AN ELEMENTARY TREATISE

ON

THE MECHANICS OF MACHINERY

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AN ELEMENTARY TREATISE

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MECHANICS OF MACHINERY

WITH SPECIAL REFERENCE TO THE

MECHANICS OF THE STEAM-ENGINE

BY

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PREFACE

THE following book is the outcome of a course of lectures on Kinematics and the Mechanics of the Steam-engine, which has been issued in the form of notes to students in the Department of Mechanical Engineering of the University of California for several years past. The first two parts embody the more important principles of what is generally called the Kinematics of Machinery, though in many instances dynamic problems which present themselves are dealt with; the real purpose of the book being the application of the principles of mechanics to certain problems connected with machinery. In the third part there is discussed the Mechanics of the Steam-engine, that machine being perhaps the most important from a designer's point of view. Here the subject is treated under two distinct heads, Kinematics and Dynamics.

No special originality is claimed for the major part of the material, and free use has been made of whatever literature could be found relating to the special subjects under consideration. To the more important of these, references are given by foot-notes.

JOSEPH N. LE CONTE.

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PART I

INTRODUCTORY

INTRODUCTORY

Kinematics. — Kinematics is that branch of Mechanics which deals with motion without reference to the cause producing it. It is important in dealing with machinery, where motion is entirely independent of the direction of the acting forces. In the following chapters we always assume rigid bodies, that is, bodies such as have their particles at an invariable distance from one another. Though there are no absolutely rigid bodies, all substances being more or less elastic, still, for the materials of machine construction, the effect of elasticity in affecting constrained motion is insignificant.

When a body has any motion whatever, its position at any given time is completely determined when the position of three of its points are known, provided these three points do not lie in one straight line. Hence no motion is possible if three such points are fixed. If one point is fixed, a knowledge of the position of two others, not in line with the first, is sufficient to determine the position of the body. Likewise if two are fixed, a knowledge of the position of a third only is required.

Kinds of Motion. — If a body has none of its points fixed, we study its motion generally by investigating the paths traced by three of its points. Such cases are not frequent in machinery, with the exception of helical screw motion, which can be reduced to a conbination of two simpler forms. If a body has one point P fixed, all its other points move on the surfaces of a system of concentric spheres with centre P, hence this motion is called Spheric Motion. In its general form it is not common in machine movements, but in the special case, where the fixed point is removed to infinity, it becomes the most important of all forms. In this case the system of concentric spheres becomes a system of parallel planes within any finite space. This form is called Uniplanar Motion, and all planes of the body at right angles to the line drawn toward the fixed point continue to move in their own planes. This fixing of one point at infinity vastly simplifies the motion, and nearly all machine motions are of this type. In treating of plane motion we have to consider the motion of but one of these planes, which may be called the Reference Plane, and the motion of all other points of the body can be studied by their projection on this plane. Hence the whole is reduced to a problem in Plane Geometry. Since one point is already fixed, we have only to consider the motion of two points, that is, of a line of the body. In uniplanar motion, then, we can treat all bodies as lines.

Kinds of Uniplanar Motion.— If we still further constrain the body by fixing a point in the reference plane or at any finite distance from it, we have the particular case of Pure Rotation. Here all points in the body describe a system of concentric circles about the line connecting the two fixed points, which line is called the Axis of Rotation. If the second fixed point is also removed to infinity in any direction not at right angles to the reference plane, we have the particular case of Rectilinear Translation. The concentric circles within any finite space become a system of straight lines parallel to the line of intersection of the two reference planes.

Combination of the Two Elementary Forms of Uniplanar Motion. —The combination of two or more rectilinear translations is an elementary principle of Mechanics, as is also the combination of two rotations. A displacement caused by a translation and a rotation taken in either order can be replaced by a single rotation. The construction for this is shown in Fig. 1, where changes of position only are considered. Let the rotation be such that if acting alone the body would move from AB to $A'B'$ about C as a centre. The translation in the same plane is such as would shift the body from $A'B'$ to $A''B''$. The result of the two is to shift the body from AB to $A''B''$. Connect AA'' and BB'' by straight lines, and at the middle points of these last erect perpendiculars intersecting at I, which then must be the point sought. Now if the body moves according to a given law, we may take a series of determinate positions, and find the point I for each successive pair of these. Finally, when these successive positions become indefinitely close together, the motion becomes continuous, and hence the centre I must move continuously also. Since, then, the position of the centre I at any time has but an instantaneous value, it is called the Instantaneous Centre.

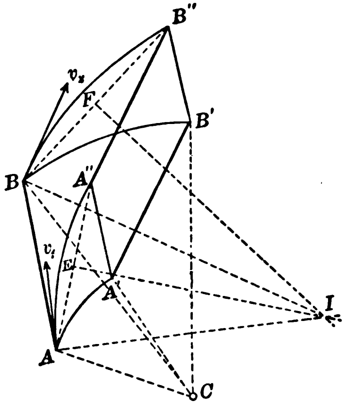


FIG. 1

The Instantaneous Centre. — When the positions AB and $A''B''$ are indefinitely close together, the arc AA'' and the chord AA'' coincide, and the line IE coincides with IA, and is perpendicular to v_1, the direction in which A is moving. The same is true with reference to IB, IF, and v_2. Hence if we know at any instant the direction of motion of two points of a body (moving with uniplanar motion), the instantaneous or virtual centre will lie at the intersection of the perpendiculars drawn to them.

Centrodes of Motion. — The instantaneous centre will itself, therefore, trace out a curve on the reference plane. The curve thus traced out with reference to axes fixed in space is known as the Space Centrode, or Fixed Centrode. If we refer the motion of the instantaneous centre to axes fixed in the moving body, we shall obtain another curve called the Body Centrode, or Moving Centrode. The body centrode can be constructed for any given position of the body by transferring every other position of the body to the given position, and carrying with each its instantaneous centre. Another curve is thus formed which must have a point in common with the first; namely, the instantaneous centre of the reference position. A simple pair of centrodes which correspond to a line moving with its extremities on a pair of rectangular coördinates is shown in Fig. 2. Let AB be one position of the body. A moves vertically downwards, and B horizontally to the right. The instantaneous centre, therefore, lies at the intersection of the perpendiculars to the directions of motion of these, or at I. Other positions of the body give p_1, p_2, etc. These points lie on a circle with centre O, and radius $OI = AB$, which is the space centrode. Choosing any reference position of the body such

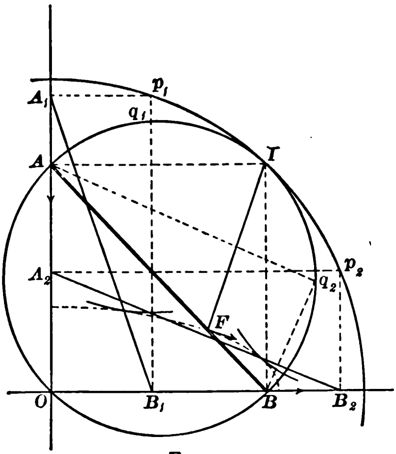


FIG. 2

as AB, we construct the body centrode by replacing on AB every other position of the body with its instantaneous centre "attached." For example, the position A_2B_2 with centre at p_2 gives, when replaced on AB, the point q_2, on the body centrode. The body centrode in this case is seen to be a circle of half the diameter of the space centrode, since it is the locus of the right angle of a right triangle of invariable hypothenuse. When a circle rolls within another of double its diameter, every point on the circumference of the small circle describes a diameter of the large one, which is a straight line hypocycloid. The pair of centrodes above is a pair of such circles. But A and B (points on the circumference of one circle) move on diameters of the other circle by the conditions of the problem; hence in this particular case the centrodes roll without slipping. Moreover, this is true for all cases.* Another more complex example of a pair of centrodes is shown in Fig. 3, and exhibits the case of the connecting

* For proof of the general theorem of rolling centrodes, see Appendix I.

rod of an ordinary steam engine. Here both are infinite curves with straight asymptotes. Any point such as F (Fig. 2) is moving at right angles to IF; hence if about I as a centre, with a radius equal to IF, we describe a circular arc, it will touch the path of F. But the same is true of a circle drawn about p_1 with

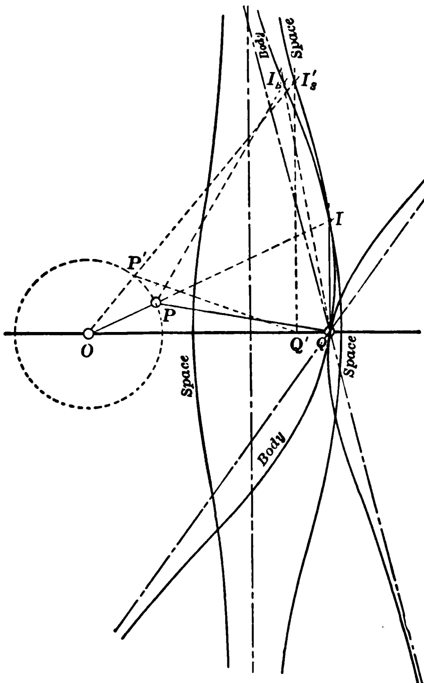


FIG. 3

a radius q_1F. Thus if we draw a series of arcs about p_1, p_2, p_3, etc., with radii equal to q_1F, q_2F, q_3F, etc., these will form the envelope of the path of F.*

Applications of the Instantaneous Centre in Machines. — In describing a machine we will use the definition given by Professor

* For important and interesting properties of rolling curves, see Appendix II.

Kennedy * as being the best, and it is as follows: "A machine is a combination of resistant bodies, whose relative motions are completely constrained, and which serve by these relative motions to transform the energies at our command into any special form of work." Considering the different parts of the definition, he points out that: —

1. A machine is a combination of bodies; that is, no one body can constitute a machine.

2. They must be resistant bodies, or bodies which have the property of resisting force in the direction in which it is applied. In the great majority of cases we deal with practically rigid bodies, or bodies which have their particles at an invariable distance from one another.

3. The motion must be constrained, or all points when referred to a given set of reference axes must follow paths determined by the construction of the machine itself, and not by the direction and magnitude of the acting forces.

4. Motion is an essential condition, as otherwise energy will not be transformed.

We are now to examine the methods used to constrain relative motion. We must have in the first place at least a pair of bodies or elements. The motion, if completely constrained, will depend on the form of the portions in contact. If these portions are so constructed that the contact between the bodies at all times during motion is a surface contact, the pair is known as a Lower Pair. But if it is such that the bodies touch upon a point or along a line, they are called Higher Pairs.

Elementary Machine Motions. — The two commonest and most important forms of motion to be found in machinery are the two elementary forms of uniplanar motion, viz., the rotation and the rectilinear translation. The first of these can be realized when the parts in contact are identical portions of any surfaces of revolution, and the second when they are identical prismatic surfaces. It will be noticed that the nature of both of these surfaces is such

* "Mechanics of Machinery," Alex. B. Kennedy. Macmillan & Co.

as to form a lower pair of elements, and while from a geometric standpoint the higher and lower pair are equally good for the constrainment of motion, practically the lower is the preferable as being less affected by the wearing away of the parts. The vast majority of machine motions are constrained in these ways, as by the journal and bearing, and the prismatic guide. The only other surface which will work as a lower pair is the cylindric screw surface, and this does not constrain a uniplanar motion.

Relative Motion. — In order that all points of a machine should be completely constrained, no part should move without all others undergoing a corresponding change in position. Take, for example, the combination of pairs or links shown in Fig. 4. Consider the body *a* fixed in space, so that the motions of

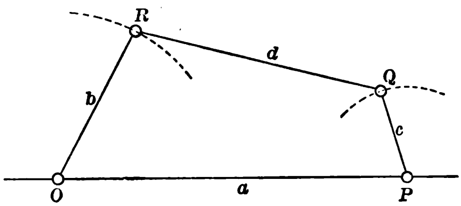


FIG. 4

the other three can be examined relatively to it. The links *b* and *c* are then completely constrained by being hinged at *O* and *P* to *a*, since all points of these links must follow definite, *i.e.* circular, paths. Concerning the link *d* we know that it has a point *R* common to *b*, which is therefore constrained relatively to *a*, and a point *Q*

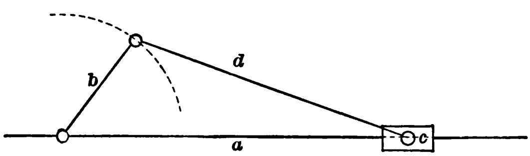


FIG. 5

common to *c* similarly constrained. Therefore the body *d* is completely constrained, as the paths of two points determine the uniplanar motion of a body. It will be noticed that the motions of *b* and *c* relatively to *a* are pure rotations, but that the motion of *d* is not. If any other link besides *a* were fixed, we would arrive at a similar result, hence the combination shown in Fig. 4 is a true machine. Fig. 5 is another very common example of a machine, being in fact the elementary combination used in

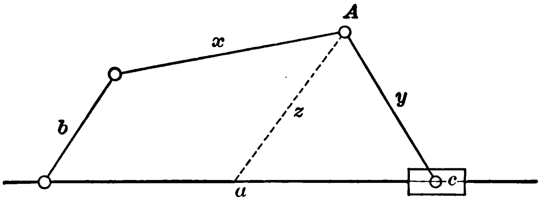


FIG. 6

the steam engine. Fig. 6 is, however, not a machine, as neither of the links x or y are constrained relatively to a. If the point A is connected to a by means of a sixth link z, x and y are then constrained.

Determination of the Instantaneous Centre in Machines. — In order to determine the instantaneous centre of a link referred to

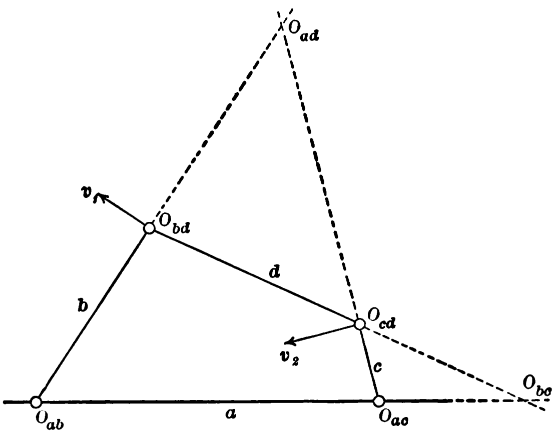


FIG. 7

any other link, we must know the direction of motion of two of its points at the position considered. In Fig. 7 we have the same combination of links as in Fig. 4. Here, as in all cases, we can write down at once the permanent centres as O_{ab}, O_{ac}, O_{bd}, and O_{cd}, the notation meaning that, for example, O_{ab} is the centre about which b is turning relatively to a (or a to b), etc. But d is not turning about a fixed centre relatively to a, and only an instantaneous centre exists for this motion. In order to determine O_{ad}, we know that O_{bd} (a point common to both b and d) is moving in a direction v_1 perpendicular to the line $O_{ab}O_{bd}$, and that the instantaneous centre O_{ad} lies somewhere on a line at right angles to this direction of motion, or somewhere on the line $O_{ab}O_{bd}$, produced if necessary. Similarly we know it must lie somewhere on $O_{ac}O_{cd}$. Therefore it lies at the intersection of these two as shown in the diagram. The same result would evidently be obtained if we were to study the motion of a relatively to d. By exactly the same process of reasoning we find, when either b or c are fixed, that the relative instantaneous centre of these two lies at the intersection of $O_{ab}O_{ac}$ and $O_{bd}O_{cd}$. We have in all six centres (permanent or instantaneous), which give the relative points of rotation of every link with respect to every other. If we select any three links in the above mechanism, such as a, b, and d, the three relative centres are O_{ab}, O_{ad}, and O_{bd}, and it will be noticed that these lie in a straight line, as do the relative centres of any other three links.

That this law is universally true may be shown as follows: let a, b, and c, be any three bodies having relative plane motion, whose three centres are at O_{ab}, O_{ac}, and O_{bc} (Fig. 8). Consider a fixed. Then b is turning about O_{ab}, and c about O_{ac}. But according to the definition of a centre, the point O_{bc} has no motion relatively to either b or c, hence it is a point common to both. Considered as a point of b, it is moving in a direction Y, at right angles to $O_{ab}O_{bc}$, and when considered a point of c it is moving in a

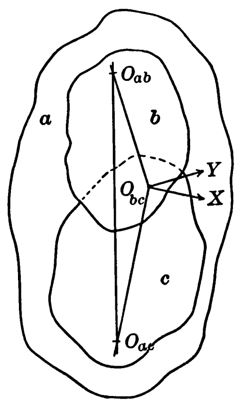


FIG. 8

direction X, at right angles to $O_{ac}O_{bc}$, both taken relatively to a. But, as a point can move in but one direction relatively to a given body, X and Y must coincide, or O_{bc} must lie on the line joining O_{ac} and O_{ab}. This proposition is of great help in finding the relative centres of more complicated combinations of links than Fig. 7, or in those where the relative motions are not easily seen by mere inspection. In the simple combination reproduced in Fig. 9 the centre O_{bc} can be found by its aid. Fig. 10

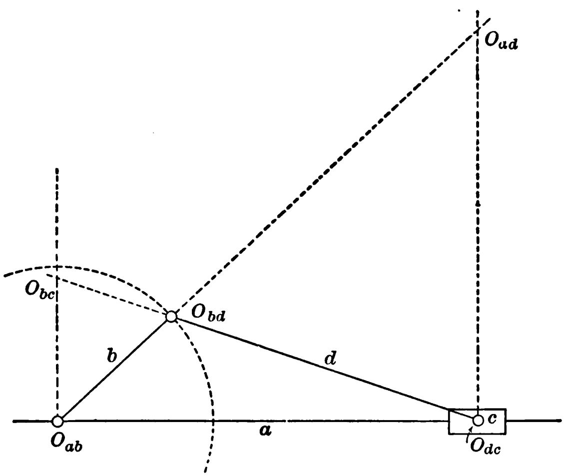


FIG. 9

shows the fifteen centres of a more complicated combination of six links. The above law can be traced throughout the construction.

Determination of Relative Linear Velocity.—A most useful property of the instantaneous centre in machine design is that which enables us to find the velocity of any point in the machine when that of any other is given. Let a, b, and c (Fig. 11) be any three bodies, of which a is fixed. The three relative centres will evidently lie in a straight line. Let v_1 be the known velocity of any point P in b. It is required to find the velocity v_2 of point Q in

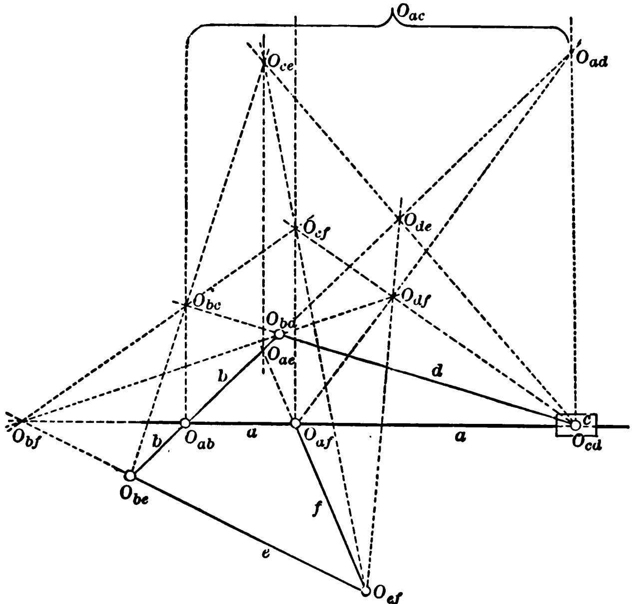


FIG. 10

the body c. The velocities of all points of a rotating body being proportional to their respective distances from the centre of rotation, we have:

$$\text{Vel. } O_{bc} = v_1 \frac{\overline{O_{ab}O_{bc}}}{\overline{O_{ab}P}},$$

the velocities being referred to a, and O_{bc} being a point of b. But also:

$$\text{Vel. } O_{bc} = v_2 \frac{\overline{O_{ac}O_{bc}}}{\overline{O_{ac}Q}},$$

where O_{bc} is a point of c. As O_{bc} is a point common to both b and c, its

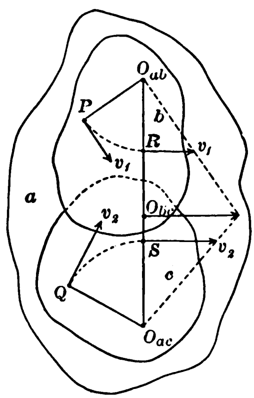


FIG. 11

velocity, when considered a point of either and referred to *a*, must be the same. Hence:

$$v_2 = v_1 \frac{\overline{O_{ab}O_{bc}}}{\overline{O_{ab}P}} \times \frac{\overline{O_{ac}Q}}{\overline{O_{ac}O_{bc}}}.$$

A simple graphical solution of the above equation is shown in Fig. 11. This consists of finding by circular projection points *R* and *S* on the line $O_{ab}O_{ac}$ which have the same velocities as *P* and *Q*,

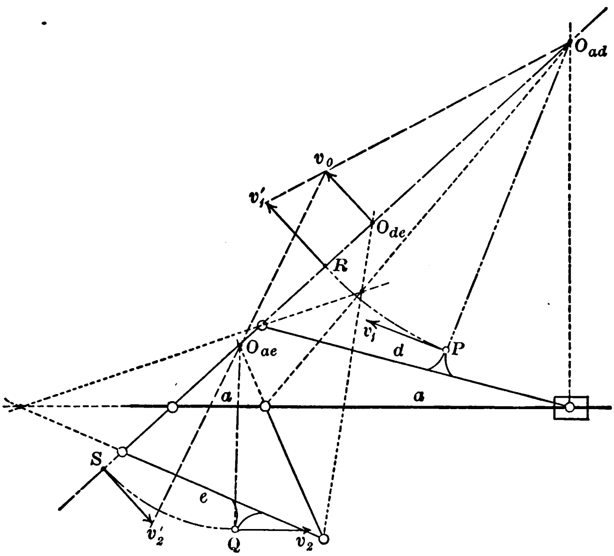


FIG. 12

and solving the equation by similar triangles. Suppose, in the mechanism of Fig. 10, we have given the velocity of a point *P* (Fig. 12) in link *d*, to find velocity of point *Q* in link *e*, both referred to *a*. We first pick out from Fig. 10 the three relative centres O_{ad}, O_{ae}, and O_{de}, all of which are instantaneous centres. *P* is turning about O_{ad} relatively to *a*, hence the velocity of *P* will be v_1 of magnitude given, and in the direction at right angles to $\overline{PO_{ad}}$. We then project *P* on a circular arc to *R*, on the straight line joining the three centres. From R we lay off the velocity of P in any direction, though it is preferable to lay it off at right angles to the line of centres, in order to preserve the direction as well as the magnitude of the velocity of R also. Connect the extremity of v_1' with O_{ad}, and draw v_0 through O_{de} parallel to v_1' to intersect this last line. v_0 is the velocity of O_{de}, or the common point of the two moving bodies. Q is turning about O_{ae}, hence

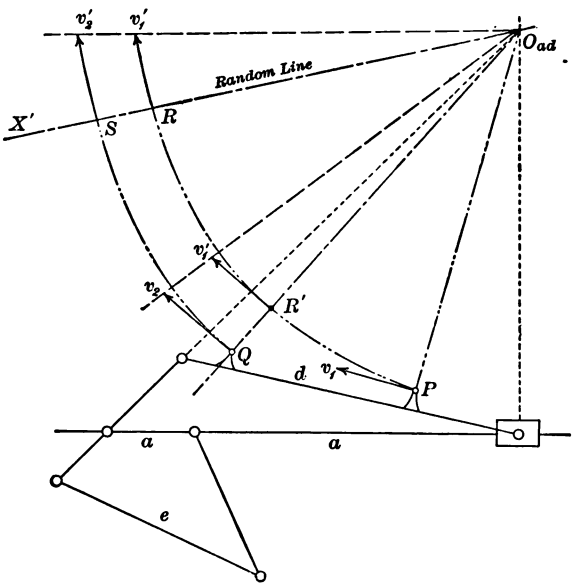


FIG. 13

project Q on a circular arc to S on the line of centres. Through the extremity of v_0 and O_{ae} draw a line and produce in the direction of S. Through S draw v_2' parallel to v_1' to intersect this last line. Then v_2' is the velocity of S in magnitude and direction, and the required velocity of Q, viz., v_2 is drawn at right angles to QO_{ae}, and equal in magnitude to v_2'. The above construction applies in all cases except where all three centres are at infinity. It is equally applicable whether the points under consideration lie in the same or in different links, and whether any or all of the links are turning about permanent or instantaneous centres. When the points lie in the same link, the two centres O_{ae} and O_{ad} coalesce, and O_{de}, the point common to both, may be any point of the moving body. In this case the line connecting the three virtual centres and upon which we project the points P and Q is not fixed in direction, and hence may be taken in any direction at random. It is preferable to take it passing through one of the points under consideration,

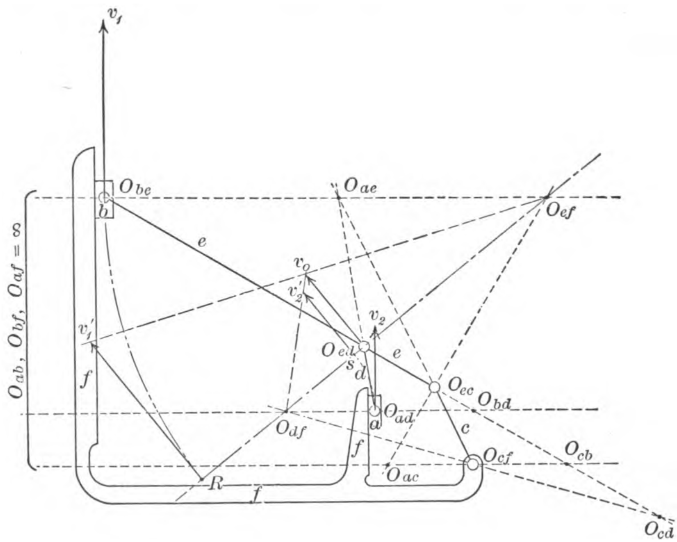


FIG. 14

as by doing so the construction is simplified. Fig. 13 shows the simple construction in this case, both where the projection line is drawn at random and where it is drawn through Q.

In the case where the three relative centres lie at infinity, the motion of the two links which carry the points must be motions of translation relatively to the fixed link, and relatively to each other. But in this case all points in the moving bodies have the same velocities, and therefore the points of the links hinged to these have the same velocities also. So the problem reduces to the comparison of the velocities of these points of attachment. Take the case exhibited in Fig. 14, which is essentially the reducing motion of the Tabor Indicator. We have two blocks, *a* and *b*, moving parallel to one another, with rectilinear translations, but with different velocities, which we wish to compare. This is evidently the same thing as comparing the velocity of the point O_{ad} of the link *d* with that of O_{be} of *e*. All the centres, as well as the construction in the particular case, are shown.

Determination of Relative Angular Velocity. — Problems in Angular Velocity can be solved in a similar manner by the use of the instantaneous centre. Referring to Fig. 15, let the body *b* turn with an angular velocity ω_1 about O_{ab}, required ω_2, that of *c* about O_{ac}. O_{bc} being a point common to both *b* and *c*, we have already seen that its velocity referred to *a*, is the same when considered a point of either. Calling its velocity v_0, we have

$$v_0 = \omega_1 \overline{O_{bc}O_{ab}} = \omega_2 \overline{O_{bc}O_{ac}},$$

or

$$\omega_2 = \omega_1 \frac{O_{bc}O_{ab}}{O_{bc}O_{ac}}.$$

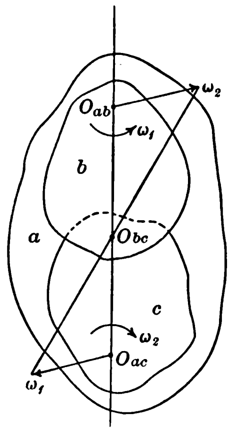


FIG. 15

This is easily solved graphically by laying off in any direction ω_1, the given angular velocity of *b* from the centre of *c* referred to *a*, then drawing a line from its extremity through O_{bc}, and producing if necessary to meet a line from the centre of *b* parallel to ω_1. This last line will give the magnitude of ω_2, and evidently, if O_{bc} lies between the other two, the angular velocities will be in opposite directions, and if not they will be in the same direction. The above proposition stated in words is as follows: the relative instantaneous centre of two moving bodies divides the distance between their centres in the inverse ratio of their angular velocities referred to a fixed body. Applying this in Fig. 16 to the combination of Fig. 10, we have the angular velocity of *d* about its instantaneous centre given, and are required to find that of *e*. The given angular velocity ω_1 is laid off from O_{ae}, and ω_2 is drawn

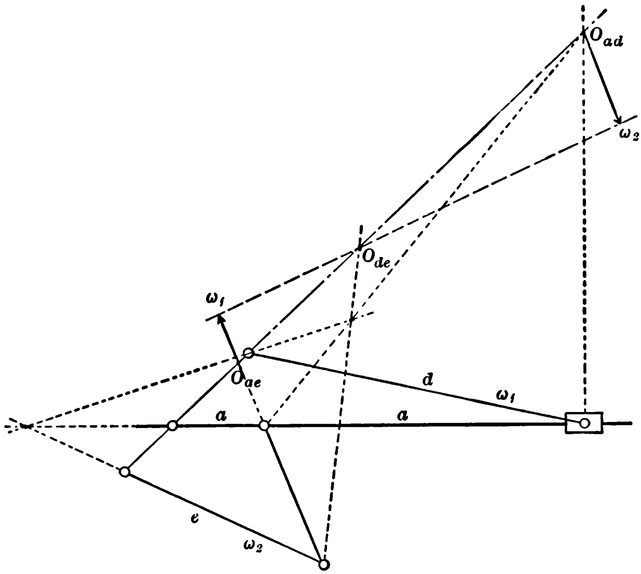


FIG. 16

parallel to it and from O_{ad} to meet the line from the tip of ω_1 through O_{de}. Generally the only angular velocities which are of importance in machine design are those taking place about permanent centres.

PART II

MACHINERY OF TRANSMISSION

CHAPTER I

TRANSMISSION OF PURE ROTATION THROUGH A RIGID INTERMEDIATE MEMBER

I. AXES OF ROTATION IN ONE AND THE SAME STRAIGHT LINE

A. Direct Connection and Rigid Couplings

In this case merely an axle or shaft is used to connect the two rotating parts, and the kinematics of the arrangement is of the utmost simplicity. Regarding the dimensions of the parts, however, a few words may be necessary.

Journals and Bearings. — A Journal in machinery is a rigid body bounded by a surface of revolution, which is usually cylindric or conic. In order that this piece may partake of no motion other than rotation, a second piece must be associated with it; namely, the Bearing. This must be bounded by the same surface of revolution with concavities and convexities reversed. The journal and bearing, therefore, form a pair of elements designated as a turning pair, and the character of the contact is that of a lower pair. Owing to the frequency with which such journals occur in machinery, they require careful designing. This should first be done according to the principles of the Strength of Materials. If we consider the journal as an overhanging beam of length l uniformly loaded, the equation of moments will be

$$M = \frac{Pl}{2} = \frac{KI}{e},$$

where K is a constant of the material, I the moment of inertia of a circle about its diameter, and e the distance of the outermost fibre from the neutral layer.

Since $I=\frac{\pi d^4}{64}$, and $e=\frac{d}{2}$,

$$K\frac{I}{e}=K\frac{\pi d^3}{32},$$

and

$$\frac{Pl}{2}=K\frac{\pi d^3}{32}.$$

This involves two variables, l and d; but as we can generally assume their ratio, we may write:

$$c=\frac{l}{d}$$

whence

$$d=2.26\sqrt{\frac{Pc}{K}},$$

d being the diameter in inches. (For cast iron, $K=4200$; wrought iron, $K=8500$; steel, $K=12000$.)

A journal designed in this way for strength may be, and generally will be, found much too small to stand the wear of continuous use and radiate the heat produced by friction at a sufficiently rapid rate. In this case it is usual to allow a certain pressure per square inch of projected area, this latter being the product of length and diameter. Let this allowable pressure be p. The projected area is $l \times d$, and $l \times d \times p = P$. Now as $l=c \times d$,

$$d=\sqrt{\frac{P}{cp}}.$$

The value of p varies greatly, depending on the total load and the nature and velocity of the rubbing surfaces. Its limits are generally between 100 and 500 pounds per square inch.*

When considerable space intervenes between journals, the connecting part may be called an Axle. If loaded it also must be designed according to Strength of Materials.

Shafting. — When of great length and used for transmitting power, the shaft and its journals are to be designed to resist torsion also. This is given by the well-known formula:

---

* For accurate design of journals, see some work on Machine Design.

$$d=\sqrt[3]{\frac{C' \times H.P.}{N}},$$

where C' is a constant of the material, N the number of revolutions per minute, and $H.P.$ the horse-power transmitted.

The following are rough values of C':

Wrought-iron shafting	$C' = 100$
Steel shafting	$C' = 75$

Also for shafting up to 4″ in diameter,

$$L = 4.8\sqrt[3]{d^2},$$

where L is the distance between hangers in feet, and d the diameter of the shaft in inches.

Pivots are end journals which sustain pressure in the direction of their axes. Their diameters need be calculated to withstand wear and heating only. They are usually flat on the thrust surface, though for light, high-speed machinery they are sometimes conical. For heavy thrusts, collars are turned on the shaft.

A line of shafting is often of very great length, and is made of many pieces coupled together. This gives rise to many forms of rigid couplings, but this is purely a problem of Machine Design.

2. AXES OF ROTATION PARALLEL

A. Parallel Bars, or Cranks

In the combination of links shown in Fig. 4 a transmission of rotation is effected between parallel shafts at O and P. In order that both shafts should revolve through a complete circle, it is necessary that the two links b and c must be of the same length, as must a and d also. It will be noticed in this case, however, that the motion is not constrained at the instant the four permanent centres come in line. To overcome this defect a second pair of cranks must be placed upon the same shaft, making some angle other than zero or 180° with the first pair. This second pair is usually at 90° with the first pair, and the combination is well illustrated in the side rods of locomotives.

B. Oldham's Coupling

In this form, two disks A and B (Fig. 17) are keyed to the ends of the shafts. These disks are connected by means of an inter-

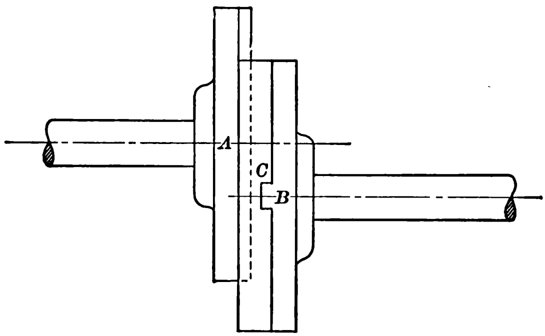


Fig. 17

mediate disk C, upon each of whose faces a prismatic groove is cut, the two crossing one another at an angle β. These grooves fit two prismatic ridges, one on each of the disks A and B, and thus driving is effected. If the shaft A turns at any angular velocity, B will follow at the same velocity. In Fig. 18, A and B are the centres of the shafts, and the lines DD and EE intersecting at C are the ridges of the disks. As the disks turn, these lines will always pass through A and B, and on account of the intermediate disk C they will always intersect at the same angle β. Hence the locus of C is a circle on AB as a chord. Let $D'D'$ and $E'E'$ be any other positions of the ridges. Then angle $DBD' =$ angle EAE' or A will turn through the same

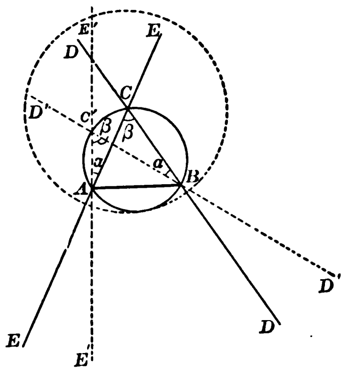


Fig. 18

angle as *B*. It will be noticed that the motion of the intermediate disk *C* can be reproduced by the rolling upon circle *ABC* of another circle of double the diameter. (Compare Fig. 2.) Driving is best effected when the angles at *C* are right angles. Then *AB* becomes a diameter.

C. Certain Forms of the Universal Joint

This will be described under intersecting shafts in the next section.

3. AXES OF ROTATION INTERSECTING

A. The Universal Joint

Hook's Universal Joint is used between intersecting shafts when it is necessary to vary the angle between the shafts. The single joint can be used only between intersecting shafts, but the double joint can be used between parallel shafts also, and this latter combination forms the most flexible of all couplings.

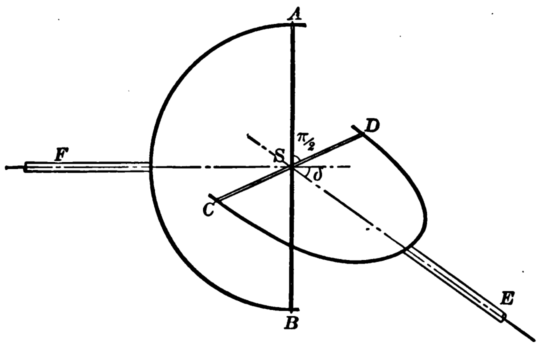


FIG. 19

Its construction is as follows: the shafts are provided with forked ends *AB* and *CD* (Fig. 19), whose four bearings at *A*, *B*, *C*, and *D*, fit the four journals of an intermediate body. This latter is generally in the form of an equal armed cross, a circular disk, or a sphere, where the common axes of the opposite pairs *AB* and *CD* intersect in a point *S* at right angles. The forks are so proportioned that S lies at the intersection of the shaft axes also. When the shaft F turns, the centre line of its bearings A and B moves in a plane at right angles to the axis of F, and will trace out a great circle $AGBH$ (Fig. 20) on the surface of a sphere whose centre is S. At the same time the driven shaft E must turn, and the axis of the bearings of its fork must remain in a plane at right angles to the axis of E, describing a great circle $ADBC$ on the sphere. When the forks of F are at A and B, those of E are at D and C. When the forks of F are at G and H, those of E are at A and B; or when the driving fork has turned through a quadrant from either of these positions, the driven fork turns through a quadrant also. Consider the forks of F to have turned through an angle α, so that A arrives at O. Then E will turn through an angle β, and D will move to P, where:

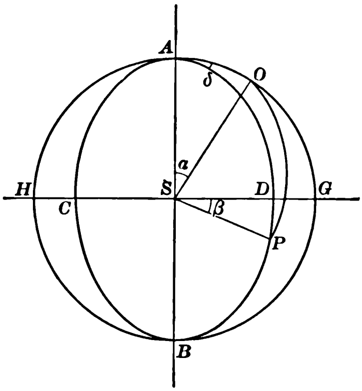


FIG. 20

$$\cos OSP = (\cos ASO)(\cos ASP) + (\sin ASO)(\sin ASP)(\cos OAP),$$

$$\cos 90^\circ = \cos\alpha \cos(90^\circ + \beta) + \sin\alpha \sin(90^\circ + \beta) \cos\delta,$$

where δ is the angle between the planes of the great circles $AGBH$ and $ADBC$, which is also the constant inclination between the intersecting shafts. Hence:

$$0 = -\cos\alpha \sin\beta + \sin\alpha \cos\beta \cos\delta,$$

$$\cos\delta = \frac{\cos\alpha \sin\beta}{\sin\alpha \cos\beta} = \frac{\tan\beta}{\tan\alpha}.$$

The angles α and β are therefore not in general equal, nor is their ratio constant during any time, but the ratio of their tangents is a constant and is equal to the cosine of the angle between the shafts. Since $\cos\delta$ is always less than unity and is positive in any practical construction, $\tan\beta$ is always less than $\tan\alpha$. Thus in the first and third quadrants $\beta < \alpha$, while in the second and fourth, $\beta > \alpha$. If F is driving at a uniform rate, E will follow at a greater and less speed alternately. The relation between the angular velocities for any given value of α is determined as follows. Let ω_1 be the angular velocity of F, and ω_2 that of E. Then:

$$\frac{d\alpha}{dt} = \omega_1, \text{ and } \frac{d\beta}{dt} = \omega_2,$$

$$\omega_2 = \frac{d\beta}{dt} = \frac{\sec^2\alpha \cos\delta}{1 + \tan^2\alpha \cos^2\delta} \frac{d\alpha}{dt},$$

$$= \frac{\omega_1 \cos\delta}{\cos^2\alpha(1 + \tan^2\alpha \cos^2\delta)},$$

$$= \frac{\omega_1 \cos\delta}{1 - \sin^2\alpha \sin^2\delta}.$$

If the driver is turning with a constant angular velocity we can determine those values of α at which the angular velocity of the follower is a maximum or minimum by putting $\frac{d\omega_2}{dt}$ equal to zero.

$$\frac{d\omega_2}{dt} = \frac{2\,\omega_1 \cos\delta \sin^2\delta(\sin\alpha \cos\alpha)}{(1 - \sin^2\alpha \sin^2\delta)^2} = 0,$$

$$\sin\alpha \cos\alpha = 0.$$

This condition is fulfilled when α is either 0, $\frac{\pi}{2}$, π, or $\frac{3\pi}{2}$. In order to discriminate between the maxima and minima, substitute these angles in $\frac{d^2\omega_2}{dt^2}$, and the algebraic sign of the result shows that:

$$\omega_2\Big]_{\text{min.}} \text{ at } \alpha = 0 \text{ and } \alpha = \pi,$$

$$\omega_2\Big]_{\text{max.}} \text{ at } \alpha = \frac{\pi}{2} \text{ and } \alpha = \frac{3\pi}{2}.$$

Substituting these values in the original expression for the angular velocity ω_2 we find:

$$\omega_2\Big]_{\text{max.}} = \frac{\omega_1}{\cos\delta},$$

$$\omega_2\Big]_{\text{min.}} = \omega_1 \cos\delta.$$

Thus as the driving fork is turned at a constant angular velocity ω_1, that of the follower will vary between $\omega_1 \dfrac{1}{\cos\delta}$ and $\omega_1 \cos\delta$. If written in terms of the angular velocity ratio, it is evident that

$$\left.\frac{\omega_1}{\omega_2}\right]_{max.} \text{ at } \alpha = 0 \text{ and } \alpha = \pi,$$

$$\left.\frac{\omega_1}{\omega_2}\right]_{min.} \text{ at } \alpha = \frac{\pi}{2} \text{ and } \alpha = \frac{3\pi}{2}.$$

Likewise:

$$\left.\frac{\omega_1}{\omega_2}\right]_{max.} = \frac{1}{\cos\delta},$$

$$\left.\frac{\omega_1}{\omega_2}\right]_{min.} = \cos\delta.$$

When δ is 90°, the angular velocity of the follower will vary between zero and infinity, or the joint will fail to work. If δ is zero, the angular velocity ratio is constantly equal to unity. For practical purposes δ should not be larger than 30°.

If ω_1 is constant, the angular acceleration of the follower can be found for any given value of α as follows:

$$\omega_2 = \frac{\omega_1 \cos\delta}{1 - \sin^2\alpha \sin^2\delta}.$$

And since ω_1 is constant

$$\frac{d\omega_2}{dt} = \frac{2\,\omega_1 \cos\delta \sin^2\delta \sin\alpha \cos\alpha}{(1 - \sin^2\alpha \sin^2\delta)^2}\frac{d\alpha}{dt},$$

$$= \frac{2\,\omega_1^2 \cos\delta \sin^2\delta(\sin\alpha \cos\alpha)}{(1 - \sin^2\alpha \sin^2\delta)^2},$$

which becomes zero at $\alpha = 0$, $\frac{\pi}{2}$, π, and $\frac{3\pi}{2}$. Since the quantity $2\,\omega_1^2 \cos\delta \sin\delta$ is a constant, we can find the value of α corresponding to the maximum or minimum value of $\frac{d\omega_2}{dt}$ by simply considering the variation of $\dfrac{\sin\alpha\cos\alpha}{(1 - \sin^2\alpha \sin^2\delta)^2}$. Put this equal to p.

Then:

$$\frac{dp}{d\alpha} = \frac{(\cos^2\alpha - \sin^2\alpha)(1 - \sin^2\alpha\sin^2\delta)^2 + 4\sin\alpha\cos\alpha\sin^2\delta\,\frac{(\sin\alpha\cos\alpha)}{(1-\sin^2\alpha\sin^2\delta)^2}}{(1-\sin^2\alpha\sin^2\delta)^4}$$

Putting this expression equal to zero, it reduces to:

$$(\cos^2\alpha - \sin^2\alpha)(1 - \sin^2\alpha\sin^2\delta) + 4\sin^2\alpha\cos^2\alpha\sin^2\delta = 0$$

$$\sin^4\alpha + \sin^2\alpha\left(\frac{2+3\sin^2\delta}{2\sin^2\delta}\right) = \frac{1}{2\sin^2\delta},$$

$$\sin\alpha = \pm\sqrt{\frac{(3\sin^2\delta - 2) \pm \sqrt{9\sin^4\delta - 4\sin^2\delta + 4}}{4\sin^2\delta}}$$

which gives two real values of $\sin\alpha$, or four values of α, corresponding to two maxima and two minima. These values of α will be symmetrically grouped about the position $\alpha = 0$ (Fig. 21). If we know the total moment of inertia of all masses rotating with the follower, we can calculate the maximum twisting moment in the follower due to its angular acceleration. As an extreme case take $\delta = 45°$, and take the number of revolutions per minute at 300. Then the values of α at which the angular acceleration will be a maximum will be:

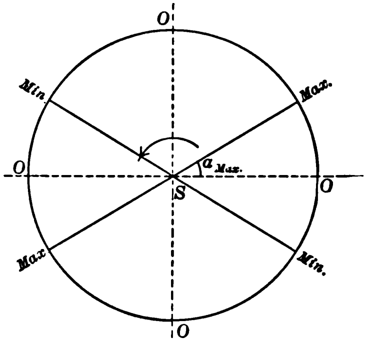


FIG. 21

$$\alpha = \sin^{-1}\pm\sqrt{\frac{\frac{3}{2} - 2 \pm \sqrt{\frac{9}{4} - 2 + 4}}{2}} = \sin^{-1}\pm\sqrt{.7808}$$

$\alpha]_{max.} = 62°\ 5'$ and $242°\ 5'$,

$\alpha]_{min.} = 117°\ 55'$ and $297°\ 55'$.

Now $\omega_1 = \frac{2\pi N_1}{60} = 31.416$ radians per second. Therefore substituting these values in the expression for angular acceleration, we find

$$\frac{d\omega_2}{dt} = \pm\ 776.94 \text{ radians per second, per second.}$$

Let a cylinder of cast iron 1 ft. in radius, 1 ft. in axial length, and of density equal to 450 lbs. per cubic foot, be keyed to the follower. If we neglect all other masses, and call the force moment Ph, then:

$$Ph = I\frac{d\omega_2}{dt},$$

$$I = M\frac{r^2}{2} = \frac{450 \times \pi}{64.4} = 21.95,$$

$$Ph = 21.95 \times 776.94 = \pm\ 17{,}054,$$

or 17,054 lbs. acting on a lever arm of 1 ft.

In order to transmit a uniform angular velocity ratio by means of a universal joint, two joints should be used (Fig. 22). Here the

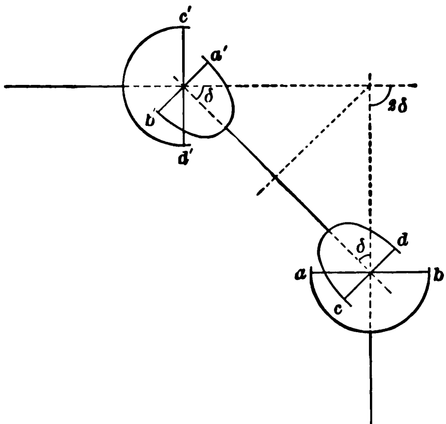


FIG. 22

angle δ is the same for both, and the axes of the bearings in the intermediate shafts' forks are parallel. Then fluctuations in angular velocity produced by the first are neutralized by the second. Transmission between shafts which are parallel but not in the same straight line may be effected by this means also. If the link which connects the two joints is made short enough, the two axes or bars $a'b'$ and cd coalesce, and we have the double universal joint. In this case, however, we must provide some means for guiding this bar in the plane bisecting the angle between the shafts, and there is no simple means of accomplishing this.

4. AXES OF ROTATION CROSSING

There appears to be no practical means of transmitting rotation between crossing shafts by means of a coupling.

CHAPTER II

TRANSMISSION OF PURE ROTATION BY MEANS OF FRICTION GEARING

1. AXES OF ROTATION IN ONE AND THE SAME STRAIGHT LINE

A. Direct Connection, Friction Couplings

The problem of Friction Couplings is purely one of Machine Design, and therefore has no place in this discussion.

2. AXES OF ROTATION PARALLEL

A. Friction Wheels

If two bodies have pure rotary motion relatively to a third, their centrodes, while rolling upon one another, may be used to transmit motion in certain cases. Suppose we have two bodies, *b* and *c* (Fig. 23), rotating about fixed centres relatively to *a*. Let the angular velocity of *b* be ω_1, and that of *c* be ω_2. Concerning the point O_{bc}, or the relative centre of *b* and *c*, we know that it must lie on the line of centres. Further, from Fig. 15 we know that its position on $O_{ab}O_{ac}$ is fixed by the angular velocity ratio, and that:

$$\frac{O_{bc}O_{ac}}{O_{bc}O_{ab}} = \frac{\omega_1}{\omega_2},$$

and since the distance $\overline{O_{bc}O_{ac}} + \overline{O_{bc}O_{ab}}$ is constant, the position O_{bc} divides a fixed length in the inverse ratio of the angular velocities. If the angular velocity ratio varies, O_{bc} will move along the line of centres, tracing out a curve with respect to the moving plane of *c* and another with respect to the plane of *b*. Such curves are known as "pitch curves," and will be again considered in the case of toothed gearing. They are the centrodes of the relative motion, and if cut out of metal and rolled one upon the

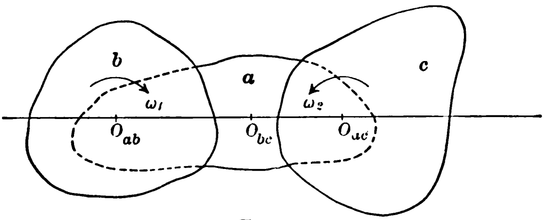


FIG. 23

other, the original motion would be reproduced. Also power could be transmitted provided (1), there is a component of the driving force normal to the curves at the point of contact, and in the direction of the follower, or (2) if this component be zero, that the coefficient of friction be great enough to prevent slipping.

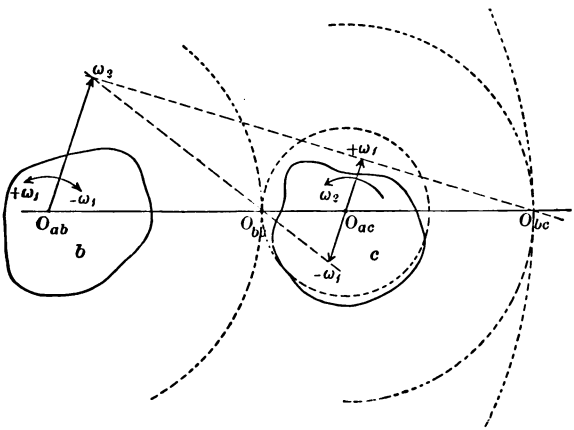


FIG. 24

If the curves both make complete and practical revolutions, *i.e.* are closed curves, the first condition is impossible of fulfilment, since it necessitates the contact radius of the driver being continually on the increase. Hence the only case in which the centrodes themselves can be used is the second, and it is applied as follows:

Let the angular velocity ratio $\frac{\omega_1}{\omega_2}$ be constant. Then point O_{bc} is fixed on the line of centres, and traces out two circles in the planes of c and b, the ratio of whose radii is the inverse ratio of the angular velocities. It will be noticed that if the angular velocities are in opposite directions, the pitch circles will touch one another on their convex or outer sides; but if ω_1 and ω_2 are in the same direction, the convex side of the circle having the larger angular velocity will touch the concave or inner side of the other. (See Fig. 24.) In this case there is no component of driving force in the direction of the normal, and the second condition is satisfied.

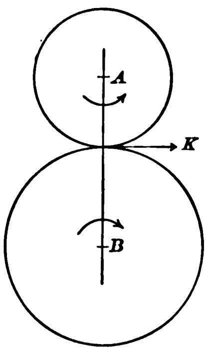


FIG. 25

The difficulty with all such Friction Gears is that there is no real tendency of the wheels to drive. If a force K (Fig. 25) is to be exerted at the circumference and the coefficient of friction is ϕ, then the two axles must be pressed together with a force P at least equal to

$$P = \frac{K}{\phi}.$$

Since P is directly in line with the axles, ϕ should be made as large as possible in order to diminish P, which causes hurtful friction in the bearings. Leather or paper may have ϕ as large as .4. Another way to diminish friction in the bearings is to have only a small component of the force P in their direction. To accomplish this grooved gearing is used. Here a series of circular ridges on one cylinder is made to fit a series of grooves on the other. The angles of the ridges and grooves are made about 30° (Fig. 26). Since a point C at the base of a ridge is moving slower than a point D at its tip, there will be rubbing between the surfaces at both C and D, causing a waste of energy. Hence in most cases the surfaces of the ridges and grooves are made curved instead of straight, as shown, and therefore touch in but a single point. The great disadvantages of all friction gearing are, the loss of energy at the bearings, due to friction there, and the fact that the centre distance of the wheels must be made adjustable in order that wear may be taken up. It is impossible to make $\frac{\omega_1}{\omega_2}$ absolutely constant, owing to the wear and slip, and hence it is used in driving light-running machinery only, except where grooved gearing is employed.

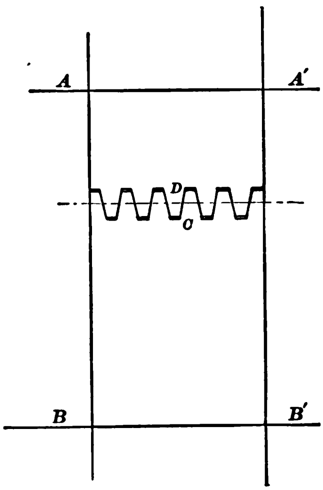


FIG. 26

3. AXES OF ROTATION INTERSECTING, AND
4. AXES OF ROTATION CROSSING.

Friction gearing is seldom used in these cases, though any of the pitch surfaces fulfilling these requirements (see chapter on Toothed Gearing) might be used in a manner similar to the preceding.

CHAPTER III

TRANSMISSION OF PURE ROTATION BY MEANS OF BELTS AND ROPES

I. AXES OF ROTATION PARALLEL, PLANES OF PULLEYS THE SAME

A. Ordinary Belting

In these cases motion is transmitted from a rotating body to an intermediate one by pure rolling, and from that similarly to a third. The case is a general one, though used extensively in but a single form. Let any two bodies *b* and *c* have purely rotary motion relatively to a third body *a*. A fourth body *d* may be made to so move that its centrodes, with respect to both *b* and *c*,

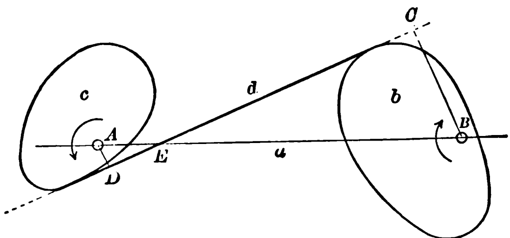


FIG. 27

shall be identical. This may be accomplished by assuming any form for the centrodes of *b*, *c*, and *d*, and forcing the three to roll together. The only important case is that in which the centrode fixed in *d* is straight, or practically so (Fig. 27). The centrodes of *b* and *c* can be cut out of rigid disks, and the straight edge of *d* will roll upon them, transmitting motion by friction, provided no points of inflection exist in the outlines of either *b* or *c*, and that these latter do not interfere. The tendency of the centrodes to slip may be overcome by forming *d* of a flexible band, which wraps around and is fastened to the outlines of the other curves. This is called Transmission through a Wrapping Connector, and power can be transmitted in but one direction. The angular velocity ratio will evidently be equal to the inverse ratio of the perpendiculars let fall from the centres of rotation upon the straight edge of the connector, or will be equal to the inverse ratio of the segments into which the line of the connector cuts the line of centres (produced if necessary). If the motion is to be continuous, *i.e.* if both *b* and *c* are to make complete revolutions, the connector cannot be fastened to their outlines, but must continuously surround them with a belt of constant length. The centrodes of *b* and *c* will then be circular, $\frac{\omega_1}{\omega_2}$ will be constant, and power can be transmitted in either direction.

(*a*) *Stresses in the Belt, and Power Transmitted*

Belts possess the great advantage of lying around a large portion of the circumference of the pulleys, and hence, for a given pull P between the shafts, create a greater frictional resistance than merely $P\phi$. They also possess the advantage of being elastic, and take up by this means any slight variation in the centre distance of the shafts, due to wear. The method of finding the total frictional resistance of a belt at the instant of slipping, in terms of the tensions on the tight and slack side, is easily shown by the principles of Mechanics to be

$$F = T_1 - T_2$$

and

$$T_1 = T_2 e^{\phi\alpha},*$$

where F is the total frictional resistance, T_1 the tension on the tight side, T_2 the tension on the slack side, α the angle of the pulley covered by the belt, and ϕ the coefficient of friction

* Values of $e^{\phi\alpha}$ for values of ϕ between 0 and .8 and of α between 0 and 360° can be taken from the accompanying Diagram No. 1.

between the belt and pulley. If the force to be transmitted is K, then K must be something less than F, or

$$K = CF = C(T_1 - T_2) = C\left(T_1 - \frac{T_1}{e^{\phi a}}\right) = CT_1\left(\frac{e^{\phi a} - 1}{e^{\phi a}}\right),$$

where C is a factor of safety. In most cases of belt transmission $a = \pi$. Now the horse-power transmitted will be

$$H.P. = \frac{v \times K}{33000} = \frac{CvT_1\left(\frac{e^{\phi a} - 1}{e^{\phi a}}\right)}{33000},$$

where v is the velocity of the belt in feet per minute. The only variables in the above equation are the horse-power and the velocity, hence one varies directly as the other. T_1 is determined as the maximum working tension that the belt will stand. If p is the maximum allowable working tension per inch of width of single belting, and w is the width of the belt in inches,

$$w = \frac{T_1}{p} = \frac{33000 \times H.P.}{v(Cp)\frac{e^{\phi a} - 1}{e^{\phi a}}}.$$

Fair average values of the constants involved will be about as follows: $\phi = .25$, $a = \pi$, $(Cp) = 55$. This last constant includes the safety factors of both strength and slip.

Hence, $$w = \frac{33000 \times H.P.}{v \times 55 \times .545} = \frac{1062 \times H.P.}{v},$$

or as it is usually written

$$w = \frac{1000 \times H.P.}{v}.$$

If D is the diameter of the pulley in inches, and N the number of revolutions per minute,

$$v = \pi N \frac{D}{12},$$

and $$w = \frac{3800 \times H.P.}{D \times N}.$$

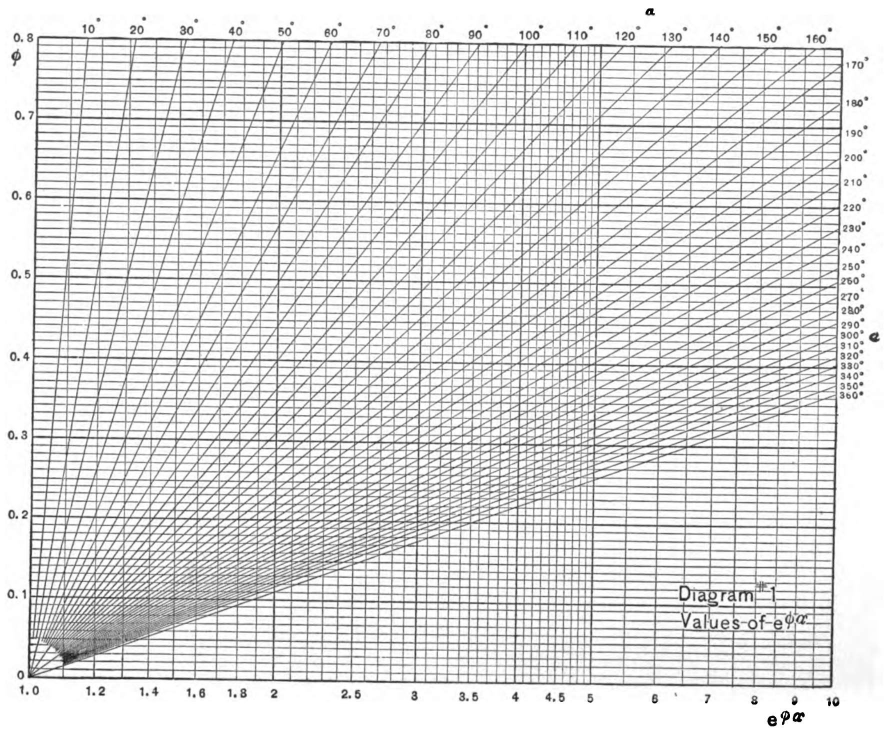


Diagram #1
Values of $e^{\phi\alpha}$
ϕ
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
α
10° 20° 30° 40° 50° 60° 70° 80° 90° 100° 110° 120° 130° 140° 150° 160°
170° 180° 190° 200° 210° 220° 230° 240° 250° 260° 270° 280° 290° 300° 310° 320° 330° 340° 350° 360°
1.0 1.2 1.4 1.6 1.8 2 2.5 3 3.5 4 4.5 5 6 7 8 9 10
$e^{\phi\alpha}$

The increase in friction over ordinary friction gears for a given pull between shafts, may now be readily computed. Let P be the pull between shafts. In the case of the belt we may take $P = T_1 + T_2$ when $\alpha = 180°$. Hence:

$$P = T_2 e^{\phi\alpha} + T_2 = T_2(e^{\phi\alpha} + 1),$$

or

$$T_2 = \frac{P}{e^{\phi\alpha} + 1}.$$

The force transmitted at the circumference of the pulley at the instant of slipping is

$$K = F = T_1 - T_2 = T_2(e^{\phi\alpha} - 1).$$

$$K = P\frac{e^{\phi\alpha} - 1}{e^{\phi\alpha} + 1}.$$

If $\phi = 0.28$, $e^{\phi\alpha} = 2.41$.

Hence for the belt,

$$K = .413\,P.$$

In the case of friction gears

$$K = \phi P = .28\,P.$$

The gain will then be in the ratio of 413 to 280.

The above formulæ can only be considered as an approximation to the truth in the case of a rapidly moving belt. In fact they give us more the general law of the variation of the quantities involved than an exact measure of them. In a rapidly moving belt the sudden change in the direction of velocity as the belt passes around the pulley creates an additional tension not hitherto considered. This may amount to a large proportion of the total tension, so that if the belt is tightened up to its safe limit at rest, it will be overstrained at high velocity. In fact the velocity may become so great as to make the tension due to this centrifugal effect equal to the safe tension, so that the belt would not touch the pulley at all, if kept within the safe limit.

Let us examine more closely into this effect. In Fig. 28, the normal pressure under an elementary length of the belt due to the opposite tensions is

$$dN_1 = T \cdot d\alpha.$$

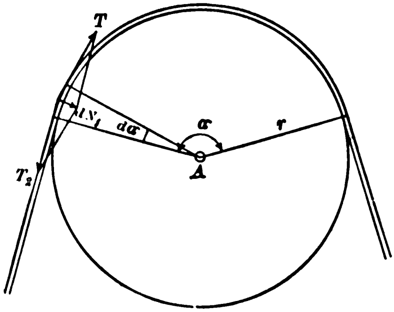


FIG. 28

But this is diminished by the centrifugal effect above mentioned, so that the true normal pressure is

$$dN_1 = T \cdot d\alpha - dm \frac{v^2}{r}.$$

The frictional force is

$$dF = \phi\left(Td\alpha - dm \frac{v^2}{r}\right).$$

But dF is equal to dT, and $dm = \frac{\delta}{g} \cdot r \cdot d\alpha$ where δ is the weight of the belt per unit of length. But the tension varies directly as δ, hence we may write

$$\delta = nT,$$

or

$$dF = dT = \phi\left(Td\alpha - \frac{nTv^2}{g}d\alpha\right),$$

$$\frac{dT}{T} = \phi\left(1 - \frac{nv^2}{g}\right)d\alpha,$$

$$\phi\alpha\left(1 - \frac{nv^2}{g}\right) = \int_{T_2}^{T_1}\frac{dT}{T} = \log_e \frac{T_1}{T_2}.$$

Hence,

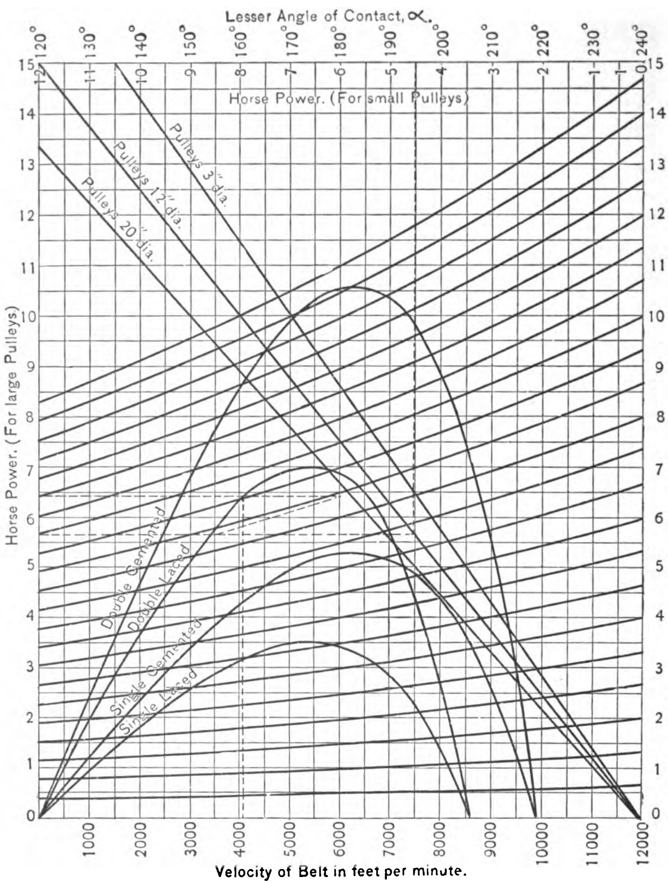
$$T_1 = T_2 e^{\left\{\phi\alpha\left(1 - \frac{nv^2}{g}\right)\right\}} = T_2 e^{\phi\alpha(1-x)},*$$

and

$$H.P. = \frac{vT_1}{33000}\left\{\frac{e^{\phi\alpha(1-x)} - 1}{e^{\phi\alpha(1-x)}}\right\}.$$

The constants of this equation can best be determined experimentally, or rather should be determined as experience dictates,

* "Rope Driving," by John J. Flather, p. 115. Wiley and Sons, 1895.



Horse-power Transmitted by Leather Belts per Inch of Width.

(After Prof. J. J. Flather.)

and no attempt should be made to get exact values of ϕ, δ, and the safety factors of slip and strength. In general we can say that as the velocity is varied, the horse-power transmitted will follow the general law of the equation as given, whose constants are to be determined experimentally. Diagram No. 2 shows a series of curves drawn by Professor Flather* which takes account of many sources of variation in belt transmission.

To use the diagram, take the speed of the belt at the bottom, and follow up to the belt curve required. Then follow horizontally to the centre line of the diagram, then parallel to the logarithmic curves to the abscissa, showing the angle of the belt at the top of the diagram. Going either to the right or left to the margin will give the horse-power which one inch width of the belt would transmit at the given velocity. Suppose we wish to find how wide should be a double-laced belt to transmit 30 H.P. at 4100 ft. per minute when covering 155° of the pulley. Following from the 4100 abscissa upwards to the double-laced belt curve, and passing horizontally to the centre line, where α equals 180°, we find that if covering 180° of the circumference the belt would transmit about 6.25 H.P. per inch of width. But by following parallel to the logarithmic curves to $\alpha = 155°$, it is seen that but 5.5 H.P. will be carried per inch of width. Dividing 30 by 5.5 gives for the width of belt about $5\frac{1}{2}$ in.

There are still other causes by which the power will be reduced. Air is drawn in by the belt at high speeds, and lessens the normal pressure between the belt and the pulley, where the two first come into contact. Also, if the diameter of the pulley is comparatively small, say less than 20 in., the necessary bending of the belt tends to injure it, and hence still less tension can be given. This is taken into account in the preceding diagram. If from the point last found on the 155° line in the previous example, we follow horizontally to one of the diagonal lines springing from the lower right-hand corner, and from the intersection with it follow upward to the scale at the top, we find the still further decreased horse-

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* *Western Electrician*, June 12, 1897.

power which each inch width of the belt will transmit. For example, if our belt should pass over a 12″ pulley, we find the horse-power per inch of width to be but 4.45, and that the belt must be about $6\frac{3}{4}$ in. If necessary, other lines can be interpolated between the three already drawn.

If the angular velocities of the two pulleys are in the same direction, the ordinary open belt is used, where the point of intersection of the line of centres with the belt line (produced), lies without the space between the centres. If the directions of rotation of the two are opposite, the crossed belt must be used, and the belt cuts the line of centres between the axes. The crossing of the two halves of the belt can be accomplished without serious interference by giving the belt a half twist.

The greatest source of loss is by friction at the bearings, due to the necessary tension on the belt. This can be roughly calculated when the diameters of the pulley and shaft are given, as well as the velocity of the belt, the tensions, and the coefficient of friction between the journal and its bearing. If ψ is the coefficient of journal friction, v the velocity of the belt in feet per minute, and R and r the radii of the pulley and shaft, then the opposing force at the belt due to journal friction will be

$$f = (T_1 + T_2)\psi\frac{r}{R}.$$

And the power wasted will be

$$W = \frac{fv}{33000} + \frac{v(T_1 + T_2)}{33000}\psi\frac{r}{R}.$$

Hence the tensions should be as small as possible to prevent slipping, ψ should be as small as possible, and r as small, and R as large as possible, for the greatest efficiency. There is also a loss due to the creeping of the belt. As the belt arrives at the driving pulley in a state of greater tension, and hence of greater stretch than when it leaves, there will be a slow creeping. The same is true of the driven pulley. This loss does not amount to more than $\frac{1}{2}$ of 1%.

In running a long horizontal belt it is best to have the tight side below, as by the greater sag of the slack side the arc of contact on both pulleys will be increased.

A variable angular velocity ratio is not practicable in belted transmission.

(*b*) *Stepped Cones*

When a follower shaft is required to have various speeds obtained from a constant speed driver, or when in any way the angular velocity ratio must be varied with belted connection, we must resort to Stepped Cones or an equivalent. This is always the case in lathes, milling machines, boring machines, etc. In the problem of stepped cones in its simplest form, we are given the distance between shafts, the diameters of the steps on one cone, and diameter of one step on the other cone; and we are required to find (1) the length of belt that will fit the given pair of steps, and (2) what must be the rest of the steps so that this same belt will fit them all with equal tension.

If the belt is crossed, the problem is a simple one. In Fig. 29 let l be the half length of the belt, α the angle between the belt and the line of centres, and R and r the radii of any pair of steps. Then

$$d \sin\alpha = R + r \quad . \quad . \quad . \quad . \quad . \qquad \text{I.}$$

Also,

$$l = d\cos\alpha + \frac{\pi}{2} R + \alpha R + \frac{\pi}{2} r + \alpha r,$$

$$= d\cos\alpha + \frac{\pi}{2}(R + r) + \alpha(R + r) \quad . \quad . \qquad \text{II.}$$

The length of belt which will fit the given pair of steps is now immediately known, for α is calculated from No. I, and l from No. II. Hence l becomes a known quantity. If now we substitute for cos α and α in No. II their values as deduced from No. I, we get

$$l = \sqrt{d^2 - (R + r)^2} + (R + r)\left(\frac{\pi}{2} + \sin^{-1}\frac{R + r}{d}\right). \qquad \text{III.}$$

In this, if the radius of any step of one cone, such as R, be given, the only unknown quantity is r, which would be the corresponding radius of the step on the other shaft, around which the already calculated belt length would fit. But equation No. III is transcendental, and of such a form that a direct solution is impossible. However, if we substitute in No. II the value of r or R deduced from No. I, both R and r eliminate from the equation. But R and r are variables. Hence α, the only other unknown quantity,

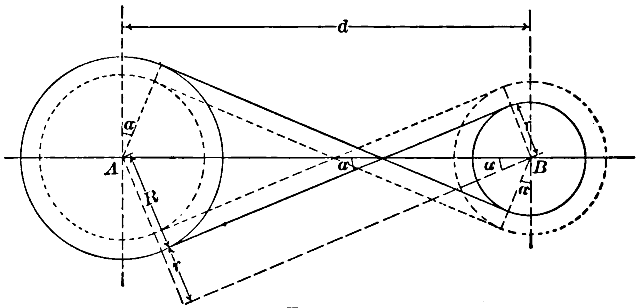

FIG. 29

must be a constant, as neither of the variables can be expressed in terms of it. If α is constant, $R + r$ is constant, or

$$R_1 + r_1 = R_2 + r_2 = R_3 + r_3 = \cdots \text{ and } R_1 - R_2 = r_2 - r_1,$$

$$R_2 - R_3 = r_3 - r_2, \text{ etc.,}$$

and the steps of the two cones are equal. This is readily seen to be true in equation No. II, for if $R + r$ is constant, l is constant, a necessary condition.

The case of the open belt is more complex, and an exact analytic solution has not yet been arrived at. In this case we have from Fig. 30,

$$d \sin \alpha = R - r \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{IV},$$

and

$$l = d \cos \alpha + \frac{\pi}{2} R + \alpha R + \frac{\pi}{2} r - \alpha r$$

$$= d \cos \alpha + \frac{\pi}{2}(R + r) + \alpha (R - r) \quad . \quad . \quad \text{V}.$$

Substituting for α and $\cos \alpha$ as before, we get

$$l = \sqrt{d^2 - (R - r)^2} + \frac{\pi}{2}(R + r) + (R - r) \sin^{-1} \frac{R - r}{d} \quad \text{VI},$$

which is transcendental also, and not capable of algebraic solution. Furthermore, if we substitute for r or R in No. V their values as deduced from No. IV, we get

$$R = \frac{l}{\pi} - \frac{d}{\pi}(\alpha \sin\alpha + \cos\alpha) + \frac{d}{2}\sin\alpha,$$

$$r = \frac{l}{\pi} - \frac{d}{\pi}(\alpha \sin\alpha + \cos\alpha) - \frac{d}{2}\sin\alpha,$$

which differ from one another only in the sign of the last term. But α is an unknown variable, and, as can readily be seen by inspection of the figure, cannot be a constant. Hence algebraic solution of the open belt is impossible.

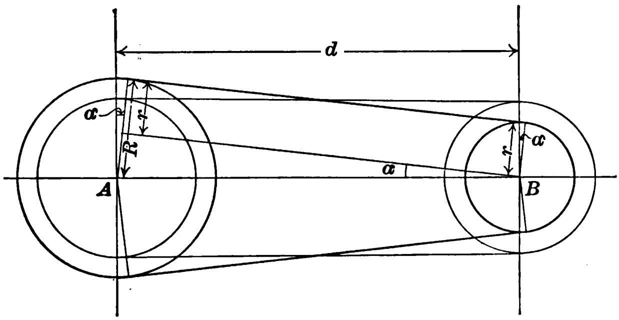

FIG. 30

Reuleaux* has deduced an exact graphical solution of the above equations, but it involves the tracing of certain higher plane curves, and is therefore not practical. His extremely beautiful and ingenious solution is as follows: in Fig. 31, draw BC and AD parallel, and at a distance apart equal to the shaft distance d. Draw AB at right angles to AD. With A as a centre and with a radius equal to AB draw the quadrant BE, which quadrant will contain all possible values of α. Draw the involute EF of the quadrant BE, with cusp at E. Let EAP be any arbitrary value

* "Constructor," p. 189.

of α. Draw the tangent at P, cutting the involute at N. Then $PN=PE$. Draw PM perpendicular to AD, NK perpendicular to PM, and QY perpendicular to AD, passing through N. Then

$$AQ=AM+MQ=d\ (\alpha \sin\alpha+\cos\alpha).$$

Now make $BC=\pi d=\pi(AB)$, and complete the rectangle $ABCD$. Draw the diagonal BD. This diagonal must make with BC an angle whose cotangent is π, or $17^\circ\ 39'\ 19''$. Then

$$YG=\frac{d}{\pi}(\alpha \sin\alpha+\cos\alpha),$$

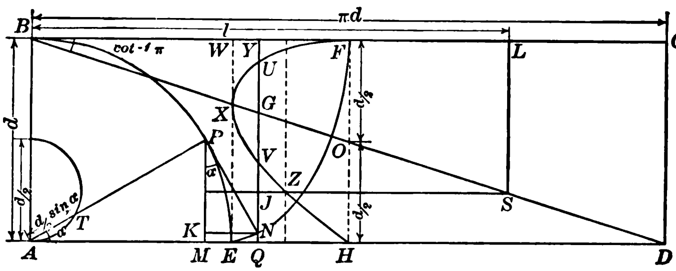

FIG. 31

which is the middle term of the equations for R and r. Again, if we lay off $BL=l$, the half length of the belt, which can be calculated for the given pair of steps, then

$$LS=YJ=\frac{l}{\pi},$$

which is the first term in the equations, and then we have

$$JG=YJ-YG=\frac{l}{\pi}-\frac{d}{\pi}(\alpha \sin\alpha+\cos\alpha).$$

Finally, if we draw a semicircle on $\frac{1}{2}AB$ as a diameter, then

$$AT=\frac{d}{2}\sin\alpha.$$

This we may add and subtract from JG to get the values of R and r corresponding to any chosen value of α. Lay off AT upwards from G and downwards from G, determining the points U and V. Then

$$JU = R$$

and

$$JV = r.$$

Now similar points may be obtained for other values of α, and thus a smooth curve $HVXUF$ is drawn, tangent to WE at X, and symmetrical as to ordinates about BD as an axis. The assumed pair of radii by which l was calculated must fit the curve, and thus may check the work. But we may even find the value of l from the diagram without calculation, for the curve $HVXUF$ may be drawn without reference to J at all. Then if we assume our first pair of steps, we will know their difference, which is equal to UV. So we merely need to try along the curve till we find an intercept UV equal to $(R - r)$, then lay off UJ equal to R, or VJ equal to r, and the point J becomes immediately fixed.

Hence Reuleaux's solution is exact and complete, but it is laborious, and involves the plotting of the involute EF, and the curve $HVXUF$.

Lately Professor W. K. Palmer* has proposed a modification of Reuleaux's method. He introduces a negligible approximation, but the method is eminently practical. Instead of plotting the curve $HVXUF$ as double ordinates QU and QV laid off along the same line, and therefore in the same direction, he lays off QU as an abscissa and QV as an ordinate with reference to A as an origin of rectangular coördinates. It will be noticed on our old diagram that as we pass along the curve from H towards V, the values of R and r become equal at X. When we pass beyond this point, the quantities R and r exchange places, and all those values formerly given to R must now be given to r, and vice versa. Also that the maximum value that can be given to R is when $l = \pi d$.

* "The Designing of Cone Pulleys," by Walter K. Palmer, Lawrence. Kansas.

Then $R = d$ and $r = 0$, or $r = d$ and $R = 0$. On our new diagram then (Fig. 32), if BFE be our new curve plotted in rectangular coördinates, it must be symmetrical with respect to a 45° line AF through the origin A. Furthermore, it must pass through the points B and E of the old diagram. Now in Fig. 31.

$$R = JU = QU - QJ,$$

$$r = JV = QV - QJ.$$

Hence to correctly represent the values of R and r on our new diagram, we must subtract the constant length QJ from both the

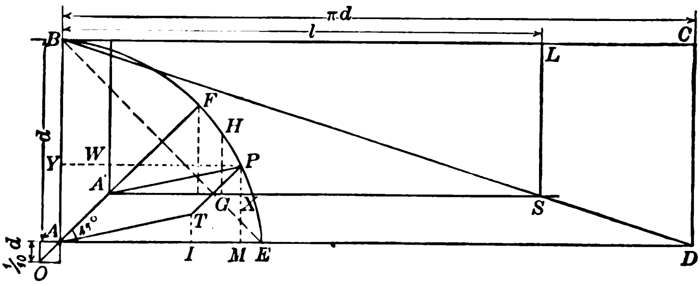

FIG. 32

ordinates and abscissæ of the new curve. This can be done by shifting the origin, from which we measure our step radii, along the 45° line from A to A'. Now

$$R = WP = YP - YW,$$

$$r = XP = MP - MX,$$

and $XM = YW$. Hence

$$R = A'X \text{ and } r = XP.$$

The point A' is determined by the length of the belt, and by the diagonal BD exactly as J was determined in Fig. 31. But we may even do away with the calculation of l, and find it directly from the diagram, for any given pair of steps will give the point A'. Make $AI = R$ and $IT = r$ for the assumed pair of steps. Draw AT.

Now shift the triangle AIT, so that its angle at A remains on the 45° line, and its sides AI and IT remain parallel to the coördinate axes, till the angle at T touches the curve at P. This can be done by drawing TP parallel to AF, and PA' parallel to TA. Then A' is the new origin from which $BL = l$ can be obtained. All other pairs of radii such as $R_1 = A'G$, $r_1 = GH$, etc., can be drawn. Palmer finds that the arc BFE can be very exactly approximated by the arc of a circle whose centre lies on the 45° line produced below A a distance equal to the diagonal of a square whose side is one-tenth of d. This removes all necessity of plotting curves of any sort, and leads us to a direct and practical result.

It is of interest to notice that in the case of crossed belts the circular arc BFE in Palmer's diagram becomes a straight line BE, since the coördinates of points along such a line measured from any set of axes parallel to AD and with origins on AF will have a constant sum. Hence $R + r =$ constant, the necessary condition for crossed belts. Otherwise the construction is the same as for the open belt.

When the distance between shafts is great as compared with the radii of the pulleys, we may consider α equal to zero in the equation

$$l = d \cos \alpha + \frac{\pi}{2}(R + r) + \alpha (R - r),$$

and any slight error due to this assumption will be made up by the elasticity of the belt. In this case our equation reduces to

$$l = d + \frac{\pi}{2}(R + r).$$

Here, as in the case of crossed belts, if we make $R + r$ constant, the equation is satisfied, or the steps on the two cones will be equal.

Professor John E. Sweet* has employed the following graphical method of designing the steps of cones, where the distance between shafts is great enough to allow equal steps. It is as follows:

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* *American Artisan*, February, 1874; *American Machinist*, October 13, 1898, p. 757.

Let A and B (Fig. 33) be the centres of the pulleys, B being the countershaft, which runs at a constant angular velocity, ω_1. Since the speeds vary inversely as the pulley diameters, we lay off the

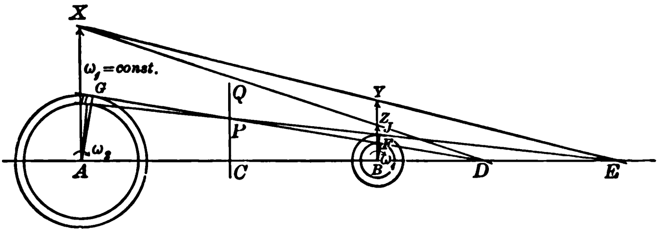


FIG. 33

velocity of B from the centre of A, and in any direction as AX. Similarly we lay off the smallest velocity of the driven machine (back gear out) from B parallel to AX and equal to BZ. Produce AB and draw XZD. Now

$$\text{Ang. vel. } A : \text{Ang. vel. } B :: BZ : AX :: BD : AD.$$

Let BF be the radius of any step of the cone, assumed to begin with. Draw a line from D tangent to circle BF, and produce toward A. Draw a circle about A as a centre, and tangent to DF at G. Then

$$GA : FB :: AD : BD :: \text{Ang. vel. } B : \text{Ang. vel. } A.$$

Now lay off BY equal to the next speed required of the machine A. Draw XYE. Take C at the middle point of AB and erect a perpendicular CQ intersecting GF at P, then P is the middle point of GF (nearly). Connect EP and produce. Draw circles about A and B tangent to this line, and these will be the next pair of steps. It will be noticed that since P is the middle point of GF, the steps will be equal, or the same belt will fit all steps. Modifications have been proposed for the above to suit cases where the shaft distance is not great enough to allow equal steps.*

* C. A. Smith, *American Machinist*, February 25, 1882; John Coffin, *American Machinist*, April 1, 1882.

The angular velocity ratio of a pair of stepped cones should form a geometrical progression, and if the driven machine is back geared the back gearing should carry the geometrical progression on. Let us suppose that we have a three-stepped cone, with back gear, which runs at ten revolutions per minute with the back gear in and the belt on the step of largest radius. Furthermore, suppose that each succeeding step increases the speed 50%. Then we would have

Back gear in	Step (1),	. . 10.
	Step (2),	$1.50 \times 10.00 = 15.$
	Step (3),	$1.50 \times 15.00 = 22.5$
Back gear out	Step (1),	$1.50 \times 22.50 = 33.75.$
	Step (2),	$1.50 \times 33.75 = 50.52.$
	Step (3),	$1.50 \times 50.62 = 75.94.$

The ratio of the back gearing would be

$$\rho = \frac{10}{33.75} = \frac{1}{3} \text{ (nearly).}$$

If these speeds are laid off as ordinates equally spaced along a horizontal line, we get a smooth curve (Fig. 34). If, however, we find by trying the different speeds of a cone-driven machine that the curve is irregular, then the cones are badly designed. This method of testing the cones of a lathe has been suggested by Professor Sweet. He also uses the following method of graphically computing the geometrical progression of speeds: lay off *PO* (Fig. 35) equal to 100 units, and *OA* equal to the number representing the percentage of increase in speed. In the preceding case $OA = 50$. Erect a

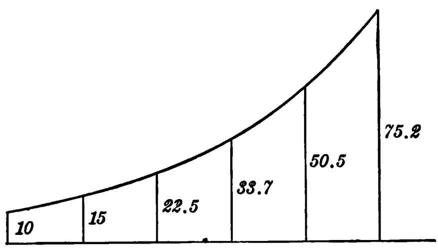


FIG. 34

perpendicular at O and another at A. Lay off $Oa = 10$, or the slowest speed of the cone with back gear in. Draw PaB; then $AB = 15$, the speed of the second step. Square back from B to b, so that Ob equals AB, and draw PbC. Then $AC = 22.5$, the third speed, and so on.

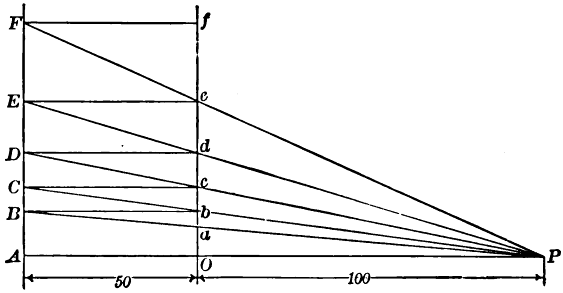


FIG. 35

Frequently the extreme speeds of the driven machine are given, and we are required to divide up the interval according to a geometrical progression. If a is the slowest speed or the first term of the series, and x the constant multiplier, then our series is

$$a, ax, ax^2, ax^3, ax^4, \text{ etc.}$$

The nth term will be $ax^{n-1} = b$. In our problem we are given a, b, and n to find x.

$$x^{n-1} = \frac{b}{a}, \quad x = \sqrt[n-1]{\frac{b}{a}},$$

and the ratio of the back gear will be

$$\rho = \left(\frac{1}{x}\right)^{\frac{n}{2}}.$$

In the preceding problem $a = 10$, $b = 75.94$, $n = 6$.

$$x = \sqrt[5]{\frac{75.94}{10}} = 1.50, \qquad \rho = \left(\frac{1}{1.50}\right)^3 = \frac{1}{3.375}.$$

Sweet's method of constructing the geometrical progression of speeds can be combined in a very elegant manner with Palmer's method of designing the cones, as the following problem will show: let the distance between the shafts of a foot lathe be 30″. Let the radius of the largest step on the driving wheel be $10\frac{1}{2}$″, and of the smallest step on the headstock be 1.5″. The maximum angular velocity ratio will then be $y = 7$ (back gear out). Furthermore, let the minimum angular velocity ratio be $y = 1$ (back gear in), and let there be three steps to the cones, so that $n = 6$. Then

$$\frac{b}{a} = \frac{y_{\text{max.}}}{y_{\text{min.}}} = \frac{7}{1} = 7.$$

The ratio of increase in speed is

$$x = \sqrt[5]{7} = 1.4758,$$

and the ratio of the back gear is

$$\rho = \left(\frac{1}{1.4758}\right)^3 = \frac{1}{3.2141} = \frac{1}{3} \text{ (nearly)}.$$

Lay off AB equal to 30″ (Fig. 36), and AK and KO each 3″, determining the centre O. Draw the arc BE about O. Lay off

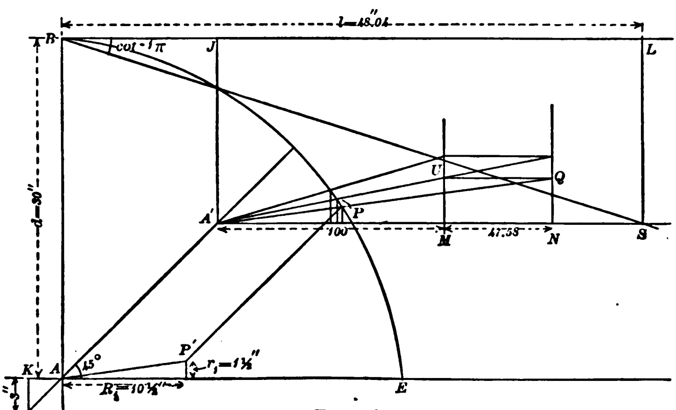


FIG. 36

from origin A, $R_1 = 10.5$″, and $r_1 = 1.5$″. Shift origin to A', and produce $A'S$. Make $A'M$ equal to 100 units of any convenient scale, and MN equal to 47.58 units. Erect perpendiculars at M and N. Produce $A'P$ to Q. Square back to U and draw AU, etc. Measurement of a large scale drawing gives us the following results:

$$d = 30'' \begin{cases} R_1 = 10''.50 & r_1 = 1''.50. \\ R_2 = 10.10 & r_2 = 2.14. \\ R_3 = 9.50 & r_3 = 2.96. \end{cases}$$

The half-length of belt is

$BL = lA = 48''.04$, or $8' \frac{1}{16}''$ for the whole length.

The ratio of the back gears is $\frac{1}{8}$.

B. Wire Ropes

For transmitting power over great distances, wire rope is used in place of belting. In this case the rope runs in grooved wheels of large diameter, known as sheaves. The bottoms of the grooves are often lined with wood or rubber to increase the coefficient of friction. The diameter of the sheaves being great compared with the diameter of the journal, the loss of power in the boxes is reduced to a minimum. The rope is supported at intervals by smaller pulleys known as idlers.

In the rope transmission problem, the horse-power to be transmitted, the velocity of the rope, and the distance between idlers are given. The tensions on the tight and slack side, which are necessary to transmit the required horse-power at the given velocity at the instant of slipping, can be calculated approximately from

$$T_1 = \frac{H.P. \times 33000}{v\left(\frac{e^{\phi a} - 1}{e^{\phi a}}\right)},$$

and

$$T_2 = \frac{T_1}{e^{\phi a}}.$$

Then if the rope is at rest, the tensions on the two sides will be equal, and will be approximately

$$T = \frac{T_1 + T_2}{2}.$$

Generally T_1 is taken equal to $2T_2$, which gives with $\alpha = \pi$ the coefficient of friction ϕ as only 0.22. This then allows an ample factor of safety; in other words it will include the factor C, and the mean tension will be $T = \frac{3}{4} T_1$.

Now to find what size of rope will bear the maximum tension T_1, we must look in some table of wire rope sizes and strengths, select that size which will bear T_1 as a working stress, and from the same table find the weight per foot of such a rope. The rope is then drawn up to the required tension T, and spliced in place.

The best way to measure the tension in such a rope is to measure the sag or deflection at the middle point of the span. The calculation of this sag we will now take up.

(*a*) *Horizontal Transmission*

Let POQ (Fig. 37) be a suspended rope, whose points of suspension P and Q are in a horizontal plane. Then O the lowest point will also be the middle point. The coördinates of P are x

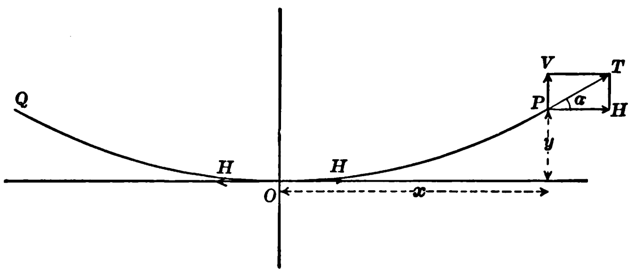


FIG. 37

and y when referred to an origin at O, with axis of X tangent to the rope (*i.e.* horizontal). The tension at P can be resolved into its vertical and horizontal components, which are

$$V = T \sin \alpha,$$

$$H = T \cos \alpha,$$

$$\frac{V}{H} = \tan \alpha = \frac{dy}{dx}.$$

TABLE

Standard Iron and Steel Hoisting Ropes, John A. Roebling's Sons Company

Wire Ropes of 19 Wires to the Strand

Trade number.	Diameter.	Circumference.	Weight per foot in pounds.	Iron: Breaking strain in tons of 2000 lbs.	Iron: Working load in tons of 2000 lbs.	Iron: Circumference of new Manilla rope of equal strength.	Iron: Minimum diameter of sheave in feet.	Cast Steel: Breaking strain in tons of 2000 lbs.	Cast Steel: Working load in tons of 2000 lbs.	Cast Steel: Circumference of new Manilla rope of equal strength.	Cast Steel: Minimum diameter of sheave in feet.
1	2¼	6¾	8.00	74	15	14	13	155	31	—	8½
2	2	6	6.30	65	13	13	12	125	25	—	8
3	1¾	5½	5.25	54	11	12	10	106	21	—	7¼
4	1⅝	5	4.10	44	9	11	8½	86	17	15	6¼
5	1½	4¾	3.65	39	8	10	7½	77	15	14	5¾
5½	1⅜	4⅜	3.00	33	6½	9½	7	63	12	13	5½
6	1¼	4	2.50	27	5½	8½	6½	52	10	12	5
7	1⅛	3½	2.00	20	4	7½	6	42	8	11	4½
8	1	3⅛	1.58	16	3	6½	5¼	33	6	9½	4
9	⅞	2¾	1.20	11.50	2½	5½	4½	25	5	8½	3½
10	¾	2¼	0.88	8.64	1¾	4¾	4	18	3½	7	3
10¼	⅝	2	0.60	5.13	1¼	3¾	3½	12	2½	5¾	2¼
10½	9/16	1⅝	0.44	4.27	¾	3½	2¾	9	1½	5	1¾
10¾	½	1½	0.35	3.48	½	3	2¼	7	1	4½	1½
10 *a*	7/16	1⅜	0.29	3.00	⅜	2¾	2	5½	¾	4¾	1¼
10⅞	⅜	1¼	0.26	2.50	¼	2½	1½	4½	½	3½	1

Wire Ropes of 7 Wires to the Strand

Trade number.	Diameter.	Circumference.	Weight per foot in pounds.	Iron: Breaking strain in tons of 2000 lbs.	Iron: Working load in tons of 2000 lbs.	Iron: Circumference of new Manilla rope of equal strength.	Iron: Minimum diameter of sheave in feet.	Cast Steel: Breaking strain in tons of 2000 lbs.	Cast Steel: Working load in tons of 2000 lbs.	Cast Steel: Circumference of new Manilla rope of equal strength.	Cast Steel: Minimum diameter of sheave in feet.
11	1½	4⅝	3.37	36	9	10	13	62	13	13	8½
12	1⅜	4¼	2.77	30	7½	9	12	52	10	12	8
13	1¼	3¾	2.28	25	6¼	8½	10¾	44	9	11	7¼
14	1⅛	3⅜	1.82	20	5	7½	9½	36	7½	10	6¼
15	1	3	1.50	16	4	6½	8½	30	6	9	5¾
16	⅞	2⅝	1.12	12.3	3	5¾	7½	22	4½	8	5
17	¾	2⅜	0.88	8.8	2¼	4¾	6¾	17	3½	7	4½
18	11/16	2⅛	0.70	7.6	2	4½	6	14	3	6	4
19	⅝	1⅞	0.57	5.8	1½	4	5¾	11	2¼	5½	3½
20	9/16	1⅝	0.41	4.1	1	3¼	4½	8	1¾	4¾	3
21	½	1⅜	0.31	2.83	¾	2¾	4	6	1½	4	2½
22	7/16	1¼	0.23	2.13	½	2½	3¼	4½	1¼	3½	2¼
23	⅜	1⅛	0.19	1.65	—	2¼	2¾	4	1	3¼	2
24	5/16	1	0.17	1.38	—	2	2½	3	¾	2¾	1¾
25	9/32	⅞	0.125	1.03	—	1¾	2¼	2	½	2¼	1½

Now the horizontal component of the tension is necessarily constant in magnitude, and if we consider the weight of the rope as uniformly distributed along its horizontal projection PQ, then approximately

$$V = wx$$

where w is the weight of the rope per running foot. So our equation is

$$\frac{dy}{dx} = \frac{wx}{H},$$

$$y = \frac{w}{H}\frac{x^2}{2}.$$

The equation represents a parabola with origin at the vertex. From the triangle of forces at P

$$T^2 = H^2 + V^2,$$

$$H = \sqrt{T^2 - w^2x^2}.$$

So our equation becomes

$$y = \frac{w}{\sqrt{T^2 - \frac{w^2s^2}{4}}}\frac{s^2}{8},$$

where y is the sag or deflection at the middle, and $s = 2x$ is the span between idlers. In most cases the term $\frac{w^2s^2}{4}$ is insignificant compared with T^2, and can be neglected, hence the formula simplifies to the extent

$$y = \frac{ws^2}{8T}.$$

As an example of the use of the formulas, take the following: In a horizontal transmission let the span be 250 ft., the power to be transmitted is 100 H.P., and the velocity of the rope is 4000 ft. per minute. We will assume $T_1 = 2\,T_2$.

$$T_1 = \frac{\text{H.P.} \times 33000}{\frac{1}{2}v} = 1650 \text{ lbs.}$$

$$T_2 = \frac{T_1}{2} = 825 \text{ lbs.}$$

$$T = \frac{T_1 + T_2}{2} = 1237 \text{ lbs.}$$

By reference to the table it is seen that the diameter of a rope, which is to stand a working load of 1650 lbs., is $\frac{9}{16}$ in., and that the weight of such a rope per foot is 0.44 lbs. The deflection at rest is

$$y = \frac{ws^2}{8T}$$

$$= \frac{0.44 \times 62{,}500}{8 \times 1237} = 2.8 \text{ ft.}$$

The deflection at full load is

$$y_1 = \frac{ws^2}{8T_1} = 2.08 \text{ ft. on the tight side,}$$

and

$$y_2 = \frac{ws^2}{8T_2} = 4.16 \text{ ft. on the slack side.}$$

(b) *Inclined Transmission*

In the case where PQ is not horizontal, as in Fig. 38, the method is slightly more complicated. As before, we consider

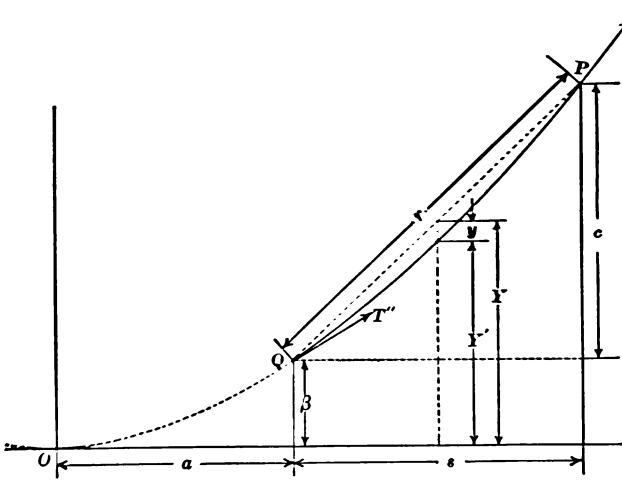


FIG. 38

the weight of the rope uniformly distributed along the chord PQ; hence, if this distance be called r, the total weight of the rope will be $w \cdot r$, where w is the weight per running foot as given from the tables. But the weight will be distributed uniformly, though at a different rate, along the horizontal projection s. Let the weight per foot distributed in this way be w_0. Then

$$w_0 = w\frac{r}{s} = w\frac{\sqrt{c^2 + s^2}}{s},$$

c being the difference in level between the points P and Q. Taking as before the origin at the vertex of the parabola, and calling the coördinates of Q, α, and β,

$$T'^2 = H^2 + w_0^2(\alpha + s)^2. \quad . \quad . \quad . \quad 1$$

$$T''^2 = H^2 + w_0^2\alpha^2. \quad . \quad . \quad . \quad 2$$

$$\beta = \frac{w_0\alpha^2}{2H}. \quad . \quad . \quad . \quad 3$$

$$\beta + c = \frac{w_0}{2H}(\alpha + s)^2. \quad . \quad . \quad . \quad 4$$

In these four equations, s and c are known from the conditions of transmission. One of the tensions, viz., T'', is known from the horse-power conditions. This leaves four unknown quantities, α, β, T', and H, to be found from the four equations. They can be combined with the following result. As equations 2, 3, and 4 contain but three unknowns, we combine them first. Taking 3 and 4:

$$\frac{w_0\alpha^2}{2H} + c = \frac{w_0(\alpha + s)^2}{2H}, \quad H = \frac{w_0 s(2\alpha + s)}{2c}. \quad . \quad . \quad 5$$

Combining 5 with 2:

$$T''^2 = \frac{w_0^2 s^2(2\alpha + s)^2}{4c^2} + w_0^2\alpha^2,$$

$$\alpha = -\frac{w_0 s^3 \pm c\sqrt{4T''^2(c^2 + s^2) - w_0^2 s^4}}{2w_0(c^2 + s^2)}. \quad . \quad 6$$

H then becomes known from 5, β from 3, and T' from 1.

For example, in an inclined transmission, let the horizontal distance between points of support be, $s = 250$ ft., and the vertical distance between the same points be, $c = 150$ ft. Eighty horse-power are to be transmitted at 4000 ft. per minute velocity of rope. Then (Fig. 39)

$$T_1'' = \frac{80 \times 33000}{\frac{1}{2}(4000)} = 1300 \text{ lbs.}$$

$$T_2'' = \frac{1300}{2} = 650 \text{ lbs.}$$

$$T = \frac{1300 + 650}{2} = 975 \text{ lbs.}$$

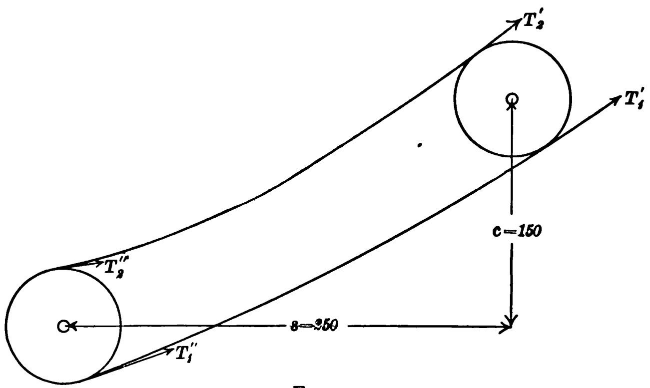


FIG. 39

Select as before a $\frac{9}{16}$-rope weighing 0.44 lbs., per foot.

$$w_0 = w\frac{\sqrt{c^2 + s^2}}{s} = .44\frac{\sqrt{250^2 + 150^2}}{250} = 0.513 \text{ lbs.}$$

Deflections at full load:

$$\alpha_1 = \frac{-.513 \times 250^3 \pm 150\sqrt{4 \times 1300^2(150^2 + 250^2) - .513^2 \times 250^4}}{2 \times .513(250^2 + 150^2)}.$$

Since in our case s is positive, and is measured in the same direction as α, we take the positive value.

$$\alpha_1 = 1207.5 \text{ ft. (or } -1391.3 \text{ ft.).}$$

Also $$H_1 = \frac{.513 \times 250(2 \times 1207.5 + 250)}{2 \times 150} = 1139.6 \text{ lbs.}$$

$$\beta_1 = \frac{.513 \times 1207.5^2}{2 \times 1139.6} = 328.3 \text{ ft.}$$

$$T_1' = \sqrt{1139.6^2 + .513^2(1207.5 + 250)^2} = 1363.1 \text{ lbs.}$$

The deflection at the middle is taken as the difference in level between the middle point of the chord r (see Fig. 38), and the point of the rope vertically below it. Calling the abscissæ of these points Y_1 and Y_1', and the deflection y_1,

$$y_1 = Y_1 - Y_1'.$$

$$Y_1 = \beta_1 + \frac{c}{2} = 403.3 \text{ ft.}$$

$$Y_1' = \frac{w_0\left(\alpha_1 + \frac{s}{2}\right)}{2H_1} = 399.74 \text{ ft.}$$

$$y_1 = 3.52 \text{ ft.}$$

Similarly for the slack side,

$$T_2'' = 650 \text{ lbs.}$$

$$\alpha_2 = 551.6 \text{ ft. (or } -735.42 \text{ ft).}$$

$$H_2 = 578.63 \text{ lbs.}$$

$$\beta_2 = 134.9 \text{ ft.}$$

$$T_2' = 709.91 \text{ lbs.}$$

$$Y_2 = 209.90 \text{ ft.}$$

$$Y_2' = 202.99 \text{ ft.}$$

$$y_2 = 6.91 \text{ ft.}$$

Similar figures might be obtained for the rope at rest.

The material of the rope is subjected to two tensions, — the working tension, and the tension due to bending around sheaves. The working tension may therefore be greater as the bending tension is less, or as the diameter of the sheaves becomes larger. The size of sheave which will keep the bending tension down to a low limit is as follows (expressed as a multiple of the rope diameter) :

Rope of 7-wire strands, Ratio = 150.

Rope of 12-wire strands, Ratio = 119.

Rope of 19-wire strands, Ratio = 90.

When the velocity of the rope is high, the effect of centrifugal force will be felt exactly as in the case of belts. If the span is less than 60 ft., the transmission cannot be effected economically. For distances up to 2000 ft., such transmission is very efficient.

2. AXES OF ROTATION PARALLEL, PLANES OF PULLEYS DIFFERENT

In belting up shafts where the medial planes of the pulleys are not identical, care has to be taken that the belt does not run off.

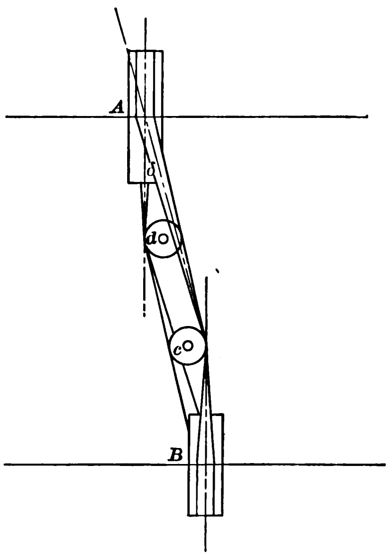


FIG. 40

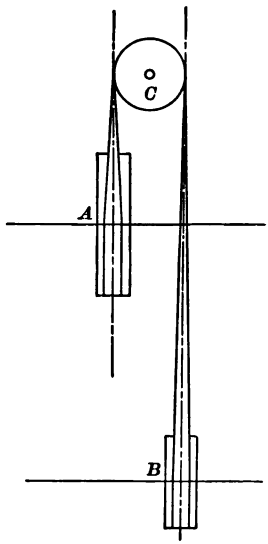


FIG. 41

If the belt approaches the pulley at an angle other than 90° with the axis, it will fall to one side because the inclination causes it to describe a helical path. On the other hand, in leading the belt off, the receding part may be given an angle of departure as great as 20° without causing it to fall off, because in this case the actual friction between the belt and pulley must be overcome. In every instance the point at which the belt is delivered from one pulley must lie in the medial plane of the next pulley. This is the only condition to be fulfilled, provided the angles of departure do not exceed 20°. By properly applying the above condition, and by

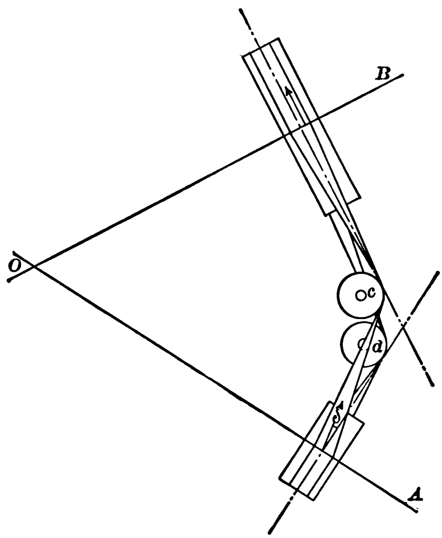


FIG. 42

the use of idlers or guide pulleys, which must themselves comply with the condition, parallel shafts can be belted up where the planes of the pulleys are different. Fig. 40 shows one method of doing this. The belt leaves pulley *A* at an angle of departure δ, but the guide pulley *c* delivers it in the medial plane of *B*. The guide pulley *d* similarly handles the part receding from *B*. It will be noticed that the guide pulleys will have their planes determined by the belt lines, and that the belt must be given a quarter twist in passing around them. The above arrangement can be run in but one direction as shown by the arrows, but if the shafts are to turn in opposite directions, the arrangement of Fig. 41 can be used, and the belt run either way, when the diameters of both guide pulleys are equal to the distance between the planes of *A* and *B*. It is not possible in any case to dispense with either of the guide pulleys.

3. AXES OF ROTATION INTERSECTING

Fig. 42 shows a method of belting a pair of shafts whose axes intersect at *O*. The belt leaves the pulley *B* at an angle of departure, but the guide pulley *c* delivers it to *B* in the medial plane of

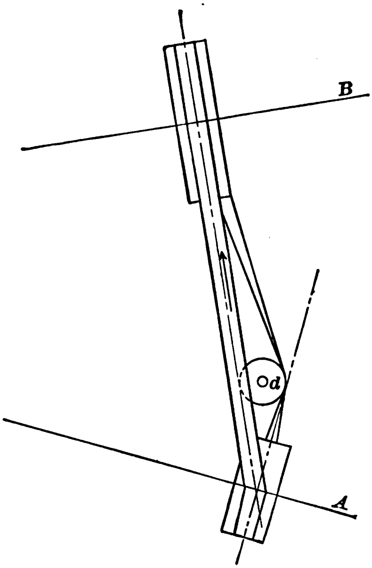


FIG. 43

the latter. The whole method is practically identical with the preceding. If *c* and *d* are so placed that the angles of departure of the belt are zero, the apparatus will run in either direction. If the medial plane of *B* is made to include the point at which the belt is delivered from *A*, the guide pulley *c* can be dispensed with, but in this case the angle between the shafts cannot be greater than 20°. In no case can both guide pulleys be dispensed with (Fig. 43).

4. AXES OF ROTATION CROSSING

In this case the method of procedure is the same as in the former. In one particular instance both the guide pulleys can be omitted, as in Fig. 44. Here the radius of pulley *A* is equal to

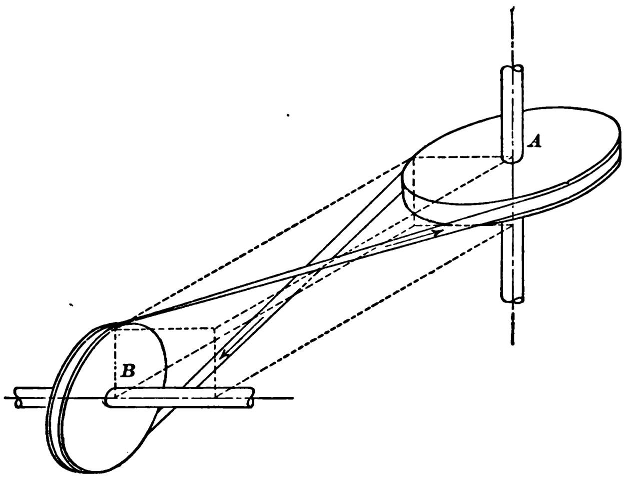


FIG. 44

the distance from its shaft to the plane of *B*, and the radius of *B* is similarly proportioned with respect to *A*. The belt will then run only in the direction shown by the arrows.

CHAPTER IV

TRANSMISSION OF PURE ROTATION BY MEANS OF TOOTHED GEARING

1. GENERAL CONSIDERATIONS

When a constant angular volocity ratio is required to be maintained at every instant of time, toothed wheel transmission is usually resorted to. In this form certain projections upon and depressions within the rotating bodies are made to mesh, so that they are driven by direct contact. Any sort of projections and depressions will cause one wheel to follow the other, and will keep up an average constant velocity ratio, provided they neither interfere nor fail to engage at any time. But if an absolutely constant angular velocity ratio is required at every instant, the outlines of the teeth must be formed according to certain laws.

In all cases of toothed gearing, the teeth are considered as fixed upon certain imaginary surfaces known as Pitch Surfaces. These are of such a form that any pair will be tangent along a straight line, and that when rotated about fixed axes, with equal component velocities at right angles to the line of contact, the required motion of the gears will be reproduced. In general such motion will be one of pure rolling normal to the line of tangency, and of sliding along it.

2. AXES OF ROTATION PARALLEL, SPUR GEARS

A. Laws of Action

In this, the most important case, the pitch surfaces are cylindrical, and tangent along an element which lies in the plane of the axes. Here, since the resultant velocity of a point on the surface is normal to an element, the motion will be one of pure rolling, with no sliding along the element. The sections of the cylinders at right angles to the axes being parallel, the relative motion is evidently uniplanar; hence we need consider the motion in but a single plane. The curves of intersection of the pitch surfaces by the normal planes, are known as Pitch Curves, which must have a point of tangency on the line of centres.

Profiles. — Two curves are said to be profiles when during any relative motion they remain tangent. The case of two free profiles is one of little practical use, but when we impose the further condition that they rotate about fixed axes, especially when these are parallel, it becomes one of great importance.

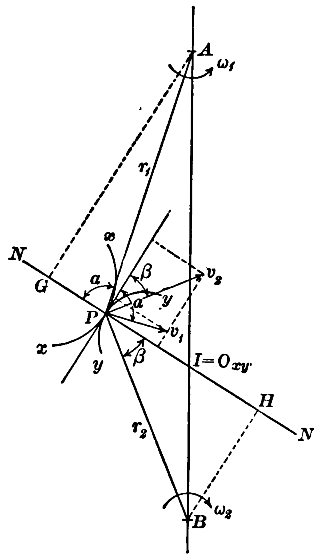


FIG. 45

Let xx and yy (Fig. 45) be two profiles, which rotate about centres A and B, at angular velocities ω_1 and ω_2. Let x be the driver and y the follower. P is the point of contact of the profiles, and $PA = r_1$ and $PB = r_2$ are the contact radii. Consider P to be a point of x. Its velocity v_1 will be at right angles to r_1 and equal to $r_1\omega_1$. The component of v_1 along the tangent to the profiles at P will be $r_1\omega_1 \cos \alpha$, and the component along the normal will be $r_1\omega_1 \sin \alpha$. Consider now P as a point of y. Its velocity v_2 is $r_2\omega_2$, and its component resolved as before are $r_2w_2 \cos \beta$, and $r_2\omega_2 \sin \beta$. If the curves are to remain continually tangent (*i.e.* are to be profiles),

$$r_1\omega_1 \sin \alpha = r_2\omega_2 \sin \beta,$$

$$\frac{\omega_1}{\omega_2} = \frac{r_2 \sin \beta}{r_1 \sin \alpha},$$

otherwise they would either leave contact or cut into one another. Now from the centres A and B drop perpendiculars AG and BH upon the normal. Then,

$$AG = r_1 \sin \alpha,$$

$$BH = r_2 \sin \beta,$$

$$\frac{\omega_1}{\omega_2} = \frac{BH}{AG}.$$

But if the normal cuts the line of centres at I,

$$\frac{BH}{AG} = \frac{BI}{AI},$$

$$\frac{\omega_1}{\omega_2} = \frac{BI}{AI}.$$

Thus in direct contact transmission, the angular velocity ratio is inversely proportional to the segments into which the line of centres is divided by the common normal to the profiles.

The above proposition can be proved by the instantaneous centre relation also. The bodies x and y are rotating about fixed centres A and B. The instantaneous centre of x referred to y must lie somewhere on the line connecting these centres. Now the point P, when considered a point of x, must be moving in the direction of the common tangent relatively to y, from the profile condition. Hence O_{xy} must lie somewhere on the line drawn at right angles to this direction, or somewhere along the common normal. It therefore lies at the intersection of the normal with the line of centres. But from Fig. 15 we know that

$$\frac{\omega_1}{\omega_2} = \frac{BO_{xy}}{AO_{xy}}.$$

Hence the proposition is established.

If the ratio $\frac{\omega_1}{\omega_2}$ is to remain constant we must have

$$\frac{BI}{AI} = \text{const.}$$

But $$BI + AI = \text{const.}$$

Hence I must be fixed on the line of centres. This important proposition can be summed up as follows: when profiles are to transmit a constant angular velocity ratio, they must be so formed that their common normal continually passes through a fixed point on the line of centres, which point divides the centre distance in the inverse ratio of the angular velocities. Profiles so related are known as Conjugate Profiles.

The velocity of sliding between the curves will be the difference of their velocities along the tangent, or

$$v_s = r_2\omega_2 \cos \beta - r_1\omega_1 \cos \alpha.$$

If the velocity of sliding is zero, there will be pure rolling between the profiles, which then become centrodes. In that case

$$r_1\omega_1 \cos \alpha = r_2\omega_2 \cos \beta.$$

But also

$$r_1\omega_1 \sin \alpha = r_2\omega_2 \sin \beta.$$

Both these conditions can be satisfied only when $v_1 = v_2$ both in magnitude and direction. But v_1 and v_2 are at right angles to r_1 and r_2, hence for pure rolling, P must lie on AB. This is easily seen to be true, since for pure rolling, P must be O_{xy}.

Conditions to be Fulfilled.—The conditions to be realized if possible are:

1. That the angular velocity ratio be constant.

2. That there should be positive driving, and that, therefore, there should be a component of driving force in the direction of the normal, and on that side of the tangent occupied by the follower. From this it follows that the contact radius of the driver must be on the increase.

3. The velocity of sliding should be zero if possible.

All three of these conditions cannot be fulfilled, for if (3) be true, P must continually lie on the line of centres, and if in addition (2) be true, the common tangent at P must cut the line of centres at some angle other than 90°. Hence the point of contact cannot remain fixed on the line of centres, but must move, thus giving a variable angular velocity ratio. Similarly if any two of the three conditions be fulfilled, the third will fail. This gives rise to three general forms of such driving:

When (1) and (2) hold, and (3) is abandoned, we have the ordinary toothed gearing.

When (1) and (3) hold, we have the case of friction wheels.

When (2) and (3) hold, we have some variable angular velocity ratio, such as may be produced through limited angles by equal ellipses or logarithmic spirals.

The second case we have already considered, and the third is of little value, though the elliptic form as applied to a pitch curve will be treated in its proper place, but the first is of great importance, and we now proceed to take up its study.

Curves of Action. — Let *aa* and *bb* (Fig. 46) be two pitch curves rotating about *A* and *B*. If there is to be pure rolling at *I*, $\frac{\omega_1}{\omega_2} = \frac{BI}{AI}$. If the motion is to be transmitted by the direct contact of two profiles *yy* and *xx* while rolling still continues between *aa* and *bb*, the common normal to the profiles must continually pass through *I*, which is called the Pitch Point. Evidently as motion continues, *P* will trace up a curve in space, whose form depends on the forms of the pitch curves and profiles. This curve is called the Curve or Locus of Action. The pitch curve, the profile, and the curve of action, are connected by the general law, that the line connecting the point of intersection *P* of the last two, with the point of intersection of the first and the line of centres, must always be normal to the profile at *P*. (See Fig. 46.)

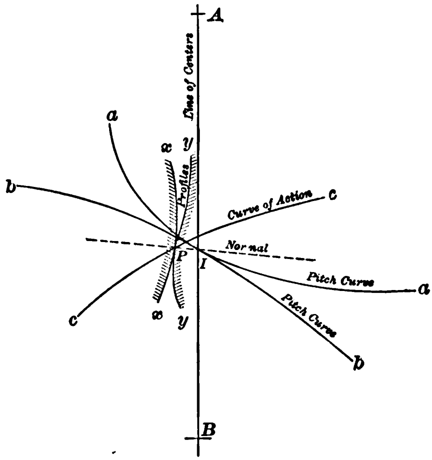


FIG. 46

Evidently then the profile cannot approach nearer to, nor recede further from, its centre of rotation than does the curve of action.

The pitch curve is determined once for all when the nature of the angular velocity ratio is chosen. The curve of action is a curve fixed in space. If it be chosen of any desired form, the profile will then be determined. The most general proposition is as follows: let *B* (Fig. 46) be a centre of rotation, and *BA* a line fixed in direction from which angular position can be measured. Let *bb* be any given pitch curve, whose equation is known, cutting *BA* at *I*. Assume any curve of action *cc*, which will be fixed in space. We are then to find what curve *xx* fixed in the plane of *bb*, and therefore carried around *B* thereby, will have as a normal the line connecting its point of intersection with *cc* to *I*. The general solution of this problem is one of considerable complexity, and even in the simpler cases, the correctness of action of an assumed pair of profiles can be shown by methods far easier than its direct solution, but as the underlying law it is well to be kept in view.

Definitions and Standard Dimensions. — Before proceeding to the deduction of any system of special profile forms, it will be best to give a few general definitions and dimensions, so that what follows may the more readily be understood. The teeth of gears are usually equally spaced along the pitch curve, that is to say, the distance measured along the pitch curve between its successive intersections with profiles facing in a given direction, will all be the same. One of these distances is called the Circular Pitch (Fig. 47), and is denoted by *P'*. It is measured usually in inches. The circular pitch must of course be the same on both of a pair

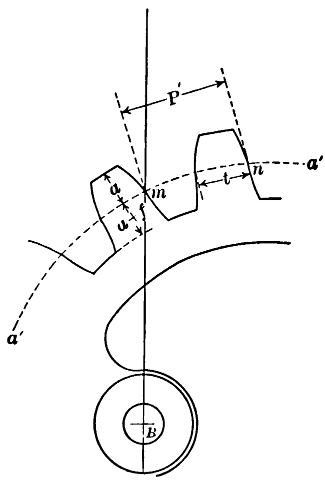


FIG. 47

of mating wheels. A fraction of a tooth being an impossibility, it must be an aliquot part of the perimeter of the pitch curve. Hence if C be this perimeter, and N the number of teeth, $P' = \frac{C}{N}$. In good cut gears the circular pitch is equally divided between the tooth and the space, so that the thickness of the tooth on the pitch curve will be $t = \frac{1}{2}P'$. In rough cast gears the tooth is made a little smaller than the space, and the difference is called the Backlash.

Another most important pitch dimension, which really should be used only in connection with circular gears, is the Diametral Pitch, denoted by P. It measures the number of teeth per inch of diameter of such a gear. Hence if D be the pitch diameter, $P = \frac{N}{D}$. But in a circular gear, $C = \pi D$, thus $P' = \frac{\pi D}{N}$, and we obtain the relations, $P = \frac{\pi}{P'}$, and $P' = \frac{\pi}{P}$. Since the size of tooth represented by a given diametral pitch is of fixed magnitude, we may apply the term to non-circular pitch lines, remembering that in these cases it means that if there were N teeth of such a size placed on a circular gear, there would be P teeth per inch of its diameter.

That portion of the tooth lying without the pitch curve is called the Addendum, and is denoted by a. The portion lying within the pitch curve is a little greater than the Addendum, and the difference is called the Clearance. The angle through which a wheel turns while a tooth is in contact with its fellow is called the Angle of Action. That portion of this angle turned off while the point of contact is approaching the line of centres is called the Angle of Approach and the rest, the Angle of Recess. Those portions of the pitch curve intercepted by these angles, are called the Arcs of Action, Approach, and Recess. The Arc of Action must evidently be greater than the circular pitch.

B. Velocity Ratio Constant, Circular Wheels

In this case the pitch curves must evidently be circles, and I will be a fixed point on the line of centres. A curve of action being assumed, all the teeth will be alike. Upon the nature of the curve of action only will depend the various forms of teeth used.

(*a*) *The Cycloidal System*

In this system the curve of action is a circle tangent to the pitch circle (Fig. 48). The solution of the general proposition in this case is of rather a complex nature (see Appendix III), but the special case where the radius of the pitch circle is infinite, that is, where it becomes a straight line as in the ordinary rack, is simple enough to be inserted here.

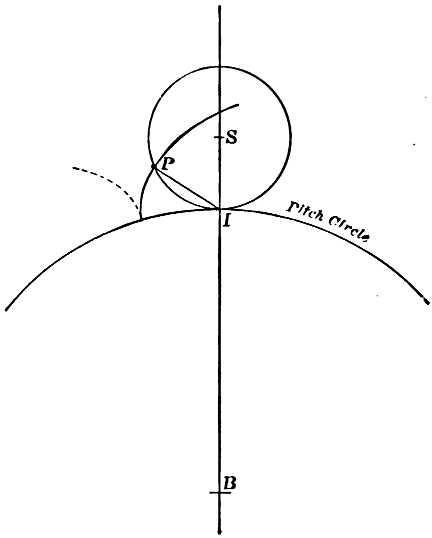


FIG. 48

Let x and y (Fig. 49) be the coördinates of the point of contact P. Let $x = f(y)$ be the equation of the profile, and $x = \phi(y)$, be that of the curve of action, which in this case is, $x = \sqrt{2ay - y^2}$, "a" being the radius of the circular line of action. The tangent of the angle which the tangent to the profile at P makes with the axis of Y is $-\frac{dx}{dy}$. The tangent of the angle between OP and the axis of X is $\frac{y}{x}$. But these are equal, hence,

$$\frac{y}{x} = \frac{y}{\sqrt{2ay - y^2}} = -\frac{dx}{dy},$$

$$-x = \int \frac{ydy}{\sqrt{2ay - y^2}} = a \text{ vers}^{-1} \frac{y}{a} - \sqrt{2ay - y^2} + C,$$

which is the equation of a cycloid, with origin on the line of cusps, and at a distance C from a cusp. C is the variable parameter which determines the position of a tooth.

A simple method of finding the nature of the profile for the general case is to imagine a reversal of the relative motion. Bring the pitch curve and profile to rest by giving the whole system an equal and opposite rotation about B (Fig. 48) thus rotating the line SB, which carries the circle S with it. If, however, the circle S be rotated about its own centre, its intersection with the profile will not be altered, hence we may allow it to roll upon the pitch circle, while S is carried around B. The path of any point on the circumference of circle S, is an epicycloid on circle B, and, I being

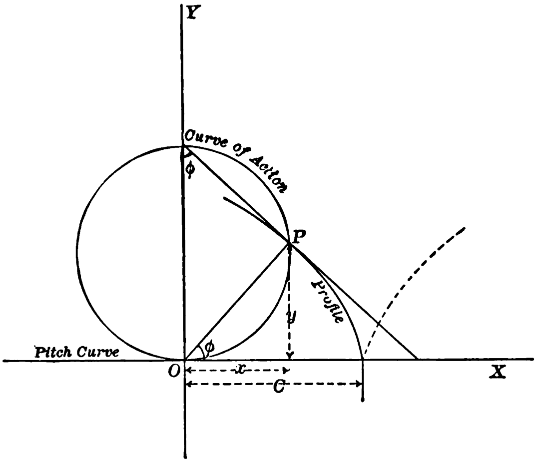


FIG. 49

the instantaneous centre, IP must be normal to the curve described. If the circle S lies within the circle B the curve described will be a hypocycloid, hence these curves may be used as profiles. The fact that any point in the plane of any curve rolled on the pitch curve will sweep up a correct profile, has led to the use of this feature as a basis for a tooth theory. It does not apply, however, in a practical way to all forms of tooth outlines in use.

The relative positions of two profiles on a pair of pitch circles are shown in Fig. 50. The profile of circle B will be an epicycloid ee_0 since the circular curve of action lies without its circumference.

The profile of A will evidently be a hypocycloid hh_0. The point of contact is P, and the common normal PI continually passes through I, a fixed point. The circular curve of action is called the Describing Circle. Action will take place, *i.e.* driving at the required constant angular velocity ratio will take place, on one side of the line of centres only. If A be the driver while rotating counter-clockwise, this will be on the left side as shown. After P has

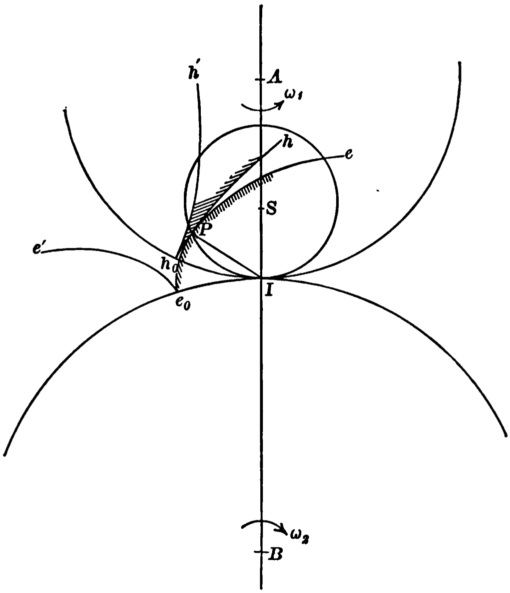


FIG. 50

passed the point I where the cusps come together, contact will no longer be between e_0e and h_0h, but between the second branches e_0e' and h_0h' beyond the cusps, and furthermore, the tendency would be for h_0h' to separate from e_0e' instead of driving it forward by pushing.

By using a describing circle within each of the pitch circles, the angle of action may be extended to both sides of the line of centres. It will then be divided into an angle of approach plus an angle of recess on each wheel. Fig. 51 is a diagrammatic representation of the manner of forming the tooth outlines. Point *P* of describing circle *S* traces the face *PC* of the tooth of *B*, and the flank *PD* of the tooth of *A*; while the point *Q* of the circle *T* traces the face *QE* = face *DX* of the tooth of *A*, and the flank *QF* = flank *CY* of the tooth of *B*. We see that in this case the locus of contact of two teeth will be *PMINQ*. The arcs of action *CIF* or *DIE* will be composed of arcs of approach *DI* or *CI*, plus arcs

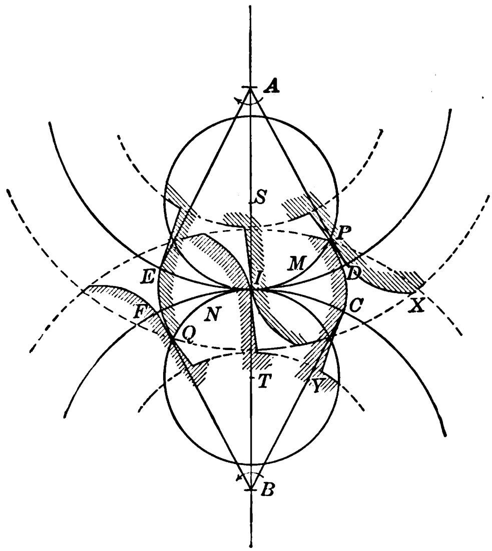


FIG. 51

of recess *IE* or *IF*. The arc of approach depends upon the length of the follower's face only, and that of recess upon the length of the driver's face.

Inside or Annular Wheels.—When one pitch circle lies within the other, the teeth of the larger wheel must be cut on the inside of an annulus or ring. In this case the direction of rotation of the wheels will be the same. The tooth outlines are generated in precisely the same manner as in outside gearing. The flanks of the teeth will be epicycloids, and the faces hypocycloids; in fact, the spaces correspond exactly with the teeth of outside gearing of the same size. Inside gearing will in general run more smoothly than outside gearing, but owing to the difficulty of construction they are seldom used. When rotations in the same direction are desired, an idle wheel is introduced.

Racks.—A rack is simply a portion of a wheel of infinite diameter, hence the pitch curve of a rack is a straight line tangent to the

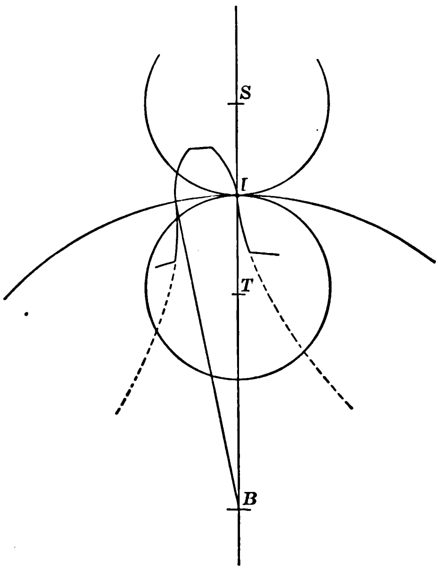


FIG. 52

pitch circle of its mating wheel. Both the faces and flanks of the rack teeth will be cycloids, and if the two describing circles are equal, the faces and flanks will be alike.

Interchangeable Wheels.—If we wish to make a set of wheels any one of which will gear with any other, we must use the same size of a describing circle for all the faces and all the flanks. The size of the describing circle depends on the properties of the hypocycloid. If the diameter of the describing circle is less than one-half that of its pitch circle (Fig. 52), the flanks of any tooth will be less converging than the radii of the pitch circle; if equal to half the diameter (Fig. 53), the flanks will be radial; and if more than half the diameter (Fig. 54), the flanks will be more converging than the radii of the pitch circle. Since converging or even radial flanks weaken the tooth at its root, which is the place where the greatest strength is needed, we should not have the common describing circle of a set of wheels much greater than

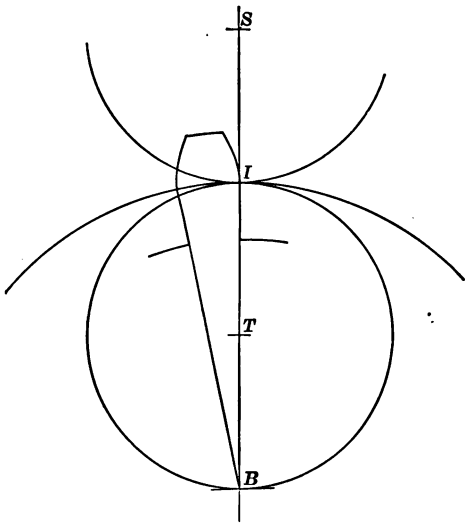


FIG. 53

half the diameter of the smallest wheel of the set. The Brown and Sharpe Mfg. Co. uses a describing circle which is half the diameter of a gear of 15 teeth, and cuts down to 12 teeth. Since the diameter of a wheel of 15 teeth is

$$d=\frac{N}{P}=\frac{15}{P},$$

the diameter of the describing circle will be

$$d'=\frac{15}{2P}.$$

Formulæ for proportioning Teeth (Fig. 55).

Let $N =$ the number of teeth,

$a =$ the addendum,

$t =$ the thickness of the tooth on the pitch curve,

$f =$ the clearance at the bottom of a space,

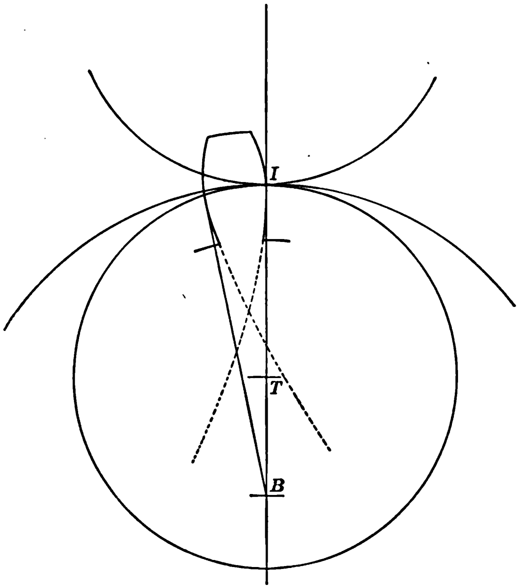


FIG. 54

$d =$ the working depth of a tooth,

$d + f =$ the whole depth of a tooth,

$D =$ the pitch diameter,

$D' =$ the outside diameter,

$P' =$ the circular pitch,

and $P =$ the diametral pitch.

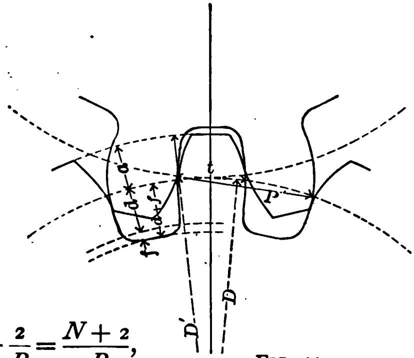


FIG. 55

Then $P = \frac{\pi}{P'}$,

$$P' = \frac{\pi}{P},$$

$$a = \frac{1}{P} = \frac{P'}{\pi} = .3183\,P',$$

$$d = 2\,a,$$

$$P = \frac{N}{D},$$

$$D' = D + 2\,a = \frac{N}{P} + \frac{2}{P} = \frac{N+2}{P},$$

$$t = \tfrac{1}{2}P' = \frac{\pi}{2\,P},$$

$$f = \tfrac{1}{10}t = \tfrac{1}{20}P' = \frac{\pi}{20\,P},$$

$$2\,a + f = d + f = \frac{2}{P} + \frac{\pi}{20\,P} = \frac{2}{P}\left(1 + \frac{\pi}{20}\right) = \frac{2.1571}{P}.$$

NOTE. — P' and t are measured on the arc of the pitch circle, and not on the chord.

In cutting gears with the rotary cutter, the important formulæ are (1), $P = \frac{N}{D}$. This gives the relation between size, pitch, and number of teeth. (2), $D' = \frac{N+2}{P}$, which gives the size to which the blank must be turned, and (3), $d + f = \frac{2.1571}{P}$, which is the depth the cutter must be run into the blank. These are all that are required in simple spur gear cutting. In cut gears the tooth is made equal to the space, and hence the backlash is zero.

Approximate Methods of drawing Teeth.—The most exact method is by using the average radius of curvature of the cycloidal arc and approximating it by means of the arc of a circle. George B. Grant * has constructed a table giving radii and positions of centres, the

* "Odontics, or the Theory and Practice of the Teeth of Gears." Lexington Gear Works, Lexington, Mass.

radius of the describing circle being one-half that of a gear of 12 teeth. This he calls the "Three-point Odontograph." To use the table draw the pitch circles, and divide them up into the tooth intervals. Draw the circle of *face* centres at a tabular distance (dis.) *inside* the pitch circle, and the circle of *flank* centres at a tabular distance *outside* the pitch circle. Draw faces and flanks with tabular radius (rad.). Note must be taken of the algebraic sign of the radius. For 12 teeth the flank radius is infinite, and the flanks are straight. For less than 12 teeth the flanks are convex, and

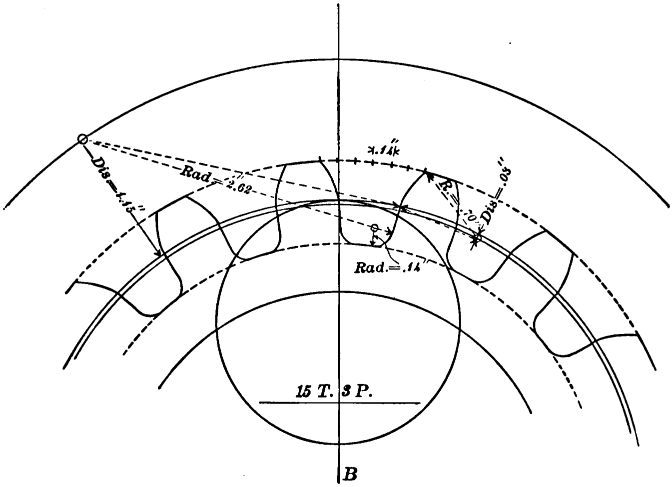


FIG. 56

for more than 12 they are concave. Fig. 56 shows the complete construction for the case of 15 teeth, 3-pitch. No odontographic table, such as is shown above, has been worked out for a system where the describing circle is half the diameter of a gear of 15 teeth, and is used by the Brown and Sharpe Mfg. Co., but sufficiently accurate methods for finding the approximate radius and position of the centre by graphical construction are given in "A Practical Treatise on Gearing," published by the company.

Draughted teeth are used principally by the pattern-maker.

THREE-POINT ODONTOGRAPH

Standard Cycloidal Teeth

From a Pinion of Ten Teeth to a Rack

Number of Teeth		For One Diametral Pitch. For any Other Pitch, divide by that Pitch.				For One-inch Circular Pitch. For any Other Pitch, multiply by that Pitch.			
		Faces		Flanks		Faces		Flanks	
Exact	Interval	Rad.	Dis.	Rad.	Dis.	Rad.	Dis.	Rad.	Dis.
10	10	1.99	.02	—8.00	4.00	.62	.01	—2.55	1.27
11	11	2.00	.04	—11.05	6.50	.63	.01	—3.34	2.07
12	12	2.01	.06	∞	∞	.64	.02	∞	∞
13.5	13–14	2.04	.07	15.10	9.43	.65	.02	4.80	3.00
15.5	15–16	2.10	.09	7.86	3.46	.67	.03	2.50	1.10
17.5	17–18	2.14	.11	6.13	2.20	.68	.04	1.95	.70
20	19–21	2.20	.13	5.12	1.57	.70	.04	1.63	.50
23	22–24	2.26	.15	4.50	1.13	.72	.05	1.43	.36
27	25–29	2.33	.16	4.10	.96	.74	.05	1.30	.29
33	30–36	2.40	.19	3.80	.72	.76	.06	1.20	.23
42	37–48	2.48	.22	3.52	.63	.79	.07	1.12	.20
58	49–72	2.60	.25	3.33	.54	.83	.08	1.06	.17
97	73–144	2.83	.28	3.14	.44	.90	.09	1.00	.14
290	145–300	2.92	.31	3.00	.38	.93	.10	.95	.12
∞	Rack	2.96	.34	2.96	.34	.94	.11	.94	.11

Radius of fillet at the root of tooth may be taken as one-sixth the distance between tips of teeth.

General Expressions for Angles of Action. — Let A and B (Fig. 57) be the centres of two pitch circles whose radii are R_1 and R_2. Let r_1 and r_2 be the radii of the corresponding describing circles. Draw the addendum circles KL and GH, where a_1 and a_2 are the addenda. Suppose the wheels revolve so that the teeth move from right to left, with A as driver. We are to find the value of α, the angle of approach of A. The point P, at the intersection of the describing circle and the addendum circle, is the first point of contact. Draw PS and PB. Then in the triangle PSB, all the sides are known, for $PS=r_1$, $SB=R_2+r_1$, and $PB = R_2 + a_2$. In any such triangle we can find the angle θ from

$$\tan \tfrac{1}{2}\theta = \sqrt{\frac{\{x - (R_2 + r_1)\}\{x - r_1\}}{x\{x - (R_2 + a_2)\}}},$$

where x is half the sum of the three sides, or

$$x = R_2 + r_1 + \frac{a_2}{2}.$$

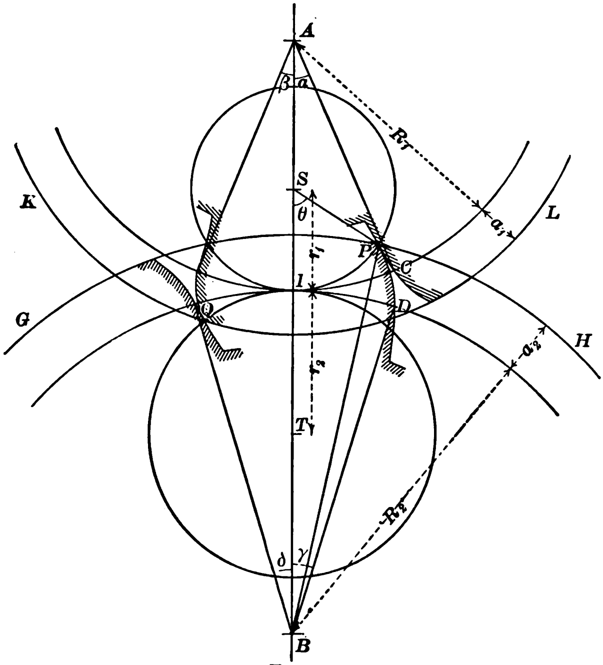


FIG. 57

Hence, $$\theta = 2 \tan^{-1} \sqrt{\frac{\frac{a_2}{2}\left(\frac{2R_2 + a_2}{2}\right)}{\left(\frac{2R_2 + 2r_1 + a_2}{2}\right)\left(\frac{2r_1 - a_2}{2}\right)}},$$

$$\theta = 2 \tan^{-1} \sqrt{\frac{a_2(2R_2 + a_2)}{(2R_2 + 2r_1 + a_2)(2r_1 - a_2)}}.$$

Since $\text{arc } ID = \text{arc } IP = \text{arc } IC,\quad \alpha = \theta\frac{r_1}{R_1},\quad \gamma = \theta\frac{r_1}{R_2}.$

Hence,
$$\alpha = \frac{2\,r_1}{R_1}\tan^{-1}\sqrt{\frac{a_2(2\,R_2 + a_2)}{(2\,R_2 + 2\,r_1 + a_2)(2\,r_1 - a_2)}}.$$

This is the most general expression for the angle of approach of the driver. For standard teeth the following arbitrary proportions can be substituted:

$$a_1 = a_2 = \frac{1}{P}.$$

$$r_1 = r_2 = \frac{15}{4\,P}.$$

Substituting these values, we get

$$\alpha = \frac{15}{2\,PR_1}\tan^{-1}\sqrt{\frac{\frac{1}{P}\left(2\,R_2 + \frac{1}{P}\right)}{\left(2\,R_2 + \frac{15}{2\,P} + \frac{1}{P}\right)\left(\frac{15}{2\,P} - \frac{1}{P}\right)}}.$$

And since $2\,PR = N$, the number of teeth,

$$\alpha = \frac{15}{N_1}\tan^{-1}\sqrt{\frac{4(N_2 + 1)}{13(2\,N_2 + 17)}}.$$

In exactly the same way we get the angle of recess of the follower:

$$\delta = \frac{15}{N_2}\tan^{-1}\sqrt{\frac{4(N_1 + 1)}{13(2\,N_1 + 17)}}.$$

Since $\gamma = \alpha\frac{R_1}{R_2}$, the angle of approach of the follower is

$$\gamma = \frac{15}{N_2}\tan^{-1}\sqrt{\frac{4(N_2 + 1)}{13(2\,N_2 + 17)}}.$$

And finally,

$$\beta = \frac{15}{N_1}\tan^{-1}\sqrt{\frac{4(N_1 + 1)}{13(2\,N_1 + 17)}}.$$

In all cases the arcs of action, which are $R_1(\alpha + \beta) = R_2(\gamma + \delta)$, must be greater than the circular pitch. When the standard addendum and describing circle are used, the least number of teeth on a wheel is not decided by the above conditions, but by the undercutting which occurs when the pitch circle becomes less than twice the describing circle. When, however, the addenda and describing circles are functions of their respective pitch circles, then definite limiting cases can be worked out. For example, let $r_1 = KR_1$, $r_2 = CR_2$, $a_1 = pR_1$, $a_2 = qR_2$, where K, C, p, and q are known constants. Then the angle of C is

$$\alpha + \beta = 2\,K \tan^{-1}\sqrt{\frac{qR_2^2(2+q)}{(2\,R_2 + 2\,KR_1 + qR_2)(2\,KR_1 - qR_2)}}$$

$$+ 2\,C\frac{R_2}{R_1}\tan^{-1}\sqrt{\frac{pR_1^2(2+p)}{(2\,R_1 + 2\,CR_2 + pR_1)(2\,CR_2 - pR_1)}}.$$

The limiting case will be when $\alpha + \beta$ equals the pitch angle, or $\frac{2\pi}{N}$, and in general it cannot be less. Hence, writing the inequality, and remembering that $R_1 = \frac{N_1}{2P}$, etc.,

$$KN_1 \tan^{-1}\sqrt{\frac{qN_2^2(2+q)}{\{N_2(2+q) + 2\,KN_1\}\{2\,KN_1 - qN_2\}}}$$

$$+ CN_2 \tan^{-1}\sqrt{\frac{pN_1^2(2+p)}{\{N_1(2+p) + 2\,CN_2\}\{2\,CN_2 - pN_1\}}} > \pi.$$

Having given all quantities except N_2, this latter can be found by a process of trial and error as that whole number giving the nearest result larger than π. The still further limitation must then be investigated as to whether the epicycloids forming the two faces of a tooth do or do not cross before reaching the addendum circle. The whole problem of limiting numbers is one which can best be studied by graphical trial processes.

(*b*) *The Involute System*

The simplest and in many respects the best curve for the outline of a tooth is the involute of a circle (Fig. 58). In this system the curve of action is a straight line passing through the pitch point. Here again the profile for the special case where the pitch curve is a straight line can be deduced very simply from the general law. Here $x = f(y)$ is the equation of the profile, and that of the curve of action is $x = \phi(y) = \frac{y}{m}$, m being equal to the tangent of θ (Fig. 59). Hence,

$$\frac{y}{x} = \frac{y}{\frac{y}{m}} = -\frac{dx}{dy}, \quad \frac{dy}{dx} = -m,$$

$$y = -mx + C,$$

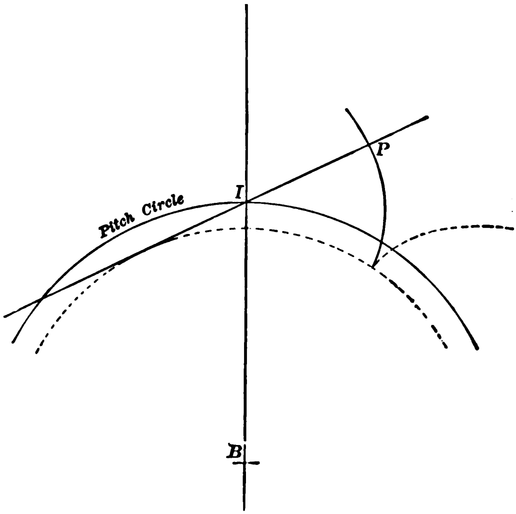


FIG. 58

which is the equation of a straight line at right angles to the curve of action, C being the arbitrary parameter giving the position of the tooth.*

Again resorting to simpler methods of procedure, imagine a reversal of relative motion, thus bringing the pitch curve and profile to rest, and rotating the curve of action about the axis of the

* For the solution of the general case, see Appendix IV.

wheel. Then, as any motion of the curve of action in the direction of its own length will not affect its intersection with the profile, we may consider it as rolling upon a circle drawn concentric with the pitch circle and tangent to the curve of action. (See Fig. 58.) The part of any point of the line will be an involute of this inner circle. The curve of action itself is a normal to the involute, and as it always passes through the point I, the tooth law is satisfied. The

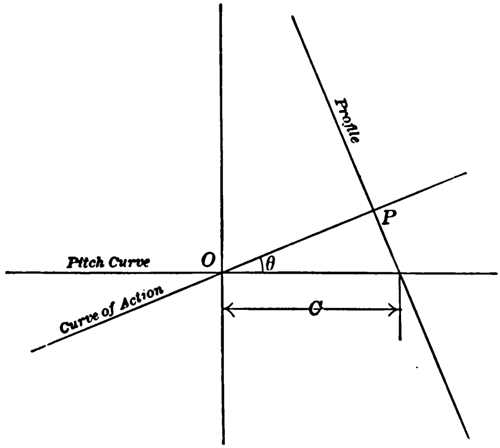


FIG. 59

relative positions of two profiles are shown in Fig. 60. In this let A and B be the centres of two pitch circles tangent at I. Through I draw a line WW making an angle θ with the common tangent to the circles. From A and B drop perpendiculars AE and BF upon WW, and with these as radii draw circles concentric with the pitch circles, and therefore tangent to the line of action. Now suppose the line WW to be pushed in the direction of its length so as to drive circle CEO by friction at E, and circle DFQ by friction at F. Then will the pitch circles roll upon one another also, for vel. $I =$ vel. $E \dfrac{IA}{EA}$ when considered a point of A, and vel. $I =$ vel. $F \dfrac{IB}{FB}$ when considered a point of B. But $\dfrac{IA}{EA} = \dfrac{IB}{FB}$, and vel. $E =$ vel. F. Hence vel. I when considered a point of A is equal to vel. I when considered a point of B, or the pitch circles roll at I. Now if we consider any point P of the line

WW, it will sweep up an involute *GPH* of the circle *OEC* in the plane of *A*, and an involute *MPN* of the circle *DFQ* in the plane of *B*. The curves will always be tangent at the common generating point *P*, and their common normal will be the line *WW*, which always passes through a fixed point *I* on the line of centres. The circles *OEC* and *QFD* are called Base Circles.

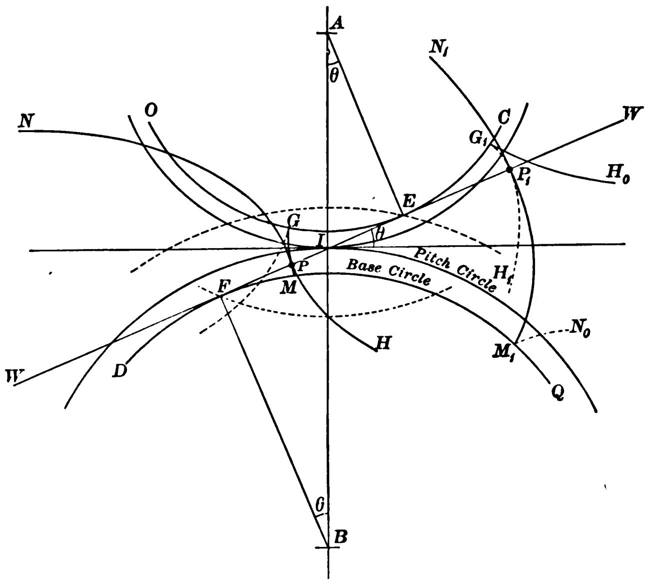


FIG. 60

In involute teeth the addenda *a* cannot be of greater lengths than those which cause the last points of contact to occur at *E* and *F*. In other words, the radii of the addendum circles cannot be greater than *BE* and *AF*. In this case the action ends at the tips and at the roots (*i.e.* at the point where the involute comes down to the base circle) of the teeth. If the tooth of *B*, for instance, were made any longer, action would continue beyond the root of *A*, that is along the second branch G_1H_1 of the involute beyond the cusp as shown at P_1 in Fig. 60. Hence we see that the maximum line of action is EF. If the tooth were made longer than the theory would indicate, not only would correct angular velocity transmission end at E, but the prolonged tooth would actually interfere with the other. This arises from the fact that the curvatures of the portions mathematically in working contact are in the same direction, and the radius of curvature of the tooth of B being FP_1, while that of A is only EP_1, the latter would lie within the former, as would also the cusp of A at G_1.

A valuable feature of the involute tooth outline is that which allows the shafts of the two wheels to be separated slightly without affecting the constancy of the angular velocity ratio. In this case the base circles remain the same as before, and hence their involutes are unchanged. What is changed by the separation of the axes is the pitch circle. Since the ratio of the radii of these always equals the ratio of the radii of the base circles, the former ratio will remain unchanged. It is readily seen that the angle θ, or the "obliquity," is changed also. Since the new pitch circles do not cut the teeth in the same positions as the old, the tooth will not equal the space on any pitch circle except the one originally designed, but there will be back lash. Care must be taken that the shafts are not so far separated as to make the angle of action less than the pitch angle.

Standard Involute Tooth.—The draughted tooth is usually one having an angle of obliquity of 15°. Other proportions are the same as for cycloidal teeth. The Brown and Sharpe Mfg. Co. makes its cutters with an obliquity of 14½°. If the obliquity is 15°, and the standard addendum $\frac{1}{P}$ is used, it will be found too long to work without interference on anything under 20 teeth on equal wheels. Therefore the involute is used as far as possible, and the remainder of the tooth outline is made an epicycloid to work on a radial flank within the base circle. This is known as the correction for interference.

The Involute Odontograph.—In the involute odontograph of George B. Grant the centres are taken on the base circles. These latter may be drawn tangent to a line of action with an obliquity

THREE-POINT ODONTOGRAPH

Standard Involute Teeth

Obliquity 15°

Number of Teeth	For One Diametral Pitch. For any Other Pitch, divide by that Pitch.		For One-inch Circular Pitch. For any Other Pitch, multiply by that Pitch.	
	Face Rad.	Flank Rad.	Face Rad.	Flank Rad.
10	2.28	.69	.73	.22
11	2.40	.83	.76	.27
12	2.51	.96	.80	.31
13	2.62	1.09	.83	.34
14	2.72	1.22	.87	.39
15	2.82	1.34	.90	.43
16	2.92	1.46	.93	.47
17	3.02	1.58	.96	.50
18	3.12	1.69	.99	.54
19	3.22	1.79	1.03	.57
20	3.32	1.89	1.06	.60
21	3.41	1.98	1.09	.63
22	3.49	2.06	1.11	.66
23	3.57	2.15	1.13	.69
24	3.64	2.24	1.16	.71
25	3.71	2.33	1.18	.74
26	3.78	2.42	1.20	.77
27	3.85	2.50	1.23	.80
28	3.92	2.59	1.25	.82
29	3.99	2.67	1.27	.85
30	4.06	2.76	1.29	.88
31	4.13	2.85	1.31	.91
32	4.20	2.93	1.34	.93
33	4.27	3.01	1.36	.96
34	4.33	3.09	1.38	.99
35	4.39	3.16	1.39	1.01
36	4.45	3.23	1.41	1.03
37–40	4.20		1.34	
41–45	4.63		1.48	
46–51	5.06		1.61	
52–60	5.74		1.83	
61–70	6.52		2.07	
71–90	7.72		2.46	
91–120	9.78		3.11	
121–180	13.38		4.26	
181–360	21.62		6.88	

In all cases the centres are on the base circles. Draw rack by special method described in text.

of 15°, or they may be drawn within the pitch circles at distances equal to $.017\,\frac{N}{P}$. It is necessary to round off the points of the teeth in some cases to prevent interference. This makes it impossible to compute the positions of the centres, hence careful draughting is resorted to. The involute is plotted, corrected for interference, and the circle most nearly coinciding with it is found by trial. This is done on a very large scale so as to reduce

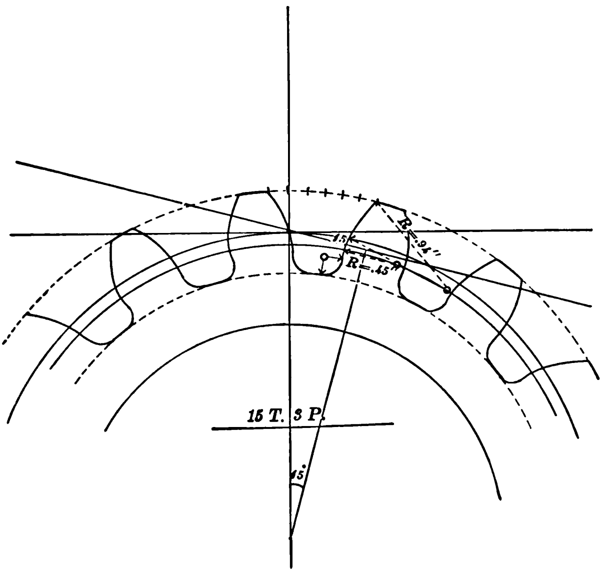


FIG. 61

the errors to a minimum. Separate curves are required for face and flank up to 36 teeth, but above that one curve is sufficient. The sides of the rack teeth are straight lines drawn at an angle of 15° with the perpendicular to the pitch line, but the tips must be corrected for interference by rounding them off. Hence draw the *outer half* of the rack *face* from a centre on the pitch line with a radius equal to 2.1 divided by the diametral pitch or .67 multiplied by the circular pitch. An example is shown in Fig. 61.

Angles of Action. — Let A and B (Fig. 62) be the centres of two pitch circles, whose radii are R_2 and R_1. The radii of the corresponding base circles are r_2 and r_1, and the addenda are a_2 and a_1. The obliquity is θ. Action will begin at P, where the addendum

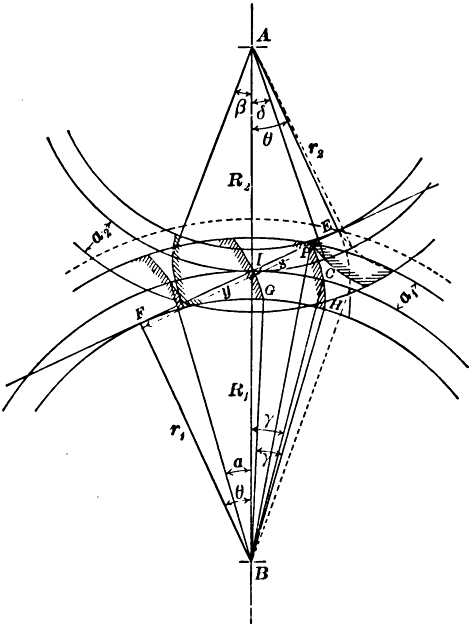


FIG. 62

circle of B cuts the line of action. The angle of approach of B is $IBC = \gamma$, and it can be calculated as follows:

$$(R_1 + a_1)^2 = (y + s)^2 + r_1^2,$$

$$y + s = \sqrt{(R_1 + a_1)^2 - r_1^2},$$

$$s = \sqrt{(R_1 + a_1)^2 - r_1^2} - y$$

Or

$$s = \sqrt{(R_1 + a_1)^2 - R_1^2 \cos^2 \theta} - R_1 \sin \theta.$$

But since $IP = s = GH$,

$$\gamma = \frac{s}{r_1}.$$

Hence, $$\gamma = \frac{\sqrt{(R_1 + a_1)^2 - R_1^2 \cos^2 \theta} - R_1 \sin \theta}{R_1 \cos \theta}.$$

In the same way the angle of recess of A is

$$\beta = \frac{\sqrt{(R_2 + a_2)^2 - R_2^2 \cos^2 \theta} - R_2 \sin \theta}{R_2 \cos \theta}.$$

Since $\alpha = \gamma \frac{R_1}{R_2}$, the angle of recess of B is

$$\alpha = \frac{\sqrt{(R_1 + a_1)^2 - R_1^2 \cos^2 \theta} - R_1 \sin \theta}{R_2 \cos \theta},$$

and that of approach of A is

$$\delta = \frac{\sqrt{(R_2 + a_2)^2 - R_2^2 \cos^2 \theta} - R_2 \sin \theta}{R_1 \cos \theta}.$$

If we use the standard values for a and θ, *i.e.* $a = \frac{1}{P}$ and $\theta = 15^\circ$, then

$$\gamma = \frac{\sqrt{\left(R_1 + \frac{1}{P}\right)^2 - .933\, R_1^2} - .2588\, R_1}{.966\, R_1}.$$

The above does not consider the effect of interference. The angles of approach and recess as limited thereby can be simply deduced as follows. In this case $s = R_2 \sin \theta$, and

$$\gamma = \frac{s}{r_1} = \frac{R_2 \sin \theta}{R_1 \cos \theta} = \frac{R_2}{R_1} \tan \theta.$$

Similarly, $$\beta = \frac{R_1}{R_2} \tan \theta.$$

And as $$\alpha = \gamma \frac{R_1}{R_2},$$

$$\alpha = \tan \theta, \qquad \delta = \tan \theta.$$

(*c*) *The Pin-tooth System*

The pin-tooth system, though totally different from either of those previously described, can be deduced from the cycloidal system. Suppose in this case one describing circle becomes equal in diameter to the pitch circle within which it lies, while the other is omitted. Since there is but one describing circle, the teeth will consist of merely epicycloidal faces on one wheel, and hypocy-

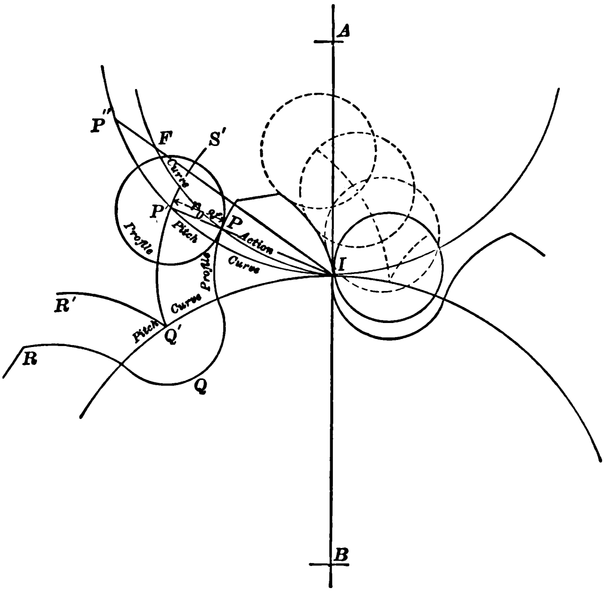


FIG. 63

cloidal flanks on the other. But the hypocycloidal portions will vanish, these reducing in fact to a point which is the describing point itself. This point will then work correctly with the epicycloid of the other wheel. The combination of P' with $R'Q'S'$ (Fig. 63) is a true cycloidal pair, the line IP being the normal to both. A mathematical line at P' would drive the epicycloid as far as I, but beyond I there would be no positive driving. If now in place of P' we put a cylindrical pin of radius r_0, the line IP' will always be normal to the pin. Finally if we replace the epicycloid by a curve PQ at a constant normal distance from it, the line IP will be normal to this also. Hence the combination of the cylindrical pin and the parallel to the epicycloid will work as a pair of gear teeth, the common normal always passing through I.

The proof of the correct action of teeth in this system can also be studied by imagining the outlines as being formed by the "conjugate method." It is evident that if any form of tooth outline be assumed for the pinion, a corresponding correct tooth outline

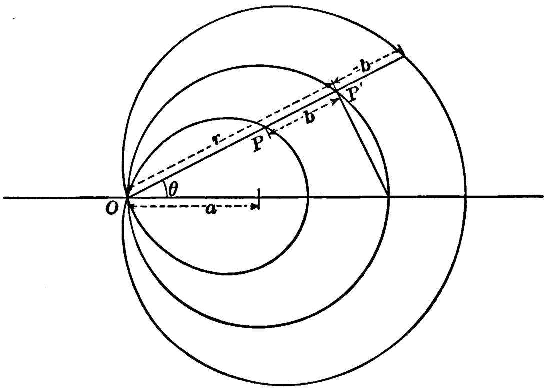


FIG. 64

can be devised for the wheel, if only the mathematical conditions are to be fulfilled. For if the required angular velocity ratio take place, this outline will be the envelope of all positions of the assumed outline when referred to the wheel. Assuming one tooth as a circle, the other must be the parallel to the curve traced out by that circle's centre (Fig. 63).

The point of contact of the pin and tooth lies on the line $P'I$ at the point P. The curve of action will be the curve IPF. This curve is evidently the locus of a point of a line which moves with one extremity on a circle, and always passes through a point on the circumference of the circle. P' is the moving point, P the tracing point, and I the fixed point. The distance $PP' = FP'' =$ constant $= r_0$ is the radius of the pin. Such a curve is known as a limacon (see Fig. 64), and was first investigated by Pascal. If we take I as the pole, and the diameter through I as the reference line, the equation of the limacon will be

$$r = 2\,a\,\cos\theta - b.$$

It is evident that action will take place wholly upon one side of the line of centres, and hence will be wholly approaching or receding, depending upon whether the pin wheel is driving or following. Since the friction of approach is greater than that of recess, it is best to give the pins to the follower.

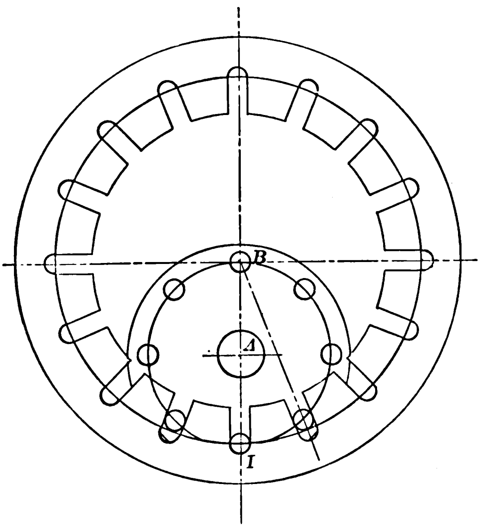


FIG. 65

In the case of inside or annular gears, the tooth curves will be parallels to the hypocycloid. One case is of particular interest; namely, when the pin wheel is half the diameter of the gear. The tooth outlines will then be parallel to the straight line hypocycloid, or will themselves be straight (Fig. 65).

If the pins are set on the circumference of the pitch circle itself, the action will be defective while the line of centres is being passed. This is best seen by considering the epicycloid as being generated by a point of a line which rolls upon its evolute. Let *Oabcd*, etc., be the epicycloid, and OX its evolute (Fig. 66). Then the parallel to the epicycloid is generated by another point of the same rolling line. But this latter will come down to the evolute at b', while the point tracing the epicycloid is still at b, and by the time the latter reaches O the parallel curve will have risen again to O', forming a cusp at b'. So the curve $O'a'b'c'd'e'f'$ is the parallel to the epicycloid $Oabcdef$. But the branch $O'b'$ would work on internal contact with the pin when near the cusp, and hence is impracticable. The only way to prevent this failure of action is to use some other curve than the epicycloid to parallel. If the pins are set a suffi-

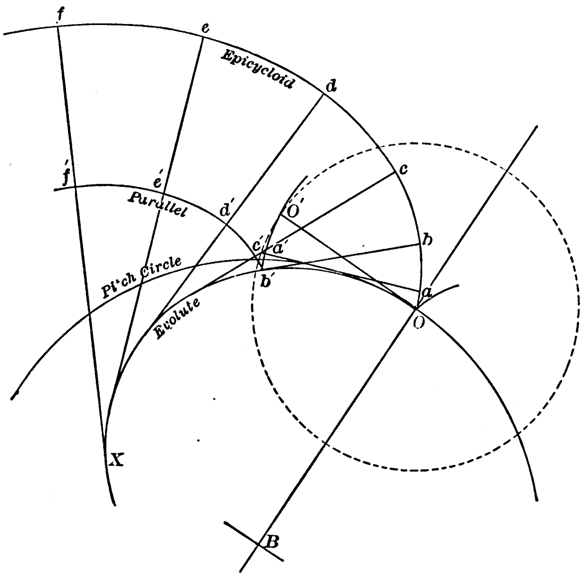


FIG. 66

cient distance within the pitch circle, the resulting curve, the epitrochoid, can be paralleled without a cusp. (See Fig. 67.)

Approximate Formula for Angles of Action. — In Fig. 68 the first point of contact will be where the addendum circle cuts the limacon IF. The circumference of the pin, the tip of the tooth, and the line QI also pass through this point. Construct the tooth curve PC. Then angle $IBC = \alpha$ is the angle required. Let r equal the radius of the pitch circle of A, R that of B, r_0 the radius of the pin, and a the addendum. The irregular action at I is disregarded, the mathematical action only being considered.

$$s = BP = R + a,$$
$$\beta = \text{angle } QIA,$$
$$\epsilon = \text{angle } QIB,$$
$$\beta + \epsilon = 180°,$$
$$\cos \tfrac{1}{2} \epsilon = \cos \tfrac{1}{2}(180° - \beta) = \sin \tfrac{1}{2} \beta.$$

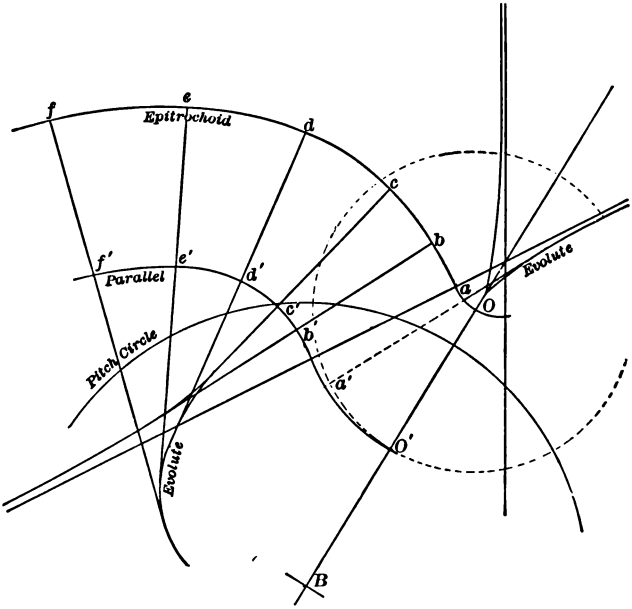


FIG. 67

Call $IP = y$. Then $IQ = y + r_0$, also $PB = s$, $BI = R$, $IA = QA = r$. Then

$$\cos \tfrac{1}{2} \epsilon = \sqrt{\frac{x(x-s)}{yR}},$$

where $$x = \tfrac{1}{2}(R + y + s);$$

hence, $$\cos^2 \tfrac{1}{2} \epsilon = \frac{\tfrac{1}{4}(R+y+s)(R+y-s)}{yR} = \sin^2 \tfrac{1}{2} \beta.$$

But we also have

$$\sin \tfrac{1}{2}\beta = \sqrt{\frac{(x'-r)(x'-y-r_0)}{r(r_0+y)}},$$

where $$x' = \tfrac{1}{2}(r+r+r_0+y);$$

hence, $$\sin^2 \tfrac{1}{2}\beta = \frac{\tfrac{1}{4}(2r-r_0-y)}{r}.$$

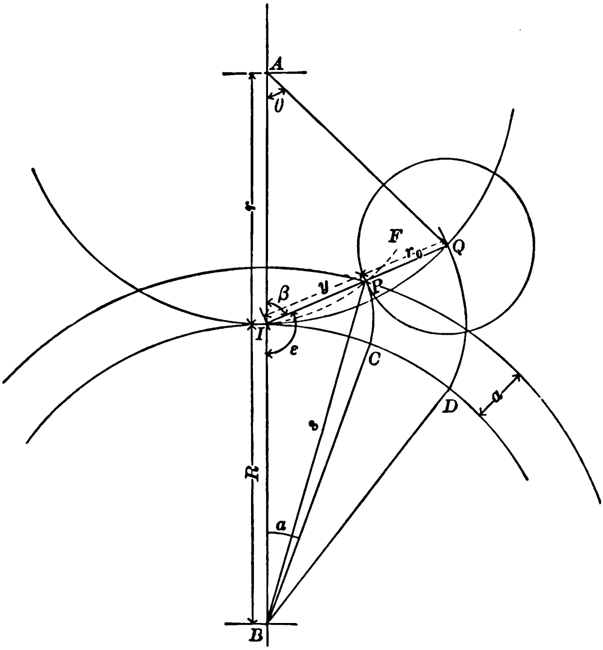


FIG. 68

Equating values of $\sin^2 \frac{1}{2}\beta$, we get

$$\frac{(R+y+s)(R+y-s)}{yR} = \frac{2r-r_0-y}{r}.$$

Reduction gives

$$y = \frac{-Rr_0}{2(R+r)} + \sqrt{r\frac{s^2-R^2}{r+R} + \frac{R^2 r_0^2}{4(r+R)^2}},$$

which is in terms of known quantities.

Now $\sin \frac{1}{2}\theta = \frac{\frac{1}{2}(y+r_0)}{r}$ or $\theta = 2\sin^{-1}\left\{\frac{\frac{1}{2}(y+r_0)}{r}\right\}$.

$$\text{Arc } QI = r\theta = \text{Arc } ID,$$

$$\text{Arc } IC = \text{Arc } ID - r_0 \text{ (nearly)}.$$

Hence, $\text{Arc } IC = r\theta - r_0$ (nearly).

$$\alpha = \frac{2r\sin^{-1}\left\{\frac{\frac{1}{2}(y+r_0)}{r}\right\} - r_0}{R}.$$

$$\alpha = \frac{2r\sin^{-1}\left\{\frac{\frac{1}{2}\left[\frac{-Rr_0}{2(R+r)} + \sqrt{r\frac{s^2-R^2}{r+R} + \frac{R^2r_0^2}{4(R+r)^2}} + r_0\right]}{r}\right\} - r_0}{R}.$$

(*d*) *Special Forms of the Above, Twisted Gears*

If a pair of spur gears be divided by a series of parallel planes perpendicular to the axes of rotation into a set of thinner gears, and if each pair of these on both wheels be given an angular displacement with respect to the pair preceding, we will have what is called a pair of stepped gears (Fig. 69). They will, of course, work together exactly as did the original wheels, but with the advantage that the number of teeth passing the line of centres during a given angular displacement will be increased as many times as there are laminæ. In this way the effective number of teeth may be increased

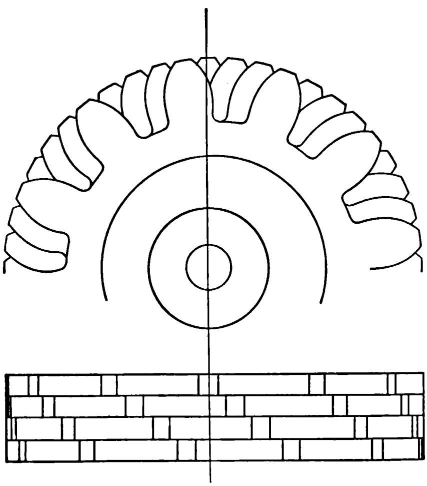


FIG. 69

without reducing the size of a tooth. Since, when the point of contact between two teeth is passing the line of centres, driving is secured with pure rolling, the pair of teeth is at that instant acting at its best. Hence by subdividing a gear in this way the action will be smoother, while the strength will not be impaired.

If the number of laminæ is made infinite, the effect will be the same as if we gave the gear a uniform twist throughout its length

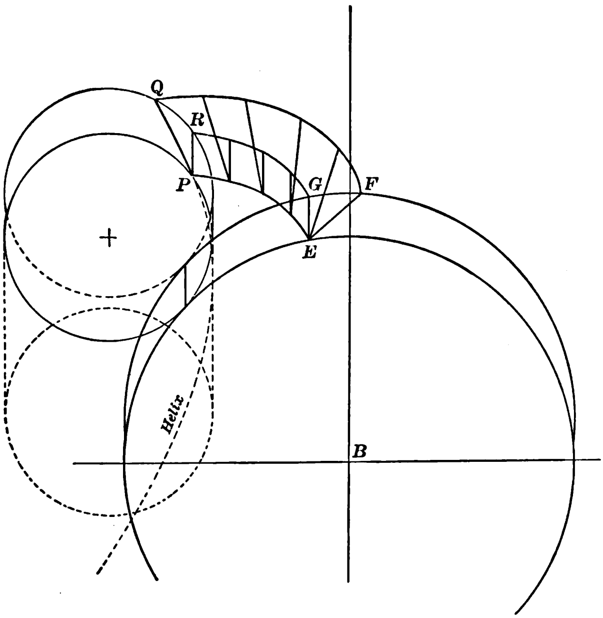


FIG. 70

and about its axis of rotation. These are called Twisted Gears, and are much used on account of their smooth action. The tooth surface can also be imagined as swept up in a manner identical with those already mentioned. We have so far considered the whole subject of spur gearing as a problem in plane geometry, but when successive sections of our gear differ either in form or in position, we must employ the geometry of three dimensions.

According to this last the cycloidal tooth would be a surface generated by an element *PR* (Fig. 70) of a right circular cylinder, rolling within and without a pitch cylinder, and the involute tooth a surface swept up by a line *PR* of a plane (Fig. 71), rolling between base cylinders, the generating line being parallel to the line of contact *TS* between the plane and the base cylinders.

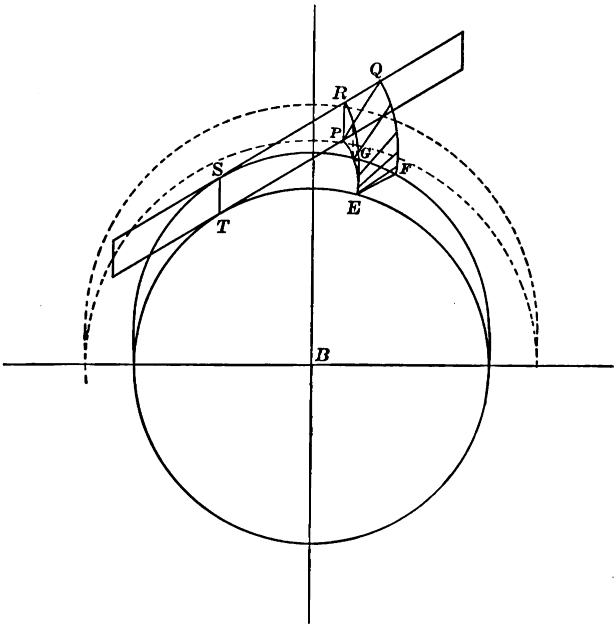


FIG. 71

Now in the twisted gear the tooth surface is swept up by a helix *PQ* (Fig. 70) of uniform pitch on the describing cylinder, where cycloidal teeth are considered, or by a line *PQ* (Fig. 71) of a plane, the generating line being oblique to the plane's lines of contact with the base cylinders, in the involute system. In either case it will be seen that the elements will no longer be straight lines but helices, and the tooth surfaces more or less complicated helicoids.

In such a gear the sections of the tooth made by planes normal to the axes of rotation, such as *CD* (Fig. 72), will give true profiles

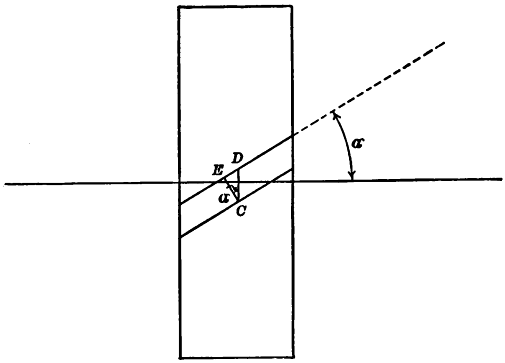


FIG. 72

according to any of the systems already mentioned; but if cut on a milling machine with a rotary cutter, the form of this cutter should not be that of the true tooth, but of the normal section of the tooth *EC*, perpendicular to its helical elements. Such teeth are usually

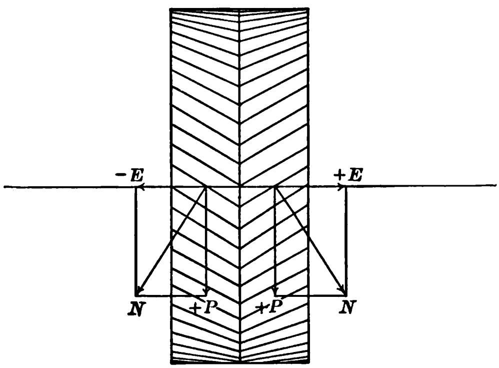


FIG. 73

cut with the same cutters as are used in cutting spur gears, and the resulting tooth form cannot be exactly correct. However, the difference is so slight as to be inappreciable, and the action is in all cases extremely smooth, for there is always a point of contact on the line of centres, provided the twist displacement is at least equal to the circular pitch.

In a plain twisted gear the normal pressure between two teeth can be resolved into a useful component tangent to the pitch surface and normal to the plane of the axes, and a useless component of end pressure. The components of end pressure can be neutralized by placing two twisted gears on the same axis with equal twists in opposite directions. (See Fig. 73.)

(*e*) *Strength of Spur Teeth*

The actual load which is applied to a tooth at the instant of rupture is the same as that applied to one end of a beam which is fixed at the other. If w is the width, d the depth, and l the length of such a beam,

$$F = K\frac{wd^2}{l},$$

where F is the breaking load and K a constant of the material. In the case of the tooth, $w = f$, the "face" of the gear, or the length of the tooth parallel to the axis,

$$d = \frac{P'}{2} = \frac{\pi}{2P}, \text{ and } l = 2a = \frac{2}{P}.$$

Hence,
$$F = K\frac{f\pi^2 P}{4P^2 \times 2} = C\frac{f}{P}.$$

The value of C is given by Grant as 11,000, but for actual running conditions a factor of safety of 10 is introduced, so that the working load is

$$F' = 1100\frac{f}{P}.$$

This is for rough cast gears, where the whole load may be borne by one tooth. If two teeth are always in contact, it may be safe to allow twice the working load. This can only be done when the tooth outlines are correctly formed.

The horse-power which a gear will transmit can best be computed from the following empirical formula:

$$H.\ P. = .12\,\frac{f\sqrt{dn}}{P^2},$$

where f is the "face" of the wheel, d the pitch diameter in inches, and n the number of revolutions per minute. This is for cast gears; for the best cut gears we allow

$$H.\ P. = .40\,\frac{f\sqrt{dn}}{P^2}.$$

C. Velocity Ratio Variable, Non-circular Wheel

The most general practical case is that of two curves of any form so related that they roll upon one another while turning about fixed centres. If one curve is chosen of definite form, the mating curve can always be constructed by the method of Fig. 74. Let *FE* be any curve rotating about *B*. Let *A* be the other centre. *I* is the intersection of the given curve with the line of centres. Step off small arcs *Ia*, *aa*, *aa*, *aa*, etc., along *FE*. With centre *B* draw circular arcs *ac*, *ac*, *ac*, etc., intersecting *AB* in *c*, *c*. With radius *Ac* draw circular arcs *cb*, *cb*, *cb*, and lay off distances *Ib*, *bb*, *bb*, equal to *Ia*, *aa*, *aa*. Then will *Ibbbb* be the required curve. If the

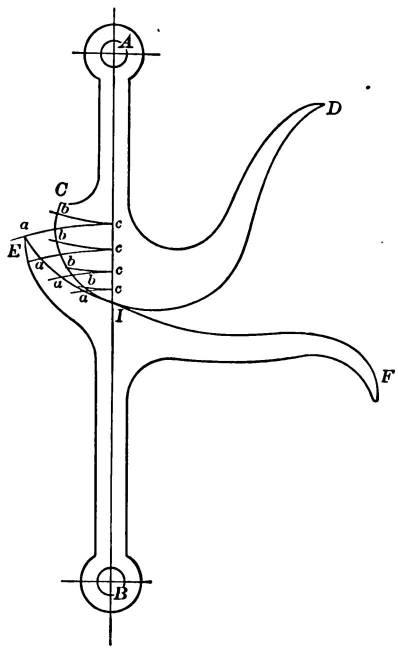


FIG. 74

curve FE is closed, the curve CD is not necessarily closed, but A can be chosen so that it will close.

(a) *Elliptic Gears*

The only pair of like closed curves which will roll upon one another while rotating about fixed centres, and make a complete and practical revolution, is a pair of equal ellipses. Let Fig. 75 represent two such ellipses, with foci at A and C, and at B and D. Suppose them to have been originally tangent at E and F (the extremities of the major axes), and to have rolled until tangent at I. Draw IA, IB, IC, and ID. Arc EI equals arc FI, and since the ellipses are exactly alike, $CI = BI$, and $AI = DI$. Also $AC = DB$. Hence triangle AIC = triangle DIB, and angle AIC = angle DIB. But by the property of the ellipse, angle $BIT = DIT_1$, where TIT_1 is the common tangent at I. Also by the equality of the ellipses, angle BIT = angle CIT. Hence, as the sum of the angles about I on each side of AIB or CID are equal, these latter must be straight lines. Furthermore by the property of the ellipse $CI + IA = GE$. But $DI = AI$. Hence $DC = GE$ constant. So if we pivot the two ellipses at their two foci A and B, at a distance apart equal to their major axes, they will roll together without slipping. The free foci may, if required, be connected by a link, since $CD = AB = GE$.

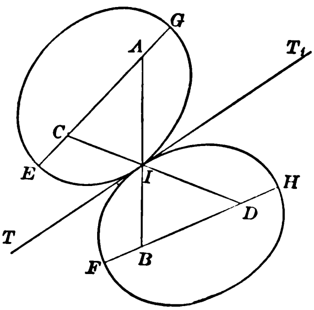


FIG. 75

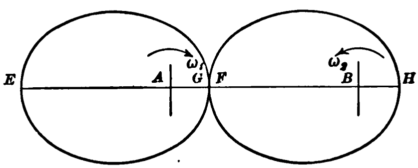


FIG. 76

Let the ellipse whose axis is A be the driver, turning with a constant angular velocity ω_1. When the ellipses are in the position shown in Fig. 76, we will have

$$\xi = \frac{\omega_1}{\omega_2} = \frac{BF}{AG} = \frac{(1+e)\,a}{(1-e)\,a} = \frac{1+e}{1-e},$$

where a is the semi-axis major, or $\frac{1}{2}\,GE$, and where e is the eccentricity. When the ellipses are as in Fig. 77, we have

$$\eta = \frac{\omega_1}{\omega_3} = \frac{BH}{AE} = \frac{(1-e)\,a}{(1+e)\,a} = \frac{1-e}{1+e}.$$

Hence, $$\frac{\xi}{\eta} = \frac{\omega_3}{\omega_2} = \frac{(1+e)^2}{(1-e)^2} = Z\,.$$

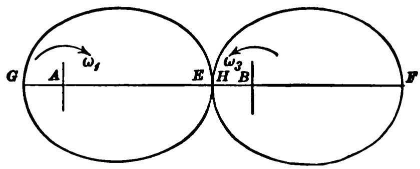


FIG. 77

This quantity Z is generally the one assumed in the design of a pair of elliptic gears. It is the ratio between the maximum and minimum angular velocities of the follower when the speed of the driver is constant. From the last equation we can find the value of e from the given value of Z, for it gives

$$e = \frac{\sqrt{z}-1}{\sqrt{z}+1}.$$

In some cases we assume the ratio of minor to major axes, or $\frac{b}{a} = K$. Then e is calculated from its defining equation, viz., $b^2 = a^2\,(1-e^2)$, which gives

$$e = \sqrt{1 - \frac{b^2}{a^2}} = \sqrt{1-K^2}.$$

In addition to Z or K we may assume the number of teeth, and the size of a tooth, that is the circular or diametral pitch. Let s represent the length of an elliptic quadrant. Then

$$\frac{4s}{N} = P' = \frac{\pi}{P},$$

$$s = \frac{\pi N}{4P},$$

TABLE OF COMPLETE ELLIPTIC FUNCTIONS, $E\left(e, \frac{\pi}{2}\right)$.

e	E	e	E	e	E	e	E
.00	1.5708	.25	1.5459	.50	1.4674	.75	1.3183
.01	1.5707	.26	1.5439	.51	1.4630	.76	1.3102
.02	1.5706	.27	1.5418	.52	1.4585	.77	1.3020
.03	1.5704	.28	1.5395	.53	1.4539	.78	1.2936
.04	1.5701	.29	1.5371	.54	1.4493	.79	1.2852
.05	1.5698	.30	1.5347	.55	1.4445	.80	1.2762
.06	1.5694	.31	1.5322	.56	1.4395	.81	1.2671
.07	1.5689	.32	1.5297	.57	1.4344	.82	1.2578
.08	1.5683	.33	1.5271	.58	1.4292	.83	1.2482
.09	1.5676	.34	1.5244	.59	1.4238	.84	1.2381
.10	1.5668	.35	1.5216	.60	1.4182	.85	1.2277
.11	1.5660	.36	1.5187	.61	1.4125	.86	1.2172
.12	1.5651	.37	1.5156	.62	1.4066	.87	1.2063
.13	1.5642	.38	1.5124	.63	1.4006	.88	1.1950
.14	1.5613	.39	1.5092	.64	1.3944	.89	1.1833
.15	1.5620	.40	1.5058	.65	1.3881	.90	1.1712
.16	1.5607	.41	1.5024	.66	1.3817	.91	1.1586
.17	1.5594	.42	1.4989	.67	1.3753	.92	1.1455
.18	1.5580	.43	1.4952	.68	1.3688	.93	1.1318
.19	1.5565	.44	1.4918	.69	1.3622	.94	1 1175
.20	1.5550	.45	1.4881	.70	1.3544	.95	1.1023
.21	1.5533	.46	1.4842	.71	1.3484	.96	1.0860
.22	1.5515	.47	1.4802	.72	1.3412	.97	1.0686
.23	1.5497	.48	1.4761	.73	1.3337	.98	1.0500
.24	1.5478	.49	1.4718	.74	1.3261	.99	1.0275
						1.00	1.0000

or the length of the elliptic quadrant is known. We must now find what length of major axis, *i.e.* what value of "a" in conjunction with the known value of e will give the required length of quadrant.

The equation of the ellipse is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$$

When written in terms of the eccentric angle ϕ (Fig. 78) this becomes

$$x=a\sin\phi,$$

$$y=b\cos\phi,$$

$$dx=a\cos\phi\ d\phi, \qquad dx^2=a^2\cos^2\phi\ d\phi^2,$$

$$dy=b\sin\phi\ d\phi, \qquad dy^2=b^2\sin^2\phi\ d\phi^2,$$

$$s=\int_0^{\frac{\pi}{2}}\sqrt{dx^2+dy^2},$$

$$s=\int_0^{\frac{\pi}{2}}\sqrt{a^2\cos^2\phi+b^2\sin^2\phi}\ d\phi,$$

$$=\int_0^{\frac{\pi}{2}}\sqrt{a^2-(a^2-b^2)\sin^2\phi}\ d\phi,$$

$$=a\int_0^{\frac{\pi}{2}}\sqrt{1-\frac{a^2-b^2}{a^2}\sin^2\phi}\ d\phi,$$

$$=a\int_0^{\frac{\pi}{2}}\sqrt{1-e^2\sin^2\phi}\ d\phi=aE.$$

The above integral is one of the forms of the "elliptic integral," and cannot be expressed in any simpler form. There are, however, methods of approximating to its value to any required degree of accuracy, and the results will be found in tables of Elliptic Functions. The accompanying table gives values of E where e advances by hundredths, and where ϕ is equal to $\frac{\pi}{2}$.

We see that s can be expressed as

$$s = a \times E,$$

where E is known as soon as e is known. But

$$s = \frac{\pi N}{4P};$$

hence,

$$a = \frac{\pi N}{4PE},$$

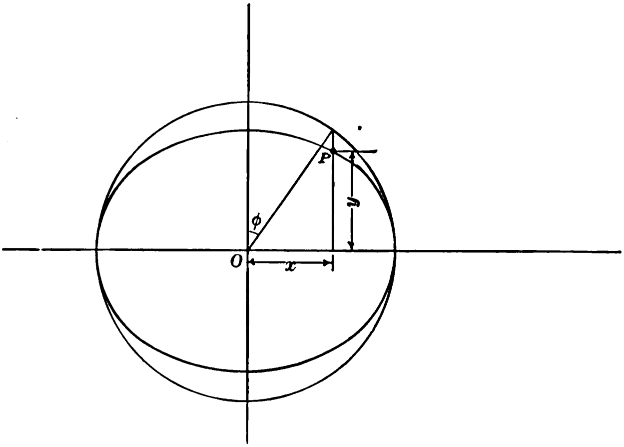


FIG. 78

which gives us the semi-axis major of the ellipse which will fulfil the requirements. A few problems will place the method in a clearer light.

Let $Z = 9$, $N = 30$ teeth, $P = 2$ (diametral pitch).

$$e = \frac{\sqrt{z} - 1}{\sqrt{z} + 1} = \frac{3 - 1}{3 + 1} = .5.$$

From the table, when $e = .5$, $E = 1.476$. Therefore

$$a = \frac{3.1416 \times 30}{4 \times 2 \times 1.476} = 8''.075,$$

$$b = a\sqrt{1 - e^2} = 8.075\sqrt{.75} = 6.993,$$

$$c = a(1 - e) = 8.075\,(1 - .5) = 4.037.$$

(See Fig. 79.)

As another example let $Z = 2$, $N = 30$, $P = 2$,

$$e = \frac{\sqrt{2} - 1}{\sqrt{2} + 1} = .1679. \quad \therefore E = 1.5576,$$

$$a = 7''.566, \qquad b = 7''.457, \qquad c = 6''.298.$$

In this case we see that the ellipse is very nearly a circle.

Sometimes the distance between shafts is given, and we are

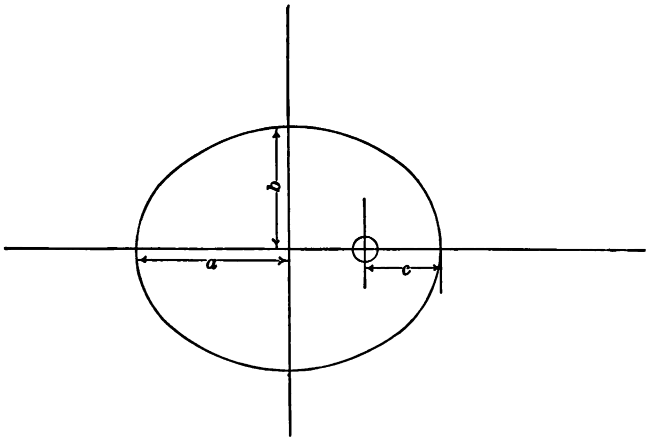


FIG. 79

required to fill up, by means of elliptic gears, this distance, while Z retains some definite value. In this case the shaft distance is evidently the major axis, $2a$, which immediately becomes known. Then

$$b^2 = a^2(1 - e^2) = a^2\left(1 - \left\{\frac{\sqrt{z} - 1}{\sqrt{z} + 1}\right\}^2\right) = \frac{4a^2\sqrt{z}}{(\sqrt{z} + 1)^2}.$$

Let the shaft distance be $10''$, and $z = 4$.

Then $a = \frac{10}{2} = 5$, $b = \frac{2a\sqrt[4]{z}}{\sqrt{z} + 1} = \frac{10 \times 1.4142}{3} = 3''.714$,

$$e = \frac{\sqrt{z} - 1}{\sqrt{z} + 1} = \frac{1}{3} = .3333,$$

$$c = a(1 - e) = 3''.33.$$

In this case the circumference becomes fixed, and a given size of tooth cannot in general be used. The circumference will be

$$4s = 4aE = 20 \times 1.5271 = 30''.542.$$

If $N = 15$ teeth, the circular pitch will be

$$P' = \frac{4s}{N} = 2''.036.$$

Having calculated its dimensions, the ellipse must now be laid out on the drawing board. When the major and minor axes are given, the best construction is that of Fig. 80. Draw circle of radius $OB = b$, and one of radius $OA = a$. Draw a number of lines through O cutting these circles, such as Oba. Through b draw bP parallel to OX, and through a draw aP parallel to OY. The intersection of these at P is a point on the ellipse. We must now lay off the tooth intervals all the way around by laying off the circular pitch P'. Bisect each of these intervals for the tooth and the space. At the centre of each tooth we must now draw a normal to the ellipse. Suppose we wish to draw a normal at Q. Draw Qr and Qs. Draw Ors, and produce to f where the circle whose radius is $(a + b)$ is intersected. Then Qf is the required normal. The same is shown at P. When the normals at every tooth have been drawn, they will envelop the evolute of the ellipse, one branch of which is shown at lmn. The point of tangency at m between the normal and the evolute will give the centre of curvature at P. If required, the radius of curvature can be calculated from

$$\rho = \frac{\mp (a^2 - e^2x^2)^{\frac{3}{2}}}{a^2(1 - e^2)^{\frac{1}{2}}},$$

but this is unnecessary.

We now construct each tooth as if on a circle whose radius is the radius of curvature at the point required. It is evident that all the teeth in a quadrant will be different, but if the ellipse is of small eccentricity, there will be a number of teeth at the extremities of the axes very nearly alike.

If the number of teeth is odd, the major axis should bisect a tooth at one end and a space at the other. If even, the tooth profile should pass through the extremity of the major axis at each end. If these rules are followed, the two wheels will be exactly alike, and can be cast from one pattern. Elliptic gears can be cut approximately on the milling machine by mounting the blank on a reversed trammel.

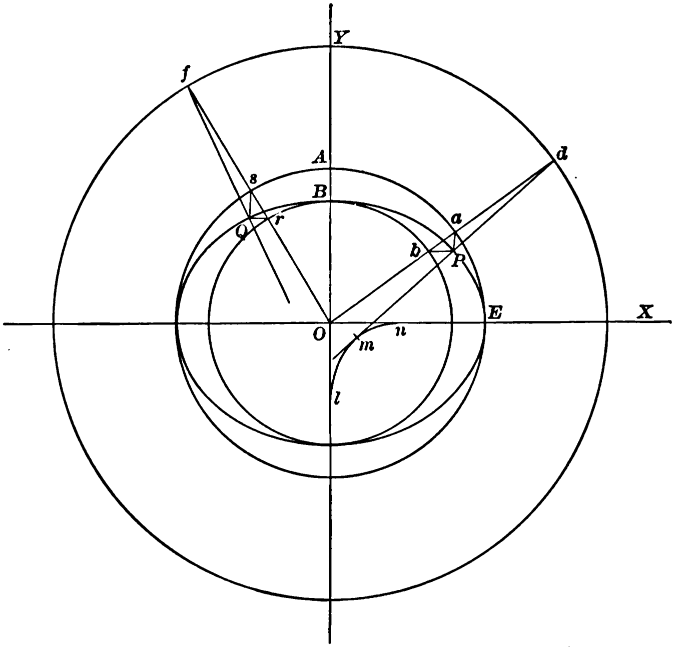


FIG. 80

As has been said, the free foci C and D (Fig. 75) can be connected by a link in case the shafts overhang, and we need then put teeth at the extremities of the major axes only. These will carry the wheels past the dead points, and the pull and push of the link will carry them the rest of the way, while the pitch ellipses roll upon one another.

Elliptic gears are frequently used for a quick return on small shapers, slotting machines, etc. The eccentricity should be less than .4 except where a link is used.

3. AXES OF ROTATION INTERSECTING, BEVEL WHEELS

Thus far we have considered only the transmission of rotation by means of gearing between parallel shafts. In this case we have seen that the pitch surfaces are cylinders. When the axes intersect, the cylinders must be replaced by cones having a common apex at the point of intersection of the shafts. Such cones will transmit motion as pitch surfaces by pure rolling along an element. As in the case of cycloidal spur teeth where the tooth surface is swept up by elements of a pair of describing cylinders rolling within and without the pitch cylinders, so in the case of cycloidal bevel teeth the tooth surfaces are swept up by elements of a pair of describing cones rolling within and without the pitch cones, all four having a common vertex. The intersection of the tooth surface with a sphere whose centre is the common apex of the cones forms a curve in space known as the spherical epi- or hypocycloid, *PX*, *PY* (Fig. 81). The involute bevel tooth surface is swept up by a line of a plane which rolls between base cones, coaxial with but with smaller vertical angle than the pitch cones. The generating line must be one which always passes through the apex of the cones. Fig. 82 shows the generation of the spherical involute. The plane *GEH* (taken for clearness as a circle with radius

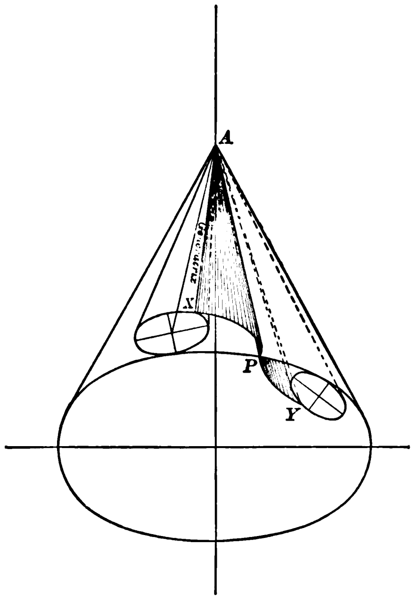


FIG. 81

equal to the slant height of the cone) rolls upon the base cone *DAC*, *AE* being the line of tangency. Any radius of the circular plane such as *AX* will sweep up the tooth surface, the point *X* sweeping up the spherical involute *XP*. Thus we see that the bevel tooth curves may be treated by spherical geometry just as spur tooth curves were treated by plane geometry. Small circles of

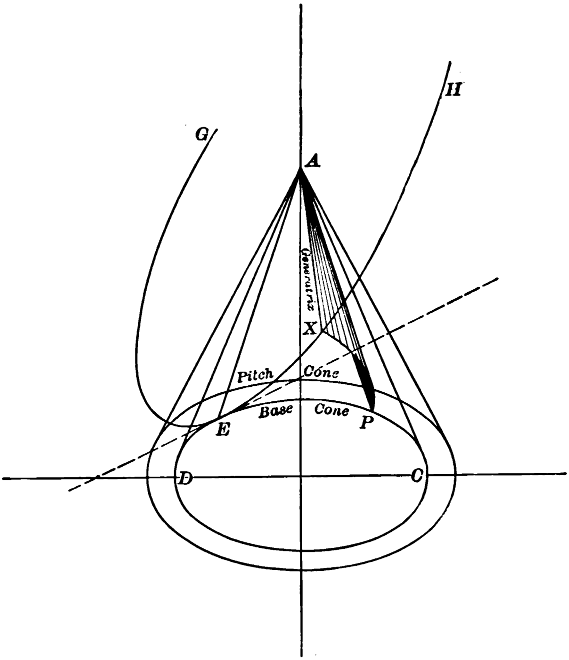


FIG. 82

the circumscribing sphere will replace circles in the plane geometry, and great circles of the sphere will replace straight lines. Thus the spherical epicycloid is traced by a point in the circumference of a small circle rolling outside of another small circle. The spherical cycloid is traced by a point of a small circle rolling upon a great circle. This would be the tooth curve of the crown wheel or bevel rack. The spherical involute is traced by a point of a great circle which rolls upon a small circle, etc. Being on the surface of a sphere, the spherical involute does not pass to infinity, but terminates in a series of cusps upon the base cone and its prolongation. (See Fig. 83.)

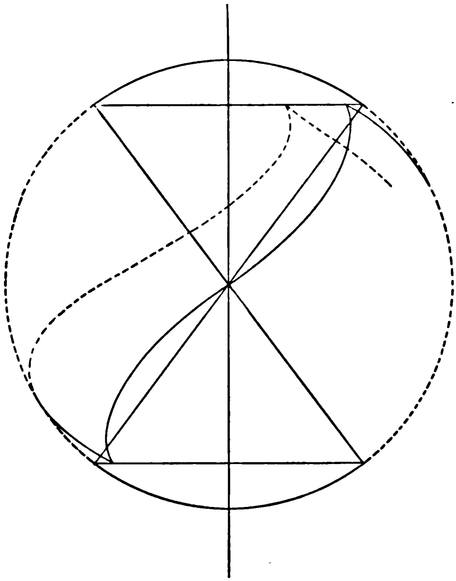


FIG. 83

For drawing bevel gears, Tredgold's approximation is used (Fig. 84). At the base of the pitch cone another is drawn, having its elements at right angles to those of the pitch cone. This "normal cone" is developed upon a plane, and the teeth are laid out upon its developed circumference as on a spur gear. This is then wrapped back upon the normal cone, and the tooth surface is supposed to be generated by a line which always passes through the vertex of the pitch cone, and always touches the tooth profile. In practice only frusta of the pitch and normal cones are used, as shown in Fig. 84. The slant heights of the frusta should seldom be more than one-third those of the cones.

Bevel wheels whose shaft angle is 90° are called Mitre Wheels.

Since all the surfaces of bevel teeth are conical by reason of their generation, they cannot be cut correctly with a rotary cutter, as such a cutter can make a cylindric surface only. However, they can be cut approximately by following the rules given in Brown and

Sharpe's "Formulas in Gearing," published by the company.* The teeth can be cut perfectly only by planing them out element by element on a special machine. With such a machine twisted

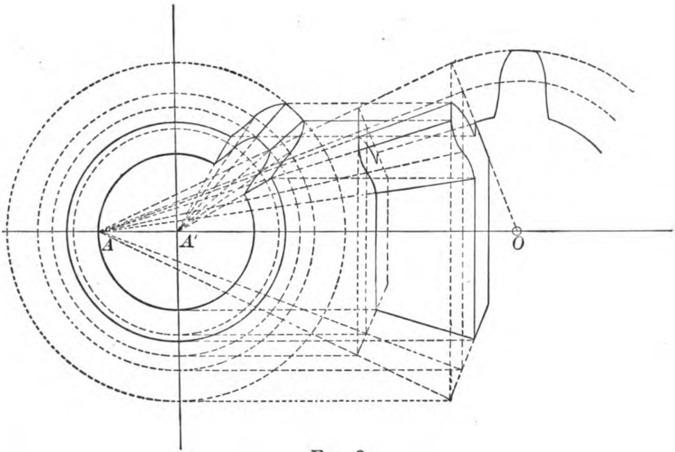


FIG. 84

bevels can be cut by giving the blank a reciprocating circular motion in unison with the reciprocating rectilinear motion of the cutter.

4. AXES OF ROTATION CROSSING, SKEW WHEELS

A. Spiral Gears

We have seen how a twisted gear is derived from a spur gear, and works according to exactly the same theory. Suppose *CD* (Fig. 85) to be an ordinary twisted gear, in mesh with a piece of twisted or oblique rack *EFGH*. If *CD* is turned with a constant angular velocity counter-clockwise as viewed from *A'*, *EF* will move with a constant velocity *PV* in the direction *EF*, and there will be no sliding parallel to the elements of the rack teeth. But if the rack is constrained by means of guides to move in any other

* For the dimensions of bevel teeth consult this book also.

direction, the gear will drive it at some other constant velocity, and there will be a sliding parallel to the elements of the rack teeth. In fact, it is self-evident that a twisted gear will drive an oblique rack in any direction except parallel to its own teeth, and that the ratio between the velocity of the pitch surface of the gear and that of the rack will be a constant.

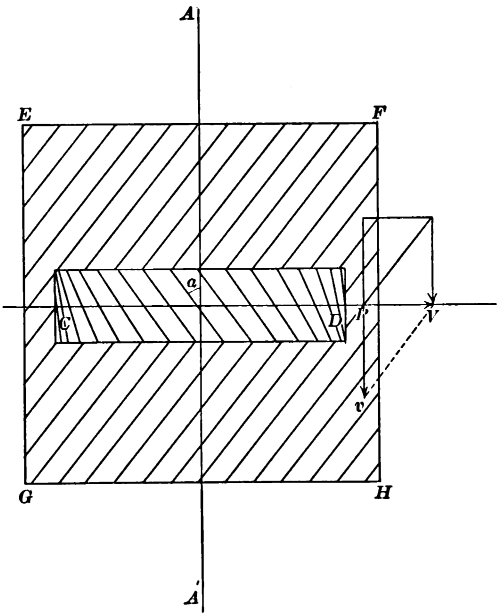


FIG. 85

Now suppose we have instead of a solid rack merely the surface of a rack, such as might be made by bending a thin sheet of metal into an oblique rack. Then a second twisted gear could mesh with this rack from below, and if the rack were driven in any direction by the upper gear, the two gears would turn with a constant velocity ratio. In Fig. 86 let the twisted gear *EFGH* mesh with a piece of rack, as shown from above, and let this mesh with another, *ABCD*, below. We will suppose for simplicity that the teeth are involute. The lines *RS* and *TU* are the lines of the root edge and the tip edge of a rack tooth when seen from above. This is shown in perspective in Fig. 87. To the right, and above in Fig. 86, are shown projections of the faces *EF*, *GH*, *AB*, and *CD* of the gears and rack, showing the tooth faces of the latter by the lines $e'f'$, $g'h'$, etc. The point of contact on the face *EF* will be where the line of obliquity intersects at right angles the tooth outline at *m'*, and when projected back gives the point *m* in its proper position. In the same way we find for the face *GH* the tooth face *g'h'*, the point of contact *l'*, and the projected point *l*. Now when viewed from below the line *RS* becomes the tip edge, and the line *TU* the root edge. Projecting the faces *AB* and *CD* of the lower gear

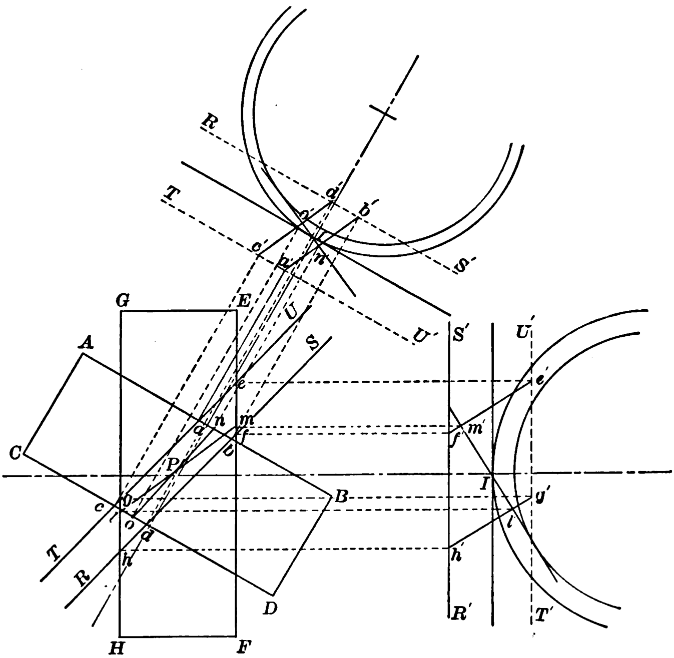


FIG. 86

upward, drawing the obliquity, and projecting the contact points back, we find the point *n* of contact on the face *AB* and the point *o* of contact on the face *CD*. Hence the upper gear touches the rack along the line *ml*, and the lower gear along the line *no*. These lines are evidently the oblique generating lines *PQ* (Fig. 71) of the teeth of the two wheels, which in twisted gears are coincident.

If now we remove the rack surface, the two gears themselves will touch in a single point P, which is the intersection of the two lines. Hence spiral gears, which are nothing more than twisted gears with shafts askew, will touch in one point only, but they will maintain a constant angular velocity ratio, since the lines ml and no will continue to intersect at some point as the action continues. Two twisted gears will therefore work together at any angle, but the wheels must be at the shortest distance between the crossing shafts. They can be cut on a milling machine by giving the blank a motion of rotation about its own axis as it passes under the cutter, and the tooth surface is approximated by the ordinary spur tooth cutter.

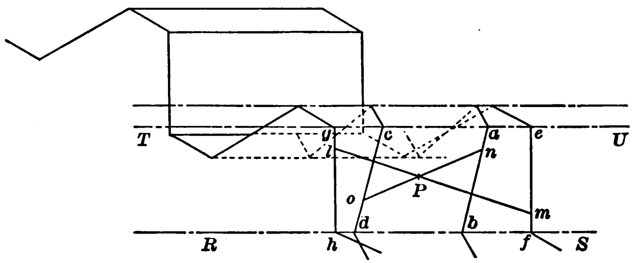


FIG. 87

The true circular pitch of a spiral gear is the pitch circumference divided by the number of teeth, viz., CD (Fig. 72). The normal circular pitch is the perpendicular distance between similar elements of two adjacent teeth measured upon the pitch surface, or CE. This will be the circular pitch of the rotary cutter. Hence we cannot calculate the pitch of the cutter in the ordinary way. If we call the angle between a tangent to a tooth on the pitch surface and a line parallel to the axis, the spiral angle, and denote this by α, then

$$CE = CD \cos \alpha.$$

If P' is the true circular, and P_n' the normal circular pitch,

$$\frac{2 \pi R}{N} = P'$$

exactly as in the case of spur gears; but

$$\frac{2\pi R\cos\alpha}{N}=P_n',$$

and
$$P=\frac{\pi}{P_n'},$$

where P is the diametral pitch of the rotary cutter. Hence,

$$P=\frac{N}{2R\cos\alpha}.$$

We will first deduce the relation between the angular velocity ratio and other constants of the gear. Consider two spiral gears whose axes are AA' and BB' (Fig. 88). Let XY be a common

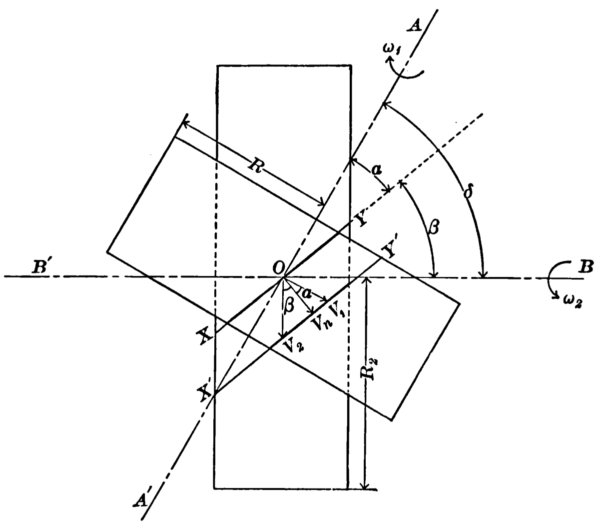


FIG. 88

element at the point of contact O. Let XY move through a differential distance to $X'Y'$. Then a point of wheel A has moved through a distance V_1 and a point of B through a distance

V_2. If we call the normal distance between XY and $X'Y'$ V_n, we will have

$$V_1 \cos\alpha = V_n = V_2 \cos\beta,$$

$$\frac{V_1}{V_2} = \frac{\cos\beta}{\cos\alpha}.$$

Now V_1 and V_2 may be considered the velocities of the circumferences of the two gears, and if ω_1 and ω_2 are their angular velocities, and R_1 and R_2 their pitch radii,

$$\frac{\omega_1}{\omega_2} = \frac{V_1 R_2}{V_2 R_1} = \frac{R_2 \cos\beta}{R_1 \cos\alpha},$$

or the angular velocity ratio is equal to the inverse ratio of the radii multiplied by the inverse ratio of the cosines of the spiral angles.

Suppose in a pair of spiral gears we are given the value of δ, or the inclination between the shafts, the angular velocity ratio $\frac{\omega_1}{\omega_2} = \eta$, and the ratio of the radii $\frac{R_1}{R_2}$. We are also given N_1, the number of teeth on one wheel, and P, the diametral pitch of the rotary cutter. Then as

$$\alpha + \beta = \delta,$$

$$\eta = \frac{\omega_1}{\omega_2} = \frac{R_2}{R_1}\frac{\cos(\delta - \alpha)}{\cos\alpha} = \frac{R_2}{R_1}\frac{(\cos\delta\cos\alpha + \sin\delta\sin\alpha)}{\cos\alpha},$$

$$\eta\frac{R_1}{R_2} = \cos\delta + \sin\delta\tan\alpha,$$

or

$$\alpha = \tan^{-1}\left\{\frac{R_1}{R_2}\frac{\eta}{\sin\delta} - \cot\delta\right\}.$$

Then we know β from

$$\beta = \delta - a.$$

Now P, the diametral pitch of the rotary cutter, is the same for both wheels, as is $\frac{\pi}{P} = P_n'$, the normal circular pitch also. Then

$$P_1' = P_n'\frac{1}{\cos\alpha} = \text{true circular pitch of } R_1,$$

and

$$P_2' = P_n'\frac{1}{\cos\beta} = \text{true circular pitch of } R_2;$$

hence,
$$R_1 = \frac{P_1'N_1}{2\pi} = \frac{N_1}{2P\cos\alpha},$$
$$R_2 = \frac{P_2'N_2}{2\pi} = \frac{N_2}{2P\cos\beta}.$$

The dimensions of the blanks and the cutting angles being now determined, we must select the proper cutter of the set whose pitch is given. If we select any tooth upon a spiral gear and pass a plane through it normal to its helical elements, the intersection of this plane with the pitch cylinder will be an ellipse whose minor axis is $2R$, and whose major axis is $\frac{2R}{\cos\alpha}$, α being the spiral angle. The spur gear whose tooth would most nearly coincide with the profile of the spiral gear selected would be the gear cut upon the osculating circle at the extremity of the minor axis of the ellipse. The radius of curvature at the point mentioned is found in the ordinary way as follows: the equation of the ellipse is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$
$$\frac{dy}{dx} = \mp \frac{bx}{a\sqrt{a^2 - x^2}},$$
$$\frac{d^2y}{dx^2} = \mp \frac{ab}{(a^2 - x^2)^{\frac{3}{2}}},$$
$$\rho = \frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}{\frac{d^2y}{dx^2}}$$
$$= \frac{\left[1 + \frac{b^2x^2}{a^2(a^2 - x^2)}\right]^{\frac{3}{2}}}{\frac{ab}{(a^2 - x^2)^{\frac{3}{2}}}}.$$

At the extremity of the minor axis $x = 0$, so we have

$$\rho\Big]_{x=0} = \frac{1}{\frac{ab}{a^3}} = \frac{a^2}{b}.$$

Now in the ellipse formed by the intersection above described $a = \frac{R}{\cos\alpha}$, and $b = R$. Hence,

$$\rho_0 = \frac{R^2}{R\cos\alpha} = \frac{R}{\cos^2\alpha}.$$

This is the radius of curvature of the "Osculating Spur Gear." The number of teeth on this gear would be

$$N_0 = \frac{2\pi\rho_0}{P_n'} = \frac{2\pi R}{P_n'\cos^2\alpha}.$$

But on the spiral gear itself we have

$$N = \frac{2\pi R}{P'} = \frac{2\pi R\cos\alpha}{P_n'}.$$

Hence, $$N_0 = \frac{N}{\cos^3\alpha}.$$

The cutter should therefore be chosen from the set, not according to the number of teeth on the spiral gear itself, but according to the number on the osculating spur gear.

If $\delta = 90°$, the following special cases may occur:

$$\alpha = \tan^{-1}\frac{R_1}{R_2}\eta.$$

If in addition $R_1 = R_2$, $\alpha = \tan^{-1}\eta$.

If $\eta = 1$, $\alpha = \tan^{-1}\frac{R_1}{R_2}$.

If $R_1 = R_2$, and $\eta = 1$, $\alpha = \tan^{-1} 1 = 45°$,

and the two wheels will be exactly alike.

If $\delta = 0$, the case reduces to a spur gear.

Critical Angles or Angles of Maximum Efficiency. — It has now been shown that the fundamental laws connecting the various quantities in the spiral gear are

$$\frac{\omega_1}{\omega_2} = \eta = \frac{R_2\cos\beta}{R_1\cos\alpha}, \quad (1)$$

$$K = R_1 + R_2, \quad (2)$$

$$\delta = \alpha + \beta, \quad (3)$$

where K is the distance between shafts. The three quantities on the left sides of the equations are evidently fixed once for all by the requirements of the design, leaving four unknown quantities, viz., α, β, R_1, and R_2, one of which must be chosen arbitrarily, or by some other independent condition. This condition may be made that of maximum efficiency.

Loss of power on spiral gears is due to friction (1) in the bearings, and (2) between the teeth. The first of these can be further separated into (*a*) journal friction due to the obliquity of action of the teeth forcing the shafts apart, as in any form of gearing, and (*b*) end-thrust friction. The sliding between the teeth may be separated into (*a*) a component of approaching and receding action, also common to all forms of gearing, and (*b*) a component of sliding along the helical elements of the teeth. Now it is useless to attempt to deduce that diameter of gear and angle of tooth which, with given coefficients of friction, would reduce all these various losses to their combined minimum value, for even when obtained the result would be too complicated and uncertain to be of any practical value. So let us see which of the above are the most wasteful of energy which are small enough to be neglected, and which may be omitted because their effects would not be greatly changed by change in radius and angle. The two losses designated (*a*) will be small compared with (*b*), particularly in the first case. Roughly speaking, their effects will be smallest in wheels of equal size, though change in radii and tooth angles will not greatly vary their magnitude. Next in importance will be the end-thrust factor, which may be varied widely by change in the angle of the tooth. And last and most important of all is the sliding between the teeth along the helical element.

Omitting first all other sources of loss of power, let us investigate that of end thrust. Let A (Fig. 89) be the driver through which power is transmitted to B. Let ON be the normal pressure or force active between the elements of the teeth in the tangent plane to the two wheels. Then OP, the projection of ON in the plane of the gear A, will be the effective driving force of A in its own plane, and NP will be the end thrust on its shaft. Similarly OQ will be the driving force on B, and NQ its end thrust.

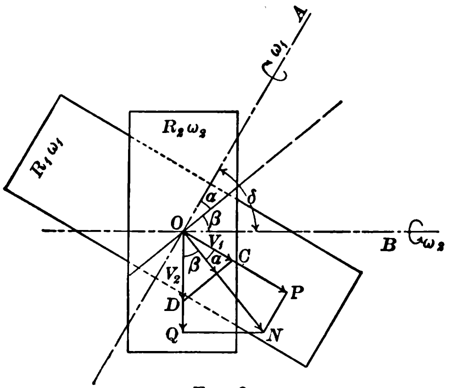


FIG. 89

Now the total power given to the system by A will be equal to $(OP) \times V_1$, where V_1 is the velocity of the surface of A in the direction of OP. The power lost due to friction through the end thrust of A will be

$$(PN) \times v' \times \phi,$$

where v' is the mean velocity of the end-thrust collar, and ϕ the coefficient of friction at that point. Hence the ratio of power lost to total power in the system due to this single cause will be

$$E = \frac{\overline{PN} \times v' \times \phi + \overline{QN} \times v'' \times \phi}{\overline{OP} \times V_1}.$$

But $v' = r_1\omega_1$, and $v'' = r_1\omega_2$, where r_1 is the mean effective frictional radius of the end-thrust collar taken the same for both shafts; also $\overline{PN} = \overline{OP} \tan \alpha$, $\overline{QN} = \overline{OQ} \tan \beta$, and $V_1 = R_1\omega_1$, which substituted give

$$E = \frac{r_1\phi(\omega_1 \times \overline{OP} \tan \alpha + \omega_2 \times \overline{OQ} \tan \beta)}{\overline{OP} \times R_1\omega_1}.$$

Furthermore, $\overline{OQ} = \overline{OP}\dfrac{\cos \beta}{\cos \alpha}$, so that the equation reduces to

$$E = r_1\phi \left\{ \frac{\omega_1 \sin \alpha + \omega_2 \sin \beta}{R_1\omega_1 \cos \alpha} \right\} \quad . \quad . \quad . \quad (4)$$

In this we must substitute for all unknown quantities in terms of α, and find what value of α will make E a minimum. Combining equation (1) with (2), and solving for R_1, we get

$$R_1 = \frac{K\omega_2 \cos\beta}{\omega_1 \cos\alpha + \omega_2 \cos\beta},$$

and substitution gives

$$E = \frac{r_1\phi}{K}\left\{\frac{(\omega_1 \sin\alpha + \omega_2 \sin\beta)(\omega_1 \cos\alpha + \omega_2 \cos\beta)}{\omega_1\omega_2 \cos\alpha \cos\beta}\right\},$$

which by reduction becomes

$$E = \frac{r_1\phi}{K}\left\{\frac{\omega_1}{\omega_2}\frac{\sin\alpha}{\cos\beta} + \frac{\omega_2}{\omega_1}\frac{\sin\beta}{\cos\alpha} + \tan\alpha + \tan\beta\right\}. \quad . \quad (5)$$

That value of α in the general case given above which makes E a minimum leads to an unsatisfactory result when the shaft angle δ and the angular velocity ratio $\frac{\omega_1}{\omega_2}$, which we may call η, are small, for then there may be no real minimum value of α within the angle δ. But fortunately in the particular case where the shafts cross at 90°, the solution is easily applicable to all cases. In this case

$$E_1 = \frac{r_1\phi}{K}\left\{\eta + \frac{1}{\eta} + \tan\alpha + \tan\beta\right\}.$$

Before applying the ordinary methods of finding that value of α which makes E_1 a minimum, we must notice a certain restriction which must be placed on all the above equations. Should for any reason the algebraic sign of either of the angles α or β change, the sign of the resulting power lost will not change. In other words, the formulæ do not necessarily hold for any other than positive angles, or angles between the limits of δ. Differentiating the above equation with respect to α, and putting the first differential coefficient equal to zero,

$$\frac{dE}{d\alpha} = \frac{r_1\phi}{K}\{\sec^2\alpha - \sec^2\beta\} = 0,$$

$$\sec^2\alpha = \sec^2\beta,$$

$$\alpha = \beta.$$

For positive values of α and β, this is easily distinguished as a minimum, and the result is independent of $r_1\ \phi$, or K. In this case then it would seem that teeth set at an angle of 45° would give a minimum of work lost by end thrust.

In the problem of minimum power lost by the sliding of the teeth along a helical element, the angle α has usually been taken as that angle giving the minimum velocity of slip in that direction, with given fixed angular velocities of both shafts. Referring again to Fig. 89, if $\overline{OC} = V_1$ is the velocity of a point on the pitch surface of A in its own plane, and tangent to its surface, and $\overline{OD} = V_2$ the velocity of a point similarly related to B, the length $\overline{CD}$ will be the velocity of these points relatively to one another along the tangent to the helical element. Calling this velocity of slip v_s, then

$$\begin{aligned} v_s &= V_1 \sin \alpha + V_2 \sin \beta, \\ &= R_1 \omega_1 \sin \alpha + R_2 \omega_2 \sin \beta, \\ &= R_1 \omega_1 \sin \alpha + (K - R_1) \sin \beta. \end{aligned}$$

But from the previous case

$$R_1 = \frac{K \omega_2 \cos \beta}{\omega_1 \cos \alpha + \omega_2 \cos \beta}.$$

Hence by substitution

$$\begin{aligned} v_s &= \frac{K \omega_1 \omega_2}{\omega_1 \cos \alpha + \omega_2 \cos \beta} \left\{ \cos \alpha \sin \beta + \cos \beta \sin \alpha \right\}, \\ &= \frac{K \omega_1 \omega_2 \sin \delta}{\omega_1 \cos \alpha + \omega_2 \cos \beta}. \end{aligned}$$

Now the numerator of this fraction is a constant, hence v_s will vary only by the variation of the denominator, or v_s will be a minimum when $(\omega_1 \cos \alpha + \omega_2 \cos \beta)$ is a maximum. Call this quantity Y, then

$$\frac{dY}{d\alpha} = -\omega_1 \sin \alpha + \omega_2 \sin \beta = 0,$$

$$\frac{\omega_1}{\omega_2} = \frac{\sin \beta}{\sin \alpha}.$$

By using again the positive pair of angles given by this last we can easily distinguish Y as a maximum. If along the axis of A (Fig. 90) we lay off the angular velocity ω_1 of A to any scale as $\overline{OL}$, and along B lay off ω_2 equal to $\overline{OM}$, and complete the parallelogram *OLFM*, the diagonal $\overline{OF}$ will give the proper values of α and β to give minimum velocity of slip. It will be noted that since the ratio of angular velocities is equal to the *inverse* ratio of the numbers of teeth, we might lay off the number of teeth of *B* along the axis of *A*, and *vice versa*. This same result has been shown by MacCord and others by different methods of procedure.

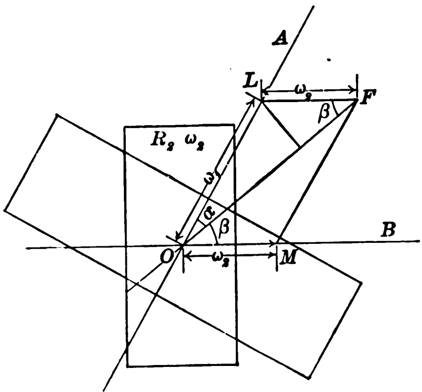


FIG. 90

But it does not appear that the angles giving the minimum velocity of slip are necessarily those giving the minimum power lost by that slip. Power is composed of two factors, force and velocity, and it is the product of these and not one alone which must be a minimum. Attacking the problem then along the lines of the first one, we see that

$$\text{Power put in by } A = PV_1,$$
$$\text{Power lost due to } v_s = N\phi v_s,$$

where N is the normal pressure between the teeth, and ϕ the coefficient of friction.

Then it is $$E = \frac{N\phi v_s}{PV_1} = \frac{N\phi v_s}{PR_1\omega_1},$$

which must be a minimum. From the figure,

$$N = \frac{P}{\cos\alpha}.$$

From the preceding, $$v_s = \frac{K\omega_1\omega_2 \sin\delta}{\omega_1\cos\alpha + \omega_2\cos\beta},$$

and $$R_1 = \frac{K\omega_2\cos\beta}{\omega_1\cos\alpha + \omega_2\cos\beta}.$$

Substituting these in the expression for E, it reduces to the simple form

$$E = \frac{\phi \sin \delta}{\cos \alpha \cos \beta},$$

and rejecting constants, $\dfrac{1}{\cos \alpha \cos \beta}$

is to be a minimum, or $Y = \cos \alpha \cos (\delta - \alpha)$ is to be a maximum. Differentiating,

$$\frac{dY}{d\alpha} = -\sin \alpha \cos \beta + \sin \beta \cos \alpha = 0,$$

$$\sin (\beta - \alpha) = 0,$$

$$\alpha = \beta.$$

By the ordinary methods of second differentiation, this may be shown to be a maximum, and therefore when the tooth bisects the shaft angle, the power lost by sliding between the teeth as above stated is a minimum.

When α and β are known, R_1 is found from

$$R_1 = \frac{K \cos \beta}{\eta \cos \alpha + \cos \beta},$$

and $$R_2 = K - R_1.$$

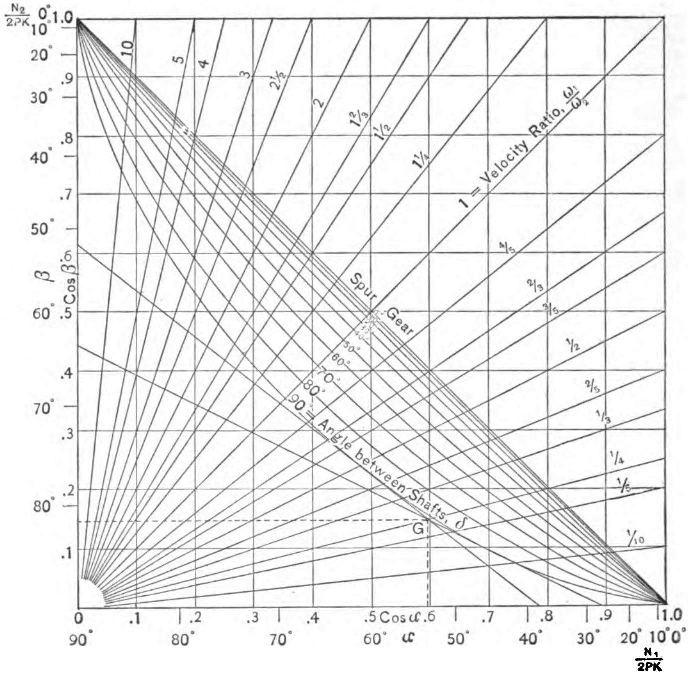
The methods of using the above formulæ can best be seen by applying them to a definite case. Let the distance between the shafts be $K = 6''$, and let the shafts cross at 90°. The angular velocity ratio is to be $\eta = \frac{\omega_1}{\omega_2} = \frac{1}{2}$, and $P = 10$. The angles are to be those giving the maximum efficiency. Then

$$\alpha = \beta = 45^\circ,$$

$$R_1 = \frac{K \cos \beta}{\eta \cos \alpha + \cos \beta} = \frac{K}{\eta + 1} = 4'',$$

$$R_2 = K - R_1 = 2''.$$

The numbers of teeth on the two wheels must now be calculated. These in general will not be whole numbers, and as a fractional



Cutting Angles of Spiral Gears.

tooth is an impossibility, we must select the nearest whole numbers to the ones found, and recalculate the angles and radii to fit the new case. The numbers of teeth will be found from

$$N_1 = 2\,PR_1 \cos\alpha,$$

$$N_2 = 2\,PR_2 \cos\beta,$$

which in the above example become

$$N_1 = 2 \times 10 \times 4 \times .7071 = 56.568,$$

and
$$N_2 = 2 \times 10 \times 2 \times .7071 = 28.284.$$

Hence we must select for our gears $N_1 = 56$ teeth, and $N_2 = 28$ teeth, as being the numbers giving the highest efficiency. Transposing the equations for tooth numbers we have

$$R_1 = \frac{N_1}{2\,P\cos\alpha},$$

and
$$R_2 = \frac{N_2}{2\,P\cos\beta} = \frac{N_2}{2\,P\sin\alpha}.$$

Hence,
$$K = \frac{N_1}{2\,P\cos\alpha} + \frac{N_2}{2\,P\sin\alpha}.$$

In our case
$$6 = \frac{2.8}{\cos\alpha} + \frac{1.4}{\sin\alpha}.$$

If we could solve the above equation for α, our inverse computation would be complete. But unfortunately the equation is of the fourth degree, and, though possible of solution, such solution is not practical. Furthermore there are four real values of α which will satisfy it. Graphic methods or continued approximations must then be resorted to. One of the best graphic methods appears to be that of Robert Bruce,* which is as follows: let OX, OY (Fig. 91) be a pair of rectangular coördinates. Lay off upon the axis of X $\frac{N_1}{2\,P}$, and upon the axis of Y $\frac{N_2}{2\,P}$, thus determining the point

* *American Machinist*, April 12, 1900.

C. Lay a graduated scale upon the paper, so that its edge passes through C, its zero point lies upon one of the axes, and shift it until it intercepts K inches between the axes. Then the angles α and β in the figure are evidently the required ones. The appended Diagram No. 3 gives a method by which the angles can be read off directly. Having obtained the nearest whole numbers of teeth on the gears, find on the diagram the point G, whose coördinates

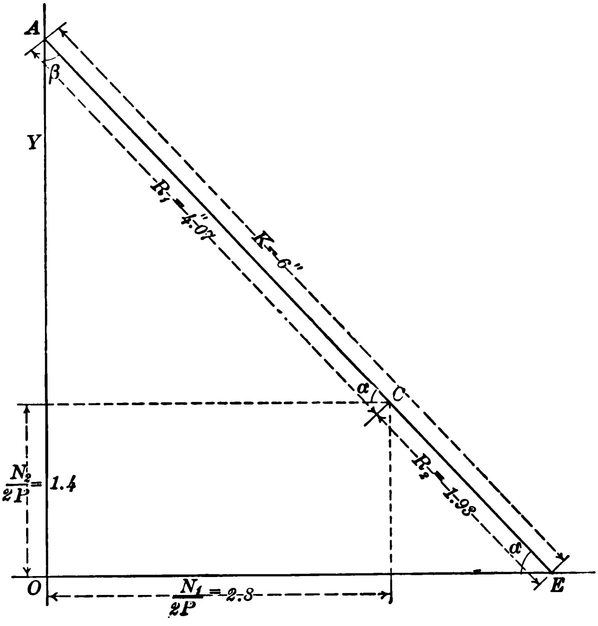


FIG. 91

are $\frac{N_1}{2\,PK}$ and $\frac{N_2}{2\,PK}$ on the inner scales. Through this point draw a line or merely lay a straight edge tangent to the curve representing the shaft angle. The outer scales on the bottom and left will give roughly the angles α and β respectively, and the inner scales the values of $\cos\alpha$ and $\cos\beta$ quite accurately. The radial lines of velocity ratio will facilitate the locating of the desired point, for if the ratio be one of those given, the point must lie on its line.

It is interesting to note that two lines can be drawn through a given point tangent to the curves as shown. As a matter of fact, four such lines could be drawn provided the whole of the curves were drawn in, but that portion shown is the only portion giving positive angles, *i.e.* angles within the angle δ. But there will be two separate positive values of the angle α, which, with a given velocity ratio, number of teeth, and shaft distance, will work correctly together, giving of course different values of the radii. Which of the two is the one required can be easily told as lying nearest to the first approximation of the angle. The same result is seen in the case of Mr. Bruce's solution. Two positions of the line AE (Fig. 91) can be found passing through C, where the length is K inches. If the point G lies on one of the curves, the two positions coincide, a limiting case, and if it lies on the concave side, the solution is impossible within the angle δ.

By the application of either method to our problem, we find

$$\alpha = 46^\circ\ 33',$$

$$\beta = 43^\circ\ 27',$$

$$R_1 = \frac{N_1}{2\,P \cos\alpha} = \frac{56}{20 \times \cos\alpha} = 4''.072,$$

$$R_2 = \frac{N_2}{2\,P \cos\beta} = \frac{28}{20 \times \cos\beta} = 1.928.$$

The cutters are then selected from

$$N_0' = \frac{N_1}{\cos^3\alpha} = 169 = \text{No. 1 (rack)},$$

$$N_0'' = \frac{N_2}{\cos^3\beta} = 73 = \text{No. 2}.$$

As a more general case take the following: let $\delta = 60^\circ$, $K = 6''$, $\eta = \frac{\omega_1}{\omega_2} = \frac{1}{2}$, $P = 10$. Suppose certain considerations in the design make it necessary that R_1 should be as nearly as possible $4\frac{1}{4}$ inches, then approximately $R_2 = 1\frac{3}{4}$.

$$\alpha = \tan^{-1}\left\{\frac{R_1}{R_2}\frac{\eta}{\sin\delta} - \cot\delta\right\},$$

$$= \tan^{-1}\left\{\frac{4.25}{1.75}\frac{0.5}{\sin 60^\circ} - \cot 60^\circ\right\} = 39^\circ\ 31',$$

$$\beta = \delta - \alpha = 60^\circ - 39^\circ\ 31' = 20^\circ\ 29',$$

$$N_1 = 2\ PR_1 \cos\alpha = 65.572,$$

$$N_2 = 2\ PR_2 \cos\beta = 32.787.$$

In this case we would have to use

$$N_1 = 66 \text{ teeth},$$

and $$N_2 = 33 \text{ teeth}.$$

From the diagram, or by repeated trials from

$$\frac{3.3}{\cos\alpha} + \frac{1.65}{\cos(\delta - \alpha)} = 6,$$

or $$2\sec\alpha + \sec\beta = 3.6364,$$

we find that $$\alpha = 38^\circ\ 43',$$

$$\beta = 21^\circ\ 17',$$

from which the exact radii are

$$R_1 = 4''.229,$$

and $$R_2 = 1''.771.$$

Method of cutting Spiral Gears. — The lead of a helix or screw is the distance between two of its consecutive intersections with an element of its pitch cylinder. If the helix is developed by rolling on a plane, it is readily seen that $\frac{C}{l} = \tan\alpha$, where C is the circumference of the pitch cylinder, l the lead, and α the spiral angle of the helix. In cutting a spiral gear on a milling machine, the blank is first set at the proper spiral angle under the rotary cutter. It is moved forward by means of the screw, and also rotated by means of the spiral head. It is evident that when the screw has moved it forward a distance equal to the lead, the spiral head must have rotated it through one complete turn. The screw is connected to the worm which actuates the spiral head by a train of four gears. The numbers of teeth on these gears we will represent by a, b, c, and d. If there are N revolutions of the screw corresponding to n revolutions of the worm, then

$$N\frac{bd}{ac}=n.$$

Now it requires 40 revolutions of the worm to rotate the spiral head once, and there are 4 threads per inch on the screws of the Brown and Sharpe milling machines. Hence to move the blank a distance equal to the lead requires, $N=4l$ turns, and to rotate the spiral head once requires $n=40$ turns, or

$$4l\frac{bd}{ca}=40.$$

But
$$l=\frac{2\pi R}{\tan\alpha};$$

hence,
$$\frac{2\pi R}{\tan\alpha}\times\frac{bd}{ca}=10,$$

R and α being given, the ratio $\frac{bd}{ca}$ must be so chosen as to satisfy the equation. In the directions which come with the machines, a number of such combinations are worked out. It is evident that but few angles can be cut with absolute accuracy. (1st gear on stud $=b$ teeth, 2d gear on stud $=c$ teeth. Gear on worm $=a$ teeth, gear on screw $=d$ teeth.)

B. Worm Gears

If the spiral angle of one gear is very nearly 90°, one tooth may be made to return on itself, and the wheel reduces to an ordinary screw. If in addition the angle α is nearly 90°, the spiral angle of the other wheel is very small, and is in fact very nearly a spur gear.

This combination is called a Worm Gear. In the case of the worm, since $\rho_0 = \frac{R}{\cos^2 \alpha}$, when α approaches 90°, ρ_0 approaches ∞, or the tooth outline of a worm is practically that of a rack.

The action of a screw upon a gear can be much improved by the following method: a copy of the screw is made in tool steel, and this is notched parallel to its axis like a tap. The wheel is notched into the required number of teeth upon a milling machine. The notched screw or "hob" is rotated between centres, and the wheel, free to revolve upon a stud, is fed up against it. The cutting edges of the hob soon work into the notches of the wheel blank, until the proper depth is reached. The hob is then replaced by the original screw. Such concave gearing is excellent for causing slow or small amounts of motion, but is not very economical for transmitting power, as the sliding between the teeth is large. A worm wheel will not work properly if it contains less than from 25 to 30 teeth.

TABLE FOR DIAMETRAL PITCH OF WORM TOOLS. (GRANT)*

Diametral Pitch	1	2	3	4	5	6	7	8
Point of Hob tool .	1.035	.517	.345	.258	.207	.173	.148	.129
Point of Worm tool	.968	.484	.323	.242	.197	.162	.138	.121
Depth of Cut . .	2.125	1.063	.708	.532	.425	.354	.304	.266
Increase . . .	.250	.125	.083	.063	.050	.042	.036	.032

Diametral Pitch	9	10	11	12	13	14	15
Point of Hob tool . . .	.115	.104	.094	.086	.078	.074	.069
Point of Worm tool . .	.105	.097	.088	.081	.073	.069	.064
Depth of Cut	.236	.213	.193	.177	.164	.152	.142
Increase	.028	.025	.023	.021	.019	.018	.017

* "The Teeth of Gears," by George B. Grant, Lexington Gear Works.

TABLE FOR CIRCULAR PITCH OF WORM TOOLS. (GRANT)

CIRCULAR PITCH	2	1¾	1½	1¼	1⅛	1	⅞	¾
Point of Hob tool.	.644	.564	.483	.402	.362	.322	.282	.241
Point of Worm tool	.620	.542	.466	.388	.349	.310	.271	.233
Depth of Cut . .	1.416	1.240	1.062	.886	.797	.708	.620	.531
Increase . . .	.166	.146	.125	.104	.094	.083	.073	.062

CIRCULAR PITCH	⅝	½	7/16	⅜	5/16	¼	3/16
Point of Hob tool . . .	.201	.161	.141	.121	.100	.080	.060
Point of Worm tool . .	.194	.155	.135	.116	.097	.078	.058
Depth of Cut	.443	.354	.310	.265	.222	.177	.133
Increase	.052	.042	.036	.031	.026	.021	.016

"The sides of the tool should come together at an angle of 30°. Make the tool the proper width at the point, and thread the *worm* to the required depth of cut. Make the diameter of the hob greater than that of the worm by the amount of the increase. Grind off half of the increase from the point of the tool, and use it to thread the *hob* to the same depth of cut." — (GRANT.)

C. HYPERBOLOIDAL GEARS

The gears we have just been considering have but one point of contact between the teeth. Gears can be constructed, however, upon shafts which cross without intersecting, having a true line contact. Nor must these wheels be placed at the shortest distance between shafts, but may be anywhere along the axes.

The pitch surfaces are hyperboloids of revolution of one sheet, for let AA' and BB' (Fig. 92) be two axes, crossing without intersecting, their shortest distance apart being at O. From the definition of pitch surfaces they must touch along a line, such as XY. Then as the axes revolve, the line XY will sweep up an hyperboloid with respect to each axis. Hence the line is called the Generatrix. If viewed directly from above as in the figure, α and β will be the angles between the generatrix and the axes. The surfaces are not necessarily tangent along XY unless the proper relation exists between α, β, R_1, and R_2. If they are tangent at any point, there must be a common normal to the surfaces at that point. But these surfaces are surfaces of revolution, and every normal to such a surface intersects its axis. Hence if two hyperboloids are tangent along the generatrix, a perpendicular let fall

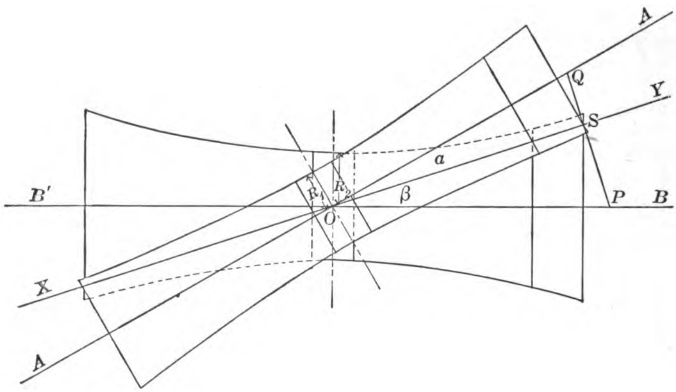


FIG. 92

from one axis upon the generatrix will, if produced, intersect the other axis. The line connecting the shortest distance between shafts is such a line, and therefore the generatrix must pass through the point O in the figure. Let PSQ be another such line. Since the axis and generatrix are both parallel to the plane of projection, the projection of the angle PSO is a right angle as shown.

Hence
$$\frac{QS}{PS}=\frac{\tan\alpha}{\tan\beta}.$$

Fig. 93 is a horizontal projection of the same two hyperboloids, and the points P', Q', and S' are found by projecting P, S, and Q of Fig. 92. Now if a straight line be divided into any two parts, the ratio of these will be equal to the ratio of their projections upon any plane, therefore

$$\frac{PS}{QS}\ (\text{Fig. } 92) = \frac{P'S'}{Q'S'}\ (\text{Fig. } 93) = \frac{\tan\alpha}{\tan\beta}.$$

But

$$\frac{P'S'}{Q'S'} = \frac{R_1}{R_2}.$$

Hence

$$\frac{\tan\alpha}{\tan\beta} = \frac{R_1}{R_2}.$$

It is evident that if the tooth surfaces are so formed that their intersections with the pitch hyperboloids are coincident with the

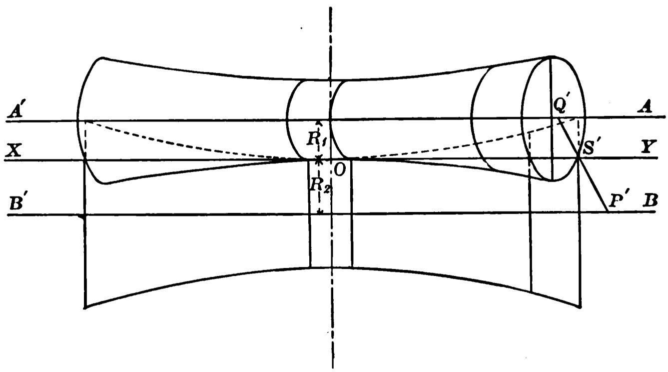


FIG. 93

generatrices, α and β will be the spiral angles of the two gears. There will be positive driving at right angles to the generatrices, and sliding along them. As in the case of spiral gears (Fig. 88), we have

$$\frac{\omega_1}{\omega_2} = \frac{R_2 \cos\beta}{R_1 \cos\alpha};$$

but

$$\frac{R_2}{R_1} = \frac{\tan\beta}{\tan\alpha}.$$

Hence

$$\frac{\omega_1}{\omega_2} = \frac{\sin\beta}{\sin\alpha}.$$

This is the same relation that we obtained when investigating the angles that give the least velocity of slip for spiral gears. (Compare Fig. 90.)

The method of producing cycloidal skew teeth, analogous to that used for spur and bevel teeth, fails in the present instance, for it has been shown by MacCord and others that tooth surfaces swept up by the generatrix of a small describing hyperboloid rolling within and without a pair of pitch hyperboloids will intersect along the generatrix instead of being tangent along it. (See also the *American Machinist*, September 5, 1889.)

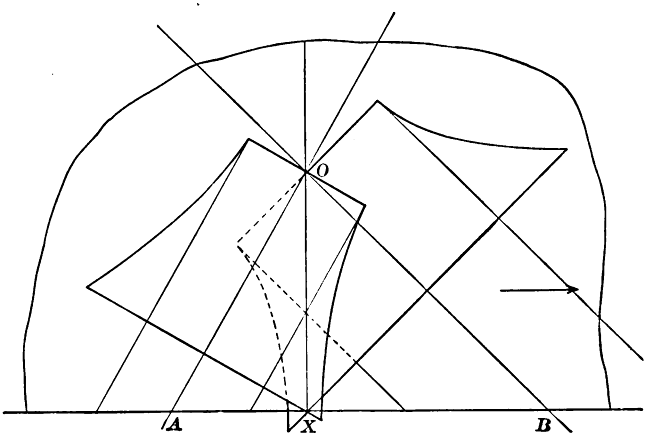


FIG. 94

Involute skew teeth can, however, be formed by a peculiar method. Let *OA* and *OB* (Fig. 94) be the axes of the pitch hyperboloids, tangent along the line *OX*, and with gorge circles tangent at *O*. Construct two cylinders, coaxial with the hyperboloids, and of diameters equal to the diameters of the gorge circles. These cylinders will be tangent at a single point, *O*. If we place a plane between the cylinders in such a way that it is tangent to both, this plane will contain the generatrix *OX*. If the plane be moved in the direction of the arrow at right angles to the generatrix, it will rotate the cylinders as if by friction, but there will be a sliding between the plane and the cylinders along the elements of the cylinders. If the generatrix is carried along with the plane it will sweep up a surface with respect to each cylinder, which is known as an Involute Spiraloid. Such surfaces will work together as perfect tooth surfaces if we make them of proper length to avoid interference. They will always be tangent along a line which is the generatrix itself.

The gears made by O. J. Beale for the Brown and Sharpe Mfg. Co. are made according to the above theory, which is due to Theodore Olivier. Beale's gears are planed out, the straight cutting edge of the tool acting as the generatrix.

Approximate skew gears can be drawn by Tredgold's approximation in the same way as bevel gears.

CHAPTER V

TRANSMISSION OF RECTILINEAR TRANSLATION

1. PRISMATIC GUIDES

If a point in a given piece of machinery is to be moved in a straight line, its motion may be constrained by means of guides. The form of these sliding pairs may be either cylindric or prismatic. The former is used usually where a rod works through packing. Prismatic guides are used in nearly all other cases, the flat guide to take up heavy pressures, and the V-shaped to prevent lateral motion. The problem of the prismatic guide is purely one of machine design.

2. PARALLEL MOTIONS

A. Classification

When a point is guided in a straight line, either wholly or in part by an assemblage of turning pairs, the mechanism is called a Parallel or Straight Line Motion. All forms of parallel motions will fall under one of the following four kinds:

Exact . . . { 1. Composed wholly of turning pairs.
2. Composed of turning and sliding pairs.

Approximate { 3. Composed wholly of turning pairs.
4. Composed of turning and sliding pairs.

B. The Cycloidal Straight Line Motions

If a circle rolls within another of double its diameter, every point on the circumference of the inner circle describes a straight line. This can be used as a parallel motion by constraining the centre of the small circle to move in a circle by means of the link OC

(Fig. 95). The point P will then move in a straight line AOB. But as two circles can be forced to roll together by means of gearing only, this form belongs to Class 2.

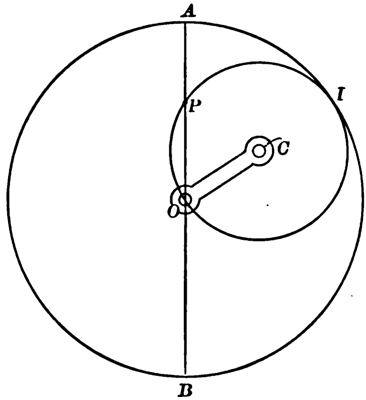


FIG. 95

In the case of the epicycloid, the rolling wheel turns in the same direction as the revolution of its centre. But by the introduction of an idle wheel, this can be reversed. At the centre O of the fixed gear (Fig. 96) an arm is hinged upon which are pivoted two gears, D and C, C being half the diameter of O. Upon C is fastened a link CP equal in length to CO. Then the triangle PCO is isosceles, and angle $OPC =$ angle $POC = \gamma$. Let angle $PCO = \beta$. Then

$$\beta = \pi - 2\gamma.$$

Let CO revolve through an angle α. Then wheel C has turned through 2α, and $\delta = \beta + 2\alpha$, or

$$\delta = \pi - 2\gamma + 2\alpha.$$

So we have

$$\theta = \tfrac{1}{2}(\pi - \delta) = \tfrac{1}{2}(\pi - \pi + 2\gamma - 2\alpha) = \gamma - \alpha = \text{Angle } P'OC'.$$

But angle $POC' = \gamma - \alpha$. Thus OP coincides with OP', and the locus of P is a straight line.

If in the hypocycloidal straight line motion we consider two diametrically opposite points of the small circle, we see that they will describe straight lines at right angles to one another. Therefore if we guide the centre of the small circle by means of a link CO (Fig. 97) and guide R through a short distance RS by means of a sliding pair, the motion of all points of circle C will be determined, and P will move in a straight line AB through quite a long distance for a very small motion of R. This also belongs to Class 2.

The angle α in Fig. 98 should not exceed 20°, otherwise the sliding at R becomes excessive. If $\alpha = 20°$, $s = \frac{2}{3}$ of $2a = 1\frac{1}{3}a$ (nearly). During such a stroke R travels through a distance $\rho = 2a(1 - \cos\alpha) = 2a \times .06 = .12a = .09s$. The use of the sliding pair at R is sometimes avoided by carrying this point on a vibrating pillar (Fig. 99). The motion now comes under Class 3. If the length of the pillar is equal to the stroke, and the angle α is equal

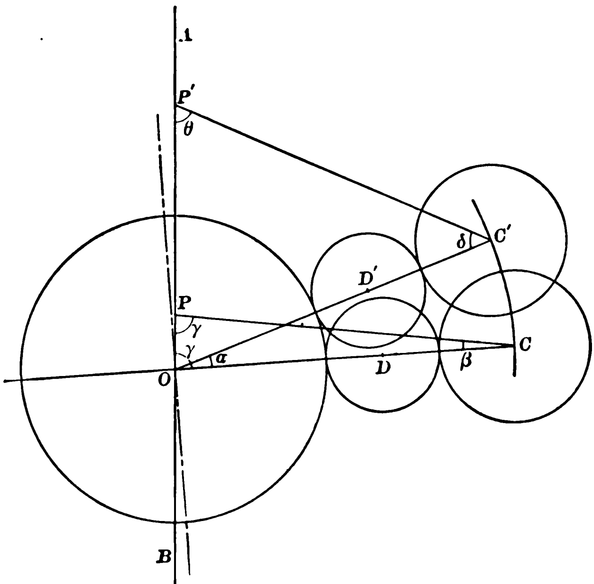


FIG. 96

to 20°, the maximum deviation from a straight line will be only $\frac{1}{4000}$ of the stroke.

It was seen that if the middle point of the link PR be chosen as the point of attachment of the link CO, the path of C was a circle. If any other point be chosen, its path will be an ellipse. The elliptic arc can then be approximated by the arc of a circle, and an approximate parallel motion is thus obtained. Fig. 100 shows the paths of the various points of the line PR, and also the radii and centres of the approximating circles. This form belongs to Class 4. If *R* is carried on the end of a vibrating pillar, it

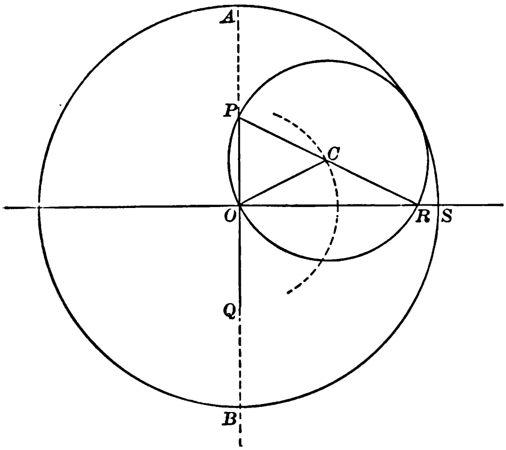


FIG. 97

belongs to Class 3. The constrainment of *P* to move in a straight line may be accomplished in other ways. Two points may be

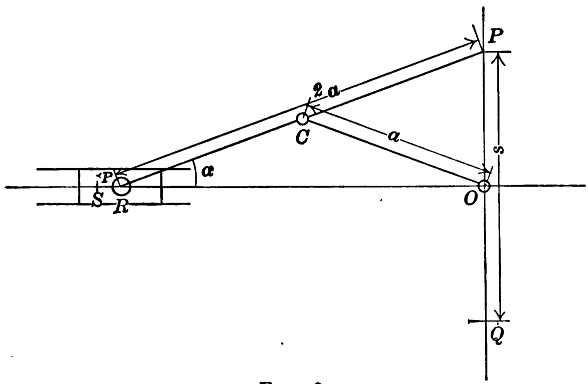


FIG. 98

chosen in the plane of the small circle, and the elliptic arcs approximated by circles. This is Robert's parallel motion.

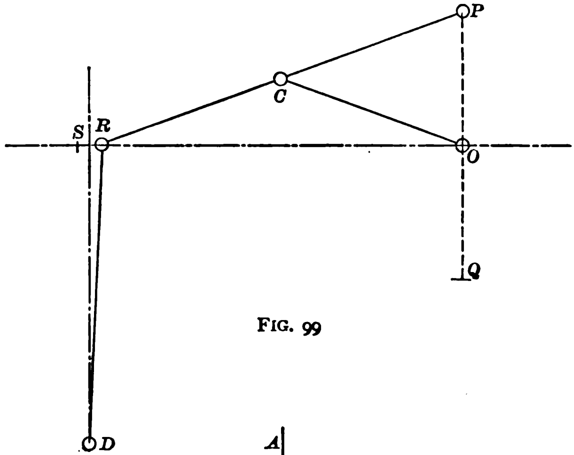


FIG. 99

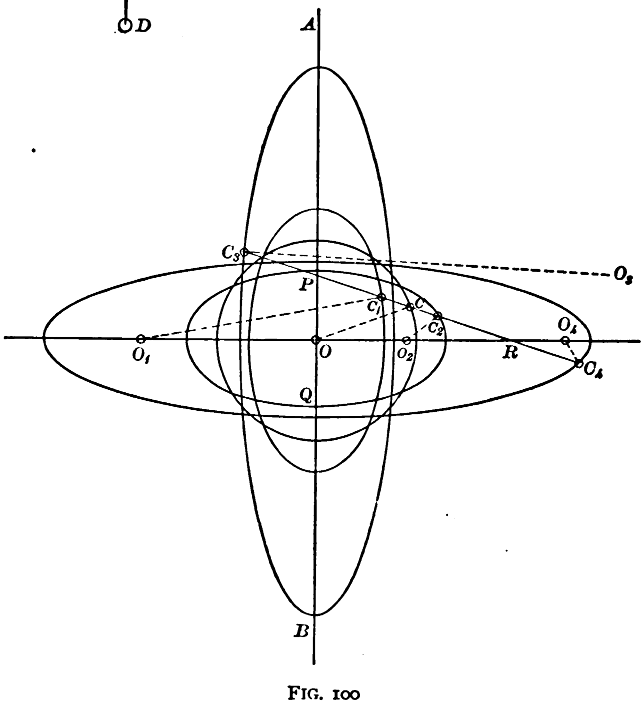


FIG. 100

C. The Conchoidal Straight Line Motions

If a line moves so that one of its points lies on a straight line, while the line itself always passes through a fixed point, all points of the line describe curves known as Conchoids. We can obtain the equation of the curve as follows: let *AB* (Fig. 101) be the

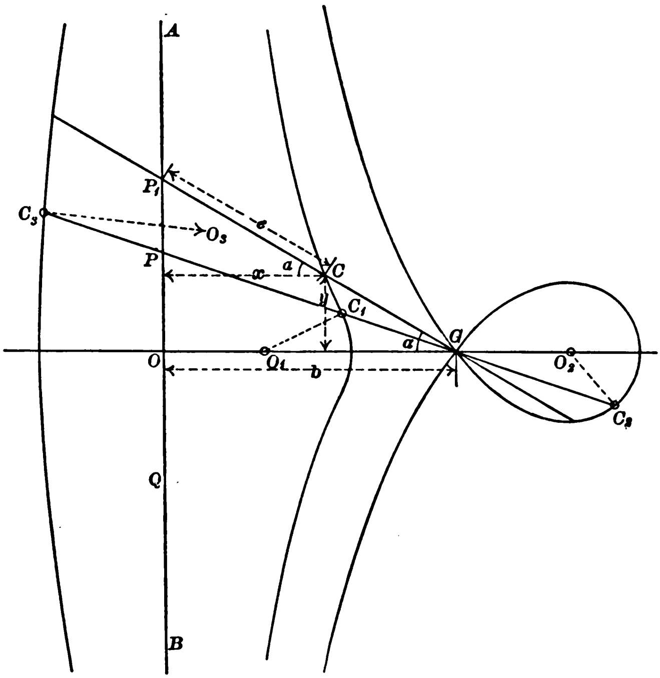


Fig. 101

straight line, and *G* the fixed point. From *G* drop a perpendicular *GO* on *AB*, and take *O* as the origin. Let P_1G be the moving line, and consider the conchoid as traced by point *C* whose coördinates are *x* and *y*. Then

$$y = (b - x) \tan \alpha,$$

$$\tan \alpha = \frac{\sqrt{e^2 - x^2}}{x},$$

$$y = (b - x) \frac{\sqrt{e^2 - x^2}}{x},$$

$$xy = (b - x) \sqrt{e^2 - x^2}.$$

If e is positive, and less than b, the curve is such as is described by C. If e is positive and equal to b, we have a cusp at G. If e is positive and greater than b, the curve is described by a point C_2. If e is negative, C_3 describes the curve. The curve can be used as a straight line motion by constraining points such as C_1, C_2, or C_3, to move in the arc of the conchoid, and also by con-

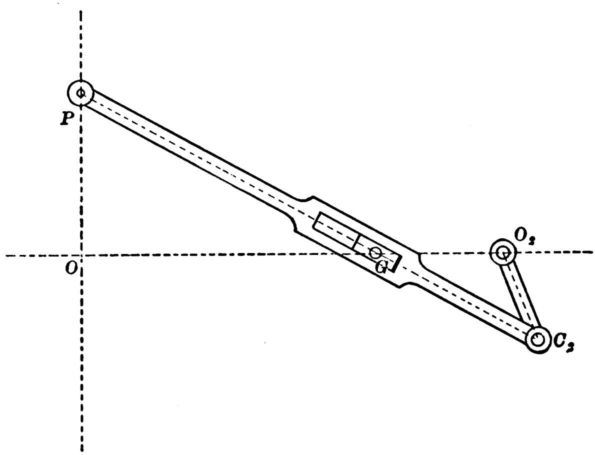


FIG. 102

straining the line to pass through G. The conchoidal arc must be approximated by means of the arc of a circle. The point C_2 is the best one to constrain, as the looped portion of the curve is very nearly a circle where it cuts the axis of X. (See Fig. 102.) At that point the radius of curvature is equal to $\rho = \frac{(e-b)^2}{e}$. All conchoidal guides belong to Class 4.

D. THE LEMNISCATE STRAIGHT LINE MOTIONS

If two points *C* and *D* of a line are made to describe circles, then any other point of the line will describe a peculiar looped curve known as a Lemniscoid. This curve is of the fourth degree, and a special form known as the Lemniscate, where the radius bars are equal and the tracing point is midway between *C* and *D*, is shown in Fig. 103. *CD* is the moving line, and *A* and *B* are the

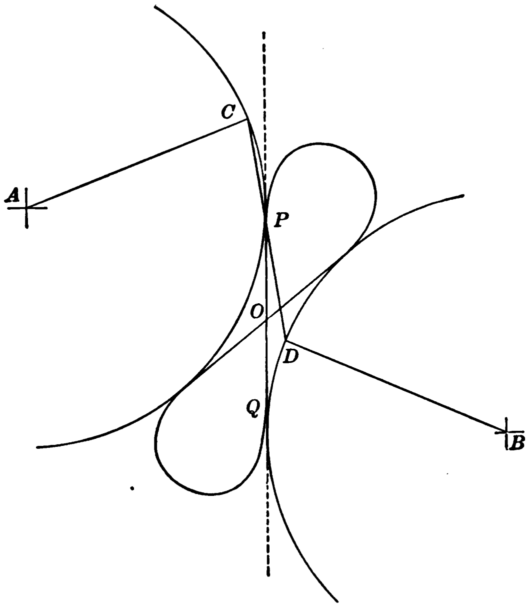


FIG. 103

fixed centres about which *C* and *D* revolve. *P* is the tracing point. That portion of the curve which is traced when the arms are parallel is of peculiar interest. Examination shows that for quite a distance on either side of this position the curve approximates a straight line. At the instant the arms are parallel, the line *CD* is turning about a centre at an infinite distance, and is therefore moving parallel to itself. We may consider the curve near the double point O as a straight line, and attach a cross-head or other portion of a machine to it. The conditions usually chosen are that the guided point should not only pass through the middle but also through the extreme positions of the rectilinear path. James Watt used this parallel motion to guide the cross-head of his steam-engine. He did not attach the cross-head directly to P, however, but duplicated the motion of P by means of an ordinary pantograph. All lemniscate straight line motions belong to Class 3.

E. INVERSORS

The first straight line motion discovered which fulfilled the conditions of Class 1 was based on the properties of inverse curves, particularly those of the circle. If P (Fig. 104) be a fixed point, UY any curve, and PB a radius vector intersecting UY at A, then the curve WX so constructed that $PA \times PB = PA' \times PB' =$

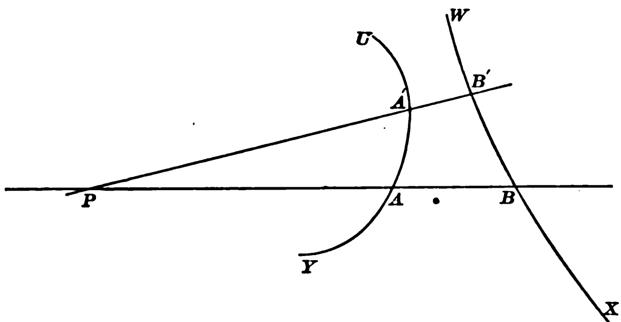


FIG. 104

constant, is called the inverse of UY with respect to P as a pole of inversion. It is shown by modern geometry that if UY is a circle, the curve WX is a circle also, and if the circle UY passes through P, WX becomes a straight line. This last proposition can be shown very simply. Let (P Fig. 105) be the pole of inversion. The circle $A'AP$, which we wish to invert, passes through it.

Draw BB' perpendicular to the diameter through P. Then by similarity of triangles,

$$PA : PB' :: PA' : PB,$$

or $$PA \times PB = PA' \times PB' = \text{constant}.$$

Hence BB' is the inverse of the circle through P.

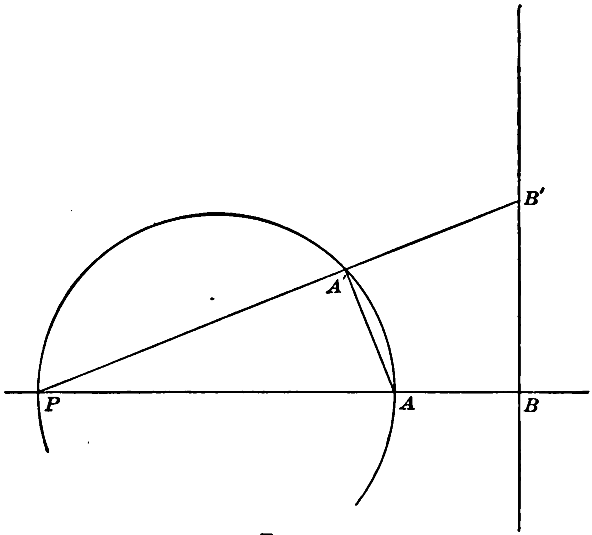


FIG. 105

Peaucellier's Straight Line Motion. — This consists of two equal links PD and PE hinged at a fixed point P. The extremities D and E are connected together by a rhombus of hinged links $ADBE$ (Fig. 106). The lines PAB and DE intersect in the centre of the rhombus at C. We have

$$PA = PC - CA = l \cos \alpha - m \cos \beta,$$

$$PB = PC + CA = l \cos \alpha + m \cos \beta,$$

$$PA \times PB = l^2 \cos^2 \alpha - m^2 \cos^2 \beta.$$

But also $$l \sin \alpha = m \sin \beta,$$

$$m^2 \cos^2 \alpha = m^2 - l^2 \sin^2 \alpha.$$

Hence, $PA \times PB = l^2 \cos^2 \alpha - m^2 + l^2 \sin^2 \alpha = l^2 - m^2 =$ constant. Therefore if we move A on any curve, B will move on the inverse curve with respect to P as a pole of inversion. Now if we constrain A to move in the arc of a circle by means of a link RA, B will move in a circle also. If RA is less than PR, B will travel in a circle whose concavity faces away from P. If RA is greater

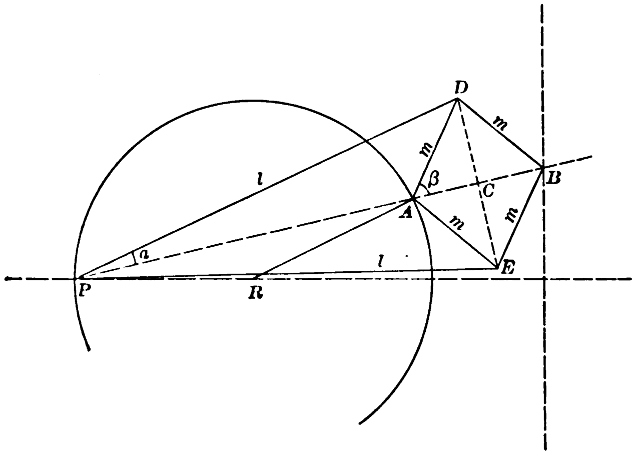


FIG. 106

than PR, B will travel in a circle whose concavity faces toward P. If AR equals PR, so that the circle in which A moves passes through P, B will describe a straight line. It is easy to find an expression for the radius of the inverse circle in terms of known constants. Call the constant product $PA \times PB$ ρ^2. Then in Fig. 107,

$$PA \times PB = PA' \times PB' = \rho^2,$$

$$PB = \frac{\rho^2}{PA} \quad PB' = \frac{\rho^2}{PA'},$$

$$PB' - PB = 2r' = \frac{\rho^2}{PA'} - \frac{\rho^2}{PA},$$

where r' is the radius of the inverse circle. If we denote by r the radius of the original circle, and the distance of its centre from P by a, we have

$$2r' = \frac{\rho^2 (PA - PA')}{PA \times PA'} = \frac{2r\rho^2}{PA \times PA'}.$$

But $$PA \times PA' = \overline{PG}^2 = a^2 - r^2.$$

Hence, $$r' = \frac{r\rho^2}{a^2 - r^2}.$$

If $a > r$, $r' = +$. If $a < r$, $r' = -$. If $a = r$, $r' = \infty$.

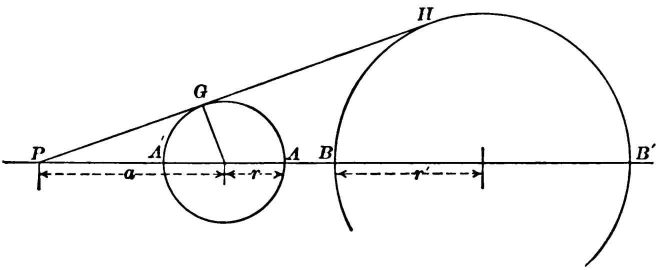


FIG. 107

Hart's Straight Line Motion. — Let the four rods GH, FK, GF, and KH (Fig. 108) be jointed together as shown. The figure $FGSKH$ is called a complete parallelogram when $FK = GH = L$, and $GF = KH = l$, which two lengths become constants of the chain. Call the distance $GK = x$ and $FH = y$. Then, as the form of the apparatus is changed, x and y will vary in length. Since the triangles GFH and KFH are equal (all three sides being equal), angle KHF is equal to angle GFH, which may be called β. Also the lines x and y are always parallel, since $l \sin \beta$ expresses their distance apart at either end. Take any fixed point in GF such as O, and through it draw a line parallel to x and y. Call $OF = a$. The line will evidently cut KH at R at a fixed distance a from H. It will also cut the other two links at P and Q. Call $FP = b$. Then $b = a\frac{L}{l}$ and b is constant, or P is a fixed point on link KF. Similarly Q is a fixed point on GH, and $QH = b$. Now $OP = x\frac{a}{l}$, and $OQ = y\frac{l-a}{l}$; hence,

$$OP \times OQ = xy\frac{a\,(l-a)}{l^2}.$$

But
$$x = L \cos \alpha - l \cos \beta,$$
$$y = L \cos \alpha + l \cos \beta,$$
$$xy = L^2 \cos^2 \alpha - l^2 \cos^2 \beta = L^2 \cos^2 \alpha - l^2 \,(l - \sin^2 \beta).$$

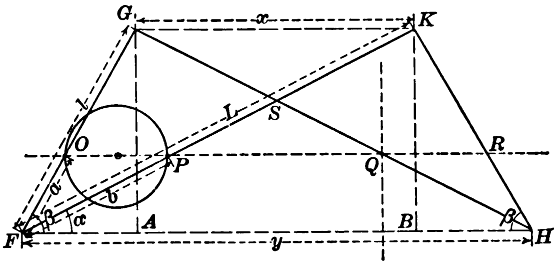


FIG. 108

From the figure
$$l \sin \alpha = L \sin \beta,$$
or
$$\sin \beta = \frac{L}{l} \sin \alpha,$$
$$xy = L^2 \cos^2 \alpha - l^2 \left(l - \frac{L^2}{l^2} \sin^2 \alpha\right) = L^2 - l^2.$$

Hence,
$$OP \times OQ = (L^2 - l^2)\,\frac{a\,(l-a)}{l^2} = \text{constant}.$$

Or, if the apparatus be hinged at O, P and Q will describe mutually inverse curves with respect to O as a pole of inversion. If P is made to move in a circle whose circumference passes through O, then will Q describe a straight line.

CHAPTER VI

TRANSMISSION OF MOTION BY CONTACT WHEN DIRECTIONAL RELATIONS ARE NOT CONSTANT

1. CAMS

A cam is a rotating body, which transmits motion to a follower by means of a curved edge. Usually the conditions do not involve any variable velocity ratio, but a series of positions of both driver and follower being given, as well as the outline of one, the form of the other is constructed.

A. Disk Cams

In the disk cam the motions of the driver and follower take place in the same plane. There are two principal kinds of disk cams. One transmits motion to a pin or roller, and the other to a straight bar. In the first case (Fig. 109), let O be the centre about which the cam revolves, and let OQ be the path of the pin or roller. A, 1, 2, 3, and 4 are given positions of the centre of the pin corresponding to positions OA, $O1'$, $O2'$, $O3'$, and $O4'$ of the cam radius. About O as a centre, and with a radius $O1$, draw an arc OB intersecting $O1'$ in B. Then B is a point on the pitch curve of the cam. In like manner points C, D, and E are ob-

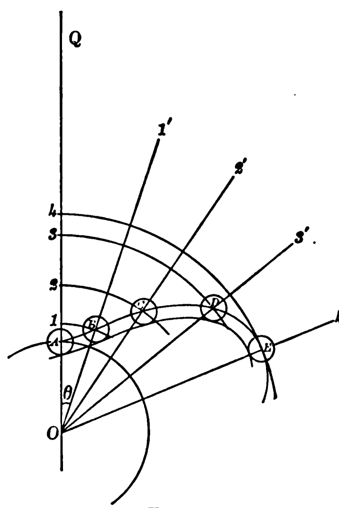


FIG. 109

tained. It is readily seen that the point A can be brought to rest at any time by giving the cam a circular outline about O as a centre. Having thus found the pitch curve, the working edge is obtained by drawing a parallel curve at a constant normal distance from it equal to the radius of the roller. If A is started from rest, care should be taken not to give it too great velocity at first; in other words it should be given, if possible, a gradually increased velocity, and the same care should be observed in bringing A to rest. If A is given a constant acceleration during the first part of its travel, the equation of the cam curve can be worked out easily if the angular velocity of the cam is constant.

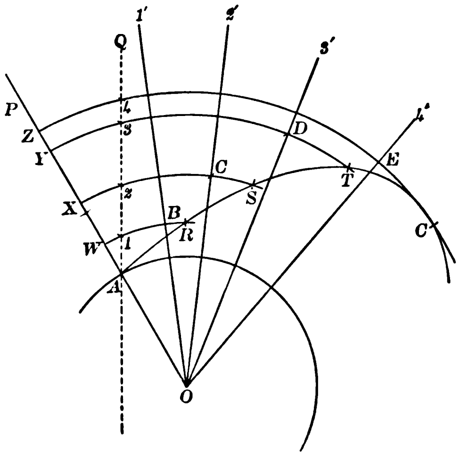


FIG. 110

The constant distance OA we may call a. Then

$$A1 = \tfrac{1}{2}ct^2,$$

where c is the constant acceleration. The angle $BOA = \theta$, hence

$$\frac{\theta}{t} = \omega = \text{constant},$$

$$r = OB = a + \tfrac{1}{2}ct^2$$

$$= a + \tfrac{1}{2}c\frac{\theta^2}{\omega^2},$$

or the polar equation of the curve is of the form

$$r = a + b\theta^2.$$

In a similar method the equation of that portion giving a constant retardation can be worked out. Care should be taken in this case that c is not greater than g, the gravitation constant, otherwise the follower will leave the cam curve unless some other force than gravitation is employed to cause its return.

If QA produced does not pass through O, we proceed as follows: let A, 1, 2, 3, and 4 (Fig. 110) be positions of the follower corresponding to positions OA, $O1'$, $O2'$, $O3'$, and $O4'$ of the cam. Draw

OP through *A*. Construct arcs about *O* as a centre cutting the radii in *B*, *C*, *D*, and *E*, and the line *OP* in *W*, *X*, *Y*, and *Z*. On arc 1 *B* lay off *BR* equal to *W* 1, on arc 2 *C* lay off *CS* equal to *X* 2, etc. Then obviously *ARSTU* will be the pitch curve of the cam.

In the second case of disk cam, the motion which can be given to a follower is not so universal as that which can be given by the first. The manner of constructing the cam curve to satisfy a given

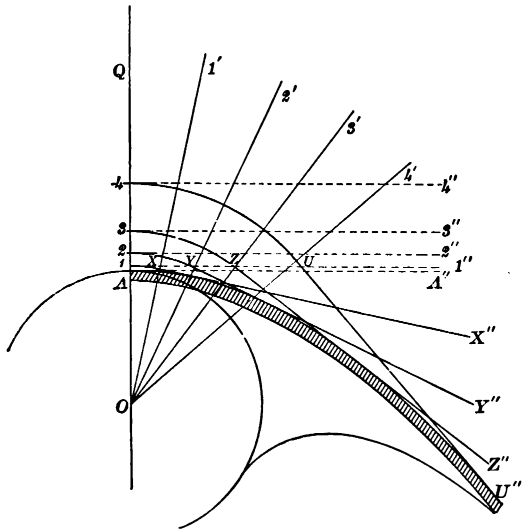


FIG. 111

condition is as follows: let *AA''*, 11'', 22'', 33'', and 44'' (Fig. 111) be successive positions of a bar which moves parallel to itself in the direction *OQ*. Let the corresponding radii of the cam which successively occupy the position *OQ* be *OA*, *O* 1', *O* 2', *O* 3', and *O* 4'. Suppose we bring the cam to rest by giving the whole an equal and opposite rotation about *O*. The lines 11'', 22'', 33'', and 44'' will take up the positions *XX''*, *YY''*, *ZZ''*, and *UU''*, where these are drawn at right angles to the several radii, from points projected by circular arcs from 1, 2, etc. But these lines must all touch the cam, hence they form the envelope of the required curve. That these conditions cannot always be fulfilled, can best be seen by referring to Fig. 112. If the line which forms the envelope intersects its previous positions in points 1, 2, 3, 4,

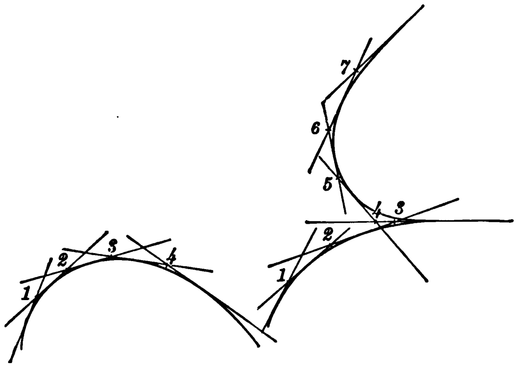


FIG. 112

etc., one beyond the other, the cam curve can be cut out of metal. But if any intersection falls between two previous ones, the construction is impossible, as a cusp is formed at the point where the first two intersections coincide.

If the straight bar rotates about a fixed centre instead of moving with pure translation, the same general method would be pursued.

B. CYLINDRICAL CAMS

Here the motion of the follower is in a plane at right angles to that of the driver. A helical slot or groove of the desired form is cut in the surface of a cylinder, and is made to engage a pin or roller whose diameter is equal to the width of the slot. The cylindrical surface of the cam must here be developed on a plane and the motion of the follower be considered as referred to it. In

Fig. 113 is shown the complete solution of a case of the cylindrical cam. The developed projection gives the best idea of the general

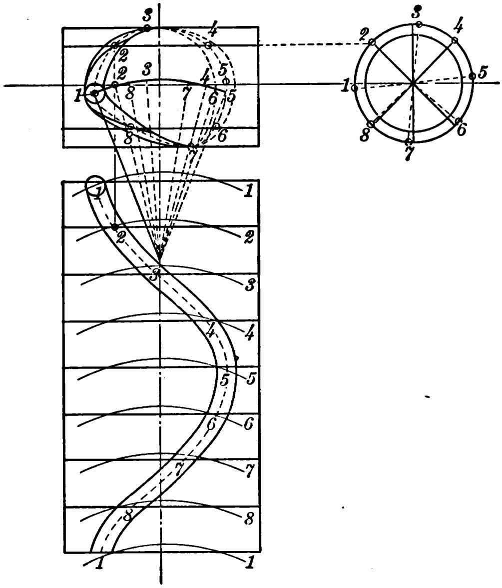


FIG. 113

form of the slot, but the curve can be reprojected on the original cylinder as shown if required.

PART III

MECHANICS OF THE STEAM-ENGINE

CHAPTER I

KINEMATICS

1. GENERAL DESCRIPTION OF THE STEAM-ENGINE CHAIN

In its most elementary form the steam-engine consists of six links as shown in Fig. 114. The bed or frame is represented by *a*, and to this the motion of all other links is referred; *b* is the crank, including the shaft, eccentric, and fly-wheel; *c* is the piston, piston-rod, and cross-head; *d* the connecting rod; *e* the eccentric rod;

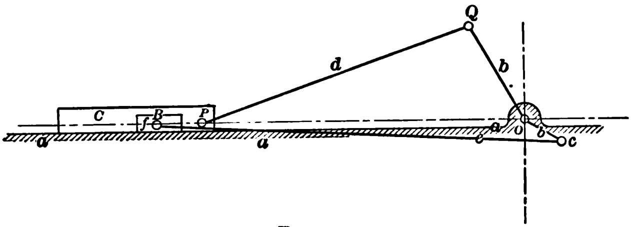


FIG. 114

and *f* the valve. This compound chain may be divided into two simple ones, the crank and engine bed being common to both, and the whole reduces to two chains like Fig. 5. It is the purpose of this chapter to discuss the relative motions of the various parts of the steam-engine chain, and to find the general proportions and positions of the parts which will give the best results.

2. THE PISTON-CRANK CHAIN

This chain consists of four links,—the engine bed, the crank, the connecting rod, and the piston. Referring all motions to the engine bed, the motion of the crank is one of pure rotation, that of the piston is pure rectilinear translation, while that of the connecting rod is a combination of both.

A. Relation between the Position of the Crank and the Positions of Other Points of the Chain

The position of a point in the connecting rod for any given angular position of the crank can be expressed as follows: Let O (Fig. 115) be the centre of the crank shaft. Since the motion of the piston is identical with that of the centre of the wrist pin, we may consider the whole of the piston and cross-head as concentrated there. Consider the position of any point R of the

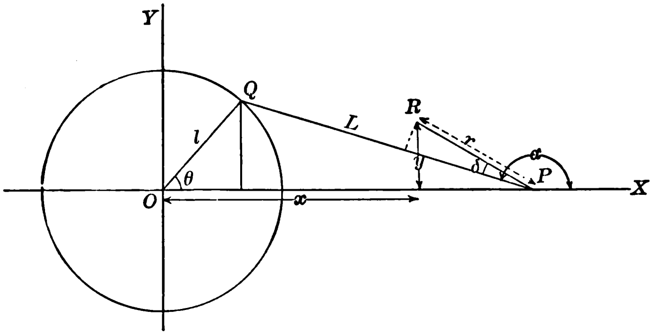


Fig. 115

connecting rod. Call its coördinates referred to an origin at O and axis of X in the line of connection, x and y, and call its fixed polar coördinates referred to pole at P and vectorial angle measured from PQ, r and δ. θ is the variable crank angle, α the corresponding angle of the connecting rod, L the length of the connecting rod, and l that of the crank. Then

$$x = l\cos\theta - L\cos\alpha + r\cos(\alpha - \delta),$$

$$y = r\sin(\alpha - \delta).$$

Since $$l\sin\theta = L\sin\alpha,\quad \sin\alpha = \frac{l}{L}\sin\theta = n\sin\theta,$$

$$\cos\alpha = -\sqrt{1 - n^2\sin^2\theta}.$$

Expanding the expression for x,

$$x = l\cos\theta - L\cos\alpha + r\cos\alpha\cos\delta + r\sin\alpha\sin\delta.$$

And substituting for α in terms of θ,

$$x = l\cos\theta + \left(\frac{l - nr\cos\delta}{n}\right)\sqrt{1 - n^2\sin^2\theta} + rn\sin\delta\sin\theta.$$

In the same way,

$$y = r\sin\alpha\cos\delta - r\cos\alpha\sin\delta$$
$$= rn\cos\delta\sin\theta + r\sin\delta\sqrt{1 - n^2\sin^2\theta}.$$

These are the most general expressions for the coördinates of a point of the connecting rod in terms of the variable angle θ. When the point R lies in the axis of the rod, $\delta = 0$, and the equation reduces to

$$x_0 = l\cos\theta + \frac{(l - nr)}{n}\sqrt{1 - n^2\sin^2\theta},$$
$$y_0 = rn\sin\theta.$$

Finally, if $r = L$, we get the position of the crank pin referred to O as an origin,

$$x_Q = l\cos\theta,$$
$$y_Q = l\sin\theta.$$

The coördinates of other points of the crank can be similarly expressed in terms of their angular position and distance from the centre of rotation.

Now if $r = 0$, the position of P is defined, or

$$x_P = l\cos\theta + \frac{l}{n}\sqrt{1 - n^2\sin^2\theta},$$
$$y_P = 0.$$

If the stroke of the piston is supposed to take place along the diameter $C'H'$ of the crank orbit (Fig. 116), its position measured from the middle of its stroke will be

$$x_P' = x_P - L$$
$$= l\cos\theta + \frac{l}{n}(\sqrt{1 - n^2\sin^2\theta} - 1).$$

These last, which are the most important of all, express the position of the centre of the wrist pin as a distance measured from the centre of the shaft or from the middle of the stroke. But piston positions are often measured from one or the other ends of the stroke, or from H and C (Fig. 116). Calling these distances in general z,

$$z_H = L + l - x_P$$
$$= l(1 - \cos\theta) + \frac{l}{n}(1 - \sqrt{1 - n^2 \sin^2\theta}),$$
$$z_C = l(1 + \cos\theta) - \frac{l}{n}(1 - \sqrt{1 - n^2 \sin^2\theta}).$$

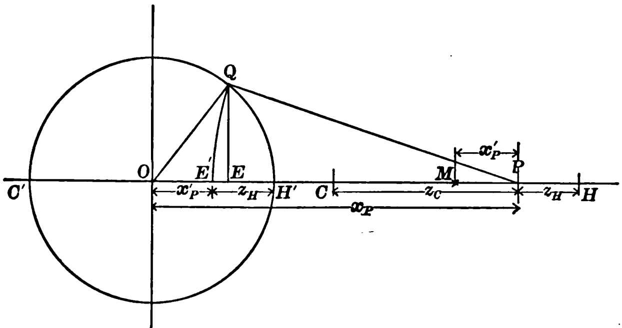


FIG. 116

If finally the connecting rod be of infinite length, $n = 0$, the motion of P becomes simply harmonic, and substituting $n = 0$ in the expression for z, and evaluating the indeterminate form, we have

$$x_P = \infty, \ x_P' = l \cos\theta,$$
$$z_H = l(1 - \cos\theta),$$
$$z_C = l(1 + \cos\theta).$$

It is often useful to lay off x_P' as a polar curve, using OR as a radius vector. Calling $\rho = x_P'$, the equation of the curve is

$$\rho = l \cos\theta + \frac{l}{n}(\sqrt{1 - n^2 \sin^2\theta} - 1),$$

and its form is shown in Fig. 117. If $n = 0$, the equation reduces to the simple form

$$\rho = l \cos\theta,$$

which is the equation of a circle whose diameter is l, whose centre lies on the axis of X to the right of the origin, and whose circumference passes through O.

The graphical solution of all the preceding cases consists in nothing more than the construction of an accurate drawing of the

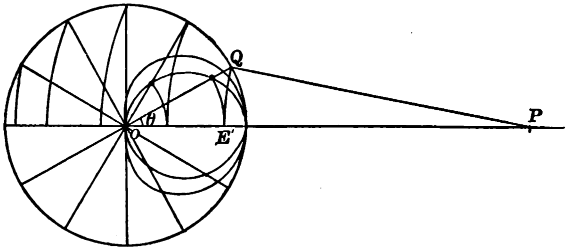


FIG. 117

chain for the chosen value of θ, from which x or z can be scaled off. If the stroke of the piston be considered to take place on $C'H'$ (Fig. 116), the position of the piston can be easily found at E' by projecting Q on a circular arc from a centre on the line of connection and with a radius equal to L. Similarly, if the length of the rod be infinite, the position will be at E, found by straight projection from Q.

B. Relation between the Position of the Crank and the Velocities of Other Points in the Chain

The component velocities of R, parallel to the axes of X and Y, can be found by evaluating $\frac{dx}{dt}$ and $\frac{dy}{dt}$, after which the resultant velocity can be found both in magnitude and direction. We have,

$$x = l\cos\theta + \left(\frac{l - nr\cos\delta}{n}\right)\sqrt{1 - n^2\sin^2\theta} + rn\sin\delta\sin\theta,$$

from which

$$\frac{dx}{dt} = -l\sin\theta\frac{d\theta}{dt} - n(l - nr\cos\delta)\frac{\sin\theta\cos\theta}{\sqrt{1-n^2\sin^2\theta}}\frac{d\theta}{dt} + rn\sin\delta\cos\theta\frac{d\theta}{dt}.$$

And also $y = rn \cos\delta \sin\theta + r\sin\delta\sqrt{1 - n^2\sin^2\theta}$,

$$\frac{dy}{dt} = rn\cos\delta\cos\theta\frac{d\theta}{dt} - rn^2\sin\delta\frac{\sin\theta\cos\theta}{\sqrt{1-n^2\sin^2\theta}}\frac{d\theta}{dt}.$$

In nearly all cases of steam-engine analysis, the fly-wheel is sufficiently large and heavy to make $\frac{d\theta}{dt}$ practically constant. Calling this ω, the formulæ reduce to

$$\frac{dx}{dt} = -l\omega\sin\theta - n\omega(l - nr\cos\delta)\frac{\sin\theta\cos\theta}{\sqrt{1-n^2\sin^2\theta}} + rn\omega\sin\delta\cos\theta,$$

$$\frac{dy}{dt} = rn\omega\cos\delta\cos\theta - rn^2\omega\sin\delta\frac{\sin\theta\cos\theta}{\sqrt{1-n^2\sin^2\theta}}.$$

These are the general expressions in terms of θ for the component velocities of any point in the connecting rod, whose position is defined as before. The resultant velocity will be

$$v = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2},$$

and the angle which its direction will make with the axis of X will be

$$\epsilon = \tan^{-1}\frac{dy}{dx}.$$

Now if the point R lies in the axis of the rod, $\delta = 0$, and the velocities are

$$\frac{dx_0}{dt} = -l\omega\sin\theta - n(l - nr)\omega\frac{\sin\theta\cos\theta}{\sqrt{1-n^2\sin^2\theta}},$$

$$\frac{dy_0}{dt} = rn\omega\cos\theta.$$

If $r = L$, we get the component velocities of the crank pin,

$$u_x = -l\omega\sin\theta,$$

and $$u_y = l\omega\cos\theta.$$

The component velocities of all other points in the crank can be similarly expressed in terms of their angular position and distances from the centre.

Finally if $r = 0$, we have the velocity of P, or that of the centre of the wrist pin,

$$v_x = -l\omega \sin\theta - nl\omega \frac{\sin\theta \cos\theta}{\sqrt{1 - n^2 \sin^2\theta}}.$$

$$v_y = 0.$$

And the velocities of all other points of the piston and cross-head will be the same.

This most important velocity can be expressed as the ordinate of a curve in rectangular coördinates, whose abscissæ are piston

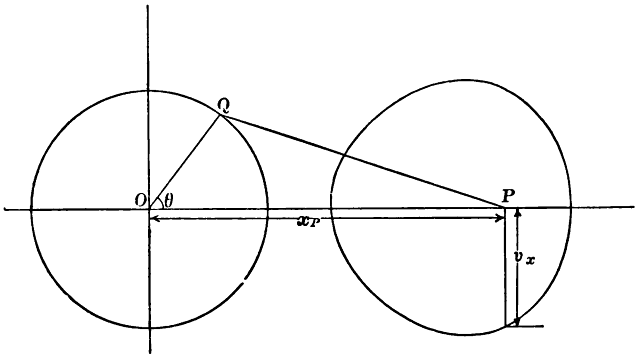


FIG. 118

positions. Taking a number of arbitrary values of θ, the corresponding values of x and v_x can be calculated, and the results plotted in graphical form along the stroke of the piston or the diameter of the crank orbit. (See Fig. 118.)

When the length of the connecting rod is infinite, compared with the throw of the crank, n becomes equal to zero. Substituting this value,

$$v_x = -l\omega \sin\theta.$$

In this case θ can be simply eliminated between the equations of position and velocity. Referred to the diameter of the crank orbit as a stroke, the position of the piston measured from O is

$$x_P' = l \cos\theta.$$

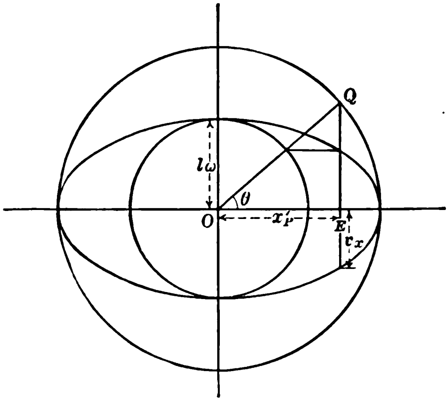


FIG. 119

Eliminating θ,

$$\frac{(x_P')^2}{l^2} + \frac{(v_x)^2}{l^2\omega^2} = 1.$$

This is the equation of an ellipse referred to its principal axes, as is evidently the necessary result of a simple harmonic motion. The angle θ is the eccentric angle (Fig. 119).

The piston velocity, viz., v_x, may be laid off along the crank OR in such a way as to form a polar curve of piston velocities. Calling the radius vector σ, the equation of the curve is

$$\sigma = -l\omega \sin\theta - nl\omega \frac{\sin\theta\cos\theta}{\sqrt{1 - n^2\sin^2\theta}}.$$

This curve is shown in Fig. 120.

When $\qquad n = 0, \qquad \sigma = -l\omega \sin\theta,$

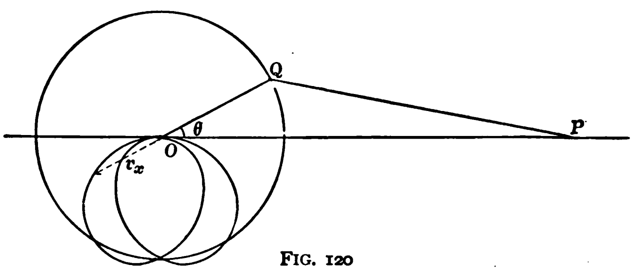


FIG. 120

the equation of a circle whose centre lies on the axis of Y, below the origin, whose diameter is l, and whose circumference passes through the origin.

The angular velocity of the crank is, as we have said, constant and equal to ω.

The angular velocity of the connecting rod is found by evaluating $\frac{d\alpha}{dt}$. Since $\alpha = \sin^{-1}\{n \sin\theta\}$,

we have
$$\frac{d\alpha}{dt} = \frac{-n\omega\cos\theta}{\sqrt{1 - n^2\sin^2\theta}}.$$

Graphical constructions for the above velocities are, in general, simpler and more useful than the analytic. Produce the axis of

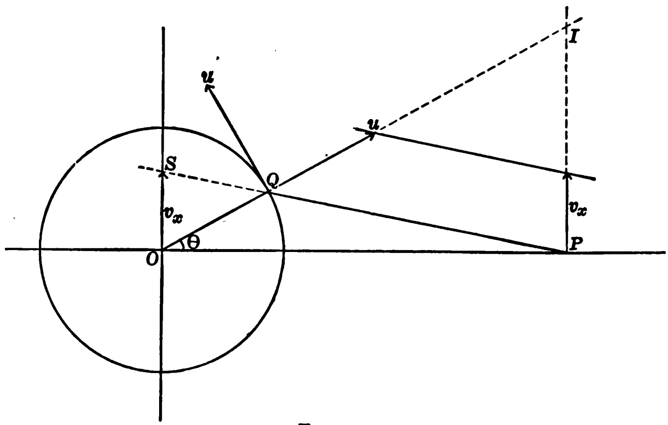


FIG. 121

the connecting rod to cut the axis of Y at S (Fig. 121). Let I be the instantaneous centre of the connecting rod referred to the bed of the engine. Then
$$\frac{velQ}{velP} = \frac{u}{v_x} = \frac{IQ}{IP}.$$

But from similarity of triangles $\frac{IQ}{IP} = \frac{OQ}{OS}$;

hence,
$$\dot{v}_x = u\frac{OS}{OQ},$$

and since u $(= l\omega)$ and OQ $(= l)$ are constants, v_x is proportional to OS. Furthermore if we choose our scale of velocities so that u is laid off equal to the crank throw, OR becomes equal to u, and $OS = v_x$.

Another simple construction is to lay off u from Q along QI. Through its extremity draw a line parallel to the connecting rod, and where this cuts IP will give v_x, since $\frac{u}{v_x} = \frac{QI}{PI}$. In either case the curve of velocities can be simply drawn in.

When the length of the connecting rod is infinite, OS is simply the projection of OQ on the axis of Y, and the velocity curve becomes a circle when the scale is taken such that $u = OQ = l$.

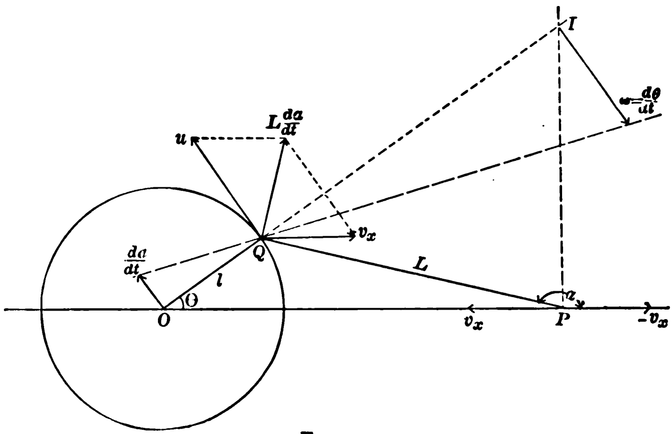


FIG. 122

This is seen also from the analytic expression, for, when $l\omega = l$ by reason of choice of scales, the equation of the ellipse reduces to that of a circle.*

The angular velocity of the connecting rod can also be graphically constructed by employing the instantaneous centre method. By reference to Fig. 16, Part I, the following construction can be verified. Lay off from I in any direction the constant angular

---

* Care should be taken here to avoid confusion. It is not intended that since $l\omega = l$, $\omega = 1$; but that $\omega = 1 \times \left\{ \frac{\text{scale of lengths}}{\text{scale of velocities}} \right\}$.

velocity of the crank. From the extremity of this line draw a line through Q. From O draw a line parallel to ω to meet the one last mentioned, and this will give the magnitude of $\frac{d\alpha}{dt}$. Its direction can be told by simple inspection of the drawing. This construction is shown in Fig. 122.

Another simple solution, where the velocities of both P and Q are given, is also shown on the same figure. Bring P to rest by giving the whole system a velocity equal and opposite to v_x. Combining $-v_x$ with u at Q, we get QN, the absolute velocity of Q referred to the piston. This is $L\frac{d\alpha}{dt}$, from which $\frac{d\alpha}{dt}$ can be obtained by simple division.

C. Relation between the Position of the Crank and the Accelerations of Other Points in the Chain

The component accelerations of R parallel to X and Y can be found by evaluating $\frac{d^2x}{dt^2}$ and $\frac{d^2y}{dt^2}$. We have, where $\omega =$ constant,

$$\frac{dx}{dt} = -l\omega \sin\theta - n\omega(l - nr\cos\delta)\frac{\sin\theta\cos\theta}{\sqrt{1-n^2\sin^2\theta}} + rn\omega\sin\delta\cos\theta,$$

$$\frac{d^2x}{dt^2} =$$

$$-l\omega^2\cos\theta - n\omega^2(l - nr\cos\delta)\left\{\frac{\cos 2\theta + n^2\sin^4\theta}{(1-n^2\sin^2\theta)^{\frac{3}{2}}}\right\} - rn\omega^2\sin\delta\sin\theta.$$

And also

$$\frac{dy}{dt} = rn\omega\cos\delta\cos\theta - rn^2\omega\sin\delta\frac{\sin\theta\cos\theta}{\sqrt{1-n^2\sin^2\theta}},$$

$$\frac{d^2y}{dt^2} = -rn\omega^2\cos\delta\sin\theta - rn^2\omega^2\sin\delta\left\{\frac{\cos 2\theta + n^2\sin^4\theta}{(1-n^2\sin^2\theta)^{\frac{3}{2}}}\right\}.$$

These are the general expressions in terms of θ, for the component accelerations of any point of the connecting rod whose position is defined by r and δ as before described. The resultant acceleration will be,

$$p = \sqrt{\left(\frac{d^2x}{dt^2}\right)^2 + \left(\frac{d^2y}{dt^2}\right)^2}$$

in magnitude, and its direction will be defined by the angle ϵ, where

$$\epsilon = \tan^{-1} \frac{\frac{d^2y}{dt^2}}{\frac{d^2x}{dt^2}}.$$

If the point R lies in the axis of the rod, $\delta = 0$, and we get,

$$\frac{d^2x_0}{dt^2} = -l\omega^2 \cos\theta + n\omega^2(l - nr)\left\{\frac{\cos 2\theta + n^2 \sin^4\theta}{(1 - n^2 \sin^2\theta)^{\frac{3}{2}}}\right\},$$

$$\frac{d^2y_0}{dt^2} = -rn\omega^2 \sin\theta.$$

If $r = L$, we get the component accelerations of the crank pin,

$$q_x = -l\omega^2 \cos\theta,$$

and

$$q_y = -l\omega^2 \sin\theta.$$

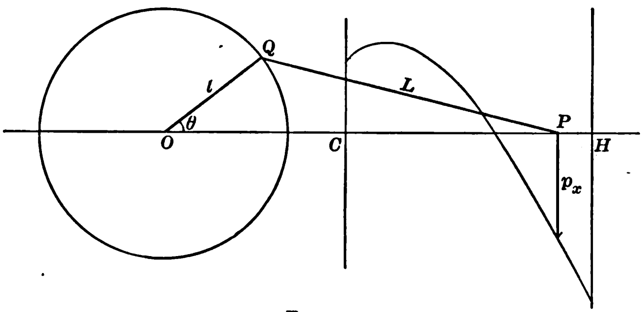


FIG. 123

The accelerations of all other points of the crank can be similarly expressed in terms of their angular positions and distances from the axis of rotation.

Finally, if $r = 0$, we have the acceleration of P or of the centre of the wrist pin,

$$p_x = -l\omega^2 \cos\theta - n\omega^2 l\left\{\frac{\cos 2\theta + n^2 \sin^4\theta}{(1 - n^2 \sin^2\theta)^{\frac{3}{2}}}\right\},$$

and

$$p_y = 0.$$

and all other points of the piston and cross-head have the same acceleration. All these accelerations are of the utmost importance in connection with the dynamics of the steam-engine.

By choosing arbitrary values of θ, and computing the corresponding values of p_x and x'_p, a curve can be drawn with reference to rectangular coördinates, which shows by its ordinates the magnitude of the piston acceleration and by its abscissæ piston positions. Such a curve is shown in Fig. 123. When the length of the connecting rod is infinite compared with that of the crank, $n = 0$, and the motion becomes simply harmonic, as in that case,

$$p_x = -l\omega^2 \cos\theta.$$

Here we can easily eliminate θ between this, and the equation for piston position, which is,

$$x'_p = l\cos\theta,$$

and we have

$$p_x = -\frac{l\omega^2 x}{l} = -x\omega^2,$$

the equation of a straight line through the origin.

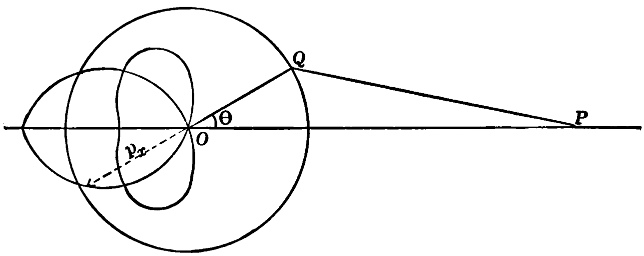


FIG. 124

We may also lay off p_x as a radius vector along the crank throw, and thus construct a polar curve of accelerations. Calling the radius vector τ, the equation of the polar curve would be,

$$\tau = -l\omega^2\cos\theta - n\omega^2 l\left\{\frac{\cos 2\,\theta + n^2\sin^4\theta}{(1 - n^2\sin^2\theta)^{\frac{3}{2}}}\right\}.$$

This curve is shown in Fig. 124.

If $n=0$, the equation reduces to

$$\tau = -l\omega^2 \cos\theta,$$

the equation of a circle whose diameter is l, whose centre lies on the axis of X to the left of the origin, and whose circumference passes through the origin.

The angular acceleration of the connecting rod can be found by simply differentiating the expression for its angular velocity,

$$\frac{d\alpha}{dt} = \frac{-n\omega\cos\theta}{\sqrt{1-n^2\sin^2\theta}},$$

$$\frac{d^2\alpha}{dt^2} = \frac{n(1-n^2)\omega^2\sin\theta}{(1-n^2\sin^2\theta)^{\frac{3}{2}}},$$

ω being taken as constant.

Graphic solutions of the general case can best be done by combining the solutions of the special cases of the two ends. The graphic solution of the acceleration of the crank pin Q follows immediately from the analytic equations for q_x and q_y. For if we lay off a distance from Q toward O equal to $-l\omega^2$ (Fig. 125), the projections of this length on the coördinate axes will give the component accelerations of Q as these are evidently $-l\omega^2\cos\theta$, and $-l\omega^2\sin\theta$. If the scale of accelerations is so chosen that $-l\omega^2$ is equal to OQ, the component accelerations will be merely the projections of the crank-throw itself upon the coördinate axes.

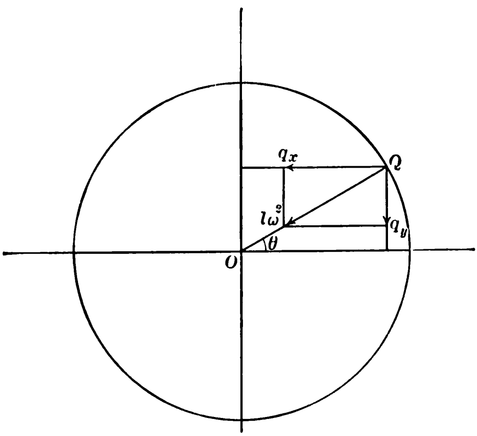


FIG. 125

The graphic solution of the acceleration of the wrist pin is more complex. We must here again employ the instantaneous centre of the connecting rod with respect to the bed of the engine. Since *OS* (Fig. 121) is proportional to the velocity of *P*, and since *OS* is measured from a fixed point *O* and in a fixed direction *OY*, the velocity of *S* along *OY* will be proportional to the acceleration of *P*. Imagine the line *Qw* (Fig. 126) as moving with

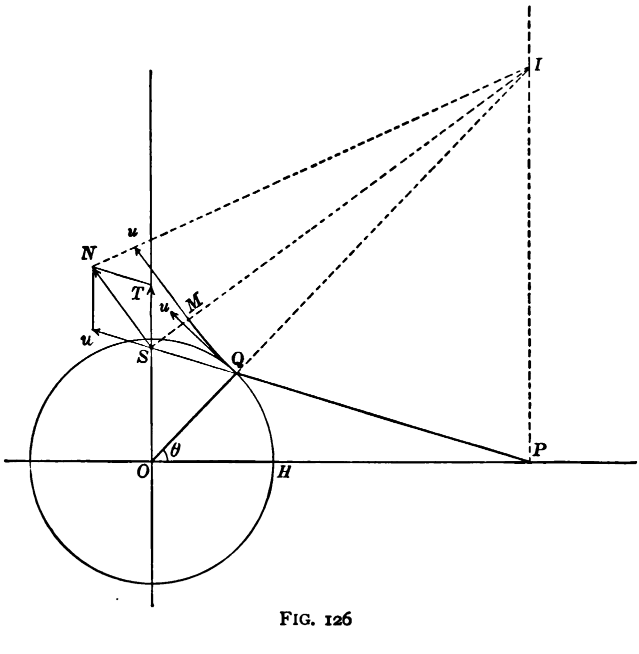


FIG. 126

PQ, being in fact a prolongation of the axis of the rod. Then *S*, when considered a point of *PQ*, is moving in a direction *SN* perpendicular to *SI*, the instantaneous radius. If we project *Q* to *M* on the arc of a circle about *I* as a centre, the velocity of *M* will evidently be equal to that of *Q*, and at right angle to *SI*. This velocity *u* can then be laid off, and from it the magnitude of the velocity of *S* follows by obvious construction. Now if this velocity

SN of *S* be resolved along the axis of *Y* and along *PQ*, the former will evidently be proportional to the acceleration of *P*. This proportionality becomes an equality when as before the scale of accelerations is so chosen that the acceleration of *Q*, viz., $-l\omega^2$, is taken equal to *OQ*.

This choosing of the proper scales of velocity and acceleration sometimes give rise to confusion in the mind of the student. All such trouble can be avoided by the following simple device. The analytic expression for the acceleration of *P* is

$$p_x = -l\omega^2 \cos\theta - n\omega^2 l \left\{ \frac{\cos 2\theta + n^2 \sin^4\theta}{(1 - n^2 \sin^2\theta)^{\frac{3}{2}}} \right\}.$$

At the head-end dead point, *i.e.* when $\theta = 0°$, this reduces to the simple form

$$p_x \Big]_{\theta=0°} = -l\omega^2(1 + n),$$

and at the crank-end dead point to

$$p_x \Big]_{\theta=180°} = +l\omega^2(1 - n),$$

which expressions can be simply calculated Now in the preceding construction when the crank pin *Q* arrives at one of the dead

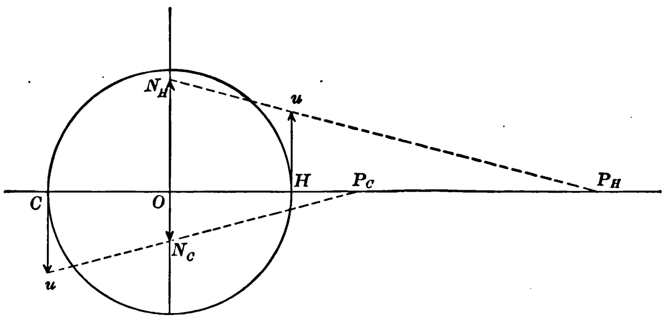


FIG. 127

points, *H*, for instance, *I* coincides with *P*, *M* and *Q* with *H*, and *S* and *w* with *O*. The construction in this case is as in Fig. 127, where *Hu* is the velocity of the crank pin, and ON_H the acceleration of the cross-head. Then if we calculate this acceleration from the simplified analytic expression, lay it off to any convenient scale as ON_H, and draw N_HP_H, the proper scale of velocities becomes immediately known as Hu, and this may now be used all around the crank orbit to get other values of the acceleration of P on the same scale as ON_H.

Having now the accelerations of both ends of PQ, it is easy to determine that of any other point R (Fig. 128). Project R on the arc of a circle about P as a centre to A, and consider first the acceleration of A. The absolute acceleration of Q is QT equal to $-l\omega^2$ directed inward along the radius vector. Bring P to zero acceleration by giving the whole system an acceleration equal and

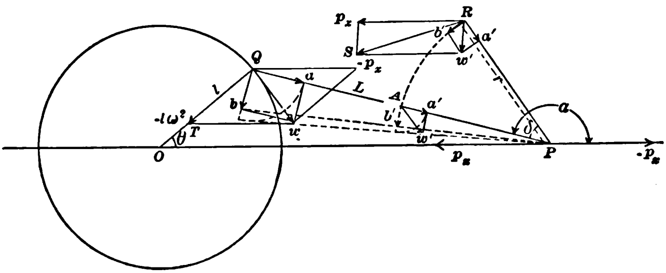


FIG. 128

opposite to that of P. Combining $-p_x$ with $-l\omega^2$ at Q, we get Qw, the resultant acceleration of Q referred to origin P in the piston. The component of Qw at right angles to QP, viz., Qb, is evidently a tangential component of acceleration about P, and $Qb = L\frac{d^2\alpha}{dt^2}$, from which the angular acceleration of the connecting rod can be found by simple division. The component of Qw along the rod, or Qa, is a normal component, or $Qa = -L\left(\frac{d\alpha^2}{dt^2}\right)$, from which the angular velocity might be determined. Now the tangential and normal components of all points of the body being simply proportional to their respective distances from P, when P has no acceleration, it follows that the components of A are equal to those at Q when multiplied by $\frac{r}{L}$. Hence the construction as given in Fig. 128. The components at R will be equal to those at A, but their directions are of course different, being along and at right angles to RP. The resultant Rw' is the absolute acceleration of R referred to the piston and origin at P, and combining this with $+p_x$, we get RS, the absolute acceleration of R when referred to the bed of the engine both in magnitude and direction. The projection of this on lines parallel to the coördinate axes gives $\frac{d^2x}{dt^2}$ and $\frac{d^2y}{dt^2}$.

3. VALVE GEARING

A. Relation between the Position of the Crank and the Position of the Valve

The chain of links forming the valve gearing consists, as is seen in Fig. 114, of four links, which are the engine bed, the eccentric, the eccentric rod, and the valve. If we express positions, velocities, and accelerations of points of this chain in terms of an angular position θ_1 of the eccentric, the results will be identical with what has preceded, and if expressed in terms of an angular position θ of the crank, a phase angle, viz., angle QOC (Fig. 114), must be introduced.

The valve of a steam-engine performs the duty of admitting steam alternately into the ends of the steam cylinder, and of exhausting the same into the atmosphere or the condenser. In the design of the valve gear the problem is to open and close the ports at certain given positions of the piston. It is therefore the position of the valve with which we are principally concerned, its velocity and acceleration being of minor importance at present.

(a) *The Plain Slide Valve*

The motion of the slide valve is usually produced by means of an eccentric, which consists of a circular plate, keyed to the shaft in a plane at right angles to its axis, and surrounded by a strap which is fastened through the eccentric rod to the valve. It is of course evident that the eccentric acts exactly as a crank, whose throw is equal to the distance between the centre of the shaft and that of the eccentric sheave. In Fig. 129 OC is the throw of the eccentric, and it is always denoted by r. The valve which it operates will move through a total distance $2r$ when directly connected.

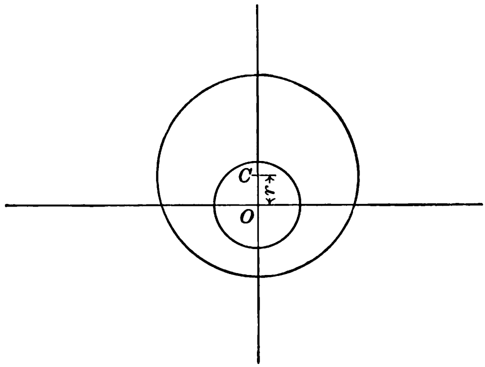


FIG. 129

In Fig. 130 let OQ be the crank, and OC the eccentric of an engine whose shaft is at O. The invariable angle QOC between the two we shall call β. If θ_1 is the angle between the line of connection of the valve and the throw of the eccentric, then the position of B can be written

$$x_B' = r\cos\theta_1 + \frac{r}{n}\{\sqrt{1 - n^2\sin^2\theta} - 1\},$$

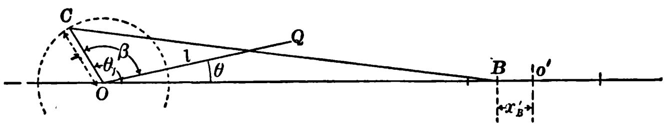


FIG. 130

when the distance x_B' is measured from the middle of the stroke of B. Now if $\theta_1 = \theta + \beta$, we have, on substituting this,

$$x_B' = r\cos(\theta + \beta) + \frac{r}{n}\{\sqrt{1 - n^2\sin^2(\theta + \beta)} - 1\}.$$

Fig. 131 is a diagrammatic representation of a plane slide valve in its central position with respect to its travel, *i.e.* when B is at O' (Fig. 130). The steam ports SS are separated from the exhaust port E by means of the bridges bb. The valve is always made longer than the sum of the widths of the three ports and two bridges. The amount by which the outside edges of the valve project beyond the outside edges of a steam port when the valve is in its *central position* of *travel* is called the Outside Lap,

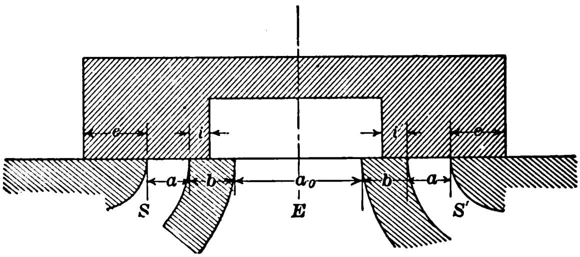


FIG. 131

and is always denoted by e. The amount by which the inside edges of the valve project beyond the inside edges of a steam port when the valve is in the same position, is called the Inside Lap, and is always denoted by i. These laps are given to the valve to work the steam expansively. They may or may not be the same at the two ends.

Fig. 132 shows the relative positions of the crank, eccentric, and valve for the head-end dead point. As the valve moves very

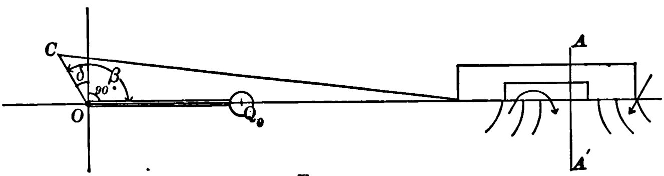


FIG. 132

nearly symmetrically across the face of the ports, in order that the steam distribution may be nearly alike on the two sides of the piston, the middle position of the valve referred to its own travel will coincide very nearly with the middle line AA of the ports. The middle position of the valve will evidently come when the eccentric throw r makes an angle of about 90° with the direction of motion of the valve, which is usually identical with that of the piston. Hence if the valve is made longer than the sum of the three ports and two bridges, the eccentric must be drawn past this position when the crank is on the dead point, as at that time the steam port must be just opening on the head end as shown. In most engines, especially if high speed, the valve is made to open a little while before the crank arrives at the dead point. The small amount by which the steam port is open when the crank turns the centre is called the Lead, and is denoted by v. If we denote by δ the angle between the eccentric and the perpendicular to the line of valve connection, then, if the piston and valve move parallel,

$$\beta = 90^\circ + \delta,$$

and δ, a constant of the valve gear, is called the Angular Advance, and its exact value will be determined hereafter.

Substituting this value for β in the equation for valve position, we get

$$x_B' = -r\sin(\theta+\delta) + \frac{r}{n}\left(\sqrt{1-n^2\cos^2(\theta+\delta)} - 1\right).$$

Now in a valve design the second term of the equation can be generally neglected, as will be seen from the following case.

Let $r = 2$, and $n = .04$ (ordinary dimensions). Then the maximum value of the first term will occur when $(\theta+\delta) = \frac{\pi}{2}$, and then $-r\sin(\theta+\delta) = -2$. The maximum value of the second term will occur when $\cos^2(\theta+\delta)$ is equal to unity, or

$$\frac{r}{n}\left(\sqrt{1-n^2\cos^2(\theta+\delta)} - 1\right) = \frac{2}{.04}\left(\sqrt{1-.0016} - 1\right) = -.04,$$

which is small enough compared with 2.00 to be neglected for purposes of design. Hence we shall always write

$$x_B' = -r\sin(\delta+\theta),$$

a simple harmonic equation.

Expanding the right hand side, we get

$$x_B' = -r\cos\delta\sin\theta - r\sin\delta\cos\theta.$$

But $r \sin \delta$ is a constant, as is $r \cos \delta$ also. Calling these A and B respectively, and dropping subscripts and accents,

$$x = -A \cos \theta - B \sin \theta.$$

If we consider this as a polar curve whose radius vector is laid off along the crank throw, we see that it represents a circle whose circumference passes through the pole, and whose diameter through

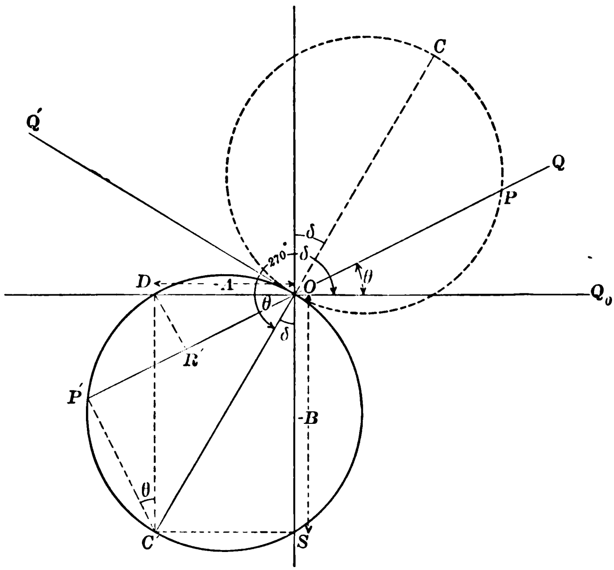


FIG. 133

the pole is inclined at an angle $(270° - \delta)$ with the reference line. For let O (Fig. 133) be the pole, and OQ_0 the reference line. Draw $OC' = r$ and inclined at an angle $(270° - \delta)$ with OQ. On OC' as a diameter describe a circle cutting the reference line at D, and cutting the perpendicular to it through O, at S. Then

$$OD = -r \sin \delta = -A,$$

and
$$OS = -r \cos \delta = -B.$$

Draw any radius vector OQ inclined at an angle θ with the reference line. Draw $C'P'$ and DR' perpendicular to OQ. Then $OP' = OR' + R'P' = -A\cos\theta - B\sin\theta = x$. Hence the position of the valve with reference to its central position for any given angular position θ of the crank measured from its head-end dead point is given by the intercept OP'. The valve will be in its central position, or x will be zero when the crank is at OQ'. The valve

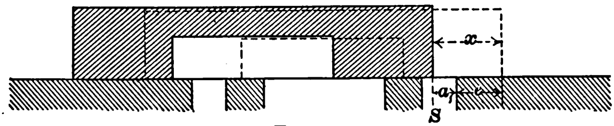


FIG. 134

will be at its maximum distance to the left, *i.e.* will have its maximum negative displacement, when the crank is at OC, and will have its greatest positive displacement when the crank is at OC'. At the dead point OQ_0, x is equal to $-A$. This polar diagram is the same as was considered in connection with piston position, with the exception of the different phase angle.

The above position of the polar locus or valve circle is correct when we take into account the algebraic sign of the valve displacement. But it is more convenient to draw another circle on $C'O$

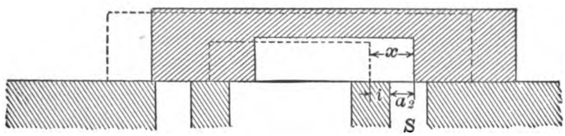


FIG. 135

produced to C, and consider the valve displacement as equal to OP, since, as we generally work with but one end of the cylinder at a time, the algebraic sign of the displacement is of but little consequence. By using the upper circle we have the advantage of measuring our radius vector in the direction of the crank throw.

Let the valve shown by the dotted lines (Fig. 134) be in its central position, and let that shown by the full lines be the same valve after having moved a distance x to the left, and opened the head-end port. In this second position call the amount that the steam port is open a_1. Then, as x is negative,

$$-x = a_1 + e,$$

$$a_1 = (-x) - e.$$

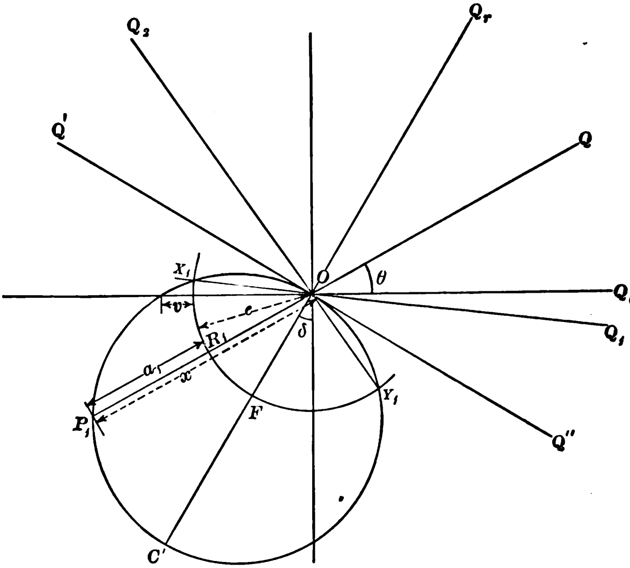


FIG. 136

Now suppose that the valve moves a distance x to the right of its central position, so as to exhaust the port S. Then from Fig. 135 we see that

$$+x = a_2 + i,$$

$$a_2 = (+x) - i,$$

where a_2 is the amount that the port is open to the exhaust, and x is positive in this case.

In Fig. 136 we draw a valve circle as in Fig. 133. We also draw a circle about O as a centre, and with radius equal to e, the outside lap. Then $OP_1 = -x$, $OR_1 = e$, and $R_1P_1 = (-x) - e = a_1$, the amount by which the steam port is open for the position OQ of the crank. At the point X_1 where the valve circle intersects the e circle, $-x = e$, and $a_1 = 0$. Hence at OQ_1 we have *admission.* When the crank has moved up to OQ_0, the dead point, the port is open the amount of the lead v. At Y_1, $-x = e$ again, and we have *cut-off* with the crank at OQ_2. At OQ_r the port is wide open by the amount FC'. When the crank has arrived at OQ' (the tangent to the circle at O), the valve will be in its central

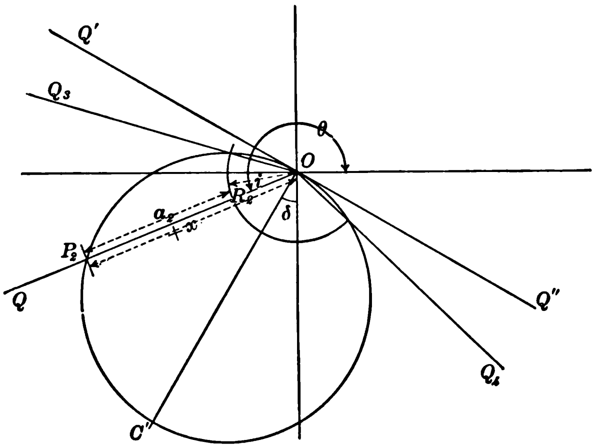


FIG. 137

position, since there $x = 0$. If it passes beyond OQ' we will have positive values of x, and positive intercepts on the circle OC'. Let OQ (Fig. 137) be such a position of the crank. The value of $+x$ is OP_2 on the circle OC^1. Now finally if we draw a circle about O and with a radius equal to i, the inside lap, then $OR_2 = i$, and $R_2P_2 = (+x) - i = a_2$, the amount by which the exhaust port is open for any angular position θ of the crank. At OQ_3, $+x = i$, and $a_2 = 0$, so we have *release.* At OQ_4, a_2 is again equal to zero, and compression begins, so that OQ_4 is called the *Compression Point.*

The regular polar or Zeuner valve diagram is shown in Fig. 138. Here an upper or negative circle is introduced as in Fig. 133, upon which $-x$ can be laid off in the direction of the crank throw. It will be noticed that while the radius vector OQ is sweeping over the upper circle OC, the steam features of the head end will be taking place, and while sweeping over the lower circle OC' the exhaust features of the head end occur. But it is evident also

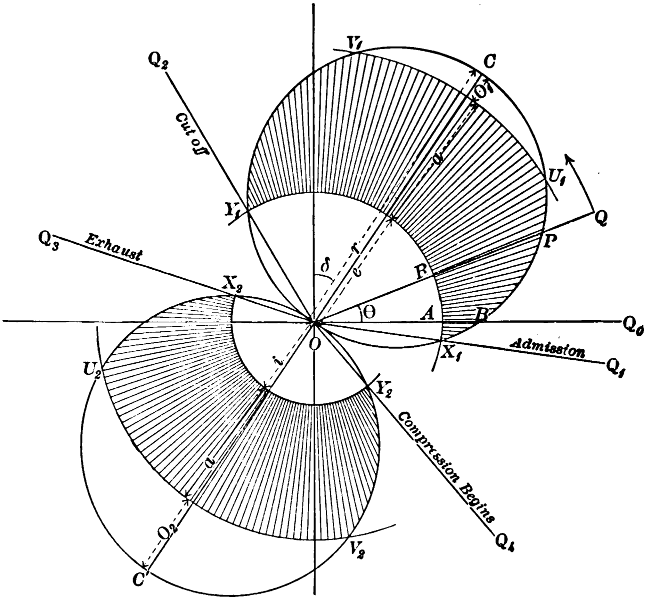


FIG. 138

that the exhaust features of the crank end come within the circle OC, while the steam features of the same end come within the circle OC'. These latter are, however, omitted in Fig. 138. The points Q_1, Q_2, Q_3, and Q_4 are the critical points of the cycle taking place in the head end of the cylinder. Variations of any of the quantities produce effects which are easily seen on the diagram. Increase in δ admits steam earlier and cuts it off earlier. Increase in e admits steam later and cuts it off earlier. Increase in r admits steam earlier and cuts it off later, etc., and similarly for the exhaust side.

Evidently neither a_1 nor a_2 can be greater than a, hence if r is greater than $a + e$, the valve will overtravel the port. If in this case a circle be drawn about O as a centre, and of radius equal to $a + e$, its points of intersection U_1 and V_1, with the valve circle, will give the crank positions at which the port is just wide open. The distance o_1 on the diagram is called the Overtravel. Similarly o_2 is the overtravel on the exhaust side. The shaded portion of the diagram shows the variation in port opening.

It is always well to make the valve as short as possible so that the pressure on its back and hence the work of moving it under this pressure may be a minimum. We must remember, however,

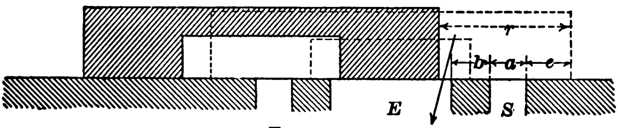


FIG. 139

that the ports are determined as to width by the general dimensions of the engine, and that the laps are also fixed by the design. The bridge b then should be as small as possible, but not so small that the valve will travel across it, and admit live steam into the exhaust port, as would be the case in Fig. 139. At the extreme travel of the valve $x = r$, hence in all cases we must have

$$e + a + b > r,$$

or

$$b > r - a - e.$$

Neither should the exhaust port be so far covered by the inside edge of the valve as to make the effective opening of that port less than a. In every case then we must make

$$a_0 + b - (i + r) > a,$$

or

$$a_0 > a + i + r - b,$$

as is evident from Fig. 140.

Problems in Valve Gearing. — In the design of any valve gear we have certain constants or dimensions given, and are required to find the values of all others. These constants may be any of the following:

Constants of the gear . . $\begin{cases} r, \ \delta. \\ e, \ i. \end{cases}$

Crank positions . . . $\begin{cases} Q_1, Q_2, Q_3, \text{ or } Q_4. (a_1 \text{ or } a_2 \text{ being zero.}) \\ Q \text{ in general. } (a_1 \text{ or } a_2 \text{ being known.}) \end{cases}$

The last of these is the most important when $Q = Q_0$, and $a_1 = v$, the lead.

It will be noticed that all four of the crank positions cannot be taken arbitrarily, as three will determine the position of a circle, and the fourth must follow from these. Also that one constant at

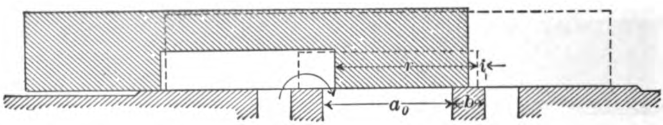


FIG. 140

least of the gear must be given, viz., e, i, r, or δ. If three crank positions are given, δ is superfluous, and hence some other constant must be chosen. Also one at least of both steam and exhaust features must be given. The following are a few examples.

1. Given e, i, γ, and cut-off; to find δ, r, Q_3, and Q_4. (γ is the angle at which the crank stands when admission occurs.) Lay off γ and the position of the crank at cut-off. (See Fig. 138.) Draw the e- and i-circles. Draw the valve circle passing through the origin and the two points of intersection of the e-circle with the given crank positions. The diameter of this through the origin will give r and δ. Produce this beyond O, and upon the prolongation draw another and similarly situated circle. The intersection of the i-circle with this last will give the release and compression positions.

2. Given e, i, v, and cut-off position of the crank; to find the

same quantities as before. Draw the e- and i-circles. Draw the crank positions at cut-off, and lay off the lead. Pass the valve circle through the origin, the lead point, and the point of intersection of the e-circle with the cut-off position of the crank, and proceed as in problem No. 1.

3. Given e, v, and r. Draw the e-circle and lay off the lead. About the origin as a centre draw a circle of radius equal to r. Erect a perpendicular to the axis of X through the lead point, and where this intersects the circle last drawn will be the extremity of the diameter of the valve circle.

4. Given cut-off position of the crank, a, and o_1. (o_1 being the overtravel on the steam side.) We are also given the compression position or the release position. Call $(a+o_1)=a'$. We have

$$e+a'=r, \quad . \quad . \quad . \quad . \quad . \qquad (1)$$

$$x=r\sin(\delta+\theta).$$

If we put $\theta=0$ in this last equation,

$$x_0=r\sin\delta=e+v. \quad . \quad . \quad . \qquad (2)$$

If we put $\theta=\theta_2$ where θ_2 is the angle of the crank at cut-off,

$$x_1=r\sin(\theta_2+\delta)=e, \quad . \quad . \qquad (3)$$

we have three unknown quantities, viz., e, r, and δ, and three independent equations. Substituting for r in (1) its value in (2), we get

$$e+v=(e+a')\sin\delta,$$

$$e=\frac{a'\sin\delta-v}{1-\sin\delta}. \quad . \quad . \quad . \qquad (4)$$

Substituting in (3) the value of e from (4) and of r from (1), we get

$$\frac{a'\sin\delta-v}{1-\sin\delta}=\left(\frac{a'\sin\delta-v}{1-\sin\delta}+a'\right)\sin(\theta_2+\delta),$$

$$a'\sin\delta-v=(a'-v)\sin(\theta_2+\delta).$$

Reduction gives

$$\sin\delta\,\{a'+(v-a')\cos\theta_2\}+\cos\delta\,\{(v-a')\sin\theta_2\}=v.$$

This is the equation of a circle in polar coördinates whose circumference passes through the origin, and whose intercepts on the axes of X and Y are $(v - a') \sin \theta_2$ and $a' + (v - a') \cos \theta_2$, respec-

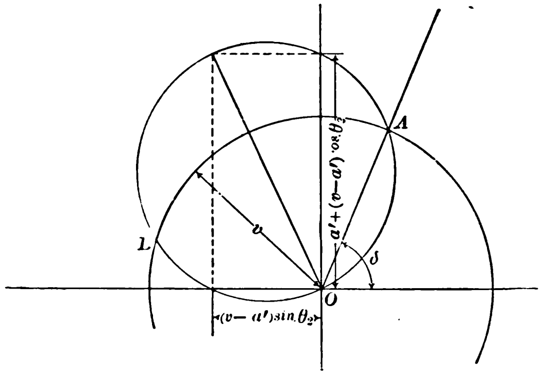


FIG. 141

tively. Hence we lay off these known quantities on the axes (Fig. 141), and pass a circle through the origin, and through their extremities. Then we draw a second circle about the origin and with a radius equal to v. The line joining the point of intersection A of the two circles with the origin will give by its angle with the axis of X the value of δ. Since a' is always greater than v, and since $\sin \theta_2$ is positive in any practical construction, the term $(v - a') \sin \theta$

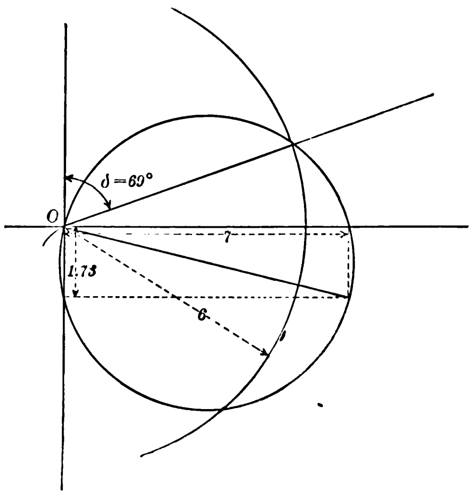


FIG. 142

will always be negative. Also it is readily seen that $a+(v-a')\cos\theta_2$ will always be positive. Hence the former is laid off to the left of the origin, and the latter upwards. The other intersection B of the circles gives a perfectly correct though impracticable value of δ.

In solving the above problem a partly analytical and partly graphical method seems to be the best. Take the following: $a'=\frac{1}{4}''$, $v=\frac{3}{16}''$, cut-off at $\frac{1}{4}$ stroke. To find δ, e, and r. If the cut-off comes at $\frac{1}{4}$ stroke, $\theta_2=60°$ (approximately).

$$a'+(v-a')\cos\theta_2=\tfrac{1}{4}+(\tfrac{3}{16}-\tfrac{1}{4})\cos 60°=\tfrac{7}{32}'',$$

$$(v-a')\sin\theta^2=(\tfrac{3}{16}-\tfrac{1}{4})\sin 60°=\tfrac{1.73}{32}''.$$

Now if we lay off $(v-a')\sin\theta_2$ vertically downwards from the origin, and $a'+(v-a')\cos\theta_2$ horizontally to the right, we will measure our δ to the right of the axis of Y as it is usually placed. Hence the construction of Fig. 142 gives δ immediately as 69°.

Now $$e=\frac{a'\sin\delta-v}{1-\sin\delta}=\frac{.233-.187}{1-.933}=.64'',$$

$$r=e+a'=.64+.25=.89.$$

Some persons prefer a purely graphical construction for such a problem, even if more complicated in reality. Therefore the following graphical method is introduced. About the origin as a centre draw a circle of radius $OA=a'$ (Fig. 143), and one of radius $OV=v$. Draw the given crank position at cut-off OQ_2. Produce Q_2O to M, so that $UM=RO$. Then $OM=v-a'$, and since angle $AOR=\theta_2$, $WO=(v-a')\sin\theta_2$, $WM=(v-a')\cos\theta_2$, and by completing the parallelogram $OMAB$, we see that $WB=WM+OA=a'+(v-a')\cos\theta_2$. Hence the circle drawn on OB as a diameter gives by its intersection with the v-circle the value of δ correctly placed.

In order to explain the theory of the construction for e and r, we must combine equations (1) and (2), remembering that δ is now a known quantity. We have

$$e + a' = r,$$

$$e + v = r \sin \delta.$$

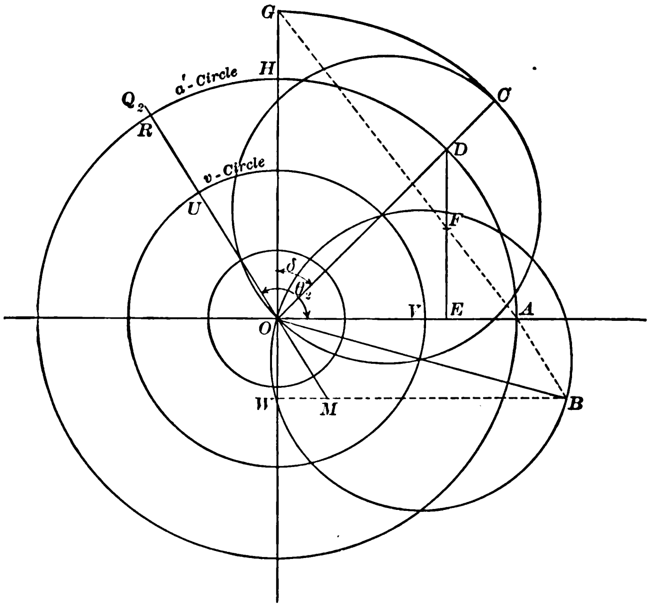


FIG. 143

Eliminating e from the above, there results

$$r = \frac{a' - v}{1 - \sin \delta}.$$

If we divide both sides of (5) by a', we get

$$\frac{r}{a'} = \frac{a' - v}{a' - a' \sin \delta}.$$

Hence from the point D, where OD cuts the a-circle, drop the perpendicular DE. On ED lay off $EF = VA = a' - v$. Draw

AF, and produce to cut the axis of Y at G. Then will $OG = r$, for by geometry

$$\frac{GO}{AO} = \frac{FE}{AE},$$

or

$$\frac{r}{a'} = \frac{a' - v}{a' - a' \sin \delta}.$$

We now project G on the arc of a circle to C, and draw our valve circle on OC. The outside lap e will be $GH = r - a'$.

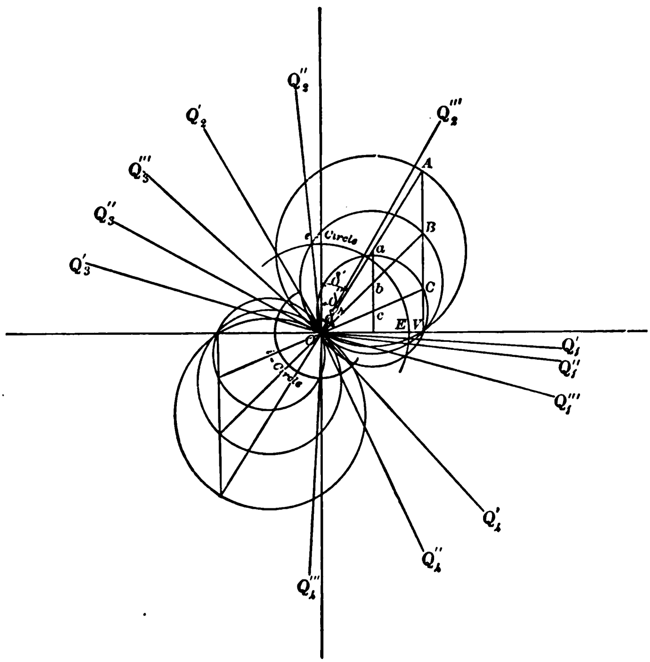


FIG. 144

Variable Cut-off with Plain Slide Valve. — If the point of cut-off is to be changed, it must be done by altering some of the constants of the driving gear, and not of the valve itself, since the laps and ports are absolutely fixed by the construction of the engine.

We must also be careful that by varying the cut-off we do not vary the admission position nor lead to any great extent. Let us first see how r and δ must be changed in order to give variable cut-off with constant lead. In this case, however the valve circle may change, all the circles must pass through the same lead point V, Fig. 144. Therefore the locus of the centres of all valve circles which give a constant lead will be a straight line at right angles to the middle point of the chord OV, and the locus of the extremities of their diameters will be the perpendicular erected at V.

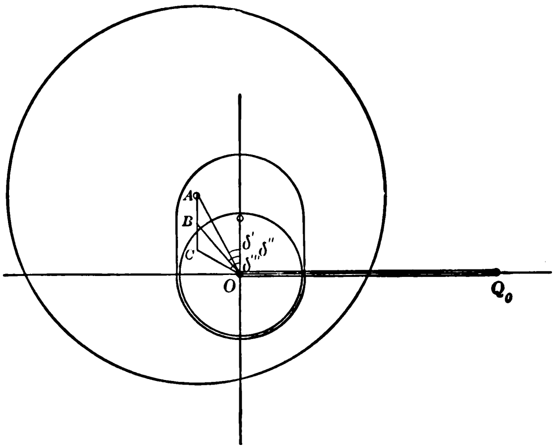


FIG. 145

In the figure are shown the positions of three valve circles, the extremities of the diameters being at A, B, and C. These all give the same lead EV, but the admissions vary slightly, being at Q_1', Q_1'', and Q_1'''. The cut-offs vary widely through the range Q_2', Q_2'', and Q_2'''. The exhaust features vary also. When the inside lap is considerable as shown, the release points do not vary much, being at Q_3', Q_3'', and Q_3''', but the compression points do, as is seen at Q_4', Q_4'', and Q_4'''. If $i = 0$, the release and compression points vary an equal amount. It will be noticed that the change is produced by increasing δ and diminishing r. It is accomplished in practice by cutting a slot in the eccentric at right angles to the crank, so that it can be shifted across the shaft as in Fig. 145. This shifting is done automatically while the engine is running by means of a shaft governor.

If we wish to keep the admission point constant, the circles must be varied as in Fig. 146. Here the locus of the extremity of the valve circle diameter is the perpendicular to the admission position OQ_1 at its intersection with the *e*-circle. The three positions

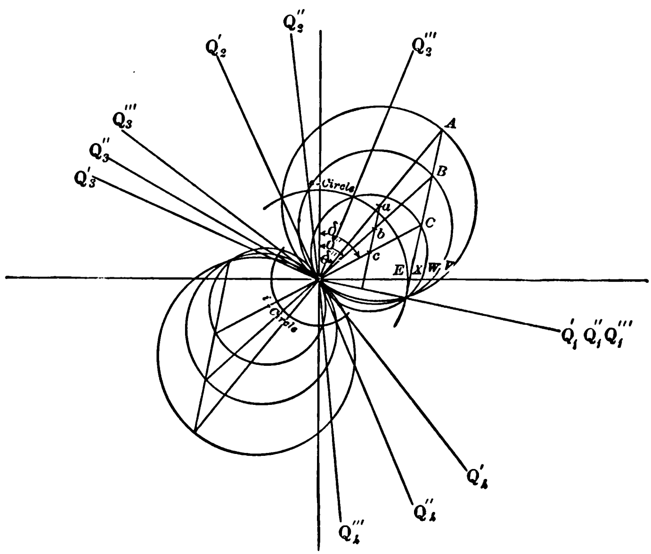


FIG. 146

A, *B*, and *C* give a constant admission at Q_1, but a variable lead *EV*, *EW*, and *EX*. The cut-offs, releases, and compression points are similar to Fig. 144. This is the method of variation in most single valve, automatic, high-speed engines. The shifting of the eccentric at the dead point is as shown in Fig. 147. Since it is difficult to get the motion of pure translation given by the straight line *ABC*, it is usually approximated by means of the arc of a circle.

Reversing Gears. — Valve gears which are so arranged as to run the engine in either direction are known as Reversing Gears. These must be applied to all marine, locomotive, and hoisting engines. Those reversing motions which are actuated by means of the so-called link are amongst the most interesting kinematic movements. By simply shifting the position of this very ingenious piece of mechanism, the engine may be run in either direction and with any degree of expansion.

The Stephenson link is the one most generally used upon locomotive and marine engines. Upon an axle O (Fig. 148) are keyed

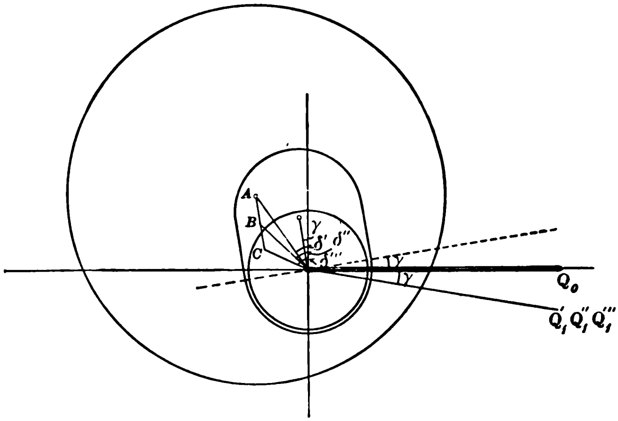


FIG. 147

two eccentrics C and C', from which the rods extend and are joined to the two ends of the expansion link BB'. The block M on the end of the valve stem fits the slot of the link. The valve stem T is held by fixed guides, and the link can be raised and lowered at will. Let O (Fig. 149) be the centre of the engine shaft, and OQ_0 the crank at its head-end dead point. In order that the engine may throw over, the eccentric must be set with an angular advance δ as shown. If, however, the engine is to throw under as in Fig. 150, the angular advance must be δ to the left of the perpendicular below the origin. Now if both eccentrics are put on as in Fig. 151, and the ends of the arms are connected by the link, then when this is depressed the engine will throw over,

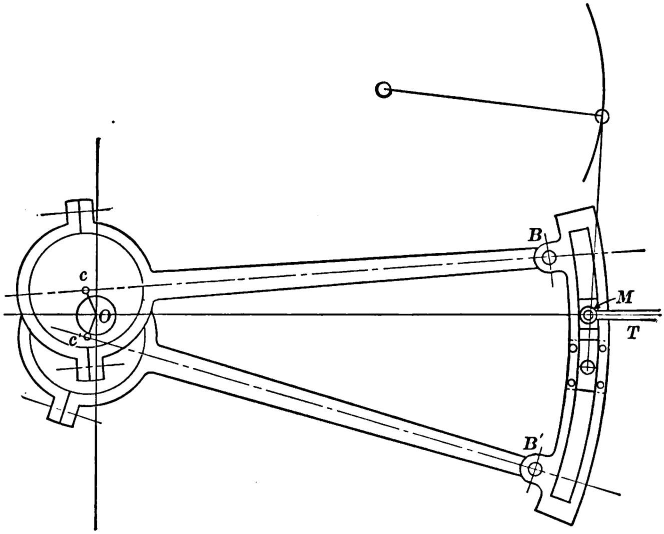


FIG. 148

and when raised it will throw under. In these cases one eccentric will control the motion of the valve. When the link is in any other than one of its two extreme positions, the motion of the valve will partake of the motion of both eccentrics.

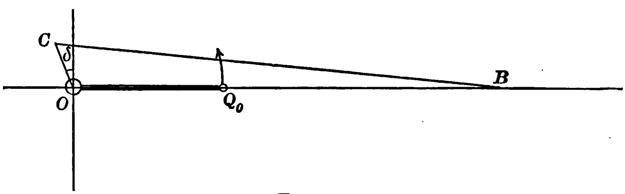


FIG. 149

By considering the length of the eccentric rods very great, and by making the radius of curvature of the link very great also (*i.e.*

by making the link straight), we will have the arrangement shown in Fig. 152. Here the points of attachment BB' (Fig. 148) are supposed to lie in the pitch line of the link, and that of the block M remains rigidly fixed in the link for any given position. When

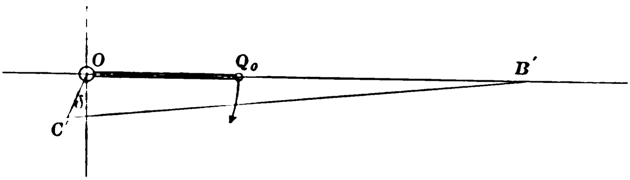


FIG. 150

the crank is on the dead point OQ_0, the eccentrics are at OC_1 and OC_1', and the link at B_1B_1'. When the crank is at the other dead point, the eccentrics are at OC_2' and OC_2, and the link at B_2B_2'. XY is the mean of these two link positions, hence we can measure movements of points of the link from it as a central position, and as the valve is rigidly attached to the link block, these movements will correspond exactly to valve movements. Let the crank turn

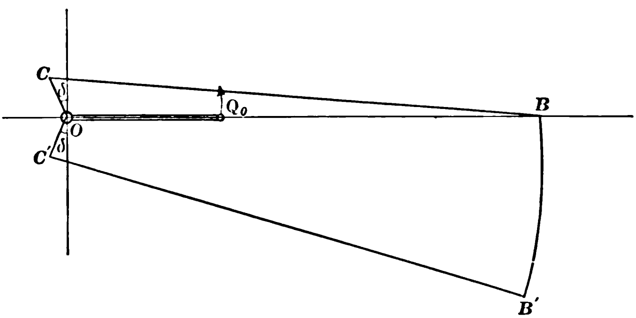


FIG. 151

through an angle θ, and consider the motion of the extremities of the link which have now moved to B and B'. Let BS equal x_1, and $B'T$ equal x_2. Since x_1 is entirely due to eccentric C, we have

$$x_1 = r \sin (\theta + \delta).$$

And for x_2, the motion of the other end of the link, we have

$$x_2 = r \sin (\theta + (180 - \delta)).$$

Expansion of these two equations gives

$$x_1 = A \cos \theta + B \sin \theta,$$

$$x_2 = A \cos \theta - B \sin \theta.$$

Now consider the motion of any other point of the link such as M, and call the distance $MN = x$. Let the whole length BB'

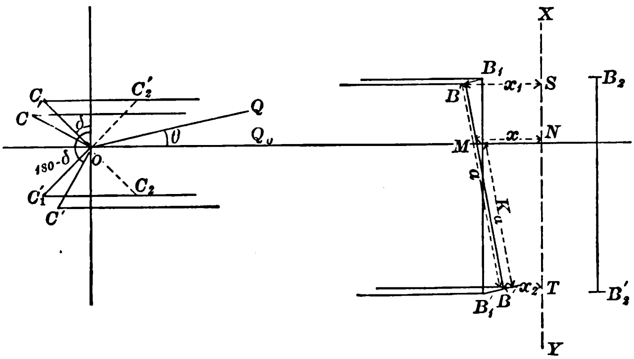


FIG. 152

of the link be represented by a, and the amount $B'M$ by Ka. Then by similarity of triangles,

$$x_1 - x_2 : x - x_2 :: a : Ka :: 1 : K.$$

$$x = Kx_1 + (1 - K)x_2.$$

Substituting the values of x_1 and x_2,

$$x = Ka \cos \theta + KB \sin \theta + (1 - K)A \cos \theta - (1 - K)B \sin \theta,$$

$$x = A \cos \theta + (2K - 1)B \sin \theta.$$

This is the equation in polar coördinates of a circle passing through the origin whose intercept on the axis of X is A, and on the axis of Y is $(2K - 1)B$. It will be observed that however the link be moved, that is, however K be varied, the intercept A will not be changed, or the lead will be constant. The system of valve circles is shown in Fig. 153. OC is the valve circle corresponding to the upper eccentric, and OC' that of the lower one. The angle $JOC = \delta$, and the angle $JOC' = 180° - \delta$, $OV = A$, the same for all the circles $OJ = B$, and $OJ' = -B$. If any radius vector be drawn as OQ, $OP_1 = x_1$, $OP_2 = x_2$. If

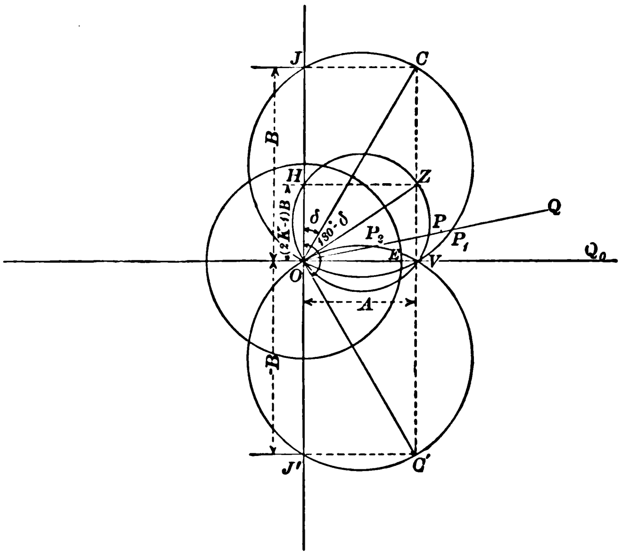


FIG. 153

we lay off $OH = (2K - 1)B$, and pass a circle through O, V, and H, we have circle OZ, representing the motion of the valve for any intermediate position of the link, and $OP = x$.

For rods of finite length the action is far more complex. Let CC' (Fig. 154) be the position of the eccentrics when the crank is on the head-end dead point. Let the pitch curve of the link slot be the arc of a circle whose radius is in general R. Then with the crank stationary, the point B will, as the link is raised and lowered, follow the circle *PA* about *C* as a centre. Let B_0SB_0' be the link at mid gear, *i.e.* when B_0 and B_0' are equidistant from the line of connection. It is readily seen now that the lead cannot be constant, as the extent of its variation, or the difference between the maximum and minimum widths of the lead will be *PS*. When the crank is on the crank-end dead point, the arms of the link will be crossed, and the locus of *B*, as the link is raised and lowered, will be the circle $A'P'$ about C_2 as a centre, and the variation in lead will be $S'P'$. The best that can be done, then, is to so choose the radius of the slot that the variation of the lead on the two ends will be the same. Let us find, then, an expression for the distance *PS*.

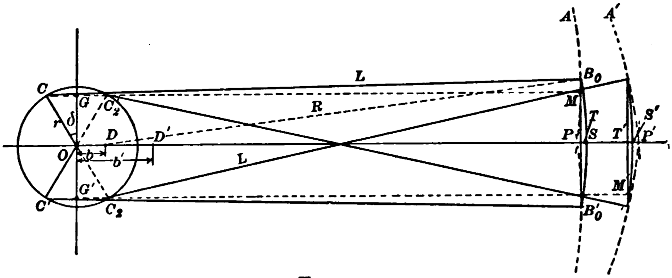


FIG. 154

When the link is in mid gear the centre of curvature of the slot will lie on the axis of *X*, as the whole apparatus is then symmetrically arranged about that axis. Let it be at *D*, and call the distance $OD = b$. Then

$$OS = b + R.$$

Call the length of the chord $B_0B_0' = a$, a constant of the link, and denote the point at which this chord cuts the axis of *X* at right angles when the link is in mid gear by *T*.

$$(OT - b)^2 + \frac{a^2}{4} = R^2,$$

$$b = OT - \sqrt{R^2 - \frac{a^2}{4}}.$$

But $OT = MG = MC - CG = \sqrt{L^2 - \left(\frac{a}{2} - r\cos\delta\right)^2} - r\sin\delta,$

where L is the length of the eccentric rod or link blade.

Hence $b = \sqrt{L^2 - \left(\frac{a}{2} - r\cos\delta\right)^2} - r\sin\delta - \sqrt{R^2 - \frac{a^2}{4}}$;

and finally,

$$OS = \sqrt{L^2 - \left(\frac{a}{2} - r\cos\delta\right)^2} - r\sin\delta - \sqrt{R^2 - \frac{a^2}{4}} + R.$$

Now the distance OP we know to be

$$OP = L\sqrt{1 - n^2\cos^2\delta} - r\sin\delta,$$

where $n = \frac{r}{L}$. Hence the variation in lead is

$$PS = OS - OP,$$

$$PS = \sqrt{L^2 - \left(\frac{a}{2} - r\cos\delta\right)^2} - \sqrt{R^2 - \frac{a^2}{4}} + R - L\sqrt{1 - n^2\cos^2\delta}.$$

At the other dead point we obtain an expression for OS' by putting $180° - \delta$ for δ; hence

$$OS' = \sqrt{L^2 - \left(\frac{a}{2} + r\cos\delta\right)^2} + r\sin\delta - \sqrt{R^2 - \frac{a^2}{4}} + R,$$

and $OP' = L\sqrt{1 - n^2\cos^2\delta} + r\sin\delta.$

$$P'S' = OP' - OS',$$

$$P'S' = -\sqrt{L^2 - \left(\frac{a}{2} + r\cos\delta\right)^2} + \sqrt{R^2 - \frac{a^2}{4}} - R + L\sqrt{1 - n^2\cos^2\delta}.$$

But these distances are to be equal; hence

$$\sqrt{R^2 - \frac{a^2}{4}} - R$$

$$= \tfrac{1}{2}\sqrt{L^2 - \left(\frac{a}{2} - r\cos\delta\right)^2} + \tfrac{1}{2}\sqrt{L^2 - \left(\frac{a}{2} + r\cos\delta\right)^2} - L\sqrt{1 - n^2\cos^2\delta}.$$

The right side of the equation contains constants only, and therefore can be calculated for any given case. Call it C. Then

$$R = \frac{-\left(\frac{a^2}{4} + C^2\right)}{2\,C}.$$

From this we may obtain an exact result for the radius of the link slot. However, as r is usually small in comparison with L and a, we may expand the first two radicals by the binomial theorem, then add them term for term, and reject all terms containing the square or higher powers of r. The remaining terms will be merely the expansion of $\left(L^2 - \frac{a^2}{4}\right)^{\frac{1}{2}}$. Likewise we may neglect the term $n^2 \cos^2 \delta$, n^2 being too small to be considered. The approximate form of the equation then is

$$\sqrt{R^2 - \frac{a^2}{4}} - R = \sqrt{L^2 - \frac{a^2}{4}} - L,$$

which is satisfied when $R = L$.

In most cases it is amply close enough to make the radius of the link slot equal to the length of the rod or blade. When

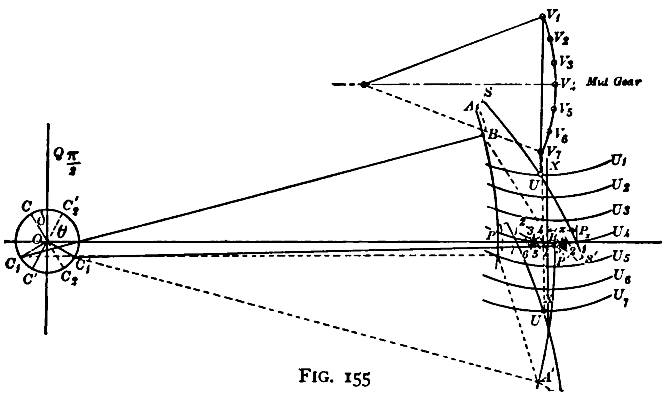


FIG. 155

the link is not hung on its pitch line, the radius of the slot will be practically the same.

In order to investigate the steam distribution, graphical methods can be best used. Let the crank stand at any angular position θ (Fig. 155), and lay in the eccentrics C_1, C_1', in their proper positions. Having the length L of the blade given, draw the locus circles AP of the point B, and $A'P'$ of the point B'. Construct a template of thin sheet metal as shown in the figure, the

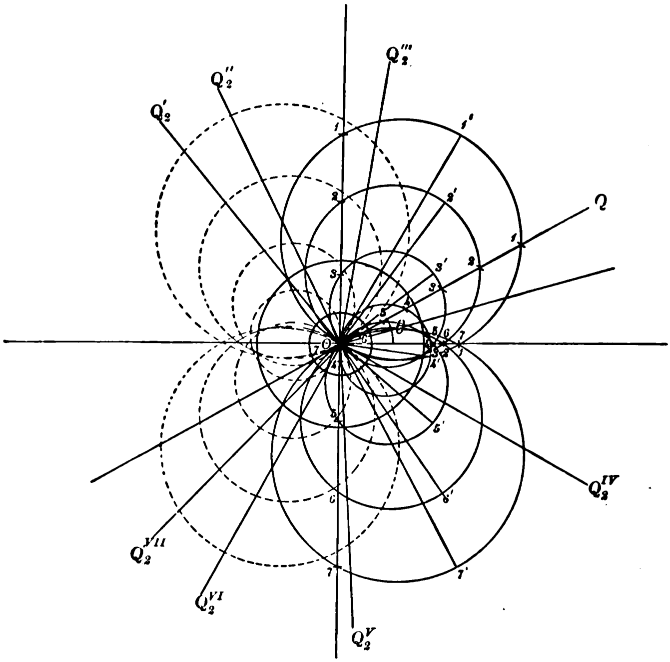


FIG. 156

corners at B and B' representing the same points as in Fig. 148, and let the edge SS' be drawn with the proper radius R. The distance BB' is, of course, a. The link is suspended from some point on its pitch line (usually the middle point U) by a radius bar from a point V. This point V can be shifted along an arc, thus varying the steam distribution. Suppose in the present instance the link is suspended from the point V_1. Then with

V_1 as a centre, and a radius equal to the length of the radius bar, strike an arc U_1. Now shift the template upon the drawing, keeping B and B' over their respective loci, until the middle point U of SS' falls upon the circular arc UU_1, and draw in the arc SS^1. Then will the distance x, measured from the middle line XX to P_x, be the distance that the valve is drawn past its central position. The line XX is obtained by taking the mean of the positions of P_x at mid gear for the two dead points. Now take a new position of V, say at V_2, strike the arc U_2, shift the template till U falls upon this, and measure x again. Having obtained a number

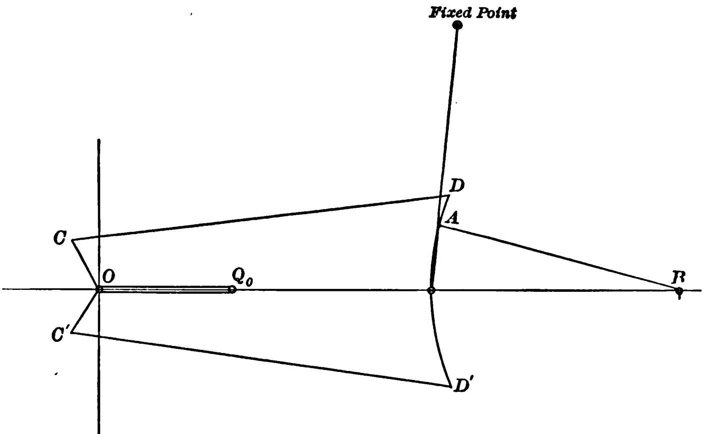


FIG. 157

of such measurements for different positions of the link, proceed to lay them off on a Zeuner diagram as intercepts $O1$, $O2$, $O3$, etc., along the crank throw OQ in its given position θ. Now take a new crank position and go through the same construction again, using the same points V_1, V_2, V_3, etc., as before. By connecting corresponding points on the Zeuner diagram with a smooth curve, an exact polar diagram of valve travel can be obtained. If the length of the link blade is sufficiently great to admit of the motion of the valve being considered as simply harmonic, the above curves will be circles, and but two positions of the crank will be necessary to determine all the circles. These two positions may conveniently be $\theta = 0°$, and $\theta = 90°$, as shown in Fig. 156.

If the blades are crossed with the crank on the crank-end dead point as shown above, the link is known as a crossed link, and the lead will be least at mid gear. If the blades are crossed on the head-end dead point, the link is an open link, and the lead will be greatest at mid gear.

In the Gooch link the link itself is not raised and lowered, but the valve rod is (Fig. 157). It has the property of giving an absolutely constant lead, but otherwise the steam distribution is not as good as that given by the Stephenson link. It requires more room also to get in the long arm *AB*.

(*b*) *The Gridiron Valve*

In this valve gear the steam chest is divided into two compartments by means of a partition parallel to the valve face. In the lower compartment the distribution valve works. It is a plain slide valve with small laps. In the upper compartment is the expansion valve, which consists of a plate with a rectangular opening in the centre. This plate slides back and forth over a single port in its seat, and cuts off the steam with its inner edges. The motion of this valve will be a simple harmonic motion, and will be given by an equation of the same form as that of the plain slide valve cutting off with its outer edges. If we represent by x_0 the distance which the expansion valve has moved past its central position, for any given angular position θ of the crank, by r_0 the throw, and by δ_0 the angular advance of the eccentric which drives it, our equation of motion will be of the form

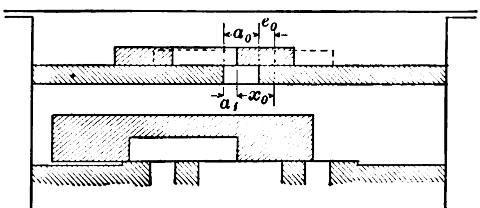


FIG. 158

$$x_0 = r_0 \sin(\theta + \delta_0).$$

The eccentric will, however, usually have a negative angular advance. Now consider the expansion valve in its central position as shown by the dotted lines in Fig. 158. Call the total width of the port a_0, and the amount by which the edge of the valve is beyond the

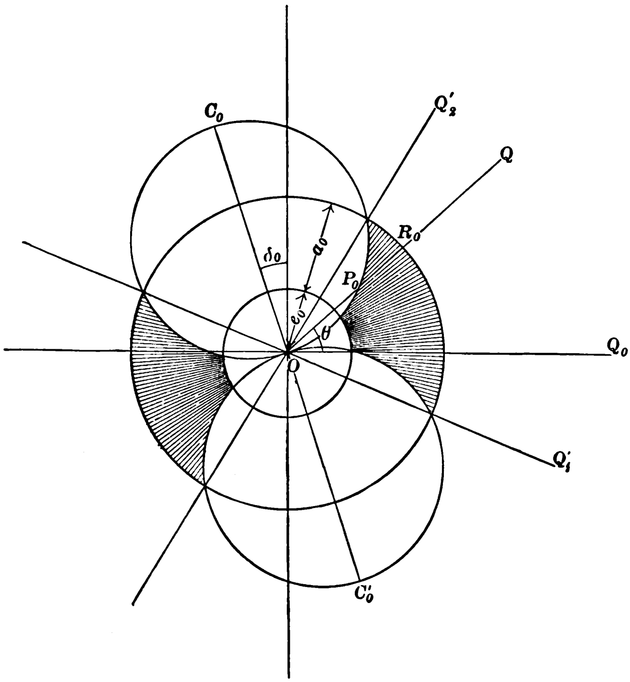


FIG. 159

edge of the port e_0. Now let the valve move a distance x to the left, throttling up the port to width a_1. Then we will have

$$x_0 + a_1 = a_0 + e_0,$$

$$a_1 = (a_0 + e_0) - x_0.$$

The value of x_0 can be found at once by means of a Zeuner diagram. In Fig. 159, OC_0 is the expansion valve circle, inclined at an angle $-\delta_0$ to the left of the perpendicular to OQ_0. OQ is any position of the crank, and hence $OP_0 = x_0$. With centre O and with radius $a_0 + e_0$, describe a circle. Then $OR_0 = a_0 + e_0$, and $P_0R_0 = (a_0 + e_0) - x_0 = a_1$. We can conveniently measure the port opening as affected by the left hand edge of the valve by means of another circle drawn on C_0O produced below the origin. The expansion valve will then admit steam into the lower

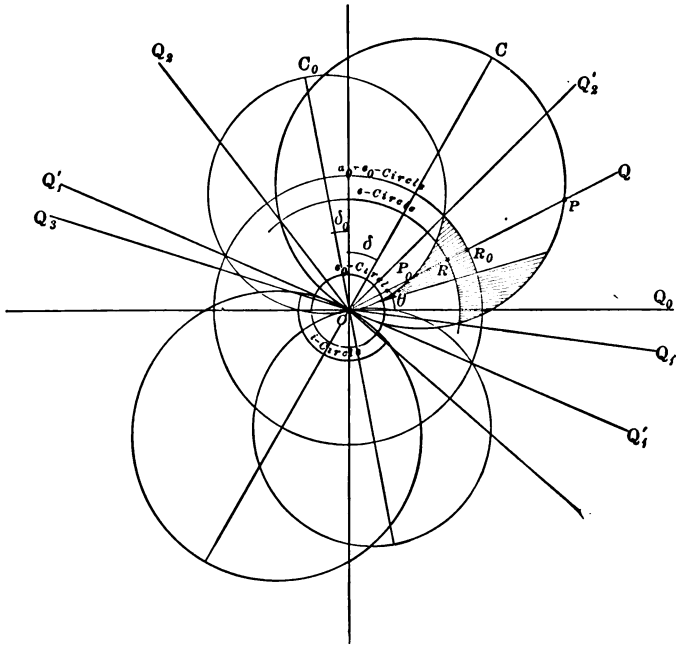


FIG. 160

steam chest by its left edge when the crank is at OQ_1' and cut it off with its right edge at OQ_2'. If we draw the distribution valve circle on the same diagram, we can follow the whole distribution of the steam. In Fig. 160, OC is the distribution valve circle at its angular advance δ, and OC_0 the expansion valve circle at its angular advance δ_0. $OR = e$, the outside lap of the main valve, and $OR_0 = a_0 + e_0$ of the expansion valve. For any crank position OQ, P_0R_0 is the port opening of the expansion valve, and PR the port opening of the distribution valve. The expansion valve opens first at OQ_1', but steam does not flow into the cylinder till the distribution valve opens at OQ_1. The expansion valve closes first at OQ_2', and hence determines the cut-off. The main valve closes at OQ_2 before the expansion valve opens again. If

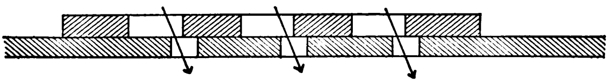


FIG. 161

the distribution valve fails to close before the expansion valve opens again, there will be double admission. It is evident that if $a_0 + e > r_0$, there will be no cut-off by the expansion valve. The shading shows the effective port opening.

This form of expansion valve is frequently made with a number of slots or openings in the valve, and a corresponding number of ports in the seat (Fig. 161); hence the name "Gridiron Valve."

(c) *The Meyer Valve*

This gear produces an early cut-off by means of an expansion valve running on the back of the distribution valve. The distribution valve consists of a flat plate (Fig. 162), in which are cut two rectangular ports D, D, as well as the exhaust hollow. It is evident that its action is exactly the same as an ordinary slide valve. The expansion valve runs on top, covering the ports D, D, and thus effecting the cut-off. If the distribution valve is driven by an eccentric of throw r and angular advance δ, and the expansion valve by one of throw r_0 and angular advance δ_0, we will have as before

$$x = r \sin(\theta + \delta),$$
$$x_0 = r_0 \sin(\theta + \delta_0).$$

Now x and x_0 are the motions of the valve referred to the fixed valve seat; but the cut-off is determined by the relative motion of the two valves, or by $(x - x_0)$. Let us first get the equation of relative motion. The upper portion of Fig. 162 represents the two valves in their central positions (a state of affairs which could never occur when the engine is running, however). The lower half shows the valves after having moved to the right; the distribution through a distance x and the expansion through a distance x_0. If y

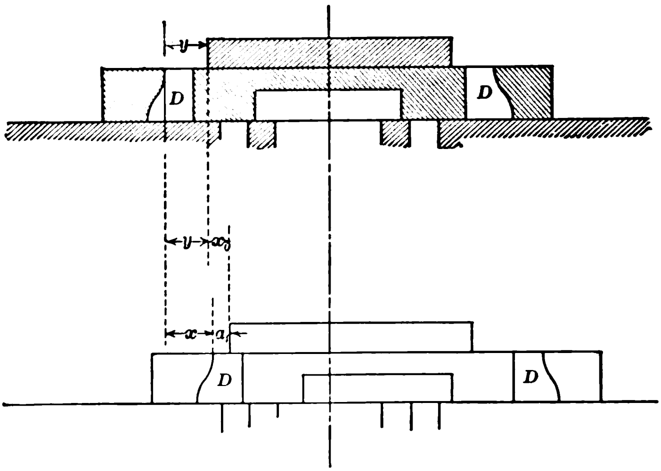


FIG. 162

is the distance from the edge of the expansion valve to the outer edge of the port D, when the upper valve is in its central position relatively to the lower one, y will be a constant of the valve gear. Call the port opening of D a_1 as before. The equation of motion is evidently

$$a_1 + x = y + x_0,$$

$$a_1 = y - (x - x_0).$$

If we draw two valve circles representing x and x_0, as shown in Fig. 163, $OP = x$, and $OP_0 = x_0$. Hence $PP_0 = x - x_0$. But this last distance is not measured from a fixed point, viz. the origin, and hence we cannot subtract it graphically from a constant quantity y by drawing a circle of radius y about the origin. We must first, therefore, find the curve which will give for every position of the crank the value of $x - x_0$ measured from the origin.

$$x - x_0 = r \sin(\theta + \delta) - r_0 \sin(\theta + \delta_0) = z,$$

$$z = r \sin\delta \cos\theta + r \sin\theta \cos\delta - r_0 \sin\delta_0 \cos\theta - r_0 \sin\theta \cos\delta_0,$$

$$z = (r \sin\delta - r_0 \sin\delta_0) \cos\theta + (r \cos\delta - r_0 \cos\delta_0) \sin\theta$$

$$= A' \cos\theta + B' \sin\theta,$$

where A' and B' are constants of the valve gear. This is the equation of a circle whose intercepts on the axes of X and Y are

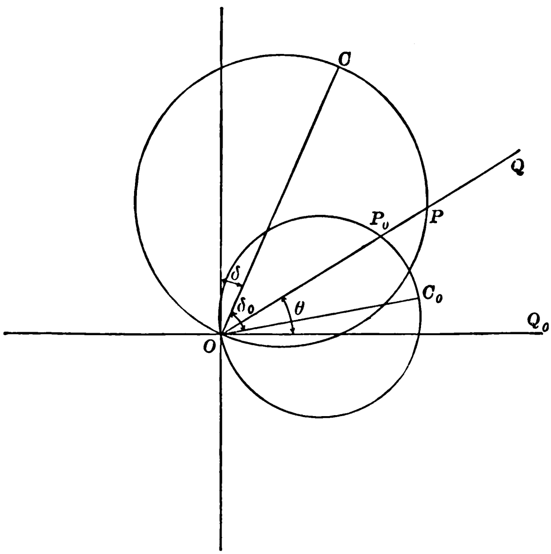


FIG. 163

A' and B' respectively. In Fig. 164 let $OC = r$ and $COY = \delta$; let $OC_0 = r_0$ and $C_0OY = \delta_0$. Connect C_0C, and complete the parallelogram OC_0CC_z, thus determining the side OC_z. From C_0, C, and C_z drop perpendiculars on the axes of X and Y. Then

$$SO = KC_0 = OH - OG = r\sin\delta - r_0\sin\delta_0 = A',$$

$$OT = KC = CH - C_0G = r\cos\delta - r_0\cos\delta_0 = B'.$$

Hence the circle on OC_z as a diameter will give by its intercepts on the axes of X and Y the values of A' and B', and by its intercept on any radius vector $(x - x_0)$ for that particular crank position. Fig. 165 shows the complete Zeuner diagram for the Meyer valve. OC is the throw of the distribution valve eccentric with its angular advance δ. OC_0 is that of the expansion valve.

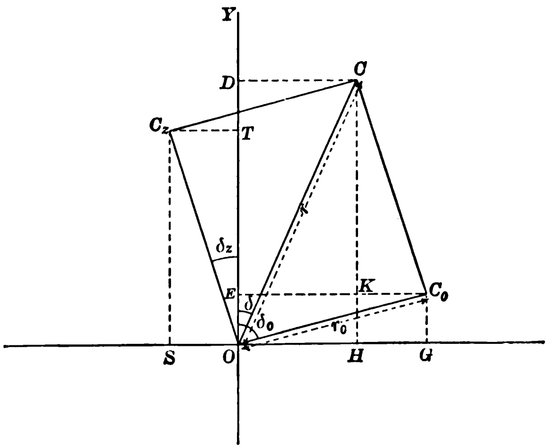


FIG. 164

From these, by completing the parallelogram, we get OC_z and δ_z of the *auxiliary circle*. On OC and OC_z as diameters, describe circles. There is no use in drawing a circle on OC_0, as it is not used. Then if OQ is any position of the crank, $OP = x$ and $OP_z = x - x_0 = z$. We now draw a y-circle about the origin, hence $OR_z = y$, and $R_zP_z = y - z = a_1$. The expansion valve opens first at Q_1', but steam is first admitted into the cylinder at OQ_1, where the distribution valve opens. Cut-off is determined by the expansion valve at OQ_2', and the distribution valve closes at OQ_2, before the expansion valve opens again at OQ_1'. The second opening of the expansion valve on $Q_1'O$ produced is on the other end of the valve, and hence does not interfere with OQ_2 in

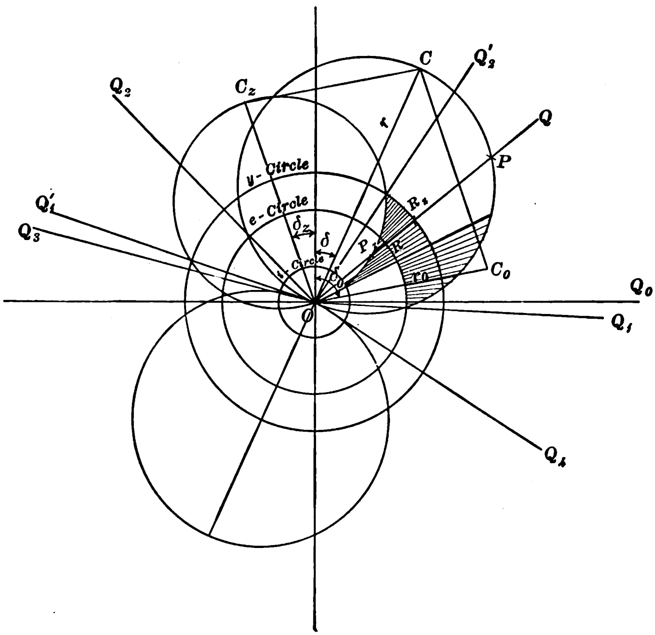


FIG. 165

such a way as to cause a double admission on the head end. Cut-off can be varied by varying the value of y. This can be done by making the cut-off plate in two parts, which can be separated by means of a right and left hand screw on the valve stem. Another way is by varying δ_0 with an automatic shaft governor, as in Fig. 166.

(*d*) *The Thompson Valve*

In this valve gear, which is used on the Buckeye engine, the arrangement of valves is the same as in the Meyer gear. The driving mechanism, however, is different. The distribution valve is driven directly by an eccentric r (Fig. 167). The point B, where

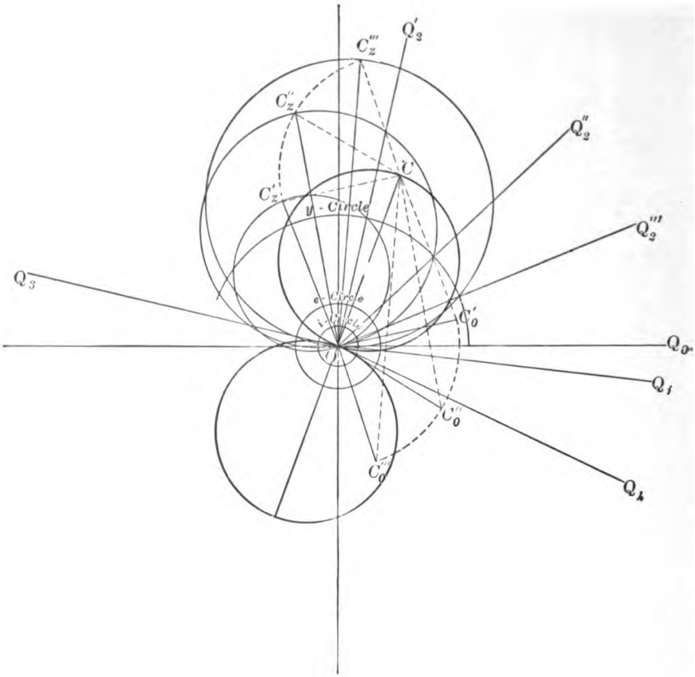


FIG. 166

the eccentric rod is hinged to the valve stem, is carried on a vibrating pillar BC. The expansion valve is driven from the extremity A of a lever AE, whose middle point D is pivoted to the middle point of BC, and whose other end is driven by an eccentric r_z. We have, as in the case of the Meyer valve,

$$a_1 = y - (x - x_0),$$

where a_1 is the port opening, y a constant, and x and x_0 distances moved through by the distribution and expansion valves, both measured from their central positions referred to the fixed valve

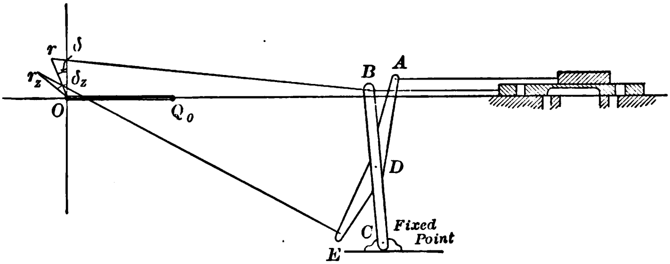


FIG. 167

seat. The value of x_0 will be determined both by r and by r_z. Since $AB = BD$ and $DC = DE$, that component of A's motion due to r will be simply x, where

$$x = r \sin(\theta + \delta).$$

That component of A's motion due to r_z will be $-x_z$, and

$$-x_z = -r_z \sin(\theta + \delta_z).$$

Now x_0 is the algebraic sum of these two as they take place in the same straight line. Hence,

$$x_0 = x - x_z,$$

$$a_1 = y - (x - x + x_z) = y - x_z.$$

The motion of the expansion valve relatively to the distribution valve when driven through the above kinematic chain will be the same as if the expansion valve were driven directly from the eccentric r_z upon a fixed seat. The form of the equation of motion is identical with that of the gridiron valve, hence the travel of the expansion valve on the top of the distribution valve will be of constant magnitude for all cut-offs when the expansion eccentric is rotated about the shaft by means of a shaft governor.

B. Relation between Position of the Crank and Velocities and Accelerations of the Valve

Since the motion of the valve is taken as simply harmonic, we can express the velocity of the valve for any angular position θ of the crank by differentiating x_B' with respect to time.

$$x_B' = - r \sin(\delta + \theta),$$

$$\frac{dx_B'}{dt} = - r \cos(\delta + \theta)\frac{d\theta}{dt}.$$

And if $\frac{d\theta}{dt} = \omega =$ constant,

$$v_B = - r\omega \cos(\delta + \theta).$$

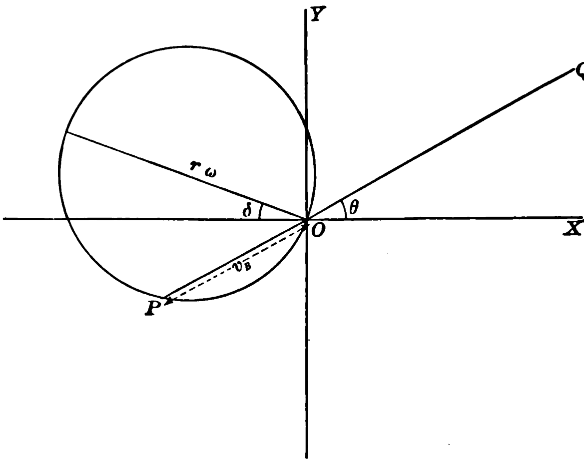


FIG. 168

If this be laid off as a polar curve along the crank throw as a radius vector, we see that it represents a circle (Fig. 168), whose diameter is $r\omega$, whose circumference passes through the pole, and whose diameter through the pole is inclined at an angle $180° - \delta$ with the axis of X.

The acceleration of the valve would be obtained by differentiating with respect to time the expression for the velocity, or

$$p_B = \frac{dv_B}{dt} = r\omega \sin(\delta + \theta) \frac{d\theta}{dt}$$

$$= r\omega^2 \sin(\delta + \theta).$$

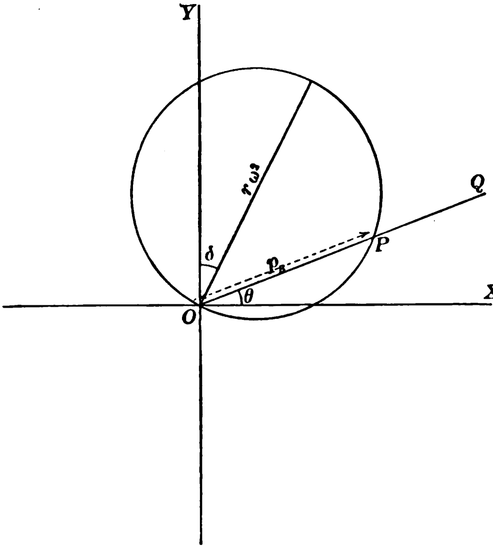


FIG. 169

This is the equation of a circle (Fig. 169) whose diameter is $r\omega^2$, whose circumference passes through the pole, and whose diameter through the pole makes an angle of $(90° - \delta)$ with the axis of X.

4. RELATION BETWEEN PISTON AND VALVE POSITIONS

In the two preceding sections we have obtained exact relations between the crank and piston position and the crank and valve position. It is the purpose of the present section to investigate the relation between the position of the piston and valve. Analytic expressions for this relation are, in the exact case, and even in the approximate ones, too cumbersome to be employed. But by the graphic line of analysis already discussed the solution is direct and complete.

Let Fig. 170 represent a polar valve diagram. About O describe any circle, as that through Q, to represent the crank orbit. If OQ is any position of the crank, $P'R'$ is the port opening. But the piston position corresponding to Q on the diameter of the crank circle as a stroke is at E, projected on a circle about P as a centre. Hence piston position E corresponds to port opening PR' of the valve, and piston positions E_1, E_2, E_3, and E_4 correspond exactly to admission, cut-off, release, and compression.

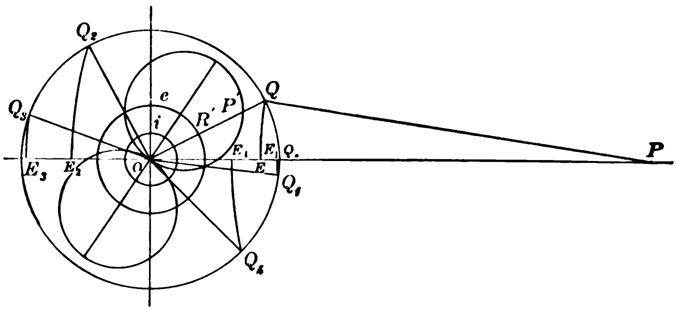


FIG. 170

Conversely, if any three of these piston positions be given, a valve can be designed to fulfil the conditions.

If the laps are the same on the two ends, then evidently the piston positions corresponding to admission, cut-off, etc., cannot be the same on the two ends. Consider, for example, the cut-off positions for equal laps in Fig. 171. The crank positions OQ_2' and OQ_2'' will be diametrically opposite, but the piston when at E_2' will have completed a larger fraction of its stroke than when at E_2''. This defect can be remedied by making the outside laps different on the two ends, giving the larger to the head end, but by doing so the admission and lead on that end are changed. For the same reason the head end will have a greater compression, as the exhaust will close when the piston is farther from the end of the stroke. If a valve is set with equal leads, the cut-off will be later and the compression will begin earlier on the head than on

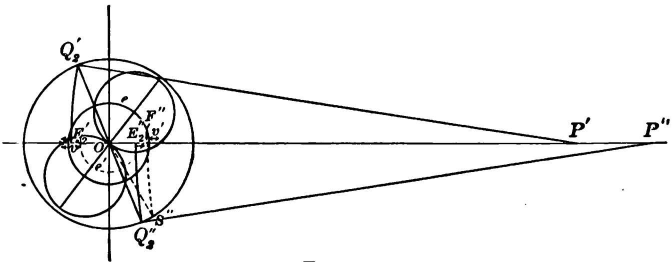


FIG. 171

the crank end. With a variable cut-off, such as is produced by a shaft governor (Figs. 144 and 146), or by a Stephenson link (Fig. 156), if the cut-offs be equalized in one position by change of laps, they will not be exactly equal for any other position.

CHAPTER II

DYNAMICS

The forces acting upon the moving parts of the steam-engine are due to two causes: (1) to steam pressure, and (2) to the inertia effects of the moving masses. Of these two the first is in general the largest and most important, and it is the force which does work outside of the machine itself. The forces due to the second cause are resident entirely within the machine, and the net work due to them within any complete revolution made under steady conditions is always zero. We will take up the discussion of these forces in the order given.

1. FORCES DUE TO STEAM PRESSURE IN THE CYLINDER

These forces are resident entirely within the cylinder of the engine, and they will act upon the piston in the direction of the main line of connection of the machine, that is, in the direction of the motion of the piston. Their magnitude then is the only important consideration, and this can best be studied by reference to a curve in rectangular coördinates, which shows the relation between piston position and total steam force. This curve can be deduced from the familiar indicator card, and for the accurate study of the steam effects it is best to use an actual card, either taken upon the engine under discussion or upon another of the same type. If no such card is at hand, an approximation to its form can be drawn after the following constants and dimensions are known:

1. The boiler pressure.
2. The back pressure.
3. The point of cut-off.
4. The release point.
5. The point of compression.
6. The clearance.

Of these, 1 and 2 are simply working conditions; 2 is immediately known in the case of the simple engine, but must be obtained by calculation in the case of the compound; 3 and 5 are known from the valve design, and 4 may be taken as occurring at the end of the stroke without serious error; 6 may be assumed from what is known to be an average value in the particular type of engine treated; 3, 5, and 6 can be expressed as fractions of the stroke or of the volume swept through by the piston.

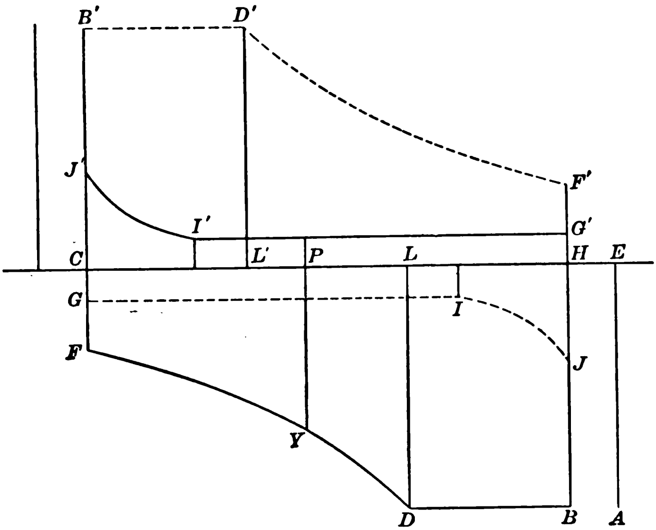


FIG. 172

The Simple Engine. — The cycle of changes of pressures and volumes taking place upon one side of the piston during one revolution of the engine is as follows: steam is admitted into the head end of the cylinder at boiler pressure (or a little less), at B (Fig. 172), and as the force exerted by the steam upon the piston is toward the origin O, or to the left, this pressure is negative and is laid off below the axis of X. Call this total pressure, *i.e.* the specific pressure multiplied by the area of the piston, P_1, and we must measure it above the vacuum or line of zero pressure. Hence

$BH = P_1$. This pressure is maintained pretty constantly up to the point of cut-off at D, so that $DL = P_1$ also. Let $HE = AB = s_c$, the clearance expressed in terms of the stroke $HC = s$ of the piston. At D expansion begins, the pressure dropping off on a curve DYF, which is nearly enough a rectangular hyperbola with asymptotes EA and EC to be taken as such by the designer. The terminal pressure $CF = P_2$ is determined by the equation of the rectangular hyperbola, viz.

$$P_2 = \frac{P_1(s_1 + s_c)}{s + s_c},$$

or by graphical construction. The release then occurs, the pressure dropping still farther to $CG = P_3$, the back pressure, which in a non-condensing engine is about 17 lbs., while in a condensing engine it is from 6 to 8 lbs., multiplied by the piston area. This back pressure is maintained at a fairly constant value during the return stroke until at I the exhaust closes, and compression begins, the pressure rising on the hyperbola IJ to P_c, where P_c is given by

$$P_c = \frac{P_3(s_1 + s_3)}{s_c}.$$

Meanwhile a similar cycle of pressures has been taking place on the crank end, and as all forces acting on the piston due to steam in this end are positive, the curves will all lie above the axis of X.

The work done by the steam in the head end in passing from H to C will evidently be proportional to the area $BDFCH$, and the work done upon the steam in the head end in passing from C back to H will be proportional to the area $GIJHC$. Hence the total or net work done in the head end during one complete revolution is proportional to the area $BDFGIJ$. Similarly on the crank end of the cylinder the net work done in one complete revolution will be proportional to the area $B'D'F'G'I'J'$, and the whole output of work will be proportional to the sum of these areas.

Now as the piston passes from H tc C, the negative forces will be given by the ordinates of the steam curve of the head end or by BDF, and during the same stroke the positive forces are given by the ordinates of the back pressure curve of the crank end. Hence the net force acting on the piston-rod will be given

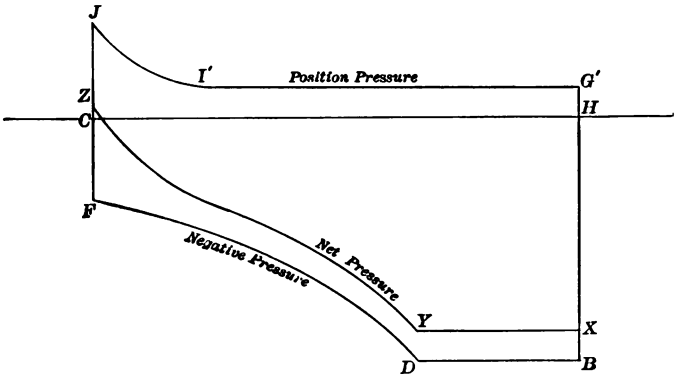


FIG. 173

by the algebraic sum of these curves or by the ordinates of *XYZ* (Fig. 173). The area *XYZCH* will be proportional to the net work done by the steam in passing from *H* to *C*, or during one forward stroke of the engine. Similarly the area under the curve

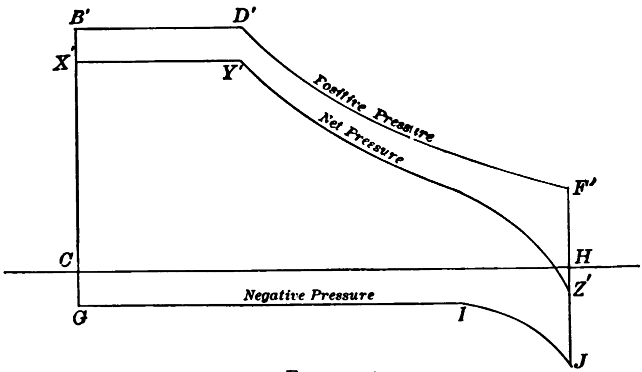


FIG. 174

X′Y′Z′HC (Fig. 174) is proportional to the work done by the steam during one backward stroke of the engine. These areas will not in general be equal to the areas of the original cards *BDFGIJ*, etc., unless the cards themselves are exactly alike, but it is evident on comparison of areas that the sum of the areas of the curves *XYZCH* and *X'Y'Z'HC* will be equal to the sum of the areas of the indicator cards.

These last curves of *Net Horizontal Steam Effort* are particularly useful in the analyses which follow.

The Multiple-cylinder Engine. — In a multiple-cylinder engine the steam is expanded through more than one cylinder, the first taking its steam from the boiler, the second taking it from the exhaust of the first, and so on. In every case all the cylinders work upon the same shaft and have the same stroke, though the arrangement of the cranks may be different in different cases. It is readily seen that the steam line of any cylinder beyond the first up to its point of cut-off will be identical as to specific pressure at any given instant of time with the back pressure line of the preceding cylinder. Here, as before, it is best to base the analysis of the acting forces upon actual indicator cards. But if unobtainable, theoretical ones must be drawn. It is not the province of the present work to go into the calculations necessary to obtain the pressures around the cycles of a multiple-cylinder engine, that being more a problem in thermodynamics. It is sufficient to say that such cycles can be drawn quite accurately to scale when the proper dimensions and constants are known. The curves of any one cylinder can then be combined to find the net horizontal force, as in the case of the simple engine. When the several pistons act upon a single rod and crank, in other words are tandem, the net horizontal pressure curves can be added directly. But when acting on different cranks each cylinder must be considered by itself.

Power developed in the Cylinder. — The power developed in the cylinder can readily be calculated from the net pressure curves. Let A be the area under the curve *XYZCH*, obtained by a planimeter in square inches, and b the length *HC* in inches. Then $\frac{A}{b}$ will give a number which, on the scale of forces used, is the total mean pressure. This constant force acting on the piston would do the same work in one stroke as the variable force before considered. Call it P_m, and let P_m' be a similar pressure for the backward stroke. Then

$$\frac{P_m \times (\text{number of forward strokes per minute}) \times L}{33000}$$

$$+\frac{P_m' \times (\text{number of backward strokes per minute}) \times L}{33000} = H.P.,$$

where L is the length of the stroke in feet, and $H.P.$ the horse-power developed.

$$H.P. = \frac{(P_m + P_m') \times L \times N}{33000},$$

N being the number of revolutions per minute.

In all the above the pressures are taken as total pressures acting on the piston-rod. If specific pressures are employed, the area of the piston will enter as a constant factor on the right side of the equation.

Forces at the Wrist Pin.—The net horizontal force active on the piston-rod being now known, its components at the wrist pin can be obtained. In Fig. 175 P_x is this horizontal force. Its

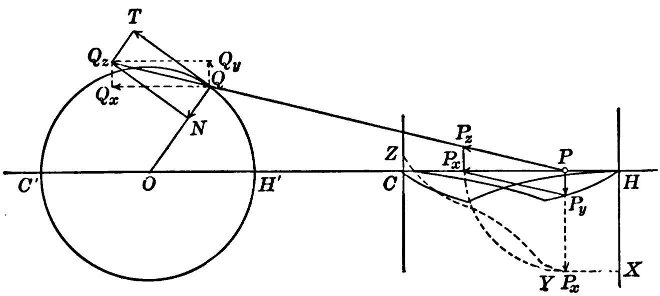


FIG. 175

component at right angles to OX will be P_y, which is the thrust upon the guide. Its component along the connecting rod is P_z. P_y can be plotted along the path of P as a curve of normal pressures. As the force is at right angles to the direction of motion, the area of the curve represents no energy. The general form of the curve is shown in Fig. 175. On the return stroke from *C* to *H* a similar curve will be produced.

Forces at the Crank Pin.—The force P_s will be transmitted directly to *Q* as a force Q_s. This can be resolved in the tan-

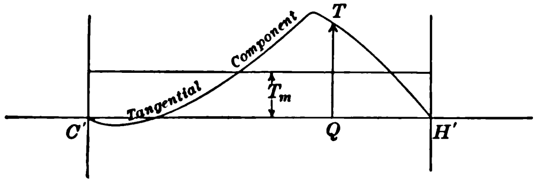


FIG. 176

gent and normal to the crank orbit, giving components *T* and *N*. The tangential component is the true turning force of the main shaft at *O*. If we develop the semicircumference *H'QC'* along a straight line, as in Fig. 176, and lay off along this as a base a curve whose ordinates are the tangential forces *T*, the area of the curve will be proportional to the total work done upon the crank pin during a semirevolution from *H'* to *C'*, as the force *T* acts in the direction of the motion of *Q*. In other

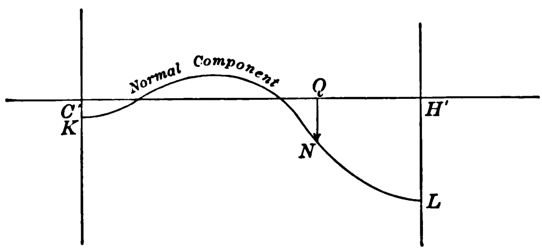


FIG. 177

words, the area of the curve *H'TC'H'* (Fig. 176) must be equal to the area *XYZCH* (Fig. 173), when the two are drawn on the same scale of pressures. If now we divide the area of this curve in square inches by the length of its base in inches, we will get a number which, on the scale of forces employed, will be the mean turning effort. If this constant force be applied at right angles to the crank, it would do the same work in passing from H' to C' as the variable force previously considered. Call this mean turning effort T_m. Then $\frac{A}{\pi l} = T_m$. But $\frac{A}{b} = P_m$, and $b = 2l$. Hence,

$$T_m = \frac{2}{\pi} P_m.$$

The normal component at the crank pin may also be laid off as the ordinates of a curve along the developed semicircumference as a base, giving results something like Fig. 177. The area of this curve represents no energy, as the direction of N is at right angles to the path of Q.

2. THE INERTIA EFFECTS OF THE RECIPROCATING PARTS

A. Analytical and Graphical Calculation of Forces Active at the Crank and Wrist Pins

The method rests primarily upon two principles of Analytic Mechanics, viz.: (1) that the sum of all forces acting upon a body when resolved in a given direction is equal to the mass of the body multiplied into the acceleration of its centre of mass in the given direction and (2) that the moment of all forces about an axis through the centre of mass is equal to the moment of inertia of the body about that axis into its angular acceleration.

In order to clearly illustrate the application of these principles to the question in hand, let us consider the case exhibited in Fig. 178. Here O is the centre of a shaft of a horizontal steam-engine and OQ is the crank, turning with a constant angular velocity ω. QP is the connecting rod with wrist pin at P. Let l represent the throw of the crank, L the length of the connecting rod measured from centre to centre of brasses, and r the distance of the centre of mass G of the rod from the centre of the wrist pin. Taking O as the origin of coördinates, and the line of connection of the engine as the axis of X, we will call the coördinates of G $\bar{x}$ and $\bar{y}$. Let θ be the angle between any position of the crank and the axis of X, and let α be the corresponding angle of the connecting rod. Q_x and Q_y are the component forces acting on the crank-pin end

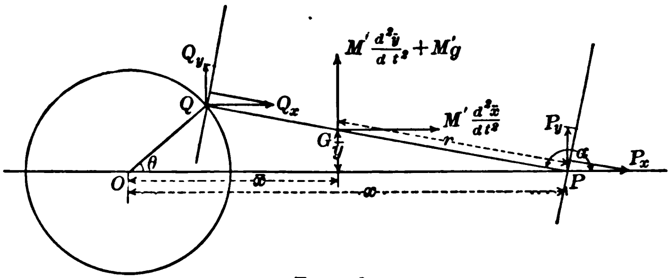


FIG. 178

of the rod parallel to the axes of X and Y respectively, and P_x and P_y are the components at the wrist-pin end. Then if M' is the mass of the rod, we will have from the first principle

$$Q_x + P_x = M' \frac{d^2\bar{x}}{dt^2}. \qquad (1)$$

$$Q_y + P_y = M' \frac{d^2\bar{y}}{dt^2} + M'g. \qquad (2)$$

Projecting these forces at right angles to the rod, and taking the moments about G, we get from the second principle

$$-\{-Q_y \cos\alpha + Q_x \sin\alpha\}(L-r) + \{P_x \sin\alpha - P_y \cos\alpha\} \cdot r$$
$$= I_0 \frac{d^2\alpha}{dt^2}. \qquad (3)$$

In these equations P_x is any external force acting on the wrist-pin end of the rod parallel to the axis of X, and is due to steam pressure on the piston, the acceleration of the cross-head and piston, and to friction. It may be either positive or negative, but can always be calculated for any value of θ. I_0 is the moment of inertia of the rod about an axis through G parallel to the main shaft, and can be experimentally determined. We have, therefore, three equations from which to determine three unknown quantities, — Q_x, Q_y, and P_y. Solving (3) for P_y, we get

$$P_y = -\frac{I_0 \dfrac{d^2\alpha}{dt^2} + (L-r)\,Q_x \sin\alpha - (L-r)\,Q_y \cos\alpha - r\,P_x \sin\alpha}{r\cos\alpha}.$$

Substituting this in (2),

$$Q_y = M'\frac{d^2\bar{y}}{dt^2} + M'g + \frac{I_0 \dfrac{d^2\alpha}{dt^2} + (L-r)\,Q_x \sin\alpha - (L-r)\,Q_y \cos\alpha - r\,P_x \sin\alpha}{r\cos\alpha}.$$

Substituting for Q_x its value from (1), and solving for Q_y, we finally get

$$Q_y = \frac{I_0}{L\cos\alpha}\frac{d^2\alpha}{dt^2} + \frac{r}{L}M'\frac{d^2\bar{y}}{dt^2} + \frac{r}{L}M'g + \frac{L-r}{L}M'\frac{d^2\bar{x}}{dt^2}\tan\alpha - P_x\tan\alpha, \quad \quad (4)$$

$$Q_x = M'\frac{d^2\bar{x}}{dt^2} - P_x, \quad \quad (5)$$

$$P_y = M'\frac{d^2\bar{y}}{dt^2} - M'g - Q_y. \quad \quad (6)$$

In these three equations there are, in addition to the unknown quantities, certain dimensions and constants of the engine as well as three accelerations which must be determined. These latter we have already deduced in Chapter I. Since G lies in the axis of the rod,

$$\frac{d^2\bar{x}}{dt^2} = -l\omega^2\cos\theta - n\omega^2(l-nr)\frac{(\cos 2\theta + n^2\sin^4\theta)}{(1-n^2\sin^2\theta)^{\frac{3}{2}}}, \quad \quad (7)$$

$$\frac{d^2\bar{y}}{dt^2} = -rn\omega^2\sin\theta, \quad \quad (8)$$

$$\frac{d^2\alpha}{dt^2} = \frac{n\omega^2(1-n^2)\sin\theta}{(1-n^2\sin^2\theta)^{\frac{3}{2}}}. \quad \quad (9)$$

It is also convenient to write down,

$$L \cos \alpha = L\sqrt{1 - n^2 \sin^2 \theta},$$

$$\tan \alpha = \frac{n \sin \theta}{\sqrt{1 - n^2 \sin^2 \theta}}.$$

All of these are known in terms of θ.

Of the constants, r and I_0 are the only ones that require any special methods of measurement. They can be determined approximately from a working drawing of the rod itself, but can better be determined experimentally as follows: let the connecting rod swing as a pendulum on a knife edge, first through the wrist pin and then through the crank-pin brasses. Let t_w be the observed time of one vibration in the first case, and t_c the time in the second. Then

$$t_w = \pi\sqrt{\frac{l_w}{g}},$$

$$t = \pi\sqrt{\frac{l_c}{g}},$$

where l_w and l_c, the lengths of equivalent simple pendulums, become immediately known. Now if h_w is the distance from the point of suspension at the wrist-pin end to the centre of gravity of the rod,

$$h_w(l_w - h_w) = q^2,$$

where q is the radius of gyration about G. Also,

$$h_c(l_c - h_c) = q^2,$$

and
$$h_w + h_c = L_0,$$

where $L_0 = L +$ (radius of crank pin) $+$ (radius of wrist pin). Combination of these equations gives

$$h_w = \frac{L_0(L_0 - l_c)}{2L_0 - l_w - l_c},$$

from which r is obtained by subtracting the radius of the wrist pin. The moment of inertia about G is also determined from

$$I_0 = M'q^2 = \frac{W'}{g} q^2,$$

where W' is the weight of the rod in pounds.

In order to study the effects of these forces it will be best to follow through an actual case. Curves showing their magnitude, direction, and general variation will give a better insight into the subject than many pages of formulæ. Take for example a small 6 × 8 horizontal engine, whose dimensions are as follows:

W'	= weight of connecting rod	24 lbs.
W''	= weight of piston and cross-head . . .	33.6 lbs.
l	= throw of crank	.333 ft.
L	= length of connecting rod	2 ft.
n	$= \frac{l}{L}$	.1666
N	= number of revolutions per minute . . .	300
ω	= number of radians per second	31.625
r	= distance of centre of mass of rod from centre of wrist pin	1.123 ft.
I_0	= moment of inertia of rod about G . . .	.4233

We must now compute the values of $\frac{d^2\alpha}{dt^2}$, $\frac{d^2\bar{x}}{dt^2}$, and $\frac{d^2\bar{y}}{dt^2}$, for every 10° of the crank orbit. Table No. 1 shows the results of these computations. The last column gives the values of $\frac{d^2x}{dt^2}$, or the acceleration of the cross-head and piston. This is easily obtained from the expression for $\frac{d^2\bar{x}}{dt^2}$ by putting $r = 0$. This then becomes

$$\frac{d^2x}{dt} = -l\omega^2 \cos\theta - \frac{ln\omega^2(\cos 2\theta + n^2 \sin^4\theta)}{(1 - n^2 \sin^2\theta)^{\frac{3}{2}}}.$$

These accelerations are expressed in feet per second per second, or in radians per second per second.

The curves of Plate I show graphically the results of Table No. 1, and the curves as numbered correspond to the columns of the table. The values in the table extend from $\theta = 0°$ to $\theta = 180°$, but it is easily seen that for the remainder of the circle columns 2 and 4 will be repeated in reverse order without change of sign, while columns 1 and 3 will be repeated in reverse order with change of sign.

TABLE No. 1

Computation of Accelerations

θ	$\frac{d^2a}{dt^2}$	$\frac{d^2\bar{x}}{dt^2}$	$\frac{d^2y}{dt^2}$	$\frac{d^2x}{dt^2}$
	(1)	(2)	(3)	(4)
0°	.000	− 357.746	.000	− 388.937
10°	+ 28.176	351.241	− 32.502	380.589
20°	55.696	332.040	64.016	356.063
30°	81.878	301.078	93.584	316.893
40°	105.985	259.805	120.310	265.466
50°	126.944	210.192	143.380	204.946
60°	144.847	154.506	162.093	138.913
70°	158.060	95.194	175.882	71.093
80°	166.265	− 34.696	184.325	− 5.009
90°	169.049	+ 24.715	187.169	+ 56.350
100°	166.265	81.084	184.325	110.771
110°	158.060	132.850	175.882	156.951
120°	144.847	178.872	162.093	194.465
130°	126.944	218.388	143.380	223.634
140°	105.985	250.859	120.310	245.298
150°	81.878	276.368	93.584	260.553
160°	55.696	294.504	64.016	270.481
170°	+ 28.176	305.383	− 32.502	276.035
180°	.000	+ 309.008	.000	+ 277.817

We now come to the calculation of the forces; namely, Q_x, Q_y, P_x, and P_y. In the expression for Q_y (equation No. 4) the third term is merely the upward reaction of the crank pin upon the rod due to its own weight. The fifth term is the horizontal thrust upon the rod at the wrist pin due to external forces. These two terms, being independent of the others, may be considered at any time during the discussion. The same applies to the second term in the expression for Q_x and the second term in the expression for P_y. For the present we will omit these, or in other words we will first consider the effect of the connecting rod on the crank pin due to its own inertia, and call the forces thus exerted on the ends of the rod Q_x', Q_y', P_x', and P_y'. Then

$$Q_x' = M' \frac{d^2\bar{x}}{dt^2}, \quad \text{(10)}$$

$$Q_y' = \frac{I_0}{L \cos \alpha} \frac{d^2\alpha}{dt^2} + M' \frac{r}{L} \frac{d^2\bar{y}}{dt^2} + M' \frac{L-r}{L} \frac{d^2\bar{x}}{dt^2} \tan \alpha, \quad \text{(11)}$$

$$P_x' = 0, \quad \text{(12)}$$

$$P_y' = M' \frac{d^2\bar{y}}{dt^2} - Q_y'. \quad \text{(13)}$$

Finally the pressures of the rod on the pins will be equal and opposite to those exerted by the pins on the rod, and these latter we will denote by (Q_x'), (Q_y'), and (P_y'). The effect of the first

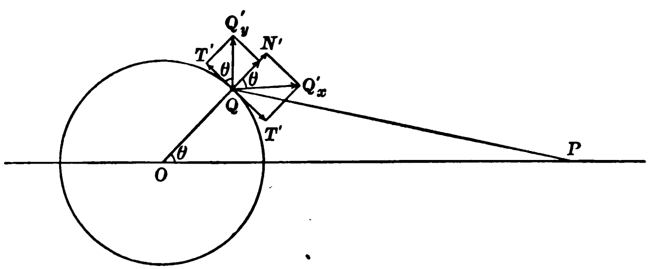


FIG. 179

two of these upon the rotation of the crank can best be studied by resolving them along the tangent and normal to the crank orbit, as in Fig. 179. Denoting these forces by (T') and (N'),

$$(T') = (Q_y') \cos \theta - (Q_x') \sin \theta,$$

$$(N') = (Q_y') \sin \theta + (Q_x') \cos \theta.$$

The values of these various forces acting upon the crank pin and guides, due to the inertia of the connecting rod alone, are given in Table No. 2, and the graphical results in Plates II and III. Since the force (T') acts in the same line as the direction of motion, the area between Curve No. 3 (Plate No. 2), the axis of X, and any ordinate represents work done upon or by the pin, and stored up as kinetic energy in the rod. Since the state of the rod as regards kinetic energy is identical at the two dead points, we should expect the net area of the curve between O and B to be zero, which a planimeter shows to be a fact. The curve on Plate III shows the variation of (P_y'), or the pressure of the wrist pin upon the guides. In this case, however, the curve is plotted along the guide itself. All forces are expressed in pounds.

TABLE No. 2

Forces acting on Crank Pin and Guides due to Connecting Rod

θ	(Q_x')	(Q_y')	(T')	(N')	(P_y')
0°	+ 266.649	.000	.000	+ 266.649	.000
10°	261.800	+ 16.204	− 29.503	260.636	+ 8.021
20°	247.488	32.329	54.267	243.620	15.386
30°	224.411	48.229	70.441	218.459	21.528
40°	193.648	63.650	75.713	189.256	26.024
50°	156.668	78.144	69.785	160.566	28.725
60°	115.162	91.360	54.053	136.701	29.458
70°	70.955	102.478	31.624	120.566	28.616
80°	+ 25.861	110.895	− 6.211	113.701	26.493
90°	− 18.422	115.989	+ 18.422	115.989	23.519
100°	60.437	117.264	39.155	125.687	20.124
110°	99.021	114.434	53.910	141.400	16.660
120°	133.324	107.433	61.746	159.702	13.385
130°	162.777	96.379	62.743	178.461	10.490
140°	186.980	81.841	58.028	195.841	7.833
150°	205.993	64.191	47.406	210.490	5.564
160°	219.511	44.151	33.589	221.374	3.564
170°	227.620	+ 22.489	+ 17.379	228.067	+ 1.736
180°	− 230.321	.000	.000	+ 2302.31	.000

The next point to be considered will be the effect of the external force P_x. This, as has been said, is due to two causes, the inertia of the piston and cross-head, and the varying pressures on the cross-head due to steam pressure and friction. We will not consider in this case the last causes, but take up the purely theoretical case of the crank and reciprocating parts, moving without friction; and neglect the pressures due to the weight of the parts. The portion of P_x due to this cause we will call P_x''. We then have

$$P_x'' = M'' \frac{d^2x}{dt^2}, \quad . \quad . \quad . \quad . \quad (14)$$

where M'' is the mass, and $\frac{d^2x}{dt^2}$ the acceleration of the piston and cross-head. Now, if the component forces acting on the crank pin parallel to the axes of X and Y due to P_x'' are (Q_x'') and (Q_y''), and the force at right angles to the guides is (P_y''), we will have

$$(Q_x'') = P_x'', \quad . \quad . \quad . \quad . \quad . \quad (15)$$

$$(Q_y'') = P_x'' \tan \alpha, \quad . \quad . \quad . \quad . \quad (16)$$

$$(P_y'') = -(Q_y''). \quad . \quad . \quad . \quad . \quad (17)$$

The values of $\frac{d^2x}{dt^2}$ we have already computed in Table No. 1, and these only need to be multiplied by M'' to give us P_x''. The components (Q_x'') and (Q_y'') may also be resolved along the tangent and normal to the crank orbit, giving (T'') and (N''). All these forces are given in Table No. 3. Plate IV gives the curves of (Q_x''), (Q_y''), (T''), and (N''), while Plate V gives the curve of (P_y'') laid out along the guide. The curve of tangential effort, namely, (T''), must have a net area equal to zero, and such is found to be the case.

Taking now the total inertia effects of connecting rod and cross-head and piston, we have the complete curves in Plates VI, VII, and VIII. The first of these shows the total horizontal and total vertical components of the forces active at the crank pin due to inertia. The second shows the total tangential and total normal components. A most remarkable resemblance is seen to exist between the two curves of tangential effort. In fact, if a mass about .535 of the mass of the connecting rod were concentrated in the cross-head, an almost identical curve of total tangential effort would be produced by the acceleration of the cross-head alone.

TABLE No. 3

Forces acting on Crank Pin and Guides due to Piston and Cross-head alone

θ	(Q_x'')	(Q_y'')	(T'')	(N'')	(P_y'')
0°	+ 405.847	.000	.000	+ 405.847	.000
10°	397.127	− 11.497	− 80.283	389.086	+ 11.497
20°	371.543	21.206	147.004	341.806	21.206
30°	330.671	27.025	189.284	272.543	27.025
40°	227.008	29.848	200.922	193.014	29.848
50°	213.857	27.522	181.515	116.381	27.522
60°	144.952	21.139	136.102	54.169	21.139
70°	74.184	11.764	73.734	14.317	11.764
80°	+ 5.227	− .870	− 5.298	.052	+ 0.870
90°	− 58.800	+ 9.939	+ 58.800	9.939	− 9.939
100°	115.587	19.234	110.491	38.581	19.234
110°	163.775	25.972	145.015	80.420	25.972
120°	202.920	29.593	160.938	127.088	29.593
130°	233.357	30.032	159.458	173.005	30.032
140°	255.963	27.581	143.402	213.807	27.581
150°	271.880	22.738	116.249	246.827	22.738
160°	282.234	16.109	81.393	270.723	16.109
170°	288.036	+ 8.339	+ 41.803	285.108	− 8.339
180°	− 289.896	.000	.000	+ 289.896	.000

Thus far we have neglected the weight of the rod in producing pressure on the crank pin and wrist pin, but this can readily be taken into account. By referring to equations (4) and (6), we see that the effect of this is to add a constant positive quantity $\frac{r}{L}M'g$ to Q_y', and a constant positive quantity $\frac{L-r}{L}M'g$ to P_y'. The effects on the crank pin and guides will be equal and opposite to these, or we will add constant negative quantities $\frac{r}{L}M'g$ and $\frac{L-r}{L}M'g$ to (Q_y') and (P_y'). No change will be produced in (Q_x'). The values of (Q_y') and (P_y'), however, change sign in passing the dead points, while the pressure due to the weight of the rod is constantly downward. Hence these two values, when corrected for the weight of the rod, will be different in the two semirevolutions above and below the axis of X. Likewise the value of (T') will be affected to the extent of having the term $-\frac{r}{L}M'g \cos \theta$ added. Plate IX shows these various corrected curves. The correction to be applied to (P_y') could be shown, if required, in Plate III, by a straight line whose ordinates would be subtracted from those of Curve No. 3, and in the same way the weight of the cross-head might be taken into account in Plate V.

The above analysis is exact and complete, but simpler approximate formulæ can be deduced from them, which in most cases will give results sufficiently close. These approximations will be taken up under the special cases to which they apply.

B. Applications

(*a*) *The Fly-wheel*

We have seen that the tangential force at the crank pin, or the turning effort, is far from constant. In fact, it is necessarily zero at the two dead points, and for a single crank reaches a maximum somewhere before the 90° position. Hence, in order to secure an approximately constant value of the angular velocity, we must provide some reservoir of energy in the shape of a large mass rotating with the main shaft. This is realized in the fly-wheel. Let $H'bTeC'$ (Fig. 180) be the curve of tangential forces or turning moments of a steam-engine. This is the sum of all such forces, due both to steam and inertia. Let T_m be the mean turning effort found by means of a planimeter. Then

$$\text{Area}\,(bTe) = \text{Area}\,(H'ab) + \text{Area}\,(C'de).$$

The ordinate T_m also represents the net resistances to turning due to the pull on the belt and to friction reduced to a lever arm equal to the crank throw. Suppose the crank pin to be travelling from H' toward C'. Then while moving from H' to B the engine will be slowing down, as the tangential force on the crank pin is less than the mean tangential force or belt pull. But at B the forces are equal, and beyond B the engine is speeding up; hence the minimum velocity of the crank pin is at B. The engine then speeds up till E is reached, and hence at E occurs the maximum velocity of the crank. Again the engine slows down, and the minimum occurs again at B. While the crank is passing from B to E, the total work done by the tangential force is proportional

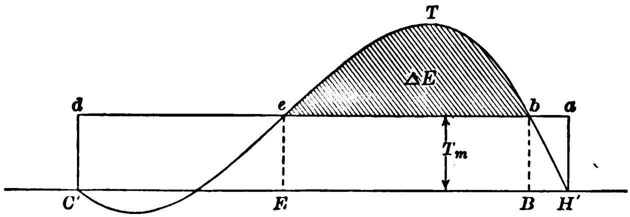


FIG. 180

to the area $BbTeE$. But of this work the belt takes out energy proportional to the rectangle $BbeE$. Hence the area bTe is proportional to the excess of energy which the engine puts in over what the belt takes out, and must be represented as change of kinetic energy in the rotating fly-wheel due to change in speed between B and E. Call this area ΔE ft. lbs.

Let v_1 and v_2 be the greatest and least velocities of the rim of the fly-wheel allowable during one stroke of the engine. Let E_1 and E_2 be the energies stored up in the rim at these velocities. Then

$$\left.\begin{aligned} E_1 &= W\frac{v_1^2}{2g} \\ E_2 &= W\frac{v_2^2}{2g} \end{aligned}\right\} \text{(very nearly),}$$

and
$$E_1 - E_2 = \Delta E = \frac{W}{2g}(v_1^2 - v_2^2).$$

If v be the mean velocity of the rim (as given by a speed counter), then, since v_1 and v_2 are very nearly equal, the value of v will be given nearly enough by

$$v = \frac{v_1 + v_2}{2}.$$

Also call δ the *Coefficient of Fluctuation* of speed, where

$$\delta = \frac{v_1 - v_2}{v}.$$

Now factoring the expression for ΔE we get

$$\Delta E = (v_1 + v_2)(v_1 - v_2)\frac{W}{2g} = \frac{Wv^2\delta}{g}.$$

If the diagram of turning efforts is drawn accurately to scale, ΔE can be measured by means of a planimeter, and expressed in foot-pounds. Hence δ, the coefficient of fluctuation, can be calculated, or if δ be assumed, the weight of rim necessary to preserve this fluctuation at the given speed of rim becomes known.

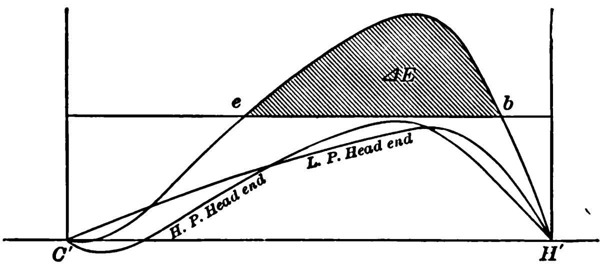


FIG. 181

If two or more cylinders are working on the same shaft, the turning efforts must be combined into a single diagram. Where the pistons work on one rod and crank, the phases of the turning efforts are identical, and the ordinates of the curves of each cylinder are merely added for each position of the crank, as in Fig. 181. Where two pistons work on different cranks at 180° apart, the effect is practically the same only that the turning effort of the head end of one cylinder is to be combined with that of the crank end of the other. In both of these cases it will be seen that the double cylinder presents no superiority so far as speed fluctuation is concerned over the single. If, however, the two cranks are at 90°, the phases are not identical, and the maximum turning effort of one cylinder comes very nearly when that of the other is zero. The resulting curve of turning efforts is therefore much smoother

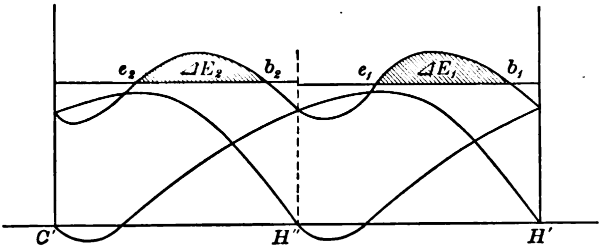


FIG. 182

than for zero or 180° cranks, and the result will be something like Fig. 182. The magnitude of δ will be greatly reduced, and hence a much lighter fly-wheel will be needed. For three cylinders working at 120° the turning effort is still more uniform.

Having the curve of total tangential forces, we can construct an approximate curve of tangential velocity of the fly-wheel by a process of mechanical integration. From the preceding we have approximately

$$v_1 = v\left(1 + \frac{\delta}{2}\right),$$

$$v_2 = v\left(1 - \frac{\delta}{2}\right).$$

Hence knowing v and δ, the extreme speeds v_1 and v_2 can be calculated, and laid off at B and E (Fig. 183). Now draw an ordinate at X at a small distance beyond B, writing the equation of energy between B and X,

$$W\frac{v_2^2}{2g} + (\text{Area } xx'b) = W\frac{v_x^2}{2g}.$$

From which

$$v_x = \sqrt{v_2^2 + \frac{2g(\text{Area } xx'b)}{W}}.$$

Similarly at point Y,

$$v_y = \sqrt{v_x^2 + \frac{2g(\text{Area } xx'yy')}{W}},^*$$

and so on. Thus we can obtain the velocity at each point along the crank orbit, and a curve of velocities can be drawn in as shown along the mean velocity as an axis of X.

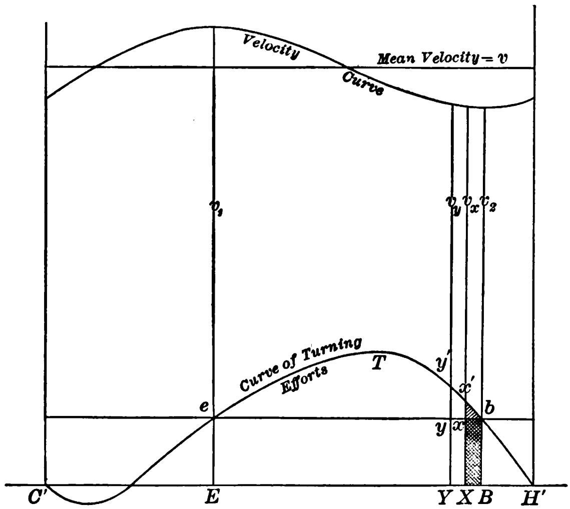


FIG. 183

Having obtained the velocity curve, the curve of space variation can be laid in by similar means, as shown in Fig. 184. Divide up the velocity curve by means of a number of ordinates spaced at equal small distances along $H''C''$, the semicircumference of the fly-wheel rim. Take one of these ordinates at A, the point of minimum velocity, v_2 (or at the point of maximum velocity, v_1). If we consider another equal wheel rotating about the same axis,

* The areas are, of course, expressed in foot-pounds.

and at the mean velocity v, we can study the space variation of the actual wheel with reference to axes fixed in this imaginary one.* Now consider a point in each of these wheels which when at A are coincident. Call the small distance AB, Δs. The time required for the mean velocity wheel to cover this distance will be

$$\Delta t = \frac{\Delta s}{v},$$

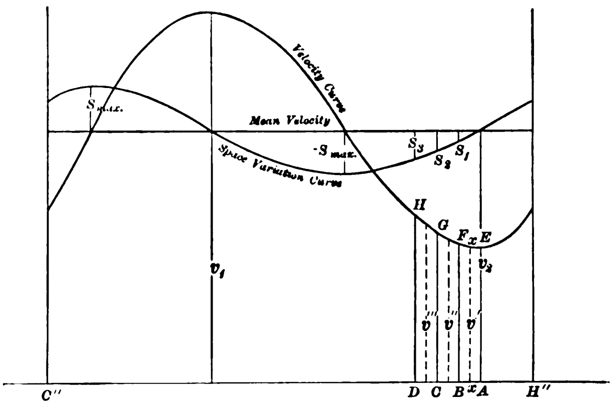


FIG. 184

v being constant. But during this same interval of time the actual wheel has covered a distance equal to

$$\Delta s' = v' \cdot \Delta t,$$

or

$$\Delta t = \frac{\Delta s'}{v'},$$

v' being the average velocity of the wheel throughout the interval Δs, which will be very nearly xx, the ordinate of the velocity curve midway between A and B. Hence,

$$\Delta s' = \frac{v'}{v} \Delta s,$$

* Strictly, we should obtain a new and exact value of v by getting the average ordinate of the velocity curve.

and the distance that the actual wheel is behind the mean velocity wheel at B will be

$$s_1 = \Delta s' - \Delta s = \Delta s\left(\frac{v'}{v} - 1\right).$$

Choosing now two points which are coincident at B, we can find in the same way the distance that the actual wheel falls behind the mean velocity wheel during the next interval Δs or BC. It will be

$$\Delta s'' - \Delta s = \Delta s\left(\frac{v''}{v} - 1\right),$$

and the total space displacement between A and C will be the sum of these, or

$$s_2 = \Delta s\left(\frac{v' + v''}{v} - 2\right),$$

and so on. These ordinates, s_1, s_2, s_3, etc., laid off with regard to their proper sign, constitute the space variation curve. It will be noticed that the actual wheel will be falling behind the mean velocity wheel whenever the velocity curve is below the mean axis, but will be gaining on it when above. Hence a check on the work consists in a closure of the space curve at A again.

Each step in the above deduction consists of a graphical method of forming a curve which is the integral of a previous periodic curve. The curve of turning efforts is a periodic acceleration curve, and from it a periodic velocity curve is obtained by a process of graphical integration; and from this a space curve is obtained by a similar means. It will be noticed also that the first or acceleration curve is quite irregular in outline, or departs in a marked way from a simple sine curve. The velocity curve is more nearly a pure sine curve, and the space curve still more nearly a simple harmonic. Furthermore it can be shown mathematically that curves formed by successive integrations from a complex harmonic, expressed as a Fourier's Series, approach indefinitely toward the fundamental harmonic as a limit. Advantage can be taken of this fact to form an approximate analytic expression for the space variation curve, and thus save the great amount of labor necessary in the graphical method. To do this consider the velocity curve a simple harmonic function of the time. Denote the relative velocity of the actual and mean velocity wheels by w, where

$$w' = v' - v.$$

Then by the preceding approximation,

$$w = w_{max} \sin Kt,$$

K being a constant, and w_{max} is a known quantity, being equal to $v\frac{\delta}{2}$. The time required for a point on the wheel to cover one complete period of the curve is $\frac{60}{2N}$ seconds. Hence our equation is

$$w = v\frac{\delta}{2} \sin \frac{\pi N t}{15} = \frac{ds}{dt},$$

$$s = v\frac{\delta}{2}\int \sin \frac{\pi N t}{15}\, dt,$$

and the equation of the space curve is

$$s = -v\frac{\delta}{2}\frac{15}{\pi N} \cos \frac{\pi N t}{15} + [C = 0.$$

The maximum values of the space variation will be

$$s_{max} = \pm v\frac{\delta}{2}\frac{15}{\pi N},$$

N in every case being the number of revolutions per minute.

If expressed in terms of angular measure, $s = \alpha R$, and $v = R\frac{2\pi N}{60}$, where R is the radius of the fly-wheel. Hence,

$$\alpha = -\frac{\delta}{4} \cos \frac{\pi N t}{15},$$

and

$$\alpha_{max} = \pm \frac{\delta}{4}.$$

If the angle is measured in degrees,

$$\alpha^{\circ}_{max} = 14^{\circ}.324\,\delta.$$

The labor of constructing the exact curve of turning efforts may be greatly abridged by making an approximation to the value of the inertia effect. In fact, the exact value is useless in connection with fly-wheel work, as the irregularities and wide variations in the steam effect are so great as to utterly mask any small error due to this approximation. We may consider, then, that a portion of the connecting rod equivalent to one half of its total mass is concentrated at the cross-head, and merely compute the tangential force due to $(M'' + \frac{1}{2}M')\,p_x$. Furthermore, no great error will be committed if we consider p_x as a simple harmonic acceleration, in which case

$$p_x = -l\omega^2 \cos\theta,$$

and since then $$(T) = (M'' + \tfrac{1}{2}M')\,p_x \sin\theta,$$

we will have $$(T) = -\left(\frac{M'' + \frac{1}{2}M'}{2}\right) l\omega^2 \sin 2\theta.$$

The use of these formulæ can best be illustrated by applying them to a specific example. Consider the curve of turning moments of the small 6 × 8 horizontal engine before mentioned. In Plate X are shown the total steam and back pressure lines of one forward stroke of the engine, deduced from the actual indicator card. Resolving the total horizontal force by the graphical method into its components at the crank pin, we obtain the curve of turning moments due to steam. Adding to this the curve of tangential accelerative forces already obtained in Plate VII, or as calculated more approximately from the preceding paragraph, we get the curve of net turning efforts (Plate XI). These are drawn on a scale of 1″ = 1000 lbs. The mean turning effort is $T_m = 692$ lbs., from which $\Delta E = 293$ ft. lbs. If the weight of the fly-wheel rim be 195 lbs., and its diameter be 30 in., we have, at 300 revolutions per minute,

$$v = 39.270 \text{ ft. per sec.},$$

$$\delta = \frac{\Delta E \times 32.2}{195 \times (39.27)^2} = .03191,$$

or 3.2%. We also have

$$v_1 = v\left(1 + \frac{\delta}{2}\right) = 39.896,$$

and $$v_2 = v\left(1 - \frac{\delta}{2}\right) = 38.646 \text{ ft. per sec.}$$

Now forming the equation of energy at each 10° of the fly-wheel semicircumference, we get the velocity at each of these points, and the final result is shown in Plate XII.

Again measuring the velocity at the middle of each 10° interval, and calculating the space variation, we get the space curve, also shown on Plate XII. It will be noticed that the mean of the two maximum values of s is

$$s_{max} = .01004 \text{ ft.},$$

and the maximum angular variation will be

$$\alpha^{\circ}_{max} = .4584^{\circ}$$

on each side of the mean.

Applying the approximate formula based on the supposition that the velocity is a simple harmonic function of the time, we have

$$s_{max} = v \frac{\delta}{2} \frac{15}{\pi N} = 39.27 \times .01595 \times \frac{1}{20\pi} = .009972 \text{ ft.},$$

and $$14.324\,\delta = .4576^{\circ}.$$

The difference is seen to be very small, in fact much less than experimental errors.

(*b*) *Counterbalancing*

In the problem of counterbalancing we generally attempt to find what mass placed at a fixed position opposite the crank will most nearly counteract certain components of the forces acting on the shaft in the direction of the crank throw. These forces will be due to two causes: (1) to the unbalanced mass of the crank itself, and (2) to normal components of the forces due to the inertia of the reciprocating parts.

We will first ascertain the counterbalance required for the crank alone. Since the only force exerted by the unbalanced crank is in the direction of the normal to the crank circle, and is constant in magnitude, its effect may be completely counterbalanced by a mass placed opposite the centre of the crank pin. This, of course, is not possible in all cases, but can be realized in the case of a centre crank engine by placing one-half of the counterbalancing weight on each crank arm. If M_0 is the mass of the unbalanced part of the crank, and s_0 is the distance of the centre of mass of this part from the centre of rotation; and if M_1 is the mass of the counterbalance, and s_1 is the distance of its centre of mass from the centre of rotation, then

$$M_0 s_0 \omega^2 = M_1 s_1 \omega^2, \quad M_1 = M_0 \frac{s_0}{s_1}, \text{ or } M_1 = M_0 \frac{s_0}{l}, \quad . \quad (18)$$

when s_1 is equal to the crank throw. This balance will evidently exist for all speeds.

On considering the effect of the reciprocating parts it will be seen from a simple inspection of the variable magnitude of the total normal component as exhibited in Curve No. 1 (Plate VII), that no fixed counterbalance can even approximately reduce the effect to zero. However, it may be of importance to counterbalance, if possible, certain components of the forces acting at the crank pin. For example, let us see how nearly we can counterbalance the total vertical component at the crank pin due to inertia, viz. $y = (Q_y)$, by means of a mass placed opposite to and rotating with the crank.* If M_2 is the mass of this counterbalance, and s_2 the distance of its centre of mass from the centre of rotation, its total normal force will be $M_2 s_2 \omega^2$, and the vertical component of this will be $-M_2 s_2 \omega^2 \sin\theta$, which is directly opposed to the vertical component of the inertia effect. This can be represented by a simple harmonic curve of semi-period equal to π, and of amplitude $M_2 s_2 \omega^2$. It will of course reach its maximum value at $\theta = 90°$. As opposed to this is the complex harmonic curve No. 2 of Plate VI. Evidently these cannot completely neutralize one another, but we may find a value of $M_2 s_2$ which will reduce the unbalanced component remaining to a minimum value. The

* It is readily seen that the vertical component due to steam is, in general, continually in one direction, and therefore cannot be balanced; while the horizontal component due to steam balances itself against the cylinder head.

equation of our complex harmonic curve is (from equations Nos. 11, 14, and 16)

$$y=(Q_v')+(Q_v'')$$

$$=\frac{I_0}{L\cos\alpha}\frac{d^2\alpha}{dt^2}+M'\frac{r}{L}\frac{d^2y}{dt^2}+M'\frac{L-r}{L}\frac{d^2x}{dt^2}\tan\alpha+M''\frac{d^2x}{dt^2}\tan\alpha. \quad (19)$$

Substitution gives

$$y=\frac{-I_0 n\omega^2(1-n^2)\sin\theta}{L(1-n^2\sin^2\theta)^2}-M'\frac{r^2}{L}n\omega^2\sin\theta$$

$$+\frac{L-r}{L}M'\left\{-l\omega^2\cos\theta\right.$$

$$\left.-\frac{n\omega^2(l-nr)(\cos 2\theta+n^2\sin^4\theta)}{(1-n^2\sin^2\theta)^{\frac{3}{2}}}\frac{(-n\sin\theta)}{\sqrt{1-n^2\sin^2\theta}}\right\}$$

$$+M''\left\{-l\omega^2\cos\theta\right.$$

$$\left.-\frac{ln\omega^2(\cos 2\theta+n^2\sin^4\theta)}{(1-n^2\sin^2\theta)^{\frac{3}{2}}}\frac{(-n\sin\theta)}{\sqrt{1-n^2\sin^2\theta}}\right\}. \quad . \quad (20)$$

It is evidently impossible to counterbalance, even approximately, any terms in the above equation except those which have a semi-period equal to π. Let us examine, then, each term separately, rejecting those which have a semi-period less than π. The first term is

$$y_1=\frac{K_1\sin\theta}{(1-n^2\sin^2\theta)^2},$$

K_1 being a constant. In this term the denominator is practically unity for all values of θ, varying between unity and .9724 when n is equal to $\frac{1}{6}$. Its effect, therefore, will be insignificant in modifying the numerator whose semi-period is π. The second term is

$$y_2=K_2\sin\theta,$$

and this has a semi-period π. The third term may be broken up into three, which can be written

$$y_3=K_3'\frac{\cos\theta\sin\theta}{\sqrt{1-n^2\sin^2\theta}}+K_3''\frac{\cos 2\theta\sin\theta}{(1-n^2\sin^2\theta)^2}+K_3'''\frac{\sin^5\theta}{(1-n^2\sin^2\theta)^2}.$$

Of these, the term whose coefficient is K_3' has a semi-period of $\frac{\pi}{2}$, since the denominator is practically unity; that whose coefficient is K_3'' has a semi-period $\frac{\pi}{3}$; while the last has a semi-period π. But the value of the coefficient K_3''' is so small in any practical case that we are justified in rejecting all three of the terms. The same applies to the fourth term of equation No. 20. Hence the only terms which we can expect to counterbalance are

$$-\frac{n\omega^2 \sin\theta}{L}\left\{\frac{I_0(1-n^2)}{(1-n^2\sin^2\theta)^2}+M'r^2\right\}, \quad . \quad . \quad (21)$$

and these must approximate to $M_2 s_2 \omega^2 \sin\theta$. The best value, then, for this counterbalance would be that which gave at its maximum ordinate (viz. at $\theta = 90^\circ$) a result equal and opposite to that obtained by putting $\theta = 90^\circ$ in equation No. 21. Hence,

$$M_2 s_2 \omega^2 = \frac{n\omega^2}{L}\left\{\frac{I_0}{1-n^2}+M'r^2\right\}. \quad . \quad . \quad . \quad (22)$$

Or the mass of the counterbalance will be

$$M_2 = \frac{n}{s_2 L}\left\{\frac{I_0}{1-n^2}+M'r^2\right\}.^*$$

In our case M_2 is found to be .3438, and its weight is 11.07 lbs., when s_2 is taken equal to l.

The results of using this counterbalance are seen in Plate XIII. The difference between the vertical components of the reciprocating parts and of the counterbalance is shown in Curve No. 1. This sums up all the more important higher harmonics rejected in the above method. It has in general a semi-period $\frac{\pi}{2}$, but by no change in the mass of the counterbalance could its effect be appreciably diminished. This counterbalance will also have an effect

---

* Still more approximately, when we neglect $1 - n^2$,

$$M_2 = \frac{n}{s_2 L}\{I_0 + M'r^2\}.$$

in diminishing, to a small extent, the horizontal component as shown.

If we wish to counterbalance the horizontal component, a similar method must be pursued. Here

$$y=(Q_x)=(Q_x')+(Q_x'')=M'\frac{d^2\bar{x}}{dt^2}+M''\frac{d^2x}{dt^2}, \quad . \quad . \quad (23)$$

$$y=M'\left\{-l\omega^2\cos\theta-\frac{n\omega^2(l-nr)(\cos 2\theta+n^2\sin^4\theta)}{(1-n^2\sin^2\theta)^{\frac{3}{2}}}\right\}$$

$$+M''\left\{-l\omega^2\cos\theta-\frac{ln\omega^2(\cos 2\theta+n^2\sin^4\theta)}{(1-n^2\sin^2\theta)^{\frac{3}{2}}}\right\}. \quad . \quad (24)$$

In this equation the term

$$y'_4=K_4\frac{\cos 2\theta}{(1-n^2\sin^2\theta)^{\frac{3}{2}}}$$

has a semi-period $\frac{\pi}{2}$, as has the term

$$y_5=K_5\frac{\sin^4\theta}{(1-n^2\sin^2\theta)^{\frac{3}{2}}},$$

also. Hence these must be rejected, and the counterbalance, whose horizontal component is

$$M_3s_3\omega^2\cos\theta,$$

must approximate to

$$(M'+M'')\,l\omega^2\cos\theta.$$

These agree exactly for all values of θ and for all speeds when

$$M_3=(M'+M'')\frac{l}{s_3}.$$

In our case M_3 is found to be 1.7888, and its weight is 57.6 lbs. when s_3 is taken equal to the crank throw l. The effect of this counterbalance is to almost entirely neutralize the horizontal component as shown in Curve No. 1 (Plate XIV) but it very much overbalances the vertical component Curve No. 2.

The problem of counterbalancing, then, is one which cannot be given a general solution, but each individual case must be worked out to best suit the existing conditions. The case of the locomotive may be cited as one where the balancing of the vertical component is important. In certain paddle-wheel ferry-boats with single horizontal engines, the unbalanced horizontal component is so great as to cause an oscillation of the whole boat backward and forward referred to its mean velocity. In some cases the counterbalance might be taken to average up the total normal component, which gives in our case a weight of 33.95 lbs. at a distance equal to the crank throw from the centre. This is very nearly the mean of the other two.

In conclusion will be given some results for a large passenger locomotive. The engine examined is one in regular service on the road of the Southern Pacific Co., and the following dimensions were obtained through the courtesy of the railroad officials at the West Oakland shops:

Southern Pacific Locomotive No. 1436; Type, *CW*.

Total weight of engine	131,400 lbs.
Weight of drivers	85,850 lbs.
Diameter of cylinders	20 in.
Stroke	24 in.
W' = weight of connecting rod	549 lbs.
W'' = weight of piston and cross-head	507 lbs.
W''' = weight of side rod	275 lbs.
l = throw of crank	1 ft.
L = length of connecting rod	8.1194 ft.
$n = \frac{l}{L}$	.12316
D = diameter of drivers	6.0000 ft.
N = revolutions per minute, at 60 miles an hour	280.1
ω = radians per second, at 60 miles an hour	29.333
r = distance of centre of mass of rod from centre of wrist pin	5.0903 ft.
I_0 = moment of inertia of rod about G	171.81

The last two dimensions were obtained by swinging the rod as a pendulum, and it was observed to make 40.50 vibrations per minute when suspended from the wrist-pin end, and 43.0 vibrations when reversed.

In the discussion of this case a new member appears; namely, the side rod. The effect of this is easily seen, however. Since its centre of mass, which as nearly as could be determined was at the middle point of a line connecting the centre of its bearings, travels uniformly in a circle relatively to the locomotive, the force necessary to maintain this path will be directed toward the centre of this circle and will be equal to $-M'''l\omega^2$, where M''' is the mass of the side rod. The horizontal component of this force will be $Q_x''' = -Ml\omega^2 \cos\theta$, and the vertical component will be $Q_y''' = -Ml\omega^2 \sin\theta$. The forces active on the two crank pins will each be equal to one-half of this, and will be opposite in sign. Hence,

$$(Q_x''') = +\frac{M'''l\omega^2}{2}\cos\theta,$$

$$(Q_y''') = +\frac{M'''l\omega^2}{2}\sin\theta,$$

$$(T''') = 0,$$

$$(N''') = \frac{Ml\omega^2}{2}.$$

In Table No. 4 are given values of total horizontal force at the crank pin of the forward driver, or

$$(Q_x) = (Q_x') + (Q_x'') + (Q_x'''),$$

and of the total vertical force,

$$(Q_y) = (Q_y') + (Q_y'') + (Q_y''').$$

The curves of Plate XV show graphically the same thing. They are of form similar to those obtained for the 6 × 8 engine, but the irregularities are even less, this being due to the smaller value of n. They are interesting as showing the enormous magnitude of these forces in a high-speed locomotive. They all increase,

TABLE No. 4

θ	(Q_x)	Horizontal Component of Counter	Unbalanc'd Horizontal Component	(Q_y)	Vertical Component of Counter	Unbalanc'd Vertical Component
0°	+ 34666	− 11717	+ 22949	0	0	0
10°	34047	11539	22508	+ 1574	− 2035	− 461
20°	32084	11010	21074	3176	4007	831
30°	29025	10147	18878	4749	5858	1109
40°	24954	8976	15978	6324	7531	1207
50°	20024	7531	12493	7838	8976	1138
60°	14572	5858	8714	9244	10147	903
70°	8770	4007	4763	10460	11010	550
80°	+ 2895	− 2035	+ 760	11408	11539	− 131
90°	− 2796	0	− 2796	12011	11717	+ 294
100°	8164	+ 2035	6126	12213	11539	674
110°	13045	4007	9038	11977	11010	967
120°	17273	5858	11415	11281	10147	1134
130°	20975	7531	13444	10158	8976	1182
140°	23934	8976	14958	8642	7531	1111
150°	26213	10147	16066	6785	5858	927
160°	27829	11010	16819	4664	4007	657
170°	28815	11539	17276	+ 2376	− 2035	+ 341
180°	− 29117	+ 11717	− 17400	0	0	0

of course, with the square of the speed. The counterbalance necessary for the reduction of the vertical component to a minimum can also be calculated, remembering that the effect of the side rod must be taken into account. Hence we have

$$M_2 = \frac{n}{s_2 L}\left\{\frac{I_0}{1-n^2} + M'r^2\right\} + \frac{M'''}{2\,s_2},$$

and if s_2 is taken equal to the crank throw, M_2 becomes 13.61, and its weight is 438.49 lbs. The result of using this counterbalance on the forward driver is seen in Plate XV, where the unbalanced vertical component is shown in Curve No. 1. This unbalanced force reaches a maximum of between 1100 and 1200 lbs. In addition to this weight must be added another sufficient to balance the crank pin, etc. As nearly as could be determined by measurement of volumes this weight would amount to 267 lbs., or the total counterbalance would be about 705 lbs., if the vertical component alone were to be balanced. The weight added to balance the crank would, of course, affect in no way the values of Curve No. 1 of Plate XIII. The counterbalance for the rear driver, when placed at a distance l from the centre of this driver, would have a mass equal to merely

$$M_3 = M_1 + \frac{M'''}{2},$$

or, as nearly as the weight of the crank pin, etc., of the rear driver could be determined, the weight of the rear counterbalance would be 202 lbs. This counterbalance would exactly neutralize all components on the rear driver.

3. GOVERNORS

The governor is intended to regulate variations in speed extending over longer intervals of time than can be kept within proper limits by the fly-wheel. It may act by opening and closing a throttle valve, thus varying the work by varying the initial pressure, the cut-off remaining constant. Or it may vary the cut-off, the initial pressure remaining constant.

A. The Fly-ball Governor

The oldest form of governor is the fly-ball, invented by Huygens, and used first on the steam-engine by James Watt. The theory of the arrangement is simple. The ball at B (Fig. 185), at the extremity of the arm CB, revolves about the axis CD, and is hinged at C. When equilibrium is attained, the resultant force acting on B must be in the direction CB. Let this force be P. Then

$$\tan \alpha = \frac{g}{R\omega^2} = \frac{h}{R},$$

$$\omega = \sqrt{\frac{g}{h}}.$$

We see that ω varies as $\sqrt{\frac{\text{const}}{h}}$, but h is not a constant, hence ω cannot be constant. The fly-ball governor is therefore static or stable. That is to say, if it is rotated at a constant angular velocity, it will take up a certain definite position, and if disturbed from that position, will return to it when the disturbing force is removed. It is evident, therefore, that we must use some other curve than the circle if the governor is to be astatic or neutral. Let DBE (Fig. 186) be the required curve. If the ball is to be in equilibrium, we must have the resultant force in the direction of the normal to the curve.

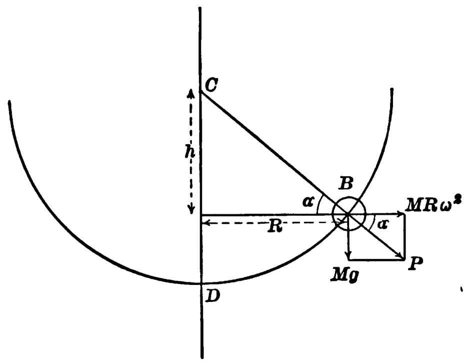


FIG. 185

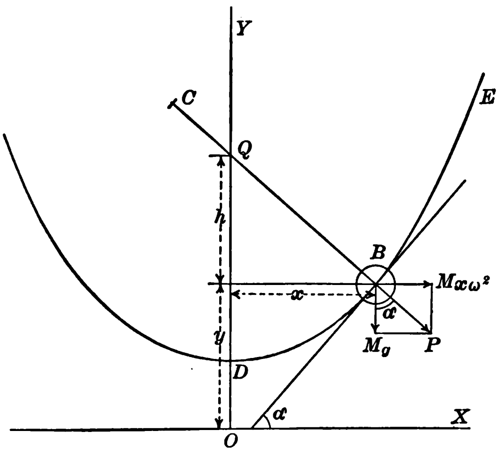


FIG. 186

$$\tan \alpha = \frac{x\omega^2}{g} = \frac{dy}{dx},$$

$$x^2 = \frac{2g}{\omega^2} y.$$

This is the equation of a parabola. We also have

$$\tan \alpha = \frac{g}{h}$$

or $\omega = \sqrt{\frac{g}{h}}.$

Hence, if ω is to be constant, h must be constant, which is true in the case of the parabola. The parabolic governor is approximated in practice by hinging the arm at the centre of curvature C of the working arc of the parabola. This always lies on the opposite side of the axis, and therefore the arms must cross. For very fine work such governors are much used, as in the driving clocks of telescopes.

A governor always has a certain amount of work to do in opening the throttle or in moving the valve gear. Suppose the governor is running at a certain speed ω, at which it is in equilibrium. Now let the speed increase to a value ω_1 such that $\omega_1 = K\omega$. The unbalanced radial force exerted by the weight will be

$$MR(\omega_1^2 - \omega^2) = MR\omega^2(K^2 - 1) = P'.$$

For a given ratio of increase in speed it is seen that the force exerted varies as the square of the speed. Hence the faster the governor is run the more sensitive it becomes, as a smaller percentage of variation of speed will cause it to overcome the frictional drag of the moving parts. The increased effect of centrifugal force due to this higher speed can be counteracted by a weight whose centre of mass lies in the axis, or by a spring.

B. Shaft Governors

In many types of slow-speed and in some types of high-speed engines the fly-ball governor has been made to actuate an automatic cut-off. But in most high-speed engines the heavy load thrown upon the governor is an objection to its use. A far more powerful governor is needed, and this is usually located in the main axis of rotation of the engine. The shaft governor is arranged in such a way that gravity has no effect upon it, and we substitute some other returning force which varies according to the same law as that due to the normal acceleration of the rotating parts. This can be realized by the use of the helical spring. Before discussing the arrangement of such governors we must look into the theory of the helical spring.

Law of the Helical Spring. — Let RQS (Fig. 187) represent one-half of a turn of the spring, the sections at R and at S being made by a plane passing through the axis of the helix. Let AB be a fixed diameter of the section R parallel to the axis, and CD a similar diameter in the section S. Now suppose also that the line AB is fixed in space. Let a force P be applied at the centre S as shown, the line CD being free to move. The whole will now take up the position RS', and CD will take up the position $C'D'$. There will be bending and also a certain amount of torsion along RS'. Suppose now that CD, instead of being free to move, is constrained to remain parallel to its original position. This must be the case in the helical spring, for if it were not, CD would twist more with respect to one section than with respect to the next equal section. This can be realized by applying P at the centre or axis of the helix as in Fig. 188. There will now be an additional bending moment along $S'R$ equal and opposite to the first one, and also an equal amount of torsion in the same direction. Hence the bending moments tend to neutralize one another, while the torsions add their effects. But the same is true of every section of the spring; so if the spring be long in proportion to its diameter, there will be no bending at all, but only uniform torsion along the wire. Hence we treat the helical spring as a problem in pure torsion. Consider a bar subject to torsion by a

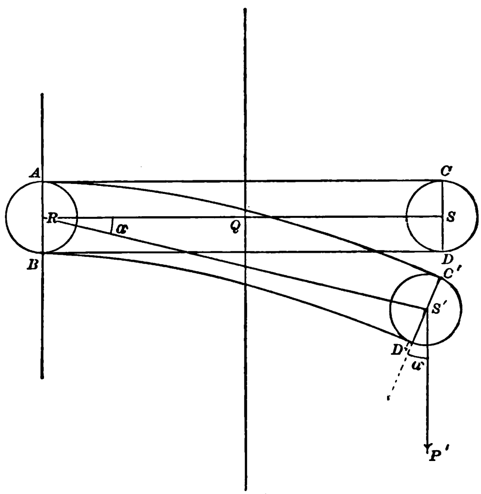


FIG. 187

force P being applied at the extremity of a lever arm a. If r is the radius of the bar, l its length, and θ the angle through which the moment Pa twists it, we have from Strength of Materials,

$$Pa = E\frac{\theta}{l}\frac{\pi r^4}{2},$$

where E is the coefficient of elasticity of the material. Now in the helical spring let r_1 equal the radius of the helix, r the radius of

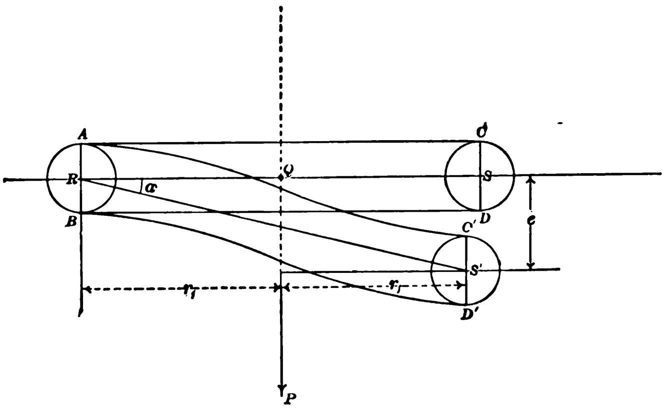


FIG. 188

the wire, and n the number of turns of the helix. If e is the elongation of the spring for full P, then

$$a = r_1,$$

since the force acts through the axis of the helix. Also when the deflection is small as compared with r_1,

$$\theta = \frac{e}{r_1},$$

as then $\alpha = \frac{e}{2r_1}$, and $\theta = 2\alpha$. (See Fig. 188.) Finally,

$$l = 2\pi r_1 n.$$

Substituting these in the equation for torsion,

$$Pr_1 = E \frac{e\pi r^4}{r_1 \times 2\,\pi r_1 n \times 2}.$$

Hence,
$$e = \frac{Pr_1^3\, 4\,\pi n}{E\pi r^4} = P\frac{r_1^3}{r^4}\frac{4\,n}{E}.$$

Let $r_1 = Cr$, where C is the ratio of the diameter of the helix to that of the wire.

$$e = P\frac{C^3 r^3}{r^4}\frac{4\,n}{E},$$

$$P = \left\{\frac{Er}{4\,C^3 n}\right\} e = Ke.$$

So we see that the elongation of the spring varies directly with the pull. K is called the constant of the spring. If P is measured in pounds and e in inches, K is that number of pounds necessary to stretch the spring one inch.

Forces Active in the Shaft Governor. — Let O (Fig. 189) be the axis of rotation of the main shaft about which the wheel rotates at an angular velocity ω. Let the mass whose centre of mass is at G be pivoted to the wheel at Q. Let us consider the forces causing moments about Q.

1. *Centrifugal Force.* — This is the force due to the change in direction of the velocity u of the centre of mass. Its magnitude will be

$$P = MR\omega^2,$$

and its direction will be radial. Its moment about Q will be

$$Ph = MR\omega^2 h.$$

Hence if Q lies on OG, the moment will reduce to zero. Permanent change in angular velocity or in radius will permanently change the magnitude of this force.

2. *Tangential Accelerative Force.* — This is due to change in the magnitude of the velocity u of the centre of mass, resulting from rotation about Q. Its magnitude will be

$$P' = M\frac{du}{dt},$$

and its direction will be at right angles to OG. The moment about Q will be

$$P'y = My\frac{du}{dt}.$$

Hence if G lies anywhere on a circle drawn on OQ as a diameter, the moment will be zero, and if Q lies at O it will have no existence. So long as R is constant, it will have no existence, and permanent change in ω and R will not affect it.

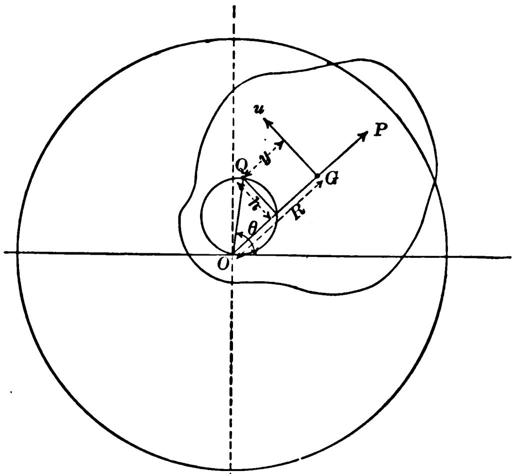


FIG. 189

3. *Angular Accelerative Moment.*—This moment is due to change in angular velocity ω. The moment will be

$$\text{Moment} = I_Q\frac{d^2\theta}{dt^2}.$$

Its effect will be felt no matter where Q and G may be. So long as ω remains constant it will have no existence, hence permanent change in ω and R will not affect it.

Of these three force moments, the first is the most important, as being the necessary result of a permanent change in speed.

We will consider its effect at some length, and calculate its variations with considerable care. In this case it will be best to begin with the simplest possible arrangement, and proceed to more complex conditions as the theory is developed.

Centrifugal Force and Moment. — The simplest or ideal case of the shaft governor is shown in Fig. 190. *O* is the centre of rotation, and the weight *G* moves with its centre of mass on a radius. A spring is attached to the centre of mass *G*, and to a

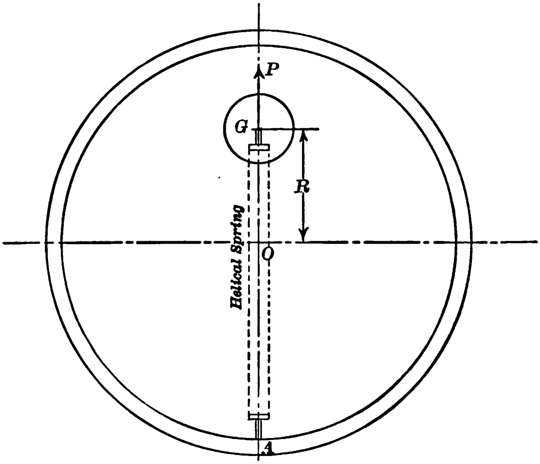


FIG. 190

diametrically opposite point on the rim of the wheel. When the whole is rotated about *O*, the centrifugal force of the weight and the pull of the spring are directly opposed. The outward force is

$$P_1 = \frac{W}{g} R\omega^2,$$

and the inward force is

$$P_2 = Ke.$$

If these are to be in equilibrium,

$$\frac{W}{g} R\omega^2 = Ke,$$

and if this equilibrium is to be maintained for every value of R, while ω remains constant, then, since $\frac{W}{g}\omega^2 = \text{const}$, and $K = \text{const}$, e must be equal to R, since on no other condition could the equation be universally satisfied. Then if $R = 0$, $e = 0$, or if the centre of mass G were to move in to O, the tension on the spring and the centrifugal force would vanish simultaneously. Such a governor would be perfectly astatic, and if the whole were balanced with an equal and similarly situated weight on the opposite side of O, and connected to the first by some form of kinematic chain, we would have the ideal case satisfied. It is readily seen that in this case

$$K = \frac{\omega^2 W}{g},$$

from which r can be calculated by

$$r = \frac{4\ C^3 n K}{E},$$

when E is known.

In a governor so proportioned, the spring is said to have its "full theoretic tension." Practically we can never have this exact equilibrium of forces, as the inertia of the heavy masses causes a racing or "hunting" action. We must therefore give some stability to the governor by employing less than this full tension. Let us call the forces acting outward from the centre of rotation positive $(+)$, while those which act inward toward the centre are negative $(-)$. If the weight could move inward till its centre of mass coincided with O, the elongation which the spring would still retain we shall call e_0. If there is tension at this point, e_0 will be negative, and the force due to e_0 will be negative. If the tension vanishes before we reach this point, and there is compression at the centre, then are e_0 and the force due to it positive. For any position of the weight,

$$e = R - e_0.$$

Now the outward force acting on the weight is

$$P_1 = +MR\omega^2,$$

and if we plot a curve between P_1 as ordinate, and R as abscissa, we will have a straight line through the origin, inclined at an angle whose tangent is $M\omega^2$ with the axis of R. The inward force acting on the weight is

$$P_2 = -K(R - e_0).$$

This is also the equation of a straight line, which cuts the P-axis at a distance Ke_0 from the origin. Suppose that when $R = R_1$, $P_1 = -P_2$, and that $e_0 = (+)$. Then our lines will be as shown in Fig. 191, where $AB = AC$. Call $P = P_1 + P_2$, then P is any unbalanced force acting on the weight. The line EAF will show by

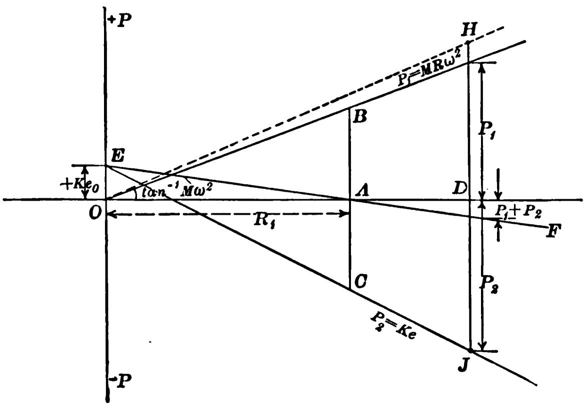


FIG. 191

its ordinates the magnitude and direction of P. At A, $P = 0$, but any positive change or increase in R will cause a negative value of P. Likewise a decrease in R will cause a positive value of P. In other words, any change in R will cause a value of P which will *oppose* the change, or the governor is *stable*. In all engines an increase in R must cause a shutting off of steam. Hence, if by throwing off load R increases to OD, then in order to maintain equilibrium DJ must equal DH, or the speed must increase to a value ω_1, where

$$\text{angle } HOD = \tan M\omega_1^2.$$

When e_0 is positive, therefore, the engine will run faster on light load than on heavy load, but the governor will be stable in every position. The amount of change in velocity between no load and full load can be made as small as we please by decreasing e_0, but by doing so the stability of the governor is sacrificed. If $e_0 = 0$, the engine would run at the same speed at all loads, but the governor would be in neutral equilibrium, and of no practical use.

If, however, e_0 is negative, $e_0 = (-)$, then our lines will be as in Fig. 192. Here E_1AF_1 is the line showing the magnitude and direction of P. Any increase in R in this case gives a positive

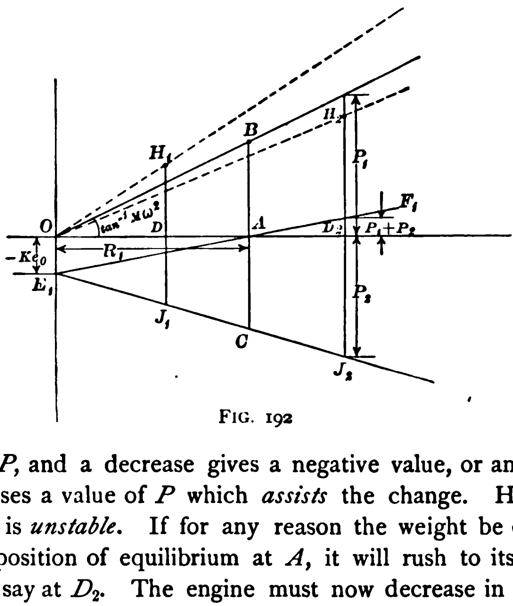


FIG. 192

value of P, and a decrease gives a negative value, or any change in R causes a value of P which *assists* the change. Hence the governor is *unstable*. If for any reason the weight be disturbed from its position of equilibrium at A, it will rush to its extreme position, say at D_2. The engine must now decrease in speed till D_2H_2 is equal to D_2J_2, or till

$$\text{angle } H_2OD_2 = \tan^{-1} M\omega_2^2.$$

The slightest disturbance at this point will cause the weight to rush to its extreme inner position at D_1, and now the speed must increase till D_1H_1 is equal to D_1J_1, or till

$$\text{angle } H_1OD_1 = \tan^{-1} M\omega_1^2,$$

The engine will fluctuate violently in speed therefore between the limits.

$$\omega_2=\sqrt{\frac{K}{M}\left(1-\frac{e_0}{R_2}\right)} \text{ (minimum)},$$

and

$$\omega_1=\sqrt{\frac{K}{M}\left(1-\frac{e_0}{R_1}\right)} \text{ (maximum)},$$

where R_1 and R_2 are the limits of the motion of the weight.

The value of e_0 must, therefore, always be positive. That is to say, the tension on the spring must vanish before G reaches O, or the engine must run faster on light than on heavy load.*

In the governor of Fig. 190 suppose we assume a reasonable range of speed fluctuation, say ω_1 at no load, and ω_2 at full load. From the valve gear design we know the values of R corresponding to these loads, or the extreme limits of the weights' motion. Let these be R_1 and R_2. Then we have

$$\frac{W}{g}R_1\omega_1^2=Ke_1 \quad . \quad . \quad . \quad . \quad . \quad . \quad (I)$$

$$\frac{W}{g}R_2\omega_2^2=Ke_2 \quad . \quad . \quad . \quad . \quad . \quad . \quad (II)$$

Subtraction gives

$$\frac{W}{g}(R_1\omega_1^2-R_2\omega_2^2)=K(e_1-e_2).$$

But

$$e_1-e_2=R_1-R_2.$$

Hence,

$$\frac{W}{g}(R_1\omega_1^2-R_2\omega_2^2)=K(R_1-R_2) \quad . \quad . \quad . \quad (III)$$

and

$$K=\frac{W}{g}\left\{\frac{R_1\omega_1^2-R_2\omega_2^2}{R_1-R_2}\right\} \quad . \quad . \quad (IV)$$

This gives us the constant of the spring.

Equation III gives us the relation between the various quantities for two definite positions R_1 and R_2. Now if we substitute R for

* Professor John Barr has used a diagram similar to the above. See Sibley, *Journal of Engineering*, 1896.

R_2, and ω for ω_2, where R and ω are any position and angular velocity, we get the general relation between R and ω,—

$$\frac{W}{g}(R_1\omega_1^2 - R\omega^2) = K(R_1 - R),$$

$$\omega^2 = \frac{Kg}{W} - \frac{R_1\left(\frac{Kg}{W} - \omega_1^2\right)}{R} \quad . \quad . \quad . \text{ (V)}$$

We have seen that if the governor is to be stable, e_0 must be positive, which means that the engine must run faster on light load than on full load, or the angular velocity must increase with an increase in R. Hence if we plot a curve which shows the rela-

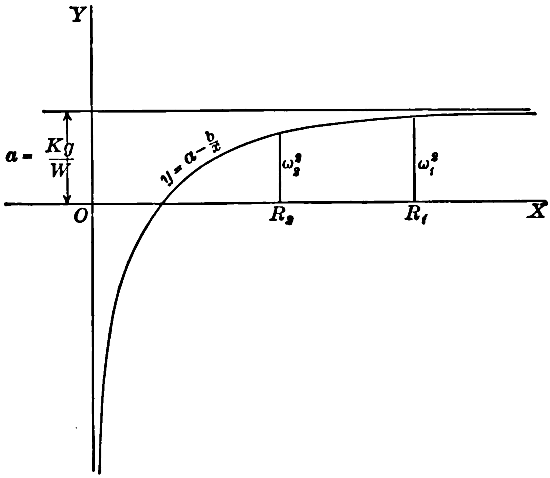


FIG. 193

tion between ω (or ω^2) and R, this curve must constantly ascend as we pass to the right of the origin. So long as this is the case, the governor is stable, but if in any part the curve falls off as we pass to the right, then will the governor race over that portion. Equation V gives us the relation between ω and R for the ideal case of a shaft governor. Taking ω^2 as the dependent variable, this equation is of the form

$$y = a - \frac{b}{x}.$$

It is the equation of a rectangular hyperbola, having $x = 0$ and $y = a$ for its asymptotes (Fig. 193). Hence the curve will always ascend as we pass to the right, provided $\frac{Kg}{W} > \omega_1^2$, which is true when $e = (+)$. Care should be taken not to use the curve too far to the right, or too near its asymptote $y = a$, otherwise the governor will not have sufficient stability.

It is of interest to note that if $e_0 = 0$, the hyperbola becomes a pair of straight lines, and $\omega^2 = a = \frac{Kg}{W}$, which is constant. If e_0 is negative, we have the hyperbola conjugate to $y = a - \frac{b}{x}$, and this will continually fall off as we pass to the right. It is impossible to use the negative branch of the hyperbola, as the angular velocities become imaginary.

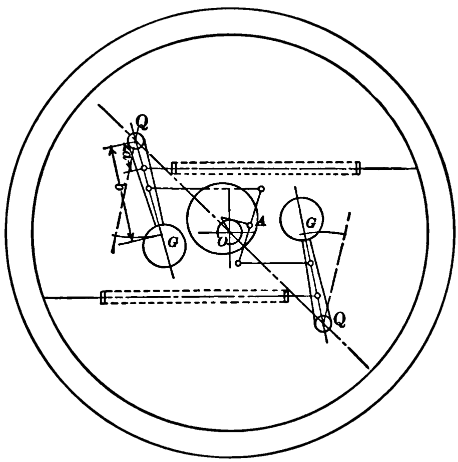


FIG. 194

The case shown in Fig. 190 is not adapted for a practical design. It is difficult to have the spring cross the shaft, to say nothing of the interference of two springs and weights as soon as another is introduced. A more practical arrangement is shown in Fig. 194. If the weights GG do not move through too great a range, we may consider their paths as approximately radii, and work out our design exactly as in the preceding case. The result may be close enough in many instances. The moment about Q due to centrifugal force is

$$MR\omega^2 b,$$

and the moment due to the returning force of the spring is

$$Kea.$$

Hence, $$MR_1\omega_1^2 b = Ke_1 a, \quad \ldots \ldots \quad (I)$$

$$MR_2\omega_2^2 b = Ke_2 a. \quad \ldots \ldots \quad (II)$$

Subtraction gives

$$\frac{W}{g} b(R_1\omega_1^2 - R_2\omega_2^2) = Ka(e_1 - e_2).$$

But $$b(e_1 - e_2) = a(R_1 - R_2),$$

$$\frac{W}{g} b(R_1\omega_1^2 - R_2\omega_2^2) = Ka(e_1 - e_2)\frac{a}{b}, \quad \ldots \quad (III)$$

$$K = \frac{W}{g}\frac{b^2}{a^2}\frac{R_1\omega_1^2 - R_2\omega_2^2}{R_1 - R_2}, \quad \ldots \quad (IV)$$

where R_1 and R_2, and ω_1 and ω_2 are the extreme values of the radius and angular velocity.

If finally we substitute R for R_2, and ω for ω_2, these being any corresponding values of radius and angular velocity,

$$\frac{W}{g} b^2(R_1\omega_1^2 - R\omega^2) = Ka^2(R_1 - R),$$

$$\omega^2 = \frac{Kg}{W}\frac{a^2}{b^2} - \frac{R_1\left\{\frac{Kg}{W}\frac{a^2}{b^2} - \omega_1^2\right\}}{R}. \quad (IV)$$

And this is the equation of a rectangular hyperbola between ω^2 and R.

If close regulation is to be attained, the fact that the weight does not move on a radius, but on the arc of a circle must be taken account of. In Fig. 195 let the centre of mass of the bar and weight be at G. Let the spring be attached to the bar at E and to the rim of the fly-wheel at A. If the spring is very long, as it usually is, we can consider it as always pulling parallel to itself. Draw QF perpendicular to the direction of the pull of the spring, and drop a perpendicular OF on QF. Call $QG = b$, $QE \doteq a$, $QF = s$, and $OF = c$. Let QG make an angle δ with QF. Given values of c, s, and δ, completely fix the position of QG. Now draw $OG = R$ and $QH = h$ perpendicular to OG. The moment about Q due to the centrifugal force of the mass rotating about O will be

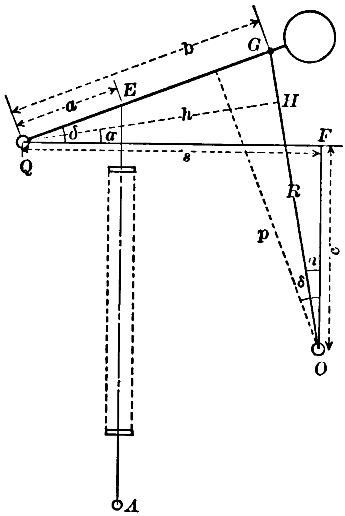


FIG. 195

$$MR\omega^2 h = \text{Moment.}$$

But $\dfrac{R}{p} = \dfrac{b}{h}$, where p is the perpendicular dropped from O upon QG.

Hence, $$M\omega^2 bp = \frac{W}{g}\,\omega^2 b(c\cos\delta + s\sin\delta) = \text{Moment.}$$

The moment about Q due to the returning force of the spring will be

$$Kea\cos\delta = \text{Moment.}$$

If these are to be in equilibrium,

$$\frac{W}{g}\,b\omega^2(c\cos\delta + s\sin\delta) = Kea\cos\delta.$$

$$\frac{W}{g}\,b\omega^2(c + s\tan\delta) = Kea.$$

When $\delta = \delta_1$, let $\omega = \omega_1$ and $e = e_1$. When $\delta = \delta_2$, let $\omega = \omega_2$ and $e = e_2$. The values of δ_1, δ_2, ω_1, and ω_2 are given by the range of action of the valve gear, and the allowable range of speeds.

$$\frac{W}{g} b\omega_1^2(c + s \tan \delta_1) = Ke_1a, \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(I)}$$

$$\frac{W}{g} b\omega_2^2(c + s \tan \delta_2) = Ke_2a. \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(II)}$$

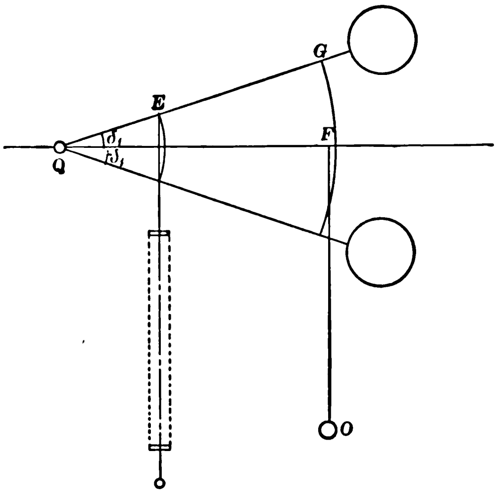


FIG. 196

Subtraction gives

$$\frac{Wb}{g}(\omega_1^2 c + \omega_1^2 s \tan \delta_1 - \omega_2^2 c - \omega_2^2 s \tan \delta_2) = Ka(e_1 - e_2).$$

But $$e_1 - e_2 = a \sin \delta_1 - a \sin \delta_2.$$

$$\frac{Wb}{g}\left\{ c(\omega_1^2 - \omega_2^2) + s(\omega_1^2 \tan \delta_1 - \omega_2^2 \tan \delta_2) \right\}$$

$$= Ka^2(\sin \delta_1 - \sin \delta_2). \quad . \quad . \quad . \quad \text{(III)}$$

$$K = \frac{Wb}{g}\left\{ \frac{c(\omega_1^2 - \omega_2^2) + s(\omega_1^2 \tan \delta_1 - \omega_2^2 \tan \delta_2)}{a^2(\sin \delta_1 - \sin \delta_2)} \right\}. \quad . \quad . \quad \text{(IV)}$$

If the governor is arranged as in Fig. 196, so that QF bisects the total angle through which the lever acts, then we have

$$\frac{Wb}{g}\left\{c(\omega_1^2-\omega_2^2)+s\tan\delta_1(\omega_1^2+\omega_2^2)\right\}=2\,Ka^2\sin\delta_1, \quad \text{(III')}$$

and
$$K=\frac{Wb}{g}\left\{\frac{c(\omega_1^2-\omega_2^2)+s\tan\delta\,(\omega_1^2+\omega_2^2)}{2\,a^2\sin\delta}\right\}. \quad \text{(IV')}$$

The general equation IV gives the constant of the spring which will give the assumed speeds at the limits. It remains to be seen whether the governor will work correctly at intermediate positions, which can be determined by plotting the curve between ω^2 and δ.

Equation III shows the general relation between the various quantities for two definite positions δ_1 and δ_2. Now let us write this equation so as to show the relation between the definite position δ_1 and any other position δ. For this latter let the speed be ω. Then

$$\frac{Wb}{g}\left\{c(\omega_1^2-\omega^2)+s(\omega_1^2\tan\delta_1-\omega^2\tan\delta)\right\}=Ka^2(\sin\delta_1-\sin\delta).$$

Solving this equation for ω^2, we get

$$\omega^2=\frac{\omega_1^2(c+s\tan\delta_1)-\dfrac{Ka^2g}{Wb}(\sin\delta_1-\sin\delta)}{c+s\tan\delta},$$

which may be written,

$$\omega^2=\frac{\left\{\omega_1^2(c+s\tan\delta_1)-\dfrac{Ka^2g}{Wb}\sin\delta_1\right\}+\dfrac{Ka^2g}{Wb}\sin\delta}{c+s\tan\delta}. \quad \text{(V)}$$

This equation is of the general form,

$$y^2=\frac{a+b\sin x}{c+s\tan x}.$$

The curve is shown in Fig. 197, where $a>b$ and $c>s$, and also in Fig. 198, where $a>b$ and $c<s$. We must always have a portion of the curve where it crosses the axis of Y which ascends

as we pass to the right. This can be known by putting $x = 0$ in the first derivative, and noticing whether the result is positive.

$$y' = \frac{a + b \sin x}{c + s \tan x},$$

$$\frac{dy'}{dx} = \frac{b \cos x(c + s \tan x) - s \sec^2 x(a + b \sin x)}{(c + s \tan x)^2},$$

$$\left.\frac{dy'}{dx}\right]_{x=0} = \frac{bc - sa}{c^2} = (+).$$

Now c^2 is always positive, hence the fraction will be positive when the numerator is positive, or when

$$bc > sa.$$

That the governor will be stable all the way from A to B (Fig. 197) can only be found by plotting the whole curve.

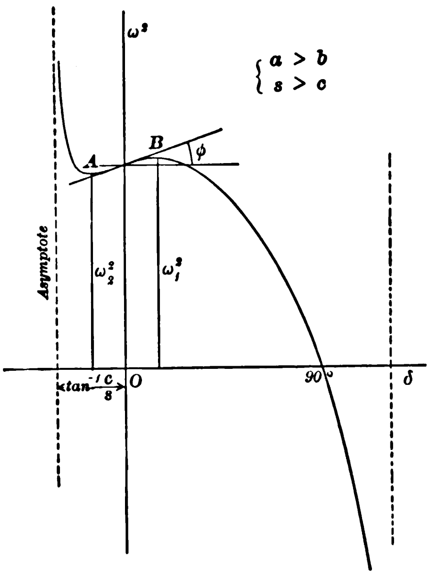


FIG. 197

The methods of permanently changing the speed of an engine may be seen by observing the equation of equilibrium for the astatic governor in one of the simpler cases. In Fig. 194 we have

$$\omega = \sqrt{\frac{Ke}{MR} \frac{a}{b}}.$$

Since e_0 is very small in most cases, it is very nearly true that

$$\frac{e}{R} = \frac{a}{b}.$$

In this case, then,

$$\omega = \frac{a}{b}\sqrt{\frac{K}{M}}.$$

Since K is unchangeable in any given spring, we may change the speed of the engine by:

1. Varying $\frac{a}{b}$, or by shifting the weight G, or the point of attachment of the spring, along the arm. Care must be taken, however, that by so doing we do not destroy the relation

$$\frac{e}{R} = \frac{a}{b},$$

so e must be changed to meet the new requirements.

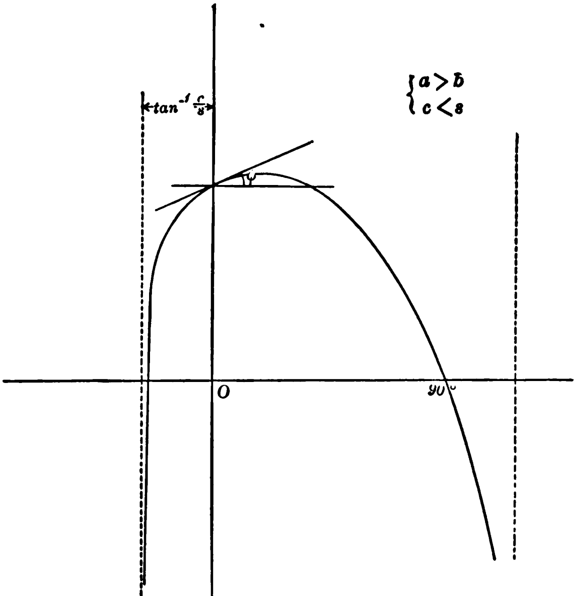


FIG. 198

2. By changing M the mass, since $\omega = \text{const}\,\frac{1}{\sqrt{M}}\cdot$ The calculation for the size of the spring wire, as determined by the formula

$$d = \frac{8\ C^3nK}{E},$$

can be considered as an approximation only, as the elasticity of steel wire varies greatly with its temper. A sample of the spring should be obtained and tested, and if this does not agree with the assumed value of K, we can vary the value of a till the proper result is obtained. If K is known to start with, we must first compute the value of a from

$$a^2 = \frac{Wb}{Kg}\left\{\frac{c(\omega_1^2 - \omega_2^2) + s(\omega_1^2 \tan \delta_1 - \omega_2^2 \tan \delta_2)}{(\sin \delta_1 - \sin \delta_2)}\right\}$$

and proceed as before.

The length L of the spring is determined by the dimensions of the governor and wheel, but we must be certain that it is great enough to prevent the wire of the spring being twisted beyond its elastic limit for the maximum value of e.

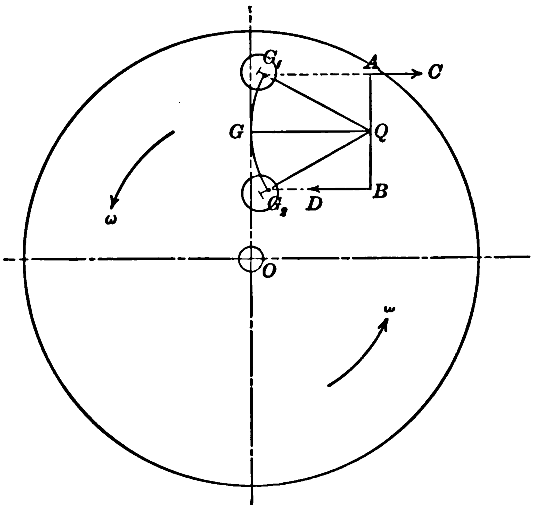


FIG. 199

Tangential Accelerative Force. — The variations of this force cannot, in general, be exactly calculated as in the case of the preceding, as the value of the coefficient $\frac{du}{dt}$ depends on too many and too complicated expressions to be put in any useful form. The best we can do is to investigate the direction of the resulting moment, and find out the relative position of the three points O, Q, and G in order that this force moment may not give instability to the governor.

In Fig. 199, let Q be the point of suspension of the arm. Draw OG tangent to the path of the centre of mass G, and GQ perpendicular to OG. If the weight shifts suddenly from G to G_1, the wheel meantime rotating counter-clockwise, its velocity will

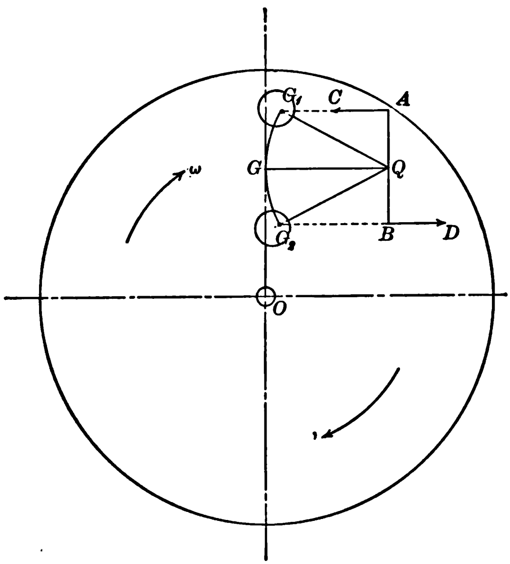


FIG. 200

have to be increased in the direction AG, and hence it will tend to hold back, giving rise to a force AC, acting through a lever arm AQ which tends to assist the change. In shifting from G to G_2, the moment $BD \times BQ$ tends to oppose the change. This effect tends to cause the governor to race from G to G_1, but gives it stability from G to G_2. In Fig. 200 the direction of rotation is reversed, and the tendency to race is now from G to G_2, while from G to G_1 greater stability is given to the governor by reason of this inertia effect. In some engines the whole action is from G to G_1 with the direction of rotation as in Fig. 200. In order to check this effect when undesirable, a dash pot may be introduced.

Angular Accelerative Moment. — The same general remarks apply here as in the preceding case, although in the ordinary form of governors already discussed the effect is small. In a certain special form of governor, however, this angular accelerative

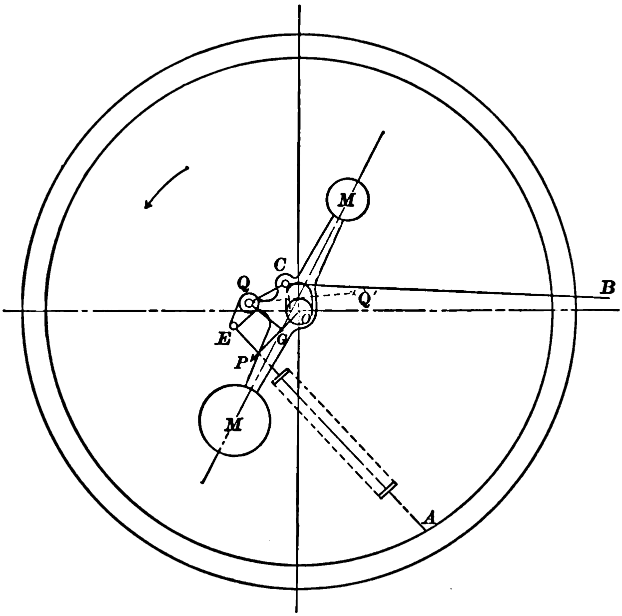


FIG. 201

moment produces a most powerful effect. Let MM (Fig. 201) be a heavy bar pivoted at Q, and with centre of gravity at G. A stud at C is directly connected to the slide valve through the rod CB. The position of the governor shown is that of the head-end dead point. Now so long as the angular velocity of the engine is constant, the governor acts exactly as do those previously described, through P causing a moment directly opposed to that of the spring. But if for any cause the engine slows down, the bar will tend to run ahead of the wheel, promptly drawing out the cut-off, and increasing the steam supply. Such effect can only occur while the speed is changing. Hence the arrangement as shown furnishes a powerful means of checking slight variations in speed. The relative positions of Q, O, G, and C are evidently such as will cause the angular accelerative effect to aid in governing, and Q cannot be located at Q', as such a location would cause the angular acceleration to destroy the governing. But if C turns about Q', a slightly better steam distribution would follow, and hence sometimes a separate eccentric is used. This is pivoted at Q', while the weight is pivoted at Q, and the two are connected by a pin of the bar working in a slot of the eccentric. This forms one of the latest and most successful of the inertia governors.

APPENDICES

APPENDIX I

At any instant during uniplanar motion, the body and space centrodes roll without slipping on one another.*

It is evident that there is always a point in common, as they are both swept up by the same point; but we are to prove that they are always tangent at this point, and that equal arcs are swept up by the describing point in equal times.

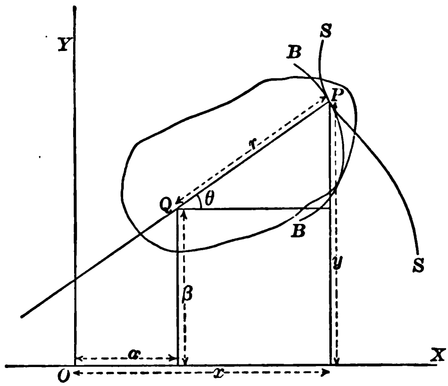


FIG. 202

Let Q be a point fixed in the body (Fig. 202), and let the coördinates of Q referred to space axes be α and β. Let P be any particle of the body at a distance r from Q, whose coördinates are x and y. If PQ makes an angle θ with the axis of x,

$$x = \alpha + r\cos\theta \quad . \quad . \quad . \quad . \qquad (1)$$

and

$$y = \beta + r\sin\theta \quad . \quad . \quad . \quad . \qquad (2)$$

Differentiating with respect to time,

$$\frac{dx}{dt} = \frac{d\alpha}{dt} - r\sin\theta\frac{d\theta}{dt},$$

$$\frac{dy}{dt} = \frac{d\beta}{dt} + r\cos\theta\frac{d\theta}{dt}.$$

* Adapted from "Uniplanar Kinematics of Solids and Fluids," by George M. Minchin, pp. 79–80.

Now if P is to be the instantaneous centre, its component velocities parallel to both X and Y must be zero; also it is seen that $\frac{d\theta}{dt}$ is the angular velocity of the body in this case. Hence,

$$\frac{d\alpha}{dt} - r\omega \sin\theta = 0 \quad . \quad . \quad . \quad (3)$$

$$\frac{d\beta}{dt} + r\omega \cos\theta = 0 \quad . \quad . \quad . \quad (4)$$

Substituting the values of $r\sin\theta$, and $r\cos\theta$ obtained from (3) and (4), in (1) and (2), we get

$$x = \alpha - \frac{\frac{d\beta}{dt}}{\omega},$$

$$y = \beta + \frac{\frac{d\alpha}{dt}}{\omega}.$$

These are the coördinates of the instantaneous centre, and the equation of the space centrode can be obtained from them by eliminating t.

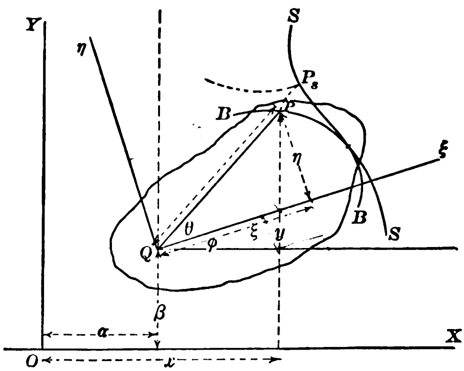


FIG. 203

Now let a set of coördinate axes be fixed in the body with origin at Q (Fig. 203.) The coördinates of P referred to these are η

and y. At the same time that P is the instantaneous centre, let these axes be parallel to the space axes, but at any other time, let them make an angle ϕ with the space axes. In general,

$$\xi = (x - \alpha)\cos\phi + (y - \beta)\sin\phi,$$

$$\eta = (y - \beta)\cos\phi - (x - \alpha)\sin\phi.$$

But if a point is on the body centrode, it will at some time be the instantaneous centre; and hence,

$$x - \alpha = -\frac{\frac{d\beta}{dt}}{\omega},$$

$$y - \beta = \frac{\frac{d\alpha}{dt}}{\omega}.$$

Substitution gives

$$-\frac{\frac{d\beta}{dt}}{\omega}\cos\phi + \frac{\frac{d\alpha}{dt}}{\omega}\sin\phi = \xi,$$

$$\frac{\frac{d\alpha}{dt}}{\omega}\cos\phi + \frac{\frac{d\beta}{dt}}{\omega}\sin\phi = \eta.$$

These equations express the locus of all points of the body which have been or will be the instantaneous centre. This locus or centrode will be swept up with a certain velocity whose components parallel to the axes of ξ and η will be

$\frac{d\xi}{dt}$ and $\frac{d\eta}{dt}$, where

$$\frac{d\xi}{dt} = -\frac{d}{dt}\left\{\frac{\frac{d\beta}{dt}}{\omega}\right\}\cos\phi + \frac{\frac{d\beta}{dt}}{\omega}\sin\phi\frac{d\phi}{dt} + \frac{d}{dt}\left\{\frac{\frac{d\alpha}{dt}}{\omega}\right\}\sin\phi + \frac{\frac{d\alpha}{dt}}{\omega}\cos\phi\frac{d\phi}{dt},$$

$$\frac{d\eta}{dt} = \frac{d}{dt}\left\{\frac{\frac{d\alpha}{dt}}{\omega}\right\}\cos\phi - \frac{\frac{d\alpha}{dt}}{\omega}\sin\phi\frac{d\phi}{dt} + \frac{d}{dt}\left\{\frac{\frac{d\beta}{dt}}{\omega}\right\}\sin\phi + \frac{\frac{d\beta}{dt}}{\omega}\cos\phi\frac{d\phi}{dt}.$$

These equations are general all along the body centrode, but at the particular time when P is the instantaneous centre, the curves are being simultaneously swept up at P, and $\phi = 0$.

$$\frac{d\xi}{dt} = -\frac{d}{dt}\left\{\frac{\frac{d\beta}{dt}}{\omega}\right\} + \frac{d\alpha}{dt} = \frac{d}{dt}\left\{\frac{-\frac{d\beta}{dt}}{\omega} + \alpha\right\},$$

$$\frac{d\eta}{dt} = \frac{d}{dt}\left\{\frac{\frac{d\alpha}{dt}}{\omega}\right\} + \frac{d\beta}{dt} = \frac{d}{dt}\left\{\frac{\frac{d\alpha}{dt}}{\omega} + \beta\right\},$$

but $$x = -\frac{\frac{d\beta}{dt}}{\omega} + \alpha, \quad \text{and} \quad y = \frac{\frac{d\alpha}{dt}}{\omega} + \beta;$$

hence, $$\frac{d\xi}{dt} = \frac{dx}{dt}, \quad \frac{d\eta}{dt} = \frac{dy}{dt}, \quad \frac{d\eta}{d\xi} = \frac{dy}{dx},$$

or the tangents to the two curves are in the same line, and they roll without slipping, since

$$\sqrt{dx^2 + dy^2} = \sqrt{d\xi^2 + d\eta^2}.$$

APPENDIX II

Rolling Curves.—The velocity of the point of contact of two curves along the curves, when rolled one upon the other, can always be expressed in terms of the angular velocity of the moving body about the instantaneous centre, and the radii of curvature of the centrodes. In Fig. 204 let p be the instantaneous centre, and let pp_1 and qq_1 be the infinitesimal elements of the space and body centrodes respectively. If we draw the common normal at p (or q), and also the normals at p_1 and q_1, we will get the centres of curvature A and B, and also the radii of curvature ρ_1 and ρ_2 of the centrodes. Let $d\alpha_1$ be the small angle pAp_1 and $d\alpha_2$ be the angle qBq_1. When the body centrode rolls upon the space till p_1 coincides with q_1, the whole system will have rotated through the differential angle $d\alpha_1 - d\alpha_2$. But $d\alpha_1 = \frac{ds}{\rho_1}$ where $pp_1 = ds$. Also $d\alpha_2 = \frac{ds}{\rho_2}$ or $d\theta = ds\left(\frac{1}{\rho_1} - \frac{1}{\rho_2}\right)$ and $\frac{d\theta}{dt} = \frac{ds}{dt}\left(\frac{1}{\rho_1} - \frac{1}{\rho_2}\right)$. But $\frac{d\theta}{dt}$ is ω, the angular velocity of the body about the instantaneous centre, and

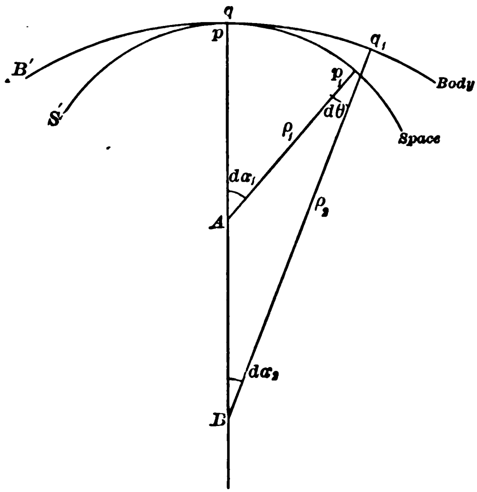


FIG. 204

$\frac{ds}{dt}$ is u, the velocity of the instantaneous centre along the space centrode. Hence, finally, $\frac{\omega}{u}=\frac{1}{\rho_1}-\frac{1}{\rho_2}$. If the curvatures of the centrodes are in opposite directions, then evidently $\frac{\omega}{u}=\frac{1}{\rho_1}+\frac{1}{\rho_2}$.

The acceleration of the point in the body corresponding to the instantaneous centre can be expressed in terms of u and ω. In Fig. 205 p is the moving centre which has its position changed by the rolling of the curve qq_1 on pp_1. Let us find the acceleration of q first in the direction of the normal and then in the direction of the tangent to the space centrode at the instant it comes in contact with p. The velocity of q will be in a direction at right angles to the chord qq_1, since the body is rotating about q_1, and the magnitude of this velocity will be $v=s\omega$ where ω is the angular velocity as before, and $s = qq_1$. The components of this velocity in the direction of the normal and tangent to the space centrode at p are

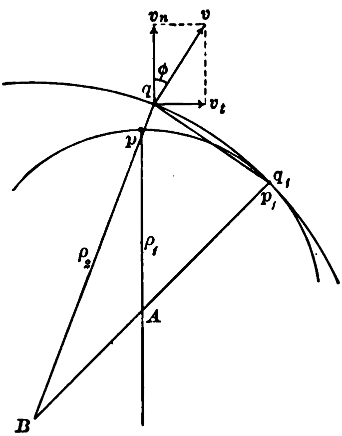


FIG. 205

$$v_n = s\omega \cos \phi,$$

and $$v_t = s\omega \sin \phi.$$

Hence, $$\frac{dv_n}{dt}=\left\{\frac{ds}{dt}\,\omega+\frac{d\omega}{dt}\,s\right\}\cos\phi - s\omega\sin\phi\,\frac{d\phi}{dt},$$

and $$\frac{dv_t}{dt}=\left\{\frac{ds}{dt}\,\omega+\frac{d\omega}{dt}\,s\right\}\sin\phi + s\omega\cos\phi\,\frac{d\phi}{dt}.$$

As q approaches and finally coincides with p, s and ϕ become equal to zero, or

$$\frac{dv_n}{dt}=\frac{ds}{dt}\,\omega = u\omega,$$

$$\frac{dv_t}{dt}=0.$$

Hence the point in the body centrode corresponding to the instantaneous centre has an acceleration in the direction of the normal to the centrodes only.

The Instantaneous Centre of Acceleration. — With origin at the instantaneous centre, the accelerations of a point of a moving body will then be three in number, viz.: $-r\omega^2$ in the direction of the radius vector, $r\dfrac{d\omega}{dt}$ in the direction at right angles to that line, and $u\omega$ in the direction of the normal to the centrodes. Let P (Fig. 206) be a point of a body whose centrodes are in contact at O. Take

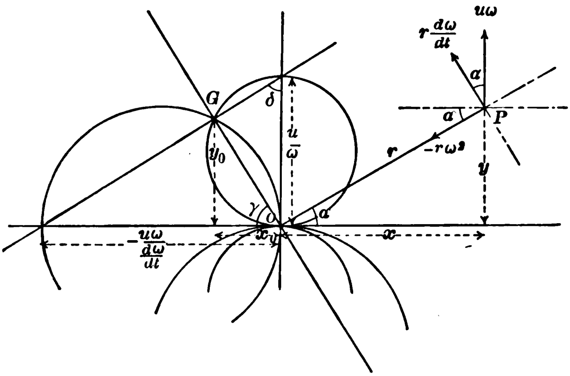


FIG. 206

the origin at O, and let the X-axis coincide with the tangent to the centrodes. If we resolve the accelerations parallel to the two axes, and denote their sums by X and Y, we get

$$X = -r\frac{d\omega}{dt}\sin\alpha - r\omega^2\cos\alpha,$$

$$Y = u\omega + r\frac{d\omega}{dt}\cos\alpha - r\omega^2\sin\alpha,$$

or since $r\cdot\sin\alpha = y$, and $r\cdot\cos\alpha = x$,

$$X = -y\frac{d\omega}{dt} - x\omega^2,$$

$$Y = u\omega + x\frac{d\omega}{dt} - y\omega^2.$$

If we make X equal to zero, we get the locus of all points which have no acceleration parallel to the axis of X, or

$$y = -\left\{\frac{\omega^2}{\frac{d\omega}{dt}}\right\} x.$$

This is the equation of a straight line through the origin the tangent of whose angle with the X-axis is

$$\tan\gamma = -\frac{\omega^2}{\frac{d\omega}{dt}}.$$

If we make Y equal to zero, we get the locus of all points having no acceleration parallel to the axis of Y, or

$$y = \frac{u}{\omega} + \frac{\frac{d\omega}{dt}}{\omega^2} x.$$

This is the equation of a straight line which cuts the X-axis at a distance $-\frac{u\omega}{\frac{d\omega}{dt}}$ from the origin, and cuts the Y-axis at a distance $\frac{u}{\omega}$ from the origin, hence the angle δ will be such that

$$\tan\delta = -\frac{\frac{u\omega}{\frac{d\omega}{dt}}}{\frac{u}{\omega}} = -\frac{\omega^2}{\frac{d\omega}{dt}} = \tan\gamma,$$

or the two lines cut one another at right angles. The point of intersection G of these is called the *Centre of Acceleration*. The axes of X and Y being independent of the direction of r, it follows that G has no acceleration at all, or a moving body has a centre of no acceleration as well as one of no velocity.

If we call the coördinates of G x_0 and y_0, these can be determined by elimination from the equations of the two straight lines; hence,

$$x_0 = -\frac{u\omega \dfrac{d\omega}{dt}}{\omega^4 + \left(\dfrac{d\omega}{dt}\right)^2},$$

$$y_0 = +\frac{u\omega^3}{\omega^4 + \left(\dfrac{d\omega}{dt}\right)^2}.$$

As an interesting extension of the above, let the accelerations of P be resolved in still other ways, viz., in the direction of the radius vector and at right angles to it. Denoting these components by A and B,

$$A = u\omega \sin\alpha - r\omega^2,$$

$$B = r\frac{d\omega}{dt} + u\omega\cos\alpha,$$

but $$\cos\alpha = \frac{x}{r}, \ \sin\alpha = \frac{y}{r}, \text{ and } x^2 + y^2 = r^2;$$

hence, $$A = u\omega\frac{y}{r} - \frac{x^2+y^2}{r}\omega^2,$$

$$B = u\omega\frac{x}{r} + \frac{x^2+y^2}{r}\frac{d\omega}{dt}.$$

Put A equal to zero, and its equation gives

$$x^2 + y^2 - \frac{u}{\omega}y = 0.$$

The equation of a circle of radius a, which passes through the origin, and whose centre lies on the axis of Y, is

$$x^2 + y^2 - 2ay = 0,$$

hence our equation represents such a circle where $2a = \frac{u}{\omega}$. Likewise if B is zero, we get

$$x^2 + y^2 + \frac{u\omega}{\dfrac{d\omega}{dt}} = 0.$$

This represents a circle passing through the origin and with centre on the axis of X. This cuts the X-axis at a distance $2a = -\frac{u\omega}{\frac{d\omega}{dt}}$ from the origin. These two circles cut the axes at the same points the straight lines do, and cut each other at the same point G, since the angles about G are at right angles.

As an example of the method of determining the instantaneous centre of acceleration, let a cylinder of mass M and radius r roll down a plane inclined at an angle α with the horizontal. The cylinder is supposed to start from rest, and, after its centre of mass has fallen through a distance h, we are required to find the coördinates of the centre of acceleration. In this case

$$\frac{\omega}{u} = \frac{1}{\infty} - \frac{1}{r} = -\frac{1}{r}, \text{ or } \omega = -\frac{u}{r}.$$

From the principle of the conservation of energy,

$$Mgh = \frac{I_0\omega^2}{2} + M\frac{u^2}{2},$$

and since $$I_0 = M\frac{r^2}{2},$$

then will $$u = 2\sqrt{\frac{gh}{3}},$$

and $$\omega = -\frac{u}{r} = -\frac{2}{r}\sqrt{\frac{gh}{3}}.$$

Also since $h = s \sin\alpha$ (see Fig. 207),

$$u^2 = 4\frac{gs\sin\alpha}{3}.$$

Differentiating, $$2u\frac{du}{dt} = \frac{4}{3}g\sin\alpha\frac{ds}{dt},$$

and as $$u = \frac{ds}{dt},$$

$$\frac{du}{dt} = \frac{2}{3}g\sin\alpha,$$

and $$\frac{d\omega}{dt} = -\frac{1}{r}\frac{du}{dt} = -\frac{2}{3}\frac{g}{r}\sin\alpha = -\frac{2}{3}\frac{g}{r}\frac{h}{s}.$$

Then $\frac{u}{\omega} = -r$ and $\frac{-u\omega}{\frac{d\omega}{dt}} = -2s$, and we can write down immediately the coördinates of the centre of acceleration,

$$x_0 = \frac{-2r^2s}{4s^2 + r^2},$$

$$y_0 = \frac{-4rs^2}{4s^2 + r^2}.$$

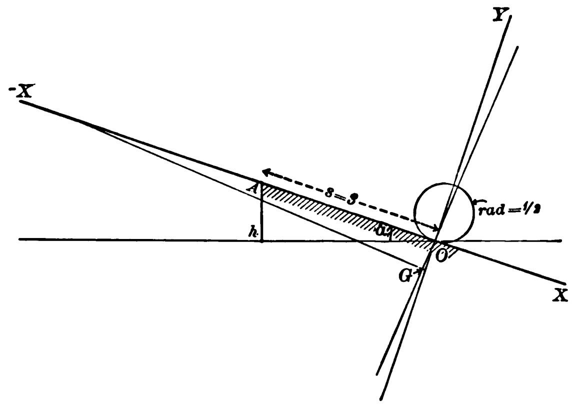


FIG. 207

In Fig. 207, where $r = \frac{1}{2}$ and $s = 3$, these values become

$$x_0 = -.0414,$$

$$y_0 = -.4965.$$

APPENDIX III*

Proof of the General Theorem.—What must be the form of the curve xx (Fig. 208), which, during its rotation about a permanent centre B, will have as a normal the line connecting its point of intersection P with a fixed curve, and a fixed point I on that curve? In the solution of this problem, then, we would consider the profile as fixed to and carried around by the pitch circle. Since, however, it is the equation of the profile that we are seeking, it will prove simpler to exchange the relative motions of the profile and curve of action, thus bringing the former to rest, and referring it to a fixed origin. From this point of view the curve of action is fixed to and carried around by the pitch circle.

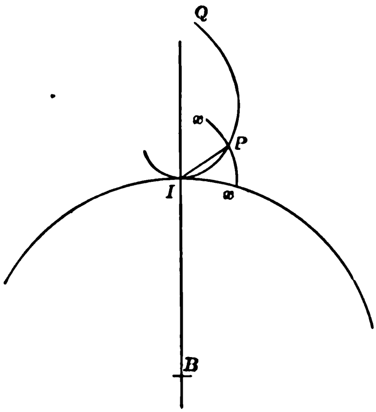


FIG. 208

Let the origin be taken at the centre of the pitch circle. Let OC (Fig. 209) make an angle ψ with the axis of X, where OC is a radius vector moving with the curve of action. The equation of the given curve of action may then be expressed in terms of the parameter ψ, or its equation will be

$$F(x, y, \psi) = O \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

* For the following beautiful method of deducing the equation of the gear-tooth profile from that of its curve of action, the author is largely indebted to Mr. A. V. Saph of the University of California.

Let the equation of the required profile be

$$x = \phi(\psi), \ y = f(\psi) \quad . \quad . \quad . \quad . \quad . \quad (2)$$

At the point P where the two curves intersect, both relations must be satisfied, or

$$F\{\phi(\psi), f(\psi), \psi\} = 0 \quad . \quad . \quad . \quad . \quad (3)$$

The slope of the tangent to the profile will be

$$\frac{dy}{dx} = \frac{f'(\psi)}{\phi'(\psi)} \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

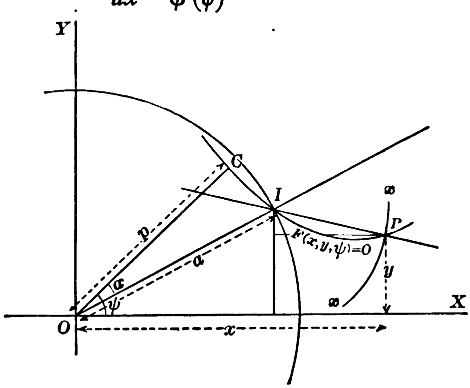


FIG. 209

If the radius of the pitch circle is a, and the constant angle COI is α, the slope of the chord PI is

$$\frac{a \sin(\psi - \alpha) - f(\psi)}{a \cos(\psi - \alpha) - \phi(\psi)} \quad . \quad . \quad . \quad . \quad (5)$$

Since these are to be at right angles,

$$\frac{f'(\psi)}{\phi'(\psi)} \frac{a \sin(\psi - \alpha) - f(\psi)}{a \cos(\psi - \alpha) - \phi(\psi)} = -1,$$

or

$$\frac{f'(\psi)}{\phi'(\psi)} = -\frac{a \cos(\psi - \alpha) - \phi(\psi)}{a \sin(\psi - \alpha) - f(\psi)} \quad . \quad . \quad . \quad (6)$$

Let

$$f'(\psi) = -\rho\{a \cos(\psi - \alpha) - \phi(\psi)\} \quad . \quad . \quad (7)$$

and

$$\phi'(\psi) = \rho\{a \sin(\psi - \alpha) - f(\psi)\} \quad . \quad . \quad . \quad (8)$$

where ρ may be a constant or a function of ψ. Differentiating equation (3), and substituting in it the values of $f'(\psi)$ and $\phi'(\psi)$ from (7) and (8), we can obtain an expression for ρ, after which (7) and (8) may be integrated to give the required equations.

As an example of the above method let us take as a curve of action a circle of radius b tangent to the pitch circle at I.* In this case we will take OC (Fig. 209) as coinciding with OI; hence, $\alpha = 0$. (See Fig. 210.)

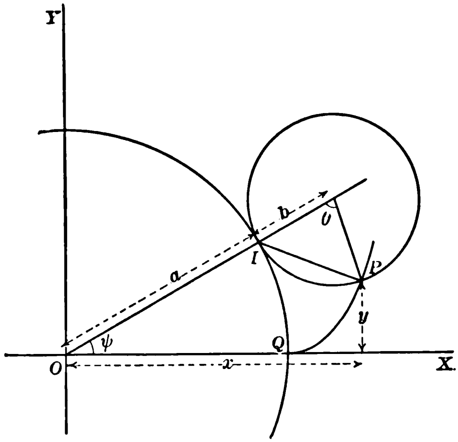


FIG. 210

The equation of the curve of action is

$$x^2+y^2-2x(a+b)\cos\psi-2y(a+b)\sin\psi+(a+b)^2-b^2=0 \quad (1)$$

and the equations of the profiles are

$$x=\phi(\psi), \qquad y=f(\psi) \quad . \quad . \quad . \quad . \quad (2)$$

At the point of intersection of the two curves at P

$$\phi(\psi)^2+f(\psi)^2-2(a+b)\cos\psi\cdot\phi(\psi)$$
$$-2(a+b)\sin\psi\cdot f(\psi)+(a+b)^2-b^2=0 \quad . \quad (3)$$

* It is to be observed that there is no rolling between the circles.

The slope of the tangent to the profile is

$$\frac{dy}{dx}=\frac{f'(\psi)}{\phi'(\psi)} \quad . \quad . \quad . \quad . \quad . \qquad (4)$$

The slope of the chord *PI* is

$$\frac{a \sin \psi - f(\psi)}{a \cos \psi - \phi(\psi)} \quad . \quad . \quad . \quad . \quad . \qquad (5)$$

Hence,
$$\frac{f'(\psi)}{\phi'(\psi)}=-\frac{a \cos \psi - \phi(\psi)}{a \sin \psi - f(\psi)} \quad . \quad . \quad . \qquad (6)$$

or
$$f'(\psi) = -\rho\{a \cos \psi - \phi(\psi)\} \quad . \quad . \quad . \qquad (7)$$

and
$$\phi'(\psi) = \rho\{a \sin \psi - f(\psi)\} \quad . \quad . \quad . \qquad (8)$$

Differentiating equation (3), and rejecting the factor 2, we have

$$f(\psi) \cdot f'(\psi)+\phi(\psi) \cdot \phi'(\psi)+(a+b) \sin \psi \cdot \phi(\psi)$$
$$-(a+b) \cos \psi \cdot \phi'(\psi)-(a+b) \cos \psi \cdot f(\psi)-(a+b) \sin \psi \cdot f'(\psi)=0,$$

and substituting for $\phi'(\psi)$ and $f'(\psi)$ their values from (7) and (8) we obtain

$$-f(\psi)\rho \cdot a \cos \psi + \rho \cdot f(\psi) \cdot \phi(\psi) + \phi(\psi)\rho a \sin \psi$$
$$-\rho \cdot \phi(\psi)f(\psi)+(a+b)\sin\psi \cdot \phi(\psi)-(a+b)\cos\psi \cdot \rho a \sin\psi$$
$$+(a+b)\cos\psi \cdot \rho \cdot f(\psi)-(a+b)\cos\psi \cdot f(\psi)$$
$$+(a+b)a\sin\psi\rho\cos\psi-(a+b)\sin\psi \cdot \rho \cdot \phi(\psi) = 0 \quad . \quad . \qquad (9)$$

$$\rho=\frac{(a+b)\cos\psi \cdot f(\psi)-(a+b)\sin\psi \cdot \phi(\psi)}{b\cos\psi \cdot f(\psi)-b\sin\psi \cdot \phi(\psi)}$$
$$=\frac{a+b}{b}=\text{constant.}$$

Differentiating equation (8), we have

$$\phi''(\psi)=\rho(a\cos\psi - f'(\psi))=\rho a\cos\psi-\rho^2 a\cos\psi-\rho^2\phi(\psi)$$
$$\phi''(\psi)+\rho^2\phi(\psi)=\rho a(1+\rho)\cos\psi \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (10)$$

A particular solution of this may be seen by inspection to be of the form $K \cos\psi$ whose second derivative is $-K\cos\psi$. Then

$$-K\cos\psi + \rho^2 K\cos\psi = \rho a(1+\rho)\cos\psi,$$

$$K = \frac{\rho a(1+\rho)}{\rho^2 - 1} = \frac{\rho a}{\rho - 1} = a + b.$$

The particular solution, then, is

$$\phi(\psi) = (a+b)\cos\psi.$$

The solutions to $\phi''(\psi) + \rho^2(\phi(\psi)) = 0$ are

$$\phi(\psi) = A_1 e^{i\rho\psi},$$

and

$$\phi(\psi) = A_2 e^{-i\rho\psi}.$$

Then

$$\phi(\psi) = (a+b)\cos\psi + A_1 e^{i\rho\psi} + A_2 e^{-i\rho\psi} \quad . \quad (11)$$

When $\psi = 0$, $\phi(\psi) = a$, and $A_1 + A_2 = -b$,

$$f'(\psi) = -\rho(a\cos\psi) + \rho\cdot\phi(\psi)$$

$$= -\frac{a+b}{b}a\cos\psi + \frac{(a+b)^2}{b}\cos\psi + \frac{a+b}{b}\{A_1e^{i\rho\psi} + A_2e^{-i\rho\psi},\}$$

$$f'(\psi) = (a+b)\cos\psi + \frac{a+b}{b}\{A_1e^{i\rho\psi} + A_2e^{-i\rho\psi}\},$$

$$f(\psi) = (a+b)\sin\psi + \frac{a+b}{b}\left\{\frac{A_1e^{i\rho\psi}}{i\rho} + \frac{A_2e^{-i\rho\psi}}{-i\rho}\right\} \quad . \quad . \quad (12)$$

When $\psi = 0$, $f(\psi) = 0$, then $A_1 - A_2 = 0$, but $A_1 + A_2 = -b$;

hence,

$$A_1 = A_2 = -\frac{b}{2}.$$

Then, finally,

$$x = \phi(\psi) = (a+b)\cos\psi - b\left\{\frac{e^{i\rho\psi} + e^{-i\rho\psi}}{2}\right\} = (a+b)\cos\psi - b\cos\rho\psi,$$

$$y = f(\psi) = (a+b)\sin\psi - b\left\{\frac{e^{i\rho\psi} - e^{-i\rho\psi}}{2i}\right\} = (a+b)\sin\psi - b\sin\rho\psi,$$

$$x = (a+b)\cos\psi - b\cos\frac{a+b}{b}\psi,$$

$$y = (a+b)\sin\psi - b\sin\frac{a+b}{b}\psi,$$

which are the equations of the epicycloid.

If there is a second pitch circle working within the first at the pitch point I, and whose profile works on the same circular curve of action, we could deduce the equation of its profile by considering the circle of radius b to touch in internal contact. In this case the sign of b would become negative, or

$$x = (a - b)\cos\psi + b\cos\frac{a-b}{-b}\psi,$$

$$y = (a - b)\sin\psi + b\sin\frac{a-b}{-b}\psi,$$

which are

$$x = (a - b)\cos\psi + b\cos\frac{a-b}{b}\psi,$$

$$y = (a - b)\sin\psi - b\sin\frac{a-b}{b}\psi,$$

the equations of the hypocycloid.

In the case of the rack we must make a equal to infinity. It is therefore necessary to move our origin from O to Q, and express our equations in terms of the parameter θ (see Fig. 210) instead of ψ. In this case

$$y' = y,$$

$$x' = x - a,$$

$$\theta = \psi\frac{a}{b}$$

(from the property of the epicycloid). Substituting these values in the equations of the epicycloid,

$$x' = (a + b)\cos\frac{b}{a}\theta - b\cos\frac{a+b}{b}\theta - a,$$

$$y' = (a + b)\sin\frac{b}{a}\theta - b\sin\frac{a+b}{b}\theta.$$

Putting $a = \infty$, and evaluating the indeterminate forms, these become

$$x' = b(1 - \cos\theta),$$

$$y' = b(\theta - \sin\theta).$$

These are the equations of the cycloid.

Another important case is that in which the curve of action is a straight line through I and making a constant angle with OI. In Fig. 211 let PIC be the straight line of action, with I as the pitch

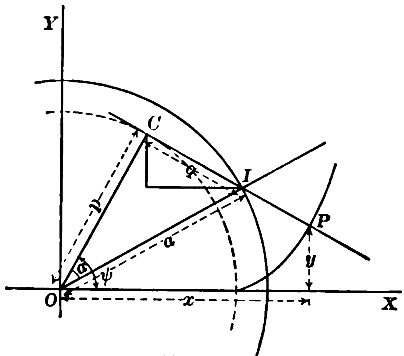


FIG. 211

point. In this case we take the radius vector OC as the perpendicular let fall upon the curve of action from the origin. α is constant, and $OC = p =$ constant. The equation of the curve of action is

$$x \cos\psi + y \sin\psi = a \cos\alpha = p \quad . \quad . \quad . \quad (1)$$

while those of the profile are

$$x = \phi(\psi), \qquad y = f(\psi) \quad . \quad . \quad . \quad . \quad (2)$$

At the point P

$$\phi(\psi) \cos\psi + f(\psi) \sin\psi = p \quad . \quad . \quad . \quad (3)$$

The slope of the tangent to the profile is

$$\frac{dy}{dx} = \frac{f'(\psi)}{\phi'(\psi)} \quad . \quad . \quad . \quad . \quad . \quad (4)$$

and that of the line PI is

$$\frac{a \sin(\psi - \alpha) - f(\psi)}{a \cos(\psi - \alpha) - \phi(\psi)} \quad . \quad . \quad . \quad . \quad (5)$$

Expanding equation (5),

$$\frac{a \cos\alpha \sin\psi - a \sin\alpha \cos\psi - f(\psi)}{a \cos\alpha \cos\psi + a \sin\alpha \sin\psi - \phi(\psi)} = \frac{p \sin\psi - q \cos\psi - f(\psi)}{p \cos\psi + q \sin\psi - \phi(\psi)},$$

where p and q are constants having the values shown in Fig. 211. Hence,

$$\frac{f'(\psi)}{\phi'(\psi)} \cdot \frac{p \sin\psi - q\cos\psi - f(\psi)}{p\cos\psi + q\sin\psi - \phi(\psi)} = -1 \quad . \quad . \quad (6)$$

or

$$f'(\psi) = -\rho\{p\cos\psi + q\sin\psi - \phi(\psi)\} \quad . \quad . \quad (7)$$

$$\phi'(\psi) = \rho\{p\sin\psi - q\cos\psi - f(\psi)\} \quad . \quad . \quad (8)$$

Differentiating the equation of the curve of action (3),

$$\phi'(\psi)\cos\psi - \phi(\psi)\sin\psi + f'(\psi)\sin\psi - f(\psi)\cos\psi = 0,$$

and substituting from equations (7) and (8),

$$\rho\{p\sin\psi - q\cos\psi - f(\psi)\}\cos\psi - \phi(\psi)\sin\psi$$
$$- \rho\{p\cos\psi + q\sin\psi - \phi(\psi)\}\sin\psi - f(\psi)\cos\psi = 0,$$

from which we find

$$\rho = \frac{\phi(\psi)\sin\psi - f(\psi)\cos\psi}{\phi(\psi)\sin\psi - f(\psi)\cos\psi - q},$$

which is a function of ψ. We now have from equation (7)

$$\frac{-\phi(\psi)\sin\psi + f(\psi)\cos\psi}{\phi(\psi)\sin\psi - f(\psi)\cos\psi - q}\{p\cos\psi + q\sin\psi - \phi(\psi)\} = f'(\psi) \quad (9)$$

and from equation (1) of the curve of action,

$$\phi(\psi) = \frac{p}{\cos\psi} - \frac{f(\psi)\sin\psi}{\cos\psi} \quad . \quad . \quad . \quad . \quad (10)$$

Substituting (10) in (9),

$$f'(\psi)\cos\psi$$
$$= \frac{-p\sin\psi + f(\psi)}{p\sin\psi - f(\psi) - q\cos\psi}\{-p^2\sin^2\psi + q\sin\psi\cos\psi + f \quad \sin(\psi)\}$$
$$= \frac{(p\sin\psi - f(\psi) - q\cos\psi)(p\sin^2\psi - f(\psi)\sin\psi)}{p\sin\psi - f(\psi) - q\cos\psi},$$

$$f'(\psi)\cos\psi + f(\psi)\sin\psi = p\sin^2\psi \quad . \quad . \quad . \quad (11)$$

In the same manner we may show that

$$-\phi'(\psi)\sin\psi + \phi(\psi)\cos\psi = p\cos^2\psi \quad . \quad . \quad (12)$$

These are the differential equations of the profile. They may be written

$$f'(\psi) + f(\psi) \tan\psi = p\,\frac{\sin^2\psi}{\cos\psi},$$

and
$$\phi'(\psi) - \phi(\psi) \cot\psi = -p\,\frac{\cos^2\psi}{\sin\psi},$$

and are of the form $y' + Py = Q$.

The integrating factor for the first will be

$$e^{\int \tan\psi\, d\psi} = \sec\psi,$$

and for the second,

$$e^{-\int \cot\psi\, d\psi} = e^{-\log\sin\psi} = \csc\psi.$$

The solutions are

$$f(\psi)\sec\psi = p\int \frac{\sin^2\psi}{\cos\psi}\sec\psi\, d\psi = p\int \tan^2\psi\, d\psi = p(\tan\psi - \psi) + A_1,$$

$$\phi(\psi)\csc\psi =$$

$$-p\int \frac{\cos^2\psi}{\sin\psi}\csc\psi\, d\psi = -p\int \cot^2\psi\, d\psi = -p(-\cot\psi - \psi) + A_2.$$

When $\psi = 0$, $f(\psi) = 0$, $\phi(\psi) = p$. Hence, $A_1 = 0$ directly, and

$$A_2 = p\left(\frac{1}{\sin\psi} - \cot\psi - \psi\right) = p\left(\frac{1-\cos\psi}{\sin\psi} - \psi\right) = p\left(\frac{2\sin^2\frac{1}{2}\psi}{2\sin\frac{1}{2}\psi\cos\frac{1}{2}\psi} - \psi\right),$$

which last expression becomes zero when $\psi = 0$; hence, $A_2 = 0$.

Hence, finally, $x = \phi(\psi) = p\,(\cos\psi + \psi\sin\psi)$,

$$y = f(\psi) = p\,(\sin\psi - \psi\cos\psi),$$

which are the equations of the involute of a circle whose radius is p.

When the radius of the pitch circle becomes infinite, we have the special case of a rack. But when $a = \infty$, $p = a\cos\alpha = \infty$ also. The radius of curvature of the involute for any value of ψ is $q = p\psi$, which is infinite when p is infinite for all values of ψ except $\psi = 0$.

Hence any finite portion of the rack tooth is a straight line. The slope of the tangent to the profile is

$$\frac{dy}{dx} = \frac{\frac{dy}{d\psi}}{\frac{dx}{d\psi}} = \tan\psi,$$

or the tangent to the profile always cuts the axis of x at an angle ψ. At the point S (Fig. 212), where the involute cuts the pitch circle,

$$x^2 + y^2 = a^2,$$

$$p^2(1 + \psi_0^2) = a^2,$$

$$\psi_0 = \frac{1}{p}\sqrt{a^2 - p^2} = \frac{q}{p} = \tan\alpha.$$

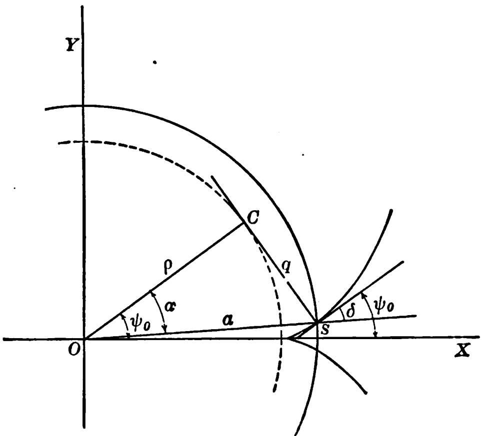


FIG. 212

Hence the tangent at S cuts the axis of x at an angle equal to $(\tan\alpha)$. The angle $SOX = \psi_0 - \alpha = \tan\alpha - \alpha$. Hence the involute cuts the radius OS at an angle δ, where

$$\delta = \tan\alpha - (\tan\alpha - \alpha) = \alpha.$$

In the case of the involute rack, therefore, the teeth are straight lines inclined at an angle α with the normals to the pitch curve.

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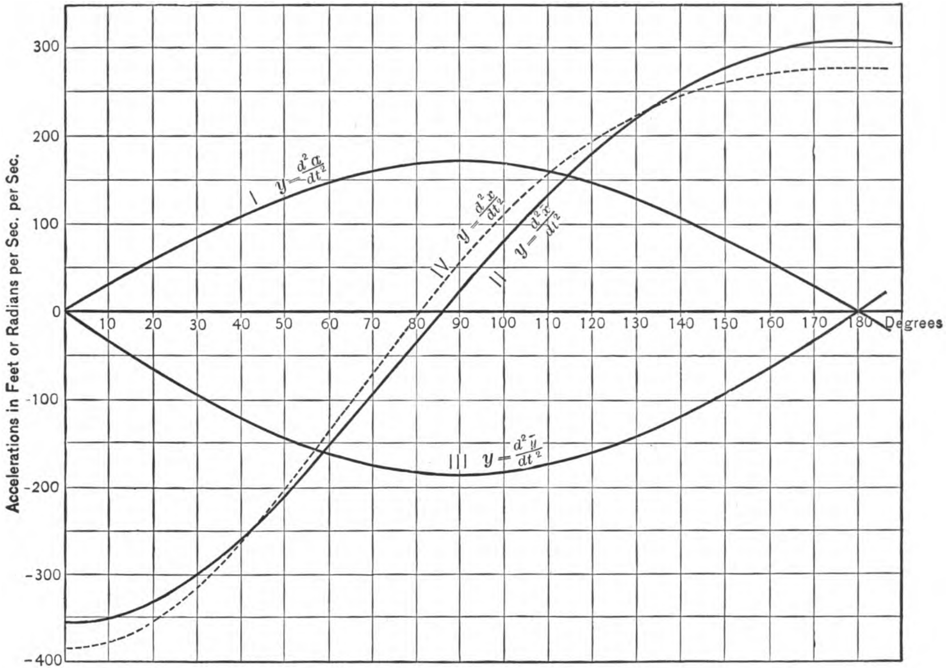
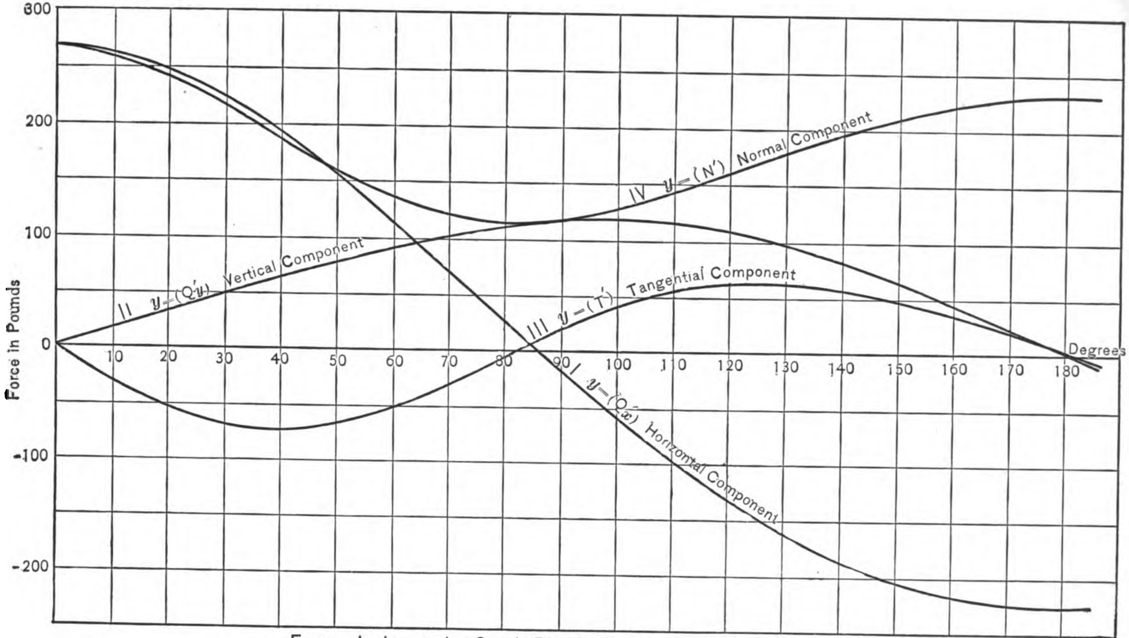


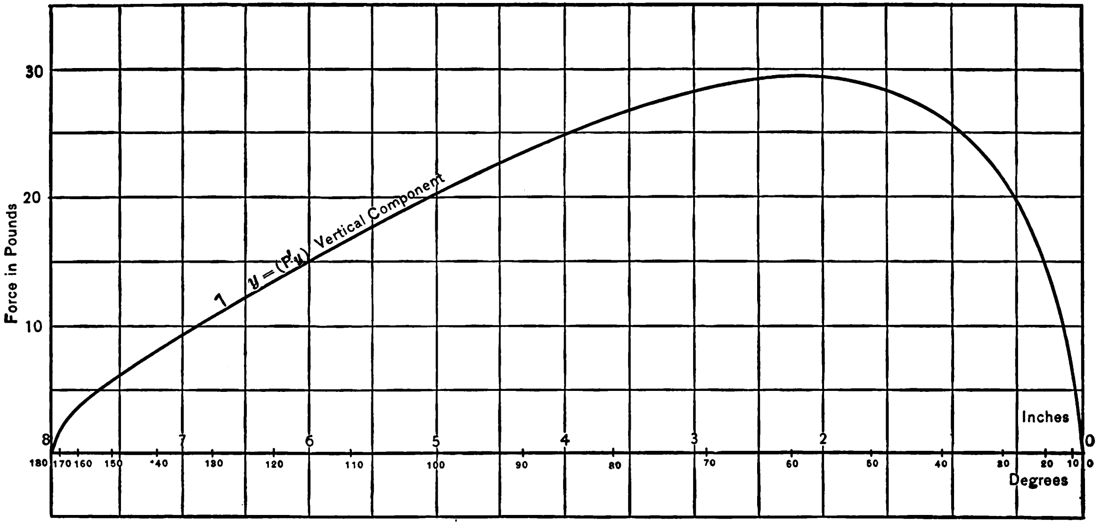
Curves of Acceleration.

PLATE II.



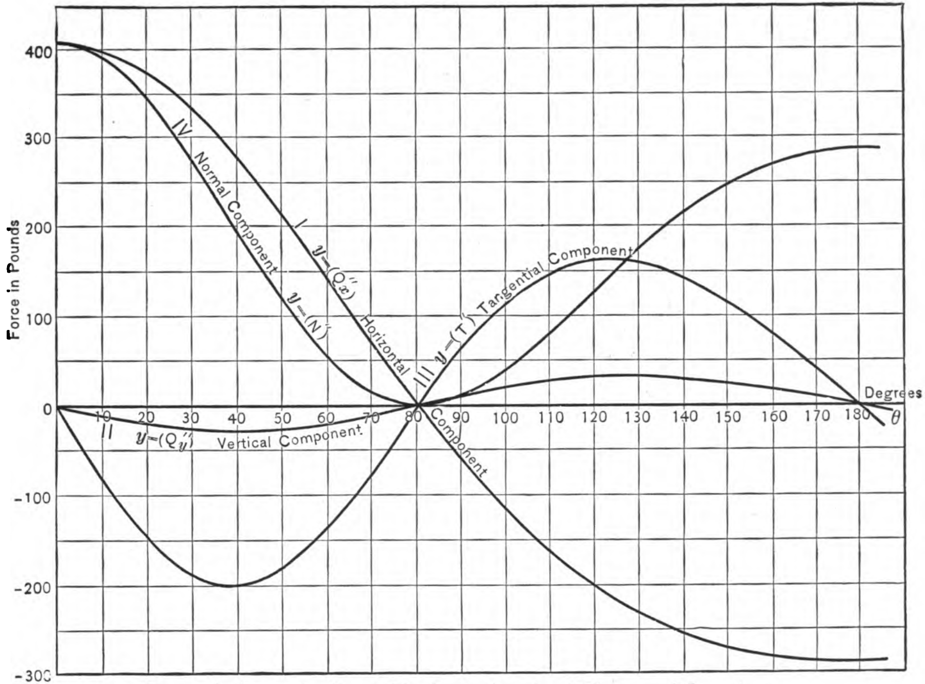
Forces Active at the Crank Pin due to Inertia of Connecting Rod.

PLATE III.



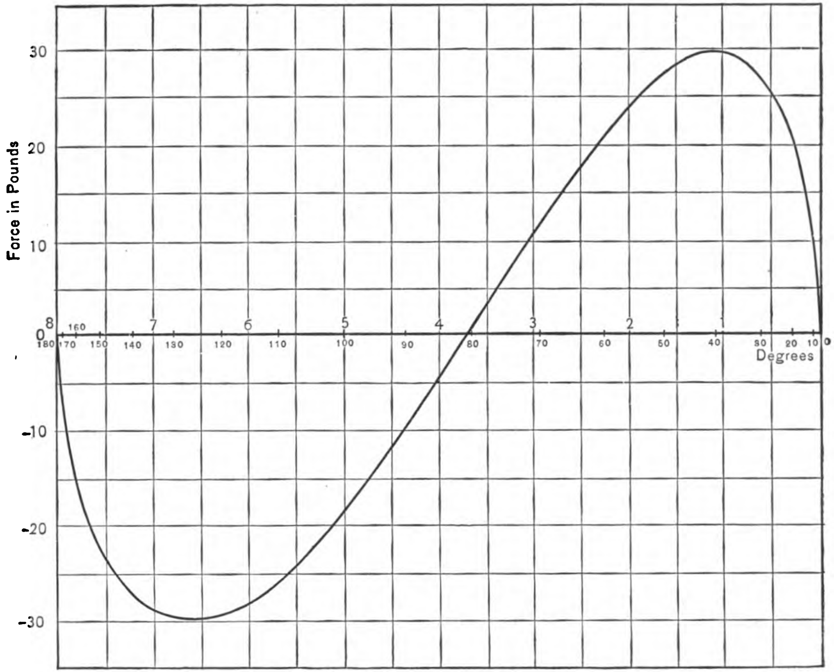
Forces Active at the Wrist Pin due to Inertia of Connecting Rod.

PLATE IV.



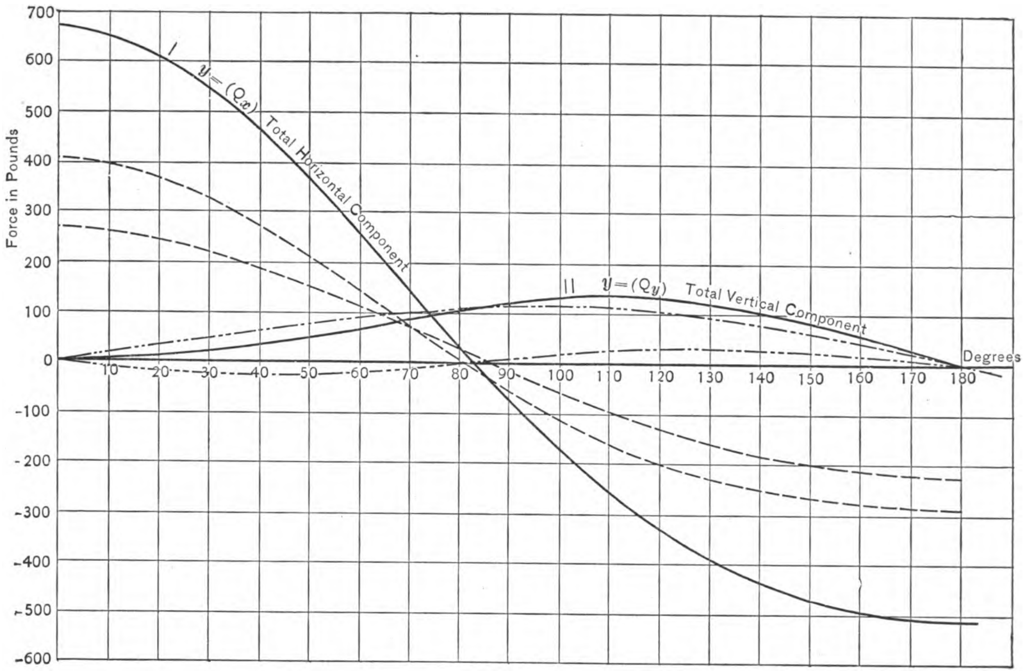
Forces Active at the Crank Pin due to Inertia of Piston and Cross-head.

PLATE V.



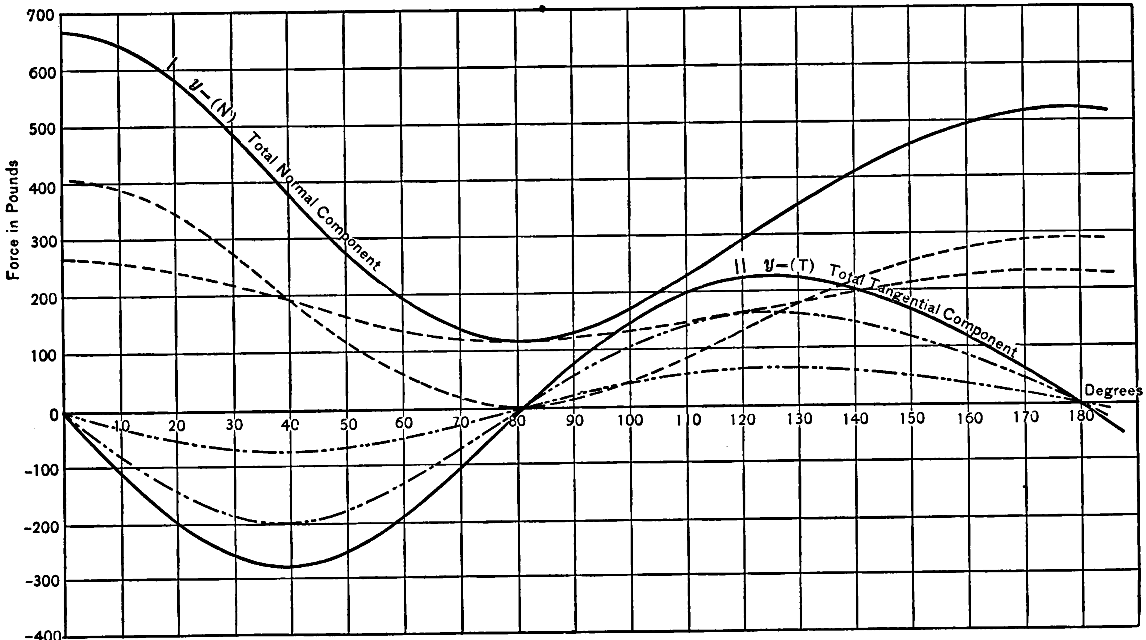
Forces Active at the Wrist Pin due to Inertia of Piston and Cross-head.

PLATE VI.



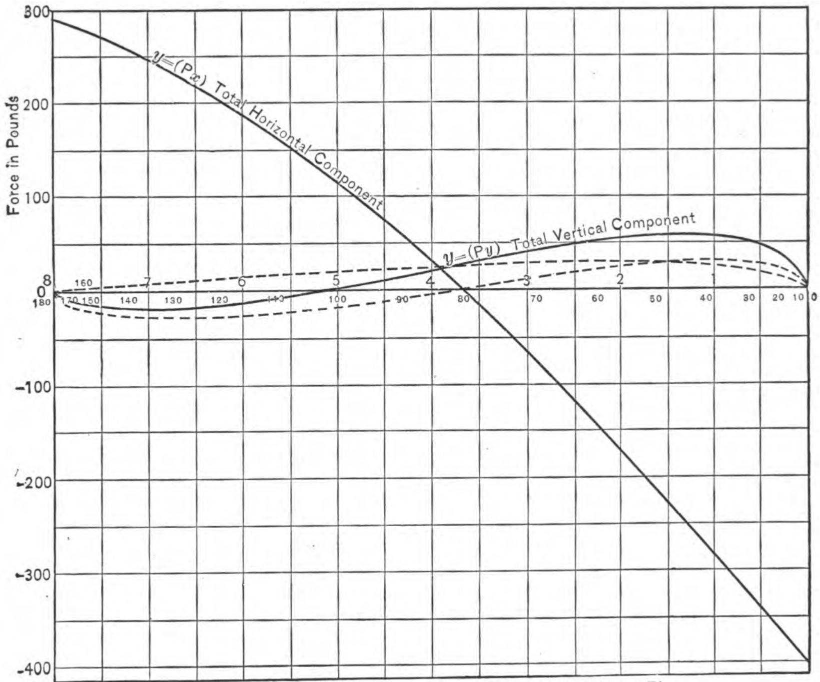
Total Horizontal and Vertical Components at the Crank Pin.

PLATE VII.



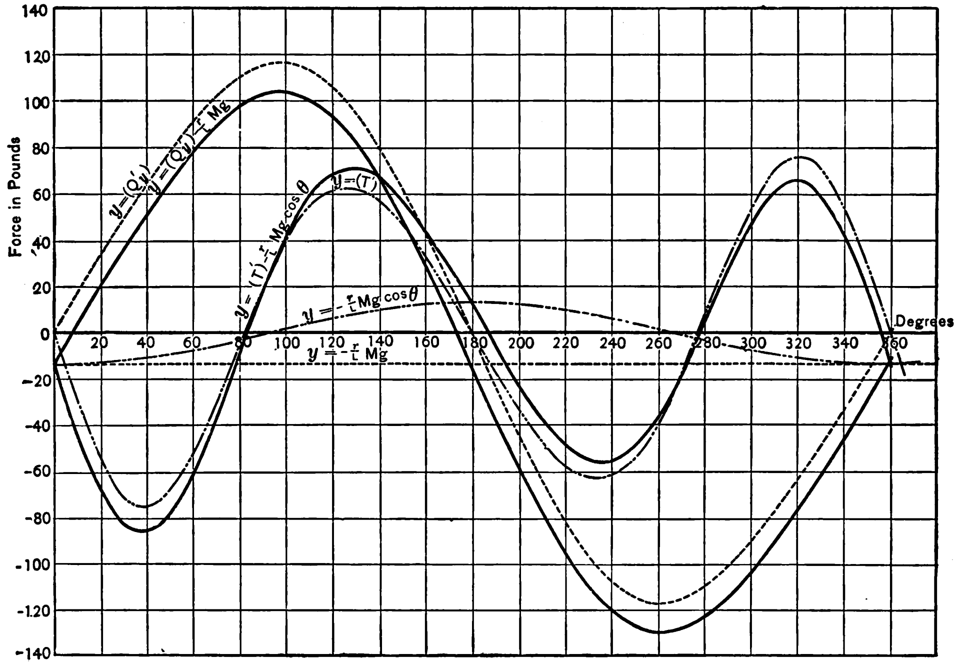
Total Tangential and Normal Components at the Crank Pin.

PLATE VIII.



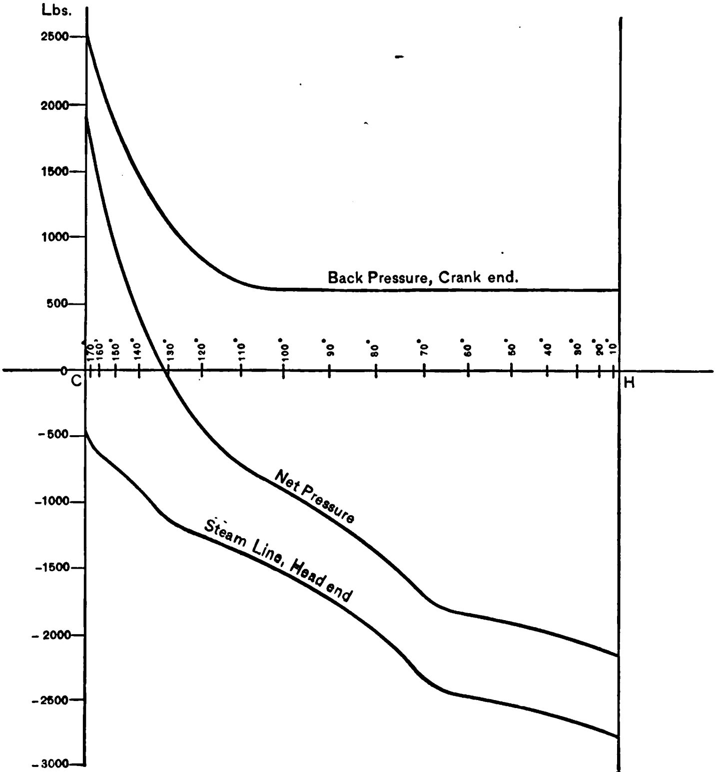
Total Horizontal and Vertical Components at the Wrist Pin.

PLATE IX.



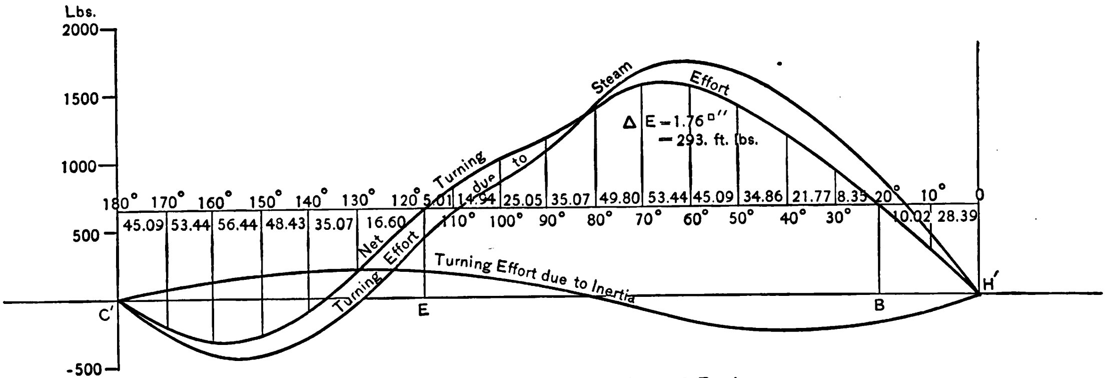
Correction for Weight of Rod at the Crank Pin.

PLATE X.



Steam lines of 6 × 8 Engine.

PLATE XI.



Net and Mean Turning Efforts of 6 × 8 Engine.

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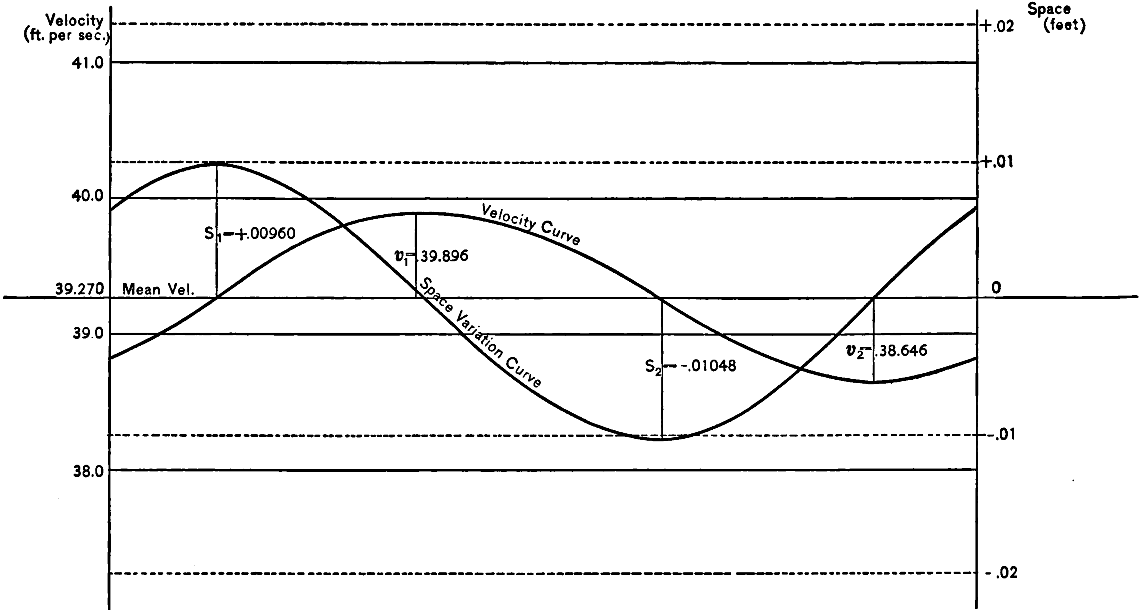

Velocity and Space Variation Curves of 6 × 8 Engine.

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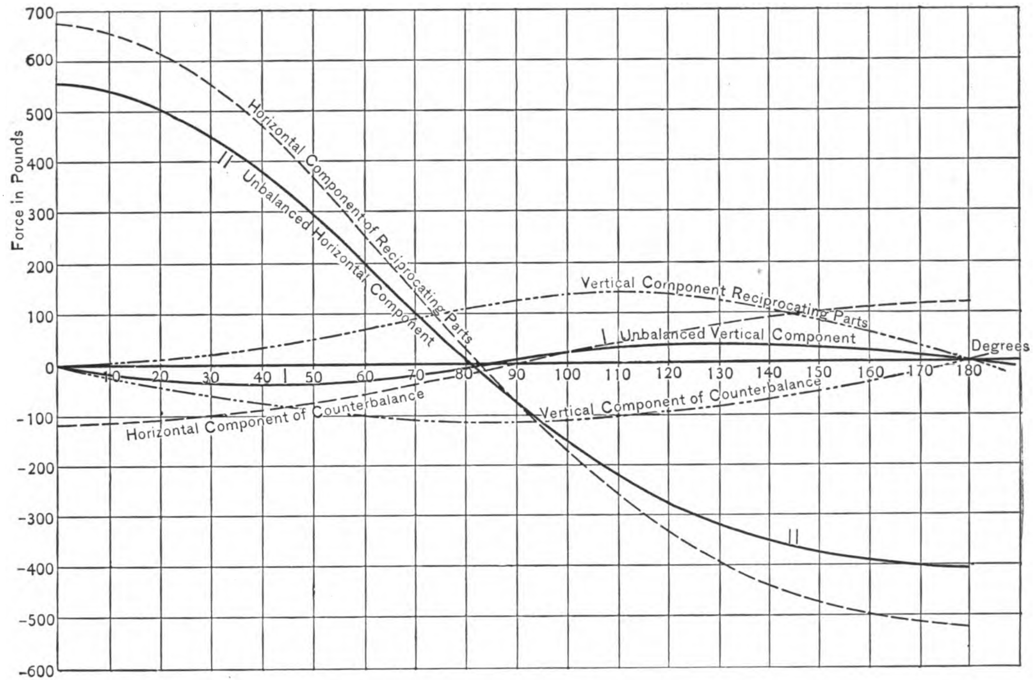
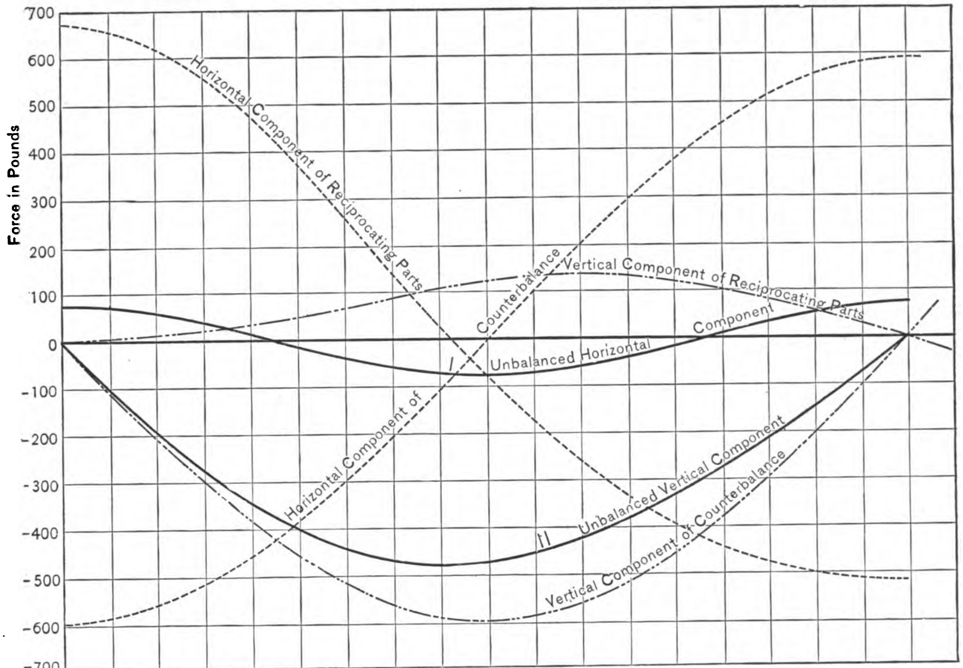


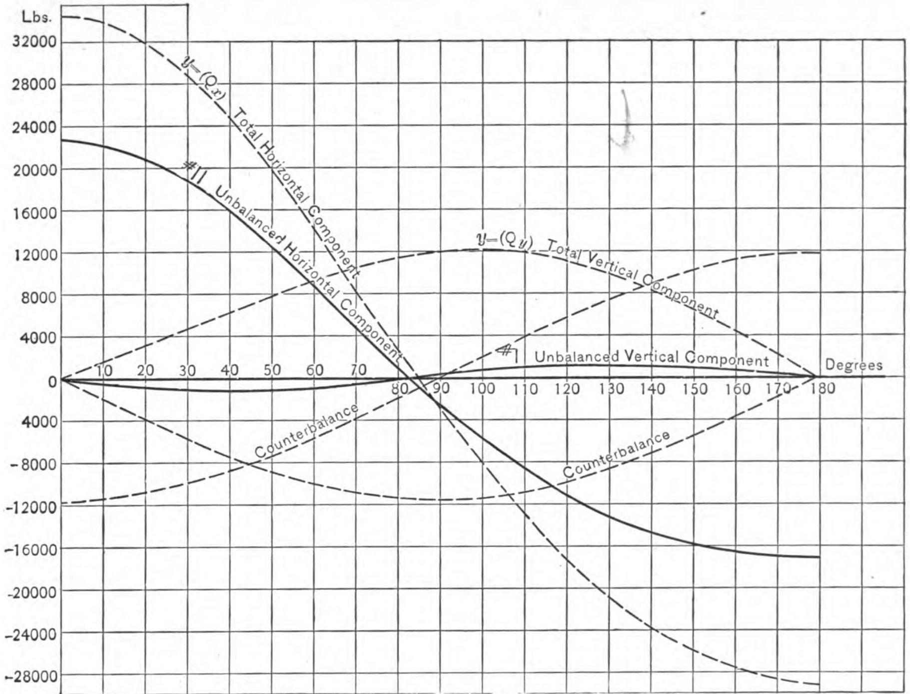
Counterbalance for Vertical Component of 6 × 8 Engine.

PLATE XIV.



Counterbalance for Horizontal Component of 6 × 8 Engine.

PLATE XV.



Counterbalances for 131,000-lb. Passenger Locomotive.