle in Fig. 5 are longer than the sides of the angle in Fig. 4 has no influence on the angle because of the fact that an angle is only the *difference in direction* of two lines. The angle in Fig. 6 which contains only 30 degrees is only one-half of the angle in Fig. 4.

One-half of a right angle is 45 degrees, as shown in Fig. 7. In Fig. 8 is shown an angle which is 120 degrees, and which can be divided into a right or 90-degree angle, and a 30-degree angle.

In order to obtain finer subdivisions for the measurement of angles than the degree, one degree is divided into 60 minutes, and one minute into 60 seconds.



Figs. 1 to 3

Any part of a degree can be expressed in minutes and seconds, for instance, 1/2 of a degree = 30 minutes, 1/3 of a degree = 20 minutes; and since 1/4 of a degree = 15 minutes, 3/4 of a degree = 45 minutes. In the same way 1/2 minute = 30 seconds, 1/4 minute = 15 seconds, and 3/4 minute = 45 seconds.

The word degree is often abbreviated "deg." or the sign (°) is used to indicate degrees; thus, 60° = 60 degrees. In the same way 60′ = 60 min. = 60 minutes, and 60″ = 60 sec. = 60 seconds; and 60° 50′ = 60 degrees 50 minutes.

When adding and subtracting degrees and minutes, care must be exercised not to make mistakes on account of there being but 60 minutes in a degree, instead of the usual 100 units met with when adding, for example, dollars and cents.

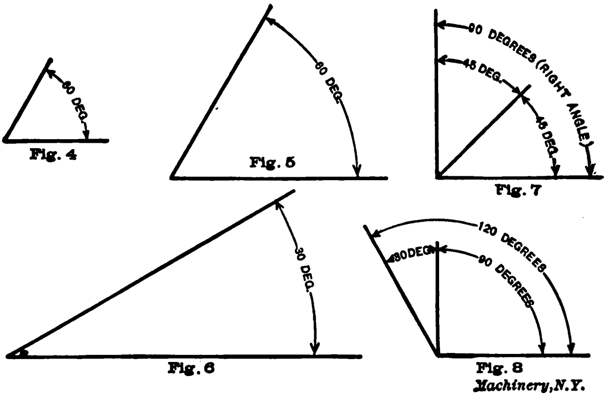
Example 1.—Add the two angles 60 deg. 32 min. and 35 deg. 16 min.

60 deg. 32 min.
35 deg. 16 min.
———————
95 deg. 48 min.

Example 2.—Add 15 deg. 43 min. to 12 deg. 27 min.

15 deg. 43 min.
12 deg. 27 min.

28 deg. 10 min.



Figs. 4 to 8

In this example the total sum of 43 and 27 minutes is 70 minutes; as 70 minutes, however, contains one whole degree (60 minutes), this is carried over and added to the degrees, leaving 10 minutes in the minute column, and 15 + 12 + 1 = 28 degrees in the degree column.

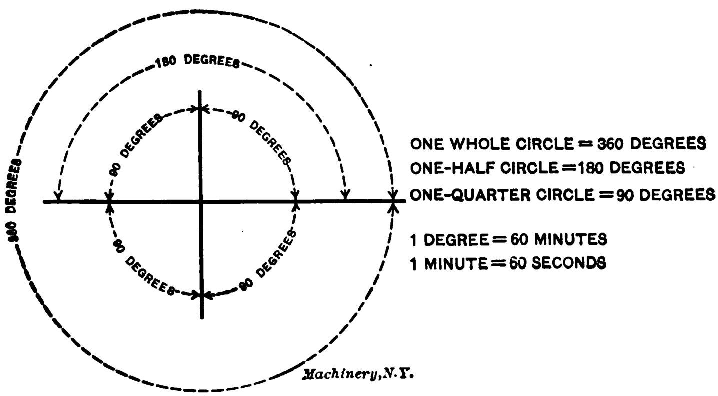


Fig. 9

Example 3.—Add 59 deg. 12 min., 16 deg. 53 min., and 103 deg. 55 min.

59 deg. 12 min.
16 deg. 53 min.
103 deg. 55 min.

180 deg. 0 min.

In adding the minutes (12 + 53 + 55 = 120 min.) we find that their sum equals 2 whole degrees. These are then carried over to the degree column and the total sum equals 59 + 16 + 103 + 2 = 180 deg.

Example 4.—Subtract 12 deg. 17 min. from 21 deg. 39 min.

21 deg. 39 min.
12 deg. 17 min.

9 deg. 22 min.

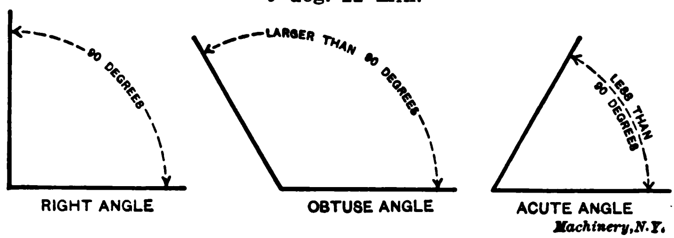


Fig. 10

Example 5.—Subtract 31 deg. 43 min. from 106 deg. 12 min.

106 deg. 12 min.
31 deg. 43 min.

74 deg. 29 min.

In this case we must borrow from the degrees. One deg. = 60 min. and 60 + 12 = 72; then 72 — 43 = 29 min. Having borrowed one degree from 106, we have 105 — 31 = 74 deg.

CHAPTER III

POSITIVE AND NEGATIVE QUANTITIES

In order to be able to use correctly the formulas for the solution of triangles under certain conditions, a working knowledge of the principles of positive and negative numbers or quantities is required. In this chapter, therefore, an explanation of the meaning of these expressions will be given, together with the rules for calculations with negative numbers, and examples to make the rules thoroughly understood.

On the thermometer scale, as is well known, the graduations extend upward from zero, the degrees being numbered 1, 2, 3, etc. Graduations also extend downward and are numbered in the same way: 1, 2, 3, etc. The degrees on the scale extending upward from the zero point may be called *positive* and preceded by a plus sign, so that, for instance, + 5 degrees means 5 degrees above zero. The degrees below zero may be called *negative* and may be preceded by a minus sign, so that — 5 degrees means 5 degrees below zero.

The ordinary numbers may also be considered positive and negative in the same way as the graduations on a thermometer scale. When we count 1, 2, 3, etc., we refer to the numbers that are larger than 0 (corresponding to the degrees *above* the zero point), and these numbers are called positive numbers. We can conceive, however, of numbers extending in the other direction of 0; numbers that are, in fact, less than 0 (corresponding to the degrees below the zero point on the thermometer scale). As these numbers must be expressed by the same figures as the positive numbers, they are designated by a minus sign placed before them. For example, — 3 means a number that is as much less than, or beyond 0 in the negative direction as 3 (or, as it might be written, + 3) is larger than 0 in the positive direction.

A negative value should always be enclosed within a parenthesis whenever it is written in line with other numbers; for example:

$$17 + (-13) - 3 \times (-0.76)$$

In this example — 13 and — 0.76 are negative numbers, and by enclosing the whole number, minus sign and all, in a parenthesis, it is shown that the minus sign is part of the number itself, indicating its negative value.

It must be understood that when we say 7 — 4, then 4 is not a negative number, although it is preceded by a minus sign. In this case the minus sign is simply the sign of subtraction, indicating that 4 is to be subtracted from 7. But 4 is still a positive number or a number that is larger than 0.

It now being clearly understood that positive numbers are all ordinary numbers greater than 0, while negative numbers are conceived of as less than 0, and preceded by a minus sign which is a part of the number itself, we can give the following rules for calculations with negative numbers.

A negative number can be added to a positive number by subtracting its numerical value from the positive number.

Examples:

$4 + (-3) = 4 - 3 = 1.$

$16 + (-7) + (-6) = 16 - 7 - 6 = 3.$

$327 + (-0.5) - 212 = 327 - 0.5 - 212 = 114.5.$

In the last example 212 is not a negative number, because there is no parenthesis indicating that the minus sign is a part of the number itself. The minus sign, then, indicates only that 212 is to be subtracted in the ordinary manner.

As an example illuminating the rule for adding negative numbers to positive ones, the case of a man having $12 in his pocket, but owing $9, may be taken. His debt is a negative quantity, we may say, and equals (— 9). Now if he adds his cash and his debts, to find out how much he really has, we have:

$12 + (-9) = 12 - 9 = 3.$

Of course, in a simple case like this, it is obvious that 9 would be subtracted directly from 12, but the example serves the purpose of illustrating the method used when a negative number is added to a positive number.

A negative number can be subtracted from a positive number by adding its numerical value to the positive number.

Examples:

$4 - (-3) = 4 + 3 = 7.$

$16 - (-7) = 16 + 7 = 23.$

$327 - (-0.5) - 212 = 327 + 0.5 - 212 = 115.5.$

In the last example, note that 212 is subtracted, because the minus sign in front of it does not indicate that 212 is a negative number.

As an illustration of the method used when subtracting a negative number from a positive one, assume that we are required to find how many degrees difference there is between 37 degrees above zero and 24 degrees below; this latter may be written (-24). The difference between the two numbers of degrees mentioned is then:

$37 - (-24) = 37 + 24 = 61.$

A little thought makes it obvious that this result is right, and the example shows that the rule given is based on correct reasoning.

When a positive number is multiplied or divided by a negative number, multiply or divide the numerical values as usual; but the product or quotient, respectively, becomes negative. The same rule holds true if a negative number is divided by a positive number.

Examples:

$4 \times (-3) = -12.$ $\qquad (-3) \times 4 = -12.$

$\frac{15}{-3} = -5.$ $\qquad \frac{-15}{3} = -5.$

When two negative numbers are multiplied by each other, the product is positive. When a negative number is divided by another negative number the quotient is positive.

Examples:

$(-4) \times (-3) = 12.$ $\qquad \frac{-4}{-3} = 1.333.$

If, in a subtraction, the number to be subtracted is larger than the number from which it is to be subtracted, the calculation can be carried out by subtracting the smaller number from the larger, and indicating that the remainder is negative.

Examples:

$3 - 5 = -(5 - 3) = -2.$

In this example 5 cannot, of course, be subtracted from 3, but the numbers are reversed, 3 being subtracted from 5, and the remainder indicated as being negative by placing a minus sign before it.

$227 - 375 = -(375 - 227) = -148.$

The examples given, if carefully studied, will enable the student to carry out calculations with negative numbers when such will be required in solving triangles.

CHAPTER IV

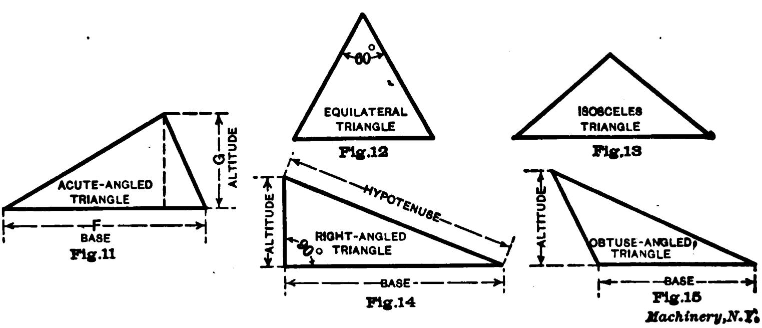
FUNCTIONS OF ANGLES

Any figure bounded by three straight lines is called a triangle. Any one of the three lines may be called the base, and the line drawn from the angle opposite the base at right angles to it is called the height or altitude of the triangle. In Fig. 11, if the side *F* is taken as the base of the triangle, then *G* is the altitude.

If all the three sides of a triangle are of equal length, as in the one shown in Fig. 12, the triangle is called *equilateral*. Each of the three angles in an equilateral triangle equals 60 degrees.

If two sides are of equal length, as shown in Fig. 13, the triangle is an *isosceles* triangle.

If one angle is a right or 90-degree angle, the triangle is called a *right* or *right-angled* triangle. Such a triangle is shown in Fig. 14; the side opposite the right angle is called the *hypotenuse*.



Figs. 11 to 15

If all the angles are less than 90 degrees, the triangle is called an *acute* or *acute-angled* triangle, as shown in Fig. 11. If one of the angles is larger than 90 degrees, as shown in Fig. 15, the triangle is called an *obtuse* or *obtuse-angled* triangle. The sum of the three angles in every triangle is 180 degrees.

Object of Trigonometry and Trigonometric Functions

The object of that part of mathematics called trigonometry is to furnish the methods by which the unknown sides and angles in a triangle may be determined when certain of the sides and angles are given.

The sides and angles of any triangle, which are not known, can be found when:

1. All the three sides,
2. Two sides and one angle, or
3. One side and two angles,

are given. In other words, if the triangle is considered as consisting of six parts, three angles and three sides, the unknown parts can be determined when any three of the parts are given, provided at least one of the given parts is a side.

In order to introduce the values of the angles in calculations of triangles, use is made of certain expressions called *trigonometrical functions* or *functions of angles*. The names of these expressions are: *sine, cosine, tangent, cotangent, secant, and cosecant.* These expressions are usually abbreviated as follows:

sin = sine,	cot = cotangent,
cos = cosine,	sec = secant,
tan = tangent,	cosec = cosecant.

In Fig. 16 is shown a right-angled triangle. The lengths of the three sides are represented by a, b and c, respectively, and the angles opposite each of these sides are called A, B and C, respectively. Angle

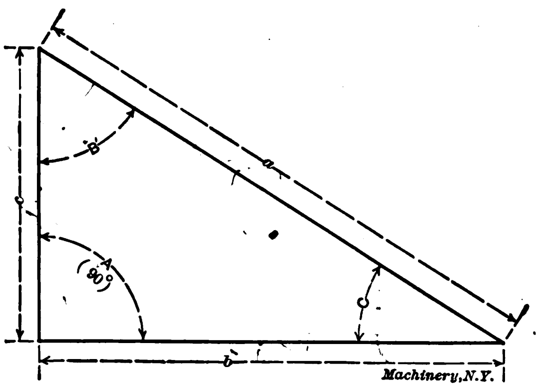


Fig. 16

A is the right angle in the triangle. The side a opposite the right angle is the *hypotenuse*. The side b is called the side *adjacent* to the angle C, but is of course also the side *opposite* to angle B. In the same way, the side c is called the side adjacent to angle B, and the side opposite to angle C. The reason for these names is made clear by studying the figure.

The meanings of the various functions of angles previously named can be explained by the aid of a right-angled triangle.

The sine of an angle equals the opposite side divided by the hypotenuse.

The sine of angle B thus equals the side b, which is opposite to the angle, divided by the hypotenuse a. Expressed as a formula we have:

$$\sin B = \frac{b}{a}.$$

If $a = 16$, and $b = 9$, then $\sin B = \frac{9}{16} = 0.5625$.

The cosine of an angle equals the adjacent side divided by the hypotenuse.

The cosine of angle B thus equals the side c, which is adjacent to this angle, divided by the hypotenuse a, or, expressed as a formula,

$$\cos B = \frac{c}{a}.$$

If $a = 24$, and $c = 15$, then $\cos B = \frac{15}{24} = 0.625$.

The tangent of an angle equals the opposite side divided by the adjacent side.

The tangent of angle B thus equals the side b divided by side c, or,

$$\tan B = \frac{b}{c}.$$

If $b = 28$, and $c = 25$, then $\tan B = \frac{28}{25} = 1.12$.

The cotangent of an angle equals the adjacent side divided by the opposite side.

The cotangent of angle B thus equals the side c divided by the side b, or, $\cot B = \frac{c}{b}$.

If $b = 28$, and $c = 25$, then $\cot B = \frac{25}{28} = 0.89286$.

The secant of an angle equals the hypotenuse divided by the adjacent side.

The secant of angle B thus equals the hypotenuse a divided by the side c adjacent to the angle, or $\sec B = \frac{a}{c}$.

If $a = 24$, and $c = 15$, then $\sec B = \frac{24}{15} = 1.6$.

The cosecant of an angle equals the hypotenuse divided by the opposite side.

The cosecant of angle B thus equals the hypotenuse a divided by the side b opposite the angle, or $\operatorname{cosec} B = \frac{a}{b}$.

If $a = 16$, and $b = 9$, then $\operatorname{cosec} B = \frac{16}{9} = 1.77778$.

The rules given above are very easily memorized, and the student should go no further before he can see at a glance the various functions in a given right-angled triangle.

If the functions of the angle C were to be found instead of the functions of angle B, as given above, they would be as follows:

$$\sin C = \frac{c}{a} \qquad \cos C = \frac{b}{a} \qquad \tan C = \frac{c}{b}$$

$$\cot C = \frac{b}{c} \qquad \sec C = \frac{a}{b} \qquad \text{cosec } C = \frac{a}{c}$$

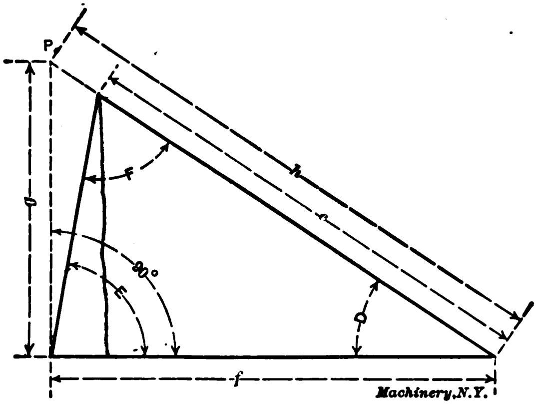


Fig. 17

It must be remembered that the functions of the angles can be found in this manner only when the triangle is right-angled. If the triangle has the shape shown by the full lines in Fig. 17, the sine of angle D, for instance, cannot be expressed by any relation between two sides of

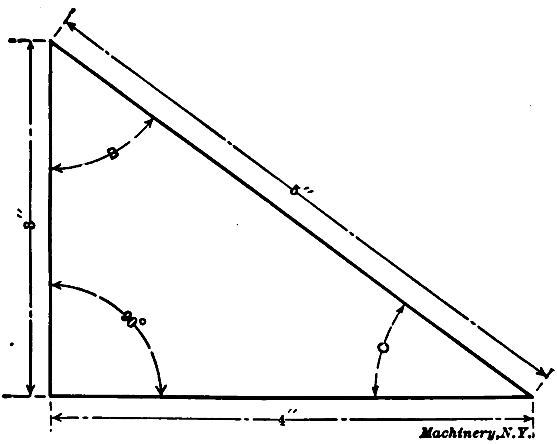


Fig. 18

this triangle. The sine of angle D, however, can be found by constructing a right-angled triangle by extending the side e to the point P, from where a line can be drawn at right angles to the vertex or point of angle E, as shown by the dotted line. The sine of angle D would then be the length of the dotted line g divided by the length of the line h, these two lines being, respectively, the side opposite angle *D*, and the hypotenuse, in a right-angled triangle. In the same way, the tangent of angle *D* would be the side *g* divided by the side *f*.

Examples for Finding the Values of the Functions of Angles

In Fig. 18 is shown a right-angled triangle where the side opposite angle *B* is four inches, the side opposite angle *C* is 3 inches, and the hypotenuse is 5 inches. Find the values of the functions of the angles *B* and *C*.

Following the rules previously given for finding the sine, cosine, tangent, etc., we have:

$$\sin B = \frac{4}{5} = 0.8 \qquad \cos B = \frac{3}{5} = 0.6$$

$$\tan B = \frac{4}{3} = 1.333 \qquad \cot B = \frac{3}{4} = 0.75$$

$$\sec B = \frac{5}{3} = 1.667 \qquad \text{cosec } B = \frac{5}{4} = 1.25$$

The functions for angle *C* are as follows:

$$\sin C = \frac{3}{5} = 0.6 \qquad \cos C = \frac{4}{5} = 0.8$$

$$\tan C = \frac{3}{4} = 0.75 \qquad \cot C = \frac{4}{3} = 1.333$$

$$\sec C = \frac{5}{4} = 1.25 \qquad \text{cosec } C = \frac{5}{3} = 1.667$$

The secant and cosecant, being merely the values of 1 divided by the cosine and sine, are not often used in calculations, or included in tables of angular functions.

By studying the results obtained in the calculations above it will be noted that in a right-angled triangle there is a definite relation between the functions of the two acute angles. The sine of angle *B* equals the cosine of angle *C*; the tangent of angle *B* equals the cotangent of angle *C*, etc. This is true of all right-angled triangles.

As the sum of the three angles in a triangle always equals 180 degrees, and as a right angle equals 90 degrees, it follows that the sum of the two acute angles in a right-angled triangle equals $180 - 90 = 90$ degrees. The angle *B* (Fig. 18) which together with angle *C* forms a 90-degree angle, is called the *complement* of angle *C*. In the same way angle *C* is the complement of angle *B*. When any two angles together make 90 degrees, the one is the complement of the other, and in all such cases, the sine of the one equals the cosine of the other, and *vice versa*, the tangent of the one equals the cotangent of the other, etc.

CHAPTER V

TABLES OF TRIGONOMETRIC FUNCTIONS

When using formulas of the type

$$A = \frac{16 \times \sin 36 \text{ deg.}}{2}$$

it is, of course, not possible to find the value of *A* unless we have some means of transforming the expression "sin 36 deg." (read: sine of 36 degrees) into plain figures. In other words, we must know the *numerical value* of "sin 36 deg.," before we can calculate *A*. Assume that "sin 36 deg." equals 0.58779. Then, if we insert this value in the formula, we have:

$$A = \frac{16 \times 0.58779}{2} = 4.70232.$$

The numerical values for the natural or trigonometric functions which must thus be found before a formula containing an expression with a trigonometric function can be calculated, are given in the tables in Part II of this treatise, MACHINERY'S Reference Series No. 55. In the following, when reference to "the tables" is made, these tables are always referred to. From these tables, when the angle is given in degrees and minutes, the corresponding numerical value of any of the trigonometric functions can be found; and if the numerical value of the function is known, the corresponding angle can be determined.

It will be seen in the tables that the number of degrees from 0 degree (0°) to 44 degrees (44°) are given above the tables, and the number of minutes in the left-hand column headed with the minute sign (′), reading downward from 0 to 60. The number of degrees from 45 degrees (45°) to 89 degrees (89°), inclusive, are given at the bottom of the tables, and the minutes for the latter degrees are given in the extreme right-hand column, reading from below and up, from 0 to 60. The four main columns in the tables are headed "Sin," "Cos," "Tan," and "Cot," at the top of the tables, and at the bottom of the same tables are the main legends "Cos," "Sin," "Cot," and "Tan." This indicates that when the sine of an angle is required the number of degrees of which angle is given at the top of the table, the sine will be found in the column headed "Sin" at the top; but when the sine of an angle, the number of degrees of which is given at the bottom, is to be found, the sine is found in the second main column, having the word "Sin" at the bottom. The same, of course, applies to the other functions, cosine, tangent, and cotangent.

By referring to the tables it will be seen further that there are two columns of figures in each of the main columns, one headed "Nat."

(natural function) and one "Log." (logarithm). For the present, we are concerned *with the figures given in the column under "Nat." only*, and will treat the subject as if the logarithms of the functions and the columns headed "d." and "c. d." did not exist. Later, we will return to the use of these.

Assume now that the sine, cosine, tangent or cotangent of an angle between 0 and 45 degrees is to be found. First find the given number of degrees at the top of the table; then find the given number of minutes in the extreme left-hand column. Then, read off the figures in the column of the natural sine, cosine, tangent or cotangent, as the case may be, which is opposite the given number of minutes. This value, just read off, is now the numerical value of the function which was to be found.

In reading off these values, care must be taken to place the decimal point properly, as this point is not always given in the tables. The sine and cosine of angles are never over 1, so that when the table gives the figures 99949 as the cosine of 1 degree 50 minutes, the decimal point should be placed in front of these figures, the value being 0.99949. The same refers to the other functions when no decimal point is given. A decimal point should then always be placed in front of the figures given in the tables.

When the sine, cosine, tangent or cotangent of an angle between 45 and 90 degrees is to be found, first find the given number of degrees at the bottom of the table; then find the number of minutes in the extreme right-hand column. Then read off the required function opposite the number of minutes, in the column marked with the required function at the bottom.

Examples of the Use of Trigonometric Tables

Example 1.—Find from the tables the sine of 56 degrees, or, as it is commonly written, sin 56°.

Find first "56°" at the bottom of its page, and then (as in this case there are no minutes) locate 0′ (0 minutes) in the extreme right-hand column, reading from the bottom up. Then, in the column "Nat. Sin." marked at the bottom, read off 0.82904 opposite 0 minutes, which is the required value of the sine of 56 degrees. (Note that the two first figures (82) in the number 82904 are not given opposite every number but only at every fifth number of minutes, but these two figures are to be prefixed, as is easily understod from the table.)

Example 2.—Find sin 50° 20′.

Find first "50°" at the bottom of its page, and then locate 20′ in the right-hand column, reading from the bottom up. Then, in the column "Nat. Sin." marked at the bottom, read off 0.76977 opposite 20 minutes. This is the required value of sin 50° 20′.

Example 3.—Find tan 36° 26′.

Locate 36° at the top of its table, and 26′ in the left-hand column. Then read off 0.73816 in the column "Nat. Tan." This is the required value of tan 36° 26′.

Example 4.—Find cos 36° 19′.

In the same manner as in the examples above, cos 36° 19′ is found to equal 0.80576.

The student should find the following functions from the tables and then compare the result found with the values given, to check the accuracy of the work:

sin 12° 10′ = 0.21076	cos 60° 0′ = 0.50000
sin 15° 50′ = 0.27284	sin 65° 10′ = 0.90753
tan 1° 20′ = 0.02328	sin 12° 3′ = 0.20877

Trigonometric Functions for Angles greater than 90 Degrees

The tables in Part II, Reference Series No. 55, give the angular functions only for angles up to 90 degrees (or 89 degrees 60 minutes, which, of course, equals 90 degrees). In obtuse triangles one angle, however, is greater than 90 degrees, and the tables can be used for finding the functions for angles larger than 90 degrees also.

The sine of an angle greater than 90 degrees but less than 180 degrees equals the sine of an angle which is the difference between 180 degrees and the given angle.

Example: Sin 118° = sin (180° — 118°) = sin 62°. In the same way sin 150° 40′ = sin (180° — 150° 40′) = sin 29° 20′.

The cosine, tangent and cotangent for an angle greater than 90 but less than 180 degrees equals, respectively, the cosine, tangent and cotangent of the difference between 180 degrees and the given angle, but in this case the angular function found has a *negative* value (is preceded by a minus sign).

Example 1.—Find tan 150°.

Tan 150° = — tan (180° — 150°) = — tan 30°. From the tables we have tan 30° = 0.57735; thus tan 150° = — 0.57735.

Example 2.—Find sin 155° 50′.

As explained above sin 155° 50′ = sin (180° — 155° 50′) = sin 24° 10′ = 0.40939.

Example 3.—Find tan 123° 20′.

As explained above tan 123° 20′ = — tan (180° — 123° 20′) = — tan 56° 40′ = — 1.5204.

[In calculations of triangles it is very important that the minus sign is not omitted in the cosines, tangents and cotangents of angles between 90 and 180 degrees.]

Finding the Angle when the Function is Given

When the value of the function of an angle is given, and the angle required in degrees and minutes, the function is located in the tables and the corresponding angle found by a process the reverse of that employed for finding the functions when the angle is given. If the value of the function cannot be found exactly in the tables, use the nearest value found.

Example 1.—The sine of a certain angle, which may be called a, equals 0.53238. Find the angle.

The function 0.53238 is located in the columns marked "Sin." either at the top or at the bottom. When located, the degrees and minutes of the angle are read off directly. If the function is located in the column marked "Sin" at the top, the number of degrees is read off at the top and the number of minutes in the left-hand column; if the function is located in the column marked "Sin." at the bottom, the degrees are read off at the bottom and the minutes in the right-hand column. Following these rules, we find the required angle to be 32° 10′.

Example 2.—The cotangent of an angle is 0.77196. Find the angle.

By observing the rules given in the previous example we find that the required angle is 52° 20′.

Example 3.—The tangent of angle $a = -3.3402$. Find a.

The positive value 3.3402 is first located and the corresponding angle found. This angle is 73° 20′. As the tangent is negative (preceded by a minus sign) the angle a, however, is not 73° 20′ but (180° — 73° 20′) = 106° 40′.

Example 4.—If $\sin a = 0.29381$, what is the value of angle a?

It will be seen that the function 0.29381 cannot be found exactly in the tables. The nearest value to be found in the sine columns is 0.29376. For practical purposes in machine construction and shop calculations it is near enough to find the angle corresponding to this nearest value. Hence, $a = 17° 5'$.

CHAPTER VI

PRACTICAL APPLICATIONS OF TRIGONOMETRIC FORMULAS

In the following are given a few problems solved by the use of formulas of which trigonometric functions are a part. These examples will show the use of these functions, as obtained from the tables, in cases where it is only required to insert their value in the given formulas.

Example 1.—The depth of the thread in the United States standard screw thread system is expressed by the formula:

$$d = 3/4 \times p \times \cos 30°$$

in which d = depth of thread,

$$p = \text{pitch of thread} = \frac{1}{\text{No. of threads per inch}}$$

Assume that it is required to find the depth of thread for 14 threads per inch. Then $p = \frac{1}{14}$, and

$$d = \frac{3}{4} \times \frac{1}{14} \times \cos 30° = \frac{3}{56} \times 0.86603 = 0.0464 \text{ inch.}$$

Example 2.—In spiral gearing, the pitch diameter of a gear is found by the formula:

$$D = \frac{N}{P \times \cos a}$$

in which D = pitch diameter of spiral gear,
N = number of teeth in gear,
P = normal diametral pitch,
a = tooth angle of gear.

Assume that in a specific case we know that $N = 20$, $P = 8$, and angle $a = 24$ degrees; find the pitch diameter. Then:

$$D = \frac{20}{8 \times \cos 24°} = \frac{20}{8 \times 0.91355} = 2.7366 \text{ inches.}$$

Example 3.—The formula for finding the lead for which to gear up the milling machine when cutting spiral gears is:

$$L = 3.1416 \times D \times \cot a$$

in which L = the lead for which to gear up the machine,
D = pitch diameter,
a = tooth angle.

Assume that in a specific case we know that $D = 5$, and angle $a = 24$ degrees. Then

$$L = 3.1416 \times 5 \times \cot 24° = 15.708 \times 2.246 = 35.28 \text{ inches.}$$

Example 4.—In a radial ball bearing, if the diameter of the balls, d, and the number of balls, N, are known, the diameter D of the outside or enveloping ball race may be found by the following formula:

$$D = \frac{d}{\sin\left(\frac{180}{N}\right)^{\circ}} + d$$

Assume that $d = \frac{1}{4}$ inch, and $N = 15$. Then:

$$D = \frac{0.25}{\sin\left(\frac{180}{15}\right)^{\circ}} + 0.25 = \frac{0.25}{\sin 12°} + 0.25 = \frac{0.25}{0.20791} + 0.25$$

$$= 1.2025 + 0.25 = 1.4525 \text{ inch.}$$

Example 5.—In a sprocket wheel for ordinary link chain, the pitch diameter D can be determined when the number of teeth required, N, the length of the inside oval of the chain link, r, and the diameter of the stock from which the chain link is made, d, are known. The formula used is:

$$D = \sqrt{\left(\frac{r}{\sin (90 \div N)^{\circ}}\right)^2 + \left(\frac{d}{\cos (90 \div N)^{\circ}}\right)^2}$$

If $r = \frac{3}{4}$ inch, $d = \frac{1}{4}$ inch, and $N = 20$ teeth, then:

$$D = \sqrt{\left(\frac{0.75}{\sin 4^\circ\ 30'}\right)^2 + \left(\frac{0.25}{\cos 4^\circ\ 30'}\right)^2} = \sqrt{9.559^2 + 0.251^2}$$

$$= \sqrt{91.437} = 9.562 \text{ inches.}$$

Example 6.—In a Bush roller chain wheel the pitch diameter D of the sprocket wheel can be found if the number of teeth in the sprocket, N, and the pitch P of the chain are decided upon. The formula is:

$$D = \frac{P}{\sin\left(\frac{180}{N}\right)^\circ}$$

Assume that the pitch diameter of a sprocket with 72 teeth, for a chain of ¾ inch pitch, is required. Then $P = ¾$, and $N = 72$; hence $\frac{180}{N} = 2½$, and $D = \frac{0.75}{\sin 2^\circ\ 30'} = \frac{0.75}{0.04362} = 17.194$ inches.

Example 7.—The following formula may be used for finding the angle to which to set the dividing head of the milling machine when cutting teeth in the ends of end mills:

$$\cos \alpha = \tan \frac{360}{N} \times \cot \beta$$

in which α = angle to which to set dividing head,
β = included angle of cutter with which teeth are milled,
N = number of teeth in end mill.

Assume that it is required to cut the teeth in the end of an end mill having 12 teeth with a 70-degree angular milling cutter.

$$\cos \alpha = \tan \frac{360}{12} \times \cot 70^\circ = \tan 30^\circ \times \cot 70^\circ$$

$$= 0.57735 \times 0.36397 = 0.21014.$$

Having found that $\cos \alpha = 0.21014$, we find that $\alpha = 77^\circ\ 52'$.

Example 8.—The angle to which to set the planer head when planing an Acme threading tool having no side clearance, but 15 degrees front clearance, can be determined by the formula:

$$\tan x = \frac{\tan 14^\circ\ 30'}{\cos 15^\circ}$$

in which x = angle to which to set planer head.

Carrying out the calculation, we have:

$$\tan x = \frac{\tan 14^\circ\ 30'}{\cos 15^\circ} = \frac{0.25862}{0.96593} = 0.26774$$

Having found that $\tan x = 0.26774$, we find from the tables that $x = 14^\circ\ 59'$, or practically 15 degrees.

CHAPTER VII

RIGHT-ANGLED TRIANGLES

If the lengths of two sides of a right-angled triangle are known, the third side can be found by a simple calculation. In every right-angled triangle the hypotenuse equals the square root of the sum of the squares of the two sides forming the right angle. If the hypotenuse equals a, and the sides forming the right angle b and c, respectively, as shown in Fig. 19, then:

$$a = \sqrt{b^2 + c^2}$$

Each of the sides b and c can also be found if the hypotenuse and one of the sides are known. The following formulas would then be used:

$$b = \sqrt{a^2 - c^2}$$

$$c = \sqrt{a^2 - b^2}$$

Assume that side b is 18 inches, and side c, 7.5 inches. What is the length of the hypotenuse a?

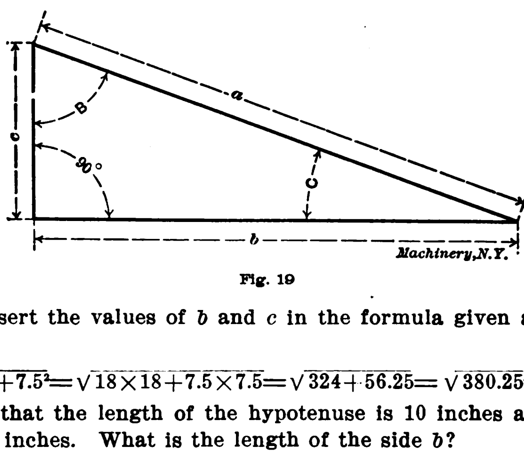


Fig. 19

If we insert the values of b and c in the formula given above for a, we have:

$$a = \sqrt{18^2 + 7.5^2} = \sqrt{18 \times 18 + 7.5 \times 7.5} = \sqrt{324 + 56.25} = \sqrt{380.25} = 19.5$$

Assume that the length of the hypotenuse is 10 inches and that the side c is 6 inches. What is the length of the side b?

Using the formula given above for b, and inserting the values of a and c we have:

$$b = \sqrt{10^2 - 6^2} = \sqrt{10 \times 10 - 6 \times 6} = \sqrt{100 - 36} = \sqrt{64} = 8.$$

Thus whenever two sides of a right-angled triangle are given, the third side can always be found by a simple arithmetical calculation. To find the angles, however, it is necessary to use the tables of sines, cosines, tangents and cotangents, as given in Part II, MACHINERY'S Reference Series No. 55; and if only one side and one of the acute angles are given, the natural trigonometric functions must be used for finding the lengths of the other sides, as explained in the following.

Solution of Right-angled Triangles by Means of the Functions of Angles

In Chapter IV it is stated that the sides and angles of any triangle, which are not known, can be found when:

1. All the three sides,
2. Two sides and one angle, or
3. One side and two angles

are given. In every right-angled triangle one angle, the right or 90-degree angle is, of course, always known. In a right triangle, therefore, the unknown sides and angles can be found when either two sides, or one side and one of the acute angles are known.

The methods of solution of right-angled triangles may be divided into four classes, according to which sides and angles are given or known:

1. Two sides known.
2. The hypotenuse and one acute angle known.
3. One acute angle and its adjacent side known.
4. One acute angle and its opposite side known.

Case 1.—When two sides are known, the third side is found by one of the formulas:

$$a = \sqrt{b^2 + c^2} \qquad (1)$$

$$b = \sqrt{a^2 - c^2} \qquad (2)$$

$$c = \sqrt{a^2 - b^2} \qquad (3)$$

which formulas are given in the first part of this chapter, and in which a is the hypotenuse, and b and c the sides forming the right angle.

The acute angles B and C, Fig. 19, are found by determining either the sine, cosine, tangent or cotangent for the angles, as explained in Chapter IV, and obtaining the angles, expressed in degrees and minutes, from the trigonometric tables. When one angle has been found, the other can also be found directly without reference to the tables, because the sum of the acute angles in a right-angled triangle equals 90 degrees, and if one of them is known, the other must equal 90 degrees minus the known angle. Expressed as formulas this would be:

$$B = 90^\circ - C$$

$$C = 90^\circ - B$$

As an example, assume that the hypotenuse of a right-angled triangle is 5 inches and one of the sides 4 inches, as shown in Fig. 20. Find angles B and C and the length of side c.

The side c is first found by Formula (3) given above, a and b being inserted in this formula as below:

$$c = \sqrt{5^2 - 4^2} = \sqrt{25 - 16} = \sqrt{9} = 3.$$

As explained in Chapter IV, the side opposite an angle divided by the hypotenuse, gives the sine of the angle.

Hence

$$\sin C = \frac{3}{5} = 0.6.$$

By referring to the trigonometric tables, it will be found that the nearest value to 0.6 in the columns of sines is 0.59995, and the angle corresponding to this value is 36° 52′. Angle C, then equals, 36° 52′.

In the same way

$$\sin B = \frac{4}{5} = 0.8.$$

From the tables we find the nearest value in the columns of sines to be 0.80003, which is the sine of 53° 8′.

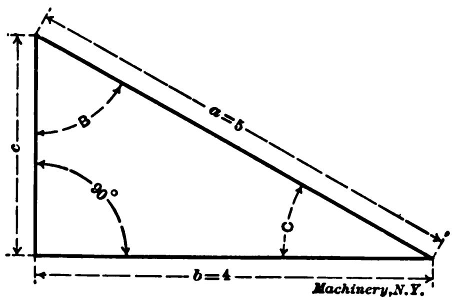


Fig. 20

This last calculation would not have been necessary, because, as has already been mentioned, angle B could have been found directly when angle C was known, by the formula

$$B = 90° - C = 90° - 36° 52' = 53° 8'.$$

It will be noted that either method for finding angle B gives the same result.

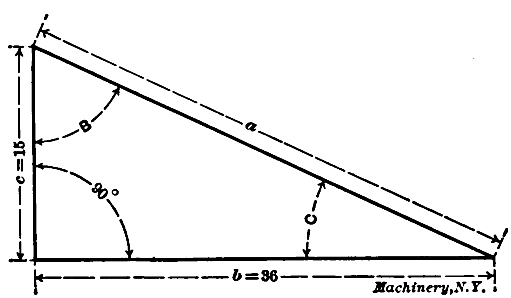


Fig. 21

As a further example, assume that the sides forming the right angle are given as shown in Fig. 21; one is 15 inches and the other is 36 inches. Find the hypotenuse and the angles B and C.

The hypotenuse is found by Formula (1), on page 25, the values of b and c being inserted.

$$a = \sqrt{36^2 + 15^2} = \sqrt{1296 + 225} = \sqrt{1521} = 39.$$

As explained in Chapter IV, the side opposite an angle divided by the side adjacent, equals the tangent of the angle.

Hence

$$\tan B = \frac{36}{15} = 2.4.$$

By referring to the tables, it will be found that the nearest value to 2.4 in the columns of tangents is 2.4004, which is the tangent of 67° 23′. Hence $B = 67° 23′$, and

$$C = 90° - B = 90° - 67° 23′ = 22° 37′.$$

Case 2.—If the hypotenuse and one acute angle are known, the side adjacent to the known angle is found by multiplying the hypotenuse by the cosine of the known angle; the side opposite the known angle is found by multiplying the hypotenuse by the sine of the known angle; and the other acute angle is found by subtracting the known angle from 90 degrees.

We can express this rule by simple formulas. Referring to Fig. 19, if a is the hypotenuse, and B the known angle, then:

$$c = a \times \cos B$$
$$b = a \times \sin B$$
$$C = 90° - B$$

If C is the known angle, then:

$$b = a \times \cos C$$
$$c = a \times \sin C$$
$$B = 90° - C$$

As an example, assume that the hypotenuse $a = 22$ inches, and angle $B = 41° 36′$. Find sides b and c, and angle C. (See Fig. 19.)

By referring to the tables, it will be found that the nearest value to case when angle B is known, we have:

$c = a \times \cos B = 22 \times \cos 41° 36′ = 22 \times 0.74780 = 16.4516$ inches.
$b = a \times \sin B = 22 \times \sin 41° 36′ = 22 \times 0.66393 = 14.6065$ inches.
$C = 90° - 41° 36′ = 48° 24′$.

Case 3.—When one acute angle and its adjacent side are known, the hypotenuse is found by dividing the known side by the cosine of the known angle; the side opposite the known angle is found by multiplying the known adjacent side by the tangent of the known angle; and the other acute angle is found by subtracting the known angle from 90°.

Referring to Fig. 19, we can express this rule by simple formulas. If B is the known angle, and c the known side, adjacent to angle B, then:

$$a = \frac{c}{\cos B} \qquad b = c \times \tan B \qquad C = 90° - B$$

If C is the known angle, and b the known side, adjacent to angle C, then:

$$a = \frac{b}{\cos C} \qquad c = b \times \tan C \qquad B = 90° - C$$

As an example, assume that angle $B = 25° 12′$, and its adjacent side $c = 12$ inches. Find the hypotenuse a, opposite side b, and angle C.

By inserting the known values in the formulas just given for the case where angle B is known, we have:

$$a = \frac{c}{\cos B} = \frac{12}{\cos 25° \; 12'} = \frac{12}{0.90483} = 13.262 \text{ inches.}$$

$b = c \times \tan B = 12 \times 0.47056 = 5.6467$ inches.

$C = 90° - 25° \; 12' = 64° \; 48'$.

Case 4.—When one acute angle and the side opposite it are known, the hypotenuse is found by dividing the known side by the sine of the known angle; the side adjacent to the known angle is found by multiplying the known opposite side by the cotangent of the known angle; and the other acute angle is found by subtracting the known angle from 90°.

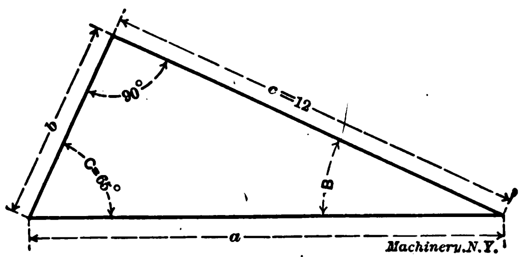


Fig. 22

By referring to Fig. 19, we can express this rule by simple formulas. If B is the known angle, and b the side opposite, which is also known, then:

$$a = \frac{b}{\sin B} \qquad c = b \times \cot B \qquad C = 90° - B$$

If C is the known angle, and c the known side, opposite to angle C, then:

$$a = \frac{c}{\sin C} \qquad b = c \times \cot C \qquad B = 90° - C$$

As an example, assume that angle C equals 65 degrees, and that the length of side c is 12 feet, as shown in Fig. 22. Find the lengths of sides a and b and angle B.

By inserting the known values in the formulas just given for the case when angle C is known, we have:

$$a = \frac{c}{\sin C} = \frac{12}{\sin 65°} = \frac{12}{0.90631} = 13.2405 \text{ inches.}$$

$b = c \times \cot C = 12 \times 0.46631 = 5.5957$ inches.

$B = 90° - 65° = 25°$.

CHAPTER VIII

PROBLEMS FROM PRACTICE

The calculations required in the design of bevel gearing offer abundant examples of the use of the trigonometric functions and the solution of right-angled triangles. The student who is anxious to obtain additional practice, and to whom the practical applications of the formulas given are of especial interest, is, therefore, referred to MACHINERY'S Reference Series No. 37, Bevel Gearing, for practical applications. In the following, however, a number of practical examples, selected for the purpose of illustration, will also be given.

Example 1.—Fig. 23 shows a section of a United States standard thread. Find a formula for the depth of the thread in terms of the pitch, and calculate the depth of screw threads with 12 and 16 threads per inch.

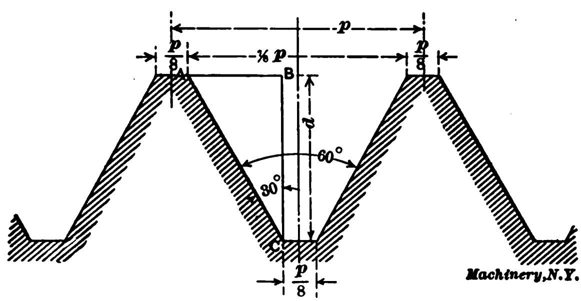


Fig. 23

In the illustration, p is the pitch of the thread. The pitch, of course, equals $\dfrac{1}{\text{No. of threads per inch.}}$.* It is required to find the depth BC of the thread, expressed in terms of the pitch. This depth can be found if we can solve the triangle ABC.

In the U. S. standard thread system there is a flat at the top and bottom of the thread as shown in Fig. 23. The width of this flat is one-eighth of the pitch, as indicated. Hence, side AB of the right-angled triangle ABC equals one-half of ⅞ pitch minus one-half of ⅛ pitch, or $\left(\dfrac{7}{16} - \dfrac{1}{16}\right)$ pitch = ⅜ pitch. The angle opposite this side is also known; it is one-half of the total thread angle, or 30 degrees. According to the rules and formulas in the previous chapter, therefore,

$$BC = AB \times \cot 30°.$$

* See MACHINERY'S Reference Series No. 18, "Shop Arithmetic for the Machinist," third edition, Chapter IV.

If we insert in this formula $BC = d$, $AB = \frac{3}{8}p$, and cot 30° = 1.7321, we have:

$$d = \tfrac{3}{8}p \times 1.7321 = 0.6495\,p$$

in which d = depth of thread,
p = pitch of thread.

We will now find the depth of the thread for 12 and 16 threads per inch. As $p = \frac{1}{\text{No. of threads per inch}}$, we have, by inserting the known values in the general formula just found:

$$d = 0.6495 \times \frac{1}{12} = 0.0541 \text{ inch, for 12 threads,}$$

$$d = 0.6495 \times \frac{1}{16} = 0.0406 \text{ inch, for 16 threads.}$$

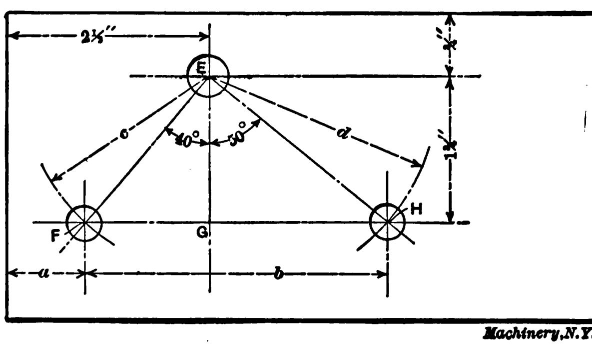


Fig. 24

Example 2.—In laying out a master jig plate, it is required that holes F and H, Fig. 24, shall be on a straight line which is 1¾ inch distant from hole E. The holes must also be on lines making, respectively, 40- and 50-degree angles with line EG, drawn at right angles to the sides of the jig plate through E, as shown in the engraving. Find the dimensions necessary for the toolmaker.

The dimensions which ought to be given the toolmaker in addition to those already given are indicated by a, b, c, and d. The two latter are the radii of the arcs which if struck with E as a center will pass through the centers of F and H. We have here two right-angled triangles EFG and EGH. We know one acute angle in each, and also the length of side EG (1¾ inch) which is mutual to both triangles and which is the side adjacent to the known angle. From the formulas in the preceding chapter we, therefore, have:

$$FG = 1.75 \times \tan 40° = 1.75 \times 0.83910 = 1.4684 \text{ inch.}$$

$$FE = \frac{1.75}{\cos 40°} = \frac{1.75}{0.76604} = 2.2845 \text{ inches.}$$

$$GH = 1.75 \times \tan 50° = 1.75 \times 1.1918 = 2.0856 \text{ inches.}$$

$$EH = \frac{1.75}{\cos 50^\circ} = \frac{1.75}{0.64279} = 2.7225 \text{ inches.}$$

But, by referring to Fig. 24 it will be seen that $FE = c$, $EH = d$, $2\frac{1}{2} - FG = a$, and $FG + GH = b$. Hence

$a = 2.5 - 1.4684 = 1.0316$ inch,

$b = 1.4684 + 2.0856 = 3.5540$ inches,

$c = 2.2845$ inches,

$d = 2.7225$ inches.

Example 3.—If the pitch p of a Bush roller chain is ¾ inch, and the sprocket wheel is to have 32 teeth, what will be the pitch diameter of the gear? (See Fig. 25.)

By referring to the engraving, it will be seen that $AD = p = \frac{3}{4}$ inch, and $AC = \frac{1}{2}\ AD = \frac{3}{8}$ inch, in this case. Line AB is the pitch radius or one-half the pitch diameter. Angle a is the angle for one

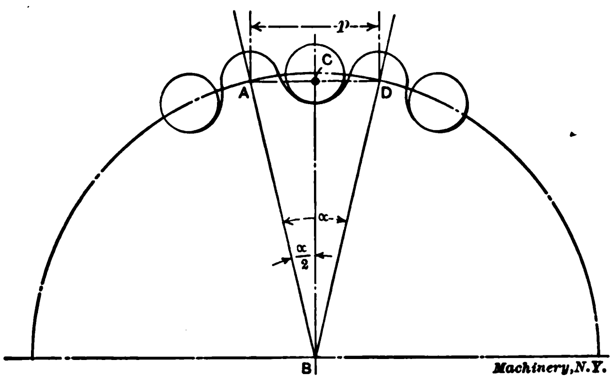


Fig. 25

tooth, and as the whole circle is 360 degrees, a in this case equals $\frac{360}{32} = 11\frac{1}{4}$ degrees, or 11 degrees 15 minutes. One-half of a, then, equals 5 degrees 37 minutes, approximately. We, therefore, have here a right-angled triangle in which we know the length of side AC and the angle opposite it. We want to find the hypotenuse AB. From the formulas in the preceding chapter, we have:

$$AB = \frac{AC}{\sin \frac{a}{2}} = \frac{0.375}{\sin 5^\circ\ 37'} = \frac{0.375}{0.09787} = 3.832 \text{ inches.}$$

The pitch diameter, then, equals $2 \times 3.832 = 7.664$ inches.

Example 4.—A common method for measuring the width of machine slide dove-tails is indicated diagrammatically in Fig. 26. At A and B are shown carefully ground cylindrical gages of standard dimensions. In the example shown it is required to find what the distance d, measured by micrometers over the gages when these are pushed into the V's of the dovetail as shown, should be, in order to make sure that the piece is planed to the dimensions given. The diameters of the gages are 0.750 inch.

In order to find dimension *d* measured over the gages, find dimension *KG*, Fig. 27, and add twice this length to the distance 3 inches from *L* to *M*, in Fig. 26. It will be seen that $KG = KE + EG$; but $KE = \frac{1}{2}$ the gage diameter $= \frac{3}{8}$ inch; and *EG* is solved from the right-angled

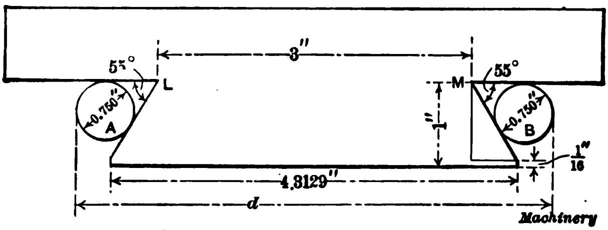


Fig. 26

triangle *EGH* in which the angle $EHG = 62°30'$, and the side $HG = \frac{1}{2}$ the gage diameter, or $\frac{3}{8}$ inch. That angle *EHG* equals 62°30′ is found as follows: Angle $GHN = 90°$; angle $GHF = 90° - 55° = 35°$. Angle $FHE = \frac{1}{2}$ of $55° = 27°30'$; hence, angle $EHG = 35° + 27°30' = 62°30'$. Then,

$EG = HG \times \tan\ 62°30' = \frac{3}{8} \times 1.921 = 0.7204$ inch.

$KE + EG = 0.375 + 0.7204 = 1.0954$ inch.

$d = 2 \times 1.0954 + 3 = 5.1908$ inches.

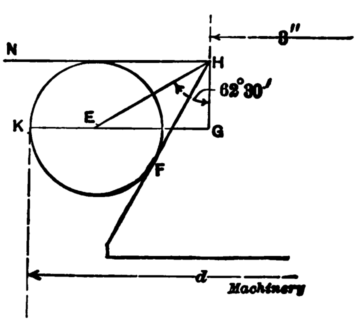


Fig. 27

Example 5.—Small reamers are sometimes provided with flats instead of actual flutes. The diameter of the reamer is, of course, measured over the sharp corners; if the reamer tapers, the taper of the flats will not be the same as the taper of the sharp corners, and the milling machine dividing head must be set to a different angle from that which the cutting edge makes with the center line. A simple formula may be deduced by the aid of trigonometry for finding the angle to which to set the dividing head when milling the flats.

Referring to Fig. 28, in which the reamer is imagined as continued to a sharp point at the end, let

a = angle made by cutting edge with center line,
a_1 = angle made by flat with center line,
N = number of sides of reamer,
T = taper per foot.

Angle β, as shown in the engraving, can be determined by the mula

$$\beta = \frac{360}{2N}$$

as is evident from the illustration.

Angle α_1 is the angle sought. It will be seen that if *FE* and *HE* were known, then

$$\tan \alpha_1 = \frac{FE}{HE}$$

But $FE = AE \times \cos \beta$. If we insert this value we have:

$$\tan \alpha_1 = \frac{AE \times \cos \beta}{HE}$$

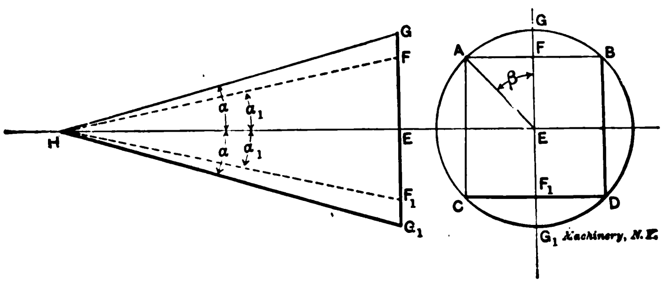


Fig. 28

As $\cos \beta = \cos \frac{360}{2N}$, we have further

$$\tan \alpha_1 = \frac{AE}{HE} \times \cos \frac{360}{2N}$$

The distance *AE*, however, is one-half of the taper in the distance *HE*.

The taper per inch then is $\frac{2\,AE}{HE}$, and the taper per foot

$$T = 12 \times \frac{2\,AE}{HE} = \frac{24\,AE}{HE}, \text{ or } \frac{T}{24} = \frac{AE}{HE}.$$

If we insert $\frac{T}{24}$ in the formula above, we have

$$\tan \alpha_1 = \frac{T}{24} \times \cos \frac{360}{2N}$$

Assume that the taper per foot is ¼ inch, and that a four-sided reamer is required. Find the angle to which to set the index-head.

$$\tan \alpha_1 = \frac{1/4}{24} \times \cos 45° = 0.00736,$$

which gives $\alpha_1 = 25$ minutes.

Example 6.—In Fig. 29 are shown two pulleys of 6 and 12 inches diameter, with a fixed center distance of 5 feet. Find the length of belt required to pass over the two pulleys. The belt is assumed to be perfectly tight.

The length of the belt is made up of the two straight portions AC and BD, tangent to the circles as shown in Fig. 29, and of the arc AEB of the larger pulley and the arc CFD of the smaller pulley. AC and BD are equal. We will first find the length AC. By drawing a line HG from H, the center of the smaller pulley, parallel to AC, we can construct a triangle HGK in which $HG = AC$, and $GK = AK - HC$. That $HG = AC$ is clear from the fact that HC and KA are parallel, both being perpendicular or at right angles to the tangent line AC. The figure $HGAC$ is, therefore, a rectangle, and, hence, opposite sides are equal. HG, therefore, equals AC, and $HC = GA$.

That $GK = AK - HC$ is evident from the fact that $GK = AK - GA$, but as $GA = HC$, it follows that $GK = AK - HC$.

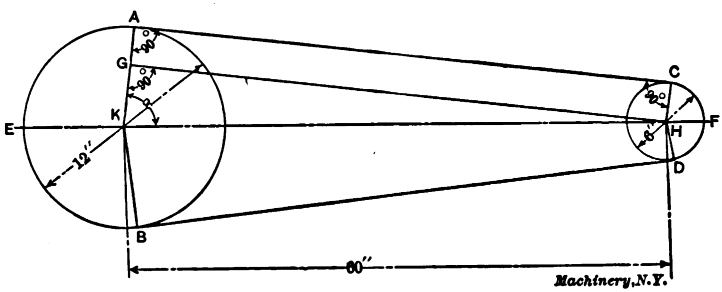


Fig. 29

Now, AK is the radius of the larger pulley, which is one-half its diameter, or 6 inches, and HC is the radius of the smaller pulley or 3 inches. Hence, $GK = 6 - 3 = 3$ inches. $HK = 5$ feet or 60 inches, as given in the problem. We then have here a right-angled triangle in which the hypotenuse $HK = 60$ inches, and one of the sides forming the right angle is 3 inches. Hence, side GH is found by the formula given for this case in the previous chapter, and by inserting the known values we have:

$$GH = \sqrt{60^2 - 3^2} = \sqrt{3600 - 9} = \sqrt{3591} = 59.925.$$

As $GH = AC$, we, therefore, have $AC = 59.925$, and as $AC = BD$, we have $AC + BD = 119.85$ inches. It now remains to find the lengths of the circular arcs AEB and CFD. In order to find these lengths we must first find the number of degrees in these arcs, and to find this, the first step is to find angle a. According to the rules given in Chapter IV,

$$\cos a = \frac{GK}{KH} = \frac{3}{60} = 0.05.$$

From this we find from the trigonometric tables that $a = 87° 8'$.

It will be seen from Fig. 29 that angle $AKE = 180° - a = 180° -$

87° 8′ = 92° 52′. Angle *EKB* = angle *AKE*, so that the arc *AEB*, therefore, is equal to twice angle *AKE* or

$$\text{arc } AEB = 2 \times 92° \ 52' = 185° \ 44'.$$

The whole circumference of the larger pulley equals 3.1416 × 12 = 37.699 inches. As the whole circumference is 360 degrees, its length in inches is to the length of arc *AEB* as 360° is to 185° 44′, or

$$\frac{37.699}{\text{arc } AEB} = \frac{360°}{185° \ 44'}$$

Transposing this expression, we have

$$\text{arc } AEB = \frac{37.699 \times 185° \ 44'}{360°}$$

Before we can carry out this calculation we must transform 44 minutes to decimals of a degree. As 44 minutes equals 44/60 of a degree, this, changed to a decimal fraction equals $\frac{44}{60} = 0.73$, and 185° 44′ equals 185.73 degrees. Then:

$$\text{arc } AEB = \frac{37.699 \times 185.73}{360} = 19.45 \text{ inches.}$$

Now, to find arc *CFD*, angle *CHF* is first determined. This angle equals angle *GKH* or α, because *AK* and *CH* are parallel lines. Hence arc *CFD* = 2 × angle α = 2 × 87° 8′ = 174° 16′. Now, proceeding as before we have:

$$3.1416 \times 6 = 18.8496 = \text{circumference of small pulley.}$$

$$\frac{18.8496}{\text{arc } CFD} = \frac{360°}{174° \ 16'}$$

Transposing this and changing 16 minutes to decimals of a degree, gives us:

$$\text{arc } CFD = \frac{18.8496 \times 174.27}{360} = 9.12 \text{ inches.}$$

The total length of the belt, then, equals

$$119.85 + 19.45 + 9.12 = 148.42 \text{ inches.}$$

CHAPTER IX

SOLUTION OF OBLIQUE-ANGLED TRIANGLES

The methods used in the solution of oblique triangles—that is, triangles, no one of whose angles is a right angle—differ according to which parts are known and which are to be found. The problems which present themselves may be divided into four classes:

1. Two angles and one side known.
2. Two sides and the angle included between them known.
3. Two sides and the angle opposite one of them known.
4. The three sides known.

1. Two Angles and One Side Known

Assume that the angles A and B in Fig. 30 are given as shown, and that side a is 5 inches. Find angle C, sides b and c, and the area of the triangle.

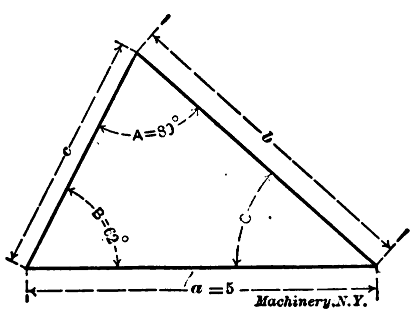


Fig. 30

As the sum of the three angles in a triangle always equals 180 degrees, angle C can be found directly when angles A and B are given, by subtracting the sum of these angles from 180 degrees. Angle $A =$ 80 degrees and $B = 62$ degrees; therefore,

$$C = 180° - (80° + 62°) = 180° - 142° = 38°.$$

For finding the sides b and c the following rule is used: *The side to be found equals the known side multiplied by the sine of the angle opposite the side to be found, and the product divided by the sine of the angle opposite the known side.*

To find side b, for example, multiply the known side a by the sine of angle B, and divide the product by the sine of angle A. Written as a formula this would be:

$$b = \frac{a \times \sin B}{\sin A} \quad (4)$$

In the same way

$$c = \frac{a \times \sin C}{\sin A} \qquad (5)$$

If we insert the known values for side a and the angles in these formulas, we have:

$$b = \frac{5 \times \sin 62^\circ}{\sin 80^\circ} = \frac{5 \times 0.88295}{0.98481} = 4.483 \text{ inches.}$$

$$c = \frac{5 \times \sin 38^\circ}{\sin 80^\circ} = \frac{5 \times 0.61566}{0.98481} = 3.126 \text{ inches.}$$

Now all the sides and angles are known, and it only remains to find the area of the triangle. This is found by the following rule: *The area of a triangle equals one-half the product of two of its sides multiplied by the sine of the angle between them.* (This rule gives the same result as that given in MACHINERY'S Reference Series Book, No. 52, Advanced Shop Arithmetic for the Machinist, Chapter VIII.)

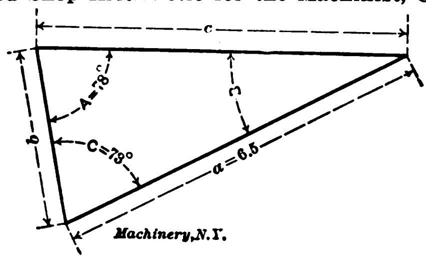


Fig. 31

In the example in Fig. 30, the area, then, equals one-half the product of sides a and b multiplied by the sine of angle C, or, expressed as a formula:

$$\text{Area} = \frac{a \times b \times \sin C}{2} \qquad (6)$$

Inserting the known values for a, b, and C in this formula we have:

$$\text{Area} = \frac{5 \times 4.483 \times \sin 38^\circ}{2} = \frac{5 \times 4.483 \times 0.61566}{2} =$$

$$\frac{13.8000}{2} = 6.9 \text{ square inches.}$$

All the required quantities in this triangle have now been found.

Examples for Practice

Example 1.—In Fig. 31 is shown a triangle of which one side is 6.5 feet, and the two angles A and C (78 and 73 degrees, respectively) are given. Call the sides a, b and c, as shown. Find angle B, sides b and c, and the area.

First find angle B. Using the same method as explained for finding angle C in the previous example, we have:

$$B = 180° - (78° + 73°) = 180° - 151° = 29°.$$

For finding sides b and c use the rule or formulas previously given, inserting the values given in this example:

$$b = \frac{a \times \sin B}{\sin A} = \frac{6.5 \times \sin 29°}{\sin 78°} = \frac{6.5 \times 0.48481}{0.97815}$$

$$= \frac{3.151265}{0.97815} = 3.222 \text{ feet.}$$

$$c = \frac{a \times \sin C}{\sin A} = \frac{6.5 \times \sin 73°}{\sin 78°} = \frac{6.5 \times 0.95630}{0.97815}$$

$$= \frac{6.21595}{0.97815} = 6.355 \text{ feet.}$$

According to the given rule and formula, the area is finally found as below:

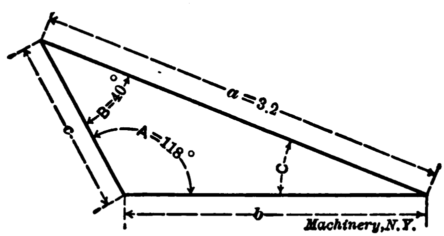


Fig. 32

$$\text{Area} = \frac{a \times b \times \sin C}{2} = \frac{6.5 \times 3.222 \times \sin 73°}{2}$$

$$= \frac{6.5 \times 3.222 \times 0.95630}{2} = \frac{20.027}{2} = 10.013 \text{ square feet.}$$

Example 2.—In Fig. 32, side a equals 3.2 inches, angle A, 118 degrees, and angle B 40 degrees. Find angle C, sides b and c, and the area.

First find angle C.

$$C = 180° - (118° + 40°) = 180° - 158° = 22°.$$

Now find side b.

$$b = \frac{3.2 \times \sin 40°}{\sin 118°} = \frac{3.2 \times 0.64279}{0.88295} = 2.330 \text{ inches.}$$

Note, when finding sin 118° from the tables, that sin 118° = sin (180° — 118°) = sin 62° as explained in Chapter V.

Next, find side c.

$$c = \frac{3.2 \times \sin 22°}{\sin 118°} = \frac{3.2 \times 0.37461}{0.88295} = 1.358 \text{ inch.}$$

Finally,

$$\text{Area} = \frac{3.2 \times 2.33 \times \sin 22^\circ}{2} = 1.396 \text{ square inch.}$$

Example 3.—In Fig. 33, side b = 0.3 foot, angle B = 35° 40′, and angle C = 24° 10′. Find angle A, sides a and c, and the area.

$$A = 180^\circ - (35^\circ\ 40' + 24^\circ\ 10') = 180^\circ - 59^\circ\ 50' = 120^\circ\ 10'.$$

To find side a, use the rule already given, from which we get the formula below:

$$a = \frac{b \times \sin A}{\sin B} = \frac{0.3 \times \sin 120^\circ\ 10'}{\sin 35^\circ\ 40'} = \frac{0.3 \times 0.86457}{0.58307} = 0.445 \text{ foot.}$$

To find side c, use again the same rule, from which we then get:

$$c = \frac{b \times \sin C}{\sin B} = \frac{0.3 \times \sin 24^\circ\ 10'}{\sin 35^\circ\ 40'} = \frac{0.3 \times 0.40939}{0.58307} = 0.211 \text{ foot.}$$

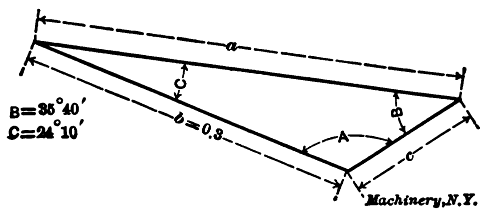


Fig. 33

Note that in this example the formulas for a and c have the same *form* as Formulas (4) and (5) on pages 36 and 37, but as the side b is the known side, instead of a, the side b is brought into the formula instead of a, and angle B instead of angle A. The formulas for a and c in this example are directly deduced from the rule on page 36, for finding the unknown sides.

To find the area, use Formula (6):

$$\text{Area} = \frac{a \times b \times \sin C}{2} = \frac{0.445 \times 0.3 \times \sin 24^\circ\ 10'}{2} =$$

$$\frac{0.445 \times 0.3 \times 0.40939}{2} = 0.027 \text{ square foot.}$$

Summary of Formulas

If the angles of a triangle are called A, B and C, and the sides opposite each of the angles, a, b and c, respectively, as shown in Fig. 30, then, if two angles and one side are known, the remaining angle, the two unknown sides and the area may be found by the formulas below:

$$A = 180^\circ - (B + C) \qquad (7)$$

$$B = 180^\circ - (A + C) \qquad (8)$$

$$C = 180^\circ - (A + B) \qquad (9)$$

$$a = \frac{b \times \sin A}{\sin B} \qquad b = \frac{a \times \sin B}{\sin A} \qquad c = \frac{b \times \sin C}{\sin B}$$

$$a = \frac{c \times \sin A}{\sin C} \qquad b = \frac{c \times \sin B}{\sin C} \qquad c = \frac{a \times \sin C}{\sin A}$$

$$\text{Area} = \frac{a \times b \times \sin C}{2} = \frac{b \times c \times \sin A}{2} = \frac{a \times c \times \sin B}{2}$$

2. Two Sides and the Included Angle Known

Assume that the sides a and b in Fig. 34 are 9 and 8 inches, respectively, as shown, and that the angle C formed by these two sides is 35 degrees. Find angles A and B, side c, and the area of the triangle.

The tangent of angle A is found by the following formula:

$$\tan A = \frac{a \times \sin C}{b - a \times \cos C} \qquad (10)$$

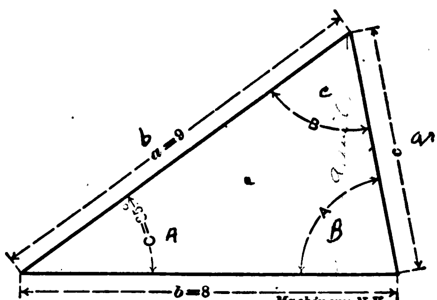


Fig. 34

If the given values of a, b and C are inserted in this formula, we have:

$$\tan A = \frac{9 \times \sin 35°}{8 - 9 \times \cos 35°} = \frac{9 \times 0.57358}{8 - 9 \times 0.81915} =$$

$$\frac{5.16222}{0.62765} = 8.22468.$$

Having now obtained the tangent of angle $A = 8.22468$, we find from the tables that the angle equals 83° 4′.

Now when both angles A and C are known, angle B is found by Formula (8) already given:

$$B = 180° - (A + C) = 180° - (83° \, 4' + 35°) =$$
$$180° - 118° \, 4' = 61° \, 56'$$

Side c is found by Formula (5):

$$c = \frac{a \times \sin C}{\sin A} = \frac{9 \times \sin 35°}{\sin 83° \, 4'} = \frac{9 \times 0.57358}{0.99269} = 5.2 \text{ inches.}$$

The area is found by Formula (6):

$$\text{Area} = \frac{a \times b \times \sin C}{2} = \frac{9 \times 8 \times 0.57358}{2} = 20.649 \text{ square inches.}$$

All the required quantities of this triangle have now been found.

Example 1.—In Fig. 35, $a = 4$ inches, $b = 3$ inches, and $C = 20$ degrees. Find A, B, c, and the area.

According to Formula (10), we have:

$$\tan A = \frac{a \times \sin C}{b - a \times \cos C} = \frac{4 \times \sin 20^\circ}{3 - 4 \times \cos 20^\circ} = \frac{4 \times 0.34202}{3 - 4 \times 0.93969}$$

$$= \frac{1.36808}{3 - 3.75876}$$

It will be seen that in the denominator of the fraction above, the number to be subtracted from 3 is greater than 3; the numbers are

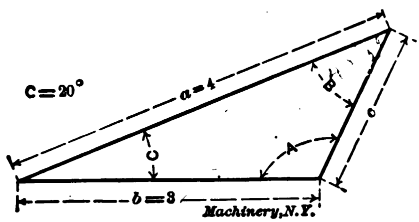


Fig. 35

therefore reversed as explained in Chapter III, 3 being subtracted from 3.75876, the remainder then being negative. Hence:

$$\tan A = \frac{1.36808}{3 - 3.75876} = \frac{1.36808}{-0.75876} = -1.80305$$

The final result is negative because a positive number (1.36808) is divided by a negative number (— 0.75876).

In Chapter V it is stated that the tangents of angles greater than 90 degrees and smaller than 180 degrees are negative. In an example in the same chapter is shown how to find an angle whose tangent is negative. Proceeding in the same manner, find in this case the value nearest to 1.80305 in the columns of tangents in the tables. It will be seen that the nearest value is 1.8028, which is the tangent of 60° 59′. As the tangent here is negative, angle A, however, is not 60° 59′, but equals 180° — 60° 59′ = 119° 1′.

Now angle B is found by the formula

$$B = 180^\circ - (A + C) = 180^\circ - (119^\circ\ 1' + 20^\circ) =$$
$$180^\circ - 139^\circ\ 1' = 40^\circ\ 59'.$$

Side c and the area are now found by the same formulas and in the same manner as previously shown.

Example 2.—In Fig. 36, $a = 7$ feet, $b = 4$ feet, and $C = 121$ degrees. Find A, B, c and the area.

Proceeding as in the previous example we have

$$\tan A = \frac{a \times \sin C}{b - a \times \cos C} = \frac{7 \times \sin 121^\circ}{4 - 7 \times \cos 121^\circ}$$

As explained in Chapter V:

$$\sin 121^\circ = \sin\ (180^\circ - 121^\circ) = \sin 59^\circ, \text{ and}$$
$$\cos 121^\circ = -\cos\ (180^\circ - 121^\circ) = -\cos 59^\circ.$$

Therefore

$$\tan A = \frac{7 \times \sin 121^\circ}{4 - 7 \times \cos 121^\circ} = \frac{7 \times \sin 59^\circ}{4 - 7 \times (-\cos 59^\circ)} =$$

$$\frac{7 \times 0.85717}{4 - 7 \times (-0.51504)} = \frac{6.00019}{4 - (-3.60528)} =$$

$$\frac{6.00019}{4 + 3.60528} = \frac{6.00019}{7.60528} = 0.78895.$$

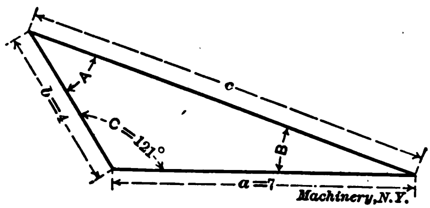


Fig. 36

The calculation with the negative number (— 0.51504) will become clear by comparing the processes above with the rules given in Chapter III. When multiplied by 7, the product 7 × (— 0.51504) becomes negative, and equals — 3.60528. As subtracting a negative quantity from a positive quantity is equal to adding the numerical value of the negative number we have:

$$4 - (-3.60528) = 4 + 3.60528 = 7.60528.$$

Having found tan A = 0.78895, we find angle A from the tables: $A = 38^\circ\ 16'$.

Angle B, side c and the area are now found in the same way as previously explained.

Summary of Formulas

If the angles of a triangle are called A, B and C and the sides opposite each of the angles a, b and c, respectively, as shown in Fig. 34, then, if any two sides and the included angle are known, the other angles, the remaining side and the area may be found. One of the angles is first found by any of the formulas below:

$$\tan A = \frac{a \times \sin C}{b - a \times \cos C} \qquad \tan A = \frac{a \times \sin B}{c - a \times \cos B}$$

$$\tan B = \frac{b \times \sin C}{a - b \times \cos C} \qquad \tan B = \frac{b \times \sin A}{c - b \times \cos A}$$

$$\tan C = \frac{c \times \sin B}{a - c \times \cos B} \qquad \tan C = \frac{c \times \sin A}{b - c \times \cos A}$$

The third angle, the remaining side, and the area are then found by using Formulas (4), (5), (6), (7), (8) and (9).

If the unknown angles are not required, but merely the unknown side of the triangle, the following formulas may be employed:

$$a = \sqrt{b^2 + c^2 - 2\,bc \times \cos A}$$
$$b = \sqrt{a^2 + c^2 - 2\,ac \times \cos B}$$
$$c = \sqrt{a^2 + b^2 - 2\,ab \times \cos C}$$

3. Two Sides and One of the Opposite Angles Known

When two sides and the angle opposite one of the given sides are known, two triangles can be drawn which have the sides the re-

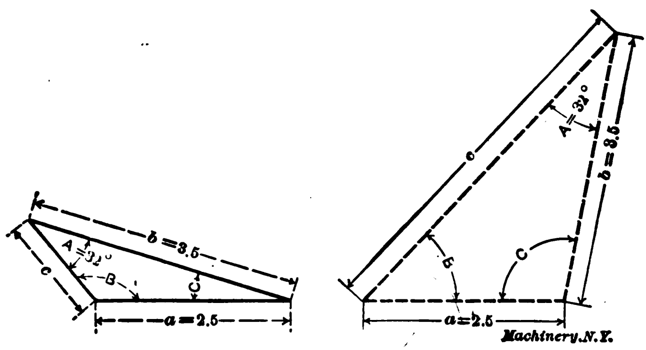


Fig. 37

quired length and the angle opposite one of the sides the required size. In Fig. 37 is shown a triangle in which side a is 2.5 inches, side b, 3.5 inches, and angle A, 32 degrees. Another triangle is shown by dotted lines in the same figure in which sides a and b have the same length as in the triangle drawn by full lines, and angle A opposite side a still remains 32 degrees; but it will be seen that in this triangle the angle B is very much smaller than in the triangle drawn by the full lines. In every case, therefore, when two sides and one of the opposite angles are given, the problem is capable of two solutions, there being two triangles which fill the given requirements. In one of these triangles, the unknown angles opposite a given side is greater than a right angle, and in one it is less than a right angle. When the triangle to be calculated is drawn to the correct shape, it is, therefore, possible to determine from the shape of the triangle which of the two solutions applies. When the triangle is not drawn to the required shape, both solutions must be found and applied to the practical problem requiring the solution of the triangle; it can then usually be determined which of the solutions applies to the practical problem in hand.

Example 1.—Assume that the sides a and b in Fig. 38 are 20 and 17 inches, respectively, as shown, and that angle A opposite the known side a is 61 degrees. Find angles B and C, side c, and the area of the triangle.

The angle B opposite the known side b may be found by the following rule: *The sine of the angle opposite one of the known sides equals the product of the side opposite this angle times the sine of the known angle, divided by the side opposite the known angle.*

From this rule we derive the following formula for the sine of angle B:

$$\sin B = \frac{b \times \sin A}{a} \qquad (11)$$

If we insert the known values for sides b and a and angle A in this formula we have:

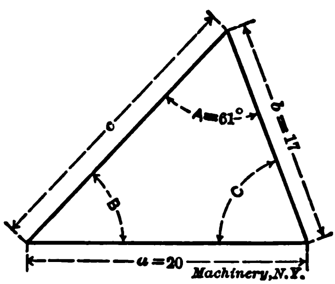


Fig. 38

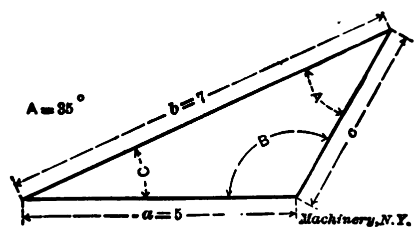


Fig. 39

$$\sin B = \frac{17 \times \sin 61^\circ}{20} = \frac{17 \times 0.87462}{20} = 0.74343.$$

Having $\sin B = 0.74343$, we find from the tables that $B = 48^\circ\ 1'$. As it is shown in Fig. 38 that angle B is less than a right angle, the solution found is the one which applies in this case.

Angle C is now found from Formula (9):

$$C = 180^\circ - (A + B) = 180^\circ - (61^\circ + 48^\circ\ 1') = 70^\circ\ 59'.$$

Side c is found by Formula (5):

$$c = \frac{a \times \sin C}{\sin A} = \frac{20 \times \sin 70^\circ\ 59'}{\sin 61^\circ} = \frac{20 \times 0.94542}{0.87462} = 21.62 \text{ inches.}$$

The area is found by Formula (6):

$$\text{Area} = \frac{a \times b \times \sin C}{2} = \frac{20 \times 17 \times \sin 70^\circ\ 59'}{2} = 160.72 \text{ square inches.}$$

All the required quantities of this triangle have now been found.

Example 2.—In Fig. 39, $a = 5$ inches, $b = 7$ inches, and $A = 35$ degrees. Find B, C, c and the area.

According to the rule and formula in the previous example:

$$\sin B = \frac{b \times \sin A}{a} = \frac{7 \times \sin 35^\circ}{5} = \frac{7 \times 0.57358}{5} = 0.80301$$

Having $\sin B = 0.80301$, we find from the tables that $B = 53^\circ\ 25'$. However, in the present case we see from the figure that B is greater than 90 degrees. The solution obtained is, therefore, not the solution applying to this case. It is explained in Chapter V that the sine of an angle also equals the sine of 180 degrees minus the angle. Therefore, 0.80301 is the sine not only of $53^\circ\ 25'$, but also of $180^\circ - 53^\circ\ 25' = 126^\circ\ 35'$. The value of angle B applying to the triangle shown in Fig. 39 is therefore $126^\circ\ 35'$, because of the two values obtained this is the one which is greater than a right angle.

When angle B is found, angle C, side c and the area are found in the same manner as in Example 1.

Example 3.—In Fig. 40, $a = 2$ feet, $b = 3$ feet and $A = 30$ degrees. Find B, C, c and the area.

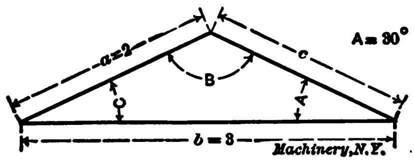


Fig. 40

The sine of angle B is found as in the previous example:

$$\sin B = \frac{b \times \sin A}{a} = \frac{3 \times \sin 30^\circ}{2} = 0.75000.$$

Having $\sin B = 0.75000$, we find from the tables that $B = 48^\circ\ 35'$. From Fig. 40 it is apparent, however, that B is greater than 90 degrees, and as 0.75000 is the sine not only of $48^\circ\ 35'$, but also of $180^\circ - 48^\circ\ 35' = 131^\circ\ 25'$, angle B in this case equals $131^\circ\ 25'$.

When the angle B is found, angle C, side c and the area are found in the same manner as in Example 1.

Summary of Formulas

If the angles of a triangle are called A, B and C, and the sides opposite each of the angles a, b and c, respectively, as shown in Fig. 37; then if any two sides and one angle opposite one of the known sides are given, the other angles, the remaining side, and the area may be found. The angle opposite the other known side is first found by any of the formulas below:

$$\sin A = \frac{a \times \sin B}{b} \qquad \sin A = \frac{a \times \sin C}{c}$$

$$\sin B = \frac{b \times \sin A}{a} \qquad \sin B = \frac{b \times \sin C}{c}$$

$$\sin C = \frac{c \times \sin A}{a} \qquad \sin C = \frac{c \times \sin B}{b}$$

The third angle, the remaining side and the area are then found by using Formulas (4) to (9) inclusive.

4. Three Sides Known

***Example* 1.—In Fig. 41 the three sides *a*, *b* and *c* of the triangle are given; $a = 8$ inches, $b = 9$ inches and $c = 10$ inches. Find the angles *A*, *B* and *C* and the area.**

Either of the angles can be found by the formulas given below:

$$\cos A = \frac{b^2 + c^2 - a^2}{2 \times b \times c} \qquad (12)$$

$$\cos B = \frac{a^2 + c^2 - b^2}{2 \times a \times c} \qquad (13)$$

$$\cos C = \frac{a^2 + b^2 - c^2}{2 \times a \times b} \qquad (14)$$

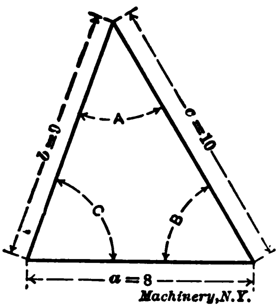


Fig. 41

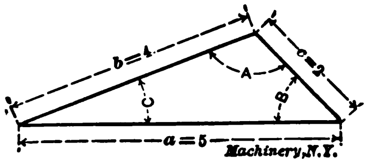


Fig. 42

If we insert the given lengths of the sides in the first of the formulas above we have:

$$\cos A = \frac{9^2 + 10^2 - 8^2}{2 \times 9 \times 10} = \frac{9 \times 9 + 10 \times 10 - 8 \times 8}{2 \times 9 \times 10} = \frac{81 + 100 - 64}{180}$$

$$= \frac{117}{180} = 0.65000$$

Having $\cos A = 0.65000$ we find from the tables that angle $A = 49° 27'$.

Having found angle *A*, the easiest method for finding angle *B* is by Formula (11). From this formula we have:

$$\sin B = \frac{b \times \sin A}{a} = \frac{9 \times \sin 49° 27'}{8} = \frac{9 \times 0.75984}{8} = 0.85482$$

Having $\sin B = 0.85482$, we find from the tables that $B = 58° 44'$.

Angle C is now found by Formula (9):

$C = 180° - (A + B) = 180° - (49° 27' + 58° 44') = 71° 49'$.

The area is finally found from Formula (6):

$$\text{Area} = \frac{a \times b \times \sin C}{2} = \frac{8 \times 9 \times \sin 71° 49'}{2} = \frac{8 \times 9 \times 0.95006}{2}$$

$= 34.20$ square inches.

Example 2.—In Fig. 42, $a = 5$ inches, $b = 4$ inches and $c = 2$ inches. Find the angles of the triangle.

Using Formula (12), given in Example 1, we have:

$$\cos A = \frac{4^2 + 2^2 - 5^2}{2 \times 4 \times 2} = \frac{16 + 4 - 25}{16} = \frac{20 - 25}{16}$$

It will be seen that in the numerator of the last fraction above, the number to be subtracted from 20 is greater than 20. The numbers are therefore reversed, as explained in Chapter III, 20 being subtracted from 25, the remainder then being negative. Hence:

$$\cos A = \frac{20 - 25}{16} = \frac{-5}{16} = -0.31250.$$

The final result is negative, because a negative number (—5) is divided by a positive number (16). In Chapter V it is stated that the cosines of angles greater than 90 degrees and smaller than 180 degrees are negative. In an example in the same chapter is shown how to find the angle whose tangent is negative; an angle whose cosine is negative is found in a similar manner: Find the value nearest to 0.31250 in the columns of cosines in the tables. It will be seen that the nearest value is 0.31261, which is the cosine of 71° 47'. As the cosine here is negative, angle A, however, is not 71° 47' but $= 180° - 71° 47' = 108° 13'$. Now angle B is found by the formula:

$$\sin B = \frac{b \times \sin A}{a} = \frac{4 \times \sin 108° 13'}{5}$$

As stated in Chapter V, $\sin 108° 13' = \sin (180° - 108° 13') = \sin 71° 47'$. Hence:

$$\sin B = \frac{4 \times \sin 71° 47'}{5} = \frac{4 \times 0.94988}{5} = 0.75990$$

and $B = 49° 27'$.

Finally, angle C is found by the formula:

$C = 180° - (A + B) = 180° - (108° 13' + 49° 27') = 22° 20'$.

CHAPTER X

SUMMARY OF FORMULAS FOR SOLUTION OF TRIANGLES

In the following will be given a summary of all the required formulas, and the methods of procedure for solving both right- and oblique-angled triangles.

Right-angled Triangles

In all the formulas for right-angled triangles reference is made to Fig. 43, in which the sides and angles are given the same names as in the formulas. Use the formulas in the order given.

1. When the hypotenuse and one of the sides forming the right angle are given, call the hypotenuse a and the known side b. Then:

$$c = \sqrt{a^2 - b^2} \qquad \sin B = \frac{b}{a} \qquad C = 90° - B$$

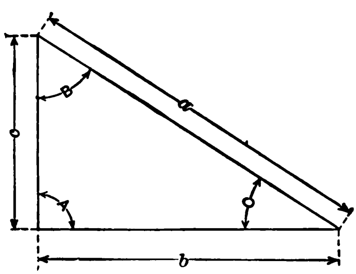


Fig. 43

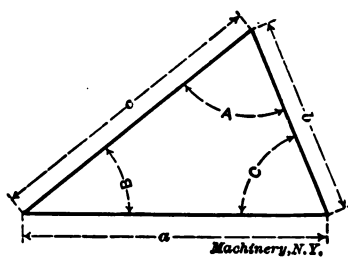


Fig. 44

2. When the two sides forming the right angle are given, call them b and c. Then:

$$a = \sqrt{b^2 + c^2} \qquad \tan B = \frac{b}{c} \qquad C = 90° - B$$

3. When the hypotenuse and one acute angle are given, call the hypotenuse a and the known angle B. Then:

$$c = a \times \cos B \qquad b = a \times \sin B \qquad C = 90° - B$$

4. When one acute angle and its adjacent side are given, call the angle B and the adjacent known side c. Then:

$$a = \frac{c}{\cos B} \qquad b = c \times \tan B \qquad C = 90° - B$$

5. When one acute angle and the side opposite it are given, call the angle B and the known opposite side b. Then:

$$a = \frac{b}{\sin B} \qquad c = b \times \cot B \qquad C = 90^\circ - B$$

The area of all right-angled triangles equals the product of the sides forming the right angle divided by 2; or, referring to Fig. 43:

$$\text{Area} = \frac{b \times c}{2}$$

Oblique-angled Triangles

In all the formulas for oblique-angled triangles reference is made to Fig. 44, in which the sides and angles are given the same names as in the formulas. Use the formulas in the order given.

1. When two angles and one side are given, call the given side a, the angle opposite it A, and the other angle B. Then if A is known:

$$C = 180^\circ - (A + B)$$

$$b = \frac{a \times \sin B}{\sin A}$$

$$c = \frac{a \times \sin C}{\sin A}$$

$$\text{Area} = \frac{a \times b \times \sin C}{2}$$

If B and C are given, but not A, then $A = 180^\circ - (B + C)$, the other formulas being as above.

2. When two sides and the included angle are given, call the given sides a and b and the given angle between them C. Then:

$$\tan A = \frac{a \times \sin C}{b - a \times \cos C}$$

$$B = 180^\circ - (A + C)$$

$$c = \frac{a \times \sin C}{\sin A}$$

$$\text{Area} = \frac{a \times b \times \sin C}{2}$$

3. When two sides and the angle opposite one of the sides are given, call the given angle A, the side opposite it a and the other given side b. Then:

$$\sin B = \frac{b \times \sin A}{a}$$

$$C = 180^\circ - (A + B)$$

$$c = \frac{a \times \sin C}{\sin A}$$

$$\text{Area} = \frac{a \times b \times \sin C}{2}$$

4. When the three sides of a triangle are given, call them a, b and c and the angles opposite them A, B and C, respectively. Then:

$$\cos A = \frac{b^2 + c^2 - a^2}{2 \times b \times c}$$

$$\sin B = \frac{b \times \sin A}{a}$$

$$C = 180° - (A + B)$$

$$\text{Area} = \frac{a \times b \times \sin C}{2}$$

The cases given include all conditions where a solution of the triangle is possible. If all the angles are given, but none of the sides, the triangle may be of any size, but the three sides will be in exact proportion to each other. The formulas below give this relationship:

$$a : b = \sin A : \sin B$$
$$b : c = \sin B : \sin C$$
$$a : c = \sin A : \sin C$$

CHAPTER XI

THE USE OF LOGARITHMS IN SOLVING TRIANGLES

Before undertaking to study the use of logarithms for solving triangles, the student should thoroughly understand the use of logarithms in ordinary numerical examples, as explained in MACHINERY'S Reference Series No. 53, "The Use of Logarithms and Logarithmic Tables." When the use of logarithms in ordinary calculations is well understood, their application to trigonometric problems is very simple. It is merely a question of finding the logarithm for the function of the angle from the tables in Part II of this treatise, and carrying out the calculation in the same manner as with logarithms in general. The heavy-faced figures in the columns headed "Log." in the tables give these logarithms. A few explanatory remarks as to the method in which they are given, will, however, be necessary.

In all cases in these tables, the characteristic is given together with the mantissa. The complete logarithm of the functions, therefore, is found directly from the tables. As however, the values of the natural functions in the three first columns from the left in the tables are always less than 1, the characteristic would always be negative. In order to avoid this negative characteristic, the logarithm as given has had

10 added to its value, so that the actual value of the logarithm for cos 3 deg., for example, is 9.99940 — 10, as is evident if we remember that the logarithm of a number less than 1 must be negative. When using these logarithms in calculations with other logarithms, the calculations can be carried out exactly as explained in Reference Series No 53, if when writing down the logarithm taken from the tables we write $\bar{1}.99940$ for 9.99940, $\bar{2}.71940$ for 8.71940, $\bar{3}.30882$ for 7.30882, and so forth, changing the form to that which was made use of in the previous Reference book. It should be remembered, however, that this change refers only to the three first columns of logarithms. In the fourth column (headed Cot.), the logarithm is given in the exact form in which it is to be used. Of course, if it appears in the divisor of an expression, it must be transformed to its *negative* value, as explained on page 10, Reference Series No. 53.

A few examples will give a better idea of the methods to be followed. The student should carefully study these examples, until all the methods employed are perfectly clear to him. The logarithms of ordinary numbers are found from Reference Series No. 53, and the logarithms for functions of angles from Reference Series No. 55.

Example 1.—Find the area of a triangle where the lengths of two sides are 53 and 82 inches, and the angle between them is 30 degrees.

The area is found by the formula:

$$\text{Area} = \frac{a \times b \times \sin C}{2} = \frac{53 \times 82 \times \sin 30^\circ}{2}$$

Proceed now to find the logarithms:

$$\begin{aligned} \log 53 &= 1.72428 \\ \log 82 &= 1.91381 \\ \log \sin 30^\circ &= \bar{1}.69897 \\ -\log 2 &= \bar{1}.69897 \\ \hline & 3.03603 \end{aligned}$$

The logarithm of the area thus is 3.03603, and from the tables in Reference Series No. 53 we find by interpolation that the area then equals 1086.5 square inches.

Example 2.—Angles A and C and side a in a triangle are known. (See Fig. 44.) $A = 37^\circ\ 42'$; $C = 68^\circ\ 12'$; $a = 12$ inches. Find side c.

The formula for finding side c is:

$$c = \frac{a \times \sin C}{\sin A} = \frac{12 \times \sin 68^\circ\ 12'}{\sin 37^\circ\ 42'}$$

When finding the logarithms, note that as $\log \sin 37^\circ\ 42' = \bar{1}.78642$, the negative value of the logarithm equals 0.21358.

$$\begin{aligned} \log 12 &= 1.07918 \\ \log \sin 68^\circ\ 12' &= \bar{1}.96778 \\ -\log \sin 37^\circ\ 42' &= 0.21358 \\ \hline & 1.26054 \end{aligned}$$

Thus $\log c = 1.26054$, and hence $c = 18.22$ inches.

Example 3.—Two sides of a triangle are 9 and 17 inches long. The angle included between them is 32 degrees. Find the angle opposite the side 9 inches long.

The formula by means of which the angle sought can be found is (see Chapter IX):

$$\tan A = \frac{a \times \sin C}{b - a \times \cos C} = \frac{9 \times \sin 32^\circ}{17 - 9 \times \cos 32^\circ}$$

As only multiplications and divisions can be carried out by means of ordinary logarithms, the subtraction in the denominator must be made independently of logarithms; but logarithms can be used for the multiplications and divisions required. The first step will be to find the value of the denominator; we must then first find the product 9 × cos 32°.

$$\begin{aligned} \log 9 &= 0.95424 \\ \log \cos 32^\circ &= \bar{1}.92842 \\ \hline &\ 0.88266 \end{aligned}$$

Hence 9 × cos 32° = 7.6323, and 17 — 7.6323 = 9.3677. Therefore,

$$\tan A = \frac{9 \times \sin 32^\circ}{9.3677}$$

$$\begin{aligned} \log 9 &= 0.95424 \\ \log \sin 32^\circ &= \bar{1}.72421 \\ -\log 9.3677 &= \bar{1}.02837 \\ \hline &\ \bar{1}.70682 \end{aligned}$$

Log tan $A = \bar{1}.70682$, or as given in the tables 9.70682. Hence A = 26° 59′.

The columns "d" (difference) and "c. d." (common differences) in the tables, give the differences between consecutive logarithms for use in interpolation in cases where subdivisions of minutes are required. The method used is the same as that used when interpolating between logarithms of ordinary numbers. It is seldom, however, in ordinary shop calculations or in machine design, that finer divisions of the angle than minutes are required.

PRICE 25 CENTS

SOLUTION OF TRIANGLES

PART II

TABLES OF TRIGONOMETRIC FUNCTIONS

SECOND EDITION

0°

′	Nat. Sin	Log.	d.	Nat. Cos	Log.	Nat. Tan	Log.	c. d.	Log. Cot	Nat.	′
[illegible]	[illegible]	—		1.00000	10.00000	00000	—		—	∞	60
1	029	6.46373	30103	000	0.00000	029	6.46373	30103	3.53627	3437.7	59
2	058	6.76476	17609	000	0.00000	058	6.76476	17609	3.23524	1718.9	58
3	087	6.94085	12494	000	0.00000	087	6.94085	12494	3.05915	1145.9	57
4	116	7.06579	9691	000	0.00000	116	7.06579	9691	2.93421	859.44	56
5	00145	7.16270	7918	1.00000	10.00000	00145	7.16270	7918	2.83730	687.55	55
6	175	7.24188	6694	000	0.00000	175	7.24188	6694	2.75812	572.96	54
7	204	7.30882	5800	000	0.00000	204	7.30882	5800	2.69118	491.11	53
8	233	7.36682	5115	000	0.00000	233	7.36682	5115	2.63318	429.72	52
9	262	7.41797	4576	000	0.00000	262	7.41797	4576	2.58203	381.97	51
10	00291	7.46373	4139	1.00000	10.00000	00291	7.46373	4139	2.53627	343.77	50
11	320	7.50512	3779	99999	0.00000	320	7.50512	3779	2.49488	312.52	49
12	349	7.54291	3476	999	0.00000	349	7.54291	3476	2.45709	286.48	48
13	378	7.57767	3218	999	0.00000	378	7.57767	3219	2.42233	264.44	47
14	407	7.60985	2997	999	0.00000	407	7.60986	2996	2.39014	245.55	46
15	00436	7.63982	2802	99999	10.00000	00436	7.63982	2803	2.36018	229.18	45
16	465	7.66784	2633	999	0.00000	465	7.66785	2633	2.33215	214.86	44
17	495	7.69417	2483	999	9.99999	495	7.69418	2482	2.30582	202.22	43
18	524	7.71900	2348	999	9.99999	524	7.71900	2348	2.28100	190.98	42
19	553	7.74248	2227	998	9.99999	553	7.74248	2228	2.25752	180.93	41
20	00582	7.76475	2119	99998	9.99999	00582	7.76476	2119	2.23524	171.89	40
21	611	7.78594	2021	998	9.99999	611	7.78595	2020	2.21405	163.70	39
22	640	7.80615	1930	998	9.99999	640	7.80615	1931	2.19385	156.26	38
23	669	7.82545	1848	998	9.99999	669	7.82546	1848	2.17454	149.47	37
24	698	7.84393	1773	998	9.99999	698	7.84394	1773	2.15606	143.24	36
25	00727	7.86106	1704	99997	9.99999	00727	7.86167	1704	2.13833	137.51	35
26	756	7.87870	1639	997	9.99999	756	7.87871	1639	2.12129	132.22	34
27	785	7.89509	1579	997	9.99999	785	7.89510	1579	2.10490	127.32	33
28	[illegible]	[illegible]	1524	997	9.99999	815	7.91089	1524	2.08911	122.77	32
[illegible]	[illegible]	[illegible]		[illegible]	0.00008	844	[illegible]		[illegible]	[illegible]	31

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