PROFESSIONAL PAPER—No. 14.

FORMULÆ FOR ATMOSPHERIC REFRACTION AND THEIR APPLICATION TO TERRESTRIAL REFRACTION AND GEODESY

BY

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INTRODUCTION.

Terrestrial refraction has usually been dealt with in survey operations by making use of a "coefficient of refraction" which involves the tacit assumption that a ray of light always is of circular form and that the curvature is the same for ascending, descending or horizontal rays. This assumption is in someways contradictory; for we know that the curvature of a ray is less at great heights than it is at sea-level, yet if a ray maintains the same curvature when ascending the assumption would give the same curvature at all heights. The object of the present investigation is to develop formulæ for refraction which will indicate precisely on what the refraction depends, and to find a means of determining such quantities, so that the refraction in any case may be computed. The formulæ are developed in Chapter I, and in Chapters II and III they are applied to some actual observations made in India by Mr. H. G. Shaw. Chapter IV shows the relation between refraction and barometric and thermometric readings at two heights: while in Chapter V the relation of celestial refraction to terrestrial refraction is developed, with a view to the determination of the latter from knowledge of the former. In Chapter VI the dip of the horizon is investigated and the results of previous chapters are briefly summarised: and suggestions are made for future investigations. For easy reference lists of the notations employed precede Chapters I and II.

Much remains to be done, but it is hoped that further enquiries may be usefully directed by the results herein obtained.

Babu Mukundananda Acharya of the Computing Office has checked the equations of Chapter I and has done many of the subsequent computations. He has also read nearly all the proofs, and I gladly take this opportunity to acknowledge the assistance he has rendered me.

The difficulty of setting up mathematical formulæ in type is well-known. I am indebted to Mr. Sarat Kumar Mukerji for always having this work done with correctness and despatch.

The two panoramic views of the snowy range as seen from Mussooree have been produced from photographs by Mr. Julian Rust of Mussooree.

DEHRA DUN,
28th August 1913.

J. DE GRAAFF HUNTER.

NOTATION IN CHAPTER I.

μ refractive index of air.
p atmospheric pressure.
τ' absolute temperature of air.
τ virtual temperature „
ρ density of air.
θ *see* equation (8).
A_1, A_2, A_3 *see* equation (10).
g value of gravity.
H barometer reading.
H_c „ „ corrected, *see* § 14.
C *see* equation (5).
C' *see* § 20.
B *see* equation (22).
K „ (21).
P, Q „ (19).
Suffix o denotes value at base station.
Suffix s denotes the standard value.
$x, y, s, \sigma, \psi, \phi, \chi$ *see* § 8.
h height above base station.
r radius of curvature of spheroidal surface through base station in azimuth of ray under consideration.
R radius of curvature of sea-level surface.
X *see* equation (32).
Y „ (33).
u „ (37).
l distance at sea-level between normals through the terminal points of the ray, *expressed in miles.*
c distance at sea-level between normals through the terminal points of the ray, *expressed in feet.*
$\omega = \omega_1 + \omega_2 + \; . \; . \; . = $ angle of refraction for adiabatic gradient.
$\Omega = f_1\,\omega_1 + f_2\,\omega_2 + \; . \; . \; . = $ angle of refraction for any gradient, *see* § 20.
γ ratio of specific heats of a gas, *see* equation (58).

$$a = -\,10^2 \frac{d\tau}{dh}.$$

$$b = \frac{10^4.\tau}{1{\cdot}042}\cdot\frac{d^2\tau}{dh^2}.$$

The symbol $\doteqdot$ indicates *approximate* equality.

CHAPTER I.

The formulæ for Barometric Heights and for Terrestrial Refraction.

1. Let p, τ', ρ be pressure, absolute temperature and density of the atmosphere at any point.

Then if the air is dry we have

$$p = C\tau'\rho \quad \ldots \quad (1)$$

When water vapour is mixed with the air in the proportion of m parts of vapour to $1-m$ parts of air, we must replace C by

$$C_1 = (1 - m)\ C + m\ C_2 \quad \ldots \quad (2)$$

where C_2 is the constant for water vapour.

Now from observation we know that

$$C_2 = \tfrac{8}{5}\ C \quad \ldots \quad (3)$$

so that (2) may be written

$$C_1 = (1 + 0{\cdot}6\,m)\ C$$

If we put

$$\tau = (1 + 0{\cdot}6\,m)\ \tau' \quad \ldots \quad (4)$$

then τ is what is usually called the *virtual temperature*, and equation (1) may be written *for moist air*

$$p = C\tau\rho \quad \ldots \quad (5)$$

Tables for τ in terms of temperature and percentage humidity have been given in "Dynamic Meteorology and Hydrography" by V. Bjerknes, published by the Carnegie Institute of Washington, and tables I and II at the end of this chapter are based on them.

Accordingly we shall always consider virtual temperatures and so take the humidity of the atmosphere into account without any additional labour or complexity of formulæ.

2. As regards barometric heights in what follows we shall be concerned only with differences of height of two stations at distances which do not exceed the greatest visible terrestrial distance. For such distances it will not be necessary to consider the variation of orthometric height between two level surfaces. In other words we shall, unless otherwise stated, consider the section by a vertical plane of the level surfaces to be concentric circles.

In the case of refraction it is also necessary to assume that the density of the air is constant over a level surface. The pressure of air at rest is of course constant over a level surface: so that it follows from (5) that the virtual temperature must also be constant over a level surface. This assumption of constancy of density is merely a statement that there is thermal equilibrium in a horizontal direction, which must be correct when there is no horizontal air-movement, except for air in separate valleys. In the case of refraction we have to deal with a column of air in which there are no land obstructions, so that the assumption here practically reduces to assuming that the air is at rest. Local winds and eddies, then, make the most unfavourable condition for the assumption: a steady wind over the whole spacc considered would probably vitiate the assumption to a slighter extent.

3. The condition for vertical equilibrium of the atmosphere at any point gives

$$dp = -\rho g \, dh \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (6)$$

g being the value of gravity at the point and dh an element of height. Substituting for ρ its value from (5) we have

$$\frac{dp}{p} = -g\frac{dh}{C\tau} = -\frac{g_0}{C}.dh.\frac{g}{g_0}.\frac{1}{\tau} \quad . \; . \; . \; . \; . \; . \; . \; . \quad (7)$$

where g_0 is gravity at surface characterised by suffix zero.

We shall put

$$\frac{g}{g_0}.\frac{1}{\tau} = \theta \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (8)$$

and so get

$$d \log p = -\frac{g_0}{C}.\theta.\,dh \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (9)$$

4. As explained in §2 we shall consider τ and g as functions of the height only: then θ is also a function of the heights and we shall accordingly put

$$\theta = \theta_0 (1 + A_1 h + A_2 h^2 \ldots\ldots) \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (10)$$

where A_1 and A_2 are constants at any particular time, which are to be found by means indicated later on (*see* § 11).

It is now possible to integrate (9) and get

$$\text{Constant} - \log p = -\frac{g_0}{C}.\theta_0 h\left(1 + \frac{A_1 h}{2} + \frac{A_2 h^2}{3} \ldots\ldots\right) \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (11)$$

Taking this between limits and measuring height from surface designated by suffix zero we get

$$\log\frac{p_0}{p} = \frac{g_0\theta_0}{C}.\,h\left(1 + \frac{A_1 h}{2} + \frac{A_2 h^2}{3} + \ldots\ldots\right) \quad . \; . \; . \; . \; . \; . \; . \quad (12)$$

Now from (10) we get

$$\frac{\theta + \theta_0}{2} = \theta_0\left(1 + \frac{A_1 h}{2} + \frac{A_2 h^2}{2} + \ldots\ldots\ldots\right) \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (13)$$

and so we can write (12)

$$\log\frac{p_0}{p} = \frac{g_0}{C}.\frac{\theta+\theta_0}{2}.\,h\left(1 - \frac{A_2}{6}.\frac{2h^2\theta_0}{\theta+\theta_0} - \frac{A_3}{4}.\frac{2h^3\theta_0}{\theta+\theta_0} - \ldots\right) \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (14)$$

This gives

$$h = \frac{C}{g_0}.\frac{2}{\theta+\theta_0}.\left(1 + \frac{A_2}{6}.\frac{2h^2\theta_0}{\theta+\theta_0} + \frac{A_3}{4}.\frac{2h^3\theta_0}{\theta+\theta_0}\ldots\right)\log_e\frac{p_0}{p} \quad . \; . \; . \; . \; . \; . \; . \quad (15)$$

It will be shown later that the quantities A_1, A_2, A_3 may be obtained if we know the refraction: but in the first instance it is proposed to neglect the A_3 term and use equation (15) for the determination of A_2 in order to calculate the refraction.

5. The approximate equation

$$h \doteq \frac{C}{g_0} \cdot \frac{2}{\theta + \theta_0} \log_e \frac{p_0}{p} \quad \ldots \ldots \ldots \ldots \quad (16)$$

is practically the same as the ordinary Laplace's form of equation for barometric heights.

The change of gravity and of humidity is all included in the θ term. The difference from the ordinary formula is simply that a harmonic mean temperature is used in place of an arithmetic mean. It may be mentioned in passing that Laplace (*see* Mécanique Céleste, Livre X, Chap. IV) took the arithmetic mean in his formula as being the simplest mean, and from this assumption he arrived at a law for the diminution of temperature with height. In the present case it is proposed to make use of actual measurements of temperature and pressure.

6. Differentiating equations (5) and (8) logarithmically and using (7) we get

$$\frac{1}{\rho} \cdot \frac{d\rho}{dh} = -\frac{1}{\tau} \cdot \frac{g}{C} - \frac{1}{\tau} \frac{d\tau}{dh}$$

aud

$$\frac{1}{\theta} \cdot \frac{d\theta}{dh} = \frac{1}{g} \cdot \frac{dg}{dh} - \frac{1}{\tau} \cdot \frac{d\tau}{dh}$$

whence

$$\frac{1}{\rho} \cdot \frac{d\rho}{dh} = -\frac{g_0 \theta}{C} - \frac{1}{g} \cdot \frac{dg}{dh} + \frac{1}{\theta} \cdot \frac{d\theta}{dh} \quad \ldots \ldots \quad (17)$$

Now values of $\frac{d^2g}{dh^2}$ are not available: and in any case they must be very small, so that we shall treat $\frac{dg}{dh}$ as constant and write (17)

$$\frac{1}{\rho} \cdot \frac{d\rho}{dh} = P + Q\,\theta + \frac{1}{\theta} \cdot \frac{d\theta}{dh} \quad \ldots \ldots \ldots \ldots \quad (18)$$

where

$$\left.\begin{aligned} P &= -\frac{1}{g} \cdot \frac{dg}{dh} \qquad Q = -\frac{g_0}{C} \\ \frac{dP}{dh} &= \frac{1}{g^2} \cdot \left(\frac{dg}{dh}\right)^2 = -P^2 \end{aligned}\right\} \quad \ldots \ldots \ldots \ldots \quad (19)$$

Differentiating (18) we have

$$-\frac{1}{\rho^2} \cdot \left(\frac{d\rho}{dh}\right)^2 + \frac{1}{\rho} \cdot \frac{d^2\rho}{dh^2} = -P^2 + Q\,\frac{d\theta}{dh} - \frac{1}{\theta^2} \cdot \left(\frac{d\theta}{dh}\right)^2 + \frac{1}{\theta} \cdot \frac{d^2\theta}{dh^2}$$

$$\therefore \quad \frac{1}{\rho} \cdot \frac{d^2\rho}{dh^2} = 2PQ\theta + Q^2\theta^2 + \frac{d\theta}{dh}\left(\frac{2P}{\theta} + 3Q\right) + \frac{1}{\theta} \cdot \frac{d^2\theta}{dh^2} \quad \ldots \ldots \ldots \quad (20)$$

Equations (18) ... (20) will be required later on.

7. The relation between the refraction of a gas and its density is expressed by Gladstone and Dale's law

$$\mu - 1 = K\rho \quad \text{. . . . .} \quad (21)$$

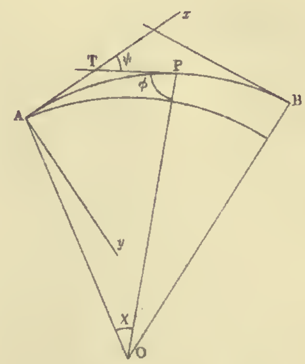
where μ is the refractive index and K is a constant for light of one colour.

The assumption of § 2 that surfaces of equal density have circular sections gives us the usual formula

$$\mu\,(r + h)\,\sin\phi = B \quad \text{. . . . .} \quad (22)$$

in which $r+h$ is the distance from the centre of curvature of the section, ϕ is the direction measured from the vertical, and B is a constant.

8. In the figure let APB be the path of the ray of light from A to B. Draw the tangents



ATx, PT at A and P.

Let AO, BO be verticals

and let $P\hat{T}x = \psi$ $\qquad OA = r$

$T\hat{P}O = \phi$ $\qquad OB = r + h$

$A\hat{O}P = \chi$

Draw Ay perpendicular to Ax.

Then if x and y are the coordinates of P and the curved distance AP is s, and if the radius of curvature at any point is σ, we have the well known formulæ

$$\left.\begin{aligned} x &= s - \frac{s^3}{6\sigma^2} + \frac{s^4}{8\sigma^3}\cdot\frac{d\sigma}{ds} + \ \text{. . . . .} \\ y &= \frac{s^2}{2\sigma} - \frac{s^3}{6\sigma^2}\cdot\frac{d\sigma}{ds} + \frac{s^4}{24\sigma^3}\left\{2\left(\frac{d\sigma}{ds}\right)^2 - 1 - \sigma\cdot\frac{d^2\sigma}{ds^2}\right\} + \ \text{. . . . . .} \end{aligned}\right\} \text{. . .} \quad (23)$$

in which we have to give to σ, $\dfrac{d\sigma}{ds}$, $\dfrac{d^2\sigma}{ds^2}$ etc. the values they have at A.

The angle of refraction $= P\hat{A}x = \tan^{-1}\dfrac{y}{x}$

$$= \frac{y}{x} - \frac{1}{3}\left(\frac{y}{x}\right)^3 + \frac{1}{5}\left(\frac{y}{x}\right)^5 - \ \text{. . . . .} \quad (24)$$

By (23) we have

$$\frac{y}{x} = \frac{s}{2\sigma}\left[1 - \frac{s}{3\sigma}\cdot\frac{d\sigma}{ds} + \frac{s^2}{12\sigma^2}\left\{2\left(\frac{d\sigma}{ds}\right)^2 - 1 - \sigma\frac{d^2\sigma}{ds}\right\} + \ldots\right]\left[1 - \frac{s^2}{6\sigma^2} + \frac{s^3}{8\sigma^3}\frac{d\sigma}{ds} + \cdots\right]^{-1}$$

$$= \frac{s}{2\sigma}\left[1 - \frac{s}{3\sigma}\cdot\frac{d\sigma}{ds} + \frac{s^2}{12\sigma^2}\left\{2\left(\frac{d\sigma}{ds}\right)^2 + 1 - \sigma\frac{d^2\sigma}{ds^2}\right\} + \frac{5s^3}{72\sigma^3}\cdot\frac{d\sigma}{ds} + \ldots\right]$$

whence $\qquad -\dfrac{1}{3}\cdot\left(\dfrac{y}{x}\right)^3 = -\dfrac{s^3}{24\sigma^3}\left[1 + \dfrac{s}{\sigma}\cdot\dfrac{d\sigma}{ds} + \ldots\right]$

∴ Refraction angle

$$P\hat{A}x = \frac{s}{2\sigma}\left[1 - \frac{s}{3\sigma}\cdot\frac{d\sigma}{ds} + \frac{s^2}{12\sigma^2}\left\{2\left(\frac{d\sigma}{ds}\right)^2 - \sigma\frac{d^2\sigma}{ds^2}\right\} + \frac{1}{36}\cdot\frac{s^3}{\sigma^3}\cdot\frac{d\sigma}{ds} + \ldots\right] \quad \text{. .} \quad (25)$$

9. Now $$\frac{1}{\sigma} = \frac{d\psi}{ds} = \frac{d\phi}{ds} + \frac{d\chi}{ds}, \text{ since } d\psi = d\phi + d\chi$$

$$= \frac{d\phi}{dh}\cos\phi + \frac{\sin\phi}{r+h}$$

since $\frac{dh}{ds} = \cos\phi$ and $(r+h)\frac{d\chi}{ds} = \sin\phi$

$$\therefore \quad \frac{1}{\sigma} = \frac{1}{r+h}\frac{d}{dh}\left(\overline{r+h}\,\sin\phi\right) \quad \ldots \quad (26)$$

$$= \frac{B}{r+h}\cdot\frac{d}{dh}\left(\frac{1}{\mu}\right) \quad \ldots \quad (27)$$

using equation (22)

From (21) we have $d\mu = K.\,d\rho$

$$\therefore \quad \frac{1}{\sigma} = -\frac{B}{r+h}\cdot\frac{1}{\mu^2}K.\frac{d\rho}{dh} = -K\sin\phi.\,\frac{1}{\mu}\cdot\frac{d\rho}{dh} \quad \ldots \quad (28)$$

Taking logarithms and differentiating and using relations $d\mu = K\,d\rho$ and $\frac{d}{ds} = \cos\phi\,\frac{d}{dh}$ we get

$$\frac{1}{\sigma}\cdot\frac{d\sigma}{ds} = \left\{\frac{1}{r+h} + \frac{2K.}{\mu}\cdot\frac{d\rho}{dh} - \frac{d}{dh}\left(\log\frac{d\rho}{dh}\right)\right\}\cos\phi \quad \ldots \quad (29)$$

Differentiating (29) we get

$$-\frac{1}{\sigma^2}\cdot\left(\frac{d\sigma}{ds}\right)^2 + \frac{1}{\sigma}\cdot\frac{d^2\sigma}{ds^2} = \left\{-\frac{1}{(r+h)^2} - \frac{2K^2}{\mu^2}\cdot\left(\frac{d\rho}{dh}\right)^2 + \frac{2K}{\mu}\cdot\frac{d^2\rho}{dh^2} - \frac{d^2}{dh^2}\left(\log\frac{d\rho}{dh}\right)\right\}\cos^2\phi$$

$$+\left\{\frac{1}{r+h} + \frac{2K}{\mu}.\frac{d\rho}{dh} - \frac{d}{dh}\left(\log\frac{d\rho}{dh}\right)\right\}\left(\frac{K}{\mu}.\frac{d\rho}{dh} + \frac{1}{r+h}\right)\sin^2\phi \quad . \quad (30)$$

$$\text{using} \quad \frac{1}{\mu}K\frac{d\rho}{dh} + \frac{1}{r+h} + \cot\phi.\,\frac{d\phi}{dh} = 0 \quad \ldots \quad (31)$$

which follows at once from the differentiation of (22).

Otherwise we may put

$$X = \frac{1}{r+h} + \frac{2K}{\mu}\cdot\frac{d\rho}{dh} - \frac{d}{dh}\left(\log\frac{d\rho}{dh}\right) \quad \ldots \quad (32)$$

$$\text{where} \quad \frac{1}{\sigma}\cdot\frac{d\sigma}{ds} = X\cos\phi$$

and by differentiating logarithmically we get

$$-\frac{1}{\sigma}\cdot\frac{d\sigma}{ds} + \frac{\frac{d^2\sigma}{ds^2}}{\frac{d\sigma}{ds}} = \cos\phi\left\{\frac{1}{X}\frac{dX}{dh} - \tan\phi\,\frac{d\phi}{dh}\right\}$$

$$Y = \frac{\frac{\sigma d^2\sigma}{ds^2}}{\left(\frac{d\sigma}{ds}\right)^2} = 1 + \frac{1}{X^2}\cdot\frac{dX}{dh} - \tan\phi\cdot\frac{1}{X}\cdot\frac{d\phi}{dh} \quad \ldots \quad (33)$$

$$\therefore \quad \frac{1}{\sigma}\frac{d^2\sigma}{ds^2} = X^2\cos^2\phi + \frac{dX}{dh}\cos^2\phi + X\sin^2\phi\left(\frac{K}{\mu}.\frac{d\rho}{dh} + \frac{1}{r+h}\right) \quad . \quad . \quad (34)$$

10. By means of equations (18)...(20) and (28)...(30) or (34) we can substitute in equation (25) and obtain the refraction in terms of certain known constants and $\frac{d\theta}{dh}, \frac{d^2\theta}{dh^2}$ etc. In ordinary cases doubtless it will be sufficient to consider only the first two terms of (25). In any case a knowledge of the quantities $\frac{d\theta}{dh}, \frac{d^2\theta}{dh^2}$ is required, and with this it is possible to calculate X and Y using equations (32) and (33).

The equation (25) can then be written

$$\hat{PAx} = \frac{s}{2\sigma}\left\{1 - \frac{s}{3}.X\cos\phi + \frac{s^2}{12}.X^2\cos^2\phi\,(2-Y) + \frac{s^3}{36}.X^3\cos^3\phi\ldots\right\} \quad . . (35)$$

This may be written in the simplified form

$$\hat{PAx} = \frac{s}{2\sigma}\left\{1 - \frac{u}{3} + \frac{u^2}{12}(2-Y) + \frac{u^3}{36} + \ldots\right\} \quad . . . . (36)$$

where $u = s\cos\phi.\,X$ . . . . (37)

$$\text{Now} \quad \frac{u^2}{12}(2-Y) = \frac{s^2}{12}\left\{X^2\cos^2\phi - \frac{dX}{dh}.\cos^2\phi - X\sin^2\phi\left(\frac{K}{\mu}.\frac{d\rho}{dh} + \frac{1}{r+h}\right)\right\}$$

We will consider the several terms separately.

$$\text{First term} = \frac{(sX\cos\phi)^2}{12} = \frac{u^2}{12}$$

The other terms are

$$-\frac{s^2\cos^2\phi}{12}.\frac{dX}{dh} \quad \text{and} \quad -\frac{sX.s.\sin^2\phi}{12}.\left(\frac{K}{\mu}.\frac{d\rho}{dh} + \frac{1}{r+h}\right)$$

Using C.G.S. units we see that $s\cos\phi$ will not exceed 10^6 and $s\sin\phi$ will not exceed 2.10^7, since we have not to deal with heights greater than 10^6 cm. and rays longer than 2.10^7 cm.

Also from (28) $\frac{K}{\mu}\frac{d\rho}{dh} = -\frac{1}{\sigma}\operatorname{cosec}\phi$

Then when ϕ is not very different from 90°, as is always the case in survey operations $-\frac{K}{\mu}\frac{d\rho}{dh}$ may be taken nearly the same as $\frac{1}{\sigma}$.

But it is a fact of all observations that the radius of curvature of the ray is only in exceptional cases as small as the radius of the earth; for the quantity $\frac{r}{\sigma}$ is what is usually called the coefficient of refraction. We may accordingly take the value $\frac{1}{r} = 1{\cdot}57 \times 10^{-9}$ C.G.S. as the upper limit of the value of $\frac{1}{\sigma}$. In cases where the ray is very close to heated ground, this value of $\frac{1}{\sigma}$ may be exceeded: but in this case the value of s will be much smaller than what we have taken.

We accordingly see that ~~that~~ the third term

$$= \frac{X}{12}\cdot s^2\sin\phi\left(\frac{\sin\phi}{r+h} - \frac{1}{\sigma}\right)$$

$$< \frac{8.10^{14}}{12}\cdot 1{\cdot}57\times10^{-9}\left(\frac{1}{r+h} - \frac{2}{\sigma}\operatorname{cosec}\phi - \frac{d}{dh}\log\frac{d\rho}{dh}\right)$$

The only part of this which need be kept accordingly is

$$+\frac{s^2\sin\phi}{12}\cdot\frac{d}{dh}\left(\log\frac{d\rho}{dh}\right)\left(\frac{\sin\phi}{r+h}-\frac{1}{\sigma}\right)$$

when $\frac{d}{dh}\left(\log\frac{d\rho}{dh}\right)$ is large enough to make this necessary; for we can see that the remainder is $< \frac{8.10^{14}}{12}\cdot(1\cdot57)^2.\ 10^{-18} = 1\cdot8.10^{-4}$, and on comparing this with the first term within the brackets in (36) it is seen to be negligible seeing that we do not have to compute to nearer than 1 part in 1000.

The remaining term is

$\frac{s^2}{12}\cos^2\phi\ \frac{dX}{dh}$, and taking $s\cos\phi$ as less than 10^6 we see that this term is less than

$$\frac{1}{12}\cdot 10^{12}\left\{-\frac{1}{\overline{r+h}|^2}+\frac{2K^2}{\mu^2}\cdot\overline{\frac{d\rho}{dh}}\Big|^2+\frac{2K}{\mu}\cdot\frac{d^2\rho}{dh^2}-\frac{d^2}{dh^2}\left(\log\frac{d\rho}{dh}\right)\right\}$$

in which the first two terms may be neglected. Writing $\mu=1$ in the third term we have as value of the term

$$\frac{s^2\cos^2\phi}{12}\left\{2K\frac{d^2\rho}{dh^2}-\frac{d^2}{dh^2}\left(\log\frac{d\rho}{dh}\right)\right\}.$$

We may accordingly put

$$\frac{u^2}{12}\left(2-Y\right)=\frac{u^2}{12}-\frac{s^2\cos^2\phi}{12}\cdot\left\{2K\cdot\frac{d^2\rho}{dh^2}-\frac{d^2}{dh^2}\left(\log\frac{d\rho}{dh}\right)\right\}+\frac{s^2\sin\phi}{12}\left(\frac{\sin\phi}{r+h}-\frac{1}{\sigma}\right)\frac{d}{dh}\log\frac{d\rho}{dh}.$$

It is also clear that we may use approximate values

$$X=2K\frac{d\rho}{dh}-\frac{d}{dh}\left(\log\frac{d\rho}{dh}\right)\quad\ldots\ldots\ldots\quad(32\text{A})$$

$$\text{and}\quad\frac{dX}{dh}=2K\frac{d^2\rho}{dh^2}-\frac{d^2}{dh^2}\left(\log\frac{d\rho}{dh}\right)$$

$$\frac{1}{\sigma}\doteqdot -K\sin\phi\frac{d\rho}{dh}$$

Hence refraction angle is PAx

$$=\Omega\doteqdot\frac{s}{2\sigma}\left\{1-\frac{u}{3}+\frac{u^2}{12}+\frac{u^3}{36}-\frac{1}{12}\cdot\frac{u^2}{X^2}\cdot\frac{dX}{dh}+\frac{s^2\sin\phi}{12}\left(\frac{\sin\phi}{r+h}-\frac{1}{\sigma}\right)\frac{d}{dh}\left(\log\frac{d\rho}{dh}\right)\right\}\quad\ldots(36\text{A})$$

X being given as above in (32A).

11. To find $\frac{dX}{dh}$ involves the previous finding of $\frac{d^2}{dh^2}\left(\log\frac{d\rho}{dh}\right)$. This quantity in turn depends partly on the value of $\left(\frac{d^3\theta}{dh^3}\right)_0$. Apart from this we can proceed to the fourth term of the series (25) and (36) with knowledge of the values of $\frac{d\theta}{dh}$, $\frac{d^2\theta}{dh^2}$ and of no higher differential coefficients. In the case of rays going to a great height, we shall probably have no means of determining the value of $\left(\frac{d^3\theta}{dh^3}\right)_0$ for each special case. In this case it may possibly be useful to estimate $\left(\frac{d^3\theta}{dh^3}\right)_0$ from the property of the existence of the isothermic layer, which has been observed to occur at a height of some 15 kilometres. This condition furnishes the relation

$$\frac{d\theta}{dh}=0\qquad\text{for a certain value of } h.$$

Differentiating $\theta = \theta_0 + h.\left(\frac{d\theta}{dh}\right)_0 + \frac{h^2}{\underline{|2}}.\left(\frac{d^2\theta}{dh^2}\right)_0 + \frac{h^3}{\underline{|3}}.\left(\frac{d^3\theta}{dh^3}\right)_0 + \ \ . \ .$

and neglecting the higher terms, we get the approximate equation

$$\left(\frac{d\theta}{dh}\right)_0 + h_1.\left(\frac{d^2\theta}{dh^2}\right)_0 + \frac{h_1^2}{2}.\left(\frac{d^3\theta}{dh^3}\right)_0 \doteq 0 \quad . \ . \ . \ . \ . \ . \ . \ . \ . \quad (38)$$

where h_1 is the height of the isothermic layer above the starting point of the ray.

This equation together with equation (15) and the readings of temperature and pressure at two known levels give us the means of determining the three quantities

$$\left(\frac{d\theta}{dh}\right)_0, \left(\frac{d^2\theta}{dh^2}\right)_0 \text{ and } \left(\frac{d^3\theta}{dh^3}\right)_0$$

12. It is to be remembered that although strictly speaking the values of $\left(\frac{d\theta}{dh}\right)_0$ etc. imply the true values at the starting point, yet to meet the practical case we can do no better than find an expression for θ in the form given in (10), taking only a finite number of terms——as many as we have the means of determining. The law we deduce will not be expected to truly represent the temperature changes with height, but will be the nearest we have means of ascertaining without actual simultaneous temperature readings at numerous heights. Actual measurement may be feasible and useful for rays proceeding close to the ground when perhaps temperatures at intervals of 10 feet might be measured up to a height of 100 feet; but, for the case of rays to snow peaks or high mountains, probably the best course will be to set up barometers at the observation station and at some other point a few thousand feet (or less) higher, and determine the quantities A_1, A_2, A_3 considering them as exactly equal to $\left(\frac{1}{\theta}.\frac{d\theta}{dh}\right)_0$, $\left(\frac{1}{\underline{|2}\,\theta}.\frac{d^2\theta}{dh^2}\right)_0$, $\left(\frac{1}{\underline{|3}\,\theta}.\frac{d^3\theta}{dh_3}\right)_0$ respectively, these quantities being evaluated as explained in § 11. It is anticipated that the value of A_3 so determined will only have an extremely small effect on the calculated refraction angle.

13. Even in cases of rays close to the ground, when the values of A_3 are likely to be larger, the formula (36) may be expected to represent the refraction by its first two terms, the higher terms being of small account by reason of the rays being short. For this case it appears that several readings of temperature at various (low) altitudes, should be sufficient.

14. To begin with we shall only consider the first two terms of (36) and see to what extent these will account for certain observed refractions.

It is first necessary to put in the values of the constants K, ρ_0, g_0, C, and express the equation in terms of the units in which we intend to work.

We first have to consider $\frac{s}{2\sigma}$, and we shall begin with the C.G.S. system and temperatures on the centigrade scale.

We have

$$\frac{1}{\sigma} = -K\sin\phi\,\frac{1}{\mu}.\frac{d\rho}{dh} \quad . \ . \ . \ . \ . \ . \ . \ . \ . \ . \quad (28)\ bis.$$

$$\frac{1}{\rho}.\frac{d\rho}{dh} = -\frac{1}{g}.\frac{dg}{dh} - \frac{g_0}{C}.\theta + \frac{1}{\theta}.\frac{d\theta}{dh} \quad . \ . \ . \ . \quad (17)\ bis.$$

$$\theta = \theta_0\,(1 + A_1 h + A_2 h^2 ...) \quad . \ . \ . \ . \ . \ . \ . \ . \quad (10)\ bis.$$

$$\theta = \frac{g}{g_0}.\frac{1}{\tau} \quad . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \quad (8)\ bis.$$

In Chwolson's Traité de Physique, Vol. I, pp. 414, 444 we find

Weight of 1 c.c. of air at standard pressure and temperature $\Big\} = \rho_s g = 10^{-6} \times 1 \cdot 3184\, g \quad gm/cm^3$ . . . . (39)

Standard pressure of 1 atmosphere at O°C. and latitude 45°, sea-level $\Big\} = 1 \cdot 0333 \quad gm/cm^2$ . . . . (40)

The suffix s will be used to indicate standard pressure, temperature, density etc., suffix 0 being retained to indicate the values at the beginning of the ray.

The value of μ is variously given. The value adopted for yellow light (D line) is 1·0002929 for dry air. For moist air we can afterwards apply a correcting factor if this appears desirable.

Taking this value and making use of (39) we have

$$K = \left(\frac{\mu - 1}{\rho}\right)_s = \frac{0 \cdot 0002929}{10^{-6} \times 1 \cdot 3184} = 2 \cdot 2216 \times 10^2 \quad . \; . \; . \; . \; . \; . \; . \quad (41)$$

$$\text{Then } C = \frac{p_s}{\tau_s \rho_s} = \frac{1 \cdot 0333 \times 10^3}{273 \times 1 \cdot 3184 \times 10^{-6}} = 2 \cdot 8709 \times 10^6 \quad . \; . \; . \; . \quad (42)$$

Using Helmert's 1884 formula for g we have

$$\frac{g}{C} = \frac{978}{2 \cdot 8709 \times 10^6}(1 + 0 \cdot 00531 \sin^2 \lambda)\left(1 - \frac{2h'}{R + h'}\right)$$

$$= 3 \cdot 4066 \times 10^{-4}(1 + 0 \cdot 00531 \sin^2 \lambda)\left(1 - \frac{2h'}{R + h'}\right) \quad . \; . \; . \; . \; . \; . \quad (43)$$

where h' is the height above sea-level and R is the radius of curvature of sea-level surface, so that

$$R + h' = r.$$

We also have
$$\frac{1}{g}\frac{dg}{dh} = -\frac{2}{R + h'} = -\frac{2}{r}.$$

We may treat μ as always having the value unity except when we are concerned with $\mu - 1$.

Then
$$\frac{K}{\mu} = 2 \cdot 2216 \times (1 - 0 \cdot 000293) \times 10^2$$
$$= 2 \cdot 2210 \times 10^2$$

$$\text{Then } -\frac{1}{\sigma} = \frac{\rho}{\rho_s} . \, 2 \cdot 221 \times 10^2 . \sin\phi \times 1 \cdot 3184 \times 10^{-6}\left\{\frac{2}{r} - 3 \cdot 4066 \times 10^{-4} \times \frac{g}{g_s} + \frac{1}{\theta} . \frac{d\theta}{dh}\right\}$$

Now
$$\frac{\rho}{\rho_s} = \frac{p}{p_s} . \frac{273}{\tau}$$
$$\left(\frac{1}{\theta} . \frac{d\theta}{dh}\right)_0 = A_1 \text{ and } \theta_0 = \frac{1}{\tau_0}$$

Let H_c be the corrected* barometric reading at the station expressed in millimetres of mercury.

Then $\dfrac{p}{p_s} = \dfrac{H_c}{760}$: also $H_c = \dfrac{H . g_0}{g_s}$ where H is the reading corrected for temperature only.

$$\therefore \; \frac{1}{\sigma} = -\frac{273}{\tau} . \frac{H_c}{760} \times 2 \cdot 221 \times 1 \cdot 3184 \times 10^{-4} . \sin\phi\left\{\frac{2}{r} - \frac{3 \cdot 4066}{\tau} \times 10^{-4} . \frac{g_0}{g_s} + A_1\right\}$$

$$= 1 \cdot 0518 \frac{H_c}{\tau^2} \times 10^{-4} \sin\phi\left\{3 \cdot 4066 \times 10^{-4}\frac{g_0}{g_s} - \left(A_1 + \frac{2}{r}\right)\tau\right\}$$

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* For temperature of mercury, latitude and altitude (as regards gravity).

We thus have $\frac{s}{2\sigma}$; which is an angle expressed in radians. To turn this into seconds we multiply by $\frac{180 \times 3600}{\pi}$ and get

$$\frac{s}{2\sigma} = 1\cdot0848 \times 10 \times \frac{H_c}{\tau^2} s. \sin\phi \left\{ 3\cdot4066 \times 10^{-4} \frac{g_0}{g_s} - \left(A_1 + \frac{2}{r}\right)\tau \right\} \quad . \quad . \quad (44)$$

15. From the figure of §8 we see that

$$AP^2 = x^2 + y^2 = s^2 \left(1 - \frac{s^2}{3\sigma^2} \cdots + \frac{s^2}{4\sigma^2} \cdots\right) = s^2\left(1 - \frac{s^2}{12\sigma^2} \cdots\right)$$

$$\therefore \quad AP = s\left(1 - \frac{s^2}{24\sigma^2} \cdots\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (45)$$

Also from ΔAOP, since OAP is equal to $\phi + \Omega$, Ω being the refraction angle and ϕ having its value at A

$$\frac{AP}{\sin \chi} = \frac{OP}{\sin(\phi + \Omega)}$$

$\therefore \; AP \sin(\phi + \Omega) = (r + h) \sin \chi.$

$$\therefore \quad s\left(1 - \frac{s^2}{24\sigma^2}\right)(\sin\phi + \Omega\cos\phi) = (r + h)\sin\chi = \left(1 + \frac{h}{r}\right). r\chi. \left(1 - \frac{\chi^2}{6}\cdots\right) \cdots \quad (46)$$

since Ω is small. A very large value for Ω is $500''$ or about $\frac{1}{400}$ of a radian. For such cases ϕ is nearly 90°, say 88°, so that $\Omega \cos\phi$ will not be greater than $\frac{1}{30} \times \frac{1}{400}$ or $\frac{1}{12000}$.

We neglect this in comparison with $\sin\phi$.

Also only in very extreme cases σ is as small as the radius of the earth: so that

$\frac{s^2}{24\sigma^2} < \frac{s^2}{24r^2} \doteq \frac{1}{24} \times \frac{1}{1600}$ for a ray of 100 miles, and we can neglect this also. For the same reason $\frac{\chi^2}{6}$ may be neglected, and equation (46) may be written in the approximate form

$$s \sin\phi \doteq (r + h). \chi = \left(1 + \frac{h_p}{R}\right). c \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (47)$$

where c is the distance, measured at the sea-level surface, between verticals through A and P: and h_p is the height of P above the sea-level surface.

16. It will be convenient for our purpose to express (44) in the British system of units. Also c is conveniently given in miles while H is measured in inches and τ on the absolute Fahrenheit scale. The portion in the brackets is clearly of dimension $\frac{\tau}{r}$ so that we can easily see that the equation now becomes

$$\frac{s}{2\sigma} = 1\cdot0848 \times 10 \times \frac{g_0}{g_s} \times \frac{1}{0\cdot03937} \times \frac{12 \times 5280}{0\cdot3937}\left(1 + \frac{h_b}{R}\right). c \times \left(\frac{9}{5}\right)^2 \left\{\frac{5}{9} \times \frac{0\cdot3937}{12}\right\}$$
$$\times \frac{H}{\tau^2} \times \left[\frac{9}{5} \times \frac{12}{0\cdot3937} \times 3\cdot4066 \times 10^{-4}\frac{g_0}{g_s} - \left(A_1 + \frac{2}{r}\right)\tau\right]$$

or $$\frac{s}{2\sigma} = 2\cdot6187 \times 10^6 \left(1 + \frac{h_b}{R}\right). \frac{g_0}{g_s}. \frac{lH}{\tau^2}\left[1\cdot8690.10^{-2}\frac{g_0}{g_s} - \left(A_1 + \frac{2}{r}\right)\tau\right] \quad . \quad . \quad . \quad (48)$$

It is to be noted that the value of A_1 is now to be found in British units: and l is the sea-level distance between A and B measured in miles. The expression contains only the ratio of gravity at observation station to normal gravity, and consequently we may express g in any unit which is most convenient.

17. When we come to deal with the actual value of A_1 it will be found that in equation (48) the term in A_1 is seldom more than one-third of the other term. To get an idea then of the magnitude of the refraction we take the expression

$$2{\cdot}62 \times 10^6 \times \frac{lH}{\tau^2} \times 1{\cdot}87 \times 10^{-2}$$

and putting $H = 30$ and $\tau = 500$ and $l = 100$ we find it is $588''$, that is $0{\cdot}00285$ radian. This is a very rough approximation to the refraction on a ray of length 100 miles: in round numbers it is $0{\cdot}003$ radian.

We now proceed to consider the term $\frac{u}{3}$ of (36) and we have by (37) and (32)

$$\frac{u}{3} = \frac{1}{3}.\, s \cos\phi\left(\frac{1}{r+h} + \frac{2K}{\mu}.\frac{d\rho}{dh} - \frac{d}{dh}\log\frac{d\rho}{dh}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (49)$$

Now $s\cos\phi$ is a quantity which can hardly exceed 5 miles, as it is approximately the AN of the figure of §8 where BN is perpendicular to OA; so that AN is less than the height of the highest mountain.

Hence
$$\frac{s\cos\phi}{3}.\frac{1}{r+h} < \frac{1}{3}.\frac{5}{4000} < \frac{1}{2400} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (50)$$

Next we have from (28)

$$\frac{K}{\mu}.\frac{d\rho}{dh} = -\frac{1}{\sigma\sin\phi}$$

$$\therefore \quad \frac{1}{3}\, s\cos\phi.\frac{2K}{\mu}.\frac{d\rho}{dh} = -\frac{4}{3}.\cot\phi.\frac{s}{2\sigma}$$

and we have just shown that $\frac{s}{2\sigma}$ is unlikely to exceed $0{\cdot}003$

$$\therefore \quad \frac{1}{3}\, s\cos\phi.\frac{2K}{\mu}.\frac{d\rho}{dh} \text{ is numerically less than } 0{\cdot}004\cot\phi.$$

Now we may assume for long rays, such as we are considering in arriving at the number $0{\cdot}003$, that ϕ will be nearly $90°$.

Taking $\phi = 90° - 3°$ we get $\frac{\pi}{2} - \phi = \frac{1}{20}$

$$\frac{1}{3}\, s\cos\phi\frac{2K}{\mu}.\frac{d\rho}{dh} < 0{\cdot}0002 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (51)$$

From (50) and (51) it is clear then that we may leave out of consideration $\frac{1}{3}\, s\cos\phi\left(\frac{1}{r+h} + \frac{2K}{\mu}.\frac{d\rho}{dh}\right)$ in evaluating $\frac{u}{3}$ and not have an error of more than $588''(0{\cdot}0004 + 0{\cdot}0002)$ or $0''{\cdot}35$. Such an error is inconsiderable. While remarking that there is no difficulty in taking account of these terms, if it is afterwards found desirable to do so, we shall for the present neglect them and write

$$\frac{u}{3} \doteqdot -\frac{1}{3}\, s\cos\phi\frac{d}{dh}\left(\log\frac{d\rho}{dh}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (52)$$

18. From equations (18) and (20) we have at once

$$\frac{d}{dh}\left(\log\frac{d\rho}{dh}\right) = \frac{\frac{d^2\rho}{dh^2}}{\frac{d\rho}{dh}} = \frac{2\,PQ\theta + Q^2\theta^3 + \left(\frac{2P}{\theta} + 3Q\right).\frac{d\theta}{dh} + \frac{1}{\theta}.\frac{d^2\theta}{dh^2}}{P + Q\theta + \frac{1}{\theta}.\frac{d\theta}{dh}} \quad . \; . \; . \quad (53)$$

Now $P = \frac{2}{r}$

$$Q = -\frac{g_o}{C} = -\,3{\cdot}4066 \times 10^{-4}.\,\frac{g_o}{g_s} \quad \text{in C.G.S. units}$$

$$= -\,1{\cdot}8690 \times 10^{-2}.\,\frac{g_o}{g_s} \quad \text{in British units}$$

Also $\frac{1}{\theta} \doteqdot \tau$, $\quad \frac{1}{\theta}\,\frac{d\theta}{dh} = A_1$, $\quad \frac{1}{\theta}\,\frac{d^2\theta}{dh^2} = 2\,A_2$ $\quad$ from (10)

$$\therefore \quad \frac{P}{Q\theta} \doteqdot -\,\frac{2}{r} \times 500 \times (1{\cdot}8690)^{-1}.\,10^2 \doteqdot \frac{1}{400}$$

since $r^{-1} \doteqdot \frac{1}{2}.\,10^{-7}$.

Hence considering that we are concerned with the second term of the refraction angle, it will doubtless be satisfactory to neglect P in comparison with $Q\theta$ in the numerator of (53) and we then get

$$\frac{d}{dh}\left(\log\frac{d\rho}{dh}\right) = \frac{1}{\tau}.\,\frac{-\,3{\cdot}4932 \times 10^{-4}\left(\frac{g_o}{g_s}\right)^2 + 5{\cdot}607 \times 10^{-2}\frac{g_o}{g_s}\,A_1\,\tau - 2A_2\,\tau^2}{1{\cdot}8690 \times 10^{-2}\frac{g_o}{g_s} - \left(A_1 + \frac{2}{r}\right)\tau} \quad . \; . \; . \quad (54)$$

This may be written with sufficient accuracy

$$\frac{d}{dh}\log\frac{d\rho}{dh} \doteqdot \frac{1}{\tau}.\,\frac{-\,3{\cdot}5 \times 10^{-4} + 5{\cdot}6 \times 10^{-2}\,A_1\,\tau - 2A_2\,\tau^2}{1{\cdot}8690 \times 10^{-2}\frac{g_o}{g_s} - \left(A_1 + \frac{2}{r}\right)\tau} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (55)$$

and the equation for refraction angle becomes

$$\Omega = 2{\cdot}6187 \times 10^6\left(1 + \frac{h_b}{R}\right).\,\frac{g_o}{g_s}.\,\frac{lH}{\tau^2}\Big\{1{\cdot}8690 \times 10^{-2}.\,\frac{g_o}{g_s} - \left(A_1 + \frac{2}{r}\right).\,\tau$$

$$-\,\frac{1}{3}\,s\cos\phi\,\frac{1}{\tau}\,(3{\cdot}5 \times 10^{-4} - 5{\cdot}6 \times 10^{-2}\,A_1\,\tau + 2A_2\,\tau^2)\Big\} \quad . \; . \; . \quad (56)$$

Equation (56) may also be written, replacing $\frac{g_o}{g_s}$ by unity in the second term,

$$\Omega = 2{\cdot}6187 \times 10^6\left(1 + \frac{h_b}{R}\right)\left(\frac{g_o}{g_s}\right)^2\frac{lH}{\tau^2}\Big\{1{\cdot}869.10^{-2} + \frac{d\tau}{dh}$$

$$. \; . \; . \; -\,\frac{1}{3}\,s.\cos\phi\left(\frac{\left(1{\cdot}869.10^{-2} + \frac{d\tau}{dh}\right)\left(1{\cdot}869.10^{-2} + 2\,\frac{d\tau}{dh}\right)}{\tau} - \frac{d^2\tau}{dh^2}\right)\Big\} \quad . \; . \; . \quad (56A)$$

19. In most cases the value of the second term of (56) is very much smaller than the first term. The refraction is accordingly least when the first term is least: that is when A_1 has its greatest positive value.

Now $$\left(\frac{d\theta}{dh}\right)_0 = A_1\,\theta_0 \quad \text{and} \quad \theta = \frac{g}{g_0}\cdot\frac{1}{\tau}$$

$$\therefore \frac{1}{\theta}\frac{d\theta}{dh} = \frac{1}{g}\cdot\frac{dg}{dh} - \frac{1}{\tau}\cdot\frac{d\tau}{dh}$$

$$\therefore A_1 = \frac{1}{g}\cdot\frac{dg}{dh} - \frac{1}{\tau}\cdot\frac{d\tau}{dh} \qquad \text{. . . . . (57)}$$

whence A_1 is greatest when $-\frac{1}{\tau}\frac{d\tau}{dh}$ is greatest.

The greatest temperature gradient which is stable is the adiabatic gradient, which corresponds to the law

$$\frac{p}{\rho^\gamma} = \text{constant} \quad \text{where } \gamma = 1\cdot408 \qquad \text{. . . . . (58)}$$

γ being the well known ratio of specific heats of a gas.

With this gradient of temperature, if we take unit volume of air at sea-level and carry it to any height, *without allowing it to gain or lose heat,* it will automatically adjust its temperature and pressure to that of the surrounding air at the given height. If the temperature falls more quickly than by the adiabatic gradient, we should find that the air, on adjusting its pressure to the upper level, would have too high a temperature and consequently too low a density. The result of this is that the air will rise by itself, and it will continue to do so until the adiabatic gradient is attained, if sufficient time be given: that is to say convection will be set up.

If the temperature falls less quickly than by the adiabatic gradient the tendency would be for the air to sink: but this tendency is prevented by the earth, and the only way in which temperature equilibrium can be reached is by conduction, which in air is a very slow process.

The diurnal variation in radiation intervenes before the conductive adjustment of temperature can be completed.

It is to be inferred, then, that when the lower layers of the atmosphere are heated by radiation from the earth there will be a tendency for the adiabatic gradient being attained: it may momentarily be exceeded, but this can only occur locally, and not through great ranges of height.

We may accordingly consider *true minimum refraction* to occur when the temperature gradient is adiabatic: but we must not expect that this minimum will always be attained.

20. For the adiabatic state we have as stated above

$$p = C'\rho^\gamma$$

whence using (5) we get

$$\tau = \frac{C'}{C}\cdot\rho^{\gamma-1} \qquad \text{. . . . . (59)}$$

Hence
$$\frac{d\tau}{dh} = \frac{C'}{C}(\gamma - 1).\, \rho^{\gamma-2} . \frac{d\rho}{dh}$$
$$= -\frac{C'}{C}.(\gamma - 1)\,\rho^{\gamma-1} . \frac{1}{\tau}.\left(\frac{g}{C} + \frac{d\tau}{dh}\right)$$
$$= -(\gamma - 1)\left(\frac{g}{C} + \frac{d\tau}{dh}\right).$$
$$\therefore \frac{d\tau}{dh} = -\frac{\gamma - 1}{\gamma}.\frac{g}{C} \quad \text{. . . . .} \quad (60)$$
$$= -\frac{0\cdot408}{1\cdot408} \times 1\cdot869 \times 10^{-2}\frac{g_o}{g_s} \text{ in British units}$$
$$= -0\cdot00542\frac{g_o}{g_s} \text{ degrees Fahrenheit per foot.} \quad \text{. . .} \quad (61)$$

Using (61), (57) and (56) we get

True minimum refraction angle
$$= 2\cdot6187 \times 10^6\left(1 + \frac{h_b}{R}\right)\frac{g_o}{g_s}.\frac{lH}{\tau^2}\left\{1\cdot869 \times 10^{-2}\frac{g_o}{g_s} - 0\cdot542 \times 10^{-2}\frac{g_o}{g_s}\right\}$$
$$\times \left\{1 - \frac{1}{3\tau}.\, s\cos\phi\left(1\cdot869 \times 10^{-2} - 2 \times 0\cdot542 \times 10^{-2}\right)\right\}$$
$$= 3\cdot475 \times 10^4\left(1 + \frac{h_b}{R}\right).\left(\frac{g_o}{g_s}\right)^2.\frac{lH}{\tau^2}\left\{1 - 2\cdot62 \times 10^{-3}.\frac{s\cos\phi}{\tau}\right\} \quad \text{. . . .} \quad (62)$$

neglecting $2\tau^2(A_2 - A_1^2)$ which vanishes if the gradient is strictly adiabatic: for in this case $\frac{d^2\tau}{dh^2} = 0$.

We shall see later to what extent this formula accords with the observations at times of so-called "minimum refraction".

Denoting the minimum refraction angle by ω we may write (62)

$$\left.\begin{aligned} &\omega = \omega_1 + \omega_2 \\ &\text{where } \omega_1 = 3\cdot475 \times 10^4\left(1 + \frac{h_b}{R}\right)\left(\frac{g_o}{g_s}\right)^2.\frac{lH}{\tau^2} \\ &\text{and } \quad \omega_2 = -2\cdot62 \times 10^{-3}.\frac{s\cos\phi}{\tau}.\,\omega_1 = -13\cdot8\,\frac{l\cot\phi}{\tau}.\,\omega_1 \end{aligned}\right\} \quad \text{. . . . .} \quad (63)$$

The general formula may be written
$$\Omega = f_1\,\omega_1 + f_2\,\omega_2 \quad \text{. . . . .} \quad (64)$$
$$\left.\begin{aligned} &\text{where } f_1 = \frac{1\cdot869 - a}{1\cdot327}, f_2 = \left\{\frac{(1\cdot869 - a)\,(1\cdot869 - 2a)}{\tau} - 10^4\frac{d^2\tau}{dh^2}\right\} \times \frac{\tau}{1\cdot042} \\ &\qquad \text{and } a = -10^2.\frac{d\tau}{dh} \end{aligned}\right\} \quad \text{. .} \quad (65)$$

Values of f_1 and $f_2 + b$ are given in table III, where $\frac{\tau}{1\cdot042}.\, 10^4\frac{d^2\tau}{dh^2} = b$, corresponding to various values of $10^2\frac{d\tau}{dh} = -a$. It is to be noted that $\frac{\tau}{1\cdot042} = 500$ when $\tau = 521$ which corresponds to ordinary temperature 62°—a very usual temperature: so that b is approximately $5 \times 10^6\frac{d^2\tau}{dh^2}$.

Now $$\tau \doteq \tau_0 + h\frac{d\tau}{dh} + \frac{h^2}{2}\cdot\frac{d^2\tau}{dh^2}.$$

Hence b may be regarded as the excess in actual temperature at a height $10^{\frac{7}{2}} = 3162$ feet, (if temperature at base is 62°F) over temperature given by the gradient *at the lower station.* The excess at 5000 feet is $2\cdot5\ b$.

TABLE I.

Correction for virtual temperature of saturated air in degrees Fahrenheit, the pressure being given in inches of mercury.

Temperature (°F)	Pressure in inches of mercury.											
	10	12	14	16	18	20	22	24	26	28	30	·32
0	0·7	0·6	0·5	0·4	0·4	0·3	0·3	0·3	0·3	0·2	0·2	0·2
10	1·1	0·9	0·8	0·7	0·6	0·6	0·5	0·5	0·4	0·4	0·4	0·4
20	1·9	1·6	1·3	1·2	1·0	0·9	0·9	0·8	0·7	0·7	0·6	0·6
25	2·4	2·0	1·7	1·5	1·3	1·2	1·1	1·0	0·9	0·9	0·8	0·7
30	3·1	2·5	2·2	1·9	1·7	1·5	1·4	1·3	1·2	1·1	1·0	0·9
32	3·3	2·8	2·4	2·1	1·9	1·7	1·5	1·4	1·3	1·2	1·1	1·0
34	3·7	3·0	2·6	2·3	2·0	1·8	1·6	1·5	1·4	1·3	1·2	1·1
36	4·0	3·3	2·8	2·5	2·2	2·0	1·8	1·6	1·5	1·4	1·3	1·2
38	4·3	3·6	3·1	2·7	2·4	2·1	1·9	1·8	1·6	1·5	1·4	1·3
40	4·7	3·9	3·3	2·9	2·6	2·3	2·1	1·9	1·8	1·7	1·5	1·4
42	5·1	4·2	3·6	3·2	2·8	2·5	2·3	2·1	1·9	1·8	1·7	1·6
44	5·5	4·6	3·9	3·4	3·0	2·7	2·5	2·3	2·1	2·0	1·8	1·7
46	6·0	5·0	4·3	3·7	3·3	3·0	2·7	2·5	2·3	2·1	2·0	1·9
48	6·5	5·4	4·6	4·0	3·6	3·2	2·9	2·7	2·5	2·3	2·1	2·0
50	7·0	5·8	5·0	4·4	3·9	3·5	3·2	2·9	2·7	2·5	2·3	2·2
52	7·6	6·3	5·4	4·7	4·2	3·8	3·4	3·1	2·9	2·7	2·5	2·3
54	8·2	6·8	5·8	5·1	4·5	4·1	3·7	3·4	3·1	2·9	2·7	2·5
56	8·9	7·4	6·3	5·5	4·9	4·4	4·0	3·6	3·4	3·1	2·9	2·7
58	9·6	8·0	6·8	5·9	5·3	4·7	4·3	3·9	3·6	3·4	3·2	3·0
60		8·6	7·3	6·4	5·7	5·1	4·6	4·2	3·9	3·6	3·4	3·2
62		9·3	7·9	6·9	6·1	5·5	5·0	4·6	4·2	3·9	3·7	3·4
64		10·0	8·5	7·4	6·6	5·9	5·4	4·9	4·6	4·2	3·9	3·7
66		10·7	9·2	8·0	7·1	6·4	5·8	5·3	4·9	4·5	4·2	4·0
68		11·5	9·9	8·6	7·6	6·9	6·2	5·7	5·3	4·9	4·5	4·3
70				9·3	8·2	7·4	6·7	6·2	5·7	5·3	4·9	4·6
72				10·0	8·8	7·9	7·2	6·6	6·1	5·6	5·3	4·9
74				10·7	9·5	8·5	7·7	7·1	6·5	6·0	5·6	5·3
76				11·5	10·2	9·2	8·3	7·6	7·0	6·5	6·1	5·7
78				12·4	10·9	9·9	8·9	8·2	7·5	7·0	6·5	6·1
80						10·6	9·6	8·8	8·1	7·5	7·0	6·5
82						11·3	10·3	9·4	8·7	8·0	7·5	7·0
84						12·1	11·0	10·1	9·3	8·6	8·0	7·5
86						13·0	11·8	10·8	10·0	9·2	8·6	8·1
88						13·9	12·6	11·6	10·7	9·9	9·2	8·6
90							13·5	12·4	11·3	10·6	9·9	9·2
92							14·5	13·2	12·2	11·3	10·5	9·9
94							15·5	14·2	13·1	12·1	11·3	10·6
96							16·6	15·1	14·0	12·9	12·1	11·3
98							17·7	16·2	14·9	13·8	12·9	12·1
100								17·3	15·9	14·8	13·8	12·9

Example:—If $t = 50$, $H = 30$, humidity $= w$, then virtual temperature $= 50 + 2\cdot3\ w$.

TABLE II.

Correction for virtual temperature of saturated air in degrees Centigrade, the pressure being given in millimetres of mercury.

Temperature (°C.)	Pressure in millimetres of mercury 250	300	350	400	450	500	550	600	650	700	750	800
−15	0·5	0·4	0·3	0·3	0·3	0·2	0·2	0·2	0·2	0·2	0·2	0·2
−10	0·8	0·6	0·6	0·5	0·4	0·4	0·4	0·3	0·3	0·3	0·3	0·2
− 5	1·2	1·0	0·9	0·8	0·7	0·6	0·6	0·5	0·5	0·4	0·4	0·4
− 2	1·6	1·3	1·1	1·0	0·9	0·8	0·7	0·7	0·6	0·6	0·5	0·5
0	1·9	1·6	1·3	1·2	1·0	0·9	0·9	0·8	0·7	0·7	0·6	0·6
1	2·0	1·7	1·5	1·3	1·1	1·0	0·9	0·8	0·8	0·7	0·7	0·6
2	2·2	1·8	1·6	1·4	1·2	1·1	1·0	0·9	0·8	0·8	0·7	0·7
3	2·4	2·0	1·7	1·5	1·3	1·2	1·1	1·0	0·9	0·8	0·8	0·7
4	2·6	2·1	1·8	1·6	1·4	1·3	1·2	1·1	1·0	0·9	0·8	0·8
5	2·7	2·3	2·0	1·7	1·5	1·4	1·2	1·1	1·1	1·0	0·9	0·9
6	3·0	2·5	2·1	1·8	1·6	1·5	1·3	1·2	1·1	1·0	1·0	0·9
7	3·2	2·6	2·3	2·0	1·7	1·6	1·4	1·3	1·2	1·1	1·1	1·0
8	3·4	2·8	2·4	2·1	1·9	1·7	1·5	1·4	1·3	1·2	1·1	1·1
9	3·7	3·1	2·6	2·3	2·0	1·8	1·7	1·5	1·4	1·3	1·2	1·1
10		3·3	2·8	2·5	2·2	2·0	1·8	1·6	1·5	1·4	1·3	1·2
11		3·5	3·0	2·6	2·3	2·1	1·9	1·8	1·6	1·5	1·4	1·3
12		3·8	3·2	2·8	2·5	2·3	2·0	1·9	1·7	1·6	1·5	1·4
13		4·1	3·5	3·0	2·7	2·4	2·2	2·0	1·9	1·7	1·6	1·5
14		4·3	3·7	3·2	2·9	2·6	2·4	2·2	2·0	1·8	1·7	1·6
15				3·5	3·1	2·8	2·5	2·3	2·1	2·0	1·8	1·7
16				3·7	3·3	3·0	2·7	2·5	2·3	2·1	2·0	1·8
17				4·0	3·5	3·2	2·9	2·6	2·4	2·3	2·1	2·0
18				4·3	3·8	3·4	3·1	2·8	2·6	2·4	2·3	2·1
19				4·5	4·0	3·6	3·3	3·0	2·8	2·6	2·4	2·3
20					4·3	3·9	3·5	3·2	3·0	2·8	2·6	2·4
21					4·6	4·1	3·8	3·4	3·2	2·9	2·7	2·6
22					4·9	4·4	4·0	3·7	3·4	3·1	2·9	2·7
23					5·2	4·7	4·3	3·9	3·6	3·3	3·1	2·9
24					5·6	5·0	4·6	4·2	3·8	3·6	3·3	3·1
25						5·4	4·9	4·5	4·1	3·8	3·6	3·3
26						5·7	5·2	4·7	4·4	4·1	3·8	3·5
27						6·1	5·5	5·0	4·7	4·3	4·0	3·8
28						6·5	5·9	5·4	5·0	4·6	4·3	4·0
29						6·9	6·3	5·7	5·3	4·9	4·6	4·3
30							6·7	6·1	5·6	5·2	4·8	4·5
31							7·1	6·5	6·0	5·5	5·2	4·8
32							7·5	6·9	6·4	5·9	5·5	5·1
33							8·0	7·3	6·7	6·3	5·8	5·5
34							8·5	7·8	7·2	6·7	6·2	5·8
35									7·6	7·1	6·6	6·2
36									8·1	7·5	7·0	6·5
37									8·6	8·0	7·4	6·9
38									9·1	8·5	7·9	7·4
39									9·7	9·0	8·3	7·8
40									10·2	9·5	8·8	8·3

TABLE III.

Values of the factors f_1 and $f_2 + b$ for different temperature gradients.

a	+0·6	+0·5	+0·4	+0·3	+0·2	+0·1	0	−0·1	−0·2	−0·3	−0·4	−0·5	−0·6	−0·7	−0·8	−0·9	−1·0
f_1	+0·956	1·032	1·107	1·182	1·258	1·333	1·408	1·484	1·559	1·635	1·710	1·785	1·861	1·936	2·011	2·087	2·162
$f_2 + b$	+0·81	1·14	1·51	1·91	2·35	2·83	3·35	3·91	4·51	5·14	5·81	6·52	7·27	8·06	8·89	9·75	10·65

NOTATION IN CHAPTER II.

l distance at sea-level between normals through the terminal points of the ray, *expressed in miles.*

c distance at sea-level between normals through the terminal points of the ray, expressed in feet.

s length of ray, expressed in feet.

h height of station B above A.

h_b height of station B above sea-level.

R radius of curvature in azimuth of ray.

$r =$ $R + h_a$.

ρ, ν principal radii of curvature of spheroid.

x, y rectangular coordinates of any point referred to origin at Nojli, *expressed in miles.*

τ absolute *virtual* temperature.

$t =$ $\tau - 459{\cdot}4 =$ *virtual* temperature, Fahrenheit.

H barometric pressure in inches of mercury, corrected for temperature of mercury.

I height of instrument above station A, expressed in feet.

S height of signal above station B, expressed in feet.

δ, δ' plumb-line deflections at A towards B, and at B towards A.

a, a' angles of elevation at A and B of the *straight line* (chord) AB.

E observed angle of elevation.

dE correction to E to allow for I and S.

$E_c =$ $E + dE + \delta$.

A azimuth of station B at A, measured from south.

ξ, η components of plumb-line deflection towards east and north.

${}_0\omega_1, {}_0\omega_2$ values of ω_1 and ω_2 corresponding to E_0, H_0, τ_0.

g_0 value of g at observatory station.

g_s value of g at sea-level, latitude 45°.

$100\,w$ percentage humidity.

$\chi =$ angle between normals at A and B.

$\epsilon =$ the error in meridional plumb-line deflection assumed in table V.

$u =$ $f_1 - 1$.

$v =$ $f_2 - 1$.

CHAPTER II.

The relative heights of Shaw's Refraction Stations.

1. During the years 1905-1909, Mr. H. G. Shaw of the Survey of India, made numerous observations of the vertical angles between the stations Nojli, Dehra Dun, Mussooree, Nag Tiba and some of the Himalayan snow-peaks (*see* Professional Paper No. 11). These observations are very suitable for testing the formulæ (63) and (65) found in Chapter I. The first three stations have been connected by spirit-levelling, and so it is possible to compare the heights as found by vertical angles with those found by spirit-levelling. The spirit-levelling would give a means of deducing the true angle of elevation were full particulars of the plumb-line deflection known: and then a comparison with the observed angle of elevation would give the difference due to refraction. However as will be seen later we can only estimate roughly the influence of irregular plumb-line deflections, so that the comparison does not in all cases give the information sought for with sufficient accuracy.

In computations of the Survey of India it has been customary to refer trigonometrically determined heights to the spheroid of Everest. While continuing this, we shall consider that the theoretical level surfaces at various heights are related to the Everest spheroid in such a way that if R is the radius of curvature of the spheroid in any azimuth, then $R + h$ is the corresponding radius of curvature of the level surface through a point at (orthometric) height h, thus taking no cognisance of the variation of orthometric height between two level surfaces. The variation is too small in the length of a terrestrial ray to have a sensible effect on the observed angles: and its inclusion would be an unnecessary complication. We also neglect the effect of change of curvature of the level surface along the projection on the surface of the ray of light: and thus measure heights from the circle of curvature instead of from the actual elliptic section of the Everest spheroid.

2. Let r be the radius of curvature of the theoretical level surface through A, the observatory station, in the proper azimuth; and let a be the angle of elevation of B, that is the observed elevation truly corrected for refraction: and let χ be the angle between the normals to the spheroid of reference which pass through A and B. Then if h is the height of B above A we have from Δ OAB, O being the centre of curvature,

$$\frac{h + r}{\cos a} = \frac{r}{\cos (a + \chi)} \qquad \ldots \ldots \ldots \ldots \quad (66)$$

whence $$h = 2r \sin\left(a + \frac{\chi}{2}\right) \sin\frac{\chi}{2} \sec(a + \chi) \quad \ldots \ldots \ldots \quad (67)$$

We can in all cases write

$$c\left(1 + \frac{h_a}{R}\right) \doteqdot 2r \sin\frac{\chi}{2} \quad \ldots \ldots \ldots \quad (68)$$

c being the arc or chord at sea-level between the normals through A and B: the difference between the two being too small to have an appreciable effect on our work.

We also have $$\chi \doteqdot \frac{c}{R} \operatorname{cosec} 1'' \quad \ldots \ldots \ldots \quad (69)$$

and log cosec $1'' = 5 \cdot 3144251$, which enable us to compute χ in seconds.

It has been usual in the Survey of India in deducing heights from the vertical angle at one end of ray, to treat r as the same for all azimuths: but this is not sufficiently approximately true and should be discontinued.

If the azimuth, measured from south, is A, then

$$\frac{1}{R} = \frac{\cos^2 A}{\rho} + \frac{\sin^2 A}{\nu} \quad \ldots \ldots \ldots \quad (70)$$

where ρ and ν are the principal radii of curvature of the sea-level spheroid.

Hence $$\frac{1}{R} = \frac{1}{\rho\nu} \left\{ \left(\frac{\rho + \nu}{2} - \frac{\rho - \nu}{2}\right) \cos^2 A + \left(\frac{\rho + \nu}{2} + \frac{\rho - \nu}{2}\right) \sin^2 A \right\}$$
$$= \frac{\rho + \nu}{2\rho\nu} \left\{ 1 - \frac{\rho - \nu}{\rho + \nu} \cos 2A \right\}.$$

Then $$R = \frac{2\rho\nu}{\rho + \nu} \left\{ 1 + \frac{\rho - \nu}{\rho + \nu} . \cos 2A + \ldots \right\}$$

and since $\frac{2\rho\nu}{\rho + \nu}$ is very nearly the same as $\frac{\rho + \nu}{2}$, $\frac{\rho - \nu}{\rho + \nu}$ being a small quantity, we may put

$$R = \frac{\rho + \nu}{2} + \frac{\rho - \nu}{2} \cos 2A \quad \ldots \ldots \ldots \quad (71)$$

with accuracy sufficient for the present purpose.

For use in what follows table IV is given:

TABLE IV.

Latitude	$\frac{\rho + \nu}{200}$	$\frac{\rho - \nu}{200}$
29° 50′	208 872	−524
30 0	208 876	−522
30 10	208 879	−520
30 20	208 883	−519
30 30	208 886	−517
30 40	208 890	−515
30 50	208 893	−514

If we wish to find a we have from (66)

$$\tan a = \frac{(r+h)\cos \chi - r}{(r+h)\sin \chi}$$

$$= \frac{h}{r+h} . \operatorname{cosec} \chi - \tan \frac{\chi}{2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (73)$$

in which the value of χ found from (69) may be used.

3. In the neighbourhood of the Himalayas large deflections of the plumb-line have been observed. Here the geoid level surfaces separate from the spheroidal surfaces to a considerable extent. The absolute amount cannot well be determined; but the increase or decrease which occurs in the separation between adjacent stations can be estimated—the estimation being better and better as we have more deflection data.

At any point P let the components of deflection be ξ η measured towards east and north respectively: and let (x, y) $(x+dx, \ y+dy)$ be the rectangular coordinates, referred to axes along a parallel and a meridian, of P and the adjacent point P'. Then in passing from P to P' the geoid separates from the spheroid by an amount $\xi\, dx + \eta\, dy$

Integrating we get Δh the rise between any two points on a level surface

$$\Delta h = \int \xi dx + \int \eta dy \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (72)$$

To apply (72) we want to know the *surface* values of ξ and η at as many points as possible. As the points where ξ and η are known are comparatively speaking few in number we shall use mean values of ξ and η at both ends of each element.

Chart No. I shows the deflections which have been observed in the district with which we are dealing. The deflections in longitude are not known at so many places as the deflection in latitude: but by assuming the direction of the resultant deflection to vary smoothly over the district, we interpolate certain values.

4. We shall now estimate the rise in the geoid over the spheroid between Nojli and Mussooree. It is certain that the geoid is above the spheroid at Nojli but as we are concerned only with relative heights, we need not pay any attention to that fact. We follow the line Nojli–A–Dehra–Rājpur–Mussooree and tabulate data in table V.

TABLE V.

	x	y	ξ	η
Nojli ...	0	0	(8·3)	13·6
A ...	15·6	19·4	(17·8)	29·0
Dehra ...	23·7	30·1	22·7	37·5
Rajpur ...	26·1	35·7	(32·8)	47·7
Mussooree ...	24·3	39·2	28·2	36·5
Nag Tiba ...			23·5	(32)

x and y are given in miles: ξ and η in seconds; values of ξ and η found by interpolation are included in brackets ( ).

Using (71) and taking mean values of ξ and η in each element we get

$$\text{Rise in element Nojli—A} = 15\cdot6 \times \frac{26\cdot1}{2} + 19\cdot4 \times \frac{42\cdot6}{2} = 204 + 413 = 617$$

$$\text{A—Dehra} = 8\cdot1 \times \frac{40\cdot5}{2} + 10\cdot7 \times \frac{66\cdot5}{2} = 164 + 356 = 520$$

$$\text{Dehra—Rājpur} = 2\cdot4 \times \frac{55\cdot5}{2} + 5\cdot6 \times \frac{85\cdot2}{2} = 67 + 239 = 306$$

$$\text{Rājpur—Mussooree} = -1\cdot8 \times \frac{61\cdot0}{2} + 3\cdot5 \times \frac{84\cdot2}{2} = -55 + 147 = 92.$$

These numbers have to be multiplied by $\frac{5280 \times \pi}{180 \times 3600} = \cdot0256$ to reduce the results to feet: after which they become 15·8, 13·3, 7·8 and 2·4 feet respectively, whence

the rise of the geoid at Dehra = 29·1 ft. + rise at Nojli
" " " Rājpur = 36·9 ft. + " "
" " " Mussooree = 39·3 ft. + " "

It will be seen that these results are not susceptible of high accuracy. Especially in the short line Dehra–Rājpur–Mussooree where the latitude deflection changes from 37·5 to 47·7 and back to 36·5 in less than 10 miles, is this the case. We have no means of knowing where the maximum deflection occurs: or what its amount is. Perhaps a more detailed examination would reveal a spot where the latitude deflection is considerably greater than that which has been found at Rājpur. The fact, which is arrived at later in § 11 that the height of Mussooree above Dehra as found by vertical angles exceeds that found by spirit-levelling by 2 feet, taking the rise of spheroid between Mussooree and Dehra to be 10·2 feet as computed above, suggests that the mean deflection on this line is some 8″ larger than we have taken it to be. It is hoped that we shall soon have additional observation data at intermediate points, which will clear up this point.

5. Table VI gives some data regarding the points with which we are concerned.

TABLE VI.

No.	Name of Station	Orthometric spirit-levelled height	Latitude	Longitude	
		feet	° ′ ″	° ′ ″	
1	Nojli Refraction Station	886·7	29 53 27·57	77 40 25·30	Greenwich terms.
2	Nojli Tower "	937·0	29 53 27·76	77 40 24·59	
3	Dehra Dun Refn. "	2234·3	30 19 28·52	78 3 23·04	
4	Mussooree " "	6929·9	30 27 40·38	78 4 17·66	
5	" Abbotsford B. S.†	6682·4			
6	" Castle B. S.†	6890·4			
7	Nag Tiba h. s.	9915*	30 35 11·09	78 9 9·57	
8	Bandarpunch ...	20720*	31 0 12·1	78 33 17·1	
9	Srikanta ...	20120*	30 57 25·2	78 48 22·0	
10	Jaonli	21760*	30 51 17·4	78 51 25·4	
11	Kedarnath	22770*	30 47 53·0	79 4 7·0	

* Height given in Synoptical Volume XXXV, determined by triangulation. † Barometer station.

Table VII gives the distances in miles (in italics), the logarithms of the sea-level distances expressed in feet between the several points, and the azimuths at station A of the lines.

TABLE VII.

No.	Station B	Station A				
		Nojli R. S.	Nojli Tower R.S.	Dehra Dun R.S.	Mussooree R S.	Nag Tiba R.S.
1	Nojli R. S.	...	*0·012* 1·8153937 287° 4′ 22″	*37·644* 5·2983245 37° 35′ 51″	*45·928* 5·3847096 31° 19′ 29″	*55·809* 5·4693389 30° 59′ 59″
2	Nojli Tower ”	*0·012* 1·8153937 107° 4′ 23″	...	*37·648* 5·2983742 37° 36′ 54″	*45·931* 5·3847381 31° 20′ 23″	*55·812* 5·4693617 31° 0′ 44″
3	Dehra Dun ”	*37·644* 5·2983245 217° 24′ 19″	*37·648* 5·2983742 217° 25′ 23″	...	*9·454* 4·6982573 5° 30′ 9″	*18·927* 4·9997102 17° 41′ 26″
4	Mussooree ”	*45·928* 5·3847096 211° 7′ 29″	*45·931* 5·3847381 211° 8′ 23″	*9·454* 4·6982573 185° 29′ 41″	...	*9·886* 4·7176760 29° 17′ 57″
5	Nag Tiba ”	*55·809* 5·4693389 210° 45′ 31″	*55·812* 5·4693617 210° 46′ 15″	*18·927* 4·9997102 197° 38′ 31″	*9·886* 4·7176760 209° 15′ 28″	...
6	Bandarpunch*	*92·923* 5·6907586 214° 14′ 7″	*92·927* 5·6907758 214° 14′ 33″	*55·376* 5·4659533 212° 16′ 28″	*47·128* 5·3959167 217° 28′ 3″	*37·370* 5·2951554 219° 40′ 26″
7	Srikanta*	*99·800* 5·7217645 222° 20′ 42″	*99·805* 5·7217874 222° 21′ 5″	*62·380* 5·5176761 225° 30′ 59″	*55·474* 5·4667208 231° 48′ 58″	*46·498* 5·3900718 236° 32′ 8″
8	Jaonli*	*96·962* 5·7092350 226° 29′ 27″	*96·968* 5·7092623 226° 29′ 49″	*60·084* 5·5013963 232° 21′ 39″	*54·063* 5·4555308 239° 42′ 0″	*45·814* 5·3836287 246° 1′ 6″
9	Kedarnath*	*104·149* 5·7402884 232° 47′ 12″	*104·156* 5·7403184 232° 47′ 31″	*68·583* 5·5588518 241° 20′ 59″	*63·759* 5·5271726 248° 24′ 22″	*56·438* 5·4742023 254° 47′ 48″

* Snow peak.

6. We now proceed to find the angle a (*see* §2) for the cases Nojli–Mussooree and Dehra-Mussooree making use of equation (72) and the differences of height obtained by spirit-levelling corrected, as far as possible, for irregular deflections of the plumb-line. The computation is as follows :—

	Nojli R.S.	Dehra R.S.
Azimuth of Mussooree ...	211° 7′ 29″	185°29′41″
$R/100$ for this azimuth by (71) ...	208630	208374
log R	7·3193768	7·3188435
log c	5·3847096	4·6982573
$\log \sin \chi \doteqdot \log \frac{c}{R}\left(\textit{see}\ (69)\right)$	$\bar{2}$·0653328	$\bar{3}$·3794138
χ	39′ 57″·5	8′14″·1
$r + h$	208710 × 10²	208443 × 10²
h	6082·5	4705·8
log h	3·7840821	3·6726335
log $(r + h)$	7·3195433	7·3189873
log $h/(r + h)$	$\bar{4}$·4645388	$\bar{4}$·3536462
log sin χ	$\bar{2}$·0653328	$\bar{3}$·3794137
log $h/(r + h)$ sin χ	$\bar{2}$·3992060	$\bar{2}$·9742325
$\frac{h}{r+h}$. cosec χ	0·0250730	0·0942394
tan $\frac{\chi}{2}$	0·0058116	0·0011977
$\tan a = \frac{h}{r+h} \operatorname{cosec} \chi - \tan \frac{\chi}{2}$	0·0192614	0·0930417
a	1° 6′ 12″·5	5° 18′ 56″·1
$a' = -a - \chi$	−1° 46′ 10″·0	−5° 27′ 10″·2

In the last line a' is the value of a *at Mussooree* to Nojli and Dehra respectively.

7. Let E be the *observed* elevation of station B from station A, δ the plumb-line deflection at A towards B and Ω the angle of refraction.

Then

$$E - \Omega + \delta = a$$

whence

$$\Omega = E + \delta - a \quad \ldots\ldots\ldots\ldots \quad (73)$$

Using table V we form table VIII.

TABLE VIII.

Station		δ	δ'	$\delta - a$	$\delta' - a'$
A	B				
Nojli	Mussooree	$15''\cdot9$	$-46''\cdot1$	$-1^\circ\ 5'\ 56''\cdot6$	$+1^\circ\ 45'\ 23''\cdot9$
Dehra	Mussooree	$39''\cdot6$	$-39''\cdot1$	$-5^\circ\ 18'\ 16''\cdot5$	$+5^\circ\ 26'\ 31''\cdot1$

δ' is the value of the deflection *at Mussooree* towards Nojli and Dehra.

8. We will now apply (63) to the determination of *minimum refraction* in the case of Mr. Shaw's observations between Nojli, Dehra and Mussooree and will compare the refraction computed in this way with the values obtainable from (73).

In the first case we will use the mean results of each season. From time to time the heights of the theodolite and of the signals were changed, so that it will be convenient to express these heights in terms of the angles they subtend, and then apply these angular corrections to the observed elevations. If we denote the heights of the instrument and of the signal by I and S respectively and express the results in seconds of arc, we have

$$dE = \frac{I - S}{c}.\ \mathrm{cosec}\ 1'' \qquad \qquad (74)$$

and in the case we are considering

$$dE = (I - S)\ 0''\cdot851 \quad \text{for Nojli–Mussooree,}$$

and $$dE = (I - S)\ 4''\cdot131 \quad \text{for Dehra–Mussooree.}$$

We will begin with the Dehra–Mussooree observations. In dealing with the actual determination of ω it is most convenient to compute ω_1 and ω_2 for selected average values of H and τ, *viz.*, H_0 and τ_0

where $$H = H_0 + dH \quad \text{and} \quad \tau = \tau_0 + d\tau \qquad \qquad (75)$$

and then apply corrections to take account of the actual values of H and τ on each occasion. The values so found, *viz.*, ${}_0\omega_1$, ${}_0\omega_2$ will be connected with ω_1 and ω_2 by relations

$$\omega_1 = {}_0\omega_1 + d\omega_1$$

$$\omega_2 = {}_0\omega_2 + d\omega_2$$

where $$\frac{d\omega_1}{{}_0\omega_1} = \frac{dH}{H_0} - \frac{2d\tau}{\tau_0}$$

and $$\frac{d\omega_2}{{}_0\omega_2} = \frac{dH}{H_0} - \frac{3d\tau}{\tau_0}$$

so that

$$\left.\begin{aligned} \omega_1 &= {}_0\omega_1\left(1 + \frac{dH}{H_0} - \frac{2d\tau}{\tau_0}\right) \\ \text{and} \quad \omega_2 &= {}_0\omega_2\left(1 + \frac{dH}{H_0} - \frac{3d\tau}{\tau_0}\right) \end{aligned}\right\} \qquad \qquad (76)$$

9. *Dehra to Mussooree*:

In this case $h_i = 6930$ feet $\quad H_0 = 27\cdot6 \quad E_0 = 5^\circ\ 18'\ 40''$

$l = 9\cdot454$ miles $\quad \tau_0 = 459\cdot4 + 85$

$g_0 = 979\cdot06 \quad = 544\cdot4$

$g_s = 980\cdot62 \quad t_0 = 85$, being the actual Fahrenheit temperature.

Then $${}_0\omega_1 = 3\cdot475 \times 10^4 \left(1 + \frac{h_b}{R}\right)\left(\frac{g_0}{g_s}\right)^2 \cdot \frac{lH_0}{\tau_0^2} = 30''\cdot5$$

$${}_0\omega_2 = -\,2\cdot62 \times 10^{-3} \times \frac{s \sin E_0}{\tau_0} \cdot {}_0\omega_1 = -\,0''\cdot68$$

Mussooree to Dehra :

$h_b = 2234$ $\quad H_0 = 23\cdot3$ $\quad E_0 = -\,5°\,26'\,20''$

$l = 9\cdot454$ $\quad \tau_0 = 459\cdot4 + 60$

$g_0 = 978\cdot79$ $\quad = 519\cdot4$

$g_s = 980\cdot62$ $\quad t_0 = 60$

from which we compute

$${}_0\omega_1 = 28''\cdot3 \qquad {}_0\omega_2 = +\,0''\cdot68$$

The remaining data and the deduction of refraction are exhibited in tables IX and XA and XB.

TABLE IX.

Serial No.	Date	Year	Instrument used	No. of observations	Observed H	H corrected for temperature	t	δt from Table I	Humidity w	$d\tau = t + w\delta t - t_0$	$\frac{dH}{H_0}$	$\frac{2d\tau}{\tau_0}$	$\omega_1 = {}_0\omega_1\left(1 + \frac{dH}{H_0} - \frac{2d\tau}{\tau_0}\right)$	$\omega_2 = {}_0\omega_2\left(1 + \frac{dH}{H_0} - \frac{3d\tau}{\tau_0}\right)$	$\omega = \omega_1 + \omega_2$
	Dehra to Mussooree.														
1	Oct. 28 — Nov. 3	1905	8″ No. 956	14	27·75	27·61	81·9	8·1	(·35)	− 0·3	+ ·0004	− ·0011	30·5	− ·7	29·8
2	March 1 — 5	1906	do.	10	27·84	27·72	75·2	6·3	·45	− 7·0	+ ·0043	− ·0257	31·4	− ·7	30·7
3	October 23 — 29	1906	do.	13	27·75	27·60	83·8	8·6	·37	+ 2·0	·0000	+ ·0073	30·3	− ·7	29·6
4	March 4 — 9	1907	12″ No. 111	12	27·81	27·69	75·7	6·5	·42	− 6·6	+ ·0033	− ·0242	31·3	− ·7	30·6
5	November 7 — 14	1907	do.	17	27·83	27·69	80·8	7·8	·33	− 1·6	+ ·0033	− ·0059	30·8	− ·7	30·1
6	March 30 — April 3	1908	8″ No. 956	12	27·65	27·47	95·3	12·8	·20	+ 12·9	− ·0047	+ ·0474	28·9	− ·6	28·3
7	October 22 — 31	1908	do.	21	27·64	27·50	82·0	8·1	·35	− 0·2	− ·0036	− ·0007	30·4	− ·7	29·7
8	February 8 — 12	1909	8″ No. 1311	10	27·85	27·74	69·6	5·3	·33	− 13·6	+ ·0051	− ·0499	32·2	− ·6	31·6
9	March 22 — 29	1909	do.	13	27·79	27·63	86·7	9·5	·18	+ 3·4	+ ·0011	+ ·0125	30·2	− ·7	29·5
	Mussooree to Dehra.														
1	November 7 — 13	1905	8″ No. 956	14	23·54	23·48	58·2	4·0	(·48)	+ 0·1	+ ·0077	+ ·0004	28·5	+ ·7	29·2
2	April 10 — 26	1906	do.	24	23·40	23·31	68·9	6·0	·34	+ 10·9	+ ·0004	+ ·0420	27·1	+ ·6	27·7
3	Oct. 11 — Nov. 7	1906	do.	21	23·52	23·45	60·6	4·5	·65	+ 3·5	+ ·0064	+ ·0135	28·1	+ ·7	28·8
4	April 8 — 23	1907	12″ No. 111	14	23·43	23·36	61·2	4·6	·48	+ 3·4	+ ·0026	+ ·0131	28·0	+ ·7	28·7
5	October 21 — 31	1907	do.	20	23·49	23·42	62·1	4·7	·39	+ 3·9	+ ·0052	+ ·0150	28·0	+ ·7	28·7
6	March 9 — 20	1908	8″ No. 956	12	23·46	23·40	58·0	4·0	·32	− 0·7	+ ·0043	− ·0027	28·5	+ ·7	29·2
7	October 22 — 31	1908	8″ No. 1316	18	23·44	23·37	59·7	4·3	·41	+ 1·5	+ ·0030	+ ·0058	28·2	+ ·7	28·9
8	April 6 — 16	1909	8″ No. 1311	15	23·44	23·35	68·0	5·9	·28	+ 9·7	+ ·0021	+ ·0374	27·3	+ ·6	27·9

In tables XA and XB the comparison of the results found in table IX with those found by means of (73) is carried out.

TABLE XA.

Dehra to Mussooree.

Serial No.	1	2	3	4	5	6	7	8	9
I	4′ 10½″	4′ 10½″	4′ 10½″	4′ 7″	4′ 7″	4′ 10½″	5′ 0″	4′ 10½″	4′ 10½″
S	2 5½	2 5½	2 5½	2 5½	2 5½	2 5½	1 7½	2 5½	2 5½
$I-S$	2′·4	2·4	2·4	2·1	2·1	2·4	3·4	2·4	2·4
dE	9″·9	9·9	9·9	8·7	8·7	9·9	14·0	9·9	9·9
E	5° 18′ 43″·9	44·2	42·7	45·8	43·3	43·3	43·3	49·7	46·6
$\delta-a$	−5° 18′ 16″·5	16·5	16·5	16·5	16·5	16·5	16·5	16·5	16·5
Deduced refraction	37″·3	37·6	36·1	38·0	35·5	36·7	40·8	43·1	40·0
ω from table IX	29″·8	30·7	29·6	30·6	30·1	28·3	29·7	31·6	29·5
Difference	+ 7·5	6·9	6·5	7·4	5·4	8·4	11·1	11·5*	10·5
Month	Oct. Nov.	Mar.	Oct.	Mar.	Nov.	Mar. April	Oct.	Feb.	Mar.

Spring mean difference + 8″·4
Autumn mean difference + 7·6

TABLE XB.

Mussooree to Dehra.

Serial No.	1	2	3	4	5	6	7	8
I	4′ 10½″	4′ 10½″	4′ 10½″	4′ 7″	4′ 7″	4′ 10½″	4′ 10½″	4′ 10½″
S	2 5½	2 5½	2 5½	2 5½	2 5½	2 5½	3 0½	2 5½
$I-S$	2′·4	2·4	2.4	2·1	2·1	2·4	1·8	2·4
dE	9″·9	9·9	9·9	8·7	8·7	9·9	7·4	9·9
E	−5° 26′ 20″·6	21·8	21·2	21·1	22·3	19·7	12·4	18·5
$\delta-a$	+5 26′ 31″·1	31·1	31·1	31·1	31·1	31·1	31·1	31·1
Deduced refraction	20″·4	19·2	19·8	18·7	17·5	21·3	26·1	22·5
ω from table IX	29″·2	27·7	28·8	28·7	28·7	29·2	28·9	27·9
Difference	−8·8	−8·5	−9·0	−10·0	−11·2	−7·9	−2·8	−5·4
Month	Nov.	April	Oct. Nov.	April	Oct.	March	Oct.	April

Spring mean difference − 8″·0
Autumn mean difference − 8·0

* This is not included in the Spring mean.

10. From tables XA, XB we see that the refraction computed by (63) is too small at Dehra and too great at Mussoorce to agree with the deductions made from the spirit-levelled height, corrected, as well as possible from available data, for plumb-line deflection. We have computed a mean refraction of $30''\cdot0$ at Dehra and $28''\cdot6$ at Mussoorce: and from spirit-levelling we deduced $38''\cdot0$ at Dehra and $20''\cdot6$ at Mussooree. That the refraction at Dehra should be practically double of what it is at Mussooree seems incredible in face of observation data. It may be stated in passing that the old method of dealing with refraction was to assume *equal* angles of refraction at the two ends of a ray.

The sum of the refraction angles at the two ends of the ray is a known quantity for if the refraction at one end is known the height can be computed: and then the refraction at the other end can be deduced from the known height. Accordingly we know, beyond doubt, that the sum of the refractions must be $38\cdot0+20\cdot6=58''\cdot6$. The value from table IX is $30\cdot0+28\cdot6=58''\cdot6$ which is in absolute agreement. So the computed refraction satisfies exactly the only *certain* test that we have. It is to be inferred that the other test fails because we have not accurately corrected the spirit-levelled height to a height above the spheroid.

In consideration of the fact that theodolites have a graduation error, this agreement of the sum of refractions found above is remarkable: and it is to be attributed to the fact that no less than four instruments have been used. With our 12-inch theodolites we have frequently found differences of $7''$ in means of measures of horizontal angles, according as different zeros are used. This represents a graduation error of $\pm\ 3''\cdot5$. We cannot expect the vertical circle to be better graduated than the horizontal circle: and as the vertical circle is not moveable, we have only observed on two zeros for vertical angles. A very possible error in the case of a 12-inch instrument, then is $3''$: and with an 8-inch instrument, we need not be surprised with as much as $5''$ error.

11. The conclusion arrived at is that, in consideration of the rapidly varying plumb-line deflection between Dehra and Mussooree, the height determined by vertical angles corrected for refraction should be accepted.

The difference of height of Mussooree and Dehra as determined from observations from Dehra exceeds that deduced from spirit-levelling, corrected for plumb-line variations as in § 6 by $\frac{8\cdot0}{4\cdot13}=1\cdot94$ feet; from observations from Mussooree the excess is the same. Accordingly we get a height $4707\cdot7$, greater by $1\cdot9$ feet than that found in § 6. We shall consider this to be the correct difference of height of Mussooree above Dehra in all that follows.

12. It is necessary first of all to revise the values of a, a' found for Nojli–Mussooree in § 7. By (74) the correction for $1\cdot94$ feet is $1\cdot94 \times 0''\cdot851 = 1''\cdot7$ and is positive, so that in this case

$$\left.\begin{aligned} \delta - a &= -\ 1^\circ\ \ 5'\ 58''\cdot3 \\ \delta' - a' &= +\ 1^\circ\ 45'\ 25''\cdot6 \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots \quad (77)$$

corresponding to a difference of height of $6084\cdot4$ feet.

We will proceed in the same way for the Nojli—Mussooree observations as we have already done in § 9 for the Dehra—Mussooree observations, making use of (77). In this case we shall have much larger refraction angles, which will make the instrumental graduation errors relatively less important: and, in addition, the slight uncertainty as to the height of Mussooree will be of very little significance.

There are two stations at Nojli which are close together. The distance of Nojli R.S. from Mussooree R.S. is 45·928 miles, while that of Nojli Tower S. is 45·931 miles. The difference of these distances is only ·003 mile or 16 feet, so that it will be convenient to treat these two stations as one and apply a small correction on this account. Now the elevation of Mussooree at Nojli is 1° 8′ 30″ approximately: so that on this account the height is decreased by 16 sin 1° 8′ 30″ = 0·32 foot which corresponds to a change of angular elevation of ·32 × ·851 = 0″·27 ≑ 0″·3 = ϵ. We will combine this with the value of $\delta - a$ and $\delta' - a'$ for the Tower station and then have the following values:—

$$\left.\begin{aligned} \delta - a + \epsilon &= -1^\circ\ 5'\ 58''\cdot 0 \\ \delta' - a' - \epsilon &= +1^\circ\ 45'\ 25''\cdot 3 \end{aligned}\right\}$$

Using this modification we can treat the two stations at Nojli as if they were on the same vertical.

13. *Nojli to Mussooree.*

$h_b = 6930$	$H_0 = 29\cdot 0$	$E_0 = 1^\circ\ 8'\ 30''$
$l = 45\cdot 928$	$\tau_0 = 459\cdot 4 + 70$	
$g_0 = 979\cdot 14$	$= 529\cdot 4$	
$g_s = 980\cdot 62$	$t_0 = 70$	

whence

$$_0\omega_1 = 164''\cdot 7 \qquad _0\omega_2 = -3''\cdot 96$$

Mussooree to Nojli.

$h_b = 887$	$H_0 = 23\cdot 3$	$E_0 = -1^\circ\ 42'\ 40''$
$l = 45\cdot 928$	$\tau_0 = 459\cdot 4 + 60$	
$g_0 = 978\cdot 79$	$= 519\cdot 4$	
$g_s = 980\cdot 62$	$t_0 = 60$	

whence

$$_0\omega_1 = 137\cdot 3 \qquad _0\omega_2 = +4''\cdot 93$$

The remaining data and the deduction of the refraction are exhibited in tables XI and XIIA and XIIB.

In tables XIIA and XIIB the heights of instruments and of signals are taken account of.

TABLE XI.

Serial No.	Date	Year	Instrument used	No. of observations	Observed H	H corrected for temperature	t	δt from Table I	Humidity w	$d\tau = t + w\delta t - t_0$	$\frac{dH}{H_0}$	$\frac{2d\tau}{\tau_0}$	$\omega_1 = {}_0\omega_1\left(1+\frac{dH}{H_0}-\frac{2d\tau}{\tau_0}\right)$	$\omega_2 = {}_0\omega_2\left(1+\frac{dH}{H_0}-\frac{3d\tau}{\tau_0}\right)$	$\omega = \omega_1 + \omega_2$
	Nojli to Mussooree. *														
1	December 1—8	1905	8″ No. 956	16	29·25	29·12	75·5	6·2	·33	7·5	·0041	·0283	160·7	−3·8	156·9
2	March 14—16	1906	do.	9	29·16	29·02	80·6	7·3	·31	12·9	·0007	·0486	156·8	−3·7	153·1
3	Nov. 22—Dec. 16	1906	do.	9	29·27	29·14	76·6	6·5	·33	8·7	·0048	·0328	160·1	−3·8	156·3
4	March 20—23	1907	12″ No. III	4	29·13	29·00	75·4	6·2	·39	7·8	·0000	·0294	159·9	−3·8	156·1
5	January 16—26	1909	8″ No. 956 and 1311	10	29·20	29·09	69·4	4·9	·36	1·2	·0031	·0045	164·5	−3·9	160·6
6	March 9—12	1909	8″ No. 1311	6	29·02	28·87	84·7	8·5	·16	16·1	− ·0045	·0607	154·0	−3·6	150·4
7	March 17—20	1906	8″ No. 956	9	29·02	28·88	81·6	7·7	·27	13·8	− ·0041	·0520	155·5	−3·6	151·9
8	March 25—28	1907	12″ No. III	9	29·10	28·96	80·0	7·3	·28	12·0	− ·0014	·0453	157·0	−3·7	153·3
9	January 28	1909	8″ No. 1311	6	29·19	29·10	61·5	3·7	·40	− 7·0	·0035	− ·0264	169·6	−4·1	165·5
10	March 13—16	1909	do.	6	29·07	28·92	83·3	8·1	·17	14·7	− ·0028	·0555	155·1	−3·6	151·5
	Mussooree to Nojli. †														
1	November 12	1905	8″ No. 956	1	23·48	23·41	60·0	4·3	(·47)	2·0	·0047	·0077	136·9	+4·9	141·8
2	April 19—25	1906	do.	5	23·38	23·29	68·3	6·0	·35	10·4	− ·0004	·0400	131·7	4·6	136·3
3	October 31—Nov. 7	1906	do.	7	23·49	23·42	60·5	4·4	·52	2·8	·0051	·0108	136·5	4·9	141·4
4	April 8—20	1907	12″ No. III	12	23·44	23·37	61·1	4·5	·51	3·4	·0030	·0131	135·9	4·9	140·8
5	March 11—April 21	1908	8″ No. 956	10	23·51	23·43	66·4	5·6	·26	7·9	·0056	·0304	133·9	4·7	138·6
6	October 24—31	1908	8″ No. 1316	11	23·43	23·37	59·3	4·4	·44	1·2	·0030	·0046	137·1	4·9	142·0
7	April 7—16	1909	8″ No. 1311	12	23·43	23·35	67·9	5·9	·29	9·6	·0021	·0370	132·5	4·7	137·2
8	April 19—25	1906	8″ No. 956	6	23·39	23·30	69·8	6·4	·35	12·0	·0000	·0462	130·9	4·6	135·5
9	November 1—7	1906	do.	7	23·49	23·42	59·8	4·3	·53	2·1	·0051	·0081	136·9	4·9	141·8
10	April 8—20	1907	12″ No. III	12	23·44	23·37	61·1	4·5	·51	3·4	·0030	·0131	135·9	4·9	140·8
11	November 21,22	1907	do.	4	23·46	23·40	57·2	4·0	·44	− 1·0	·0043	− ·0039	138·4	5·0	143·4
12	March 9—April 21	1908	8″ No. 956	14	23·48	23·41	62·5	4·8	·27	3·8	·0047	·0146	135·9	4·9	140·8
13	October 24—31	1908	8″ No. 1316	12	23·43	23·37	59·3	4·4	·44	1·2	·0030	·0046	137·1	4·9	142·0
14	April 6—16	1909	8″ No. 1311	13	23·44	23·36	67·7	5·9	·28	9·4	·0026	·0362	132·7	4·7	137·4

* Nos. 1—6 from Refraction station: Nos. 7—10 from Tower station.

† Nos. 1—7 to Nojli Refraction station: Nos. 8—14 to Tower station.

TABLE XII A.

Nojli to Mussooree.

	Nojli Refraction Station						Nojli Tower Station			
Serial No.	1	2	3	4	5	6	7	8	9	10
I	4′·9	4′·9	4′·9	4′·6	4′·9	4′·9	54′·7	54′·4	54′·7	54′·7
S	1·6	1·6	1·6	1·6	1·6	1·6	1·6	1·6	1·6	1·6
I−S	3′·3	3·3	3·3	3·0	3·3	3·3	53·1	52·8	53·1	53·1
d E	2″·8	2·8	2·8	2·5	2·8	2·8	45·2	44·9	45·2	45·2
E	1° 8′ 29″·8	26·9	25·4	23·9	49·5	19·9	1° 7′ 45″·4	46·0	63·8	44·8
δ−α	−1° 5′ 58″·3	58·3	58·3	58·3	58·3	58·3	−1° 5′ 58″·0	58·0	58·0	58·0
Deduced refraction = Ω	154·3	151·4	149·9	148·1	174·0	144·4	152·6	152·9	171·0	152·0
ω from table XI	156·9	153·1	156·3	156·1	160·6	150·4	151·9	153·3	165·5	151·5
Ω−ω	−2·6	−1·7	−6·4	−8·0	+13·4	−6·0	+0·7	−0·4	+5·5	+0·5
Month	Decr.	March	Nov. Decr.	March	January	March	March	March	January	March

Nojli Refraction Station: March mean Ω − ω − 5″·2; Nov. and Dec. mean „ − 4·5; January „ +13·4

Nojli Tower Station: March mean Ω−ω + 0″·3; January „ + 5·5

TABLE XII B.

Mussooree to Nojli.

	Nojli Refraction Station							Nojli Tower Station						
Serial No.	1	2	3	4	5	6	7	8	9	10	11	12	13	14
I	4′·9	4′·9	4′·9	4′·6	4′·9	4′·9	4′·9	4′·9	4′·9	4′·6	4′·6	4′·9	4′·9	4′·9
S	1·6	1·6	1·6	1·6	1·6	1·6	1·6	52·7	52·7	52·7	53·5	53·5	53·5	53·5
I−S	3·3	3·3	3·3	3·0	3·3	3·3	3·3	−47·8	−47·8	−48·1	−48·9	−48·6	−48·6	−48·6
d E	2″·8	2·8	2·8	2·6	2·8	2·8	2·8	−40·7	−40·7	−40·9	−41·6	−41·3	−41·3	−41·3
E	−1° 42′ 54″·0	65·5	54·3	63·4	63·5	44·4	61·9	23·8	8·6	20·4	3·3	13·2	*−1·3	16·7
δ−α	+1 45 25·6	25·6	25·6	25·6	25·6	25·6	25·6	25·3	25·3	25·3	25·3	25·3	25·3	25·3
Deduced refraction = Ω	154·4	142·9	154·1	144·8	144·9	164·0	146·5	140·8	156·0	144·0	160·4	150·8	165·3	147·3
ω from table XI	141·8	136·3	141·4	140·8	138·6	142·0	137·2	135·5	141·8	140·8	143·4	140·8	142·0	137·4
Ω−ω	12·6	6·6	12·7	4·0	6·3	22·0	9·3	5·3	14·2	3·2	17·0	10·0	23·3	9·9
Month	Nov.	April	Oct. Nov.	April	March April	Oct.	April	April	Nov.	April	Nov.	March	Oct.	April

Nojli Refraction Station: March and April mean Ω−ω 6″·6; October and November mean „ 15·8

Nojli Tower Station: March and April mean Ω−ω 7″·1; October and November mean „ 18·2

March and April mean Ω−ω 6″·8
October and November mean „ 17·0

* This means an observed depression of 1° 42′ 0″ −1″·3 = 1° 41′ 58″·7.

14. There are several points to be noted in the results of tables XIIA, XII B. From table XIIA we see that the values of $\Omega-\omega$ at Nojli Tower are smaller and more consistent than those at Nojli R.S. Considering them by season we have remarkable accordance of the March results at Nojli Tower, and the mean value of $\Omega-\omega$ in this case is $+0''\cdot3$: while at Nojli R.S. the accordance is fairly satisfactory and the mean value of $\Omega-\omega$ is $-5''\cdot2$. In table XIIB, as might be expected, there is very little difference between the results found for Nojli R.S. and those for Nojli Tower. Combining the two cases we find the mean value of $\Omega-\omega$ determined from March and April observations is $+6''\cdot8$: while the value from October and November observations is $+17''\cdot0$. Again in table XIIA we see that the observations in January yield values of $\Omega-\omega$ of $+13''\cdot4$ and $+5''\cdot5$ at Nojli R.S and Nojli Tower respectively.

The conclusions which may be drawn are:

(1). That ω represents the refraction more closely in the months of March and April than it does in January or October and November.

(2). That in January the refraction at Nojli is largely different from what it is in the other months in which we have observations: but that this abnormality rapidly decreases with height above the ground level.

(3). That the observations at Nojli Tower are much more accordant than those at Nojli R. S.; and that ω very closely represents the refraction in spring at Nojli Tower.

It is to be expected that the adiabatic gradient should be more nearly attained with a rising temperature, such as occurs in spring, than with a falling temperature, such as occurs in autumn. That the spring results are more nearly represented by ω than the other results, then, is satisfactory. Just as there is a diurnal change in refraction, with a minimum at about 2 P.M. so there is also a seasonal variation with a minimum in the spring. Observations from Nojli Tower naturally would be expected to be superior to those from Nojli R. S.: for the Tower stands 50 feet above the ground and irregularities of the temperature gradient are certain to be felt much less on this account. A simple law of temperature which represents, in the main, the temperature between Nojli and Mussooree cannot be expected to represent the temperature *very close* to the ground, where it is liable to being disturbed by radiation. This was foreseen when the observations at two heights at Nojli were initiated.

To represent the refraction in the Nojli R. S.—Mussooree ray, it is clear that some temperature readings at several heights between ground level and 50 feet higher are desirable. The facts that in spring the values of the refraction given by ω—corresponding to the adiabatic gradient of temperature—satisfy the observations practically perfectly at Nojli Tower, while they are in defect by $5''\cdot2$ at Nojli R. S., show that the temperature gradient in the first 50 feet above the ground must have been much steeper than the adiabatic gradient; which prevailed from that height onwards for some considerable height. This of course is another fact which can be foreseen by considering the effects of the radiation of heat from the hot earth in the early afternoon when convection is sure to be occurring. That the adiabatic gradient did not persist as far as the height of Mussooree is made clear by the fact that the value of ω at Mussooree is less than that of Ω by a mean amount of $6''\cdot8$ in the spring.

It was not expected that the adiabatic gradient would often be reached, as measurements of fall of temperature made by baloons and kites do not show this to occur. These observations, however, have not been made at the time (say 2 P.M.) of minimum refraction when a steeper gradient is likely to occur than what has been observed at 8 P.M. Meteorologists presumably wish to find an average value of the diurnally varying gradient, and do not in the first place wish to find the *maximum* gradient. On the other hand refraction results may give the gradient at any instant in a way which will be useful to meteorology.

15. We will now consider the Mussooree-Nag Tiba observations.

Mussooree to Nag Tiba.

$h_b = 9915$ $\quad H_0 = 23 \cdot 3$ $\quad E_0 = 3^\circ\ 12'\ 0''$

$l = 9 \cdot 886$ $\quad \tau_0 = 459 \cdot 4 + 60$

$g_0 = 978 \cdot 79$ $\quad = 519 \cdot 4$

$g_s = 980 \cdot 62$ $\quad t_0 = 60$

from which we compute

$${}_0\omega_1 = 29'' \cdot 6 \qquad {}_0\omega_2 = -\ 0'' \cdot 44$$

Nag Tiba to Mussooree.

$h_b = 6930$ $\quad H_0 = 20 \cdot 9$ $\quad E_0 = -\ 3^\circ\ 20'\ 0''$

$l = 9 \cdot 886$ $\quad \tau_0 = 459 \cdot 4 + 60$

$g_0 = (978 \cdot 62)$ $\quad = 519 \cdot 4$

$g_s = 980 \cdot 62$ $\quad t_0 = 60$

from which we compute

$${}_0\omega_1 = 26'' \cdot 6 \qquad {}_0\omega_2 = +\ 0'' \cdot 41$$

The deduction of refraction is carried out in table XIII.

TABLE XIII.

Serial No.	Date	Year	Instrument used	No. of observations	Observed H	H corrected for temperature	t	δt from Table I	Humidity w	$d\tau = t + w\delta t - t_0$	$\frac{dH}{H_0}$	$\frac{2d\tau}{\tau_0}$	$\omega_1 = {}_0\omega_1\left(1+\frac{dH}{H_0}-\frac{2d\tau}{\tau_0}\right)$	$\omega_2 = {}_0\omega_2\left(1+\frac{dH}{H_0}-\frac{3d\tau}{\tau_0}\right)$	$\omega = \omega_1 + \omega_2$
	Mussooree to Nag Tiba.														
1	April 14 — 19	1906	8″ No. 956	8	23·31	23·22	68·4	6·0	·32	10·3	− ·0035	+ ·0396	28·3	−0·4	27·9
2	April 24 — 26	1906	do.	6	23·43	23·34	72·8	7·0	·36	15·3	+ ·0017	+ ·0589	28·1	−0·4	27·7
3	Oct. 18 — Nov. 7	1906	do.	16	23·50	23·43	60·9	4·5	·55	3·4	+ ·0057	+ ·0131	29·4	−0·4	29·0
4	April 8 — 23	1907	12″ No. III	17	23·44	23·37	62·2	4·7	·47	4·4	+ ·0030	+ ·0169	29·2	−0·4	28·8
5	Oct. 22 — Nov. 22	1907	do.	28	23·49	23·42	60·1	4·3	·40	1·8	+ ·0052	+ ·0069	29·5	−0·4	29·1
6	March 9 — April 21	1908	8″ No. 956	16	23·48	23·40	65·8	5·4	·27	7·3	+ ·0043	+ ·0281	28·9	−0·4	28·5
7	October 21 — 31	1908	8″ No. 1316	20	23·44	23·37	59·9	4·3	·41	1·7	+ ·0030	+ ·0065	29·5	−0·4	29·1
8	April 6 — 29	1909	8″ No. 1311	18	23·44	23·36	68·3	6·0	·20	9·5	+ ·0026	+ ·0365	28·6	−0·4	28·2
	Nag Tiba to Mussooree.														
1	May 1 — 7	1906	8″ No. 956	14	21·03	20·96	64·4	5·7	·31	6·2	+ ·0029	+ ·0238	26·0	+0·4	26·4
2	May 10 — 15	1907	do.	14	21·07	21·01	60·8	5·0	·33	2·5	+ ·0053	+ ·0096	26·5	+0·4	26·9
3	April 27 — May 1	1908	do.	11	21·04	20·98	61·9	5·2	·34	3·7	+ ·0038	+ ·0142	26·3	+0·4	26·7
4	October 5 — 14	1908	8″ No. 1316	15	21·00	20·95	57·0	4·3	·52	−0·8	+ ·0024	− ·0031	26·7	+0·4	27·1
5	May 6 — 11	1909	8″ No. 1311	10	broken	(20·98)*	65·6	6·0	·26	7·2	+ ·0038	+ ·0277	26·0	+0·4	26·4

* Estimated.

As we have no spirit-levelled value of the height of Nag Tiba we have only one check on our deduced refraction—that the difference of height of Nag Tiba and Mussooree should be the same when deduced from observations at either end: or, what is the same thing, that

$$90^\circ + a + 90^\circ + a' + \chi = 180$$

that is $$a + a' + \chi = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (78)$$

To compute χ we have by (69)

$$\log \chi = \log c - \log R + \log \operatorname{cosec} 1''$$

and from (71) and table IV, with $A = 29^\circ\ 18'$ (*see* table VII), and latitude $30^\circ\ 31'$

$$\frac{R}{100} = 208886 - 517 \cos 58^\circ\ 36'$$
$$= 208886 - 270$$
$$= 208616$$
$$\log \chi = 4{\cdot}71768 + 5{\cdot}31442 - 7{\cdot}31935$$
$$= 2{\cdot}71275$$
$$\chi = 516''{\cdot}1 = 8'\ 36''{\cdot}1.$$

16. In tables XIV A and XIV B the heights of instruments and of signals are taken account of. Here we have $dE = (I - S)\ 3''{\cdot}95$.

TABLE XIV A.

Mussooree to Nag Tiba.

Serial No.	1	2	3	4	5	6	7	8
I	′ 4·9	′ 4·9	′ 4·9	′ 4·6	′ 4·6	′ 4·9	′ 4·9	′ 4·9
S	4·2	5·0	5·0	5·0	5·0	5·0	10·5	5·0
$I-S$	′ 0·7	−0·1	−0·1	−0·4	−0·4	−0·1	−5·6	−0·1
dE	″ 2·8	−0·4	−0·4	−1·6	−1·6	−0·4	−22·1	−0·4
E	° ′ ″ 3 12 5·1	9·8	11·9	12·5	17·8	12·5	43·4	11·7
δ	″ 45·7	45·7	45·7	45·7	45·7	45·7	45·7	45·7
$E_c = E + dE + \delta$	° ′ ″ 3 12 53·6	55·1	57·2	56·6	61·9	57·8	67·0	57·0
ω from table XIII	″ 27·9	27·7	29·0	28·8	29·1	28·5	29·1	28·2
$E_c - \omega$	° ′ ″ 3 12 25·7	27·4	28·2	27·8	32·8	29·3	37·9	28·8
Month	April	April	Oct. Nov.	April	Oct. Nov.	March April	Oct.	April

March and April mean $E_c - \omega$ $3^\circ\ 12'\ 28''{\cdot}1$.

October and November mean $E_c - \omega$ $3^\circ\ 12'\ 33''{\cdot}0$.

Before taking the mean for March and April numbers 1 and 2 have been combined as they refer to observations of the same year (*see* table XIII).

TABLE XIV B.

Nag Tiba to Mussooree.

Serial No.	1	2	3	4	5
I	4′·9	4·9	4·9	4·9	4·9
S	2·5	2·5	2·5	2·5	2·5
$I-S$	2′·4	2·4	2·4	2·4	2·4
dE	9″·5	9·5	9·5	9·5	9·5
E	−3° 19′ 66″·0	63·6	64·4	51·0	62·5
δ*	−39″·5	39·5	39·5	39·5	39·5
$E_c = E + dE + \delta$	−3° 20′ 36″·0	33·6	34·4	21·0	32·5
ω from table XIII	26·4	26·9	26·7	27·1	26·4
$E_c - \omega$	−3° 20′ 62″·4	60·5	61·1	48·1	58·9
Month	May	May	April	Oct.	May

April and May mean $E_c - \omega$ $\quad - 3° 21'\ 0''\cdot 7$

October $E_c - \omega$ $\quad - 3° 20' 48''\cdot 1$

If ω represents the actual refraction by (78) we should have

$$(E_c - \omega) \text{ from table XIVA} + \chi + (E_c - \omega) \text{ from table XIVB} = 0,$$

and putting in the values of mean spring we get

$$3° 12' 28''\cdot 1 + 8' 36''\cdot 1 - 3° 21' 0''\cdot 7$$
$$= 3° 21' 4''\cdot 2 - 3° 21' 0''\cdot 7$$
$$= 3''\cdot 5$$

We shall discuss this residual $3''\cdot 5$ later, in §§ 21, 22, but will first proceed with the observations from Nag Tiba to Nojli.

17. *Nag Tiba to Nojli.*

$h_b = 887$ $\qquad H_0 = 20\cdot 90$ $\qquad E_0 = -\ 2° 6' 0''$

$l = 55\cdot 809$ $\qquad \tau_0 = 459\cdot 4 + 60$

$g_0 = (978\cdot 62)$† $\qquad = 519\cdot 4$

$g_s = 980\cdot 62$ $\qquad t_0 = 60$

whence we compute

$${}_0\omega_1 = 149\cdot 6 \qquad {}_0\omega_2 = +\ 8''\cdot 14$$

* Interpolated value: *see* table V.

† Exterpolated value.

The deduction of refraction is carried out in table XV.

TABLE XV.

Nag Tiba to Nojli.

Serial No.	Date	Year	Instrument used	No. of observations	Observed H	H corrected for temperature	t	δt from Table I	Humidity w.	$d\tau = t + w\delta\tau - t_0$	$\frac{dH}{H_0}$	$\frac{2d\tau}{\tau_0}$	$\omega_1 = {}^0\omega_1\left(1 + \frac{dH}{H_0} - \frac{2d\tau}{\tau_0}\right)$	$\omega_2 = {}^0\omega_2\left(1 + \frac{dH}{H_0} - \frac{3d\tau}{\tau_0}\right)$	$\omega = \omega_1 + \omega_2$
1*	May 1—7	1906	8″ No. 956	7	21·02	20·95	63·5	5·1	·37	5·4	·0024	·0208	146·8	7·9	154·7
2	May 1—7	1906	do.	3	21·03	20·96	64·8	5·8	·34	6·8	·0029	·0262	146·1	7·8	153·9
3*	May 13	1907	do.	2	21·11	21·04	64·2	5·5	·29	5·8	·0067	·0223	147·3	7·9	155·2
4*	April 30	1908	do.	1	21·08	21·03	57·0	4·3	·40	−1·3	·0062	−·0050	151·3	8·3	159·6
5	April 30	1908	do.	1	21·08	21·03	57·0	4·3	·40	−1·3	·0062	−·0050	151·3	8·3	159·6

In table XVI the heights of instruments and of signals are taken account of. Here we have $dE = (I - S)\ 0''{\cdot}700$.

TABLE XVI.

Nag Tiba to Nojli.

Serial No.	1	2	3	4	5
I	4′·9	4′·9	4′·9	4′·9	4′·9
S	50′·3+2′·4	1′·6	50′·3+2′·4	50′·3+3′·2	1′·6
$I-S$	−47′·8	+ 3′·3	−47′·8	−48′·6	+ 3′·3
dE	−33″·5	+ 2·3	−33·5	−34·0	+ 2·3
Correction for distance to tower	− 0·4	...	− 0·4	− 0·4	...
δ	−39·6	−39·6	−39·6	−39·6	−39·6
E	−2° 6′ 12″·7	−50·2	− 6·7	− 5·1	−41·1
E_c	−2° 7′ 26·2	27·5	20·2	19·1	18·4
ω from table XV	2′ 34″·7	33·9	35·2	39·6	39·6
$E_c - \omega$	−2° 9′ 60″·9	61″·4	55·4	58·7	58·0
Month	May	May	May	April	April

May 1906, mean $E_c - \omega$		− 2° 9′ 61″·2
May 1907	,,	55″·4
April 1908	,,	58″·4
Combined mean		− 2° 9′ 58″·3

* To Nojli Tower.

In this case we have from table IV corresponding to a mean latitude 30° 14′,

$$\frac{R}{100} = 208881 - 520 \cos 62^\circ$$
$$= 208637$$
$$\log \chi = \log c - \log R + \log \operatorname{cosec} 1''$$
$$\chi = 2913\cdot 2$$
$$= 48'\ 33''\cdot 2$$

Now
$$h = c\left(1 + \frac{h_a}{R}\right) \sin\left(a + \frac{\chi}{2}\right) \sec\,(a + \chi)$$

whence if we assume that the refraction is ω and $E_c - \omega = a$

we have $h = -\,c\,(1 + 4\cdot 8.10^{-4}) \sin 1^\circ\ 45'\ 41''\cdot 8 \sec 1^\circ\ 21'\ 25''\cdot 2$

or $h = -\ 9065\cdot 3$

18. We will next determine the height of Nag Tiba above Mussooree and for the present will suppose that the discrepancy of 3″·5 found at the end of § 16 is to be distributed equally between the two refraction angles. In doing this we are not likely to be in error by as much as 1″ as will appear later (*see* § 24). This corresponds to about 3 inches in height: which in turn corresponds to an error of 0″·2 in elevation of Nojli. Then we have

$$a = E_c - \omega - 1''\cdot 75 = 3^\circ\ 12'\ 26''\cdot 35$$
$$\chi = 8'\ 36''\cdot 1$$
$$h = c\,(1 + 3\cdot 3 \times 10^{-4}).\ \sin 3^\circ\ 16'\ 44''\cdot 4 \sec 3^\circ\ 21'\ 2''\cdot 4$$
$$= 2991\cdot 9$$

∴ the height of Nag Tiba above Nojli is $2991\cdot 9 + 6084\cdot 4 = 9076\cdot 3$

But the height we deduced from table XVI is 9065·3 which is accordingly too small by 11·0 feet: *i.e.*, we computed the dip of Nojli from Nag Tiba too small by 7″·7. If we have assigned the true deflection to the plumb-line at Nag Tiba this must be the difference $\Omega - \omega$. If on the other hand the assumed deflection is too large by ϵ in azimuth 30° (which is roughly the azimuth of both Mussooree and Nojli) we have for the Nag Tiba—Nojli ray

$$\Omega - \omega - \epsilon = 7''\cdot 7 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (79)$$

and in the Nag Tiba–Mussooree and reverse ray

$$(\Omega - \omega)_1 + (\Omega - \omega)_2 - \epsilon = 3''\cdot 5 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (80)$$

the suffixes 1 and 2 being used to distinguish the two ends of the ray.

19. We proceed to discuss the residuals $\Omega - \omega$ which have been found. It will be convenient to collect results, etc., in one table.

TABLE XVII.

Results of spring observations.

Year	1906					1907				
Ray from	Dates	No. of obs.	ω_1	ω_2	$\Omega - \omega$	Dates	No. of obs.	ω_1	ω_2	$\Omega - \omega$
Nojli Tower to Mussooree	Mar. 17–20	9	″ 155·5	″ −3·6	″ 0·7	Mar. 25–28	9	″ 157·0	″ −3·7	″ −0·4
Mussooree to Nojli	Ap. 19-25	11	131·3	+4·6	6·0	Ap. 8–20	24	135·9	4·9	3·6
Mussooree to* Nag Tiba	Ap. 14–26	14	28·2	−0·4	} 0·3 + ϵ	Ap. 8–23	17	29·2	−0·4	} 3·4 + ϵ
Nag Tiba to* Mussooree	May 1–7	14	26·0	0·4		May 10–15	14	26·5	0·4	
Nag Tiba to Nojli	May 1–7	10	146·5	7·9	† 4·9 + ϵ	May 13	2	147·3	7·9	† 10·7 + ϵ

Year	1908					1909				
Ray from	Dates	No. of obs.	ω_1	ω_2	$\Omega - \omega$	Dates	No. of obs.	ω_1	ω_2	$\Omega - \omega$
Nojli Tower to Mussooree						Mar. 13–16	6	″ 155·1	″ −3·6	″ 0·5
Mussooree to Nojli	Mar. 9 Ap. 21	24	″ 134·9	″ 4·8	″ 8·2	Ap. 6–16	25	132·6	4·7	9·6
Mussooree to* Nag Tiba	Mar. 9 Ap. 21	16	28·9	−0·4	} 4·3 + ϵ	Ap. 6–29	18	28·6	−0·4	} 6·0 + ϵ
Nag Tiba to* Mussooree	Ap. 27 May 1	11	26·3	0·4		May 6–11	10	26·0	0·4	
Nag Tiba to Nojli	Ap. 30	2	151·3	8·3	† 7·7 + ϵ					

* *See* also table XVIII. † These figures contain an additional 0″·1 which is shown to be necessary in § 24.

The brackets indicate that the sum of the values of $\Omega - \omega$ at Mussooree to Nag Tiba and at Nag Tiba to Mussooree are given. If we regard the observations at Mussooree and Nag Tiba to be sufficiently close, in point of time, for us to treat them as simultaneous, we can deduce, with the aid of table III, values of a and b which will account for the values of $\Omega - \omega$ given in table XVII. As only two observations were taken from Nag Tiba to Nojli in each of the years 1907, 1908, it is not desirable to take these along with the ten observations in 1906 and take a mean, as this would imply the observations in the several years being of the same weight. We will accordingly consider the year 1906 alone in which a fair number of observations were taken in the four cases, and take the mean of the years 1907, 1908 together. The solutions will enable us to determine ϵ, and so we shall have a determination of the plumb-line deflection at Nag Tiba.

20. We will take f_1, f_2, a, b to refer to Mussooree and f_1', f_2', a', b to refer to Nag Tiba, b being treated as a constant. From (65) we have

$$f_2 + b = f_1\,(3 \cdot 38\, f_1 - 2 \cdot 38)$$

and replacing f_1 by $u + 1$ and f_2 by $v + 1$ we get

$$v + b = 3 \cdot 38\, u^2 + 4 \cdot 38 u \quad \ldots\ldots\ldots\ldots \quad (81)$$

Also since, as may be seen from end of § 20 Chapter I

$$a - a' = 2.10^{-5} bh \quad \ldots\ldots\ldots\ldots \quad (82)$$

$$\therefore \quad u' = f_1' - 1 = f_1 - 1 + \frac{a - a'}{1 \cdot 327}$$

$$u' = u + 1 \cdot 507.10^{-5}\, bh \quad \ldots\ldots\ldots\ldots \quad (83)$$

21. The conditions that we have to satisfy in 1906 at Mussooree and Nag Tiba can at once be written down from table XVII.

$$131 \cdot 3 u + 4 \cdot 6 v = 6 \cdot 0 \quad \ldots\ldots\ldots\ldots \quad (84)$$

$$\left.\begin{array}{l} 28 \cdot 2 u \;\; - 0 \cdot 4 v \;\; + 26 \cdot 0 u' + 0 \cdot 4 v' = 0 \cdot 3 + \epsilon \\ 146 \cdot 5 u' + 7 \cdot 9 v' = \;\; 4 \cdot 9 + \epsilon \end{array}\right\} \quad \ldots\ldots \quad (85)$$

whence eliminating ϵ from the last pair.

$$28 \cdot 2 u - 0 \cdot 4 v - 120 \cdot 5 u' - 7 \cdot 5 v' + 4 \cdot 6 = 0 \quad \ldots\ldots\ldots \quad (86)$$

It is possible to form a biquadratic equation for u, by means of the equations (81), (83), (84), (86): but it is more convenient to proceed as follows:

From (84) and (81) we have

$$131 \cdot 3 u + 4 \cdot 6\, (3 \cdot 38 u^2 + 4 \cdot 38 u - b) = 6 \cdot 0$$

whence

$$b = 3 \cdot 38 u^2 + 4 \cdot 38 u + \frac{131 \cdot 3}{4 \cdot 6} u - \frac{6 \cdot 0}{4 \cdot 6}$$

$$b = 3 \cdot 38 u^2 + 32 \cdot 93 u - 1 \cdot 304 \quad \ldots\ldots\ldots\ldots \quad (87)$$

Also (83) becomes for this case, with $h = 2992$

$$u' = u + \cdot 0451\, b \quad \ldots\ldots\ldots\ldots \quad (88)$$

By (87) we see that to each value of u there is a corresponding value of b: hence also of v, u' and v'.

Thus if $u = 0$ then $b = -1 \cdot 304$ $v = +1 \cdot 304$ $u' = -\cdot 059$ $v' = +1 \cdot 058$

and if $u = 0 \cdot 1$ then $b = +2 \cdot 023$ $v = -1 \cdot 551$ $u' = +\cdot 191$ $v' = -1 \cdot 064$

Substituting in the left hand side of (86) which we denote by C, we get

(1) when $u = 0$

$$\begin{aligned} C &= -0\cdot4 \times 1\cdot304 \quad -120\cdot5 \times \overline{-\cdot059} \quad -7\cdot5 \times 1\cdot058 - 4\cdot6 \\ &= -\cdot52 \quad + \quad 7\cdot11 \quad -7\cdot94 + 4\cdot6 \\ &= +3\cdot25 \end{aligned}$$

(2) when $u = \cdot1$

$$\begin{aligned} C &= 28\cdot2 \times 0\cdot1 \quad -0\cdot4 \times \overline{-1\cdot551} \quad -120\cdot5 \times \cdot191 \quad -7\cdot5 \times \overline{-1\cdot064} - 4\cdot6 \\ &= 2\cdot82 \quad +0\cdot62 \quad - \quad 23\cdot02 \quad +7\cdot98 + 4\cdot6 \\ &= -7\cdot00 \end{aligned}$$

These values of C lead us to try values of u, $\cdot03$ and $\cdot04$.

If $u = \cdot03$ then $b = -\cdot313$ $v = +\cdot447$ $u' = +\cdot016$ $v' = +\cdot384$

and if $u = \cdot04$ then $b = +\cdot018$ $v = +\cdot162$ $u' = +\cdot041$ $v' = +\cdot168$

Substituting we get

(3) when $u = \cdot03$

$$\begin{aligned} C &= 28\cdot2 \times \cdot03 \quad -0\cdot4 \times \cdot447 \quad -120\cdot5 \times \cdot016 \quad -7\cdot5 \times \cdot384 + 4\cdot6 \\ &= \cdot85 \quad - \cdot18 \quad - \quad 1\cdot93 \quad -2\cdot88 \quad +4\cdot6 \\ &= +0\cdot46 \end{aligned}$$

(4) when $u = \cdot04$

$$\begin{aligned} C &= 28\cdot2 \times \cdot04 \quad -0\cdot4 \times \cdot162 \quad -120\cdot5 \times \cdot041 \quad -7\cdot5 \times \cdot168 + 4\cdot6 \\ &= 1\cdot13 \quad - \cdot06 \quad - \quad 4\cdot95 \quad -1\cdot26 \quad +4\cdot6 \\ &= -0\cdot54 \end{aligned}$$

Finally

(5) if $u = \cdot035$ then $b = -\cdot148$ $v = +\cdot305$ $u' = \cdot028$ $v' = +\cdot274$

$$\begin{aligned} \text{and } C &= 28\cdot2 \times \cdot035 - 0\cdot4 \times \cdot305 \quad -120\cdot5 \times \cdot028 \quad --7\cdot5 \times \cdot274 + 4\cdot6 \\ &= \cdot99 \quad - \cdot12 \quad - \quad 3\cdot37 \quad -2\cdot05 \quad +4\cdot6 \\ &= +\cdot05 \end{aligned}$$

We see then that the values (5) above satisfy (84) and (86).

From the second equation of (85) we have

$$\begin{aligned} \epsilon &= 146\cdot5\,u' + 7\cdot9\,v' - 4\cdot9 \\ &= 4\cdot10 + 2\cdot16 - 4\cdot9 \\ \epsilon &= 1''\cdot36 \qquad \ldots\ldots\ldots\ldots \quad (89) \end{aligned}$$

This is a value determined from 1906 observations: we proceed to make another determination from 1907,8 observations.

22. Taking mean figures for these two years we have the conditions

$$135\cdot4\,u + 4\cdot85\,v = 5\cdot90 \qquad \ldots\ldots\ldots\ldots \quad (90)$$

$$\left.\begin{aligned} 29\cdot05\,u - 0\cdot4\,v + 26\cdot4\,u' + 0\cdot4\,v' &= 3\cdot85 + \epsilon \\ 149\cdot3\,u' + 8\cdot1\,v' &= 9\cdot20 + \epsilon \end{aligned}\right\} \ldots\ldots \quad (91)$$

whence

$$29\cdot05\,u - 0\cdot4\,v - 122\cdot9\,u' - 7\cdot7\,v' + 5\cdot35 = 0 \qquad \ldots\ldots \quad (92)$$

Here we have

$$b = 3\cdot38\,u^2 + 4\cdot38\,u + \frac{135\cdot4}{4\cdot85}\,u - \frac{5\cdot90}{4\cdot85}$$

$$b = 3\cdot38\,u^2 + 32\cdot30\,u - 1\cdot217 \qquad \ldots\ldots\ldots\ldots \quad (93)$$

If $u = \cdot04$ then $b = +\cdot080$ $v = +\cdot100$ $u' = \cdot044$ $v' = +\cdot120$

and if $u = \cdot05$ then $b = +\cdot405$ $v = -\cdot178$ $u' = \cdot068$ $v' = -\cdot091$

so that if C' represents the left hand side of (92) we have

(1) if $u = \cdot 04$

$$C' = 29{\cdot}05 \times {\cdot}04 - 0{\cdot}4 \times {\cdot}100 - 122{\cdot}9 \times {\cdot}044 - 7{\cdot}7 \times {\cdot}120 + 5{\cdot}35$$
$$= 1{\cdot}16 - {\cdot}04 - 5{\cdot}41 - {\cdot}92 + 5{\cdot}35$$
$$= + {\cdot}14$$

(2) if $u = \cdot 05$

$$C' = 29{\cdot}05 \times {\cdot}05 - 0{\cdot}4 \times \overline{-\ {\cdot}178} - 122{\cdot}9 \times {\cdot}068 - 7{\cdot}7 \times \overline{-\ {\cdot}091} + 5{\cdot}35$$
$$= 1{\cdot}45 + {\cdot}07 - 8{\cdot}36 + {\cdot}70 + 5{\cdot}35$$
$$= - {\cdot}79.$$

The solution here is $u = \cdot 0415$ corresponding to which we have

$$v = +{\cdot}058 \quad u' = {\cdot}048 \quad v' = + {\cdot}088$$

From (92) we get

$$\epsilon = 149{\cdot}3 \times {\cdot}048 + 8{\cdot}1 \times {\cdot}088 - 9{\cdot}20$$
$$= 7{\cdot}17 + {\cdot}71 - 9{\cdot}20$$
$$\epsilon = - 1''{\cdot}32 \quad \ldots\ldots\ldots\ldots \quad (94)$$

23. Our two determinations of the plumb-line deflection differ by $2''{\cdot}7$.

If we compute ϵ corresponding to cases (3) and (4) of § 21

we get when $u = \cdot 03$ $\quad C = + \cdot 46 \quad \epsilon = + \cdot 63$

and when $u = \cdot 04$ $\quad C = - \cdot 54 \quad \epsilon = + 2{\cdot}43.$

We see then that while C changes by $1''$, ϵ changes by $1''{\cdot}8$. Any error in any of the residuals $\Omega - \omega$ appears in C: so that if any one angle of elevation is wrong by $1''$, the deduced value of ϵ will be wrong by $1''{\cdot}8$.

To account then for our discrepancy of $2''{\cdot}7$ we have only to suppose that the combined errors in the mean observations from Mussooree to Nag Tiba, Nag Tiba to Mussooree and Nag Tiba to Nojli, amounted to $1''{\cdot}5$: that is an average error of $0''{\cdot}5$ at each station.

Also the observations at Mussooree were not made on the same date as the other observations: which may account for small changes.

Accordingly there is no great reason for surprise at the discrepancy between our two determinations.

24. The mean of the results (89) and (94) gives

$$\epsilon_1 = + 0''{\cdot}02$$

If we weight the equations in proportion to the numbers of observations at Nag Tiba to Nojli we have

$$\epsilon_2 = + 0''{\cdot}6.$$

This corresponds to a change of meridional deflection of amount $0''{\cdot}6 \sec 30° = 0''{\cdot}7$.

Taking the solutions we have found in §§ 21, 22 we can put down the corresponding refractions in the observations from Mussooree to Nag Tiba.

In 1906 we have $u = \cdot 035$, $v = \cdot 305$.

$$\therefore \quad \Omega = 28{\cdot}2\,(1+u) - 0{\cdot}4\,(1+v)$$
$$= 27{\cdot}8 + 1{\cdot}0 - 1{\cdot}2$$
$$= 27{\cdot}6$$

In 1907, 1908 $\quad u = \cdot 042 \quad v = \cdot 058$

$$\Omega = 29{\cdot}05\,(1+{\cdot}042) - 0{\cdot}4\,(1+{\cdot}058)$$
$$= 28{\cdot}65 + 1{\cdot}22 - {\cdot}02$$
$$= 29{\cdot}9$$

Hence deduced values of a are

for 1906 $3^\circ\ 12'\ 54''\cdot4 - 27''\cdot6$

$= 3^\circ\ 12'\ 26''\cdot8$

and for 1907, 1908 $3^\circ\ 12'\ 57''\cdot2 - 29''\cdot9$

$= 3^\circ\ 12'\ 27''\cdot3$

The agreement of these results is very good and their weighted mean is $3^\circ\ 12'\ 27''\cdot1$, from which it appears that the value we assumed in § 18 was too small by $0''\cdot75$. This corresponds to a height too low by 0·19 foot, which in turn corresponds to a value of $\Omega-\omega$, in case of Nag Tiba-Nojli ray, too small by $0''\cdot13$.

On this account the values of $\Omega-\omega$ in table XVII relating to Nag Tiba–Nojli have been increased by 0·1, so that the solutions for ϵ will stand.

We have from this that the meridional deflection at Nag Tiba is

$$32'-0''\cdot7=31''\cdot3 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (95)$$

While bearing in mind the liability to error which this result has, we must remember also that the value given in table V (32″) was only arrived at by interpolation : moreover this method of interpolation cannot be advocated for determining plumb-line deflection in a mountainous country where the changes of deflection are extremely rapid. The result (95) at least bears some relation to facts and on this account is to be preferred to the interpolated value.

We shall accordingly use the result (95) in discussing the observations to the Snow-Peaks.

The final value of the difference of height of Nag Tiba and Mussooree is 2992·1. We will conclude this chapter with a statement of results found.

Height of Mussooree above Nojli R.S. ... 6084·4 feet

,, Nag Tiba ,, ... 9076·5 ,,

Meridional component of plumb-line deflection at Nag Tiba = $31''\cdot3$.

Ray	a			δ
Nojli R.S. to Mussooree	1°	6′	14″·2	15·9
Nojli Tower to Mussooree	1	5	31·1	15·9
Mussooree to Nojli R.S.	−1	46	11·7	−46·1
Mussooree to Nag Tiba	3	12	27·1	+45·7
Nag Tiba to Mussooree	−3	21	3·2	−38·9
Nag Tiba to Nojli R.S.	−2	10	6·1	−38·8
Dehra to Mussooree	5	19	4·1	39·6
Mussooree to Dehra	−5	27	18·2	−39·1

TABLE XVIII.

Results of autumn and winter observations and of spring observation between Mussooree and Nag Tiba.

Year	1905					1906					1907					1908,1909				
Ray from	Dates	No. of obs.	ω_1	ω_2	$\Omega-\omega$	Dates	No. of obs.	ω_1	ω_2	$\Omega-\omega$	Dates	No. of obs.	ω_1	ω_2	$\Omega-\omega$	Dates	No. of obs.	ω_1	ω_2	$\Omega-\omega$
			″	″	″			″	″	″			″	″	″			″	″	″
Nojli Tower to Mussooree	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	Jan. * 28	6	169·6	−4·1	+5·5
Mussooree to Nojli	Nov. 12	1	136·9	+4·9	12·6	Oct.31 to Nov.7	14	136·7	+4·9	13·5	Nov. 21,22	4	138·4	5·0	17·0	Oct.† 24–31	23	137·1	+4·9	22·7
Mussooree to Nag Tiba	...	...	...	...	...	Oct. 18 to Nov.7	16	29·4	−0·4	1·1	Oct. 22 to Nov.22	28	29·5	−0·4	5·7	Oct.† 21–31	20	29·5	−0·4	10·8
Nag Tiba to Mussooree	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	Oct.† 5–14	15	26·7	+0·4	15·7

Year	1906					1907					1908					1909				
Mussooree to Nag Tiba	Ap. 14–26	14	28·2	−0·4	−0·5	Ap. 8–23	17	29·2	−0·4	+0·7	Mar.9 to Ap. 21	16	28·9	−0·4	+2·2	Ap. 6–29	18	28·6	−0·4	1·7
Nag Tiba to Mussooreo	May 1–7	14	26·0	+0·4	1·4	May 10–15	14	26·5	+0·4	3·3	Ap. 27 to May 1	11	26·3	+0·4	2·7	May 6–11	10	26·0	+0·4	4·9

* 1909. † 1908.

CHAPTER III.

The heights of the snow-peaks Bandarpunch, Srikanta, Jaonli and Kedarnath.

1. When observations have been made at any station to two points whose heights are known, then, provided we have the necessary information about plumb-line deflections, it is possible to compute the true angles of elevation a, and so to obtain the quantity $\Omega - \omega$, as has already been done in tables XVII and XVIII. We can next form two equations as follows:—

$$\left.\begin{aligned} u\,\omega_1 + v\,\omega_2 &= \Omega - \omega \\ u\,\omega'_1 + v\,\omega_2' &= \Omega' - \omega' \end{aligned}\right\} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (96)$$

where $\omega_1, \omega_2, \Omega, \omega$ refer to one ray, and the same letters dashed refer to the other ray.

The solution of (96) gives the values of u and v. Theoretically we can apply these values of u and v to find the refraction on any third ray. In the case of the observations taken at Mussooree, we can form equations (96) by means of the rays to Nojli and Nag Tiba, determine the values of u and v, and then deduce the refraction in the case of rays to the snow-peaks. Unfortunately, when the actual numerical quantities are put in, it is soon apparent that the method is not satisfactory in this case. The reason is mainly that the short ray to Nag Tiba has very small refraction and that the quantity $\Omega' - \omega'$ must be known with greater precision than we can determine it, otherwise the error introduces a quantity which is large compared with the refraction. Both the equations (96) are based on fallible measurements. The observed angles of elevation are in error by unknown amounts, depending on the graduation errors of the instrument and the errors of intersection. The deduction of ω depends on observations of temperature, pressure and humidity. Errors in these quantities, especially in the temperature, give rise to errors in ω. The temperature is supposed to be that of the undisturbed outer air: in the cases we are to deal with, the temperature has been observed in the observatory tent. Moreover the temperature readings cannot be quite simultaneous with the observations of vertical angles to several points, since all have been made by one observer and temperature readings were not repeated after each ray had been observed. Accordingly errors of $3''$ or more probably exist in most of our determinations of $\Omega - \omega$. The refraction on the Mussooree–Nag Tiba ray is only some thirty seconds.

In the case of observations from Mussooree to Nag Tiba a change of one second in $\Omega' - \omega'$ gives a very large change in the values of u and v deduced from (96): and the two sets of values of u and v when applied to the snow-peak observations give results differing by many seconds. If we had observations to two stations of known heights which differed considerably at distances of some forty miles, the solution would be satisfactory.

2. The method of solution by (96) assumes that the temperature law may be represented with sufficient accuracy by three terms, as on page 16, throughout the range of heights that the rays under consideration traverse; but more especially in the neighbourhood of the observing station. As will be seen shortly (*see* table XXXV) this appears to be upheld by facts with certain restrictions as to the hours of the day. Thus observations from the plains about midday may be corrected by a method of determining u and v about to be described. On the other hand observations from the hills in the forenoon seem, on the whole, to be satisfactorily explained. The reason of this variation of the suitable times for applying the method is perhaps as follows. In the plains at night the earth chills the lower layers of the air by actual contact and conduction: the earth itself being chilled by radiation. The result is that in the lower layers of the air and to a height to which the effect of diurnal *conduction* reaches, the law of temperature is very different from what prevails higher. An inversion of temperature usually occurs, that is to say the temperature rises, instead of falling, with height, up to a certain height. This state of things can hardly occur over a mountain. It persists in the plains because the cold air at the bottom is prevented from sinking, as it naturally would, by the presence of the flat earth. On the mountain top there is no obstacle to the cold air sinking down the mountain side. Now the inversion of temperature is probably most marked at sunrise, when its causes have been in operation the longest possible time. With the rising of the sun, the earth begins to be warmed, and the air in contact with it is warmed in due course. As the warming continues the lower strata of the air become as warm and then warmer than the upper layers, and this warming no doubt continues in some cases until the adiabatic gradient is reached or exceeded, when the lower air rises and the temperature gradient is modified by convection.

Now it is quite impossible that a temperature law of so few terms as we have taken which represents the general trend of the temperature at considerable heights, can also represent a discontinuity or inversion near the earth's surface. On this account we may foresee that the u and v process of calculating refraction will not hold so long as this discontinuity exists. It is not at present possible to say with what accuracy it may be applied at noon or 2 P. M.: but figures are given in table XXXV based on observations at both these hours. Now in the case of the observations from mountain tops, as we have seen, there appears to be no objection to using the method in the earlier hours of the day. The observations up till noon agree fairly well among themselves while those which follow give somewhat different results. It may be that the earlier observations give the correct results while the later ones, in the case we are considering, are vitiated by the fact that the ray from which we derive a relation between u and v, namely the ray down to the plains at Nojli, enters strata of air considerably disturbed by the convective adjustments consequent on the heat radiated by the earth after noon.

3. Turning now to the special cases of the Nojli, Mussooree and Nag Tiba observations we are faced with the fact that the value of $\Omega - \omega$ determined by the Mussooree–Nag Tiba ray does not give a well conditioned equation of the form of (96). Further, in the case of the observations at Nojli, we have only one condition, formed from the ray to Mussooree, since no observations were taken to Nag Tiba. Failing a better method, the idea of the isothermal layer has been made use of. The isothermal layer has frequently been observed to occur in Europe at heights varying

between $9 \cdot 6$ and $10 \cdot 9$ km.* In the tropics there are fewer observations, but such as have been made indicate that the layer is higher—between 17 and 20 km.—in the tropics than it is in the temperate zones. I have assumed as a working hypothesis that it always occurs at a height of 35,000 feet above Mussooree *i.e.*, 41,000 feet = $12 \cdot 5$ km. above sea level.

The first condition for an isothermal layer is $\frac{d\tau}{dh} = 0$ at the height of the layer. In the notation of Chapter I

$$\frac{d\tau}{dh} = \left(\frac{d\tau}{dh}\right)_0 + h\left(\frac{d^2\tau}{dh^2}\right)_0$$

$$= -a\ .10^{-2} + 2\ bh\ 10^{-7} \quad \ldots\ldots\ldots\ldots \quad (97)$$

Hence if this vanishes at height H we have

$$a = 2\ bH.\ 10^{-5} \quad \ldots\ldots\ldots\ldots \quad (98)$$

Putting $H = 35{,}000$ we get $\quad b = \frac{a}{\cdot 7} \quad \ldots\ldots\ldots\ldots \quad (99)$

Now $\quad u = f_1 - 1 = \frac{\cdot 542 - a}{1 \cdot 327}$ from (65)

whence for values at Mussooree $a = \cdot 542 - 1 \cdot 327\ u$

and $\quad b = \cdot 774 - 1 \cdot 89\ u \quad \ldots\ldots\ldots\ldots \quad (100)$

We also have from (81)

$$v = 3 \cdot 38\,u^2 + 4 \cdot 38\,u - b$$

$$= 3 \cdot 38\,u^2 + 6 \cdot 27\,u - \cdot 774 \quad \ldots\ldots\ldots\ldots \quad (101)$$

making use of (100).

This equation (101) together with one equation of the form (96) suffices to determine u and v.

4. The selection of the height of the isothermal layer as 35,000 feet above Mussooree may appear arbitrary. It is known that the height of the isothermal layer varies with season In Gold's Report, p. 104, the mean of 13 European stations showed a variation between $9 \cdot 1$ km. in March and $11 \cdot 9$ km. in October. Further (p. 106 *idem*) it is pointed out that variations of as much as 5 km. have been observed between the heights on successive days. Yet the mean values show a remarkable constancy. Now it can be seen at once, with a few computations, that the effect of choosing the height 40,000 feet instead of 35,000 feet does not change the deduced refraction appreciably: in the case of the rays to the snow-peaks the change is only $0'' \cdot 6$. If then the height assumed is correct to *two or three miles*, it will be satisfactory. The fault in our assumption is much more likely to be that the temperature changes before the isothermal layer is reached, are somewhat irregular and cannot be accurately represented by such a simple law as we have taken. We must consider the equation, of which (101) is a special case, as only affording some control over the values of u and v, and make use of it only when no other means of getting values of u and v is obtainable. If, for instance, we had data of the temperature gradient in latitude 30° at various hours and seasons, up to height of some 20,000 feet, we should make use of these in preference to (101). Such observations as have been made do not give simultaneous values of the gradient at specified hours. The only ones known to the author are those of Field†. In his report Field makes the following statements with regard to observations made at Belgaum:—p. 18 (3)—"The temperature gradient (in May) by day was always above "the adiabatic rate for unsaturated air, and even at night was above it." P. 20 (3)—"The tem-"perature gradients (in August and September) were considerably smaller than in May, and, "except near the surface, were always below the adiabatic gradient for unsaturated air, even "during the hottest parts of rainless days."

* *See* Gold, p. 103, B.A. Report 1909, Winnipeg. † *See* Memoirs of the Meteorological Department Vol. XX Parts I, II, VII.

These two extracts explain why we get smaller values of $\Omega - \omega$ in May than we do in the autumn (*see* tables XVII and XVIII); and agree with the otherwise observed fact that refraction is less in spring than in autumn. More observations of the same kind would throw further light on refraction.

5. The procedure followed is now indicated in the case of the observations taken at Mussooree in November 1906.

The first step is to form the condition

$$u\,\omega_1 + v\,\omega_2 = \Omega - \omega$$

at the several hours of observation for the ray to Nojli. It is hardly necessary to state that the solution of this equation and (101) can only be applied to observations to other points, *taken at the same time.* We can however select several days on which at a particular hour observations were taken to several of the points with which we are dealing. The means of the elevations, temperatures, etc., can then be used as though they represented a more precise set of observations than any single set of the actual observations.

The dates selected at the various hours are given in table XIX. In table XX the mean values of observed pressure, temperature and humidity are shown. From these the values of ω_1, ω_2 given in table XXII are computed and the values of $\Omega - \omega$ given in the same table are deduced, taking the true value of a for Mussooree to Nojli to be $-\ 1^\circ\ 46'\ 11''\cdot 7$ (*see* Chapter II § 24).

We can now form the equations

$$\left.\begin{array}{lll} 139\cdot 6\,u + 5\cdot 0\,v = 25\cdot 1 & \text{at} & 8 \text{ hours} \\ 136\cdot 5\,u + 4\cdot 9\,v = 34\cdot 5 & ,, & 10\ ,, \\ 135\cdot 5\,u + 4\cdot 8\,v = 23\cdot 1 & ,, & 12\ ,, \\ 136\cdot 7\,u + 4\cdot 9\,v = 13\cdot 6 & ,, & 14\ ,, \end{array}\right\} \quad \ldots\ldots\ldots \quad (102)$$

Eliminating v by means of (101) we get quadratic equations giving the values of u. The *smaller* root is taken and then, by (102), the corresponding value of v is found.

Values of u and v, so deduced, are given in table XXI. With these values we form the quantities $u\,\omega_1 + v\,\omega_2$ or $\Omega - \omega$ for the appropriate times for the observations to the snow-peaks, and so arrive at the values of a or $E_c - \Omega$ for these observations.

Mussooree observations.

TABLE XIX.

Hour		8	10	12	14	16½
1906	Nov.	1,2,3,5,7,8	1,2,3,5,6,7,8	2,3,5,7,8	Oct. 31 Nov. 1,2,5,7	...
1907	April	8,15,19,20,23	15,20,22,23	20,22	...	...
1907	Nov.	19,20,22	18,19,20	18,23	21,22	...
1908	Mar.	10,17,19,20	10,16,17	10,17	10,17	...
1908	April	16	16	16	16	..
1908	Oct.	24,26,27,28,29,30	24,26,27,28,29,30	24,26,27,29,30	26,27,29,30,31	26, 30
1909	April	8,10,12,15,16	8,15	8	...	...

TABLE XX.

Hour		8			10			12			14			16½		
		H	*t*	*w*	*H*	*t*	*w*	*H*	*t*	*w*	*H*	*t*	*w*	*H*	*t*	*w*
1906	Nov.	23·51	55·6	·46	23·54	61·5	·41	23·52	62·8	·46	23·49	60·2	·53	...	...	...
1907	April	23·43	58·4	·49	23·45	61·3	·53	23·48	63·7	·50	...	...	...	...	...	...
1907	Nov.	23·49	51·6	·38	23·51	57·7	·41	23·50	62·6	·37	23·46	57·2	·44	...	...	...
1908	Mar.	23·49	54·2	·39	23·44	54·2	·35	23·44	56·0	·39	23·43	57·5	·38	...	...	...
1908	April	23·45	61·0	·35	23·48	66·5	·30	23·49	72·0	·24	...	...	...	...	...	...
1908	Oct.	23·46	56·4	·41	23·47	59·0	·44	23·47	61·4	·43	23·42	58·5	·45	23·41	54·7	·52
1909	April	23·43	61·1	·37	23·45	61·5	·35	23·42	58·0	·18	...	...	...	...	...	

TABLE XXI.

Hour		8		10		12		14		16½	
		u	*v*	*u*	*v*	*u*	*v*	*u*	*v*	*u*	*v*
1906	Nov.	·1668	·361	·2240	·800	·1596	·306	·1028	−·092	...	...
1907	April	·1562	·301	·1596	·313	·0692	−·325	...	...	...	...
1907	Nov.	·2696	1·155	·2211	·776	·2423	·950	·1208	·032	...	...
1908	March	·2165	·744	·1937	·567	·1211	·029	·0864	−·207	...	...
1908	April	·2304	·850	·1448	·205	·1124	−·022	...	...	...	...
1908	October	·2932	1·355	·2720	1·182	·1893	·528	·1505	·244	·1309	0·108
1909	April	·1639	·345	·0962	−·139	·0954	−·144	...	...	...	...
Mean ...		·2138	·730	·1873	·529	·1413	·189	·1151	−·006	·1309	0·108

TABLE XXII.

Point observed	Hour	Nojli 8	Nojli 10	Nojli 12	Nojli 14	Nojli 16½	Nag Tiba 8	Nag Tiba 10	Nag Tiba 12	Nag Tiba 14	Nag Tiba 16½
	δ	−45″·1					+45·7	-			
November 1906. 8″ No. 956.	n	16	17	16	14	...	...	...	...	10	...
	ω_1	139″·5	136″·5	135″·5	136″·7	...	...	...	...	29·5	...
	ω_2	+5·0	+4·9	+4·8	+4·0	...	...	...	...	−0·4	...
	E_c	−1° 43′ 22″·0	15·8	28·3	36·5	...	3° 12′	...	...	57·4	...
	$E_c-\omega$	−1 45 46·6	37·2	48·6	58·1	...	3 12	...	...	28·3	...
	$\Omega-\omega$	+25·1	34·5	23·1	13·5	...	...	...	...	3·1	...
	$E_c-\Omega$	−1 46 11·7	11·7	11·7	11·7	...	3 12	...	...	25·2	...
April 1907 12″ No. 111	n	18	16	6			8	5	4	...	...
	ω_1	137·5	135·9	134·7	...	...	29·6	20·3	20·0	...	...
	ω_2	+4·9	+4·8	+4·7	...	...	−0·4	−0·4	−0·4	...	...
	E_c	−1° 43′ 26″·4	27·8	44·5	...	...	3° 12′ 59″·5	56·6	53·2	...	...
	$E_c-\omega$	−1 45 48·8	48·5	63·0	...	...	3 12 30·3	27·7	24·6	...	...
	$\Omega-\omega$	+22·9	23·2	7·8	...	...	4·5	4·6	2·1	...	...
	$E_c-\Omega$	−1 45 11·7	11·7	11·7	...	...	3 12 25·8	23·1	22·5	...	...
November 1907 12″ No. 111	n	7	8	4	4	...	3	3	...	4	...
	ω_1	141·9	138·5	135·8	138·5	...	30·6	20·0	...	29·9	...
	ω_2	+5·2	+5·0	+4·8	+5·0	...	−0·5	−0·4	...	−0·4	...
	E_c	−1° 43′ 0″·3	13·7	13·6	31·3	...	3° 12′ 65″·7	62·0	...	64·4	...
	$E_c-\omega$	−1 45 27·4	37·2	34·2	54·8	...	3 12 35·6	32·5	...	34·9	...
	$\Omega-\omega$	+44·3	34·5	37·5	16·9	...	7·7	6·3	...	3·6	...
	$E_c-\Omega$	−1 46 11·7	11·7	11·7	11·7	...	3 12 27·9	25·2	...	31·3	...
March 1908 8″ No. 956	n	15	8	6	6	...	5	3	4	4	...
	ω_1	140·4	140·3	130·1	138·2	...	30·3	30·2	30·0	29·8	...
	ω_2	+5·1	+5·1	+5·0	+5·0	...	−0·5	−0·5	−6·5	−0·4	...
	E_c	−1° 43′ 12″·0	16·2	30·6	37·6	...	3° 12′ 59″·2	59·2	59·0	55·7	...
	$E_c-\omega$	−1 45 37·5	41·6	54·7	60·8	...	3 12 20·4	29·5	29·5	26·3	...
	$\Omega-\omega$	+34·2	30·1	17·0	10·9	...	6·2	5·6	3·6	2·7	...
	$E_c-\Omega$	−1 46 11·7	11·7	11·7	11·7	...	3 12 23·2	23·9	25·9	23·6	...

Point observed	Hour	Bandarpunch 8	Bandarpunch 10	Bandarpunch 12	Bandarpunch 14	Bandarpunch 16½	Srikanta 8	Srikanta 10	Srikanta 12	Srikanta 14	Srikanta 16½
	δ	+46·1					+44·7				
November 1906. 8″ No. 956.	n	7	8	7	...	...	8	9	6	10	...
	ω_1	143·4	140·2	139·2	...	...	168·7	165·1	163·9	165·3	...
	ω_2	−9·0	−8·8	−8·7	...	...	−9·7	−9·4	−9·3	−9·4	...
	E_c	2° 52′ 68″·3	65·2	65·4	...	...	2° 14′ 17″·3	12·1	13·0	8·6	...
	$E_c-\omega$	2 50 53·9	53·8	54·9	...	...	2 11 38·3	36·4	38·4	32·7	..
	$\Omega-\omega$	20·7	24·4	19·6	...	...	24·6	29·5	23·3	16·1	...
	$E_c-\Omega$	2 50 33·2	20·4	35·3	...	...	2 11 13·7	6·9	15·1	16·6	...
April 1907 12″ No. 111	n	6	5	4	...	...	6	5	4	...	...
	ω_1	141·2	139·6	138·4	...	...	166·2	164·3	162·9	...	...
	ω_2	−8·9	−8·7	−8·6	...	...	−9·5	−9·3	−9·2	...	...
	E_c	2° 52′ 65″·6	62·5	57·1	...	...	2° 14′ 18″·5	11·1	5·4	...	...
	$E_c-\omega$	2 50 54·3	51·6	47·3	...	...	2 11 41·8	36·1	31·7	...	...
	$\Omega-\omega$	10·4	10·6	12·4	...	.	23·1	23·3	14·3	...	...
	$E_c-\Omega$	2 50 34·9	32·0	34·9	...	...	2 11 18·7	12·8	17·4	...	...
November 1907 12″ No. 111	n	3	5	2	4	...	.4	4	2	4	...
	ω_1	145·6	142·3	139·5	142·2	...	171·5	167·5	164·2	167·4	...
	ω_2	−0·3	−8·0	−8·7	−8·9	...	−9·9	−0·6	−0·3	−9·6	...
	E_c	2° 52′ 78″·0	71·7	68·3	67·9	...	2° 14′ 30″·8	20·2	20·3	16·5	...
	$E_c-\omega$	2 56 62·6	58·3	57·5	54·6	...	2 11 49·2	43·3	45·4	38·7	...
	$\Omega-\omega$	28·5	24·6	25·6	16·0	...	34·8	20·6	31·6	19·9	...
	$E_c-\Omega$	2 50 34·1	33·7	31·9	37·7	..	2 11 14·4	12·7	14·4	18·8	...
March 1908 8″ No. 956	n	7	4	4	4	...	6	4	3	4	...
	ω_1	144·2	144·0	142·9	142·0	...	169·7	169·5	168·2	167·1	...
	ω_2	−9·1	−9·1	−0·0	−8·9	...	−9·8	−0·8	−9·7	−9·6	...
	E_c	2° 52′ 71″·3	64·9	61·5	59·5	...	2° 14′ 17″·7	14·3	10·1	8·2	...
	$E_c-\omega$	2 50 56·2	50·0	47·6	45·4	...	2 11 37·8	34·6	31·6	30·7	...
	$\Omega-\omega$	24·5	22·7	17·1	14·1	...	29·5	27·3	20·1	16·4	...
	$E_c-\Omega$	2 50 31·7	27·3	30·5	32·3	..	2 11 8·3	7·3	11·5	14·3	...

Point observed	Hour	Jaonli 8	Jaonli 10	Jaonli 12	Jaonli 14	Jaonli 16½	Kedarnath 8	Kedarnath 10	Kedarnath 12	Kedarnath 14	Kedarnath 16½
	δ	+42·8					+39·7				
November 1906. 8″ No. 956.	n	7	9	6	10	...	7	8	5	...	...
	ω_1	164·4	160·8	159·7	161·0	...	103·9	189·7	188·3	...	...
	ω_2	−10·9	−10·5	−10·4	−10·5	...	−13·3	−12·8	−12·7	...	...
	E_c	2° 38′ 12″·1	8·9	8·6	7·3	...	2° 18′ 50″·2	46·8	48·6	...	...
	$E_c-\omega$	2 35 38·6	38·6	39·3	26·9	...	2 15 49·6	49·9	52·4	...	...
	$\Omega-\omega$	23·5	27·6	22·3	15·6	...	28·6	32·3	26·2	...	...
	$E_c-\Omega$	15·1	11·0	17·0	21·3	...	2 15 21·0	17·6	25·2	...	...
April 1907 12″ No. 111	n	7	5	3	...	...	7	5	...	...	...
	ω_1	161·9	166·1	158·7	...	...	101·0	188·9	...	...	...
	ω_2	−10·7	−10·5	−10·3	...	...	−13·0	−12·8	...	...	...
	E_c	2° 38′ 13″·0	7·7	1·2	...	...	2° 18′ 47″·3	44·0	...	...	...
	$E_c-\omega$	2 35 41·8	38·1	32·8	...	...	2 15 49·3	47·9	...	...	...
	$\Omega-\omega$	22·1	22·3	14·3	...	...	25·9	26·1	...	...	...
	$E_c-\Omega$	2 35 19·7	15·8	18·5	...	...	2 15 23·4	21·8	...	...	...
November 1907 12″ No. 111	n	3	4	...	4	...	4	4	2	4	...
	ω_1	167·0	163·2	...	163·1	...	197·0	192·5	188·7	102·4	...
	ω_2	−11·1	−10·8	...	−10·8	...	−13·6	−13·1	−12·8	−13·1	...
	E_c	2° 38′ 21″·3	14·3	...	12·2	...	2° 18′ 66″·2	50·6	46·6	49·5	...
	$E_c-\omega$	2 35 45·4	41·9	...	39·9	...	2 15 62·8	51·2	50·7	50·2	...
	$\Omega-\omega$	22·2	27·7	...	19·4	...	37·4	32·4	33·6	22·8	...
	$E_c-\Omega$	2 35 13·2	14·2	...	20·5	...	2 15 25·4	18·8	17·1	27·4	...
March 1908 8″ No. 956	n	6	4	3	4	...	6	4	3	...	...
	ω_1	165·3	165·1	163·8	162·8	...	195·0	194·8	103·3	...	...
	ω_2	−11·0	−11·0	−16·8	−10·7	...	−13·4	−13·4	−13·2	...	...
	E_c	2° 38′ 18″·0	11·6	7·3	5·7	...	2° 18′ 52″·7	52·6	44·1	...	...
	$E_c-\omega$	2 35 43·7	37·5	34·3	33·6	...	2 15 51·1	51·2	44·6	...	...
	$\Omega-\omega$	27·6	25·7	19·5	16·3	...	32·3	30·1	23·0	...	...
	$E_c-\Omega$	2 35 16·1	11·8	14·8	17·3	...	2 15 18·8	21·1	21·0	...	...

TABLE XXII.—(*Continued*).

Point Observed	Hour	Nojli 8	10	12	14	16½	Nag Tiba 8	10	12	14	16½
	δ	−46″·1					+45·7				
April 1908 8″ No. 956	n	4	4	4	...	...	2	1	1	...	...
	ω_1	136·4	133·6	130·7	...	...	29·4	28·8	28·2	...	...
	ω_2	+4·9	+4·7	+4·6	...	...	−0·4	−0·4	−0·4	...	...
	E_c	−1° 43′ 14″·6	33·1	41·8	...	...	3° 12′ 60″·9	57·3	58·1	...	...
	$E_c-\omega$	−1 45 36·1	51·4	57·1	...	...	3 12 31·9	28·9	30·3	...	...
	$\Omega-\omega$	+ 35·6	20·3	14·6	...	...	6·4	4·1	3·2	...	...
	$E_c-\Omega$	−1 46 11·7	11·7	11·7	...	...	3 12 25·5	24·8	27·1	...	...
October 1908 8″ No. 1316	n	23	15	18	20	6	10	12	10	10	4
	ω_1	138·9	137·5	136·1	137·5	139·5	30·0	29·6	29·3	29·6	30·1
	ω_2	+5·0	+4·9	+4·0	+4·9	+5·0	−0·4	−0·4	−0·4	−0·4	−0·4
	E_c	−1° 43′ 0″·3	6·1	22·3	27·4	28·4	3° 12′ 71″·7	68·1	68·0	68·4	67·5
	$E_c-\omega$	−1 45 24·2	28·5	43·3	49·8	52·9	3 12 42·1	38·9	39·1	39·2	37·8
	$\Omega-\omega$	+ 47·5	43·2	28·4	21·9	18·8	8·2	7·6	5·3	4·4	3·9
	$E_c-\Omega$	−1 46 11·7	11·7	11·7	11·7	11·7	3 12 33·9	31·3	33·8	34·6	33·9
April 1909 8″ No. 1311	n	17	7	4	...	...	8	3	1	...	...
	ω_1	136·2	136·1	132·8	...	...	29·4	29·3	28·6	...	...
	ω_2	+4·9	+4·9	+4·7	...	...	−0·4	−0·4	−0·4	...	...
	E_c	−1° 43′ 26″·6	38·3	42·2	...	...	3° 12′ 60″·2	56·3	56·1	...	...
	$E_c-\omega$	−1 45 47·7	59·3	50·7	...	...	3 12 31·2	27·4	27·9	...	...
	$\Omega-\omega$	+ 24·0	12·4	12·0	...	...	4·7	2·9	2·8	...	...
	$E_c-\Omega$	−1 46 11·7	11·7	11·7	...	...	3 12 26·5	24·5	25·1	...	...

Point Observed	Hour	Bandarpunch 8	10	12	14	16½	Srikanta 8	10	12	14	16½
	δ	+46·1					+44·7				
April 1908 8″ No. 956	n	2	2	2	...	..	2	2	2	...	...
	ω_1	140·1	137·3	134·5	...	...	164·9	161·6	158·3	...	...
	ω_2	−8·8	−8·5	−8·2	...	...	−9·4	−9·1	−8·6	...	...
	E_c	2° 52′ 66″·8	62·6	58·0	...	...	2° 14′ 14″·5	9·1	4·6	...	...
	$E_c-\omega$	2 50 57·5	53·8	51·7	...	...	2 11 39·0	36·6	35·1	...	...
	$\Omega-\omega$	24·8	18·1	15·3	...	...	30·0	21·5	18·0	...	...
	$E_c-\Omega$	2 50 32·7	35·7	36·4	..	...	2 11 9·0	15·1	17·1	...	...
October 1908 8″ No. 1316	n	13	12	10	10	4	12	13	10	10	4
	ω_1	142·7	141·3	139·8	141·2	143·2	167·0	166·3	164·6	166·2	168·5
	ω_2	−9·0	−8·9	−8·7	−8·0	−0·0	−9·7	−9·5	−9·4	−9·5	−9·7
	E_c	2° 52′ 83″·8	80·0	77·4	79·2	89·4	2° 14′ 36″·6	31·5	30·4	28·5	35·1
	$E_c-\omega$	2 50 70·1	67·6	66·3	66·9	75·2	2 11 58·4	54·7	55·2	51·8	56·3
	$\Omega-\omega$	29·6	27·9	21·9	19·1	17·6	36·1	34·0	26·2	22·7	21·0
	$E_c-\Omega$	2 50 40·5	30·7	44·4	47·8	57·4	2 11 22·3	20·7	29·0	29·1	35·3
April 1909 8″ No. 1311	n	6	3	1	...	...	8	3	1		
	ω_1	139·0	139·9	136·5	...	...	164·7	164·6	160·7	...	...
	ω_2	−6·7	−8·7	−6·4	...	...	−9·4	−9·4	−9·0	...	...
	E_c	2° 52′ 60″·8	63·9	60·7	...	...	2° 14′ 16″·8	16·0	9·9	...	...
	$E_c-\omega$	2 50 58·0	52·7	52·6	...	...	2 11 41·5	40·8	38·2	...	...
	$\Omega-\omega$	19·0	14·7	14·2	...	...	23·8	17·1	16·6	...	...
	$E_c-\Omega$	2 50 38·7	38·0	38·4	...	...	2 11 17·7	23·7	21·6	...	...

Point Observed	Hour	Jaonli 6	10	12	14	16½	Kedarnath 6	10	12	14	16½
	δ	+42·8					+39·7				
April 1908 8″ No. 956	n	2	2	2	...	...	2	2	2	...	...
	ω_1	160·7	157·4	154·2	...	...	189·5	185·7	161·9	...	...
	ω_2	−10·5	−10·2	−9·9	...	...	−12·9	−12·5	−12·1	...	...
	E_c	2° 37′ 64″·1	62·0	59·6	...	...	2° 16′ 41″·9	44·5	35·7	...	...
	$E_c-\omega$	2 35 33·9	34·8	35·5	...	...	2 15 45·3	51·3	45·9	...	...
	$\Omega-\omega$	28·1	20·7	17·6	...	...	32·7	24·3	20·7	...	...
	$E_c-\Omega$	2 35 5·8	14·1	17·9	...	...	2 15 12·6	27·0	25·2	...	...
October 1908 8″ No. 1316	n	13	12	10	10	4	12	12	...	9	4
	ω_1	163·6	162·0	160·4	162·0	164·2	193·0	191·1	...	191·0	193·7
	ω_2	−10·6	−10·7	−10·5	−10·7	−10·9	−13·2	−13·0	...	−13·0	−13·3
	E_c	2° 38′ 34″·6	30·4	26·7	26·0	31·0	2° 18′ 70″·3	65·3	...	64·1	64·8
	$E_c-\omega$	2 35 61·6	59·1	56·8	55·6	57·7	2 15 70·5	67·2	...	66·1	64·4
	$\Omega-\omega$	33·3	31·4	24·8	21·8	20·3	38·7	26·6	...	25·6	23·9
	$E_c-\Omega$	2 35 28·5	27·7	32·0	33·8	37·4	2 15 31·8	30·6	...	40·5	40·5
April 1909 8″ No. 1311	n	6	3	1	...	...	...	...	1	...	...
	ω_1	160·4	160·4	156·5	...	...	...	...	164·7	...	...
	ω_2	−10·5	−10·5	−10·1	...	...	...	...	−12·4	...	...
	E_c	2° 38′ 16″·3	13·5	9·8	...	...	2° 18	...	43·5	...	...
	$E_c-\omega$	2 35 46·4	43·6	43·4	...	...	2 15	...	51·2	...	...
	$\Omega-\omega$	22·7	16·9	16·4	...	...	...	...	19·4	...	...
	$E_c-\Omega$	2 35 23·7	26·7	27·0	...	...	2 15	...	31·8	...	...

It is to be remarked that the observations to Nojli R. S. and Nojli Tower have been combined. Thus if there were observations to both stations on any day, the observations to the Tower were reduced to terms of the lower station by correcting for height of Tower and increasing the depression by 0″·3 to take account of the fact that the Tower station is somewhat further from Mussooree than the lower station. This done, the means of the results reduced from Tower and R.S. on that day gave a result of improved precision. The means of the results of the several days were next taken, each day of course being treated as of equal weight.

So long as the height of the observed object does not change the value of $E_c - \Omega$ should be the same by all the deductions. Considering first the case of Nag Tiba, we find values somewhat different from 3° 12′ 27″·1, the value decided on in §24, Chapter II. The seconds of these values range from 22″·5, observed with a 12-inch theodolite at 12 hours in April 1907 to 34″·8 observed with an 8-inch theodolite at 14 hours in October 1908. This range of 12″·3 is to be considered as attributable to errors of observation to Nag Tiba, differences of graduation errors of the various instruments, similar errors in the case of the observations to Nojli on which the deduction of $\Omega - \omega$ depends, as well as any shortcomings of the methods of computing the refraction. The variation 12″·3 could well be attributed to the first two causes, so that it cannot fairly be used as an argument against the method of deducing the refraction.

So far as the observations at Mussooree are concerned we can consider to what extent the results—values of $E_c - \Omega$, or a,—are satisfactory by considering the variation in the angle a or in the heights which are deducible from them. But to institute a comparison with the results of the observations from Nag Tiba and Nojli, it is necessary to consider the deduced height. Before making these deductions the observations from Nag Tiba and Nojli will be considered.

6. Tables, similar to tables XIX to XXII, which refer to Nag Tiba and Nojli are now given without further explanation.

Nag Tiba observations.

TABLE XXIII.

Hour	8	10	12
1906 May ...	1,4,7	6,7	1,6
1907 May ...	13,14	...	13,14
1908 April ...	28	28	...

TABLE XXIV.

Hour	8			10			12		
	H	t	w	H	t	w	H	t	w
1906 May ...	21·00	56·5	·33	21·02	66·0	·25	21·04	66·5	·32
1907 May ...	21·08	53·4	·33	...	...	...	21·11	63·2	·32
1908 April ...	20·95	55·8	·37	20·98	62·4	·35	...	...	...

TABLE XXV.

Hour	8		10		12	
	u	v	u	v	u	v
1906 May ...	·1197	−·024	·1110	−·086	·0909	−·230
1907 May ...	·1234	·004	...	...	·0899	−·235
1908 April ...	·1349	·088	·1238	·006	...	...
Mean ...	·1260	·023	·1174	−·040	·0904	−·233

TABLE XXVI.

Point observed		Nojli			Mussooree			Bandarpunch			Srikanta			Jaonli			Kedarnath		
	Hour	8	10	12	8	10	12	8	10	12	8	10	12	8	10	12	8	10	12
	δ	−39″·0			−38·9			+39·2			+37·0			+34·4			+31·1		
May 1906	n	10	7	3	5	...	...	3	3	2	3	2	2	3	2	3	4	2	...
	ω_1	151·2	145·6	145·2	26·9	...	...	101·4	07·6	97·4	126·1	121·5	121·1	124·3	119·7	119·4	153·1	147·4	...
	ω_2	+8·3	+7·8	+7·8	+0·4	...	...	−5·1	−4·8	−4·8	−5·7	−5·4	−5·4	−6·6	−6·3	−6·2	−8·6	−8·1	...
	E_c	−2° 7′ 8″·7	17·2	21·7	−3° 20′ 33″·8	...	...	2° 54′ 28″·5	20·6	28·3	2° 5′ 32″·1	28·8	30·0	2° 30′ 45″·9	41·5	41·0	2° 7′ 57″·4	57·5	...
	$E_c-\omega$	−2 9 48·2	50·6	54·7	−3 20 61·1	...	...	2 52 52·2	56·8	55·7	2 3 31·7	32·7	34·3	2 28 48·2	48·1	47·8	2 5 32·9	38·2	...
	$\Omega-\omega$	+17·0	15·5	11·4	+ 3·2	...	...	12·3	11·2	9·5	15·2	14·0	12·2	15·0	13·8	12·3	18·5	17·1	...
	$E_c-\Omega$	−2 10 6·1	6·1	6·1	−3 21 4·3	...	...	2 52 39·9	45·6	46·2	2 3 16·5	18·7	22·1	2 28 33·2	34·3	35·5	2 5 14·4	21·1	...
May 1907	n	3 + 4	...	2+4	3	...	4	4	...	3	4	...	3	3	..	4	3	...	3
	ω_1	153·7	...	147·7	27·3	...	26·3	103·0	...	99·0	128·2	...	123·2	126·4	...	121·4	155·6	...	149·6
	ω_2	+8·4	...	+8·0	+0·4	...	+0·4	−5·2	...	−4·9	−5·8	...	−5·5	−6·8	...	−6·4	−8·8	...	−8·3
	E_c	−2° 7′ 5″·0	...	19·0	−3° 20′ 30″·9	...	31·0	2° 54′ 34″·4	...	34·5	2° 5′ 41″·6	...	40·3	2° 30′ 56″·3	...	55·2	2° 7′ 72″·0	...	71·9
	$E_c-\omega$	−2 9 47·1	...	54·7	−3 20 58·6	...	57·7	2 52 56·6	...	60·4	2 3 39·2	...	42·6	2 28 56·7	...	60·2	2 5 45·2	...	50·6
	$\Omega-\omega$	+19·0	...	11·4	+3·4	...	2·5	12·7	...	10·1	15·8	...	12·4	15·6	...	12·4	19·2	...	15·4
	$E_c-\Omega$	−2 10 6·1	...	6·1	−3 21 2·0	...	0·2	2 52 43·9	...	50·3	2 3 23·4	...	30·2	2 28 41·1	...	47·8	2 5 26·0	...	35·2
April 1908	n	4	3	...	2	2	...	2	2	...	2	2	...	2	2	...	2	2	...
	ω_1	151·1	147·3	...	26·9	26·2	...	101·3	98·7	...	126·1	122·8	...	124·2	121·1	...	153·0	149·1	...
	ω_2	+8·3	+7·9	...	+0·4	+0·4	...	−5·1	−4·0	...	−5·7	−5·5	...	−6·7	−6·4	...	−8·6	−8·2	...
	E_c	−2° 7′ 5″·6	12·6	...	−3° 20′ 31″·8	32·9	...	2° 54′ 35″·1	38·9	...	2° 5′ 44″·5	45·7	...	2° 30′ 58″·0	58·7	...	2° 7′ 70″·1	70·4	...
	$E_c-\omega$	−2 9 45·0	47·8	...	−3 20 59·1	59·5	...	2 52 58·9	65·1	...	2 3 44·1	48·4	...	2 28 61·4	64·0	...	2 5 45·7	49·5	...
	$\Omega-\omega$	+21·1	18·3	...	+3·7	3·2	...	13·2	12·2	...	16·5	15·2	...	16·2	15·0	...	19·9	18·4	...
	$E_c-\Omega$	−2 10 6·1	6·1	...	−3 21 2·8	2·7	...	2 52 45·7	52·9	...	2 3 27·6	33·2	...	2 28 45·2	49·0	...	2 5 25·8	31·1	...

Nojli Tower Observations.

TABLE XXVII.

Hour	8	10	12	14	16½
1906 March	19	19	19	..	...
1907 March	25, 26, 28	25	25	25	25
1909 January	...	28	28	28	...

TABLE XXVIII.

Hour	8			10			12			14			16½		
	H	*t*	*w*	*H*	*t*	*w*	*H*	*t*	*w*	*H*	*t*	*w*	*H*	*t*	*w*
1906 Mar.	28·93	59·8	·65	28·97	71·6	·46	28·96	77·4	·32	...	...	...	...	...	...
1907 Mar.	29·05	64·2	·68	29·11	74·0	·45	29·11	77·2	·24	29·07	78·2	·24	29·04	77·2	·30
1909 Jan.	...	...	...	29·30	49·5	·86	29·25	56·2	·50	29·19	61·5	·40	...	...	...

TABLE XXIX.

Hour	8		10		12		14		16½	
	u	*v*	*u*	*v*	*u*	*v*	*u*	*v*	*u*	*v*
1906 Mar.	·4297	2·53	·1074	0·315	·0004	−0·659	...	...	...	...
1907 Mar.	·5228	3·40	·2580	1·118	−·0081	−0·709	−·0305	−0·845	·0018	−0·520
1909 Jan.	...	...	·3034	0·46	·0615	−0·277	·0234	−0·522	...	...
Mean	·4763	2·965	·2229	0·631	·0179	−0·548	−·0036	−0·684	·0018	−0·520

TABLE XXX.

Point Observed		Mussooree					Bandarpunch					Srikanta				Jaonli				Kedarnath			
	Hour	8	10	12	14	18½	8	10	12	14	16½	8	10	12	14	8	10	12	14	8	10	12	14
	δ	+15″·9					+15·9					+15·6				+15·4				+14·6			
March 1906	n	2	2	1	...	...	2	2	...	...	...	...	...	...	...	1	2	1	...	...	2	...	...
	ω_1	186·5	161·4	157·0	...	...	341·4	326·6	...	...	...	...	...	...	...	356·2	341·0	333·7	...	...	368·3	...	...
	ω_2	−4·1	−3·9	−3·7	...	...	−25·4	−23·6	...	...	...	...	...	...	...	−27·5	−25·8	−25·0	...	...	−28·3	...	...
	E_c	1° 8′ 77″·5	24·7	7·8	...	...	1° 44′ 64″·3	30·4	...	...	...	...	...	...	...	1° 43′ 06″·1	50·3	28·0	...	1° 36′	35·5	...	...
	$E_c-\omega$	1 5 93·1	47·2	33·6	...	...	1 39 48·3	27·4	...	...	...	...	...	...	...	1 38 69·4	35·1	17·3	...	1 32	57·5	...	..
	$\Omega-\omega$	62·0	16·1	2·5	...	...	62·4	27·6	...	...	...	...	...	...	...	83·5	26·5	16·6	...	...	30·4	...	...
	$E_c-\Omega$	1 5 31·1	31·1	31·1	...	...	1 38 25·9	59·6	...	...	...	...	...	...	...	1 37 45·9	66·6	80·7	...	1 32	27·1	...	...
March 1907	n	10	3	3	2	3	6	2	2	2	1	...	2	1	...	...	2	2	2	...	...	2	2
	ω_1	166·2	160·6	150·2	158·4	158·6	336·7	325·2	322·4	320·7	321·1	...	349·3	346·3	...	...	339·4	336·4	334·6	...	...	381·3	359·4
	ω_2	−4·0	−3·6	−3·8	−3·7	−3·7	−24·6	−23·6	−23·3	−23·1	−23·2	...	−23·2	−22·9	...	...	−25·6	−25·2	−25·1	...	...	−27·7	−27·5
	E_c	1° 6′ 66″·6	45·1	7·9	4·1	6·2	1° 44′ 50″·5	16·7	1·4	3·5	11·9	1° 28′	25·0	3·0	...	1° 43′	49·8	30·3	20·1	1° 38′	...	12·3	10·3
	$E_c-\omega$	1 5 104·4	66·3	32·5	29·4	33·3	1 39 36·6	15·1	2·3	5·9	14·0	1 22	58·9	30·6	...	1 38	36·0	19·1	19·6	1 32	...	38·7	38·4
	$\Omega-\omega$	73·3	37·2	1·4	−1·7	2·2	91·7	57·5	13·9	9·7	12·6	...	64·2	13·4	...	...	58·9	15·1	11·0	...	...	16·7	12·3
	$E_c-\Omega$	1 5 31·1	31·1	31·1	31·1	31·1	1 36 6·9	17·6	48·4	56·2	61·4	1 21	54·7	86·2	...	1 37	37·1	64·0	66·6	1 31	...	82·0	88·1
January 1909	n	...	2	2	2	...	...	2	2	2	...	...	2	2	2	...	2	2	2	...	2	2	2
	ω_1	...	177·1	173·1	169·4	...	...	356·8	350·5	343·1	...	...	385·3	376·4	388·5	...	374·4	385·7	358·0	...	402·1	392·8	384·5
	ω_2	...	−4·4	−4·2	−4·1	...	...	−27·1	−28·3	−25·5	...	...	−26·7	−25·6	−25·1	...	−29·4	−28·5	−27·6	...	−32·3	−31·2	−30·3
	E_c	1° 6′	75·5	31·6	22·5	...	1° 44′	67·4	45·0	39·4	...	1° 28′	65·0	54·3	45·3	1° 43′	93·7	73·7	52·8	1° 38′	83·2	64·7	52·9
	$E_c-\omega$	1 5	82·8	42·9	37·2	...	1 39	35·7	20·8	21·6	...	1 23	6·4	3·7	1·9	1 38	48·7	36·5	22·4	1 32	73·4	63·1	58·7
	$\Omega-\omega$	...	51·7	11·6	6·1	...	...	96·4	28·6	21·3	...	...	104·6	30·3	21·7	...	100·1	30·4	22·8	...	107·1	32·8	24·8
	$E_c-\Omega$	1 5	31·1	31·1	31·1	...	1 38	−0·7*	52·0	60·5	...	1 21	21·8	93·4	100·2	1 37	8·6	66·1	59·6	1 31	26·3	90·3	93·9

* This means that the quantity is $1° 38' 0'' - 0''\cdot7 = 1° 37' 59''\cdot3$.

7. For the computation of heights values of χ are given in table XXXI: also the angular values in seconds of one foot, which enable us to compute at once changes of height due to small changes of angular elevation. The values of χ have been computed with the help of table VII as was done in Chapter II, §6.

TABLE XXXI.

Values of χ and the angular values of one foot.

	Nojli R.S.	Mussooree	Nag Tiba	Bandar-punch	Srikanta	Jaonli	Kedarnath
Nojli Tower	...	39′ 57″·6 0″·851	48′ 33″·4 0″·700	1° 20′ 49″·5 0″·420	1° 26′ 45″·0 0″·391	1° 24′ 15″·2 0″·403	1° 30′ 26″·8 0″·375
Mussooree	39′ 57″·5 0″·851	...	8′ 36″·1 3″·95	40′ 58″·8 0″·829	48′ 10″·4 0″·704	46′ 55″·1 0″·723	55′ 18″·3 0″·613
Nag Tiba	48′ 33″·2 0″·700	8′ 36″·1 3″·95	...	32′ 29″·1 1″·045	40′ 22″·0 0″·840	39′ 44″·6 0″·853	48′ 56″·1 0″·692

8. The computations of heights, for stated values of a, from the formula

$$h = c\left(1 + \frac{h_a}{R}\right)\sin\left(a + \frac{\chi}{2}\right)\sec\left(a + \chi\right)$$

which is derived from (67) and (68), are given in the next three tables. Log c is taken from table VII, and a value of a is selected in each case which is close to the values obtained in tables XXII, XXVI and XXX.

(*a*) *From Nojli Tower.*

Here $h_a = 937 \cdot 0$ feet and $1 + \frac{h_a}{R} = 1 \cdot 000045$.

TABLE XXXII.

	Bandarpunch	Srikanta	Jaonli	Kedarnath
a	1° 38′ 50″	1° 22′ 30″	1° 38′ 0″	1° 32′ 30″
$a + \frac{\chi}{2}$	2 19 14·8	2 5 52·5	2 20 7·6	2 17 43·4
$a + \chi$	2 59 39·5	2 49 15·0	3 2 15·2	3 2 56·8
$\log \sin \left(a + \frac{\chi}{2}\right)$	$\bar{2}$·6073922	$\bar{2}$·5635686	$\bar{2}$·6101266	$\bar{2}$·6026176
$\log \sec (a + \chi)$	0·0005934	0·0005265	0·0006106	0·0006153
$\log \left(1 + \frac{h_a}{R}\right)$	0·0000196	0·0000196	0·0000196	0·0000196
$\log c$	5·6907758	5·7217874	5·7092623	5·7403184
Sum $= \log h$	4·2987810	4·2859021	4·3200191	4·3435709
h	19896·7	19315·3	20893·9	22058·2
h_a	937·0	937·0	937·0	937·0
Height above sea-level	20833·7	20252·3	21830·9	22995·2

(*b*) *From Mussooree.*

Here $h_a = 6084 \cdot 4 + 886 \cdot 7 = 6971 \cdot 1$ feet and $1 + \frac{h_a}{R} = 1 \cdot 000334$.

TABLE XXXIII.

	Bandarpunch	Srikanta	Jaonli	Kedarnath
a	2° 50′ 30″	2° 11′ 10″	2° 35′ 10″	2° 15′ 20″
$a + \frac{\chi}{2}$	3 10 59·4	2 35 15·2	2 58 37·6	2 42 59·2
$a + \chi$	3 31 29	2 59 20	3 22 5	3 10 38
$\log \sin \left(a + \frac{\chi}{2}\right)$	$\bar{2}$·7445134	$\bar{2}$·6546194	$\bar{2}$·7154770	$\bar{2}$·6757155
$\log \sec (a + \chi)$	0·0008223	0·0005912	0·0007508	0·0006681
$\log \left(1 + \frac{h_a}{R}\right)$	0·0001450	0·0001450	0·0001450	0·0001450
$\log c$	5·3959167	5·4667208	5·4555308	5·5271726
Sum = $\log h$	4·1413974	4·1220764	4·1719036	4·2037012
h	13848·3	13245·7	14856·1	15984·6
h_a	6971·1	6971·1	6971·1	6971·1
Height above sea-level	20819·4	20216·8	21827·2	22955·7

(c) *From Nag Tiba.*

Here $h_a = 9076\cdot5 + 886\cdot7 = 9963\cdot2$ feet and $1 + \frac{h_a}{R} = 1\cdot0004775.$

TABLE XXXIV.

Peak	Bandarpunch	Srikanta	Jaonli	Kedarnath
a	2° 52′ 40″	2° 3′ 20″	2° 28′ 30″	2° 5′ 20″
$a + \frac{\chi}{2}$	3 8 54·6	2 23 31·0	2 48 22·3	2 29 48·1
$a + \chi$	3 25 9	2 43 42	3 8 15	2 54 16
$\log \sin \left(a + \frac{\chi}{2}\right)$	$\bar{2}$·7397625	$\bar{2}$·6205023	$\bar{2}$·6898215	$\bar{2}$·6391054
$\log \sec (a + \chi)$	0·0007738	0·0004926	0·0006515	0·0005582
$\log \left(1 + \frac{h_a}{R}\right)$	0·0002073	0·0002073	0·0002073	0·0002073
$\log c$	5·2951554	5·3900718	5·3836287	5·4742023
Sum = $\log h$	4·0358990	4·0112740	4·0743090	4·1140732
h	10861·7	10263·0	11866·1	13003·9
h_a	9963·2	9963·2	9963·2	9963·2
Height above sea-level	20824·9	20226·2	21829·3	22967·1

9. We can now readily deduce the heights for all the values of $E_c - \Omega$ given in tables XXII, XXVI and XXX by means of the angular values of 1 foot given in table XXXI and the values of height found in tables XXXII to XXXIV. An example will make the process clear. The selected value of a in table XXXIII for Bandarpunch is 2° 50′ 30″. The value of $E_c - \Omega$ found for November 1906 at 8 hours from table XXII is 2° 50′ 33″·2: so that the excess is $da = 3''\cdot2$. Now 3″·2 corresponds to $\frac{3\cdot2}{0\cdot829} = 3\cdot9$ feet, since 0″·829 corresponds to 1 foot. Hence the deduced height is 20819·4 + 3·9 = 20823·3 feet. In table XXXV all the values of da and the excess dh of deduced height above the various heights stated are given. Thus for the case we have just considered we find the entries 3″·2 and 23·3 feet.

TABLE XXXV.—Values of *da* (*italics*) in seconds and *dh* in feet.

Peak		Bandarpunch					Srikanta					Jaonli					Kedarnath				
		dh = Height − 20800					*dh* = Height − 20200					*dh* = Height − 21800					*dh* = Height − 22900				
Observed at	Hour	8	10	12	14	16½	8	10	12	14	16½	8	10	12	14	16½	8	10	12	14	16½
Nojli	March 1906	*−24·1* −23·7	*+ 9·8* +57·0	... ...	... ...	... ...	... ...	... ...	... ...	... ...	... ...	*−14·1* − 4·1	*+ 6·6* +47·3	*+ 0·7* +32·6	... ...	... ...	... ...	*− 2·9* +87·5	... ...	... ...	... ...
	March 1907	*−43·1* −68·9	*−32·4* −43·4	*− 1·6* +29·9	*+ 6·2* +48·5	*+11·4* +60·8	... ..	*− 35·3* − 38·0	*− 3·8* +42·6	... ...	... ...	... ...	*−22·9* −25·9	*+ 4·0* +40·8	*+ 8·6* +52·2	... ...	... ...	... ...	*− 8·0* +73·9	*− 3·9* + 84·8	... ...
	January 1909	... ...	*−50·7* −87·0	*+ 2·0* +38·5	*+10·5* +58·7	... ...	... ...	*− 68·2* −122·1	*+ 3·4* +61·0	*+10·2* +78·4	... ...	... ...	*−51·4* −96·6	*+ 6·1* +46·0	*− 0·4* +29·9	... ...	... ...	*−63·7* −74·7	*+ 0·3* +96·0	*+ 3·9* +105·6	... ...
Mussooree	November 1906	*+ 3·2* +23·3	*− 0·6* +18·7	*+ 5·3* +25·8	... ...	... ...	*+ 3·7* +22·1	*− 3·1* + 12·4	*+ 5·1* +24·0	*+ 6·6* +26·2	... ...	*+ 5·1* +34·3	*+ 1·0* +28·6	*+ 7·0* +36·9	*+11·3* +42·8	... ...	*+ 1·0* +57·3	*− 2·4* +51·8	*+ 6·2* +65·8	... ...	... ...
	April 1907	*+ 4·9* +25·3	*+ 2·0* +21·8	*+ 4·9* +25·3	... ...	... ...	*+ 8·7* +29·2	*+ 2·8* + 20·8	*+ 7·4* +27·3	... ...	... ...	*+ 9·7* +40·6	*+ 5·8* +35·2	*+ 8·5* +39·0	... ..	... ...	*+ 3·4* +61·2	*+ 1·8* +58·6	... ...	... ...	... ...
	November 1907	*+ 4·1* +24·3	*+ 3·7* +23·9	*+ 1·9* +21·7	*+ 7·7* +28·7	... ...	*+ 4·4* +23·1	*+ 2·7* + 20·6	*+ 4·4* +23·1	*+ 8·8* +29·3	... ...	*+ 3·2* +31·6	*+ 4·2* +33·0	... ...	*+10·5* +41·7	... ...	*+ 5·4* +64·5	*− 1·2* +53·7	*− 2·9* +51·0	*+ 7·4* + 67 8	... ...
	March 1908	*+ 1·7* +21·5	*− 2·7* +16·1	*+ 0·5* +20·0	*+ 2·3* +22·2	... ...	*− 1·7* +14·4	*− 2·7* + 13·0	*+ 1·5* +18·9	*+ 4·3* +22·9	... ...	*+ 6·1* +35·6	*+ 1·8* +29·7	*+ 4·8* +33·8	*+ 7·3* +37·3	... ...	*− 1·2* +53·7	*+ 1·1* +57·5	*+ 1·0* +57·3	... ...	... ...
	April 1908	*+ 2·7* +22·7	*+ 5·7* +26·3	*+ 6·4* +27·1	... ...	... ..	*− 1·0* +15·4	*+ 5·1* + 24·0	*+ 7·1* +26·9	... ...	... ...	*− 4·2* +21·4	*+ 4·1* +32·9	*+ 7·9* +38·1	... ...	... ...	*− 7·4* +43·6	*+ 7·0* +67·1	*+ 5·2* +64·2	... ...	... ...
	October 1908	*+10·5* +32·1	*+ 9·7* +31·1	*+14·4* +36·8	*+17·8* +40·9	*+27·4* +52·5	*+12·3* +34·3	*+ 10·7* + 32·0	*+19·0* +43·8	*+19·1* +43·9	*+25·3* +52·7	*+18·5* +52·8	*+17·7* +51·7	*+22·0* +57·6	*+23·8* +60·1	*+27·4* +65·1	*+11·8* +74·9	*+10·6* +73·0	... ...	*+ 20·5* + 89 1	*+20·5* +89·1
	April 1909	*+ 8·7* +29·9	*+ 8·0* +29·1	*+ 8·4* +29·5	... ...	... ...	*+ 7·7* +27·7	*+ 13·7* + 36·3	*+11·6* +33·3	... ...	... ..	*+13·7* +46·1	*+16·7* +50·3	*+17·0* +50·7	... ...	... ...	... ...	... ...	*+11·8* +74·9	... ...	... ...
Nag Tiba	May 1906	*− 0·1* +24·8	*+ 5·6* +30·3	*+ 6·2* +30·8	... ...	... ...	*− 3·5* + 22·0	*− 1·3* + 24·7	*+ 2·1* +28·7	... ...	... ...	*+ 3·2* +33·1	*+ 4·3* +34·3	*+ 5·5* +35·7	... ...	... ...	*− 5·6* +59·0	*+ 1·1* +68·7	... ...	... ...	... ...
	May 1907	*+ 3·9* +28·6	... ...	*+10·3* +34·8	... ...	... ...	*+ 3·4* +30·2	... ...	*+10 2* +38·3	... ...	... ...	*+11·1* +42·3	... ...	*+17·8* +50·2	... ...	... ...	*+ 6·0* +75·8	... ...	*+15·2* +89·1	... ...	... ...
	April 1908	*+ 5·7* +30·4	*+12·9* +37·2	... ...	... ...	... ...	*+ 7·6* +35·2	*+ 13·2* + 41·9	... ...	... ...	... ...	*+15·2* +47·1	*+19·0* +51·6	... ...	... ...	... ...	*+ 5·8* +75·5	*+11·1* +83·1	... ...	... ...	... ...

10. On inspecting table XXXV the following points are apparent regarding the heights deduced:—

(*a*). *From Nojli.* The heights deduced at the earlier hours are much less than those deduced from the later hours; and the heights found from 12 and 14 hours are in fair agreement with the heights found from Mussooree and Nag Tiba, due regard being paid to the extreme length of the rays. These Nojli heights are on the whole a little greater than the Mussooree and Nag Tiba heights, a fact which may be due to their having been observed earlier in the year. This may also account for the January heights being greater than the March heights.

(*b*). *From Nag Tiba.* The heights deduced from 8 hour observations are distinctly less than those deduced from 12 hour observations.

(*c*). *From Mussooree.* Here also the 8 hour heights are less than the 12 hour heights, though this is not so marked as in the case of Nag Tiba. The fact is brought out with more certainty on taking means. If this be done for all occasions on which observations were made at the three hours 8, 10, 12, we get the following results:—

TABLE XXXVI.

Mean value of *dh* for	From observations at Mussooree at			
	8 hours	10 hours	12 hours	14 hours
Bandarpunch ...	25·6	23·9	26·6	30·7
Srikanta ...	23·7	22·7	28·2	32·0
Jaonli ...	38·5	38·1	42·7	46·8
Kedarnath ...	54·8	57·5	59·6	70·0
Mean ...	35·7	35·6	39·3	44·9

The last column is found by taking the mean of the differences of the 14 hour observations from the mean of 8, 10, 12 hours, year by year, and adding to the general mean of 8, 10, 12 hours for each peak. The observations at 16½ hours in October 1908 give values of the height believed to be too high: and the reason for this is probably that the lower air is beginning to be cooled by the earth, and the temperature gradient is accordingly being disturbed. When the temperature is falling, is the most unfavourable time for the application of our formulæ.

In the Nag Tiba heights there are too many gaps in the observations to allow us to treat them in quite the same way. We can however take out the mean differences of observations at 10 hours and 12 hours from those at 8 hours and then fill up the gap. Then to Bandarpunch the two differences 5·5 and 6·8 are found between 8 and 10 hours. Of these the mean is 6·2 which, added to 28·6 gives 34·8. Again we have the differences 6·0, 6·2 between 8 hours and 12 hours, giving a mean 6·1 which added to 30·4 gives 36·5.

Proceeding in this way we form the next table.

TABLE XXXVII.

Mean value of *dh* for	From observations at Nag Tiba at		
	8 hours	10 hours	12 hours
Bandarpunch ...	27·9	34·1	34·0
Srikanta ...	29·1	33·8	36·5
Jaonli ...	40·8	43·7	46·1
Kedarnath ...	70·1	78·8	83·4
Mean ...	42·0	47·6	50·0
Excess over Mussooree determination	6·3	12·0	10·7

Both tables XXXVI and XXXVII show the same tendency for the heights determined at the later hours to be greater than those determined at the earlier hours. Moreover the heights determined from Nag Tiba exceed those determined from Mussooree by a mean amount of 9·7 feet. This discrepancy may be partly due to an incorrect value in the assumed plumb-line deflection at Nag Tiba. Another possible explanation is given later (*see* § 13). It may also be due to a true difference in height on account of snow*.

11. The height of a snow peak is liable to change; increasing with deposit of fresh snow and diminishing when the snow melts, slides down the hill side or is blown away by wind. It is not to be expected that the height given by observation should remain the same from season to season and from year to year. On the other hand no figures appear to be available showing the amount by which the height varies. This variation has been masked by the seasonal variation of refraction: and it is only after the refraction has been taken account of with considerable precision that the variation in actual height can be discussed.

The figures in table XXXV do give some evidence as to such a change. The fairly good accordance between observations from Mussooree *at the same season* but reduced from observations at different hours, leads us to suppose that the relative height of the same peak at various seasons is fairly well established, provided we use the observations taken at the same hours. For the observations at the several hours appear to differ in a regular manner when the means of sufficient observations to reduce observation error to a small probable amount are taken. As will be shown in § 13 there is some reason to attribute the differences in height determined from observations at different hours to differences in illumination of the peak observed, so that it is possible that the differences occurring in table XXXV at the same season are due to this fact and not to refraction. We will assume that such is the case.

Consider then the mean of the heights found at 8, 10, 12 hours at Mussooree and Nag Tiba and the mean of heights found at 12 and 14 hours at Nojli at the several seasons.

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* Dr. Walker, Director General of Observatories, in a letter dated March 1913 says:—"In some parts of the northern Himalayas apparently most of the snow falls in April, but on the N.W. frontier most of the snow-fall is in February, probably".

Forming the means of the observations at 8, 10, 12 hours for Mussooree and at 8, 10, 12 hours at Nag Tiba; and at 12, 14 hours at Nojli we get the following values arranged in chronological order:

TABLE XXXVIII.

Serial No.	Observed from	Date	Bandarpunch		Srikanta		Jaonli		Kedarnath		Mean		No. of observations for condition $u\omega_1 + v\omega_2 = \Omega - \omega$
			No. of obsn.	Values of *dh*	No. of obsn.	Values of *dh*	No. of obsn.	Values of *dh*	No. of obsn.	Values of *dh*	No. of obsn. to peaks	Values of *dh*	
1	Nag Tiba	May 1906	8	28·6	7	25·1	8	34·4	6	66·9	29	38·8	20
2	Mussooree	Nov. 1906	22	22·6	23	19·5	22	33·3	20	58·3	87	33·4	49
3	Nojli	March 1907	4	39·2	1	51·3	4	46·5	4	79·4	13	54·1	5
4	Mussooree	April 1907	15	24·1	15	25·8	15	38·3	12	61·0	57	38·3	40
5	Nag Tiba	May 1907	7	32·8	7	34·6	7	46·4	6	83·2	27	49·3	13
6	Mussooree	Nov. 1907	10	23·3	10	22·3	7	33·8	10	56·4	37	33·9	19
7	Mussooree	March 1908	15	19·2	13	15·4	13	33·0	13	56·2	54	30·9	29
8	Mussooree	April 1908	6	25·4	6	22·1	6	30·8	6	58·3	24	34·1	12
9	Nag Tiba	April 1908	4	34·8	4	40·3	4	50·7	4	82·3	16	52·0	7
10	Mussooree	Oct. 1908	35	33·3	35	36·7	35	54·0	24	75·1	129	49·7	56
11	Nojli	Jany. 1909	4	48·6	4	69·7	4	38 0	4	100·8	16	64·3	4
12	Mussooree	April 1909	12	29·5	12	32·4	12	49·0	1	68·9	37	44·9	28

Considering the figures in the last column we find changes which in the main could be attributed to snow fall. Thus No. 3 is higher than No. 2 on account of winter snows. A decrease takes place between Nos. 3 and 4 due to melting of snow in April. The increase in No. 5 may be attributed to April snow-fall alluded to at the end of § 10. The same feature occurs in Nos. 7, 8, 9. Nos. 9 and 10 show a smaller decrease during the summer than occurred between Nos. 5 and 6 and the reality of this is upheld by the following Nos. 11, 12, which show a maximum in January. It is to be remarked that the summer of 1908 was characterised by a specially heavy rain-fall during the monsoon at Mussooree.

No doubt January observations would usually and rightly show a greater height than those of other months. Unfortunately we only have January observations in the year 1909.

The increased height in October 1908, January 1909 and April 1909 is certainly the most notable feature of the variation. But even here the change in height is not very large. It is a rather remarkable fact that the heights of the mean peak we have deduced between May 1906 and April 1909 from three stations only show a variation in *dh* from 30·9 to 64·3 = 33·4 feet. Excluding the last winter, the range is from 30·9 to 54·1 or 23·2 feet. The range in Mussooree values alone is 18·8. All of these ranges are considerably smaller than might well have been expected, so that there is reason to suppose that the effects of refraction have been eliminated satisfactorily. It is to be remarked that 33 feet only corresponds to an angular change of some 13″ in the case of observations from Nojli to the peaks.

12. A curious fact exhibited by the Nojli results, is that apparently the *diurnal change* in refraction is less in absolute amount in the case of the rays to the snow-peaks than it is in the case of the ray to Mussooree: notwithstanding the fact that the total refraction in the former case is at midday about double that in the latter. This is explained to some extent, but not completely, by the fact that the rays to the peaks are not so oblique to the strata of the atmosphere as the Mussooree ray.

Let us denote by Ω and $\Omega + d\Omega$ the refractions at different hours on a certain ray proceeding to a given height h above observation station, and let dashed letters represent corresponding quantities on another ray proceeding to *the same height.* Then if ϕ, ϕ' be the zenith distances of these rays we have from (64).

$$\left.\begin{aligned} \Omega &= f_1\,\omega_1 + f_2\,\omega_2 \\ \Omega' &= f_1\,\omega_1' + f_2\,\omega_2' \end{aligned}\right\} \qquad (103).$$

Also by (63) since $s \cos \phi = h$ is the same for both rays $\dfrac{\omega_2}{\omega_1} = \dfrac{\omega_2'}{\omega_1'}$.

Hence by (103)

$$\frac{\Omega}{\Omega'} = \frac{\omega_1\left(f_1 + f_2\dfrac{\omega_2}{\omega_1}\right)}{\omega_1'\left(f_1 + f_2\dfrac{\omega_2'}{\omega_1'}\right)} = \frac{\omega_1}{\omega_1'}$$

and since $l = h \tan \phi$, we have from (63)

$$\frac{\Omega}{\Omega'} = \frac{\tan \phi}{\tan \phi'} = k, \text{ say} \qquad (104).$$

So

$$\Omega = k\,\Omega'$$

and in the same way

$$\Omega + d\Omega = k\,(\Omega' + d\Omega')$$

so that by subtraction

$$d\Omega = k d\Omega' \qquad (105).$$

If then we know the *difference* of refraction on one ray, we can at once compute the corresponding difference of refraction of another ray of different inclination to vertical, which proceeds to the same height.

We can apply this result to the case of the ray from Nojli to Mussooree and of the rays in the *direction* of one of the peaks, up to the same height as Mussooree. That is to say, we can find what variation in refraction is attributable to that portion of the ray which does not exceed the height of Mussooree. The Mussooree and Nag Tiba results indicate that the diurnal change in the portion of the atmosphere above Mussoorce is very small compared with that found in the plains.

Taking $d\Omega$ to apply to Mussooree and adding suffixes 1, 2, 3, 4 to characterise the four peaks we form the following table: $d\Omega$ being the difference in values of E_c given in table XXX.

Now taking approximate values of E_c to nearest minute we compute the values of k_1, k_2, k_3, k_4 from (104), and get

$$k_1 = \tan 1^\circ 45' \;\; \cot 1^\circ 8' = 1\cdot544$$
$$k_2 = \tan 1^\circ 28' \;\; \cot 1^\circ 8' = 1\cdot294$$
$$k_3 = \tan 1^\circ 44' \;\; \cot 1^\circ 8' = 1\cdot530$$
$$k_4 = \tan 1^\circ 38' \;\; \cot 1^\circ 8' = 1\cdot441$$

With these values we compute $\dfrac{d\Omega}{k}$ also given in table XXXIX.

TABLE XXXIX.

Date	Between hours	$d\,\Omega$	$d\,\Omega_1$	$d\,\Omega/k_1$	$d\,\Omega_2$	$d\,\Omega/k_2$	$d\,\Omega_3$	$d\,\Omega/k_3$	$d\,\Omega_4$	$d\,\Omega/k_4$
March 1906	8 and 10	52·8	33·9	34·2	...	...	47·8	34·5	...	...
March 1907	8 and 14	82·5	47·0	53·4	...	...	...	...	...	...
	10 and 12	37·2	15·3	24·1	22·0	28·7	19·5	24·3	...	...
January 1909	10 and 14	53·0	28·0	34·3	19·7	40·9	40·9	34·6	30·3	36·8

It is seen that in all cases except two the value of $\frac{d\Omega}{k}$ is greater than the apparent values of $d\Omega_1$ etc. In the higher part of the ray the discrepancy will be increased seeing that although little diurnal change occurs in the upper layers, yet the direction of the ray entering these layers has been altered by the diurnal change in the lower layers, and if the ray continues with the curvature it had at the time of smaller refraction, yet the variation on the whole ray must surely be greater than that portion of the variation arising from its passage through the lower layers.

13. The apparent values of $d\Omega_1$ etc. seem to be inexplicably small. However it may be that in taking the difference of E_c at two hours we have not actually arrived at the variation in the refraction Ω. This would be the case if a different and lower point on the peak was observed at the earlier hour than was done at the later hour. Is it not possible that this has been done owing to the difference in illumination at the two hours? The idea of the apparent positions of graduations of a theodolite as viewed by a microscope being altered according as the illumination is from one side or the other is one well known to users of these instruments. The snow-peaks we are discussing, as seen from Nojli lie between 32° and 53° east of north. The sun in March rises in the east almost behind them; while in the afternoon it shines on to the faces directed more or less towards Nojli. The actual numerical effect of this change in illumination, could only be determined if we knew with some precision the actual shape of the peaks. We only know the appearance of these peaks at distances of 40 miles and upwards, and at that distance gain no knowledge of the orientation and inclination of the faces of the peaks.

Another thing to be considered is that the peaks are visible only when they show sufficient contrast with the back ground. In some cases they are strongly illuminated and appear as white objects against the blue or grey sky. When the illumination is not from overhead shadows are cast and parts of the snow are well illuminated and appear white while other portions in shade, appear darker than the back ground. In this case probably there are intermediate portions which are illuminated so as to appear of the same brightness as the back ground. Such portions will not be visible as distinct from the back ground. It appears then that it is possible that portions of the peaks may be lost in the back ground under certain conditions of illumination. One condition for visibility is that there should be sufficient contrast between the hill and the back ground.

In the case of very long rays a certain amount of diffused light will come from the atmosphere *in front of* the peak. This will tend to make the peak fainter and more liable to be lost in the back ground. This is suggested as a reason for the fact that the observations from

Nojli show smaller heights at the early hours than those at and after midday. The same kind of reason may cause the peaks to appear lower from Mussooree than they do from Nag Tiba. However this question is mixed up with that of the variation in height due to snow-fall, and nothing can be said with certainty. It might be studied if simultaneous observations were made to the peaks from Mussooree and Nag Tiba.

14. As has been mentioned in § 2, it was hardly to be expected that the u, v method of calculating the refraction could give entirely satisfactory results in the case of observations from the plains. Yet it is believed on the strength of the results given in table XXXIX that the discrepancies between the heights deduced from observations at different hours from Nojli are by no means entirely due to failure to correct for refraction: and the suggestion made in § 13 seems to show that there may be considerable uncertainty as to the point to be intersected, on account of unsatisfactory illumination. At best a distant snow-peak is a difficult object to intersect with great precision.

15. Enough has been said to show that the u, v method of calculating refraction does account for most of the observations with fair precision, and reduces discrepancies to magnitudes of the same order as the instrumental errors. In our formulæ no notice has been taken of the effect of humidity on the change of refractive index μ. The effect is not a large one but, such as it is, it is taken into account by the factors u and v, if these are determined from observations of rays to known points. It appears, then, that it is desirable to find means of calculating u and v. We have already found values of u and v in a way which is not wholly satisfactory. Two other methods appear to be possible: (1) actual measurements of temperature and pressure at two or more heights, (2) observations to celestial objects whose refraction can be determined. Both of these methods have inherent difficulties, and will be considered later.

CHAPTER IV.

The Diurnal Change in Refraction, and the Calculation of Refraction from Barometric Readings.

1. Allusion has already been made in Chapter III to the diurnal change in refraction. On glancing at the values of E_c in tables XXII, XXVI and XXX and taking account of the sign it is seen that the diurnal change from Mussooree and Nojli in general shows a decrease from the 8 hour value to a minimum at 14 hours.

In the case of Mussooree the descending ray to Nojli shows this effect more markedly than the ascending rays to Nag Tiba and the peaks: while at Nag Tiba the effect of the change in the case of the ascending rays is sometimes to increase E_c as the hour becomes later. The observations from Nojli show a very much larger variation. The natural deduction is that the diurnal change is mostly to be attributed to the lower layers of the atmosphere and that at heights above 10,000 feet practically no such change occurs, or possibly a small change occurs in the opposite direction. In consideration of the formula for refraction these facts may be accounted for by conceiving that the lower layers of the atmosphere have a much larger diurnal change in temperature than the upper layers have. This is a fact which is otherwise well established. It is a question of interest whether at greater heights a state is reached where the diurnal change in temperature is practically nothing. This question can ouly be answered by observation. An analogous case is that of underground temperature. In this case even the annual change in temperature at depths of 25 feet at Dehra Dun is only one-tenth of the change in temperature of the air as recorded daily at 4 P. M. The case of the earth and the atmosphere are essentially different on account of the impossibility of convective adjustment of temperature in the case of the former, except by means of rain percolating through it.

2. In charts Nos. II, III the observed elevations, at various hours, of Mussooree from Nojli and of Bandarpunch from Nojli are plotted against temperature. Means of several days have been used in each case, and the same days have been selected for the several hours. On joining up the observations thus plotted of different hours of the same season, a remarkable tendency of the points to lie on straight lines is apparent. Further these straight lines seem to have nearly the same slope for observations to one point. The mean slope in the case of the observations to Mussooree is greater than the slope in the case of observations to Bandarpunch.

3. There appears to be no reason for expecting this simple relation between diurnal change of refraction and temperature. If the relation is truly a straight line relation in the case of observations to both Mussooree and Bandarpunch, it seems to imply that on any given day both u and v must also be related to the temperature by a straight line law. As has been pointed out in Chapter III § 13, there is some reason to believe that the actual summit of Bandarpunch is very likely invisible at the earlier hours, which renders this deduction from chart III open to question.

4. If we take a set of observations on the same days at one season for several hours we can find the most probable straight line which relates them. Thus if x and y are the arguments, the most probable line is

$$y = mx + c$$

subject to the condition that $\Sigma\,(y - mx - c)^2$ shall be a minimum.

This gives

$$\Sigma\, x\,(y - mx - c) = 0$$

and $$\Sigma\,(y - mx - c) = 0$$

whence $$c = \frac{1}{n}\Sigma y - \frac{m}{n}\Sigma x$$

and $$m = \left(\frac{1}{n}\Sigma x\,\Sigma y - \Sigma xy\right) \div \left\{\frac{1}{n}(\Sigma x)^2 - \Sigma x^2\right\} \qquad \ldots\ldots (106)$$

where n is the number of points under discussion. This process has been applied to the various observation quantities, which are now tabulated.

Diurnal Change:

Most probable linear relation between E_c and t

$$E_c - mt - c = 0 \qquad \ldots\ldots\ldots (107)$$

When actual values are substituted we find value of this expression is $\Delta E - m\,\Delta t$ where ΔE and Δt are the errors in E_c and t. We now tabulate observed values of E_c, t, deduced values of m and c for each season and the quantities ΔE, assuming that $\Delta t = 0$ and Δt assuming that $\Delta E = 0$.

TABLE XL.

Nojli R. S. to Mussooree.

Season	Dates	Hour	No. of observations	$E_t - 1° 8'$	t	m	c	ΔE	Δt
Dec. 1905	Dec. 1, 2, 3, 4, 5, 7, 8	8	19	$185''\cdot8$	$49°\cdot8$	−4·930	430·8	$+0''\cdot5$	$-0°\cdot1$
		10	14	98·9	67·0			−1·6	+0·3
		12	13	58·9	75·7			+1·3	−0·3
		14	16	48·5	77·5			−0·2	0·0
Mar. 1906	Mar. 14,15, 16	8	6	119·3	65·0	−4·442	406·4	+1·6	−0·4
		10	6	59·6	77·0			−4.8	+1·1
		12	5	43·4	81·5			−1·0	+0·2
		14	7	41·7	83·0			+4·0	−0·9
Dec. 1906	Dec. 4,5	8	5	168·9	52·2	−4·811	421·4	−1·3	+0·3
		10	4	94·8	68·9			+4·9	−1·0
		12	4	52·9	75·7			−4·3	+0·9
		14	5	44·9	78·4			+0·7	−0·1
Mar. 1907	Mar. 20,23	8	5	116·4	59·7	−4·321	372·4	+2.0	−0·5
		10	6	65·6	70·5			−2·1	+0·5
		12	6	44·8	74·4			−6·1	+1·4
		14	4	42·4	77·8			+6·2	−1·4
Jan. 1909	Jan. 16,23, 26	8	7	172·7	48·8	−4·578	394·4	+1·7	−0·4
		10	7	106·1	62·1			−4·0	+0·9
		12	5	73·7	70·0			−0·2	0·0
		14	8	66·3	72·2			+2·5	−0·5
Mar. 1909	Mar. 9,10,12	8	7	131·6	63·6	−4·034	388·0	+0·2	0·0
		10	6	74·1	77·7			−0·4	+0·1
		12	6	42·5	86·3			+2·7	−0·7
		14	6	38·6	86·0			−2·4	+0·6

The quantities ΔE, Δt are of the order of errors of observation. It is to be remembered that *half* the values given above are sufficient to explain the discrepancies provided that errors both in E and t occur simultaneously. The value of m varies by some 20 °/₀, being consistently lower in March than in December and Jauuary.

TABLE XLI.

Nojli R. S. to Bandarpunch.

Season	Dates	Hour	No. of observations	$E_c - 1° 44'$	t	m	c	ΔE	Δt
				"	°			"	°
Nov. Dec. 1905	No., 28, 29, 30 Dec. 1,2,5,7	8	10	131·8	50·2	−3·469	305·7	− 0·4	+ 0·1
		10	10	69·0	68·0			− 0·8	+ 0·2
		12	10	39·1	77·0			+ 0·6	− 0·2
Mar. 1906	Mar. 16	8	2	68·4	66·4	−3·109	279·0	− 4·2	+ 1·3
		10	2	47·2	79·2			+14·4	− 4·6
		12	1	20·2	83·3			+ 0·2	− 0·1
		14	2	6·2	84·4			−10·4	+ 3·3
Dec. 1906	Dec. 5	8	1	137·7	52·9	−3·891	344·3	− 0·8	+ 0·2
		10	2	75·3	69·5			+ 1·4	− 0·4
		12	2	56·3	74·5			+ 1·9	− 0·5
		14	3	36·4	78·5			− 2·5	+ 0·6
Mar. 1907	Mar. 23	8	2	87·9	60·3	−3·594	303·9	+ 0·8	− 0·2
		10	2	59·1	67·6			− 1·8	+ 0·5
		12	2	42·9	72·9			+ 1·0	− 0·3
Jan. 1909	Jan. 23	8	2	127·2	45·9	−3·093	268·7	+ 0·4	− 0·1
		10	2	74·4	62·6			− 0·7	+ 0·2
		12	2	48·3	70·4			− 2·7	+ 0·9
		14	2	48·7	72·1			+ 3·0	− 1·0

Here the values of ΔE and Δt are much larger than they were in the case of the observations to Mussooree. This is natural as observations to a snow-peak 93 miles distant arc obviously less precise than observations to a helio at half the distance. Moreover the number of observations to the peak is much smaller than the number to Mussooree. Further if the suggestion made in Chapter III §13 is correct, it is quite likely that even though the refraction does truly bear a linear relation to the temperature, yet the effects of illumination, which are dependent on the position of the sun, are not directly related to thc temperature: and this would cause the deduced values of ΔE and Δt to be increased.

5. In the case of the observations to Mussooree the precision of the linear relationship deduced is so remarkable that it seems natural to suppose that it has a physical reality: and there seems to be no reason to imagine that the relationship would be of a different type along another ray. If then we suppose that the diurnal change has a linear relationship with the temperature on two independent rays, we at once see that for this to be the case both u and v must be linear functions of the temperature on a given day; unless the irregularities of the temperature gradient are sufficiently marked to cause the expression $u\omega_1 + v\omega_2$ to fail to represent the refraction. This is a point which cannot be decided from observations which are at present available. If observations were taken from Nojli to Mussooree at various hours, and simultaneous readings of the pressure and temperature were made both at Nojli and Mussooree, it would be possible to compute the refraction from the deduced value of u and v and so to find whether the expression $u\omega_1 + v\omega_2$ did represent the refraction with sufficient accuracy. It would be still better if a set of simultaneous temperature readings could be made by means of kites or otherwise. The method of finding u and v from barometric observations is explained in §10.

6. If u and v are very nearly linear functions of the temperature, it at once follows that both $\frac{dt}{dh}$ and $\frac{d^2t}{dh^2}$ are also very nearly linear functions of the temperature. Hence we have

$$\left.\begin{aligned} \frac{dt}{dh} &= \left(\frac{dt}{dh}\right)_0 + \alpha\ (t - t_0) \\ \frac{d^2t}{dh^2} &= \left(\frac{d^2t}{dh^2}\right)_0 + \beta\ (t - t_0) \end{aligned}\right\} \quad \ldots\ldots\ldots \quad (108).$$

where α, β are constants, on any day.

∴ temperature at any height h is

$$t_h = t + h.\frac{dt}{dh} + \frac{h^2}{2}.\frac{d^2t}{dh^2} \quad \ldots\ldots$$

and the difference of temperature from the standard case represented by suffix zero is

$$(t - t_0)\left(1 + h\alpha + \frac{h^2}{2}.\ \beta\right)$$

We may express this result by saying that the difference in temperature at two times at any given height h is $\left(1 + h\alpha + \frac{h^2}{2}.\ \beta\right)$ times the difference in temperature at Nojli at the same two times.

This is a result which very likely has a good deal of truth in the lower layers of the atmosphere—those layers in fact from which the main part of the diurnal variation in refraction arises. It is not at all supposed to be true to great heights. The result implies that the daily maximum temperature occurs at the same moment at all heights. It is an observed fact that the maximum temperature at Mussooree usually occurs at least an hour earlier than it does at Dehra Dun: but the change in temperature for several hours near the time of maximum temperature is small.

7. The observations from Mussooree to the peaks show a diurnal change of the same sign as those from Nojli. If we plot the results against temperature it is seen at once that the variation is a small one and its relation to temperature is not very noticeable. The conclusion is that it is the layer of air below the level of Mussooree which gives rise to the bulk of the diurnal change in refraction, and that the diurnal change in temperature gradient at heights above Mussooree is very small. The Nag Tiba observations show a further diminution to have occurred in diurnal change of temperature gradient.

It is probable that the determining factor in the height to which the diurnal change in temperature gradient is appreciable, is the height above the plain or plateau and has not much to do with absolute height above sea-level.

8. We will now pass on to the consideration of certain barometric observations made simultaneously at Mussooree and at Dehra Dun.

Referring to (16) we will denote the quantity $\frac{C}{g_0} \cdot \frac{2}{\theta + \theta_0} \log_e \frac{p_0}{p}$ by h_a: this being an approximate expression for difference of heights. Then by (15) we have

$$h = h_a \left(1 + \frac{A_2}{6} \cdot \frac{2h^2\theta_0}{\theta + \theta_0} \quad . \quad . \quad . \quad . \quad . \quad . \right)$$

and for the present neglecting the higher terms we get

$$A_2 h^2 = 6 \cdot \frac{h - h_a}{h_a} \cdot \frac{\theta + \theta_0}{2\,\theta_0} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (109).$$

Hence if we have barometric and temperature observations at two stations whose relative height h is otherwise known we can find A_2 by means of (109). This in connection with (10) enables us to find A_1, if we neglect the higher terms involving A_3 etc., for we have

$$\theta \doteqdot \theta_0 \, (1 + A_1 \, h + A_2 \, h^2) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (110).$$

It is to be noted in passing that the quantity $h - h_a$ occurs as an error in the usual (Laplace) equation for barometric heights.

Having calculated the height h_a corresponding to a given set of values θ θ_0 p p_0 it is easy to find the change δh_a in h_a corresponding to small changes, as $\delta\theta$ in θ etc. For by logarithmic differentiation we have

$$\frac{\delta h_a}{h_a} = -\frac{\delta\theta + \delta\theta_0}{\theta + \theta_0} + \frac{1}{\log_e \frac{p_0}{p}} \cdot \left(\frac{\delta p_0}{p_0} - \frac{\delta p}{p}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (111)$$

In both equations (109) and (111) since g differs from g_0 by a small amount (in the Dehra-Mussooree case by 0·03 °/₀) we can with sufficient accuracy write equation (8) as

$$\theta \doteqdot \frac{1}{\tau} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (112).$$

9. With a view to determining the quantities u and v and so studying the diurnal change in refraction observations of temperature, humidity and pressure were made in the winter of 1911 simultaneously at Dehra Dun and Mussooree. The fact that the formula (63) gives the refraction very closely at 2 P. M. suggested that at that time the temperature gradient must be nearly adiabatic—or more precisely that the gradient was much more nearly adiabatic at 2 P.M. than it was at the early hours in the morning. In this case, neglecting the small variation of g with height, we see by (61) that $\frac{d\tau}{dh}$ is constant, whence it follows that $\frac{d^2\tau}{dh^2}$ vanishes: so it is to be expected that the quantity $A_2 h^2$ will be a small quantity and consequently by (109) that $h - h_a$ should be a small quantity. In other words we see that the error in height deduced by the Laplace barometric formula or by (16) will naturally be much smaller at the time of minimum refraction than at other times. This is borne out by the simultaneous observation above referred to, of which the details will be given shortly. The Mussooree observations were begun at Abbotsford, the office of No. 17 Party, by kind permission of Lieut.-Colonel G. P. Lenox Conyngham, R.E., and extended from 12th—25th October, when the office was closed. Readings were taken at each of the following hours :—10, 12, 13, 14, 15, 16. In the light of the results found from these readings it appeared desirable to take further observations, including some at hour 8. Colonel W. J. Bythell, R.E., Superintendent, Northern Circle, kindly allowed this to be done in his office, The Castle, and readings were taken at hours 8, 10, 12, 14, 16 between dates 20th November and 7th December 1911.

A standard Fortin barometer No. $\frac{5}{1869}$ was first of all compared with the standard at Dehra, and then carried up to Mussooree. It was feared that some air had found its way into the tube on the journey: and so another barometer of a more portable type was compared with the Dehra standard, then taken to Mussooree and compared with the Fortin, brought back and compared again with the Dehra standard. Luckily this barometer stood the double journey well and showed no measurable change relative to the Dehra standard. The results showed that the Fortin had an error of $0''\cdot031$. Although this error could no doubt have been removed, it appeared best not to attempt to alter it, but rather to apply a correction. The height of the cistern of the Fortin was fixed in both its positions by spirit-levelling and these heights are given in table VI. The barometer stations at Mussooree and Dehra are unfortunately not intervisible, so that the physical idea of a continuous air column between the two stations is not realised.

10. We now proceed to the statement of the observations in tables XLII, XLIII. The pressure at any time is $H + \delta H$ and the virtual temperature $\tau + \delta\tau$, the quantities H, τ being given at the head of the table and the small increments δH, $\delta\tau$ being given for each case. This is a form suitable for use with the approximate equation (111).

TABLE XLII.

Dehra and Abbotsford.

At Dehra $H_0 = 27·500$; $\tau_0 = 459·4 + 70 = 529·4$ Spirit-levelled height of barometer cistern 2232·8					At Mussooree $H = 23·500$; $\tau = 459·4 + 50 = 509·4$ Spirit-levelled height of barometer cistern 6682·4			At Dehra $H_0 = 27·500$; $\tau_0 = 459·4 + 70 = 529·4$ Spirit-levelled height of barometer cistern 2232·8					At Mussooree $H = 23·500$; $\tau = 459·4 + 50 = 509·4$ Spirit-levelled height of barometer cistern 6682·4		
Date	Hour	δH_0	Humidity	$\delta \tau_0$	δH	Humidity	$\delta \tau$	Date	Hour	δH_0	Humidity	$\delta \tau_0$	δH	Humidity	$\delta \tau$
1911 12 Oct.	10	+0·132	0·69	+10·7	+0·149	0·72	+11·4	1911 20 Oct.	10	+0·254	0·70	+6·6	+0·201	0·88	+6·0
	12	·113	·57	13·2	·130	·70	16·3		12	·238	·59	9·0	·188	·86	7·2
	13	·092	·56	14·9	·103	·70	18·6		13	·220	·60	10·0	·177	·84	8·1
	14	·071	·52	14·1	·091	·82	16·1		14	·197	·53	9·9	·159	·80	10·2
	15	·065	·55	16·0	·080	·77	16·9		15	·182	·59	9·1	·141	·74	12·9
	16	·057	·59	15·7	·087	·72	16·5		16	·177	·56	10·8	·132	·79	8·0
13 Oct.	10	·150	·70	9·9	·172	·72	10·5	21 Oct.	10	·315	·75	6·1	·237	1·00	6·5
	12	·135	·60	12·5	·125	·70	14·3		12	·281	·70	8·4	·222	0·79	8·7
	13	·110	·54	11·8	·111	·63	16·0		13	·258	·62	11·2	·206	·78	11·5
	14	·087	·43	13·7	·104	·61	16·3		14	·234	·63	11·2	·193	·85	9·5
	15	·076	·52	15·2	·092	·60	15·9		15	·216	·69	9·4	·183	·89	9·2
	16	·067	·54	13·0	·086	·64	14·9		16	·205	·68	9·4	·175	·89	9·0
16 Oct.	10	·215	·60	6·1	·171	·60	7·4	22 Oct.	10	·297	·74	5·5	·233	·95	5·2
	12	·196	·47	10·8	·159	·54	11·0		12	·271	·63	9·1	·216	·80	9·7
	13	·176	·45	12·1	·151	·55	13·9		13	·240	·58	11·2	·203	·81	11·6
	14	·163	·45	12·2	·145	·50	13·4		14	·208	·53	10·6	·189	·75	11·7
	15	·137	·50	11·6	·135	·51	12·5		15	·186	·55	11·0	·172	·74	11·9
	16	·130	·58	12·7	·123	·56	11·4		16	·175	·58	10·3	·159	·75	11·1
17 Oct.	10	·251	·60	5·3	·196	·66	6·2	23 Oct.	10	·215	·70	7·1	·174	·80	4·6
	12	·226	·51	9·4	·181	·60	10·8		12	·174	·55	10·2	·151	·73	10·4
	13	·196	·51	10·4	·169	·58	11·7		13	·145	·49	10·7	·136	·69	12·5
	14	·167	·48	11·6	·155	·52	11·2		14	·120	·47	11·4	·123	·68	14·2
	15	·153	·52	11·7	·145	·53	11·3		15	·096	·48	11·1	·106	·67	11·9
	16	·146	·54	10·9	·134	·58	10·4		16	·087	·54	10·7	·089	·70	9·8
18 Oct.	10	·257	·65	5·0	·207	·65	5·2	24 Oct.	10	·076	·56	4·0	·040	·60	3·9
	12	·242	·56	8·6	·191	·65	9·8		12	·063	·48	7·6	·027	·54	7·0
	13	·218	·50	9·3	·176	·57	10·4		13	·043	·50	9·5	·014	·53	9·7
	14	·206	·47	8·7	·166	·58	12·8		14	·026	·45	10·1	·001	·57	11·7
	15	·198	·45	8·5	·157	·64	12·7		15	·010	·48	10·3	− ·007	·61	10·6
	16	·195	·53	9·7	·154	·64	11·5		16	− ·002	·53	8·5	− ·006	·58	8·3
19 Oct.	10	·256	·64	4·9	·198	·79	6·5	25 Oct.	10	+ ·143	·62	4·0	+ ·102	·70	5·3
	12	·235	·58	8·7	·179	·69	8·9		12	·130	·55	7·4	·087	·64	7·4
	13	·212	·55	8·1	·168	·74	12·2		13	·108	·53	8·0	·075	·60	9·2
	14	·166	·58	10·5	·151	·70	14·9		14	·092	·52	8·5	·065	·63	11·7
	15	·156	·60	10·4	·135	·70	15·1		15	·087	·50	9·0	·056	·60	10·5
	16	·148	·63	9·7	·131	·69	12·3		16	·085	·57	8·2	·060	·58	9·3

TABLE XLIII.

Dehra and Castle.

At Dehra $H_0 = 27·500$; $\tau_0 = 459·4 + 70 = 529·4$ Spirit-levelled height of barometer cistern 2232·8					At Mussooree $H = 23·500$; $\tau = 459·4 + 50 = 509·4$ Spirit-levelled height of barometer cistern 6890·4		
Date	Hour	δH_0	Humidity	$\delta\tau_0$	δH	Humidity	$\delta\tau$
1911							
20 Nov.	10	+ ·375	·89	− 13·3	+0·035	0·93	− 6·5
	12	·339	·69	− 9·2	− ·008	·92	− 4·9
	14	·285	·70	− 6·1	− ·039	·95	− 3·5
	16	·267	·66	− 6·8	− ·048	·98	− 5·8
21 Nov.	8	·229	·88	− 17·0	− ·092	·89	− 4·8
	10	·263	·81	− 8·4	− ·070	·94	− 4·6
	12	·226	·70	− 5·3	− ·089	·88	− 1·7
	14	·158	·63	− 3·6	− ·129	·88	− 1·4
	16	·140	·64	− 3·3	− ·144	·95	− 1·5
22 Nov.	8	·213	·88	− 16·7	− ·106	·67	− 6·5
	10	·238	·82	− 9·0	− ·085	·75	− 5·3
	12	·228	·66	− 5·5	− ·091	·89	− 1·1
	14	·181	·69	− 5·0	− ·105	·95	− 2·6
	16	·166	·65	− 4·6	− ·121	·94	− 3·9
23 Nov.	8	·245	·86	− 17·8	− ·082	·54	− 6·3
	10	·259	·79	− 10·2	− ·061	·54	− 4·7
	12	·177	·71	− 6·1	− ·067	·75	− 0·8
	14	·170	·51	− 4·2	− ·108	·94	+ 0·2
	16	·154	·73	− 7·1	− ·131	·94	− 3·7
29 Nov.	8	·312	·90	− 20·1	+ ·014	·50	− 3·9
	10	·337	·81	− 10·8	− ·014	·69	− 2·0
	12	·303	·70	− 8·2	− ·027	·88	− 0·6
	14	·252	·61	− 5·6	− ·060	·92	+ 0·8
	16	·223	·63	− 5·5	− ·077	·72	− 0·9
30 Nov.	8	·255	·80	− 18·4	− ·069	·52	− 3·7
	10	·274	·78	− 11·4	− ·054	·66	− 0·4
	12	·239	·67	− 7·5	− ·070	·80	+ 1·1
	14	·191	·61	− 4·3	− ·102	·92	+ 2·5
	16	·172	·76	− 4·6	− ·115	·94	− 1·6

At Dehra $H_0 = 27·500$; $\tau_0 = 459·4 + 70 = 529·4$ Spirit-levelled height of barometer cistern 2232·8					At Mussooree $H = 23·500$; $\tau = 459·4 + 50 = 509·4$ Spirit-levelled height of barometer cistern 6890·4		
Date	Hour	δH_0	Humidity	$\delta\tau_0$	δH	Humidity	$\delta\tau$
1911							
1 Dec.	8	+0·286	0·89	− 17·4	−0·072	0·71	− 6·2
	10	·329	·88	− 12·3	− ·042	·81	− 3·0
	12	·300	·75	− 8·6	− ·055	·87	− 1·2
	14	·260	·75	− 8·0	− ·075	·92	− 2·1
	16	·236	·78	− 7·1	− ·083	·93	− 3·8
4 Dec.	8	·300	·83	− 21·4	− ·042	·67	− 6·5
	10	·317	·48	− 11·7	− ·022	·65	− 4·6
	12	·293	·55	− 6·9	− ·044	·68	− 1·2
	14	·245	·51	− 4·7	− ·061	·86	− 1·6
	16	·224	·54	− 5·7	− ·086	·86	− 3·7
5 Dec.	8	·233	·92	− 24·2	− ·108	·59	− 8·6
	10	·263	·65	− 14·3	− ·070	·49	− 6·1
	12	·235	·56	− 8·0	− ·091	·44	− 2·9
	14	·182	·54	− 6·9	− ·112	·74	− 2·1
	16	·174	·63	− 6·6	− ·124	·80	− 3·5
6 Dec.	8	·246	·86	− 23·2	− ·083	·56	− 6·8
	10	·262	·72	− 14·0	− ·069	·68	− 5·2
	12	·243	·62	− 7·8	− ·081	·75	+ 0·1
	14	·199	·50	− 6·4	− ·106	·76	− 0·9
	16	·183	·55	− 6·1	− ·112	·76	− 4·0
7 Dec.	8	·233	·84	− 21·1	− ·105	·52	− 6·0
	10	·258	·81	− 7·6	− ·068	·38	− 4·9
	12	...	...	...	...	...	...
	14	·205	·57	− 4·7	− ·091	·67	+ 1·1
	16	...	...	...	...	...	...

11. The quantities δH, $\delta \tau$ in the last two tables are rather larger than can be allowed for (111) to give accuracy to one-tenth of a foot, and a direct computation of h_a has accordingly been made. In table XLIV the values of the quantity $h_a - h$ in feet so deduced for the various hours and days, are given: together with mean values and probable errors.

TABLE XLIV.

Hour	10	12	13	14	15	16	Hour	8	10	12	14	16
At Abbotsford.							*At Castle.*					
1911 12th Oct.	−37·5	− 3·1	+24·5	+ 3·3	+21·5	+ 1·9	1911 20th Nov.	...	−49·2	− 9·0	− 5·9	−27·0
13th "	−53·7	+14·1	+ 9·7	+ 3·5	+10·9	− 4·4	21st "	−53·4	− 6·6	+ 6·2	− 5·5	− 4·9
16th "	−15·1	+14·7	+21·7	+13·7	− 7·3	− 0·4	22nd "	−59·3	−19·6	+12·5	−22·6	−22·9
17th "	−16·8	+12·5	+ 4·1	− 6·1	− 7·7	− 8·7	23rd "	−59·7	−29·4	...	−13·6	−33·1
18th "	−29·3	+ 9·2	+ 8·2	+15·8	+17·2	+17·4	29th "	−104·6	+ 2·5	+ 2·1	+ 8·2	− 8·0
19th "	−14·5	+12·8	+14·4	+ 9·2	+18·7	+ 0·3	30th "	−55·8	− 8·1	+ 0·2	+10·6	−13·4
20th "	−15·2	− 1·0	+ 1·7	+ 8·6	+23·3	+14·5	1st Decr.	−29·0	+15·9	+27·4	+ 9·8	− 8·6
21st "	+ 3·9	+ 6·3	+25·6	+ 7·8	− 7·2	− 9·7	4th "	−69·4	−23·9	+15·1	− 4·7	−10·5
22nd "	−17·7	+10·6	+11·2	− 6·7	− 6·5	− 8·4	5th "	−80·7	−39·6	+ 0·1	−19·3	−18·4
23rd "	−27·5	− 3·3	− 3·9	− 3·8	−19·1	−18·8	6th "	−84·2	−36·2	+11·1	− 1·7	−23·7
24th "	−24·4	+ 6·2	+21·0	+30·3	+19·4	−11·4	7th "	−58·2	−11·7	...	+ 3·3	...
25th "	−24·0	+ 3·8	+ 6·0	+14·6	+17·1	+ 1·8						
Mean	−22·7	+ 6·9	+12·0	+ 7·5	+ 6·7	− 2·2	Mean	−65·4	−18·7	+ 7·3	− 3·8	−17·1
Range	57·6	18·0	29·5	37·0	42·4	36·2	Range	75·6	65·1	36·4	33·2	28·2
Prob. error of 1 obsn.	9·2	4·3	6·1	6·8	9·7	6·8	Prob. error of 1 obsn.	13·2	12·5	6·8	7·3	6·0
Prob. error of mean	2·75	1·28	1·86	2·06	2·92	2·04	Prob. error of mean	4·40	3·95	2·39	2·31	2·00

Considering the mean at each hour, it is seen that the approximate height h_a agrees most closely with the spirit-levelled height h in the case of deductions from observations at and after noon. Here $h_a - h$ is a small positive quantity, while at 8 hours and 10 hours we find larger negative quantities. Moreover if we examine the range of variation of $h_a - h$ at the various hours, it is found to be largest at the hours 8 and 10. In the Abbotsford results the range is decidedly smallest at 12 hours, but in the Castle results the actual smallest range is at 16 hours. The results indicate that the best time for finding height by barometer is between noon and 4 P. M. The probable errors of the means of the observations are also given and from their smallness it is to be inferred that the quantity h_a differs from h by amounts greater than which is due to observation error. The residual $h_a - h$ should accordingly by (109) give a fairly good value of A_2. It will be seen here that we require to know the zero corrections of both barometers with considerable precision, otherwise a faulty value of A_2 will be obtained.

12. We now proceed with the calculation of mean values of A_1 and A_2 corresponding to the means of the observations in the two cases Abbotsford and Castle. For this purpose we have (109) which we write

$$A_2 h^2 = 6 \frac{h - h_a}{h_a} \left\{ 1 + \tfrac{1}{2} \left(\frac{g}{g_0} . \frac{\tau_0}{\tau} - 1 \right) \right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (113).$$

and from (110) we have

$$A_1 h = \frac{g}{g_0} . \frac{\tau_0}{\tau} - 1 - A_2 h^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (114).$$

To solve these equations we require the values of $\frac{h - h_a}{h_a}$ and $\frac{g}{g_0} . \frac{\tau_0}{\tau} - 1$. The former is deducible from table XLIV and the mean values of $\tau_0 \tau$ are deduced from tables XLII, XLIII. From §9 Chap. II we take approximate values $g_0 = 979 \cdot 1$, $g = 978 \cdot 8$. We also have from (8) and (10)

$$\tau = \frac{\tau_0 g}{g_0} \left(1 - A_1 h + \overline{A_1^2 - A_2}\, h^2 \ldots \right)$$

$$\doteqdot \tau_0 \left\{ 1 - \overline{A_1 + 6 \cdot 2 \times 10^{-8}}\, h + \overline{A_1^2 - A_2}\, h^2 \ldots \right\}$$

whence at lower station

$$\left.\begin{aligned} \frac{d\tau}{dh} &= - \tau_0 \left(A_1 + 6 \cdot 2 \times 10^{-8} \right) \\ \tfrac{1}{2} \frac{d^2\tau}{dh^2} &= \tau_0 \left(A_1^2 - A_2 \right) \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (115).$$

Also

$$\frac{g\tau_0}{g_0\tau} - 1 = \left(1 - \cdot 0003 \right) \left(1 + \frac{\tau_0 - \tau}{\tau} \right) - 1$$

$$= - \cdot 0003 + \frac{\tau_0 - \tau}{\tau}$$

Bearing in mind the expressions for a, b given on page 15 and making use of equations (81) and (83) we make the computations now exhibited in tables XLV, XLVI.

TABLE XLV.

Dehra to Abbotsford $h = 4449 \cdot 6$.

Hour	10	12	13	14	15	16
H_0	27·714	27·692	27·668	27·645	27·630	27·623
H	23·673	23·655	23·641	23·629	23·616	23·610
τ_0	535·7	539·0	540·0	540·4	540·5	540·2
τ	516·0	519·5	521·5	522·2	522·0	520·4
$\frac{g}{g_0} \cdot \frac{\tau_0}{\tau} - 1$ ...	·0379	·0372	·0352	·0346	·0351	·0377
$\frac{h - h_a}{h_a}$ ...	·00513	−·00155	−·00269	−·00168	−·00150	·00050
$A_2 h^2$ ...	·03136	−·00947	−·01642	−·01025	−·00916	·00306
$A_1 h$ ...	·0065	·0467	·0516	·0449	·0443	·0346
$\tau_0 A_1$ ...	·00078	·00566	·00626	·00545	·00538	·00420
$a = -\frac{d\tau}{dh} \times 10^2$ ...	·081	·569	·629	·548	·541	·423
$(A_1^2 - A_2) h^2$ ...	− ·03132	·01165	·01908	·01227	·01112	−·00186
$\frac{1}{2} 10^7 \frac{d^2\tau}{dh^2} = \frac{\tau_0 (A_1^2 - A_2) h^2}{1 \cdot 98}$	−8·47	3·17	5·20	3·35	3·04	−0·51
$b = \frac{\tau_0}{521} \cdot \frac{1}{2} \cdot 10^7 \cdot \frac{d^2\tau}{dh^2}$	−8·71	+3·28	+5·39	+3·47	+3·15	−0·53
u ...	+0·347	−0·020	−0·066	−0·005	+0·001	+0·090
v by (81) ...	+10·64	−3·37	−5·66	−3·49	−3·15	+0·95
u' by (83) ..	−0·237	+0·200	+0·295	+0·228	+0·212	+0·054
v' ...	+7·86	−2·27	−3·80	−2·30	−2·07	+0·78

TABLE XLVI.

Dehra to Castle. $h = 4657 \cdot 6$.

Hour	8	10	12	14	16
H_0	27·755	27·789	27·767	27·712	27·694
H	23·426	23·453	23·438	23·410	23·396
τ_0	509·7	518·2	522·0	524·0	523·7
τ	503·5	505·1	508·0	508·5	506·2
$\frac{g}{g_0} \cdot \frac{\tau_0}{\tau} - 1$ ...	·0120	·0256	·0273	·0302	·0343
$\frac{h - h_a}{h_a}$	·01424	·00403	− ·00157	·00082	·00369
$A_2 h^2$	·08595	·02449	− ·00955	·00499	·02252
$A_1 h$	− ·0740	+ ·0011	·0369	·0252	·0118
$\tau_0 A_1$	− ·00810	+ ·00012	·00414	·00283	·00133
$a = -\frac{d\tau}{dh} \times 10^2$ ...	− 0·807	+ 0·015	+ 0·417	+ 0·287	+ 0·136
$(A_1^2 - A_2) h^2$ ...	− ·08047	− ·02449	+ ·01091	− ·00435	− ·02238
$\frac{1}{2} \cdot 10^7 \frac{d^2\tau}{dh^2} = \frac{\tau_0 (A_1^2 - A_2) h^2}{2 \cdot 17}$ ...	− 18·90	− 5·85	+ 2·624	− 1·050	− 5·40
$b = \frac{\tau_0}{521} \cdot \frac{1}{2} \cdot 10^7 \cdot \frac{d^2\tau}{dh^2}$ ...	− 18·49	− 5·82	+ 2·629	− 1·056	− 5·43
u	1·017	0·3971	0·0942	0·1922	0·3060
v by (81) ...	+ 26·44	+ 8·09	− 2·186	+ 2·023	+ 7·09
u' by (83) ...	− 0·2809	− 0·0114	+ 0·2788	+ 0·1181	− 0·0752
v'	+ 17·53	+ 5·77	− 1·145	+ 1·620	+ 5·12

13. With the values of u, v, u', v' just found together with the mean temperatures and pressures we can compute the refraction on the Dehra Dun–Mussooree ray. The process is precisely the same as that used in Chapter II, so that only the results are now given. Unfortunately we have no simultaneous measures of the vertical angles. Values of the refraction during October and November in several other years are available and these are also given. They have been arrived at in the same way as was done in tables X A, X B; except that in accordance with § 11 Chapter II the value of a computed in § 6 Chapter II has been increased by 8″. The results are now given in tables XLVII and XLVIII in which the argument is the angle of refraction.

TABLE XLVII.

Dehra to Mussooree.

Hour		8	10	12	14	16½
Oct., Nov. 1905	Observed	41″·3	33″·3	30″·6	29″·3	35″·7
October 1906	Observed	36·9	30·4	27·2	28·1	32·1
November 1907	Observed	40·4	31·9	27·1	27·5	30·2
October 1908	Observed	42·6	34·9	32·9	32·8	36·7
Mean		40·3	32·6	29·5	29·4	33·7
October 1911	Computed		34·3 (42·6−8·3)	32·3 (30·6+1·7)	32·5 (30·8+1·7)	32·4 (33·8−1·4)
Nov., Dec. 1911	Computed	47·8 (70·6−22·8)	40·2 (47·4−7·2)	37·3 (36·4+0·9)	37·0 (39·3−2·3)	36·9 (43·1−6·2)

TABLE XLVIII.

Mussooree to Dehra.

Hour		8	10	12	14	16½
November 1905	Observed	38″·1	32″·0	29″·5	28″·4	30″·7
Oct., Nov. 1906	Observed	33·4	30·6	27·7	27·8	27·3
October 1907	Observed	31·1	30·3	26·5	25·5	27·1
October 1908	Observed	39·8	36·6	34·8	34·1	35·5
Mean		35·6	32·4	29·6	29·0	30·2
October 1911	Computed		28·4 (22·2+6·2)	33·5 (34·4−0·9)	34·0 (34·9−0·9)	31·3 (30·1+1·2)
Nov., Dec. 1911	Computed	35·6 (21·7+13·9)	34·7 (29·7+5·0)	37·9 (38·0−0·1)	35·0 (33·1+1·9)	32·1 (27·6+4·5)

In the last portion of each of the two preceding tables the quantities included in brackets are $\Omega_1 + \Omega_2$ and show the component parts of the refraction angles computed by means of the values of u, v, u', v' found in tables XLV and XLVI. In the case of table XLVIII the actual values observed at Abbotsford and Castle have been used as though they had been observed at the Mussooree refraction station which is somewhat higher than either of the barometer stations (*see* table VI). This is not quite accurate, but it has only a very small effect on the deduced refraction.

14. The figures obtained for 1911 cannot be expected to exactly hold for any one of the years 1905 to 1908, which show a good deal of variation amongst themselves. In table XLVII we find the results computed for October 1911 lying within the range of deduced values: while those computed for November and December 1911 are in the main some 5″ higher than the highest deduced values, those of 1908. This fact is perhaps evidence against the correctness of having increased the value of a deduced from the spirit-levelled values of height by so much as 8″, as was done in § 11 Chapter II. On the other hand we may suppose that the refraction computed is greater because the season is later, since the computed refraction in October agrees well with the deduced values. As can be seen in table XA there is a distinct seasonal change in the refraction, and that found in February 1909 exceeds that of October 1908 by 2″·3. It is possible that the value in December was higher than either of the values in October or February. Moreover if we were to increase the deduced values at Dehra by 8″ we should have to diminish those at Mussooree: and table XLVIII gives no evidence that the deduced values are too large.

In addition to these possible explanations, we can see by means of equation (36) that when $\frac{\Omega_2}{\Omega_1}$ is large, as at 8 hours in November, December 1911, the two-term expression cannot always be relied on to give the refraction: for the third order term now may become important. Its value however cannot be assigned unless we have means of computing Y which involves $\frac{d^3\tau}{dh^3}$: and this cannot be done in the present case.

The most noteworthy point of table XLVIII is that the computed diurnal variation in refraction shows a maximum instead of a minimum at 14 and 12 hours respectively. This state of affairs certainly did not occur in the years 1905-1908, and it seems improbable that it actually occurred in 1911. It is more probably partially due to faulty determination of u' and v' due either to faulty pressure and temperature data; or to the neglecting of higher terms in the refraction formula: or to failure of the 3 term-temperature law to represent actual facts with sufficient precision. This is no doubt one of the difficulties inherent in the method. A temperature reading at the surface may, owing to very local influences, have an abnormal value. We may have to fit our law to two abnormal terminal values of temperature. In this respect the pressure condition is superior as really depending on the integrated pressure-differences all the way between the terminal points. We might perhaps get better results if we measured our temperature at some considerable distance above ground level. Of course the best method of all would be to obtain simultaneous readings of temperature at a number of heights.

For the present it is thought that the results exhibited in tables XLVII and XLVIII, while not entirely upholding the refractions computed from the barometric readings, by no means prove these computed values to be far wrong. We still have some doubt about the values deduced from vertical angles on account of the difference of height between the spheroid and geoid: and, until it is cleared up by further observations of plumb-line deflection, this doubt will continue. When this has been done it will be desirable to test the theory by simultaneous observations of temperature, pressure and vertical angle, both at Mussooree and Dehra. Every care will have to be taken to obtain temperature readings as little disturbed as possible.

CHAPTER V.

Celestial Refraction and its relation to Nocturnal Terrestrial Refraction.

1. In the last section of Chapter III the possibility of deducing the terrestrial refraction by means of observations to celestial objects was mentioned. Up to the present no observations have been made with this end in view, so that the success, or otherwise, of the method cannot yet be stated. It is desirable that the method should be tried, and so the necessary formulæ will now be developed, ready for application as soon as observations are available.

The formulæ of Chapter I are adapted to terrestrial refraction. Similar formulæ may readily be deduced for celestial refraction. The essential difference of the two refractions is that whereas the terrestrial refraction is $B\hat{A}x$ (*see* figure in § 8 Chapter I) the celestial refraction is $B'\hat{T'}x$, where B' is the point where the ray cuts the upper limit of the atmosphere and $B'T'$ is the tangent to the ray at B'. Owing to the very great distance of celestial objects, with the exception of the moon, we are only concerned with the direction the ray has on leaving the atmosphere: for this direction afterwards remains unchanged. A difficulty which at once arises is to find the height of the atmosphere from which the appropriate value of s may be derived.

2. The celestial refraction accordingly is the value of the angle ψ, when s is given the value it has at the point where the ray leaves the atmosphere.

It is at once clear that, in the notation of Chapter I,

$$\frac{d\psi}{ds} = \frac{1}{\sigma} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (116)$$

Denoting the reciprocal of σ by β_s we have

$$\psi = \int \beta_s ds \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (117)$$

the integration being taken from station of observation up to the limit of the atmosphere.

Now $$\beta_s = -\frac{K}{\mu} \sin\phi \frac{d\rho}{dh} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (28)\ \textit{bis}$$

hence we may write

$$\beta_s = -\gamma \sin\phi \quad \dots\dots\dots\dots \quad (118)$$

where

$$\gamma = \frac{K}{\mu}\cdot\frac{d\rho}{dh} = \frac{d}{dh}(\log\mu) \quad \dots\dots\dots\dots \quad (119)$$

by virtue of (21).

Substituting for β from (118) and putting $dh \sec\phi$ for ds in (117) we have

$$\psi = -\int \gamma \tan\phi\, dh \quad \dots\dots\dots\dots \quad (120)$$

Now γ is clearly a function of h (*see* (119)) and ϕ is given by (22) in terms of its initial value and μ and h : μ being a function of h. Hence we can expand $\gamma\tan\phi$ in a series of powers of h as follows:—

$$\gamma\tan\phi = y + \frac{dy}{dh}\cdot h + \frac{d^2y}{dh^2}\cdot\frac{h^2}{\underline{|2}} + \quad \dots\dots\dots \quad (121)$$

where

$$y = (\gamma\tan\phi)_0$$

$$\frac{dy}{dh} = \left(\frac{d(\gamma\tan\phi)}{dh}\right)_0 \text{ etc.}$$

the suffix zero indicating that initial values are to be used, for which $h = 0$.

3. From Chapter I § 9 putting R for $r + h$ we have

$$-\gamma\sin\phi = \frac{d\phi}{dh}\cos\phi + \frac{\sin\phi}{R}$$

Hence

$$\frac{d\phi}{dh} = -\tan\phi\left(\gamma + \frac{1}{R}\right) \quad \dots\dots\dots\dots \quad (122)$$

Putting $\tan\phi = u$ we have at once

$$\left.\begin{aligned}
\frac{du}{dh} &= -(u+u^3)\left(\gamma+\frac{1}{R}\right)\\
\frac{d^2u}{dh^2} &= (1+3u^2)(u+u^3)\left(\gamma+\frac{1}{R}\right)^2 - (u+u^3)\left(\frac{d\gamma}{dh}-\frac{1}{R^2}\right)\\
\frac{d^3u}{dh^3} &= -(1+12u^2+15u^4)(u+u^3)\left(\gamma+\frac{1}{R}\right)^3\\
&\quad + 3(1+3u^2)(u+u^3)\left(\gamma+\frac{1}{R}\right)\left(\frac{d\gamma}{dh}-\frac{1}{R^2}\right)\\
&\quad - (u+u^3)\left(\frac{d^2\gamma}{dh^2}+\frac{2}{R^3}\right)
\end{aligned}\right\} \quad \dots\dots \quad (123)$$

Also by Leibnitz's theorem

$$\frac{d\,(\gamma \tan\phi)}{dh} = u\frac{d\gamma}{dh} + \gamma\frac{du}{dh}$$

$$\frac{d^2\,(\gamma \tan\phi)}{dh^2} = u\frac{d^2\gamma}{dh^2} + 2\frac{du}{dh}\cdot\frac{d\gamma}{dh} + \gamma\frac{d^2u}{dh^2}$$

$$\frac{d^3\,(\gamma \tan\phi)}{dh^3} = u\frac{d^3\gamma}{dh^3} + 3\frac{du}{dh}\cdot\frac{d^2\gamma}{dh^2} + 3\frac{d^2u}{dh^2}\cdot\frac{d\gamma}{dh} + \gamma\frac{d^3u}{dh^3}$$

whence

$$\psi = -\Big[hu\gamma + \frac{h^2}{\underline{|2}}\Big(u\frac{d\gamma}{dh} + \gamma\frac{du}{dh}\Big) + \frac{h^3}{\underline{|3}}\Big(u\frac{d^2\gamma}{dh^2} + 2\frac{du}{dh}\cdot\frac{d\gamma}{dh} + \gamma\frac{d^2u}{dh^2}\Big)$$
$$+ \frac{h^4}{\underline{|4}}\Big(u\frac{d^3\gamma}{dh^3} + 3\frac{du}{dh}\cdot\frac{d^2\gamma}{dh^2} + 3\frac{d^2u}{dh^2}\cdot\frac{d\gamma}{dh} + \gamma\frac{d^3u}{dh^3}\Big) + \ldots\Big] \quad . \quad (124)$$

in which values of $\frac{du}{dh}$, etc. are to be substituted from (123).

Now if we put $R = \infty$ in (123) we evidently arrive at the case of an atmosphere arranged in plane strata. In such a case the refraction may be found at once, for we have

$$\mu \sin\phi = \sin(\phi + \psi_a) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (125)$$

by the ordinary law of refraction, ψ_a being the angle of refraction in this case. It is clear then that the terms in (124) which are independent of R must amount to ψ_a, given by (125).

We can accordingly write

$$\psi - \psi_a = -(u+u^3)\Big[-\frac{\gamma}{R}\cdot\frac{h^2}{\underline{|2}} + \frac{h^3}{\underline{|3}}\Big\{-\frac{2}{R}\cdot\frac{d\gamma}{dh} + \gamma\Big((1+3u^2)\Big(\frac{2\gamma}{R}+\frac{1}{R^2}\Big)+\frac{1}{R^2}\Big)\Big\}$$
$$+\frac{h^4}{\underline{|4}}\Big\{-\frac{3}{R}\cdot\frac{d^2\gamma}{dh^2} + 3\frac{d\gamma}{dh}\Big((1+3u^2)\Big(\frac{2\gamma}{R}+\frac{1}{R^2}\Big)+\frac{1}{R^2}\Big)$$
$$-\Big(1+12u^2+15u^4\Big)\Big(\frac{3\gamma^2}{R}+\frac{3\gamma}{R^2}+\frac{1}{R^3}\Big)\gamma$$
$$+3\gamma\,(1+3u^2)\Big(\frac{1}{R}\cdot\frac{d\gamma}{dh}-\frac{\gamma}{R^2}-\frac{1}{R^2}\Big)-\frac{2\gamma}{R^3}\Big\}$$
$$+ \ldots\ldots\ldots\Big] \quad . \quad . \quad (126).$$

Consider the portion of this within the square brackets. The part independent of u is

$$= -\frac{1}{R}\Big\{\frac{h^2}{\underline{|2}}\cdot\gamma + \frac{h^3}{\underline{|3}}\cdot 2\frac{d\gamma}{dh} + \frac{h^4}{\underline{|4}}\cdot 3\frac{d^2\gamma}{dh^2} + \ldots\Big\}$$

$$+\frac{h^3}{\underline{|3}}\cdot\gamma\Big(\frac{2\gamma}{R}+\frac{2}{R^2}\Big)+\frac{h^4}{\underline{|4}}\Big\{\frac{3d\gamma}{dh}\Big(\frac{2\gamma}{R}+\frac{2}{R^2}\Big)-\frac{3\gamma^3}{R}-\frac{3\gamma^2}{R^2}-\frac{\gamma}{R^3}+\frac{3\gamma}{R}\cdot\frac{d\gamma}{dh}-\frac{3\gamma^2}{R^2}-\frac{3\gamma}{R^3}-\frac{2\gamma}{R^3}\Big\}$$

$$= -\frac{1}{R}\int h\left(\gamma + h\frac{d\gamma}{dh} + \frac{h^2}{\underline{|2}}\frac{d^2\gamma}{dh^2} \;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\right)dh$$

$$+ \frac{h^3}{\underline{|3}}\cdot 2\gamma\left(\frac{\gamma}{R} + \frac{1}{R^2}\right) + \frac{h^4}{\underline{|4}}\left\{\frac{3d\gamma}{dh}\left(\frac{3\gamma}{R} + \frac{2}{R^2}\right) - \frac{3\gamma^3}{R} - \frac{6\gamma^2}{R^2} - \frac{6\gamma}{R^3}\right\}$$

$$= -\frac{1}{R}\int h\, d\log\mu + \frac{\gamma}{R}\cdot\frac{h^3}{\underline{|3}}\left(2\gamma + \frac{9h}{4}\cdot\frac{d\gamma}{dh}\;\cdot\;\cdot\;\right) + \frac{h^3}{R^2\underline{|3}}\left(2\gamma + \frac{3h}{2}\cdot\frac{d\gamma}{dh}\;\cdot\;\cdot\right)$$

$$+ \;\cdot\;\cdot\;\cdot\;\cdot$$

since

$$\frac{d\log\mu}{dh} = \gamma_s = \gamma + h\frac{d\gamma}{dh} + \;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot$$

Now

$$\int hd\log\mu = h\log\mu - \int\log\mu\, dh$$

$$\doteqdot h\log\mu - K\int\rho\, dh$$

$$\text{since}\quad \log\mu \doteqdot \mu - 1 = K\rho \;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot \quad (21)\ bis.$$

But $h\log\mu$ vanishes at both limits, h being zero at lower limit and μ being unity at upper limit. Also $\int\rho\, dh \doteqdot \frac{1}{g_m}\int g\rho\, dh$, where g_m is a value of gravity at some point between the earth and the limit of the atmosphere.

$$\therefore \qquad \int hd\log\mu = -\frac{K}{g_m}.\ \Pi$$

where Π is the atmospheric pressure. We can finally write for terms independent of u

$$\frac{K}{Rg_m}\Pi + \frac{h^3}{\underline{|3}}\left\{\frac{\gamma}{R}\left(2\gamma + \frac{9h}{4}\cdot\frac{d\gamma}{dh} + \;\cdot\;\cdot\;\cdot\right) + \frac{1}{R^2}\left(2\gamma + \frac{3h}{2}\cdot\frac{d\gamma}{dh} + \;\cdot\;\cdot\;\cdot\right)\right\} \;\cdot\;\cdot\;(127).$$

Next the coefficient of u^2 is

$$\frac{h^3}{\underline{|3}}3\gamma\left(\frac{2\gamma}{R}+\frac{1}{R^2}\right) + \frac{h^4}{\underline{|4}}\left\{9\frac{d\gamma}{dh}\left(\frac{2\gamma}{R}+\frac{1}{R^2}\right) - 12\gamma\left(\frac{3\gamma^2}{R}+\frac{3\gamma}{R^2}+\frac{1}{R^3}\right) + 9\gamma\left(\frac{1}{R}\cdot\frac{d\gamma}{dh} - \frac{\gamma}{R^2} - \frac{1}{R^3}\right)\right\}$$

$$+ \;\cdot\;\cdot\;\cdot$$

$$= \frac{h^3}{\underline{|3}}3\gamma\left(\frac{2\gamma}{R}+\frac{1}{R^2}\right) + \frac{h^4}{\underline{|4}}\left\{9\frac{d\gamma}{dh}\left(\frac{3\gamma}{R}+\frac{1}{R^2}\right) - 3\gamma\left(\frac{12\gamma^2}{R}+\frac{15\gamma}{R^2}+\frac{7}{R^3}\right)\right\} + \;\cdot\;\cdot\;\cdot$$

$$= \frac{h^3}{\underline{|3}}\left[\frac{3\gamma}{R}\left(2\gamma + \frac{9h}{4}\cdot\frac{d\gamma}{dh} + \;\cdot\;\cdot\right) + \frac{3}{R^2}\left(\gamma + \frac{3h}{4}\cdot\frac{d\gamma}{dh}\cdot\;\cdot\;\cdot\right)\right] \;\cdot\;\cdot\;\cdot\;\cdot\; (128)$$

4. In (127) and (128) it is to be noted that the terms involving $h\frac{d\gamma}{dh}$ are of importance compared with γ. For supposing γ to vanish at the limit of the atmosphere, we see that the average value of $\frac{d\gamma}{dh}$ all along the curve is $\frac{1}{r}\cdot\frac{1}{h}$: so that $h\frac{\frac{d\gamma}{dh}}{\gamma}$ is of the order $\frac{1}{\gamma r} = -\frac{1}{k}$ (where k is the coefficient of horizontal refraction) which is greater than unity in practically all cases. The presumption is that the higher terms are also larger and so the expressions occurring in (127) and (128) cannot be considered to be fully represented by the terms written down. We can write an expression for the refraction in the form replacing R by its initial value

$$\psi = \psi_a - \tan\phi \sec^2\phi \frac{K}{rg_m}.\,\Pi + \frac{h^3}{r^3} \tan\phi \sec^2\phi \left\{P + Q\tan^2\phi + \ldots\right\} \quad \ldots (129)$$

where P, Q, etc., are functions of γ, $\frac{d\gamma}{dh}$ etc., but we cannot say what these functions are, though theoretically any number of terms may be written down.

From the point of view of determining γ, $\frac{d\gamma}{dh}$ etc. the equation (129) is useless. It shows, however, the nature of celestial refraction, and the reason that celestial refraction maintains a much greater constancy than terrestrial refraction. For we see that the principal terms of the refraction are independent of γ, $\frac{d\gamma}{dh}$ etc. The quantities P and Q are small and only have significance when their multipliers are large, *i.e.* when ϕ exceeds 80°. Up to this point the refraction as given by Bessel is very closely represented by

$$\psi \doteqdot \psi_a - \tan\phi \sec^2\phi \frac{K}{rg_m}.\,\Pi \quad \ldots\ldots\ldots (130).$$

as is indicated in table XLIX.

We will first evaluate $\frac{K}{rg_m}.\,\Pi$. In the first place if we treat g as not varying appreciably with height

$$\frac{K\Pi}{g} = (\mu-1)\frac{\Pi}{g\rho} = (\mu-1)\,H$$

where H is the height of the hypothetical homogeneous atmosphere of density ρ whose pressure at the lowest point is Π. Bessel's tables are constructed for the case of temperature 50° *F.* and pressure 30 inches of mercury: to which corresponds the value ·000283 of $\mu-1$. Now the weight of 1 *c.c.* of air at 0° *C.* and 760 *m.m.* is $10^{-3} \times 1{\cdot}2928$ *gm*: and for 50° *F.* and 30″ it is accordingly $10^{-3} \times 1{\cdot}2504$ *gm.* The weight of a column of mercury 30″ high at freezing point is $30 \times 2{\cdot}54 \times 13{\cdot}596 = 1035.$ *gm.* Hence $H = \frac{1035}{10^{-3}\times 1{\cdot}250} = 8{\cdot}280$ kms. Hence $\frac{\mu-1}{r}.\,H = \frac{2{\cdot}83\times 10^{-4}\times 8{\cdot}280}{6{\cdot}371} = 3{\cdot}68 \times 10^{-7}$, taking the mean radius of the earth as 6371 kms. This is an angle expressed in radians: to turn it into seconds we multiply by $2{\cdot}063\times 10^5$ and get 0″·0759.

Table XLIX, next given, explains itself.

TABLE XLIX.

ϕ	I ψ_a from (125)	II $0''{\cdot}0759 \tan\phi \sec^2\phi$	I − II = III	IV Bessel's refraction	IV − III
45	58″·4	0″·1	58″·3	58″·2	− 0″·1
60	1′ 41″·2	0·5	1′ 40″·7	1′ 40″·6	− 0·1
70	2 40·6	1·7	2 38·9	2 38·8	− 0·1
75	3 38·3	4·3	3 34·0	3 34·1	+ 0·1
80	5 32·8	14·3	5 18·5	5 19·2	0·7
83	8 0·0	41·7	7 18·3	7 23·9	5·6
85	11 20·0	2′ 35″·8	9 25·0	9 52	27
86	14 20·5	3 43·0	10 37·5	11 46	68
87	19 38·0	8 48·8	10 49·2	14 27	218

This table shows that the first two terms of (129) give, up to $\phi = 80°$, practically the same refraction as given by Bessel. Up to this point the only assumption made is that P and Q are not sufficiently large to make the higher terms of importance for values of ϕ not exceeding 80°. In deducing his formula Bessel assumed the law of density

$$\rho = \rho_0 \, e^{-\frac{l-H}{l} \cdot \frac{rh}{HR}} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad *(131)$$

and determined the best value of l to fit observations. Without any such assumption (129) enables us to write down the refraction up to zenith distance of 80°. For greater zenith distances Bessel's values can be closely reproduced by choosing suitable values of P and Q. It is doubtful whether the quantities P and Q can be considered sufficiently nearly constant for all time to represent the refraction even up to 85°, the limit originally adopted by Bessel. In any case they can be determined just as well as the quantity l in Bessel's formula (131).

The values of $\frac{h^3}{r^3} \tan\phi \sec\phi \left\{ P + Q \tan^2\phi + \; . \; . \; . \right\}$ are given in table L under heading II corresponding to $\frac{Ph^3}{r^3} = {\cdot}01288$ and $\frac{Qh^3}{r^3} = {\cdot}00005506$, and it is seen that they are nearly the same as the quantities under heading I, which have been taken from the last column of table XLIX. The last column of table L shows the residuals.

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* *See* Chauvenet's Astronomy Vol. I, page 145.

TABLE L.

ϕ	I	II	II – I
80°	0″·7	2″·5	1″·8
83	5·6	8·2	2·6
85	27	27·8	0·8
86	68	66·3	–1·7
87	218	218·3	0·3

It is to be remarked that (129) cannot be used for all values of ϕ up to 90°. The quantity ψ_a ceases to have a possible value when ϕ is greater than $\sin^{-1}\frac{1}{\mu}$ ($\doteq$ 88° 38′ for standard air density). Moreover the last term of (129) which comprises an infinite series may become divergent for certain values of P, Q etc.

5. We must accordingly find another expansion for the case when ϕ exceeds the value at which (126) becomes very slowly convergent. Equation (129) shows the refraction in terms of that which would occur were the radius of curvature of the earth infinite (gravity remaining the same) together with a correction for curvature. For values of ϕ greater than $\sin^{-1}\frac{1}{\mu}$ it is clear that the ray would not escape from such an atmosphere arranged in plane strata, so that this form of expression breaks down in this case when applied to the actual spherically arranged atmosphere.

We see at once that it is necessary to expand the expression for ψ in terms of s instead of h: for it is the multiplier sec ϕ which causes the series to diverge.

We have by Maclaurin's theorem

$$\beta_s = \beta + s\frac{d\beta}{ds} + \frac{s^2}{\underline{|2}}\cdot\frac{d^2\beta}{ds^2} + \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (132)$$

whence substituting in (117) and integrating

$$\psi = s\beta + \frac{s^2}{\underline{|2}}\cdot\frac{d\beta}{ds} + \frac{s^3}{\underline{|3}}\cdot\frac{d^2\beta}{ds^2} + \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (133)$$

Now

$$\left.\begin{aligned} \frac{d\beta}{ds} &= \cos\phi\,\frac{d\beta}{dh} \\ \frac{d^2\beta}{ds^2} &= \left\{\cos\phi\,\frac{d^2\beta}{dh^2} - \frac{d\beta}{dh}\cdot\sin\phi\,\frac{d\phi}{dh}\right\}\cos\phi \\ &\text{etc.} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (134)$$

We have the values of $\frac{d\phi}{dh}$ from (122). Also from (118)

$$\frac{d\beta}{dh} = \sin\phi\,\left\{\gamma^2 + \frac{\gamma}{R} - \frac{d\gamma}{dh}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (135)$$

$$\frac{d^2\beta}{dh^2} = \sin\phi\left\{2\gamma\frac{d\gamma}{dh} + \frac{d\gamma}{dh}\cdot\frac{1}{R} - \frac{\gamma}{R^2} - \frac{d^2\gamma}{dh^2}\right.$$
$$\left. - \left(\gamma^2 + \frac{\gamma}{R} - \frac{d\gamma}{dh}\right)\left(\gamma + \frac{1}{R}\right)\right\}$$
$$= \sin\phi\left\{-\frac{d^2\gamma}{dh^2} + \frac{d\gamma}{dh}\left(3\gamma + \frac{2}{R}\right) - \gamma\left(\overline{\gamma + \frac{1}{R}}\Big|^2 + \frac{1}{R^2}\right)\right\} \quad . \quad . \quad . \quad . \quad (136)$$

Substituting in (134) and putting $h = 0$ we get

$$\left(\frac{d\beta}{ds}\right)_0 = \cos\phi\sin\phi\left\{-\frac{d\gamma}{dh} + \gamma\left(\gamma + \frac{1}{r}\right)\right\}$$
$$= A\cos\phi\sin\phi \quad . \quad . \quad . \quad . \quad (137)$$
$$\left(\frac{d^2\beta}{ds^2}\right)_0 = \cos^2\phi\sin\phi\left\{-\frac{d^2\gamma}{dh^2} + \frac{d\gamma}{dh}\left(3\gamma + \frac{2}{r}\right) - \gamma\left(\overline{\gamma + \frac{1}{r}}\Big|^2 + \frac{1}{r^2}\right)\right\}$$
$$+ \sin^3\phi\left(\gamma + \frac{1}{r}\right)\left\{-\frac{d\gamma}{dh} + \gamma\left(\gamma + \frac{1}{r}\right)\right\}$$
$$= (C\cos^2\phi + D\sin^2\phi)\sin\phi \quad . \quad . \quad . \quad . \quad (138)$$

where A, C, D are written in place of the several coefficients, and $D = A\left(\gamma + \frac{1}{r}\right)$.

We can now write (133) as follows

$$\psi = s\sin\phi\left\{-\gamma + \frac{s\cos\phi}{\underline{|2}}A + \frac{s^2}{\underline{|3}}(C\cos^2\phi + D\sin^2\phi) \quad . \quad . \right\} \quad . \quad . \quad (139)$$

where the coefficients A, C, D are independent of ϕ and are functions of h.

It appears to be most probable that the value of β at the upper limit of the atmosphere is zero. That the ray from a celestial object should suffer an immediate finite refraction on reaching the attenuated atmosphere seems most unlikely. It is clear from the formula that the value of β depends on $\frac{d\rho}{dh}$, and this is a vanishing quantity when ρ, h and τ vanish only with certain conditions of temperature-height relation. When there is thermal equilibrium and the adiabatic gradient occurs at the upper limit of the atmosphere, then $\frac{d\rho}{dh}$ vanishes. It seems probable that this gradient should occur there. Assuming then that β_s vanishes at the limit we have at once *for the appropriate value of s*

$$\left(\frac{d\psi}{ds}\right)_s = \beta_s = \beta + s\frac{d\beta}{ds} + \frac{1}{\underline{|2}}s^2\frac{d^2\beta}{ds^2} + \ldots = 0 \quad . \quad . \quad . \quad . \quad . \quad (140)$$

This expresses the condition that a small change in s will not affect the angle ψ given by (139). The physical reality of this—that the last portion of the atmosphere contributes little to the refraction—seems to be clear.

If we neglect the higher differential coefficients of β with respect to s, (140) enables us to write (133) in other forms. Neglecting $\frac{d^2\beta}{ds^2}$ and higher differential coefficients and subtracting (140) multiplied by $\frac{s}{2}$ from (133) we get

$$\psi \doteq \tfrac{1}{2}s\beta \quad . \quad . \quad . \quad . \quad . \quad . \quad (141)$$

Neglecting $\frac{d^3\beta}{ds^3}$ and multiplying by $\frac{s}{3}$ we get a more accurate expression

$$\psi \doteqdot \frac{2}{3} s \beta + \frac{1}{6} s^2 \frac{d\beta}{ds} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (142)$$

Neglecting $\frac{d^4\beta}{ds^4}$ and multiplying by $\frac{s}{4}$ we get a still more accurate expression

$$\psi \doteqdot \frac{3}{4} s \beta + \frac{1}{4} s^2 \frac{d\beta}{ds} + \frac{1}{24} s^3 \frac{d^2\beta}{ds^2} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (143)$$

We can at once put in the values of β, $\frac{d\beta}{ds}$ etc., as was done in (139) in these last three equations. They have the advantage over (139) of being more rapidly convergent. Moreover each is correct to the next higher term than occurs in the equation: for this term has been replaced by terms of lower order by means of (140).

6. It remains to find an expression for s.

We have obviously

$$s = \int dh \sec \phi \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (144)$$

and $$\mu R \sin \phi = B \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (22)\ bis$$

where $$R = r + h$$

$$\therefore \qquad \sec \phi = \frac{\mu R}{\sqrt{\mu^2 R^2 - B^2}}$$

Then, writing x for μR we get

$$s = \int \frac{x}{\sqrt{x^2 - B^2}} dx \cdot \frac{dR}{dx} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (145)$$

and $$\frac{dR}{dx} = \left(\frac{dR}{dx}\right)_0 + x\left(\frac{d^2R}{dx^2}\right)_0 + \; . \; . \; .$$

Now $$\frac{dR}{dx} = \left(\frac{d\mu R}{dR}\right)^{-1} = \left(R\frac{d\mu}{dh} + \mu\right)^{-1}$$

$$\frac{d^2R}{dx^2} = \frac{dR}{dx} \cdot \frac{d}{dR}\left(\frac{dR}{dx}\right) = -\left(R\frac{d\mu}{dh} + \mu\right)^{-3}\left(R\frac{d^2\mu}{dh^2} + 2\frac{d\mu}{dh}\right)$$

Hence

$$s = \int \frac{x\,dx}{\sqrt{x^2 - B^2}}\left\{\left(r\frac{d\mu}{dh} + \mu\right)^{-1} - x\left(r\frac{d\mu}{dh} + \mu\right)^{-3}\left(r\frac{d^2\mu}{dh^2} + 2\frac{d\mu}{dh}\right) \; . \; . \right\} \; . \; . \quad (146)$$

Now $$\int \frac{x\,dx}{\sqrt{x^2 - B^2}} = \sqrt{x^2 - B^2} = B \cot \phi$$

$$\int \frac{x^2 dx}{\sqrt{x^2 - B^2}} = \int \sqrt{x^2 - B^2}\, dx + B^2 \int \frac{dx}{\sqrt{x^2 - B^2}}$$

$$= \frac{1}{2} x \sqrt{x^2 - B^2} + \frac{B^2}{2} \log \tan \frac{\phi}{2} - B^2 \log \tan \frac{\phi}{2}$$

$$=\frac{1}{2}B^2\left(\operatorname{cosec}\phi\ \cot\phi - \log\tan\frac{\phi}{2}\right)$$

$$=\frac{1}{2}\mu^2 R^2\left(\cos\phi - \sin^2\phi\log\tan\frac{\phi}{2}\right)$$

Substituting in (146)

$$s=\left(r\frac{d\mu}{dh}+\mu\right)^{-1}\left\{B\cot\phi - \frac{\frac{rd^2\mu}{dh^2}+2\frac{d\mu}{dh}}{\left(r\frac{d\mu}{dh}+\mu\right)^2}\cdot\frac{B^2}{2}\left(\operatorname{cosec}\phi\cot\phi - \log\tan\frac{\phi}{2}\right) - \ldots\ldots\right\} \quad \ldots (147)$$

taken between limits of ϕ.

The lower limit is ϕ_0, the observed zenith distance; the upper limit is ϕ_1 where

$$\sin\phi_1 = \frac{B}{r+h} = \frac{\mu r}{r+h}\sin\phi_0 \quad \ldots\ldots\ldots\ldots (148)$$

h being the height to which the atmosphere extends above the observatory. It is of course clear that the refractive index is unity at the upper limit, and that the μ occurring in (148) is the value at the observatory.

Now

$$\gamma = \frac{d}{dh}(\log\mu) \quad \ldots\ldots\ldots\ldots \text{(119) } bis$$

$$=\frac{1}{\mu}\cdot\frac{d\mu}{dh}$$

$$\therefore \quad \frac{d^2\mu}{dh^2} = \mu\frac{d\gamma}{dh} + \gamma\frac{d\mu}{dh}$$

Hence since μ is very nearly unity we can write with sufficient accuracy in (147)

$$\mu = 1$$

$$\frac{d\mu}{dh} = \gamma$$

$$\frac{d^2\mu}{dh^2} = \frac{d\gamma}{dh} + \gamma^2$$

and

$$\left.\begin{array}{l} s = \dfrac{1}{1+r\gamma}\left\{BL - \dfrac{B^2}{2}\cdot\dfrac{r\left(\frac{d\gamma}{dh}+\gamma^2\right)+2\gamma}{(1+r\gamma)^2}\cdot M \ \ldots\right\} \ \ldots \\ \text{where } L = \Big[\cot\phi\Big]_{\phi_0}^{\phi_1} \text{ and } M = \Big[\operatorname{cosec}\phi\cot\phi - \log_e\tan\frac{\phi}{2}\Big]_{\phi_0}^{\phi_1} \end{array}\right\} \quad \ldots (149)$$

7. The fact that the refraction cannot be much affected by the last portion of the upper atmosphere shows that a highly accurate value for s is not necessary.

Supposing $\gamma = -\frac{2k}{r}$ at the earth's surface—in other words the coefficient of horizontal refraction is k — then the average value of $\frac{d\gamma}{dh}$ is $-\frac{2k}{rh}$, h being the height of the atmosphere. Then in expression $r\left(\frac{d\gamma}{dh} + \gamma^2\right) + 2\gamma$ which occurs in (149) $\frac{r\frac{d\gamma}{dh}}{2\gamma}$ is of order $\frac{2k}{rh}\cdot\frac{r}{2\gamma} = \frac{r}{h}$, showing that $r\frac{d\gamma}{dh}$ is the important part of this expression.

Comparing the second term with the first in (149) we have ratio

$$\doteqdot \frac{B^2 r\frac{d\gamma}{dh}}{2(1+r\gamma)^2}\cdot\frac{M}{BL} = \frac{\mu r^2 \sin\phi\,\frac{d\gamma}{dh}}{2(1-2k)^2}\cdot\frac{M}{L} \text{ which is of order } -\frac{\sin\phi}{2}\cdot\frac{r}{h}\cdot\frac{2k}{(1-2k)^2}\cdot\frac{M}{L}.$$

Taking k as $\frac{1}{10}$ and $\frac{r}{h} = 100$ this becomes $\doteqdot -15{\cdot}6\ \sin\phi\cdot\frac{M}{L}$ which is small for large values of ϕ.

Values of $15{\cdot}6 \sin\phi\,\frac{M}{L}\cdot\frac{r}{100h}$ are as follows for various values of h and ϕ:

ϕ	h = 20 miles	h = 30 miles	h = 40 miles
80°	62·6448	41·8076	31·3957
83	62·6416	41·8040	31·3914
85	62·6373	41·7996	31·3864
86	62·6345	41·7961	31·3827

The above shows that the expression found for s is not satisfactory for determining s.

8. It is possible to obtain a value of s as follows. First of all assume s equal to the chord of the ray of light, from which it cannot differ greatly. Then if Ω is the terrestrial refraction up to the limit of the atmosphere and Ω_1 the terrestrial refraction at the other end of the ray, we have by projection on the chord

$$s = (r+h)\cos(\phi_1 - \Omega_1) - r\cos(\phi + \Omega)$$

$$\sin\overline{\phi_1 - \Omega_1} = \sin\overline{\phi + \Omega}\cdot\frac{r}{r+h}$$

whence

$$s = \sqrt{r^2\cos^2\overline{\phi+\Omega} + 2rh + h^2} - r\cos\overline{\phi+\Omega} \quad \ldots\ldots \quad (150)$$

from which s can be computed if Ω and h are known.

Now

$$\beta = \frac{1}{\sigma} = -\gamma \sin \phi. \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (118)\ bis$$

$$\frac{1}{\sigma^2} \cdot \frac{d\sigma}{ds} = -\frac{d\beta}{ds}$$

$$\frac{2}{\sigma^3}\left(\frac{d\sigma}{ds}\right)^2 - \frac{1}{\sigma^2} \cdot \frac{d^2\sigma}{ds^2} = \frac{d^2\beta}{ds^2}$$

Hence by (25)

Terrestrial refraction angle $= \Omega$

$$= \frac{s}{2}\beta + \frac{s^2}{6} \cdot \frac{d\beta}{ds} + \frac{s^3}{24} \cdot \frac{d^2\beta}{ds^2} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (151)$$

$$= s \sin \phi \left\{ -\frac{\gamma}{2} + \frac{s \cos \phi}{\underline{|3}} A + \frac{s^2}{\underline{|4}}\left(C \cos^2\phi + D \sin^2\phi\right) . \; . \; . \right\} . \; . \; . \quad (152)$$

and the coefficients of the several powers of s are those occurring in (133) and (139) multiplied by the numerical quantities $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$ . . . respectively, and are accordingly known. In (154) s represents the length of the terrestrial ray.

Hence by (140) and (151), eliminating $\frac{d^2\beta}{ds^2}$ and neglecting higher terms, we may write

$$\Omega = \frac{s}{12}\beta + \frac{s^2}{12} \cdot \frac{d\beta}{ds} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (153)$$

or less accurately neglecting $\frac{d^2\beta}{ds^2}$ and eliminating $\frac{d\beta}{ds}$, we have

$$\Omega \doteq \frac{s}{3}\beta$$

$$\doteq \frac{2}{3}\psi \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (154)$$

That is the terrestrial refraction on a ray to the limit of the atmosphere is approximately two-thirds of the celestial refraction. The accuracy of this relation depends on the values of $\frac{d^2\beta}{ds^2}$ and higher coefficients, but it should give a value of Ω good enough for substituting in (150), from which s may be found with sufficient accuracy. It is to be remembered that a small change in s should make only a second order change in ψ, since $\beta_s = 0$ at the upper limit of the atmosphere.

It is now possible, though troublesome, to find the quantities h, γ, A, C, provided we have four observed values of ϕ and ψ. We can calculate s for several assumed values of h and substitute in (139), and then solve for γ, A, C by means of these equations. The fourth equation will also be satisfied when the correct value of h is taken. If h is found to be a quantity which does not vary appreciably, the process will be much simplified: for a value can be found once for all, after which the determination of γ, A, C, is simple.

9. It is then an easy matter to determine γ, $\frac{d\gamma}{dh}$, $\frac{d^2\gamma}{dh^2}$, but these are not actually required for the determination of the terrestrial refraction, which is given by (152).

The relation between the celestial and terrestrial refractions is accordingly established.

Four observations of stars at various altitudes are theoretically sufficient to determine these unknowns. For the true zenith distance at a given time is known from the star catalogue, and the actual apparent zenith distance can be observed. The differences of these are clearly the refractions, and each observation accordingly gives an equation of condition.

Equations (139) and (143) are perfectly general and have no limits as regards the value of ϕ. However, stars do not in general become visible the moment they rise above the horizon, owing to haze, dust etc. In India the pole star is often hard to discern at latitudes not lower than that of Madras, *viz.*, 13° and it is probable that these equations will not have a chance of application for value of ϕ greater than 85°.

10. Now consider the case of values of ϕ, say not greater than 45°.

In this case we may expand (150) as follows:

$$s = r\cos(\phi+\Omega)\left\{1+\frac{1}{2}\cdot\frac{2rh+h^2}{r^2\cos^2(\phi+\Omega)}-\frac{1}{8}\cdot\left(\frac{2rh+h^2}{r^2\cos^2(\phi+\Omega)}\right)^2\cdot\;\cdot\;\cdot-1\right\}$$

$$= \frac{h}{r}\left(r+\frac{h}{2}\right)\sec(\phi+\Omega) - \frac{1}{2}\frac{h^2}{r^3}\left(r+\frac{h}{2}\right)^2\sec^3(\phi+\Omega)$$

$$= h\left(1+\frac{h}{2r}\right)\sec(\phi+\Omega) - \frac{1}{2}\frac{h^2}{r}\sec^3(\phi+\Omega)$$

$$= h\sec(\phi+\Omega) - \frac{h^2}{2r}\sec(\phi+\Omega)\tan^2(\phi+\Omega)$$

$$= h\sec(\phi+\Omega)\left\{1-\frac{h}{2r}\tan^2\overline{\phi+\Omega}\right\}.$$

$$\doteqdot h\left\{\sec\phi+\Omega\tan\phi\sec\phi\right\}\left\{1-\frac{h}{2r}\tan^2\phi\right\}$$

$$\doteqdot h\sec\phi\left\{1+\Omega\tan\phi-\frac{h}{2r}\tan^2\phi\right\}.$$

Now $\Omega \doteqdot \frac{2}{3}\psi \doteqdot \frac{2}{3}\cdot(\mu-1)\tan\phi.$

$$\therefore \quad s \doteqdot h\sec\phi\left\{1+\tan^2\phi\left(\frac{2}{3}\cdot\overline{\mu-1}-\frac{h}{2r}\right)\right\}$$

$$\doteqdot h\sec\phi\left\{1-\frac{h}{2r}\tan^2\phi\right\} \quad \text{since } \frac{2}{3}\overline{\mu-1} \text{ is very small.}$$

Substituting in (143)

$$\psi = h\tan\phi\left\{1-\frac{h}{2r}\tan^2\phi\right\}\left\{-\frac{3}{4}\gamma+\frac{hA}{4}+\frac{h^2}{24}\left(C+D\tan^2\phi\right)\right\}$$

$$= h\tan\phi\left[-\frac{3}{4}\gamma+\frac{hA}{4}+\frac{h^2C}{24}+\tan^2\phi\left\{\frac{h^2}{24}A\left(\gamma+\frac{1}{r}\right)-\frac{h}{2r}\left(-\frac{3}{4}\gamma+\frac{hA}{4}\right)\right\}\right]$$

$$= h\tan\phi\left[-\frac{3}{4}\gamma+\frac{hA}{4}+\frac{h^2C}{24}+\tan^2\phi\left\{\frac{3h}{8r}\gamma+\frac{h^2A}{24}\left(\gamma+\frac{1}{r}-\frac{3}{r}\right)\right\}\right]$$

Now from (129) for values of ϕ less than 45°

$$\psi \doteqdot (\mu - 1) \tan \phi$$

Hence by comparison we see that

$$h\left(-\frac{3}{4}\gamma + \frac{hA}{4} + \frac{h^2C}{24}\right) \doteqdot \mu - 1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (155)$$

Or more accurately

$$h\left\{-\frac{3}{4}\gamma + \frac{hA}{4} + \frac{h^2C}{24} - \frac{3h}{8r}\gamma + \frac{h^2A}{24}\left(\frac{2}{r} - \gamma\right)\right\} = \mu - 1$$

that is

$$h\left\{-\frac{3}{4}\gamma\left(1 + \frac{h}{2r}\right) + \frac{hA}{4}\left(1 + \frac{h}{3r}\right) + \frac{h^2}{24}\left(C - \gamma A\right)\right\} = \mu - 1 \quad . \quad . \quad . \quad . \quad (156)$$

Put $\quad r\gamma = x,\ Ar^2 = y,\ Cr^3 = z$ and $\frac{h}{r} = a$, where y has no connection with the y occurring in §2.

$$a\left\{-\frac{3}{4}x\left(1 + \frac{a}{2}\right) + \frac{ay}{4}\left(1 + \frac{a}{3}\right) + \frac{a^2}{24}\left(z - xy\right)\right\} = \mu - 1 \quad . \quad . \quad . \quad . \quad (157)$$

Also, replacing $\frac{s}{r}$ by κ and substituting values of $\beta, \frac{d\beta}{ds}, \frac{d^2\beta}{ds^2}$ from (118) and (134) we write (143)

$$\psi = \kappa \sin\phi\left\{-\frac{3}{4}x + \frac{\kappa \cos\phi}{4}y + \frac{\kappa^2}{24}\left(z\cos^2\phi + y\,\overline{x+1}\,\sin^2\phi\right)\right\} \quad . \quad . \quad . \quad (158)$$

Eliminating z between (157) and (158) by multiplying (159) by $\frac{\kappa^3}{a^3}\cos^2\phi\sin\phi$ we get

$$\psi - \left.\overline{\mu - 1\frac{\kappa}{a}}\right|^3 \cos^2\phi\sin\phi$$

$$= \kappa\sin\phi\left\{-\frac{3}{4}x\left(1 - \frac{\kappa^2}{a^2}\cos^2\phi\,\overline{1 + \frac{a}{2}}\right) + \frac{\kappa\cos\phi}{4}y\left(1 - \frac{\kappa}{a}\cos\phi\,\overline{1 + \frac{a}{3}}\right) + \frac{\kappa^2}{24}\left(xy + y\sin^2\phi\right)\right\}$$

or $\quad \psi - \left.\overline{\mu - 1}\,\frac{\kappa}{a}\right|^3 \cos^2\phi\sin\phi$

$$= \kappa\sin\phi\left\{-\frac{3}{4}x\left(1 - \frac{\kappa^2}{a^2}\cos^2\phi\,\overline{1 + \frac{a}{2}}\right) + \frac{\kappa y}{4}\left(\cos\phi - \cos^2\phi\,\frac{\kappa}{a}\,\overline{1 + \frac{a}{3}} + \kappa\frac{\sin^2\phi}{6}\right) + xy\frac{\kappa^2}{24}\right\} \quad . \quad . \quad (159)$$

11. As an example of the method proposed we proceed to the solution of this equation for the case when the refraction is that given in Bessel's tables: that is to say we treat these refractions as if they were the results of observations to stars at various altitudes. In the first place it is necessary to guess a value of a and then solve (159) from the knowledge of ψ at two elevations. If the solution thus found also satisfies (159) for other values of ϕ, then it is a fair inference that the correct value of h has been obtained: otherwise it is necessary to select another value of a and repeat the process. The two results will indicate in what neighbourhood the true value of a lies and the process may be continued until the desired accuracy is obtained. It is obvious that we must deal with large values of ϕ, since the refraction is nearly independent of the distribution of the atmosphere for small values (*see*(130)). For convenience we now give values of β corresponding to four values of a for certain values of ϕ.

TABLE LI.

Values of κ.

a \ ϕ	75°	80°	83°	85°	86°	90°
·0050	·01869	·02695	·03619	·04625	·05322	·10690
·0075	·02770	·03918	·05146	·06418	·07260	·12944
·0090	·03292	·04619	·06008	·07396	·08297	·14119
·0100	·03640	·05075	·06555	·08020	·08727	·14849
Bessel's refraction	·001038	·001548	·002149	·002870	·003429	·009847

These values have been computed by the help of (150) and (154). The last line gives Bessel's refractions expressed in radians, except for $\phi = 90°$ when the refraction is Argelander's, to suit the solution of (159).

If we select $a = \cdot 0075$ and solve (159) for the two values of ϕ, 80° and 85°, we get

$$\left.\begin{aligned} x &= -\quad 0\cdot 144 \\ y &= -\quad 47\cdot 8 \\ z &= +8267 \end{aligned}\right\} \qquad (160)$$

Taking $a = \cdot 01$ we get

$$\left.\begin{aligned} x &= -\quad 0\cdot 121 \\ y &= -\quad 31\cdot 9 \\ z &= +4150 \end{aligned}\right\} \qquad (161)$$

Taking $a = \cdot 009$ we get

$$\left.\begin{aligned} x &= -\quad 0\cdot 128 \\ y &= -\quad 36\cdot 6 \\ z &= +5278 \end{aligned}\right\} \qquad (162)$$

With these values we can substitute in (158) and find corresponding values of ψ. The results are exhibited in table LII.

TABLE LII.

ϕ	I Bessel's refraction	II ψ for $a = \cdot 0075$	III ψ for $a = \cdot 009$	IV ψ for $a = \cdot 01$	(I−II).10^6	(I−III).10^6	(I−IV).10^6
75°	·001038	·001037	·001038	·001037	1	0	1
80	·001548	·001544	·001548	·001545	4	0	3
83	·002149	·002150	·002150	·002148	−1	−1	1
85	·002870	·002874	·002868	·002867	−4	2	3
86	·003423	·003425	·003417	·003459	−2	6	−36
90	·009847	·010283	·009813	·009650	−436	34	197

The last three columns show that the solution is not quite perfect in any case even for the values $\phi = 80°$ and $\phi = 85°$ from which the equations of condition were formed. It is to be remembered that the number 4×10^{-6} radian is only $0''\cdot 8$ so that it is unnecessary to make the solution with greater accuracy. But for the other values of ϕ we find that the solution fits somewhat better except for $\phi = 86°$ and $a = \cdot 01$ and in the three cases for $\phi = 90°$. For $a = \cdot 009$ we find the solution is practically perfect throughout and the conclusion is that $a = \cdot 009$ (corresponding to a height of atmosphere 35·7* miles) is very near to the truth.

It is to be noted that x is the same as $-2k$, k being the coefficient of horizontal refraction. Hence our result is that corresponding to Bessel's refractions we have a coefficient of refraction of 0·064. Considering that Bessel's refractions must refer to the night, and are for sea level, this coefficient is unduly small. Harkness gives the ratio of nocturnal refraction to minimum refraction as 1·3 : and if we divide 0·064 by 1·3 we get 0·049 which is without doubt too small for minimum refraction at sea level. Such a value as 0·075 might be expected in that case. If we ignored the refraction given by Argelander, and took a lower value of h we should no doubt arrive at a larger value of x. The solution for h depends almost entirely on the assumed horizontal refraction. There is some difficulty in deciding what is the best value to take for h. It is clear that above the height at which the pressure is $0''\cdot 03$ the refraction effect can only be of the order $\frac{\cdot 03}{30} = \frac{1}{1000}$ of what it is at sea level. From this it may be argued that little difference in refraction will be made according as we take a height of 30 miles or 35 miles. The effect on the deduced coefficient of refraction, however, is considerable.

Bessel's refractions were founded on observations made by Bradley, so that they presumably refer to English climatic conditions. It is quite possible they are not equally applicable to India, for values of ϕ greater than 80°.

12. It remains to consider the most favourable way in which observations to stars may be made. It is clear that if we observe the altitude of a star on the prime vertical and note the time (local sidereal), we can also find the true altitude of the star from a star catalogue. The difference is the refraction. Taking a star as soon as it becomes visible after rising, we can continue observing it at short intervals and so get a set of observations which probably may be considered simultaneous so far as refraction is concerned. In any case, even if several stars were selected at suitable altitudes, the observations would necessarily be separated by short intervals of time and would not be strictly simultaneous. In this case the time needs to be carefully observed and the times at which altitudes are taken must be accurately noted, for an error of one second of time will cause an error of 15 seconds of arc in the deduced refraction. It would be desirable to observe the transits over several cross wires and reduce to the central wire. If we observe a star of suitable altitude on the meridian, it will not be necessary to know the time but in this case it is important to know the *astronomic* latitude. Several intersections of the star could be made and the readings reduced just as is done in a circummeridian observation for latitude. The observations could only extend over a short time as it would be necessary also to observe two or three other stars of various altitudes. Herein lies a difficulty, as it would not usually be possible to select some four stars of convenient altitude which transited quickly one after the other. Moreover if a star of too small altitude was selected, it might happen that, owing to haze, it would not become visible.

The method of observing on (or near) the prime vertical is accordingly to be preferred.

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* Dr. Walker, Director General of Observatories informs me that a pressure of $0''\cdot 03$ is to be expected at about 34 miles. The refraction due to the portion of the atmosphere above this is presumably negligible.

For in the case of haze being greater than was anticipated, it will only be necessary to wait till the star has a slightly greater elevation, when it is sure to become visible. Moreover there is a great advantage in it being necessary to observe to only one star. It is of course not essential that the star should rise on the prime vertical. In the general case it will be necessary to know both the local sidereal time and the astronomical latitude.

The procedure would probably be as follows:—First observe a rising star at several altitudes: second, observe the vertical angles to the luminous signal whose height is required: third, observe another rising star at several altitudes. At least three altitudes of each star must be observed, and extra observations would be of value. An observation of a star would probably consist of transits over several cross wires, whose intervals were known.

Two conditions necessary for the success of the method are first, that the refraction should remain sensibly constant during the time occupied in taking the several observations to either stars: and second, that throughout the complete time occupied by observations to stars and luminous signal, the refraction should be a linear function of the time. As to the constancy, or otherwise, of refraction during one night, the Survey of India does not appear to have any observations from which an opinion could be formed. Harkness* makes the following statement:—"The "investigations of the U.S. Coast and Geodetic Survey show (Transcontinental Triangulation "pp. 254-256) that the daily course of the coefficient of refraction consists of a day minimum, "usually lasting with little change from about 10 A.M. until 4 P.M., a night maximum lasting with "more or less unsteadiness from 9 or 10 P.M. until sunrise, and the junction lines uniting the "maximum and minimum." Apparently the refraction in the U.S. has a much longer period of approximate minimum than it has in India where the time of minimum refraction could not be said to commence before noon. If the nocturnal maximum lasts for about as long in India as it does in the United States then the two conditions referred to earlier in this paragraph will probably be satisfied during the period of maximum refraction.

It is well to recall at this point the simplicity of the relation between the change of refraction during the hours 8 A.M. to 2 P.M. and the temperature, found to occur in the observations we have discussed, to which attention has been drawn in the early part of Chapter IV. This simple relation was not so clearly followed from 2 P.M. to 4.30 P.M., and apparently the change of refraction from minimum to maximum is not so simple or regular as the change from maximum to minimum. This is to be expected as the thermal adjustment of the air naturally takes place more rapidly and evenly when assisted by convection, as occurs when the lower layers are gaining heat from the earth, than when these layers are chilled and convective adjustment is impossible. When the lower layers are too hot convection rapidly restores thermal equilibrium. It seems however quite likely that when maximum refraction has been attained this is itself an indication that the air has approached fairly closely to a state of thermal equilibrium. In this case it also seems very likely that when the maximum is more or less reached, any change in refraction will be of a slow and minor character.

Further comment on the proposed method is for the present withheld. It is hoped that observations which will test it will shortly be undertaken, and if satisfactory results can be obtained these will be published in a later paper.

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* "Terrestrial Refraction and the Trigonometrical Measurement of Heights" by William Harkness, in "The Astronomical Journal" Boston, Vol. XXII, No. 526, p. 178.

CHAPTER VI.

The Dip of the Horizon. Concluding Remarks.

1. The dip of the horizon is of great practical interest to navigators. This is a particular case in which the ray throughout its length is not far above the water, to which it is eventually tangential. If the water is perfectly calm there seems no objection to assuming the law of spherical surfaces of equal temperature. When there are waves this simple state of affairs can hardly exist, and no doubt the temperatures of those layers, which come in contact with the water as the water rises, fluctuate more or less. Taken over several wave lengths, however, the mean temperature at a given height will probably be pretty constant.

The dip of the horizon depends on the height from which it is observed. Moreover we must remember that we see only the crests of the waves at the horizon. Accordingly we are concerned with the difference of level of the points of observations and the *crests* of the waves, not of the mean water-level.

The expression for the dip is easily derived if we can assume the spherical distribution of temperature. For then we have

$$\mu\,(r + h)\sin\phi \doteq \mu_0\, r \sin\phi_0 \qquad (163)$$

where r is the radius of the sea-level surface or, more strictly, of the sphere which envelops the crests of the waves, and h is the height of the point of observation above this sphere. μ_0, μ are the refractive indices of the air at the two levels.

Now the ray of light to the horizon is obviously tangential to the water at the horizon. Hence $\phi_0 = 90°$: and if we put $\phi = 90° + a$ then a is the dip and we may write (163)

$$\cos a = \frac{\mu_0}{\mu}\left(1 + \frac{h}{r}\right)^{-1}$$

Remembering that a and $\frac{h}{r}$ are small and that $\mu = 1 + K\rho$, we have

$$1 - \frac{a^2}{2} = \frac{1 + K\rho_0}{1 + K\rho}\left(1 - \frac{h}{r}\right)$$

$$\doteq 1 + K\,.\,\overline{\rho_0 - \rho} - \frac{h}{r}$$

Hence, putting $\rho = \rho_0 + \Delta\rho$, we have

$$\frac{a^2}{2} = K\Delta\rho + \frac{h}{r}$$

or

$$a = \sqrt{2\left(\frac{h}{r} + K\,\Delta\rho\right)} \quad \ldots\ldots\ldots \quad (164)$$

the result being expressed in radians.

We accordingly have to evaluate $\Delta\rho$.

Now

$$\rho = \frac{p}{C\tau} \quad \ldots\ldots\ldots\ldots\ldots \quad (5)\ \textit{bis}$$

$$\therefore \quad \Delta\rho = \frac{1}{C}\left(\frac{p}{\tau} - \frac{p_0}{\tau_0}\right).$$

Also

$$p = p_0 - \rho_m\, gh$$

where ρ_m is some value of the density not very different from ρ_0 or ρ

$$\therefore \quad \Delta\rho = \frac{p_0}{C}\left(\frac{1}{\tau} - \frac{1}{\tau_0}\right) - \frac{\rho_m\, gh}{C\tau}$$

$$= -\frac{p_0}{C\tau_0}.\frac{\Delta\tau}{\tau} - \frac{\rho_m}{\tau}.\frac{gh}{C}$$

where

$$\tau = \Delta\tau + \tau_0$$

Hence turning to (164) we see that

$$\frac{h}{r} + K\Delta\rho = \frac{h}{r}\left(1 - r\frac{K\rho_m}{\tau}\frac{g}{C}\right) - K\rho_0\frac{\Delta\tau}{\tau}.$$

2. Now

$$r\frac{K\rho_m}{\tau}\frac{g}{C} = 3{\cdot}960 \times 5{\cdot}280 \times 10^6 \times 2{\cdot}93 \times 10^{-4}\frac{\rho_m}{\rho_s}.\frac{519{\cdot}4}{\tau}.\frac{1{\cdot}869 \times 10^{-2}}{519{\cdot}4}.\frac{g}{g_s}$$

$$= {\cdot}2204\cdot\frac{g}{g_s}\cdot\frac{\rho_m}{\rho_s}.\frac{519{\cdot}4}{\tau}$$

where suffix s indicates the standard values. If $g = g_s$, $\rho_m = \rho_s$ and the temperature is 60° F. this reduces to ·2204.

Next

$$K\rho_o\frac{\Delta\tau}{\tau} = 2{\cdot}93 \times 10^{-4}\frac{\rho_o}{\rho_s} \times \frac{519{\cdot}4}{\tau}.\frac{1}{519{\cdot}4}\Delta\tau$$

$$= 10^{-7} \times 5{\cdot}641\frac{\rho_o}{\rho_s}\ \frac{519{\cdot}4}{\tau}\Delta\tau$$

Put

$$F_1 = \frac{g}{g_s}.\frac{\rho_m}{\rho_s}\ \frac{519{\cdot}4}{\tau} \quad \text{and} \quad F_2 = \frac{\rho_o}{\rho_s}\ \frac{519{\cdot}4}{\tau} \quad \ldots\ldots\ldots \quad (165)$$

then F_1 and F_2 are both nearly unity, and if we also put

$$\left.\begin{aligned} h\,(1 - \cdot 2204\,F_1) &= h'\,(1 - \cdot 2204) \\ \text{and} \qquad F_2\,\Delta\tau &= \Delta\tau' \end{aligned}\right\} \qquad \qquad (166)$$

$$\frac{h}{r} + K\Delta\rho = 4\cdot 783 \times 10^{-8}\,h' \times \cdot 7796 - 5\cdot 641 \times 10^{-7}\,\Delta\tau'$$

whence

$$a = 10^{-4}\sqrt{7\cdot 46\,h' - 112\cdot 8\,\Delta\tau'}$$

or expressing the result in seconds

$$a'' = 56''\cdot 33\sqrt{h'\left(1 - 15\cdot 13\,\frac{\Delta\tau'}{h'}\right)} \qquad \qquad (167)$$

Reverting to (164) we see that

$$\frac{r}{h}\,K\Delta\rho = -\,rK.\frac{d\rho}{dh} \qquad \text{if } h \text{ is sufficiently small}$$

$$= \frac{\mu}{\sin\phi}.\frac{r}{\sigma} \qquad \text{by (28)}$$

$$\doteqdot 2k$$

since $\mu \doteqdot 1$ and ϕ differs little from 90°. It is clear then that we can also write (167) as follows

$$a'' = a_0''\sqrt{1 - 2k}$$

where a_0 is the dip in the case of no refraction. This is only true for small values of h'. a'' obviously vanishes when $k = \frac{1}{2}$, in which case the curvature of the ray is the same as that of the earth and the ray continues right round the sea surface.

Also $$a_0'' = \text{cosec } 1''\sqrt{\frac{2h'}{r}} = 63''\cdot 8\sqrt{h'}$$

whence $$a'' = 63''\cdot 8\sqrt{h'\,(1 - 2k)} \qquad \qquad (168)$$

In *Chauvenet's Astronomy* page 175 it is stated "For an altitude of a few feet the difference "of pressure will not sensibly affect the value of D' [$= a$ in (164)] and may be disregarded". This appears to me to be incorrect, for we are not concerned so much with difference of pressure as *rate of change* of pressure, a quantity which has a definite value even for a horizontal ray. The consideration of this brings in the quantity ·2204 in (166) which changes the coefficient of h' from unity to ·7796.

3. Values of the dip, a, calculated from (167) for certain values of h' and $\frac{\Delta\tau'}{h'}$ are now given in table LIII.

TABLE LIII.

Dip of the horizon.

$\frac{\Delta\tau'}{h'}$	h'=100 *feet*		80 *feet*		60 *feet*		40 *feet*		20 *feet*	
	′	″	′	″	′	″	′	″	′	″
+0·06609	0	0	0	0	0	0	0	0	0	0
·06	2	51	2	33	2	13	1	48	1	16
·05	4	38	4	9	3	35	2	56	2	4
·04	5	54	5	17	4	34	3	44	2	38
·03	6	56	6	12	5	22	4	23	3	6
·02	7	50	7	1	6	4	4	58	3	30
·01	8	39	7	44	6	42	5	28	3	52
·00	9	23	8	24	7	16	5	56	4	12
−0·01	10	4	9	1	7	48	6	22	4	30
− ·02	10	43	9	35	8	18	6	47	4	48
− ·03	11	19	10	7	8	46	7	10	5	4
− ·04	11	54	10	38	9	13	7	31	5	19
− ·05	12	27	11	8	9	38	7	52	5	34
− ·06	12	58	11	36	10	3	8	12	5	48
− ·07	13	28	12	3	10	26	8	31	6	2
− ·08	13	58	12	29	10	49	8	50	6	15
− ·09	14	26	12	54	11	11	9	8	6	27
− ·10	14	53	13	19	11	32	9	25	6	39

It is seen that when the temperature increases upwards at the rate of $0^\circ\cdot066$ F. per foot the dip is zero. For more rapid increase of temperature the sea surface will appear *concave* and there will be no true horizon. Such rates of change of temperature of course cannot persist to any considerable height, so that this phenomenon is only witnessed near to the earth (or sea) surface.

4. The object of the present investigation was to express the refraction undergone by a terrestrial ray by means of a formula deduced from known physical laws. Heretofore in survey operations refraction has been estimated by assuming that, for all rays starting from a given point, the angle of refraction bears a constant ratio to the angle subtended by the ray at the centre of the earth. This ratio has been termed the "coefficient of refraction," and denoted by k. Considering the section of the earth in any given azimuth to be practically circular, it follows from the assumption that the path of the light is a circle whose radius is $R/2k$, R being the radius of the circular section of the earth. Seeing that the radius of curvature of a section of the earth varies according to the azimuth in which the ray lies, and that this variation is about a half per cent of the radius in latitude 30°, it is evident that no very high precision can be

expected. Moreover it is a well known fact of observation that refraction becomes smaller in the higher layers of the atmosphere: still it has been customary to assume the same "coefficient of refraction" for a ray proceeding horizontally as for one reaching a great height. If we take the ratio of mean curvature of the earth to that of a horizontal ray of light we arrive at a "coefficient of horizontal refraction" which is a consistently defined quantity. The assumption that the ray is of circular form may be otherwise stated by saying that the angles of refraction of a ray at its two ends are equal. This is tantamount to saying that the coefficient of refraction at both ends of a ray is the same. If we consider a ray ascending from say 3,000 feet to 20,000 feet we arrive at the false conclusion that the coefficient of refraction at these two widely different heights is the same. The assumption accordingly is not a good one in the case of rays which have a wide range of height.

Two methods have been used to determine the coefficient of refraction. In both methods the effect of plumb-line deviation has been ignored in general. The first is to observe the elevation of a point whose height has previously been found by spirit-levelling. From this the true angle of elevation can be deduced and the difference of this and the observed elevation when divided by the angle subtended at the earth's centre is the coefficient required. The second method consists in observing the ray from both ends. If E_1, E_2 are the observed elevations, Ω the refraction, *assumed equal at both ends,* and χ the angle at the centre of the earth we know that $E_1 - \Omega + E_2 - \Omega + \chi = 0$, whence $\Omega = \dfrac{\chi + E_1 + E_2}{2}$ and the coefficient follows by dividing by χ.

We may replace E_1 and E_2 by $90° - \phi_1$ and $\phi_2 - 90°$ and get $\Omega = \dfrac{\chi - \phi_1 + \phi_2}{2}$. We also have the relation $\mu_1 r_1 \sin \phi_1 = \mu_2 (r+h) \sin \phi_2$, so that if the density of the air is known at both stations *it is not necessary to observe both angles ϕ_1 and ϕ_2.* Refraction is known to vary largely from time to time on the same ray. Accordingly observations at the two ends of the ray should be made simultaneously if we are to deduce the refraction coefficient by assuming equal refractions at both ends. Simultaneous observations are as a rule difficult to arrange for, and have seldom been made in practice. In cases where they have been, the deduced difference of height, on the assumption of equal refractions at both ends of the ray, by no means always remains the same for observations at different times.

It has long been known that refraction is usually least during the middle hours of the day—in India the time of "minimum refraction" is believed to be 1.30—3 P.M. The important feature of "minimum refraction" is that its value varies from day to day in general very much less than the refraction at any other hour of the day. On this account the plan of observing reciprocal angles on different days at the time of minimum refraction, in the hope that the refraction would then be the same, has come into operation. This plan constituted a very great advance on the old method of observing vertical angles at any time of the day. It cannot be said, however, that the results are as precise as could be wished for. In deducing the height of points in triangulation the method has had a fair measure of success. In deducing the heights of inaccessible mountains, making use of some assumed value of the coefficient of refraction, large discrepancies have been found. This is largely attributable to the fact that owing to plumb-line deflection being neglected, a wrong coefficient of refraction has been deduced.

Even if the original assumption as to the form of the ray being circular were correct, in mountain triangulation or in cases where large deflections of the plumb-line occur, very large errors may be made in the determination of the coefficient. The expression $\frac{\chi+E_1+E_2}{2\chi}$ is in error on account of the angles E_1 and E_2 being burdened with the error of plumb-line deflection: and only where the deflection at both stations is the same in the azimuth of the ray can a correct coefficient be deduced independent of the plumb-line deflections. In the Himalayas where deflections of as much as one minute of arc occur the error may be very serious. This is the case of the Nojli–Mussooree ray, the difference of plumb-line deflection along the ray at the two ends is 30″ which occurs as an error in the combined refractions which amount to some 300″: the error accordingly being 10°/₀.

5. Enough has been said to show that the "coefficient of refraction" method of dealing with refraction can only be regarded as a make-shift: that such should be the case is not surprising considering that the method is based on an arbitrary assumption which is in itself somewhat contradictory. The present paper is an endeavour to put terrestrial refraction on a more accurate and scientific basis: to find out on what the refraction depends and to explain as far as possible its variation. While much work remains to be done on the subject a certain measure of success has been obtained. The formulæ derived in Chapter I show that the refraction depends very largely on the rate at which the temperature changes with height, and with the change of this rate: it also depends on the height above the horizontal plane through the observing station to which the ray extends. Thus the refraction on a ray of given length differs according as the ray is ascending or descending. This is in accordance with observation.

The only assumption made is that layers of equal density in the air are concentric with the (circular) section of the earth in the azimuth of the ray. This is an assumption which has always been made in investigations of refraction. Except in the case of rays close to disturbing matter radiating heat it is a natural assumption, being one condition for thermal equilibrium. Its general truth is well substantiated by the fact that refraction in a horizontal plane is in general so small as to be scarcely measurable. Exceptions certainly occur in the case of rays passing very close to heated ground, especially when this ground does not lie symmetrically with respect to the vertical plane through the ray. Another condition of *static* thermal equilibrium of the atmosphere is that the temperature gradient should be adiabatic. This is the gradient which would obtain if sufficient time was allowed without disturbance. In actual fact the diurnal change of temperature does not allow sufficient time. Most observations on the fall of temperature of the air have given a less rapid fall of temperature than the adiabatic rate. The rate however can hardly be exceeded, except momentarily, for convection must be set up if it is; and this adjusts the gradient. The adiabatic gradient is found then to give a natural minimum refraction. In Chapter II the formulæ are applied to the heights of several stations which have been connected by spirit levelling, differences of height of geoid and spheroid being allowed for as well as existing plumb-line deflection data permit. It is found that in spring the refraction found by assuming the adiabatic gradient closely approximates to that observed at the time of "minimum refraction". This is notably the case in observations from a station in the plains up to the hills. In reconciling this with observations made on the temperature gradient, it is to be remarked that such observations are not simultaneous, depending as they do on the ascent of a balloon or kite: moreover not many of them have been made at the time of minimum refraction. Certain Indian observations have shown a gradient not very different from the adiabatic gradient for unsaturated air. This may be a peculiarity of the country or of Indian latitudes. In any case we have an explanation, on thermal grounds, of the observed fact of minimum refraction.

It is not however proposed that the adiabatic gradient should be assumed. Steps should be taken to determine the temperature gradient in any particular case. The desirability of this has been brought out by the present investigation, but the observations made heretofore are devoid of any data concerning temperature gradient. Various methods are suggested in the later chapters for the practical determination of the gradient. One method depending on observations to points of known height is given in Chapter III and gives a means of determining the height of certain snow-peaks which have been observed to many times. It is not claimed that this method is as good as the direct determination of temperature gradient: it was adopted because no such determination was available, in order to proceed with the application of the formula to great heights. In this chapter an estimation of the plumb-line deflection at Nag Tiba is made, based on vertical angle observations, giving a meridional deflection of 31″·3. Observations made subsequently to the printing of that chapter give the deflection as 30″·5 (*see* appendix).

Chapter IV gives an interesting empirical relation between temperature and refraction on a given day. Observations elsewhere are needed to show that this relation is in any sense universal. The very close agreement however seems to indicate that there must be some physical reality in it. A method of computation of temperature gradients by means of barometric observations at two heights follows, and application is made to the computation of refraction. The degree of reliance to be placed in the results cannot be stated owing to the uncertainty at present existing in the difference in height of the geoid and spheroid at the two stations under consideration. There is distinct evidence however that the angles of refraction at the two ends of the ray are unequal to an extent not previously expected. The ray in question has the rather unusually large elevation for terrestrial observations of more than five degrees. The result is also arrived at that barometric heights deduced from the ordinary Laplace formula are most precise when the observations are made at midday, which is in accordance with refraction theory. Chapter V discusses a method of finding the terrestrial refraction at night by certain relations with celestial refraction therein developed. In this case no observations are available by means of which the corresponding method can be tested. Application is made to Bessel's refractions, with the result that the height of the atmosphere so far as it has any appreciable effect on refraction is estimated to be 35·7 miles and the corresponding coefficient of nocturnal refraction is found to be ·064. The accuracy with which Bessel's formula for refraction represents celestial refraction at all hours and in all countries cannot be estimated by the author.

The following remarks, relative to Bessel's tables, are made in *Chauvenet's Astronomy*, p.132:—
"These tables extend only to 85° of zenith distance beyond which no refraction table can be "relied upon. There occur at times anomalous deviations of the refraction from the tabular "value at all zenith distances; and these are most sensible at great zenith distances. "Fortunately, almost all valuable astronomical observations can be made at zenith distances less "than 85°, and indeed less than 80°; and within this last limit we are justified by experience in "placing the greatest reliance in Bessel's table. In an extreme case where an observation is "made within 5° of the horizon we can compute an approximate value of the refraction by the aid "of the following* supplementary table, which is based upon actual observations made by "Argelander".

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* Not given here.

Bessel gives* the following probable errors of refraction computed from his tables:—

TABLE LIV.

φ	P.E. ±	φ		P.E. ±
°	″	°	′	″
45	0·27	85	0	1·71
60	0·34		30	2·00
65	0·37	86	0	2·40
70	0·46		30	2·63
75	0·66	87	0	3·87
80	0·92		30	5·30
81	1·00	88	0	7·74
82	1·11		30	10·58
83	1·25	89	0	16·84
84	1·43		30	20·01

Differences of this kind would modify the solution of Chapter V by a considerable amount.

6. The formulæ and methods developed in this paper have explained satisfactorily the refraction observed to occur on the rays between the stations Nojli, Mussooree and Nag Tiba. It would be interesting however if such observations were repeated along with accurate pressure and temperature readings at each end of the ray, and if possible at some third station. Such observations would enable the temperature law to be well established, and the refraction could then be directly computed. If kite observations of temperature and pressure were also made simultaneously, these would be of great use.

Observations at night to lamps and stars might also be made and the refraction also computed from the star observations in the manner indicated in Chapter V. In this way we should find the relative precision of the alternative method.

Such observations should be continued throughout the year, as far as practicable. We have distinct evidence of a seasonal change in minimum refraction and some regular change might perhaps be brought to light. The diurnal change appears greater in winter than in spring. Its variation might be investigated and the empirical relation found to exist in Chapter IV with temperature might be checked.

The rays which go to greater heights than Nag Tiba (10,000 feet) are burdened with uncertainty due to the fact that no fixed recognisable marks have been intersected. The snow-peaks vary in height on account of snow falls etc: and change of illumination may affect the value of the vertical angles observed. It is desirable that signals should be set up on one or more of

* Tabulæ Regiomontanæ p. LXIII.

the snow mountains with which we have been concerned, at various heights; and some of these should be placed as high as possible. If signals were put up at heights 10000, 14000, 18000 feet observations to these would give valuable information concerning the variation in height due to snow-fall and the secular change in height of the mountains as a whole. One of the objects of Mr. Shaw's observations was to determine whether the Himalayas are gradually rising.

7. Very many anomalies have been, and are often, found to occur in the case of rays which pass close along the ground or water. The formulæ of Chapter I may account for many of these, if the temperature-height law is determined. In this case we might find the law by actually observing temperatures at several heights above the ground level. Table XIIA of Chapter II shows that the first fifty feet above ground level have a considerable disturbing effect on the refraction, and probably the irregularities due to the higher portion of the atmosphere are not very great. The Nojli–Mussooree ray, however, is not close to the ground, except at its start: and we have sometimes to deal with rays which never get far above the ground. I have recently made a light mast 150 feet high, which can easily be erected or taken down, with the object of attaching to it some apparatus for the measurement of temperature at several heights, and so being able to compute the refraction. A suitable apparatus has just been designed by Mr. R. W. Paul in collaboration with the National Physical Laboratory through the courtesy of Dr. R. T. Glazebrook, C.B., F.R.S. The masts have the advantage of affording excellent signals for intersecting and also in increasing the height of the ray above the ground and so partially avoiding the highly disturbed air close to the ground. A description of the mast and the means of erecting it will shortly be issued in another paper of this series.

In the case of a ray passing over country of a uniform character (as regards radiating power) it is anticipated that the formula will be satisfactory. Over country of a less uniform character, equally good results cannot be expected: but the degree of precision in either case can only be found by trial. General Walker made numerous observations for refraction between some stations in the Punjab, which are described in Appendix 3 of Professional Volume II of the Survey of India. Refractions of very widely varying amounts were found, the highest deduced coefficient of refraction being $+1 \cdot 21$ occurring between 6 and 8 A.M. and the lowest $-0 \cdot 09$ between 1·30 and 3 P.M. The phenomenon of a coefficient of refraction being greater than one-half causes the surface of the earth to appear *concave* instead of convex and General Walker describes other curious effects. These abnormal results may perhaps be largely capable of explanation by the formula of Chapter I: at least it is of interest for similar observations to be made in the light of these formulæ.

8. The same method might have useful application in the case of observations for precise levelling across the wide Indian rivers, provided the rays are kept at a sufficient height above the water. A crossing has recently been made at Goalundo over the Ganges by Captain V. R. Cotter, I.A., of which some particulars are now given.

Goalundo is a few miles below the junction of the Ganges with the Brahmaputra, and the actual observations were made about three miles below this junction. Pillars were set up on both banks of the Ganges in each case some ten yards from the bank, the distance between the pillars being 1·365 miles, and simultaneous observations of vertical angles were made with two twelve-inch theodolites. As far as could be seen the conditions at both ends were the same, though the mean of observations, reduced in the usual way, actually showed the pillar on one side to be higher by 2·356 feet than that on the other side. The theodolites in each case were about 4·8 feet higher than the signals so that the reciprocal rays were not absolutely identical: they passed at a mean height of some ten feet above the water.

If we assume the mean of all the very numerous observations to give the correct difference of level, namely 2·3560 feet, we can deduce the appropriate dips of the two straight lines joining the theodolites and signals, obtaining the angles 1′ 46″·26 and 3′ 58″·00, the contained arc being 71″·01. This done we can at once find the refraction in all the cases observed. The stations are denoted by *A, B*. The details are given below :—

TABLE LV.

Date	Time	Refraction at A	Refraction at B	Date	Time	Refraction at A	Refraction at B
1913	h m	″	″	1913	h m	″	″
Feb. 3	1 6	13·73	13·55	Feb. 6	1 9	13·34	12·84
	1 28	7·42	10·64		1 21	17·53	14·84
	1 59	11·58	11·21		1 34	12·10	11·82
	2 44	19·17	14·76				
Feb. 4	10 14	6·13	8·18	Feb. 7	10 8	13·12	9·70
	10 40	10·26	11·14		10 22	14·37	13·70
	11 9	13·88	13·37		10 38	16·85	13·67
	12 48	16·01	14·58		10 53	18·18	14·45
	1 11	16·55	11·14		11 23	10·95	13·80
	1 57	14·54	11·64		11 35	11·00	9·83
	2 18	19·19	13·26		11 48	11·10	11·85
					1 1	10·55	9·71
Feb. 5	11 39	11·38	15·53		1 11	16·88	15·51
	12 2	13·75	14·00		1 24	14·47	10·76
	12 22	25·29	21·99				
	12 43	29·72	27·07	Feb. 8	10 5	7·43	8·54
	1 35	17·60	20·31		10 19	5·56	8·64
					10 31	7·66	12·05
Feb. 6	10 13	7·07	8·94		10 44	10·48	15·12
	10 33	7·83	10·80		11 21	7·67	9·70
	10 47	13·19	12·33		11 32	8·80	10·66
	11 10	16·34	21·14		11 46	12·51	13·43
	11 25	15·46	21·62		11 57	7·94	10·25
	11 43	15·53	16·94		12 32	6·59	6·16
	12 30	11·37	12·32		12 44	8·19	5·64
	12 44	12·01	11·94		12 54	9·89	8·16
	1 1	14·02	13·24		1 2	15·97	6·53

A fact immediately noticeable is that the refraction reached its largest value generally (in nine cases out of twelve) after noon when minimum refraction usually occurs. Moreover the variation in refraction was not the same at *A* and *B* although special pains were taken to secure simultaneous observations. Further, the range in the refraction was greater at *A* than at the higher station *B*. This was to be expected. If we classify the results according to time and take means we find the following figures.

TABLE LVI.

Between hours	No. of Observations	Refraction at		Coefficient of refraction at	
		A	B	A	B
10 and 11	13	10″·63	11″·33	0·150	0·160
11 ,, 12	12	11·88	14·01	·167	·197
12 ,, 1	9	14·76	13·54	·208	·191
1 ,, 3	16	14·67	12·61	·206	·178
Variation ...	...	4·13	2·68	·058	·037

This table shows how at A the refraction reached a maximum between hours 12 and 1. It also shows that the variation in refraction was smaller at B than at A. The conclusion to be derived from this, as might be foreseen from thermal considerations, is that it is advantageous to raise the height of the instruments and signals, so as to reach higher and less disturbed layers of air. The temperature of the air is naturally disturbed near to the water. Table LVI shows a decrease of variation in coefficient of refraction from ·058 to ·037 due to a rise of 2·4 feet only. It is to be presumed that much greater steadiness in refraction would have been obtained at a mean height of 20 feet instead of 10 feet above the water.

The figures given in table LV may be explained in more ways than one, though the true explanation may be due to a combination of the causes suggested. When the observations were in progress it was thought that the reciprocal rays were symmetrical with regard to the river, and that the refraction at the two ends would be the same. That this was not strictly the case is shown by the figures of table LV. At the end of Chapter I it is shown how largely temperature gradient affects refraction. In the present case we have the confluence of two rivers. The temperatures of the water of each cannot be expected to agree absolutely. Nor can it be expected that these waters will mix properly within a distance only about twice the width of the united river. Accordingly it is highly probable that one side of the river is distinctly cooler than the other, and this will affect the temperature of the lower layers of the air differently on the two sides. It is not, however, necessary to assume that this actually occurred, although it appears highly probable that it did. The discrepancy may be due to difference in height of the two stations. The signal at B was 3·713 feet below the horizontal plane through instrument at A: and the signal at A was 8·316 feet below the horizontal plane through the instrument at B. If we assume the barometric pressure to have been 29″·5 and the absolute temperature $\tau = 521°$ we find from (63) that

$$\omega_1 = +\ 5''\!\cdot 15$$
$$_A\omega_2 = +\ 0''\!\cdot 962 \times 10^{-4}$$
$$_B\omega_2 = +\ 2''\!\cdot 154 \times 10^{-4}$$

where the prefixes A and B indicate the station to which ω_2 refers.

As an example, let us consider the case, between hours 1 and 3, given in table LVI. In the notation of Chapter II § 20, we have at once

$$\left.\begin{aligned} u\,\omega_1 + v\,{}_A\omega_2 &= 14\cdot67 - 5\cdot15 = 9\cdot52 \\ u'\,\omega_1 + v'\,{}_B\omega_2 &= 12\cdot61 - 5\cdot15 = 7\cdot46 \end{aligned}\right\} \quad \ldots\ldots \quad (169)$$

The theodolite at B was 2·36 feet higher above sea-level than that at A.

Solving these equations, making use of (81) and (83) and assuming as is subsequently justified that u can be neglected in comparison with b, we find the following values:—

$$\left.\begin{aligned} -b \doteq v \doteq v' &= 3\cdot22 \times 10^4 \\ u &= 1\cdot247 \\ u' &= 0\cdot102 \end{aligned}\right\} \quad \ldots\ldots \quad (170)$$

Equation (170) indicates a very rapid *variation* in temperature gradient with height: and temperature increasing with height at both A and B; at B very much less than that at A.

From (170) we have at once in notation of Chapters I, II.

$$\begin{aligned} a &= 0\cdot542 - 1\cdot327u \\ &= -1\cdot113 \end{aligned}$$

whence

$$\frac{d\tau}{dh} = +\ \cdot01113$$

Also

$$\begin{aligned} \frac{d^2\tau}{dh^2} &= 2.10^{-7}b \\ &= -6\cdot44 \times 10^{-3} \end{aligned}$$

and

$$\begin{aligned} \tau' - \tau &= h\frac{d\tau}{dh} + \frac{h^2}{2}\frac{d^2\tau}{dh^2} \\ &= \cdot0263 - \cdot0179 \\ &= \cdot0084 \end{aligned}$$

if we put $h = 2\cdot36$; τ', τ being the temperatures at B and A respectively.

This indicates how extremely small are the temperature differences necessary to account for the refractions, amounting in this case to less than $0^\circ\cdot01$ between the stations A and B. It is of course the temperature gradients which are important.

These results are of type which may well be expected to hold in the case of the layers of the atmosphere close to the ground. As is shown the temperature gradient is rapidly changing between A and B and no doubt it approximates to a normal value at a somewhat greater height.

It is at the same time recognised that perfect thermal symmetry cannot be expected in the case under discussion, so that the figures found above are only regarded as indicating the nature of the causes of the observed refractions, which at first sight appeared extraordinary. The remedy in such a case lies not in attempting to measure the temperature gradient and its variation, but in building higher pillars and so observing through less disturbed layers of air. For this purpose perhaps a height of 20 feet might prove sufficient. A few observations made at several heights in such a case would decide how high it is desirable to go.

No doubt the observations would have shown smaller irregularities had the rays been at a greater height above the water than they were. In fact it is certainly a safe rule to always arrange for the ray to be as high as possible above the earth's (or water's) surface, when vertical angles are being observed.

9. Lieut. F. J. M. King, R.E., obtained some curious irregularities when observing vertical angles in Burma in 1911.

The work being carried on was principal triangulation (Great Salween Series). The early observations agreed with observations of the previous year in the case of those stations which had been previously observed to. Between the 13th and 18th of December there was a large change in the vertical angles which subsequently showed a tendency to return to the early values. The early values, however, were not reached. The figures are now given.

TABLE LVII.

Observations taken at Loi Lung Hill Station, height 7631·3 feet.

	Zenith distance and temperature			
Date / Observed Station	Loi Paning	Loi Chang	Loi Pemong	Loi Wan Wa
11th Dec. 1911 ...	...	...	...	90° 33′ 97″ 54°·0
12th ,, ,, ...	...	90° 32′ 104″ 57°·0	90° 25′ 122″ 57°·0	...
13th ,, ,, ...	91° 0′ 49″ 53°·4	...	90° 25′ 117″ 56°·3	90° 33′ 96″ 51°·0
18th ,, ,, ...	...	90° 32′ 19″ 55°·0	...	90° 33′ 22″ 58°·4
19th ,, ,, ...	91° 0′ 18″ 70°·0	90° 32′ 25″ 78°·0	90° 25′ 51″ 58°·0	90° 33′ 41″ 65°·0
20th ,, ,, ...	91° 0′ 24″ 70°·0	90° 32′ 44″ 75°·0	90° 25′ 72″ 68°·0	90° 33′ 76″ 79°·0
Height of station ...	5709·7	6525·6	6924·0	6522·6
Distance in miles from Loi Lung ...	24·506	50·885	38·124	33·137

The barometric height (mean of two aneroids) varied between 22·39 and 22·61 inches. The observations were all made between 1 and 3 P. M.

It is clear that the changes in refraction are far larger than can be accounted for by the temperature and pressure changes observed at Loi Lung. On the other hand there appears to have been a sudden change of temperature after the 18th of December, which indicates a disturbance of the atmospheric conditions.

It is not possible now to account for these irregularities quantitatively, but perhaps suitable temperature gradient measurements might enable future observations of a similar character to be explained. In such a case simultaneous temperature measurements at several heights down the mountain side might give the temperature law, if suitable precautions against radiation etc., were taken.

10. Several writers have pointed out a difference between the refraction over water and over land. Colonel Clarke (*see* p. 550 *Ordnance Survey, Account of Principal Triangulation, 1858*) obtained mean values of ·0809 and ·0750 respectively for these two cases. Sir Henry James, in the introduction to the same volume, said :—

"The great amount of refraction in the morning, its diminution towards the middle of the "day and increase again towards evening is obviously caused by the greater amount of aqueous "vapour in the lower portion of the atmosphere in the morning and evening as compared with "the amount in the middle of the day". Both these features may be accounted for by the temperature gradient, which is intimately related to the humidity gradient, as can be seen by glancing at any charts showing these two quantities plotted against height.

11. It was supposed early in the 19th century that the refraction at sea had an intimate connection with the difference of temperatures of the air and water. Lieut. Raper (*see* p. 61, *Practice of Navigation, tenth edition, 1870*)) stated :—

"When the sea is *warmer* than the air the horizon appears *below* its mean place, or that at "which it appears when the air and water are at the same temperature, or the apparent dip is "too small; when the sea is *colder* than the air the horizon appears above its mean place or the "apparent dip is too great".

If we consider the air in contact with the water as having the same temperature as the water, and that "the temperature of air" refers to air at the height of the sextant, this statement can be put in terms of temperature gradient: and is supported qualitatively by the formulæ at the beginning of this chapter. In this connection see *Chauvenet's Astronomy, p. 175* where the following statement occurs: "We may however assume the temperature of the water to be that of the lowest stratum of the air . . . . while τ denotes the temperature of the air at the height of the eye".

Lieut. Raper goes on to say in a footnote "Admiral W. F. W. Owen informs me that he "found on one occasion, in observing a star's altitude a change of 4′ in the place of the sea "horizon, in the tropics soon after sunset. Mr. Fisher observed a variation in the place of the "horizon of 18′ in the arctic region. In summer the ice horizon was *elevated** not depressed: "in the winter it was depressed several minutes.—*Appendix to Captain Parry's Voyage in 1821-23*

* This is equivalent to General Walker's statement that the earth appeared concave when the coefficient of refraction is greater than one-half. I think the term horizon in this case has no strict meaning. With the ray of light of greater curvature than that of the earth it would be possible to see all round the earth where there were no obstacles above sea-level, were it not for absorption of light. This limitation would not give a sharp horizon.

"*p. 187*. These observations, however, do not all follow the rule above. A table for correcting "the place of the sea horizon for the difference of temperature of the sea and the air according "to the height of the eye, would be useful: but there are scarcely any data for the construction "of such a table, and the theory itself appears not to be complete".

Captain W. R. Martin, R.N. in his article on Navigation in Encyclopædia Britannica, tenth edition, Vol. XXXI, page 110 d writes as follows:—

"The effect of refraction in displacing the apparent sea horizon was partially investigated "by the French about a century ago, and the conclusion arrived at was that it mainly depended "on the difference of the surface temperature of the sea and that of the stratum of air resting on "it: when the sea surface was warmer the horizon was apparently *depressed* and observed altitudes "were too great, and *vice versâ*, differences of 3′ to 4′ being found where these temperatures "differed 9° F. The subject has now been thoroughly reinvestigated, and the original conclusion "confirmed. . . . . . . The interesting experiments referred to were carried out in 1899 "by the Austrian Navy in the Red Sea and at Pola, the requisite observations being made "throughout the day with an alt-azimuth instrument from shore stations at elevations of "21, 33, 52 and 138 feet respectively, the sea surface and air temperatures being simultaneously "recorded from a steam boat standing off shore. From the experience of over 1,000 observations "it was found that at an elevation of 33 feet, with a difference of 6° F. in sea and air "temperatures, a displacement of $1\frac{1}{2}'$ was observed which increased to $2\frac{3}{4}'$ when the difference "of 12° F. occurred in the temperatures. Such results were only found to be true when a wind "of force at least 2 or 3 was blowing: with winds of less force the warm air can apparently "remain at a higher level without mixing with the lower cooler air, and abnormal results follow, "amounting in one case to a displacement of $9\frac{1}{3}'$. When navigating the Red Sea or localities "such as near the edge of the Gulf Stream, where very great differences of sea and air "temperatures prevail, due consideration must be made for this source of error, and the "practical navigator can also see how greatly this may affect his estimation with regard to "currents".

The statement that these results are only true when a wind of force 2 or 3 is blowing seems to rob the result of much generality; it appears, however, from the expression "temperature . . . of the stratum of air resting on it" that perhaps the *variation* of air temperature with height was not recognised. In my opinion the temperatures which affect the problem are the temperature at the height of the eye and that at the level of the crests of the waves: and the difference of height of the eye and wave crests is the other quantity concerned. Measured in this way the differences of temperature which occur in Captain Martin's statement would probably be reduced, and possibly the dips would then accord with those given in table LIII. Some observations on the Indian coast might serve to make this clear.

12. In triangulating in big mountain ranges such as the Himalayas, it is very desirable that observations for latitude and azimuth be made at all, or most of, the stations. These observations are required to enable the plumb-line deflections to be found and need not be precise to more than 1 or 2 seconds. Without this information it is not possible to deal properly with the vertical angles observed. It is possible that those deflections may be deduced from the vertical angles themselves provided the necessary temperature gradient measurements are made, by means of which the refraction can be computed: but in the first instance, until the success of this method has been established, it is desirable that both observations be performed.

APPENDIX.

Some recent Latitude determinations.

In Chapter II §§ 10, 11 it was pointed out that a discrepancy of 1·9 feet occurs in the height of Mussooree above Dehra as determined by vertical angles and by spirit-levelling, the difference in height of the geoid and spheroid being allowed for as well as deflection of plumb-line data allowed. This discrepancy was then wholly attributed to failure to compute the height of the geoid above the spheroid, and as a result, larger deflections of the plumb-line between Dehra, Rajpur and Mussooree than were deducible by simple interpolation were expected to occur. In Chapter IV, § 14 it was suggested, in view of the simultaneous temperature and pressure readings taken at Dehra and Mussooree in 1911, that possibly only a portion of the discrepancy is to be explained in this way, the residue being attributable to the refraction in the vertical angles being insufficiently allowed for. That this should have occurred was inevitable, as no simultaneous pressure and temperature readings at both ends of the ray, at the time the vertical angles were observed, were available. The original view, that the discrepancy is due to the rise of the geoid being insufficiently allowed for, may require modification to the extent that perhaps only part of the discrepancy is due to this cause.

To test this theory, latitude has since been observed at a number of stations between Dehra and Mussooree. It is worthy of note that this has resulted in two stations being found at which the deflection in latitude is greater than has ever been observed before elsewhere.

The results are given in tabular form : stations III to VI are between Dehra and Rajpur: stations VIII and IX between Rajpur and Mussooree. The observations were made by Lieuts. Almond and McKay, R.E., of the Survey of India.

Station Number	Longitude			Latitude			Deflection in Latitude
	°	′	″	°	′	″	″
III	78	4	7·39	30	21	46·61	41·04
IV	78	4	30·87	30	22	8·93	42·15
V	78	5	21·38	30	22	51·83	44·37
VI	78	6	2·00	30	23	30·79	45·89
VIII	78	5	35·94	30	24	37·72	53·17
IX	78	5	21·53	30	25	10·05	52·50
Nag Tiba Lat. Station	78	9	9·90	30	35	11·57	30·52

The deflections at III and VI are very closely the same as are obtainable by linear interpolation between the old observed values at Dehra and Rajpur. Those at VIII and IX far exceed the values found by interpolation between the Rajpur and Mussooree deflection (*see* table V). They go some way to explain the discrepancy of 1·9 feet referred to above. Further data as regards the deflection of plumb-line in longitude, however, are still required before the rise of the geoid above the spheroid can be evaluated with sufficient precision. At present we have the deflection in longitude only at Dehra and Mussooree, and the deflections at intermediate points can only be arrived at by guessing. It is hoped that before long azimuth observations may be made at the two or three intermediate points between Dehra and Mussooree.

Even yet it is likely that the point of maximum deflection in latitude between Rajpur and Mussooree has not been found, since stations VIII and IX show practically equal deflections. They may of course be on the ridge of maximum deflection.

The observation at Nag Tiba well upholds the value of the deflection derived from vertical angle observations in § 24, Chapter II. The discrepancy is only 0″·8 and it was pointed out that an error of 1″ in any of the vertical angles would give rise to an error of 1″·8 in the deduced deflection. The case was not a straightforward one; so that the close agreement of the deduction with the result subsequently observed must be regarded as evidence that the refraction on the rays concerned has been satisfactorily computed.

Chart I is in the pocket at the end of the volume.

CHART II

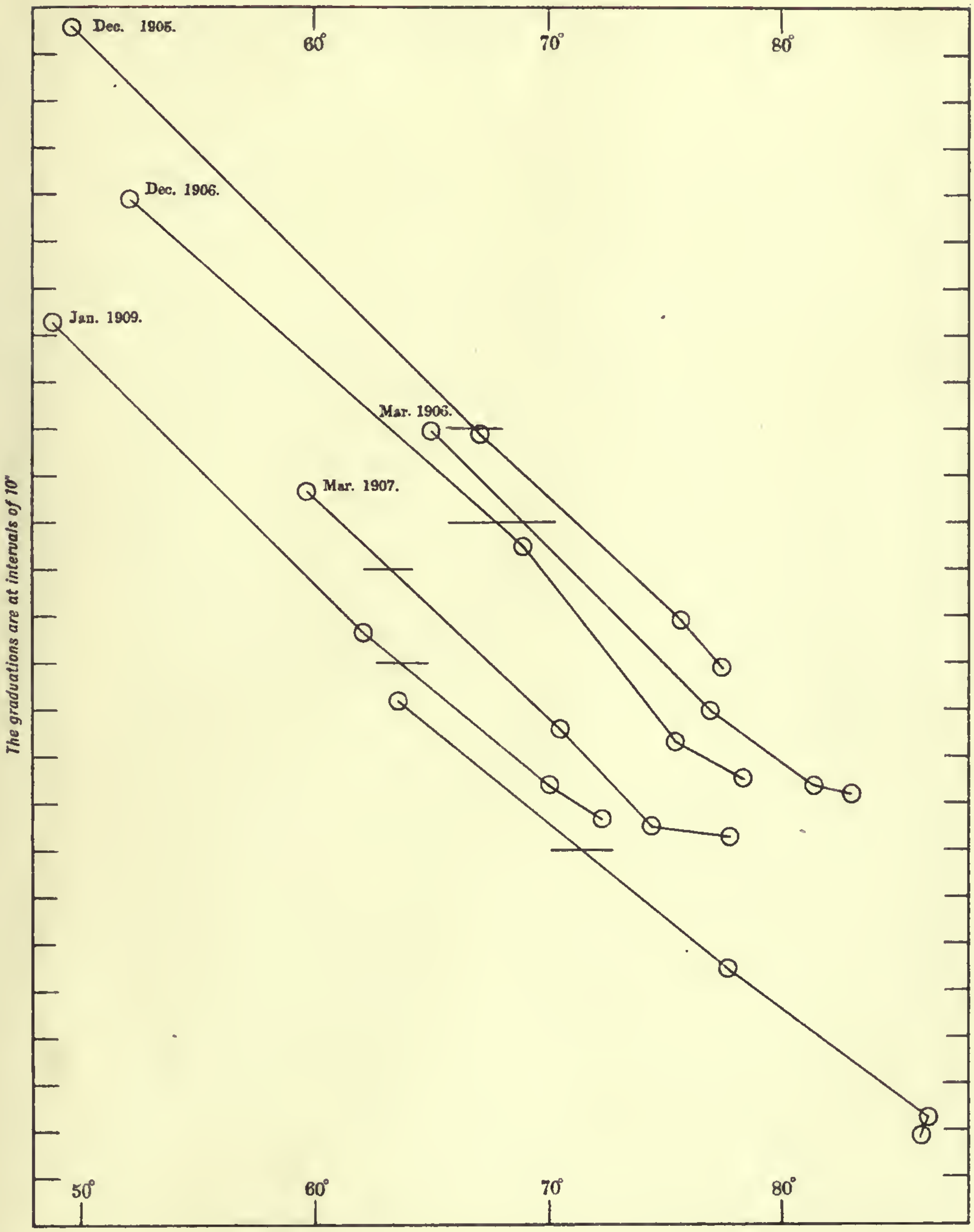

CHART II.

Duirnal Change of Refraction.

Graphical representation of table XL. Ordinates represent observed angles of elevation of Mussooree from Nojli. The axes from which these are measured differ for the several seasons and years; but the height corresponding to $E_e - 1° 8' = 100''$ is indicated in each case by a short horizontal cross line. The points are indicated by small circles and are joined up by straight lines in the order 8, 10, 12 and 14 hours.

Attention is drawn to the approximate straightness of the lines and their tendency to be parallel to each other.

CHART III

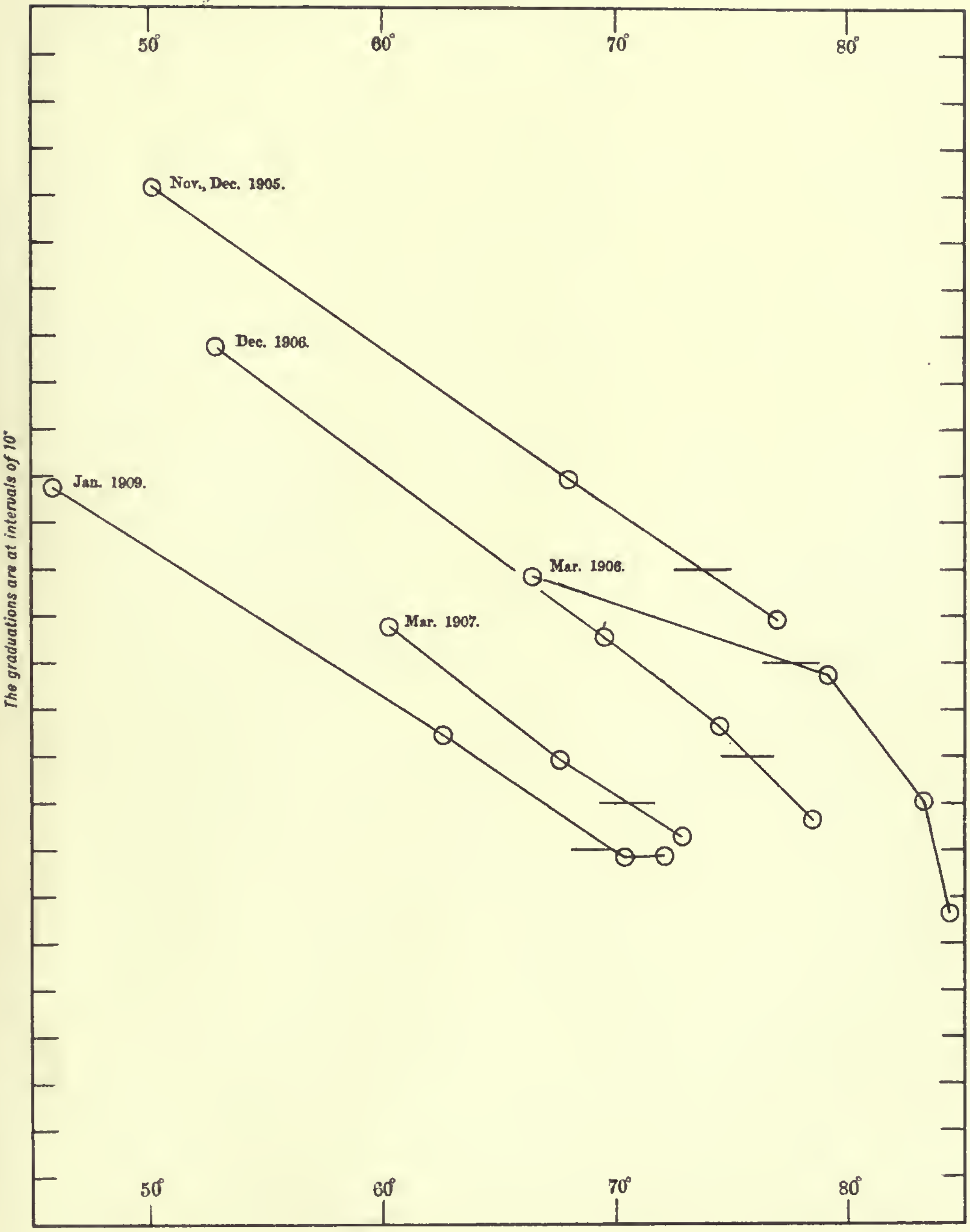

CHART III.

Duirnal Change of Refraction.

Graphical representation of table XLI. Ordinates represent observed angles of elevation of Bandarpanch from Nojli. The axes from which these are measured differ for the several seasons and years: but the height corresponding to $E_c - 1° 44' = 50''$ is indicated in each case by a short horizontal cross line. The points are indicated by small circles and are joined up by straight lines in the order 8, 10, 12 and 14 hours.

The lines are nearly straight except in the case of March 1906: and are approximately parallel. The slope of these lines, however, is less than that of the corresponding lines in Chart II.