A TREATISE

ON

PLANE AND SPHERICAL TRIGONOMETRY,

AND ITS APPLICATIONS TO

ASTRONOMY AND GEODESY,

WITH

NUMEROUS EXAMPLES.

BY

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PREFACE.

THE present treatise on Plane and Spherical Trigonometry is designed as a text-book for Colleges, Scientific Schools, and Institutes of Technology. The aim has been to present the subject in as concise a form as is consistent with clearness, to make it attractive and easily intelligible to the student, and at the same time to present the fullest course of Trigonometry which is usually given in the best Technological Schools.

Considerable care has been taken to instruct the student in the theory and use of Logarithms, and their practical application to the solution of triangles. It is hoped that the work may commend itself, not only to those who wish to confine themselves to the numerical calculations which occur in Trigonometry, but also to those who intend to pursue the study of the higher mathematics.

The examples are very numerous and are carefully selected. Many are placed in immediate connection with the subject-matter which they illustrate. The numerical solution of triangles has received much attention, each case being treated in detail. The examples at the ends of the chapters have been carefully graded, beginning with those which are easy, and extending to those which are more and more difficult. These examples illustrate every part of the subject, and are intended to test, not only the student's knowledge of the usual methods of computation, but his ability to grasp them in the many forms they may assume in practical applications. Among these examples are some of the most elegant theorems in Plane and Spherical Trigonometry.

The Chapters on De Moivre's Theorem, and Astronomy, Geodesy, and Polyedrons, will serve to introduce the student to some of the higher applications of Trigonometry, rarely found in American text-books.

In writing this book, the best English and French authors have been consulted. I am indebted especially to the works of Todhunter, Casey, Lock, Hobson, Clarke, Eustis, Snowball, M'Clelland and Preston, Smith, and Serret.

It remains for me to express my thanks to my colleagues, Prof. R. W. Prentiss for reading the MS., and Mr. I. S. Upson for reading the proof-sheets.

Any corrections or suggestions, either in the text or the examples, will be thankfully received.

E. A. B.

RUTGERS COLLEGE,
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A

TREATISE ON TRIGONOMETRY.

PART I.

PLANE TRIGONOMETRY.

CHAPTER I.

MEASUREMENT OF ANGLES.

1. Trigonometry is that branch of mathematics which treats (1) of the solution of plane and spherical triangles, and (2) of the general relations of angles and certain functions of them called the *trigonometric functions.*

Plane Trigonometry comprises the solution of plane triangles and investigations of plane angles and their functions.

Trigonometry was originally the science which treated only of the sides and angles of plane and spherical triangles; but it has been recently extended so as to include the analytic treatment of all theorems involving the consideration of angular magnitudes.

2. The Measure of a Quantity.—All measurements of lines, angles, etc., are made in terms of some fixed standard or *unit,* and *the measure of a quantity* is the number of times the quantity contains the unit.

It is evident that the same quantity will be represented by different numbers when different units are adopted. For example, the distance of a mile will be represented by the number 1 when a mile is the unit of length, by the number 1760 when a yard is the unit of length, by the number 5280 when a foot is the unit of length, and so on. In like manner; the number expressing the magnitude of an angle will depend on the unit of angle.

EXAMPLES.

1. What is the measure of $2\frac{1}{2}$ miles when a yard is the unit?

$$2\tfrac{1}{2} \text{ miles} = \tfrac{5}{2} \times 1760 \text{ yards}$$
$$= 4400 \text{ yards} = 4400 \times 1 \text{ yard}.$$

∴ the measure is 4400 when a yard is the unit.

2. What is the measure of a mile when a chain of 66 feet is the unit? *Ans.* 80.

3. What is the measure of 2 acres when a square whose side is 22 yards is the unit? *Ans.* 20.

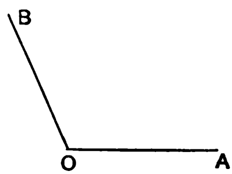
4. The measure of a certain field is 44 and the unit is 1100 square yards; express the area of the field in acres. *Ans.* 10 acres.

5. If 7 inches be taken as the unit of length, by what number will 15 feet 2 inches be represented? *Ans.* 26.

6. If 192 square inches be represented by the number 12, what is the unit of linear measurement? *Ans.* 4 inches.

3. Angles. — *An angle* is the opening between two straight lines drawn from the same point. The point is called the *vertex* of the angle, and the straight lines are called the *sides* of the angle.

An angle may be generated by revolving a line from coincidence with another line about a fixed point. The initial and final positions of the line are the sides of the angle; the amount of revolution measures the magnitude of the angle; and the angle may be traced out by any number of revolutions of the line.

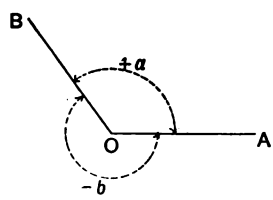


Thus, to form the angle AOB, OB may be supposed to have revolved from OA to OB; and it is obvious that OB may go on revolving until it comes into the same position OB as many times as we please; the angle AOB, having the same bounding lines OA and OB, may therefore be greater than 2, 4, 8, or any number of right angles.

The line OA from which OB moves is called the *initial line*, and OB in its final position, the *terminal line*. The revolving line OB is called the *generatrix*. The point O is called the *origin*, *vertex*, or *pole*.

4. Positive and Negative Angles. — We supposed in Art. 3 that OB revolved in the direction opposite to that of the hands of a watch. But angles may, of course, be described by a line revolving in the same direction as the hands of a watch, and it is often necessary to distinguish between the two directions in which angles may be measured from the same fixed line. This is conveniently effected by adopting the convention that angles measured in one direction shall be considered positive, and angles measured in the opposite direction, negative. In all branches of mathematics angles described by the revolution of a straight line in the direction *opposite* to that in which the hands of a watch move are usually considered *positive*, and all angles described by the revolution of a straight line in the *same* direction as the hands of a watch move are considered *negative*.

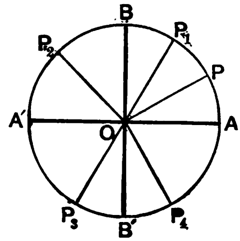
Thus, the revolving line OB starts from the initial line OA. When it revolves in the direction *contrary* to that of the hands of a watch, and comes into the position OB, it traces out the *positive* angle AOB (marked $+a$); and when it revolves in the *same* direction as the hands of a watch, it traces the *negative* angle AOB (marked $-b$).



The revolving line is always considered negative.

5. The Measure of Angles. — An angle is measured by the arc of a circle whose centre is at the vertex of the angle and whose ends are on the sides of the angle (Geom., Art. 236).

Let the line OP of fixed length generate an angle by revolving in the positive direction round a fixed point O from an initial position OA. Since OP is of constant length, the point P will trace out the circumference ABA′B′ whose centre is O. The two perpendicular diameters AA′ and BB′ of this circle will inclose the four right angles AOB, BOA′, A′OB′, and B′OA.



The circumference is divided at the points A, B, A′, B′ into four *quadrants*, of which

AB	is called	the	*first quadrant.*
BA′	"	"	*second quadrant.*
A′B′	"	"	*third quadrant.*
B′A	"	"	*fourth quadrant.*

In the figure, the angle AOP_1, between the initial line OA and the revolving line OP_1, is less than a right angle, and is said to be *an angle in the first quadrant.* AOP_2 is greater than one and less than two right angles, and is said to be *an angle in the second quadrant.* AOP_3 is greater than two and less than three right angles, and is said to be *an angle in the third quadrant.* AOP_4 is greater than three and less than four right angles, and is said to be *an angle in the fourth quadrant.*

When the revolving line returns to the initial position OA, the angle AOA is an angle of four right angles. By supposing OP to continue revolving, the angle described will become greater than an angle of four right angles. Thus, when OP coincides with the lines OB, OA′, OB′, OA, in the second revolution, the angles described, measured from the beginning of the first revolution, are angles of five right angles, six right angles, seven right angles, eight right angles, respectively, and so on. By the continued revolution of OP the angle between the initial line OA and the revolving line OP may become of any magnitude whatever.

In the same way OP may revolve in the *negative* direction about O any number of times, generating a *negative* angle; and this negative angle may obviously have any magnitude whatever.

The angle AOP may be the geometric representative of any of the Trigonometric angles formed by any number of complete revolutions, either in the *positive* direction added to the *positive* angle AOP, or in the *negative* direction added to the *negative* angle AOP. In all cases the angle is said to be in the quadrant indicated by its terminal line.

There are three methods of measuring angles, called respectively the Sexagesimal, the Centesimal, and the Circular methods.

6. The Sexagesimal Method. — This is the method in general use. In this method the right angle is divided into 90 equal parts, each of which is called a *degree.* Each degree is subdivided into 60 equal parts, each of which is called a *minute.* Each minute is subdivided into 60 equal parts, each of which is called a *second.* Then the magnitude of an angle is expressed by the number of degrees, minutes, and seconds which it contains. Degrees, minutes, and seconds are denoted respectively by the symbols $^\circ$, $'$, $''$: thus, to represent 18 degrees, 6 minutes, 34.58 seconds, we write

$$18^\circ\ 6'\ 34''.58.$$

A degree of arc is $\frac{1}{360}$ of the circumference to which the arc belongs. The degree of arc is subdivided in the same manner as the degree of angle.

Then 1 circumference $= 360^\circ = 21600' = 1296000''$.
1 quadrant or right angle $= 90^\circ$.

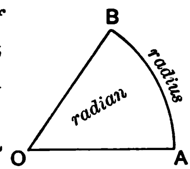
Instruments used for measuring angles are subdivided accordingly.

7. The Centesimal or Decimal Method. — In this method the right angle is divided into 100 equal parts, each of which is called a *grade*. Each grade is subdivided into 100 equal parts, each of which is called a *minute*. Each minute is subdivided into 100 equal parts, each of which is called a *second*. The magnitude of an angle is then expressed by the number of grades, minutes, and seconds which it contains. Grades, minutes, and seconds are denoted respectively by the symbols ^{g}, $^\backslash$, $^{\backslash\backslash}$: thus, to represent 34 grades, 48 minutes, 86.47 seconds, we write

$$34^g\ 48^\backslash\ 86^{\backslash\backslash}.47.$$

The centesimal or decimal method was proposed by the French mathematicians in the beginning of the present century. But although it possesses many advantages over the established method, they were not considered sufficient to counterbalance the enormous labor which would have been necessary to rearrange all the mathematical tables, books of reference, and records of observations, which would have to be transferred into the decimal system before its advantages could be felt. Thus, the centesimal method has never been used even in France, and in all probability never will be used in practical work.

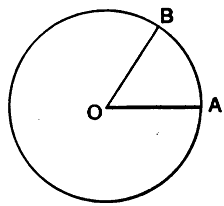
8. The Circular Measure. — The *unit of circular measure* is the angle subtended at the centre of a circle by an arc equal in length to the radius.



This unit of circular measure is called a *radian*.

Let O be the centre of a circle whose radius is r.

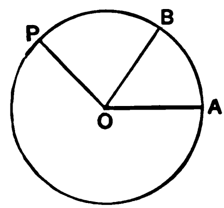
Let the arc AB be equal to the radius $OA = r$.



Then, since angles at the centre of a circle are in the same ratio as their intercepted arcs (Geom., Art. 234), and since the ratio of the circumference of a circle to its diameter is $\pi = 3.14159265$ (Geom., Art. 436),

$$\therefore \text{ angle AOB} : 4 \text{ rt. angles} :: \text{arc AB} : \text{circumference},$$
$$:: r : 2\pi r :: 1 : 2\pi.$$
$$\therefore \text{ angle AOB} = \frac{4 \text{ rt. angles}}{2\pi} = \frac{2 \text{ rt. angles}}{\pi}.$$
$$\therefore \text{ a radian} = \text{angle AOB} = \frac{180^\circ}{3.14159265} = 57^\circ.2957795$$
$$= 3437'.74677 = 206264''.806.$$

Therefore, *the radian is the same for all circles*, and $= 57^\circ.2957795$.



Let ABP be any circle; let the angle AOB be the radian; and let AOP be any other angle.

Then arc AB = radius OA.

$\therefore$ angle AOP : angle AOB :: arc AP : arc AB;

or angle AOP : radian :: arc AP : radius.

$$\therefore \text{ angle AOP} = \frac{\text{arc AP}}{\text{radius}} \times \text{radian}.$$

The *measure* of any quantity is the number of times it contains the *unit* of measure (Art. 2).

$$\therefore \text{ the circular measure of angle AOP} = \frac{\text{arc AP}}{\text{radius}}.$$

NOTE 1. — The student will notice that a radian is a little less than an angle of an equilateral triangle, *i.e.*, of 60°.

Angles expressed in circular measure are usually denoted by Greek letters, $\alpha, \beta, \gamma, \ldots, \phi, \theta, \psi, \ldots$.

The circular measure is employed in the various branches of Analytical Mathematics, in which the angle under consideration is almost always expressed by a *letter*.

NOTE 2. — The student cannot too carefully notice that unless an *angle* is obviously referred to, the letters $\alpha, \beta, \ldots, \theta, \phi, \ldots$ stand for *mere numbers*. Thus, *π stands for a number, and a number only*, *viz.*, 3.14159 ..., but in the expression '*the angle π*,' that is, 'the angle 3.14159 ...,' there must be *some unit* understood. The *unit understood* here is a radian, and therefore 'the angle π' stands for 'π radians' or 3.14159 ... radians, that is, *two right angles*.

Hence, *when an angle is referred to, π is a very convenient abbreviation for two right angles*.

So also 'the angle α or θ' means 'α radians or θ radians.'

The units in the three systems, when expressed in terms of one common standard, two right angles, stand thus:

The unit in the Sexagesimal Method $= \frac{1}{180}$ of 2 right angles.

" " " " Centesimal " $= \frac{1}{200}$ " " " "

" " " " Circular " $= \frac{1}{\pi}$ " " " "

If D, G, and θ denote the number of degrees, grades, and radians respectively in any angle, then

$$\frac{D}{180} = \frac{G}{200} = \frac{\theta}{\pi}, \tag{1}$$

because each fraction is the ratio of the angle to two right angles.

9. Comparison of the Sexagesimal and Centesimal Measures of an Angle. — Although the centesimal method was never in general use among mathematicians, and is now totally abandoned everywhere, yet it still possesses some interest, as it shows the application of the decimal system to the measurement of angles.

From (1) of Art. 8 we have

$$\frac{D}{180} = \frac{G}{200}.$$

$$\therefore\ D = \frac{9}{10}G, \text{ and } G = \frac{10}{9}D.$$

EXAMPLES.

1. Express 49° 15′ 35″ in centesimal measure.

First express the angle in degrees and decimals of a degree thus:

$$\begin{array}{r} 60)\ \underline{35''\quad} \\ 60)\ \underline{15'.58\dot{3}\quad} \\ 49^\circ.2597\dot{2} \\ \underline{10} \\ 9)\ \underline{492.5972\dot{2}\quad} \\ 54^g.733024\cdots. \end{array}$$

$$\therefore\ 49^\circ\ 15'\ 35'' = 54^g\ 73^{\backprime}\ 30^{\backprime\backprime}.24\cdots.$$

2. Express $87^g\ 2^{\backslash}\ 25^{\backslash\backslash}$ in degrees, etc.

First express the angle in grades and decimals of a grade thus:

$$\begin{array}{r} 87^g\ 2^{\backslash}\ 25^{\backslash\backslash} = 87^g.0225 \\ .9 \\ \hline 78.32025 \\ 60 \\ \hline 19.215 \\ 60 \\ \hline 12.9 \end{array}$$

$$\therefore\ 87^g\ 2^{\backslash}\ 25^{\backslash\backslash} = 78^\circ\ 19'\ 12''.9.$$

Find the number of grades, minutes, and seconds in the following angles:

3. $51^\circ\ 4'\ 30''$. *Ans.* $56^g\ 75^{\backslash}\ 0^{\backslash\backslash}$.
4. $45^\circ\ 33'\ 3''$. $50^g\ 61^{\backslash}\ 20^{\backslash\backslash}.37$.
5. $27^\circ\ 15'\ 46''$. $30^g\ 29^{\backslash}\ 19^{\backslash\backslash}.75\cdots$.
6. $157^\circ\ 4'\ 9''$. $174^g\ 52^{\backslash}\ 12^{\backslash\backslash}.962\cdots$.

Find the number of degrees, minutes, and seconds in the following angles:

7. $19^g\ 45^{\backslash}\ 95^{\backslash\backslash}$. *Ans.* $17^\circ\ 30'\ 48''.78$.
8. $124^g\ 5^{\backslash}\ 8^{\backslash\backslash}$. $111^\circ\ 38'\ 44''.592$.
9. $55^g\ 18^{\backslash}\ 35^{\backslash\backslash}$. $49^\circ\ 39'\ 54''.54$.

10. Comparison of the Sexagesimal and Circular Measures of an Angle.

From (1) of Art. 8 we have

$$\frac{D}{180} = \frac{\theta}{\pi}.$$

$$\therefore\ D = \frac{180\,\theta}{\pi}, \text{ and } \theta = \frac{D\pi}{180}.$$

EXAMPLES.

1. Find the number of degrees in the angle whose circular measure is $\frac{1}{2}$.

Here $$\theta = \frac{1}{2}.$$

$$\therefore\ D = \frac{180}{\pi} \times \frac{1}{2} = \frac{90}{\pi}$$

$$= \frac{90 \times 7}{22} = 28^\circ\ 38'\ 10''\tfrac{10}{11},$$

where $\frac{22}{7}$ is used for π.

2. Find the circular measure of the angle $59^\circ\ 52'\ 30''$.

Express the angle in degrees and decimals of a degree thus:

$$\begin{array}{r} 60)\underline{52.5} \\ 59.875 \end{array}.$$

$$\therefore\ \theta = \tfrac{59.875}{180}\pi = (.333\cdots)\,\pi = 1.0453\cdots.$$

3. Express, in degrees, the angles whose circular measures are $\frac{\pi}{2}$, $\frac{\pi}{3}$, $\frac{\pi}{4}$, $\frac{\pi}{6}$, $\frac{2}{3}\pi$.

Note 1. — The student should especially accustom himself to express readily in circular measure an angle which is given in degrees.

4. Express in circular measure the following angles:

60°, $22^\circ\ 30'$, $11^\circ\ 15'$, 270°. *Ans.* $\frac{\pi}{3}$, $\frac{\pi}{8}$, $\frac{\pi}{16}$, $\frac{3\pi}{2}$.

5. Express in circular measure $3^\circ\ 12'$, and find to seconds the angle whose circular measure is .8.

$\left(\text{Take } \pi = \frac{22}{7}.\right)$ *Ans.* $\frac{4\pi}{225}$, $45^\circ\ 49'\ 5''\tfrac{5}{11}$.

6. One angle of a triangle is 45°, and the circular measure of another is 1.5. Find the third angle in degrees.

Ans. $49^\circ\ 5'\ 27''\tfrac{3}{11}$.

Note 2. — Questions in which angles are expressed in different systems of measurement are easily solved by expressing each angle in right angles.

7. The sum of the measure of an angle in degrees and twice its measure in radians is $23\frac{2}{7}$; find its measure in degrees ($\pi = \frac{22}{7}$).

Let the angle contain x right angles.

Then the measure of the angle in degrees $= 90x$.

" " " " " " " radians $= \frac{\pi}{2}x$.

$$\therefore\ 90x + \pi x = 23\tfrac{2}{7};$$

$$\therefore\ 90x + \tfrac{22}{7}x = \frac{163}{7};$$

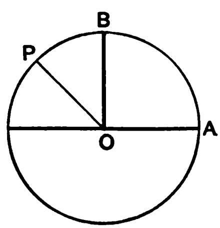
$$\therefore\ 652x = 163, \quad \therefore\ x = \frac{1}{4}.$$

$\therefore$ the angle is $\frac{1}{4}$ of $90° = 22\frac{1}{2}°$.

8. The difference between two angles is $\frac{\pi}{9}$, and their sum is $56°$; find the angles in degrees. *Ans.* $38°$, $18°$.

11. General Measure of an Angle. — In Euclidian geometry and in practical applications of trigonometry, angles are generally considered to be less than two right angles; but in the theoretical parts of mathematics, angles are treated as quantities which may be of any magnitude whatever.

Thus, when we are told that an angle is in some particular quadrant, say the second (Art. 5), we know that the position in which the revolving line *stops* is in the second quadrant. But there is an unlimited number of *angles* having the same final position, OP.



The revolving line OP may pass from OA to OP, not only by describing the arc ABP, but by moving through a whole revolution *plus* the arc ABP, or through any number of revolutions *plus* the arc ABP.

For example, the final position of OP may represent *geometrically* all the following angles:

Angle AOP $=130°$, or $360°+130°$, or $720°+130°$, or $-360°+130°$, or $-720°+130°$, etc.

Let A be an angle between 0 and 90°, and let n be any *whole* number, positive or negative. Then

(1) $2n \times 180° + A$ represents algebraically an angle in the *first* quadrant.

(2) $2n \times 180° - A$ represents algebraically an angle in the *fourth* quadrant.

(3) $(2n+1)\ 180° - A$ represents algebraically an angle in the *second* quadrant.

(4) $(2n+1)\ 180° + A$ represents algebraically an angle in the *third* quadrant.

In circular measure the corresponding expressions are

(1) $2n\pi+\theta$, (2) $2n\pi-\theta$, (3) $(2n+1)\pi-\theta$, (4) $(2n+1)\pi+\theta$.

EXAMPLES.

State in which quadrant the revolving line will be after describing the following angles:

(1) 120°, (2) 340°, (3) 490°, (4) $-100°$,

(5) $-380°$, (6) $\frac{2}{3}\pi$, (7) $10\pi+\frac{\pi}{4}$.

12. Complement and Supplement of an Angle or Arc. — The *complement* of an angle or arc is the remainder obtained by subtracting it from a right angle or 90°.

The *supplement* of an angle or arc is the remainder obtained by subtracting it from *two right angles* or 180°.

Thus, the complement of A is $(90° - A)$.

The complement of 190° is $(90° - 190°) = -100°$.

The supplement of A is $(180° - A)$.

The supplement of 200° is $(180° - 200°) = -20°$.

The complement of $\frac{5}{4}\pi$ is $\left(\frac{\pi}{2} - \frac{5}{4}\pi\right) = -\frac{3}{4}\pi$.

The supplement of $\frac{3}{4}\pi$ is $(\pi - \frac{3}{4}\pi) = \frac{1}{4}\pi$.

EXAMPLES.

1. If 192 square inches be represented by the number 12, what is the unit of linear measurement? *Ans.* 4 inches.

2. If 1000 square inches be represented by the number 40, what is the unit of linear measurement? *Ans.* 5 inches.

3. If 2000 cubic inches be represented by the number 16, what is the unit of linear measurement? *Ans.* 5 inches.

4. The length of an Atlantic cable is 2300 miles and the length of the cable from England to France is 21 miles. Express the length of the first in terms of the second as unit. *Ans.* $109\frac{11}{21}$.

5. Find the measure of a miles when b yards is the unit. *Ans.* $\frac{1760a}{b}$.

6. The ratio of the area of one field to that of another is 20 : 1, and the area of the first is half a square mile. Find the number of square yards in the second. *Ans.* 77440.

7. A certain weight is 3.125 tons. What is its measure in terms of 4 cwt.? *Ans.* 15.625.

Express the following 12 angles in centesimal measure:

8. $42^\circ\ 15'\ 18''$. *Ans.* $46^{g}\ 95^{`}$.

9. $63^\circ\ 19'\ 17''$. $70^{g}.35^{`}\ 70^{``}.98\cdots$.

10. $103^\circ\ 15'\ 45''$. $114^{g}\ 73^{`}\ 61^{``}.1$.

11. $19^\circ\ 0'\ 18''$. $21^{g}\ 11^{`}\ 66^{``}.\dot{6}$.

12. $143^\circ\ 9'\ 0''$. $159^{g}\ 5^{`}\ 55^{``}.\dot{5}$.

13. $300^\circ\ 15'\ 58''$. $333^{g}\ 62^{`}\ 90^{``}.\dot{1}23456789\dot{0}$.

14. $27^\circ\ 41'\ 51''$. $30^{g}.775$.

15. $67^\circ.4325$. $74^{g}.925$.

16. $8^\circ\ 15'\ 27''$. $9^{g}\ 17^{`}\ 50^{``}$.

17. $97^\circ\ 5'\ 15''$. $107^{g}\ 87^{`}\ 50^{``}$.

18. $16^\circ\ 14'\ 19''$. $18^{g}\ 4^{`}\ 29^{``}\cdots$.

19. $132^\circ\ 6'$. $146^{g}\ 77^{`}\ 77^{``}.\dot{7}$.

Express the following 11 angles in degrees, minutes, and seconds:

20.	$105^g\ 52^{`}\ 75^{``}$.	*Ans.* $94°\ 58'\ 29''.1$.
21.	$82^g\ 9^{`}\ 54^{``}$.	$73°\ 53'\ 9''.096$.
22.	$70^g\ 15^{`}\ 92^{``}$.	$63°\ 8'\ 35''.808$.
23.	$15^g\ 0^{`}\ 15^{``}$.	$13°\ 30'\ 4''.86$.
24.	$154^g\ 7^{`}\ 24^{``}$.	$138°\ 39'\ 54''.576$.
25.	$324^g\ 13^{`}\ 88^{``}.7$.	$291°\ 43'\ 29''.9388$.
26.	$10^g\ 42^{`}\ 50^{``}$.	$9°\ 22'\ 57''$.
27.	$20^g\ 77^{`}\ 50^{``}$.	$18°\ 41'\ 51''$.
28.	$8^g\ 75^{`}$.	$7°\ 52'\ 30''$.
29.	$170^g\ 45^{`}\ 35^{``}$.	$153°\ 24'\ 29''.34$.
30.	$24^g\ 0^{`}\ 25^{``}$.	$21°\ 36'\ 8''.1$.

Express in circular measure the following angles:

31. $315°$, $24°\ 13'$. *Ans.* $\frac{7}{4}\pi$, $\frac{1453\pi}{10800}$.

32. $95°\ 20'$, $12°\ 5'\ 4''$. $\frac{143\pi}{270}$, $\frac{2719\pi}{40500}$.

33. $22\frac{1}{2}°$, $1°$, $57°.295$. $\frac{\pi}{8}$, $\frac{\pi}{180}$, 1 radian.

34. $120°$, $45°$, $270°$. 2.09439, $\frac{\pi}{4}$, $\frac{3}{2}\pi$.

35. $360°$, $3\frac{1}{2}$ rt. angles. 2π, $\frac{7}{4}\pi$.

Express in degrees, etc., the angles whose circular measures are:

36. $\frac{5}{8}\pi$, $\frac{2}{3}\pi$, $\frac{1}{2}$. *Ans.* $112°.5$, $120°$, $\frac{90}{\pi}$ degrees.

37. $\frac{1}{4}$, $\frac{1}{6}$, $\frac{2}{3}$. $\frac{45}{\pi}$ degrees, $\frac{30}{\pi}$ degrees, $\frac{120}{\pi}$ degrees.

38. $\frac{5}{6}$, $.7854$. $47°\ 43'\ 38''\frac{2}{11}$, $45°$.

39. $4\frac{1}{2}$, $\frac{1}{12}\pi$, 2.504. *Ans.* 257° 49′ 43″.39, 15°, 143°.468.

40. .0234, 1.234, $\frac{2}{3}$. 1° 20′ 27″, 70° 42′ 11″, 38° 11′ 50″.

41. Find the number of radians in an angle at the centre of a circle of radius 25 feet, which intercepts an arc of $37\frac{1}{2}$ feet. *Ans.* $1\frac{1}{2}$.

42. Find the number of degrees in an angle at the centre of a circle of radius 10 feet, which intercepts an arc of 5π feet. *Ans.* 90°.

43. Find the number of right angles in an angle at the centre of a circle of radius $3\frac{2}{11}$ inches, which intercepts an arc of 2 feet. *Ans.* $4\frac{4}{5}$.

44. Find the length of the arc subtending an angle of $4\frac{1}{2}$ radians at the centre of a circle whose radius is 25 feet. *Ans.* $112\frac{1}{2}$ ft.

45. Find the length of an arc of 80° on a circle of 4 feet radius. *Ans.* $5\frac{37}{63}$ ft.

46. The angle subtended by the diameter of the Sun at the eye of an observer is 32′: find approximately the diameter of the Sun if its distance from the observer be 90 000 000 miles. *Ans.* 838 000 miles.

47. A railway train is travelling on a curve of half a mile radius at the rate of 20 miles an hour: through what angle has it turned in 10 seconds? *Ans.* $6\frac{4}{11}$ degrees.

48. If the radius of a circle be 4000 miles, find the length of an arc which subtends an angle of 1″ at the centre of the circle. *Ans.* About 34 yards.

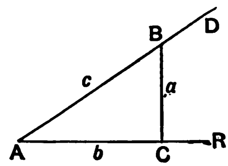
49. On a circle of 80 feet radius it was found that an angle of 22° 30′ at the centre was subtended by an arc 31 ft. 5 in. in length: hence calculate to four decimal places the numerical value of the ratio of the circumference of a circle to its diameter. *Ans.* $3.141\dot{6}$.

50. Find the number of radians in 10″ correct to four significant figures (use $\frac{355}{113}$ for π). *Ans.* .00004848.

CHAPTER II.

THE TRIGONOMETRIC FUNCTIONS.

13. Definitions of the Trigonometric Functions. — Let RAD be an angle; in AD, one of the lines containing the angle, take any point B, and from B draw BC perpendicular to the other line AR, thus forming a right triangle ABC, right-angled at C. Then denoting the angles by the capital letters A, B, C, respectively, and the three sides opposite these angles by the corresponding small italics, *a*, *b*, *c*,* we have the following definitions:



$\frac{a}{c} = \frac{\text{opposite side}}{\text{hypotenuse}}$ is called the *sine* of the angle A.

$\frac{b}{c} = \frac{\text{adjacent side}}{\text{hypotenuse}}$ is called the *cosine* of the angle A.

$\frac{a}{b} = \frac{\text{opposite side}}{\text{adjacent side}}$ is called the *tangent* of the angle A.

$\frac{b}{a} = \frac{\text{adjacent side}}{\text{opposite side}}$ is called the *cotangent* of the angle A.

$\frac{c}{b} = \frac{\text{hypotenuse}}{\text{adjacent side}}$ is called the *secant* of the angle A.

$\frac{c}{a} = \frac{\text{hypotenuse}}{\text{opposite side}}$ is called the *cosecant* of the angle A.

If the cosine of A be subtracted from unity, the remainder is called the *versed sine* of A. If the sine of A be sub-

* The letters *a*, *b*, *c* are *numbers*, being the number of times the lengths of the sides contain some chosen unit of length.

tracted from unity, the remainder is called the *coversed sine* of A; the latter term is hardly ever used in practice.

The words sine, cosine, etc., are abbreviated, and the functions of an angle A are written thus: sin A, cos A, tan A, cot A, sec A, cosec A, vers A, covers A.

The following is the verbal enunciation of these definitions:

The sine of an angle is the ratio of the opposite side to the hypotenuse; or $\sin A = \frac{a}{c}$.

The cosine of an angle is the ratio of the adjacent side to the hypotenuse; or $\cos A = \frac{b}{c}$.

The tangent of an angle is the ratio of the opposite side to the adjacent side; or $\tan A = \frac{a}{b}$.

The cotangent of an angle is the ratio of the adjacent side to the opposite side; or $\cot A = \frac{b}{a}$.

The secant of an angle is the ratio of the hypotenuse to the adjacent side; or $\sec A = \frac{c}{b}$.

The cosecant of an angle is the ratio of the hypotenuse to the opposite side; or $\operatorname{cosec} A = \frac{c}{a}$.

The versed sine of an angle is unity minus the cosine of the angle; or $\operatorname{vers} A = 1 - \cos A = 1 - \frac{b}{c}$.

The coversed sine of an angle is unity minus the sine of the angle; or $\operatorname{covers} A = 1 - \sin A = 1 - \frac{a}{c}$.

These ratios are called *Trigonometric Functions.* The student should carefully commit them to memory, as upon them is founded the whole theory of Trigonometry.

These functions are, it will be observed, not *lengths,* but *ratios* of one length to another; that is, they are *abstract numbers*, simply numerical quantities; and they remain unchanged so long as the angle remains unchanged, as will be proved in Art. 14.

It is clear from the above definitions that

$$\operatorname{cosec} A = \frac{1}{\sin A}, \text{ or } \sin A = \frac{1}{\operatorname{cosec} A},$$

$$\sec A = \frac{1}{\cos A}, \text{ or } \cos A = \frac{1}{\sec A},$$

$$\tan A = \frac{1}{\cot A}, \text{ or } \cot A = \frac{1}{\tan A}.$$

The powers of the Trigonometric functions are expressed as follows:

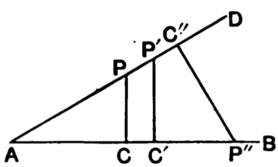
$(\sin A)^2$ is written $\sin^2 A$,

$(\cos A)^3$ is written $\cos^3 A$,

and so on.

NOTE. — The student must notice that '$\sin A$' is a *single symbol*, the *name* of a *number*, or *fraction* belonging to the angle A. Also $\sin^2 A$ is an abbreviation for $(\sin A)^2$, *i.e.*, for $(\sin A) \times (\sin A)$. Such abbreviations are used for *convenience*.

14. The Trigonometric Functions are always the Same for the Same Angle. — Let BAD be any angle; in AD take P, P′, any two points, and draw PC, P′C′ perpendicular to AB. Take P″, any point in AB, and draw P″C″ perpendicular to AD.



Then the three triangles PAC, P′AC′, P″AC″ are equiangular, since they are right-angled, and have a common angle at A: therefore they are similar.

$$\therefore \frac{PC}{AP} = \frac{P'C'}{AP'} = \frac{P''C''}{AP''}.$$

But each of these ratios is the sine of the angle A. Thus, $\sin A$ is the same *whatever* be the position of the point P on *either* of the lines containing the angle A.

Therefore sin A is always the same. A similar proof may be given for each of the other functions.

In the right triangle of Art. 13, show that

$$a = c \sin A \quad = c \cos B = b \tan A = b \cot B,$$
$$b = a \cot A \quad = a \tan B = c \cos A = c \sin B,$$
$$c = a \operatorname{cosec} A = a \sec B = b \sec A = b \operatorname{cosec} B.$$

NOTE. — These results should be carefully noticed, as they are of frequent use in the solution of right triangles and elsewhere.

EXAMPLES.

1. Calculate the value of the functions, sine, cosine, etc., of the angle A in the right triangles whose sides a, b, c are respectively (1) 8, 15, 17; (2) 40, 9, 41; (3) 196, 315, 371; (4) 480, 31, 481; (5) 1700, 945, 1945.

Ans. (1) $\sin A = \frac{8}{17}$, $\cos A = \frac{15}{17}$, $\tan A = \frac{8}{15}$, etc.;
(2) $\sin A = \frac{40}{41}$, $\cos A = \frac{9}{41}$, etc.;
(3) $\sin A = \frac{28}{53}$, $\tan A = \frac{28}{45}$, etc.;
(4) $\sin A = \frac{480}{481}$, $\tan A = \frac{480}{31}$, etc.;
(5) $\sin A = \frac{340}{389}$, $\tan A = \frac{340}{189}$, etc.

In a right triangle, given:

2. $a = \sqrt{m^2 + n^2}$, $b = \sqrt{2mn}$; calculate sin A. *Ans.* $\frac{\sqrt{m^2+n^2}}{m+n}$.

3. $a = \sqrt{m^2 - mn}$, $b = n$; calculate sec A. $\frac{m-n}{n}$.

4. $a = \sqrt{m^2 + mn}$, $c = m + n$; calculate tan A. $\sqrt{\frac{m^2+mn}{mn+n^2}}$.

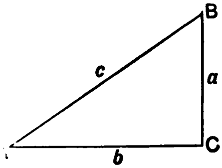
5. $a = 2mn$, $b = m^2 - n^2$; calculate cos A. $\frac{m^2-n^2}{m^2+n^2}$.

6. $\sin A = \frac{3}{5}$, $c = 200.5$; calculate a. 120.3.

7. $\cos A = .44$, $c = 30.5$; calculate b. 13.42.

8. $\tan A = \frac{11}{3}$, $b = \frac{27}{11}$; calculate c. $\frac{9}{11}\sqrt{130}$.

15. Functions of Complemental Angles. — In the rt. $\triangle$ABC we have



$$\sin A = \frac{a}{c}, \text{ and } \cos B = \frac{a}{c}. \quad \text{(Art. 13.)}$$

$$\therefore \sin A = \cos B.$$

But B is the complement of A, since their sum is a right angle, or 90°; *i.e.*, $B = 90° - A$.

$$\therefore \sin A = \cos B = \cos(90° - A) = \frac{a}{c}.$$

Also,

$$\cos A = \sin B = \sin(90° - A) = \frac{b}{c},$$

$$\tan A = \cot B = \cot(90° - A) = \frac{a}{b},$$

$$\cot A = \tan B = \tan(90° - A) = \frac{b}{a},$$

$$\sec A = \operatorname{cosec} B = \operatorname{cosec}(90° - A) = \frac{c}{b},$$

$$\operatorname{cosec} A = \sec B = \sec(90° - A) = \frac{c}{a},$$

$$\operatorname{vers} A = \operatorname{covers} B = \operatorname{covers}(90° - A) = 1 - \frac{b}{c},$$

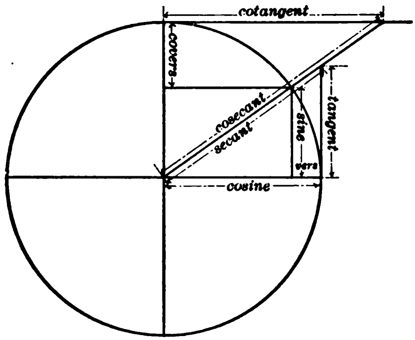
$$\operatorname{covers} A = \operatorname{vers} B = \operatorname{vers}(90° - A) = 1 - \frac{a}{c}.$$

Therefore the *sine, tangent, secant, and versed sine of an angle are equal respectively to the cosine, cotangent, cosecant, and coversed sine of the complement of the angle.*

16. Representation of the Trigonometric Functions by Straight Lines. — The Trigonometric functions were formerly defined as being certain *straight lines* geometrically connected with the *arc* subtending the angle at the centre of a circle of given radius.

Thus, let AP be the arc of a circle subtending the angle AOP at the centre.

Draw the tangents AT, BT′ meeting OP produced to T′, and draw PC, PD ⊥ to OA, OB.



Then	PC was called the sine of the arc AP.			
	OC	"	cosine	"
	AT	"	tangent	"
	BT′	"	cotangent	"
	OT	"	secant	"
	OT′	"	cosecant	"
	AC	"	versed sine	"
	BD	"	coversed sine	"

Since any arc is the measure of the angle at the centre which the arc subtends (Art. 5), the above functions of the arc AP are also functions of the angle AOP.

It should be noticed that the old functions of the *arc* above given, when divided by the radius of the circle, become the modern functions of the *angle* which the arc subtends at the centre. If, therefore, the radius be taken as *unity*, the old functions of the arc AP become the modern functions of the angle AOP.

Thus, representing the arc AP, or the angle AOP by θ, we have, when $OA = OP = 1$,

$$\sin\theta = \frac{PC}{OP} = \frac{PC}{1} = PC,$$

$$\tan\theta = \frac{AT}{OA} = \frac{AT}{1} = AT,$$

and similarly for the other functions.

Therefore, in a circle whose radius is unity, the *Trigonometric functions* of an arc, or of the angle at the centre measured by that arc, may be defined as follows:

The sine is the perpendicular let fall from one extremity of the arc upon the diameter passing through the other extremity.

The cosine is the distance from the centre of the circle to the foot of the sine.

The tangent is the line which touches one extremity of the arc and is terminated by the diameter produced passing through the other extremity.

The secant is the portion of the diameter produced through one extremity of the arc which is intercepted between the centre and the tangent at the other extremity.

The versed sine is the part of the diameter intercepted between the beginning of the arc and the foot of the sine.

Since the lines PD or OC, BT′, OT′, and BD are respectively the sine, tangent, secant, and versed sine of the arc BP, which (Art. 12) is the complement of AP, we see that *the cosine, the cotangent, the cosecant, and the coversed sine of an arc are respectively the sine, the tangent, the secant, and the versed sine of its complement.*

EXAMPLES.

1. Prove $\tan A \sin A + \cos A = \sec A$.
2. " $\cot A \cos A + \sin A = \operatorname{cosec} A$.
3. " $(\tan A - \sin A)^2 + (1 - \cos A)^2 = (\sec A - 1)^2$.
4. " $\tan A + \cot A = \sec A \operatorname{cosec} A$.
5. " $(\sin A + \cos A) \div (\sec A + \operatorname{cosec} A) = \sin A \cos A$.
6. " $(1 + \tan A)^2 + (1 + \cot A)^2 = (\sec A + \operatorname{cosec} A)^2$.

7. Given $\tan A = \cot 2A$; find A.

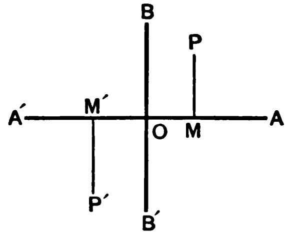
8. " $\sin A = \cos 3A$; find A.

9. " $\sin A = \cos(45° - \frac{1}{2}A)$; find A.

10. " $\tan A = \cot 6A$; find A.

11. " $\cot A = \tan(45° + A)$; find A.

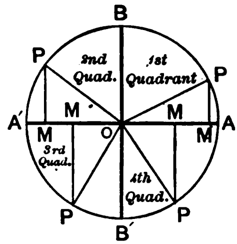
17. Positive and Negative Lines. — Let AA′ and BB′ be two perpendicular right lines intersecting at the point O. Then the position of any point in the line AA′ or BB′ will be determined if we know the distance of the point from O, and if we know also upon which side of O the point lies. It is therefore convenient to employ the algebraic signs + and −, so that if distances measured along the fixed line OA or OB from O in one direction be considered positive, distances measured along OA′ or OB′ in the opposite direction from O will be considered negative.



This *convention*, as it is called, is extended to lines parallel to AA′ and BB′; and it is customary to consider distances measured from BB′ towards the *right* and from AA′ *upwards* as *positive*, and consequently distances measured from BB′ towards the *left* and from AA′ *downwards* as *negative*.

18. Trigonometric Functions of Angles of Any Magnitude. — In the definitions of the trigonometric functions given in Art. 13 we considered only acute angles, *i.e.*, angles in the first quadrant (Art. 5), since the angle was assumed to be one of the acute angles of a right triangle. We shall now show that these definitions apply to angles of any magnitude, and that the functions vary in *sign* according to the *quadrant* in which the angle happens to be.

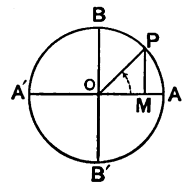
Let AOP be an angle of any magnitude formed by OP revolving from an initial position OA. Draw PM ⊥ to AA′. Consider OP as always positive. Let the angle AOP be denoted by A; then whatever be the magnitude of the angle A, the definitions of the trigonometric functions are



$$\sin A = \frac{MP}{OP},\quad \cos A = \frac{OM}{OP},\quad \tan A = \frac{MP}{OM},$$

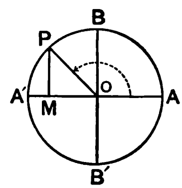
$$\sec A = \frac{OP}{OM},\quad \cot A = \frac{OM}{MP},\quad \operatorname{cosec} A = \frac{OP}{MP}.$$

I. When A lies in the 1st quadrant, MP is *positive* because measured from M *upwards*, OM is *positive* because measured from O towards the *right* (Art. 17), and OP is *positive*.



Hence in the *first* quadrant all the functions are *positive*.

II. When A lies in the 2d quadrant, as the *obtuse* angle AOP, MP is *positive* because measured from M *upwards*, OM is *negative* because measured from O towards the *left* (Art. 17), and OP is *positive*.



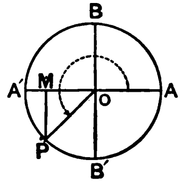
Hence in the *second* quadrant

$$\sin A = \frac{MP}{OP} \text{ is } \textit{positive};$$

$$\cos A = \frac{OM}{OP} \text{ is } \textit{negative};$$

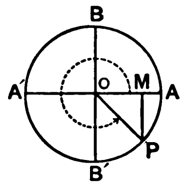
$$\tan A = \frac{MP}{OM} \text{ is } \textit{negative};$$

and therefore sec A and cot A are *negative*, and cosec A is *positive* (Art. 13).



III. When A lies in the 3d quadrant, as the *reflex* angle AOP, MP is *negative* because measured from M *downwards*, OM is *negative*, and OP is *positive*.

Hence in the *third* quadrant the ***sine***, ***cosine***, ***secant***, and ***cosecant***, are ***negative***, but the ***tangent*** and ***cotangent*** are ***positive***.



IV. When A lies in the 4th quadrant, as the reflex angle AOP, MP is *negative*, OM is *positive*, and OP is *positive*.

Hence in the *fourth* quadrant the ***sine***, ***tangent***, ***cotangent***, and ***cosecant*** are ***negative***, but the ***cosine*** and ***secant*** are ***positive***.

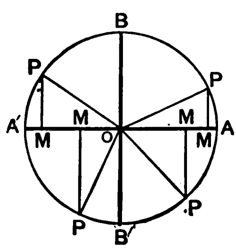
The signs of the different functions are shown in the annexed table.

QUADRANT.	I.	II.	III.	IV.
Sin and cosec	+	+	−	−
Cos and sec	+	−	−	+
Tan and cot	+	−	+	−

NOTE. — It is apparent from this table that the signs of all the functions in any quadrant are known when those of the sine and cosine are known. The tangent and cotangent are + or −, according as the sine and cosine have like or different signs.

19. Changes in the Value of the Sine as the Angle increases from 0° to 360°. — Let A denote the angle AOP described by the revolution of OP from its initial position OA through 360°. Then, PM being drawn perpendicular to AA′,

$$\sin A = \frac{MP}{OP},$$



whatever be the magnitude of the angle A.

When the angle A is 0°, P coincides with A, and MP is zero; therefore $\sin 0° = 0$.

As A increases from 0° to 90°, MP increases from zero to OB or OP, and is *positive;* therefore $\sin 90° = 1$.

Hence in the 1st quadrant $\sin A$ is *positive*, and increases from 0 to 1.

As A increases from 90° to 180°, MP decreases from OP to zero, and is *positive;* therefore $\sin 180° = 0$.

Hence in the 2d quadrant $\sin A$ is *positive*, and decreases from 1 to 0.

As A increases from 180° to 270°, MP increases from zero to OP, and is *negative;* therefore $\sin 270° = -1$.

Hence in the 3d quadrant $\sin A$ is *negative*, and decreases algebraically from 0 to -1.

As A increases from 270° to 360°, MP decreases from OP to zero, and is *negative;* therefore $\sin 360° = 0$.

Hence in the 4th quadrant $\sin A$ is *negative*, and increases algebraically from -1 to 0.

20. Changes in the Cosine as the Angle increases from 0° to 360°. — In the figure of Art. 19

$$\cos A = \frac{OM}{OP}.$$

When the angle A is 0°, P coincides with A, and $OM = OP$; therefore $\cos 0° = 1$.

As A increases from 0° to 90°, OM decreases from OP to zero and is *positive;* therefore $\cos 90° = 0$.

Hence in the 1st quadrant $\cos A$ is *positive*, and decreases from 1 to 0.

As A increases from 90° to 180°, OM increases from zero to OP, and is *negative;* therefore $\cos 180° = -1$.

Hence in the 2d quadrant $\cos A$ is *negative*, and decreases algebraically from 0 to -1.

As A increases from 180° to 270°, OM decreases from OP to zero, and is *negative;* therefore $\cos 270° = 0$.

Hence in the 3d quadrant cos A is *negative*, and increases algebraically from -1 to 0.

As A increases from 270° to 360°, OM increases from zero to OP, and is *positive;* therefore $\cos 360° = 1$.

Hence in the 4th quadrant cos A is *positive*, and increases from 0 to 1.

21. Changes in the Tangent as the Angle increases from 0° to 360°. — In the figure of Art. 19

$$\tan A = \frac{MP}{OM}.$$

When A is 0°, MP is zero, and OM = OP; therefore $\tan 0° = 0$.

As A increases from 0° to 90°, MP increases from zero to OP, and OM decreases from OP to zero, so that on both accounts tan A increases numerically; therefore $\tan 90° = \infty$.

Hence in the 1st quadrant tan A is *positive*, and increases from 0 to ∞.

As A increases from 90° to 180°, MP decreases from OP to zero, and is *positive*, OM becomes *negative* and decreases algebraically from zero to -1; therefore $\tan 180° = 0$.

Hence in the 2d quadrant tan A is *negative*, and increases algebraically from $-\infty$ to 0.

When A passes into the 2d quadrant, and is only just greater than 90°, tan A changes from $+\infty$ to $-\infty$.

As A increases from 180° to 270°, MP increases from zero to OP, and is *negative*, OM decreases from OP to zero, and is *negative;* therefore $\tan 270° = \infty$.

Hence in the 3d quadrant tan A is *positive*, and increases from 0 to ∞.

As A increases from 270° to 360°, MP decreases from OP to zero, and is *negative*, OM increases from zero to OP, and is *positive;* therefore $\tan 360° = 0$.

Hence in the 4th quadrant tan A is *negative*, and increases algebraically from $-\infty$ to 0.

The student is recommended to trace in a manner similar to the above the changes in the other functions, *i.e.*, the cotangent, secant, and cosecant, and to see that his results agree with those given in the following table.

22. Table giving the Changes of the Trigonometric Functions in the Four Quadrants.

QUADRANT.	I.	II.	III.	IV.
sin varies from	+ 0 to 1	+ 1 to 0	− 0 to − 1	− − 1 to 0
cos " "	+ 1 to 0	− 0 to − 1	− − 1 to 0	+ 0 to 1
tan " "	+ 0 to ∞	− − ∞ to 0	+ 0 to ∞	− − ∞ to 0
cot " "	+ ∞ to 0	− 0 to ∞	+ ∞ to 0	− 0 to − ∞
sec " "	+ 1 to ∞	− − ∞ to − 1	− −1 to − ∞	+ ∞ to 1
cosec " "	+ ∞ to 1	+ 1 to ∞	− − ∞ to − 1	− − 1 to − ∞
vers " "	+ 0 to 1	+ 1 to 2	+ 2 to 1	+ 1 to 0

NOTE 1. — The *cosecant, secant,* and *cotangent* of an angle A have the same *sign* as the *sine, cosine,* and *tangent* of A respectively.

The *sine* and *cosine* vary from 1 to − 1, passing through the value 0. They are *never greater than unity.*

The *secant* and *cosecant* vary from 1 to − 1, passing through the value ∞. They are *never numerically less than unity.*

The *tangent* and *cotangent* are unlimited in value. They have all values from − ∞ to + ∞.

The *versed sine* and *coversed sine* vary from 0 to 2, and are always positive.

The trigonometric functions change sign in passing through the values 0 and ∞, and through no other values.

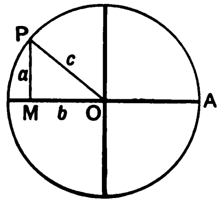
In the 1st quadrant the *functions* increase, and the *cofunctions* decrease.

NOTE 2. — From the results given in the above table, it will be seen that, if the value of a trigonometric function be given, we cannot fix on one angle to which it belongs exclusively.

Thus, if the given value of sin A be ½, we know since sin A passes through all values from 0 to 1 as A increases from 0° to 90°, that *one* value of A lies between 0°

and 90°. But since we also know that the value of sin A passes through all values between 1 and 0 as A increases from 90° to 180°, it is evident that there is *another* value of A between 90° and 180° for which sin A = ½.

23. Relations between the Trigonometric Functions of the Same Angle. — Let the radius start from the initial position OA, and *revolve* in either direction, to the position OP. Let θ denote the angle traced out, and let the lengths of the sides PM, MO, OP be denoted by the letters a, b, c.*



The following relations are evident from the definitions (Art. 13):

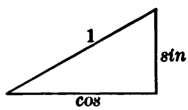
$$\operatorname{cosec}\theta = \frac{1}{\sin\theta},\quad \sec\theta = \frac{1}{\cos\theta},\quad \cot\theta = \frac{1}{\tan\theta}.$$

I. $$\tan\theta = \frac{\sin\theta}{\cos\theta}.$$

For $$\tan\theta = \frac{a}{b} = \frac{\frac{a}{c}}{\frac{b}{c}} = \frac{\sin\theta}{\cos\theta}.$$

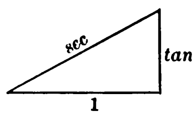
II. $\sin^2\theta + \cos^2\theta = 1.$

For $$\sin^2\theta + \cos^2\theta = \frac{a^2}{c^2} + \frac{b^2}{c^2} = \frac{a^2+b^2}{c^2} = 1.$$



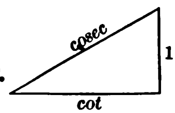
III. $\sec^2\theta = 1 + \tan^2\theta.$

For $$\sec^2\theta = \frac{c^2}{b^2} = \frac{b^2+a^2}{b^2} = 1 + \frac{a^2}{b^2} = 1 + \tan^2\theta.$$



IV. $\operatorname{cosec}^2\theta = 1 + \cot^2\theta.$

For $$\operatorname{cosec}^2\theta = \frac{c^2}{a^2} = \frac{a^2+b^2}{a^2} = 1 + \frac{b^2}{a^2} = 1 + \cot^2\theta.$$



Formulæ I., II., III., IV. are very important, and must be remembered.

* *a, b, c are numbers*, being the number of times the lengths of the sides contain some chosen unit of length.

24. Use of the Preceding Formulæ.

I. *To express all the other functions in terms of the sine.*

Since $\sin^2\theta + \cos^2\theta = 1, \quad \therefore \cos\theta = \pm\sqrt{1-\sin^2\theta}.$

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \pm\frac{\sin\theta}{\sqrt{1-\sin^2\theta}}.$$

$$\cot\theta = \frac{1}{\tan\theta} = \pm\frac{\sqrt{1-\sin^2\theta}}{\sin\theta}.$$

$$\sec\theta = \frac{1}{\cos\theta} = \pm\frac{1}{\sqrt{1-\sin^2\theta}}.$$

$$\operatorname{cosec}\theta = \frac{1}{\sin\theta}.$$

II. *To express all the other functions in terms of the tangent.*

Since $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$,

$$\sin\theta = \tan\theta\cos\theta = \frac{\tan\theta}{\sec\theta} = \pm\frac{\tan\theta}{\sqrt{1+\tan^2\theta}}.$$

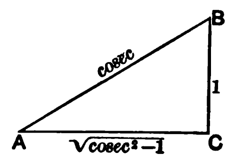
$$\cos\theta = \frac{1}{\sec\theta} = \pm\frac{1}{\sqrt{1+\tan^2\theta}}.$$

$$\cot\theta = \frac{1}{\tan\theta}. \quad \sec\theta = \pm\sqrt{1+\tan^2\theta}.$$

$$\operatorname{cosec}\theta = \frac{1}{\sin\theta} = \pm\frac{\sqrt{1+\tan^2\theta}}{\tan\theta}.$$

Similarly, any one of the functions of an angle may be expressed in terms of any other function of that angle. The sign of the radical will in all cases depend upon the quadrant in which the angle θ lies.

25. Graphic Method of finding All the Functions in Terms of One of them.



To express all the other functions in terms of the cosecant.

Construct a right triangle ABC, having the side BC = 1. Then

$$\operatorname{cosec} A = \frac{AB}{BC} = \frac{AB}{1} = AB.$$

$$\therefore\ AC = \pm\sqrt{\operatorname{cosec}^2 A - 1}.$$

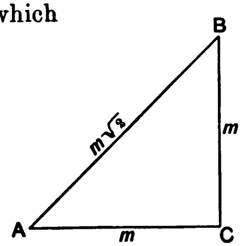
Now
$$\sin A = \frac{BC}{AB} = \frac{1}{\operatorname{cosec} A},$$

$$\cos A = \frac{AC}{AB} = \pm\frac{\sqrt{\operatorname{cosec}^2 A - 1}}{\operatorname{cosec} A},$$

$$\tan A = \frac{BC}{AC} = \pm\frac{1}{\sqrt{\operatorname{cosec}^2 A - 1}},$$

and similarly the other functions may be expressed in terms of cosec A.

26. To find the Trigonometric Functions of 45°. — Let ABC be an isosceles right triangle in which



$$CA = CB.$$

Then $CAB = CBA = 45°.$

Let $AC = m = CB.$

Then

$$\overline{AB}^2 = \overline{AC}^2 + \overline{CB}^2 = m^2 + m^2 = 2m^2.$$

$$\therefore\ AB = m\sqrt{2}.$$

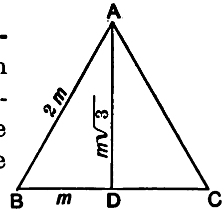
$$\therefore\ \sin 45° = \frac{BC}{AB} = \frac{m}{m\sqrt{2}} = \frac{1}{\sqrt{2}}.$$

$$\cos 45° = \frac{AC}{AB} = \frac{m}{m\sqrt{2}} = \frac{1}{\sqrt{2}}.$$

$$\tan 45° = \frac{BC}{AC} = \frac{m}{m} = 1. \quad \cot 45° = 1.$$

$$\sec 45° = \sqrt{2}. \quad \operatorname{cosec} 45° = \sqrt{2}.$$

27. To find the Trigonometric Functions of 60° and 30°. — Let AB be an equilateral triangle. Draw AD perpendicular to BC. Then AD bisects the angle BAC and the side BC. Therefore BAD = 30°, and ABD = 60°.



Let $\quad BA = 2m. \quad \therefore BD = m.$

Then $\quad AD = \sqrt{4m^2 - m^2} = m\sqrt{3}.$

$$\therefore \sin 60° = \frac{AD}{AB} = \frac{m\sqrt{3}}{2m} = \tfrac{1}{2}\sqrt{3}. \quad \therefore \operatorname{cosec} 60° = \frac{2}{\sqrt{3}}.$$

$$\cos 60° = \frac{BD}{BA} = \frac{1}{2}. \quad \therefore \sec 60° = 2.$$

$$\tan 60° = \frac{AD}{BD} = \frac{m\sqrt{3}}{m} = \sqrt{3}. \quad \therefore \cot 60° = \frac{1}{\sqrt{3}}.$$

$$\operatorname{vers} 60° = 1 - \cos 60° = 1 - \frac{1}{2} = \frac{1}{2}.$$

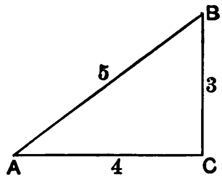
Also $\quad \sin 30° = \frac{BD}{AB} = \frac{m}{2m} = \frac{1}{2}. \quad \therefore \operatorname{cosec} 30° = 2.$

$$\cos 30° = \frac{AD}{AB} = \frac{m\sqrt{3}}{2m} = \tfrac{1}{2}\sqrt{3}. \quad \therefore \sec 30° = \frac{2}{\sqrt{3}}.$$

$$\tan 30° = \frac{DB}{DA} = \frac{m}{m\sqrt{3}} = \frac{1}{\sqrt{3}}. \quad \therefore \cot 30° = \sqrt{3}.$$

EXAMPLES.

1. Given $\sin\theta = \frac{3}{5}$; find the other trigonometric functions.



Let BAC be the angle, and BC be perpendicular to AC. Represent BC by 3, AB by 5, and consequently AC by $\sqrt{25 - 9} = 4$.

Then
$$\cos\theta = \frac{AC}{AB} = \frac{4}{5},$$

$$\tan\theta = \frac{BC}{AC} = \frac{3}{4},$$

$$\sec\theta = \frac{AB}{AC} = \frac{5}{4},$$

$$\operatorname{cosec}\theta = \frac{AB}{BC} = \frac{5}{3}.$$

2. Given $\sin\theta = \frac{3}{5}$; find $\tan\theta$ and $\operatorname{cosec}\theta$. *Ans.* $\frac{3}{4}, \frac{5}{3}$.

3. Given $\cos\theta = \frac{1}{3}$; find $\sin\theta$ and $\cot\theta$. $\frac{2}{3}\sqrt{2}, \frac{1}{2\sqrt{2}}$.

4. Given $\sec\theta = 4$; find $\cot\theta$ and $\sin\theta$. $\frac{1}{\sqrt{15}}, \frac{\sqrt{15}}{4}$.

5. Given $\tan\theta = \sqrt{3}$; find $\sin\theta$ and $\cos\theta$. $\frac{1}{2}\sqrt{3}, \frac{1}{2}$.

6. Given $\sin\theta = \frac{12}{13}$; find $\cos\theta$. $\frac{5}{13}$.

7. Given $\operatorname{cosec}\theta = 5$; find $\sec\theta$ and $\tan\theta$. $\frac{5}{2\sqrt{6}}$ $\frac{1}{2\sqrt{6}}$.

8. Given $\sec\theta = \frac{41}{9}$; find $\sin\theta$ and $\cot\theta$. $\frac{40}{41}, \frac{9}{41}$.

9. Given $\cot\theta = \frac{2}{\sqrt{5}}$; find $\sin\theta$ and $\sec\theta$. $\frac{\sqrt{5}}{3}, \frac{3}{2}$.

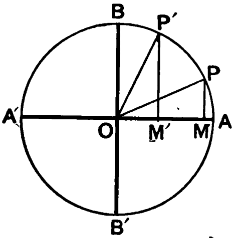
10. Given $\sin\theta = \frac{3}{4}$; find $\cos\theta$, $\tan\theta$, and $\cot\theta$.
$\frac{\sqrt{7}}{4}, \frac{3\sqrt{7}}{7}, \frac{\sqrt{7}}{3}$.

11. Given $\sin\theta = \frac{b}{c}$; find $\tan\theta$. $\frac{b}{\sqrt{c^2 - b^2}}$.

12. Given $\sin\theta = \frac{2mn}{m^2 + n^2}$; find $\tan\theta$. $\frac{2mn}{m^2 - n^2}$.

28. Reduction of Trigonometric Functions to the 1st Quadrant. — All mathematical tables give the trigonometric functions of angles between 0° and 90° only, but in practice we constantly have to deal with angles greater than 90°. The object of the following six Articles is to show that the trigonometric functions of any angle, positive or negative, can be expressed in terms of the trigonometric functions of an angle less than 90°, so that, if a given angle is greater than 90°, we can find an angle in the 1st quadrant whose trigonometric function has the same absolute value.

29. Functions of Complemental Angles. — Let AA′, BB′ be two diameters of a circle at right angles, and let OP and OP′ be the positions of the radius for any angle AOP = A, and its complement AOP′ = 90° − A (Art. 12).



Draw PM and P′M′ at right angles to OA.

Angle OP′M′ = BOP′ = AOP = A.

Also OP = OP′.

Hence the triangles OPM and OP′M′ are equal in all respects.

$$\therefore\ P'M' = OM. \qquad \therefore\ \frac{P'M'}{OP'} = \frac{OM}{OP}.$$

$$\therefore\ \sin(90° - A) = \cos AOP = \cos A.$$

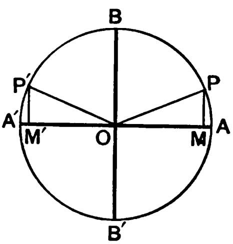
Also, $$OM' = PM. \qquad \therefore\ \frac{OM'}{OP'} = \frac{PM}{OP}.$$

$$\therefore\ \cos(90° - A) = \sin AOP = \sin A.$$

Similarly, $$\tan(90° - A) = \tan AOP' = \frac{P'M'}{OP'} = \frac{OM}{OP} = \cot A.$$

The other relations are obtained by inverting the above.

30. Functions of Supplemental Angles. — Let OP and OP′ be the positions of the radius for any angle AOP = A, and its supplement AOP′ = 180° − A (Art. 12).



Since OP = OP′, and POA = P′OA′, the triangles POM and P′OM′ are geometrically equal.

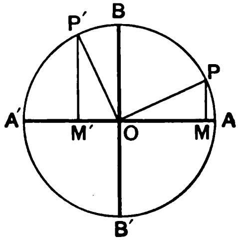
$$\therefore\ \sin(180° - A) = \sin AOP' = \frac{P'M'}{OP'} = \frac{PM}{OP} = \sin A,$$

$$\cos(180° - A) = \cos AOP' = \frac{OM'}{OP'} = \frac{-OM}{OP} = -\cos A,$$

$$\tan(180° - A) = \tan AOP' = \frac{P'M'}{OM'} = \frac{PM}{OM} = -\tan A.$$

Similarly the other relations may be obtained.

31. To prove sin(90° + A) = cos A, cos(90° + A) = − sin A, and tan (90° + A) = − cot A.



Let OP and OP′ be the positions of the radius for any angle AOP = A, and AOP′ = 90° + A.

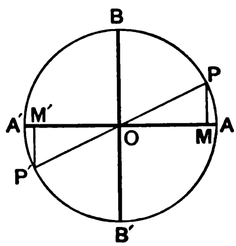
Since OP = OP′, and AOP = P′OB = OP′M′, the triangles POM and P′OM′ are equal in all respects.

$$\therefore \sin(90° + A) = \sin AOP' = \frac{P'M'}{OP'} = \frac{OM}{OP} = \cos A,$$

$$\cos(90° + A) = \cos AOP' = \frac{OM'}{OP'} = \frac{-PM}{OP} = -\sin A,$$

$$\tan(90° + A) = \tan AOP' = \frac{P'M'}{OM'} = \frac{OM}{-PM} = -\cot A.$$

32. To prove sin(180° + A) = − sin A, cos (180° + A) = − cos A, and tan (180° + A) = tan A.



Let the angle AOP = A; then the angle AOP′, measured in the positive direction, = (180° + A).

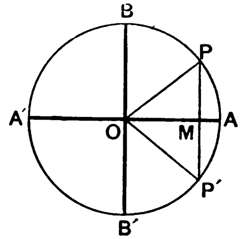
The triangles POM and P′OM′ are equal.

$$\therefore \sin(180° + A) = \sin AOP' = \frac{P'M'}{OP'} = \frac{-PM}{OP} = -\sin A,$$

$$\cos(180° + A) = \cos AOP' = \frac{OM'}{OP'} = \frac{-OM}{OP} = -\cos A,$$

$$\tan(180° + A) = \tan AOP' = \frac{P'M'}{OM'} = \frac{-PM}{-OM} = \tan A.$$

33. To prove $\sin(-A) = -\sin A$, $\cos(-A) = \cos A$, $\tan(-A) = -\tan A$.



Let OP and OP′ be the positions of the radius for any equal angles AOP and AOP′ measured from the initial line AO in opposite directions. Then if the angle AOP be denoted by A, the numerically equal angle AOP′ will be denoted by $-A$ (Art. 4).

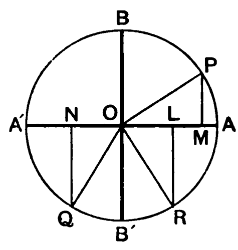
The triangles POM and P′OM are geometrically equal.

$$\therefore \sin(-A) = \sin AOP' = \frac{P'M}{OP'} = \frac{-PM}{OP} = -\sin A,$$

$$\cos(-A) = \cos AOP' = \frac{OM}{OP'} = \frac{OM}{OP} = \cos A,$$

$$\tan(-A) = \tan AOP' = \frac{P'M}{OM} = \frac{-PM}{OM} = -\tan A.$$

34. To prove $\sin(270° + A) = \sin(270° - A) = -\cos A$, and $\cos(270° + A) = -\cos(270° - A) = \sin A$.



Let the angle $AOP = A$; then the angles AOQ and AOR, measured in the positive direction, $= (270° - A)$ and $(270° + A)$ respectively.

The triangles POM, QON, and ROL are geometrically equal.

$$\therefore \quad RL = QN \quad = OM. \quad \therefore \frac{RL}{OR} = \frac{QN}{OQ} = \frac{-OM}{OP}.$$

$$\therefore \sin(270° + A) = \sin(270° - A) = -\cos A.$$

Also, $$\frac{OL}{OR} = \frac{-ON}{OQ} = \frac{PM}{OP}.$$

$$\therefore \cos(270° + A) = -\cos(270° - A) = \sin A.$$

35. Table giving the Values of the Functions of Any Angle in Terms of the Functions of an Angle less than 90°. — The foregoing results, and other similar ones, which may be proved in the same manner, are here collected for reference.

QUADRANT II.

$$\sin(180° - A) = \sin A. \qquad \sin(90° + A) = \cos A.$$
$$\cos(180° - A) = -\cos A. \qquad \cos(90° + A) = -\sin A.$$
$$\tan(180° - A) = -\tan A. \qquad \tan(90° + A) = -\cot A.$$
$$\cot(180° - A) = -\cot A. \qquad \cot(90° + A) = -\tan A.$$
$$\sec(180° - A) = -\sec A. \qquad \sec(90° + A) = -\operatorname{cosec} A.$$
$$\operatorname{cosec}(180° - A) = \operatorname{cosec} A. \qquad \operatorname{cosec}(90° + A) = \sec A.$$

QUADRANT III.

$$\sin(180° + A) = -\sin A. \qquad \sin(270° - A) = -\cos A.$$
$$\cos(180° + A) = -\cos A. \qquad \cos(270° - A) = -\sin A.$$
$$\tan(180° + A) = \tan A. \qquad \tan(270° - A) = \cot A.$$
$$\cot(180° + A) = \cot A. \qquad \cot(270° - A) = \tan A.$$
$$\sec(180° + A) = -\sec A. \qquad \sec(270° - A) = -\operatorname{cosec} A.$$
$$\operatorname{cosec}(180° + A) = -\operatorname{cosec} A. \qquad \operatorname{cosec}(270° - A) = -\sec A.$$

QUADRANT IV.

$$\sin(360° - A) = -\sin A. \qquad \sin(270° + A) = -\cos A.$$
$$\cos(360° - A) = \cos A. \qquad \cos(270° + A) = \sin A.$$
$$\tan(360° - A) = -\tan A. \qquad \tan(270° + A) = -\cot A.$$
$$\cot(360° - A) = -\cot A. \qquad \cot(270° + A) = -\tan A.$$
$$\sec(360° - A) = \sec A. \qquad \sec(270° + A) = \operatorname{cosec} A.$$
$$\operatorname{cosec}(360° - A) = -\operatorname{cosec} A. \qquad \operatorname{cosec}(270° + A) = -\sec A.$$

NOTE. — These relations* may be remembered by noting the following rules:

When A is associated with an *even* multiple of 90°, any function of the angle is *numerically* equal to the *same function* of A.

When A is associated with an *odd* multiple of 90°, any function of the angle is numerically equal to the corresponding *cofunction* of the angle A.

The ***sign*** to be prefixed will depend upon the quadrant to which the angle belongs (Art. 5), regarding A as an acute angle.

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* Although these relations have been proved only in case of A, *an acute angle*, they are true whatever A may be.

Thus, $\cos(270° - A) = -\sin A$; the angle $270° - A$ being in the 3d quadrant, and its cosine negative in consequence.

For an angle in the

First quadrant all the functions are ***positive.***
Second quadrant all are ***negative*** except the ***sine*** and ***cosecant.***
Third quadrant all are ***negative*** except the ***tangent*** and ***cotangent.***
Fourth quadrant all are ***negative*** except the ***cosine*** and ***secant.***

36. Periodicity of the Trigonometric Functions. — Let AOP be an angle of any magnitude, as in the figure of Art. 18; then if OP revolve in the positive or the negative direction through an angle of 360°, it will return to the position from which it started. Hence it is clear from the definitions that the trigonometric functions remain unchanged when the angle is increased or diminished by 360°, or any multiple of 360°. Thus the functions of the angle 400° are the same both in numerical value and in algebraic sign as the functions of the angle of $400° - 360°$, *i.e.*, of the angle of 40°. Also the functions of $360° + A$ are the same in numerical value and in sign as those of A.

In general, if n denote any integer, either positive or negative, *the functions of* $n \times 360° + A$ *are the same as those of* A.

Thus the functions of $1470° =$ the functions of 30°.

If θ denotes any angle in circular measure, the functions of $(2n\pi + \theta)$ are the same as those of θ. Thus

$$\sin(2n\pi + \theta) = \sin\theta,\ \cos(2n\pi + \theta) = \cos\theta,\ \text{etc.}$$

By this proposition we can reduce an angle of any magnitude to an angle less than 360° without changing the values of the functions. It is therefore unnecessary to consider the functions of angles greater than 360°; the formulæ already established *are true for angles of any magnitude whatever.*

EXAMPLES.

Express $\sin 700°$ in terms of the functions of an acute angle.

$$\sin 700° = \sin(360° + 340°) = \sin 340° = \sin(180° + 160°)$$
$$= \sin 160° = -\sin 20°.$$

Express the following functions in terms of the functions of acute angles:

1. $\sin 204°$, $\sin 510°$. *Ans.* $-\sin 24°$, $\sin 30°$.
2. $\cos(-800°)$, $\cos 359°$. $\cos 80°$, $\cos 1°$.
3. $\tan 500°$, $\tan 300°$. $-\tan 40°$, $-\cot 30°$.

Find the value of the sine, cosine, and tangent of the following angles:

4. $150°$. *Ans.* $\frac{1}{2}$, $-\frac{1}{2}\sqrt{3}$, $-\frac{1}{\sqrt{3}}$.
5. $-240°$. $\frac{1}{2}\sqrt{3}$, $-\frac{1}{2}$, $-\sqrt{3}$.
6. $330°$. $-\frac{1}{2}$, $\frac{1}{2}\sqrt{3}$, $-\frac{1}{\sqrt{3}}$.
7. $225°$. $-\frac{1}{\sqrt{2}}$, $-\frac{1}{\sqrt{2}}$, 1.

Find the values of the following functions:

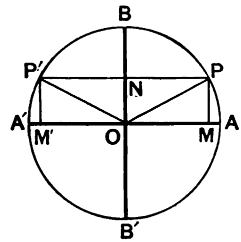
8. $\sin 810°$, $\sin(-240°)$, $\cos 210°$. *Ans.* 1, $\frac{1}{2}\sqrt{3}$, $-\frac{1}{2}\sqrt{3}$.
9. $\tan(-120°)$, $\cot 420°$, $\cot 510°$. $\sqrt{3}$, $\frac{1}{\sqrt{3}}$, 1.
10. $\sin 930°$, $\tan 6420°$. $-\frac{1}{2}$, $\frac{1}{\sqrt{3}}$.
11. $\cot 1035°$, $\operatorname{cosec} 570°$. -1, -2.

37. Angles corresponding to Given Functions. — When an angle is given, we can find its trigonometric functions, as in Arts. 26 and 27; and to each value of the angle there is but one value of each of the functions. But in the converse proposition — being given the value of the trigonometric functions, to find the corresponding angles — we have seen (Art. 36) that there are many angles of different magnitude which have the same functions.

If two such angles are in the *same quadrant*, they are represented geometrically by the same position of OP, so that they differ by some multiple of four right angles.

If we are given the value of the sine of an angle, it is important to be able to find all the angles which have that value for their sine.

38. General Expression for All Angles which have a Given Sine a. — Let O be the centre of the unit circle. Draw the diameters AA′, BB′, at right angles.



From O draw on OB a line ON, so that its measure is a.

Through N draw PP′ parallel to AA′. Join OP, OP′, and draw PM, P′M′, perpendicular to AA′.

Then since $MP = M'P' = ON = a$, the sine of AOP is equal to the sine of AOP′.

Hence the angles AOP and AOP′ are supplemental (Art. 30), and if AOP be denoted by α, AOP′ will $= \pi - \alpha$.

Now it is clear from the figure that the only *positive* angles which have the sine equal to a are α and $\pi - \alpha$, and the angles formed by adding any multiple of four right angles to α and $\pi - \alpha$. Hence, if θ be the general value of the required angle, we have

$$\theta = 2n\pi + \alpha, \text{ or } \theta = 2n\pi + \pi - \alpha, \quad . \quad . \quad . \quad . \quad (1)$$

where n is zero or any positive integer.

Also the only *negative* angles which have the sine equal to a are $-(\pi + \alpha)$, and $-(2\pi - \alpha)$, and the angles formed by adding to these any multiple of four right angles taken negatively; that is, we have

$$\theta = 2n\pi - (\pi + \alpha), \ \theta = 2n\pi - (2\pi - \alpha), \quad . \quad . \quad (2)$$

where n is zero or any negative integer.

Now the angles in (1) and (2) may be arranged thus:

$$2n\pi + \alpha, \ (2n+1)\pi - \alpha, \ (2n-1)\pi - \alpha, \ (2n-2)\pi + \alpha,$$

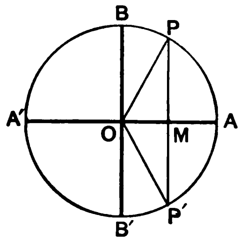
all of which, and no others, are included in the formula

$$\theta = n\pi + (-1)^n \alpha, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

where n is zero, or any positive or negative integer. Therefore (3) is the general expression for all angles which have a given sine.

NOTE. — The same formula determines all the angles which have the same *cosecant* as a.

39. An Expression for All Angles with a Given Cosine a. — Let O be the centre of the unit circle. Draw AA′, BB′, at right angles.



From O draw OM, so that its measure is a.

Through M draw PP′ parallel to BB′. Join OP, OP′.

Then since $OM = a$, the cosine of AOP is equal to the cosine of AOP′.

Hence, if $AOP = \alpha$, $AOP' = -\alpha$.

Now it is clear that the only angles which have the cosine equal to a are α and $-\alpha$, and the angles which differ from either by a multiple of four right angles.

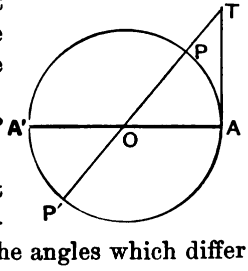
Hence if θ be the general value of all angles whose cosine is a, we have

$$\theta = 2n\pi \pm \alpha,$$

where n is zero, or any positive or negative integer.

NOTE. — The same formula determines all the angles which have the same *secant* or the same *versed sine* as a.

40. An Expression for All Angles with a Given Tangent a. — Let O be the centre of the unit circle. Draw AT, touching the circle at A, and take AT so that its measure is a.



Join OT, cutting the circle at P and P′.

Then it is clear from the figure that the only angles which have the tangent equal to a are α and $\pi + \alpha$, and the angles which differ from either by a multiple of four right angles. Hence if θ be the general value of the required angle, we have

$$\theta = 2n\pi + \alpha, \text{ and } 2n\pi + \pi + \alpha. \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Also, the only *negative* angles which have the tangent equal to a are $-(\pi - \alpha)$, and $-(2\pi - \alpha)$, and the angles which differ from either by a multiple of four right angles taken negatively; that is, we have

$$\theta = 2n\pi - (\pi - \alpha), \text{ and } 2n\pi - (2\pi - \alpha), \quad . \quad . \quad (2)$$

where n is zero or any negative integer.

Now the angles in (1) and (2) may be arranged thus:

$$2n\pi + \alpha, \ (2n+1)\pi + \alpha, \ (2n-1)\pi + \alpha, \ (2n-2)\pi + \alpha,$$

all of which, and no others, are included in the formula

$$\theta = n\pi + \alpha, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

where n is zero, or any positive or negative integer. Therefore (3) is the general expression for all angles which have a given tangent.

NOTE.—The same formula determines all the angles which have the same *cotangent* as α.

EXAMPLES.

1. Find six angles between -4 right angles and $+8$ right angles which satisfy the equation $\sin A = \sin 18°$.

We have from (3) of Art. 38,

$$\theta = n\pi + (-1)^n \frac{\pi}{10}, \text{ or } A = n \times 180° + (-1)^n 18°.$$

Put for n the values $-2, -1, 0, 1, 2, 3$, successively, and we get $A = -360° + 18°, -180° - 18°, 18°, 180° - 18°, 360° + 18°, 540° - 18°$;

that is, $\quad -342°, -198°, 18°, 162°, 378°, 522°$.

NOTE.—The student should draw a figure in the above example, and in each example of this kind which he works.

2. Find the four smallest angles which satisfy the equations (1) $\sin A = \frac{1}{2}$, (2) $\sin A = \frac{1}{\sqrt{2}}$, (3) $\sin A = \frac{\sqrt{3}}{2}$, (4) $\sin A = -\frac{1}{2}$.

Ans. (1) 30°, 150°, −210°, −330°;
(2) 45°, 135°, −225°, −315°;
(3) 60°, 120°, −240°, −300°;
(4) −30°, −150°, 210°, 330°.

41. Trigonometric Identities. — A trigonometric identity is an expression which states in the form of an equation a relation which is true for all values of the angle involved. Thus, the relations of Arts. 13 and 23, and all others that may be deduced from them by the aid of the ordinary formulæ of Algebra, are universally true, and are therefore called *identities;* but such relations as $\sin\theta = \frac{1}{2}$, $\cos\theta = \frac{1}{3}$, are not *identities.*

EXAMPLES.

1. Prove that $\sec\theta - \tan\theta \cdot \sin\theta = \cos\theta$.

Here
$$\sec\theta - \tan\theta \sin\theta = \frac{1}{\cos\theta} - \frac{\sin\theta}{\cos\theta}\sin\theta \qquad \text{(Art. 24)}$$
$$= \frac{1 - \sin^2\theta}{\cos\theta}$$
$$= \frac{\cos^2\theta}{\cos\theta} \qquad \text{(Art. 24)}$$
$$= \cos\theta.$$

2. Prove that $\cot\theta - \sec\theta \operatorname{cosec}\theta\,(1 - 2\sin^2\theta) = \tan\theta$.

$$\cot\theta - \sec\theta \operatorname{cosec}\theta\,(1 - 2\sin^2\theta)$$
$$= \frac{\cos\theta}{\sin\theta} - \frac{1}{\cos\theta} \cdot \frac{1}{\sin\theta}(1 - 2\sin^2\theta) \qquad \text{(Art. 24)}$$
$$= \frac{\cos^2\theta - 1 + 2\sin^2\theta}{\sin\theta\cos\theta}$$

$$= \frac{\cos^2\theta - (\sin^2\theta + \cos^2\theta) + 2\sin^2\theta}{\sin\theta\cos\theta} \qquad \text{(Art. 24)}$$

$$= \frac{\sin^2\theta}{\sin\theta\cos\theta} = \frac{\sin\theta}{\cos\theta} = \tan\theta.$$

Note. — It will be observed that in solving these examples we first express the other functions in terms of the sine and cosine, and in most cases the beginner will find this the simplest course. It is generally advisable to begin with the most complicated side and work towards the other.

Prove the following identities:

3. $\cos\theta\tan\theta = \sin\theta.$

4. $\cos\theta = \sin\theta\cot\theta.$

5. $(\tan\theta + \cot\theta)\sin\theta\cos\theta = 1.$

6. $(\tan\theta - \cot\theta)\sin\theta\cos\theta = \sin^2\theta - \cos^2\theta.$

7. $\sin^2\theta \div \operatorname{cosec}^2\theta = \sin^4\theta.$

8. $\sec^4\theta - \tan^4\theta = \sec^2\theta + \tan^2\theta.$

9. $(\sin\theta - \cos\theta)^2 = 1 - 2\sin\theta\cos\theta.$

10. $1 - \tan^4\theta = 2\sec^2\theta - \sec^4\theta.$

11. $\dfrac{1+\cos\theta}{1-\cos\theta} = (\operatorname{cosec}\theta + \cot\theta)^2.$

12. $(\sin\theta + \cos\theta)^2 + (\sin\theta - \cos\theta)^2 = 2.$

13. $\sin^4\theta - \cos^4\theta = \sin^2\theta - \cos^2\theta.$

14. $\sin^2\theta + \operatorname{vers}^2\theta = 2(1 - \cos\theta).$

15. $\cot^2\theta - \cos^2\theta = \cot^2\theta\cos^2\theta.$

EXAMPLES.

In a right triangle ABC (see figure of Art. 15) given:

1. $a = p^2 + pq,\ c = q^2 + pq$; calculate cot A.

   Ans. $\dfrac{\sqrt{q^2 - p^2}}{p}.$

2. $b = lm \div n,\ c = ln \div m$; calculate cosec A. $\dfrac{n^2}{\sqrt{n^4 - m^4}}.$

3. $\sin A = \frac{3}{5}$, $c = 20.5$; calculate a. *Ans.* 12.3.

4. Given $\cot \frac{1}{2} A = \tan A$; find A.

5. " $\sin A = \cos 2A$; find A.

6. " $\cot A = \tan 6A$; find A.

7. " $\tan A = \cot 8A$; find A.

8. " $\sin 2A = \cos 3A$; find A.

9. " $\sin A = \frac{2}{3}$; find cos A and tan A. $\frac{\sqrt{5}}{3}$, $\frac{2}{\sqrt{5}}$.

10. " $\cos A = \frac{4}{5}$; find sin A and tan A $\frac{3}{5}$, $\frac{3}{4}$.

11. " $\operatorname{cosec} A = \frac{4}{3}$; find cos A and tan A. $\frac{\sqrt{7}}{4}$, $\frac{3}{\sqrt{7}}$.

12. " $\sin A = \frac{1}{\sqrt{3}}$; find cos A and tan A. $\sqrt{\frac{2}{3}}$, $\frac{1}{\sqrt{2}}$.

13. " $\cos A = b$; find tan A and cosec A. $\frac{\sqrt{1-b^2}}{b}$, $\frac{1}{\sqrt{1-b^2}}$.

14. " $\sin A = .6$; find cos A and cot A. $\frac{4}{5}$, $\frac{4}{3}$.

15. " $\tan A = \frac{4}{5}$; find sin A. $\frac{4}{\sqrt{41}}$.

16. " $\cot A = \frac{8}{15}$; find sec A and sin A. $\frac{8}{17}$, $\frac{15}{17}$.

17. " $\sin A = \frac{12}{13}$; find cos A. $\frac{5}{13}$.

18. " $\cos A = .28$; find sin A. .96.

19. " $\tan A = \frac{4}{3}$; find sin A. $\frac{4}{5}$.

20. " $\sin A = \frac{1}{3}$; find cos A. $\frac{2}{3}\sqrt{2}$.

21. Given $\tan A = \frac{4}{3}$; find $\sin A$ and $\sec A$. *Ans.* $\frac{4}{5}$, $\frac{5}{3}$.

22. " $\tan\theta = \frac{a}{b}$; find $\sin\theta$ and $\cos\theta$.

$$\frac{a}{\sqrt{a^2+b^2}}, \quad \frac{b}{\sqrt{a^2+b^2}}.$$

23. " $\cos\theta = \frac{1}{a}$; find $\sin\theta$ and $\cot\theta$.

$$\frac{\sqrt{a^2-1}}{a}, \quad \frac{1}{\sqrt{a^2-1}}.$$

24. If $\sin\theta = a$, and $\tan\theta = b$, prove that

$$(1-a^2)(1+b^2) = 1.$$

Express the following functions in terms of the functions of acute angles less than 45°:

25. $\sin 168°$, $\sin 210°$. *Ans.* $\sin 12°$, $-\sin 30°$.

26. $\tan 125°$, $\tan 310°$. $-\cot 35°$, $-\cot 40°$.

27. $\sec 244°$, $\operatorname{cosec} 281°$. $-\operatorname{cosec} 26°$, $-\sec 11°$.

28. $\sec 930°$, $\operatorname{cosec}(-600°)$. $-\sec 30°$, $\sec 30°$.

29. $\cot 460°$, $\sec 299°$. $-\tan 10°$, $\operatorname{cosec} 29°$.

30. $\tan 1400°$, $\cot(-1400°)$. $-\tan 40°$, $\cot 40°$.

Find the values of the following functions:

31. $\sin 120°$, $\sin 135°$, $\sin 240°$. *Ans.* $\frac{\sqrt{3}}{2}$, $\frac{1}{\sqrt{2}}$, $-\frac{\sqrt{3}}{2}$.

32. $\cos 135°$, $\tan 300°$, $\operatorname{cosec} 300°$. $-\frac{1}{\sqrt{2}}$, $-\sqrt{3}$, $-\frac{2}{\sqrt{3}}$.

33. $\sec 315°$, $\cot 330°$, $\tan 780°$. $\sqrt{2}$, $-\sqrt{3}$, $\sqrt{3}$.

34. $\sin 480°$, $\sin 495°$, $\sin 870°$. $\frac{\sqrt{3}}{2}$, $\frac{1}{\sqrt{2}}$, $\frac{1}{2}$.

35. $\tan 1020°$, $\sec 1395°$, $\sin 1485°$. $-\sqrt{3}$, $\sqrt{2}$, $\frac{1}{\sqrt{2}}$.

36. $\sin(-240°)$, $\cot(-675°)$, $\operatorname{cosec}(-690°)$.

Ans. $\frac{\sqrt{3}}{2}$, 1, 2.

37. $\cos(-300°)$, $\cot(-315°)$, $\operatorname{cosec}(-1740°)$.

$\frac{1}{2}$, 1, $\frac{2}{\sqrt{3}}$.

38. $\tan^3 660°$, $\cos^3 1020°$. $-3\sqrt{3}$, $\frac{1}{8}$.

Find the value of the sine, cosine, and tangent of the following angles:

39. $-300°$. *Ans.* $\frac{\sqrt{3}}{2}$, $\frac{1}{2}$, $\sqrt{3}$.

40. $-135°$. $-\frac{1}{\sqrt{2}}$, $-\frac{1}{\sqrt{2}}$, 1.

41. $750°$. $\frac{1}{2}$, $\frac{\sqrt{3}}{2}$, $\frac{1}{\sqrt{3}}$.

42. $-840°$. $-\frac{\sqrt{3}}{2}$, $-\frac{1}{2}$, $\sqrt{3}$.

43. $1020°$. $-\frac{\sqrt{3}}{2}$, $\frac{1}{2}$, $-\sqrt{3}$.

44. $(2n+1)\pi - \frac{\pi}{3}$. $\frac{\sqrt{3}}{2}$, $-\frac{1}{2}$, $-\sqrt{3}$.

45. $(2n-1)\pi + \frac{\pi}{6}$. $-\frac{1}{2}$, $-\frac{\sqrt{3}}{2}$, $\frac{1}{\sqrt{3}}$.

Prove, drawing a separate figure in each case, that

46. $\sin 340° = \sin(-160°)$.

47. $\sin(-40°) = \sin 220°$.

48. $\cos 320° = -\cos(140°)$.

49. $\cos(-380°) = -\cos 560°$.

50. $\cos 195° = -\cos(-15°)$.

51. $\cos 380° = -\cos 560°$.

52. $\cos(-225°) = -\cos(-45°)$.

53. $\cos 1005° = -\cos 1185°$.

54. Draw an angle whose sine is $\frac{1}{2}$.

55. " " " " cosecant is 2.

56. " " " " tangent is 2.

57. Can an angle be drawn whose tangent is 427?

58. " " " " " " cosine is $\frac{5}{4}$?

59. " " " " " " secant is 7?

60. Find four angles between zero and + 8 right angles which satisfy the equations

(1) $\sin A = \sin 20°$, (2) $\sin A = -\frac{1}{\sqrt{2}}$, (3) $\sin A = -\frac{\pi}{7}$.

Ans. (1) $20°$, $160°$, $380°$, $520°$;

(2) $\frac{5\pi}{4}, \frac{7\pi}{4}, \frac{13\pi}{4}, \frac{15\pi}{4}$;

(3) $\frac{8\pi}{7}, \frac{13\pi}{7}, \frac{22\pi}{7}, \frac{27\pi}{7}$.

61. State the *sign* of the sine, cosine, and tangent of each of the following angles:

(1) $275°$; (2) $-91°$; (3) $-193°$; (4) $-350°$;

(5) $-1000°$; (6) $2n\pi + \frac{3\pi}{4}$.

Ans. (1) −, +, −; (2) −, −, +; (3) +, −, −;

(4) +, +, +; (5) +, +, +; (6) +, −, −.

Prove the following identities:

62. $(\sin^2\theta + \cos^2\theta)^2 = 1$.

63. $(\sin^2\theta - \cos^2\theta)^2 = 1 - 4\cos^2\theta + 4\cos^4\theta$.

64. $(\sin\theta + \cos\theta)^2 = 1 + 2\sin\theta\cos\theta$.

65. $(\sec\theta - \tan\theta)(\sec\theta + \tan\theta) = 1$.

66. $(\operatorname{cosec}\theta - \cot\theta)(\operatorname{cosec}\theta + \cot\theta) = 1.$

67. $\sin^3\theta + \cos^3\theta = (\sin\theta + \cos\theta)(1 - \sin\theta\cos\theta).$

68. $\sin^6\theta + \cos^6\theta = \sin^4\theta + \cos^4\theta - \sin^2\theta\cos^2\theta.$

69. $\sin^2\theta\tan^2\theta + \cos^2\theta\cot^2\theta = \tan^2\theta + \cot^2\theta - 1.$

70. $\sin\theta\tan^2\theta + \operatorname{cosec}\theta\sec^2\theta = 2\tan\theta\sec\theta - \operatorname{cosec}\theta + \sin\theta.$

71. $\cos^3\theta - \sin^3\theta = (\cos\theta - \sin\theta)(1 + \sin\theta\cos\theta).$

72. $\sin^6\theta + \cos^6\theta = 1 - 3\sin^2\theta\cos^2\theta.$

73. $\tan\alpha + \tan\beta = \tan\alpha\tan\beta(\cot\alpha + \cot\beta).$

74. $\cot\alpha + \tan\beta = \cot\alpha\tan\beta(\tan\alpha + \cot\beta).$

75. $1 - \sin\alpha = (1 + \sin\alpha)(\sec\alpha - \tan\alpha).$

76. $1 + \cos\alpha = (1 - \cos\alpha)(\operatorname{cosec}\alpha - \cot\alpha).$

77. $(1 + \sin\alpha + \cos\alpha)^2 = 2(1 + \sin\alpha)(1 + \cos\alpha).$

78. $(1 - \sin\alpha - \cos\alpha)^2(1 + \sin\alpha + \cos\alpha)^2 = 4\sin^2\alpha\cos^2\alpha.$

79. $2\operatorname{vers}\alpha - \operatorname{vers}^2\alpha = \sin^2\alpha.$

80. $\operatorname{vers}\alpha(1 + \cos\alpha) = \sin^2\alpha.$

CHAPTER III.

TRIGONOMETRIC FUNCTIONS OF TWO ANGLES.

42. Fundamental Formulæ. — We now proceed to express the trigonometric functions of the sum and difference of two angles in terms of the trigonometric functions of the angles themselves.

The *fundamental formulæ* first to be established are the following:

$$\sin(x+y)=\sin x\cos y+\cos x\sin y \quad . \quad . \quad . \quad . \quad (1)$$

$$\cos(x+y)=\cos x\cos y-\sin x\sin y \quad . \quad . \quad . \quad . \quad (2)$$

$$\sin(x-y)=\sin x\cos y-\cos x\sin y \quad . \quad . \quad . \quad . \quad (3)$$

$$\cos(x-y)=\cos x\cos y+\sin x\sin y \quad . \quad . \quad . \quad . \quad (4)$$

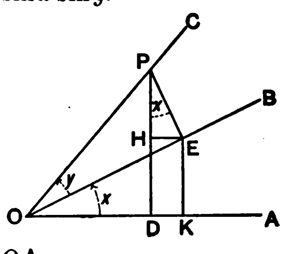
Note. — Here x and y are angles; so that $(x+y)$ and $(x-y)$ are also angles. Hence, $\sin(x+y)$ is the sine of an angle, and is not the same as $\sin x+\sin y$.
$\sin(x+y)$ is a single fraction.
$\sin x+\sin y$ is the sum of two fractions.

43. To prove that

$$\sin(x+y)=\sin x\cos y+\cos x\sin y,$$

and $$\cos(x+y)=\cos x\cos y-\sin x\sin y.$$

Let the angle AOB $=x$, and the angle BOC $=y$; then the angle AOC $=x+y$.



In OC, *the bounding line of the angle* $(x+y)$, take any point P, and draw PD, PE, perpendicular to OA and OB, respectively; draw EH, EK, perpendicular to PD and OA.

Then angle EPH = 90° − HEP = HEO = AOE = x.

$$\sin(x+y)=\frac{DP}{OP}=\frac{EK+PH}{OP}=\frac{EK}{OP}+\frac{PH}{OP}$$

$$=\frac{EK}{OE}\cdot\frac{OE}{OP}^{*}+\frac{PH}{PE}\cdot\frac{PE}{OP}$$

$$=\sin x\cos y+\cos x\sin y.$$

$$\cos(x+y)=\frac{OD}{OP}=\frac{OK-HE}{OP}=\frac{OK}{OP}-\frac{HE}{OP}$$

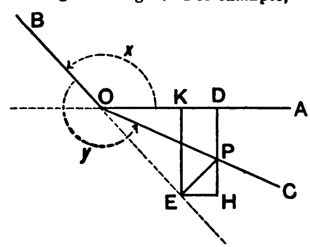
$$=\frac{OK}{OE}\cdot\frac{OE}{OP}-\frac{HE}{PE}\cdot\frac{PE}{OP}$$

$$=\cos x\cos y-\sin x\sin y.$$

NOTE. — These two formulæ have been obtained by a construction in which $x+y$ is an acute angle; but the proof is perfectly general, and applies to angles of any magnitude whatever, by paying due regard to the algebraic signs. For example,

Let AOB = x, as before, and BOC = y; then AOC, measured in the positive direction, is the angle $x+y$.

In OC, take any point P, and draw PD, PE, perpendicular to OA and OB produced; draw EH and EK perpendicular to PD and OA.



Then, angle EOK = 180° − x;

EPH = EOK = 180° − x;

and COE = y − 180°.

$$\sin(x+y)=-\frac{DP}{OP}=-\frac{EK-PH}{OP}=-\frac{EK}{OP}+\frac{PH}{OP}$$

$$=-\frac{EK}{OE}\cdot\frac{OE}{OP}+\frac{PH}{PE}\cdot\frac{PE}{OP}$$

$$=-\sin(180^\circ-x)\cos(y-180^\circ)+\cos(180^\circ-x)\sin(y-180^\circ)$$

$$=\sin x\cos y+\cos x\sin y. \qquad \text{(Art. 35)}$$

$$\cos(x+y)=\frac{OD}{OP}=\frac{OK+EH}{OP}=\frac{OK}{OP}+\frac{EH}{OP}$$

$$=\frac{OK}{OE}\cdot\frac{OE}{OP}+\frac{EH}{PE}\cdot\frac{PE}{OP}$$

---

* The introduced line OE is the *only* line in the figure which is at once a side of two right triangles (OEK and OEP) into which EK and OP enter. A similar remark applies to PE.

$$= \cos(180° - x)\cos(y - 180°) + \sin(180° - x)\sin(y - 180°)$$

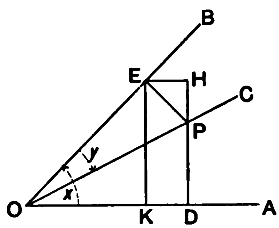
$$= \cos x \cos y - \sin x \sin y. \qquad \text{(Art. 35)}$$

The student should notice that the words of the two proofs are very nearly the same.

44. To prove that

$$\sin(x - y) = \sin x \cos y - \cos x \sin y,$$

and
$$\cos(x - y) = \cos x \cos y + \sin x \sin y.$$



Let the angle AOB be denoted by x, and COB by y; then the angle $AOC = x - y$.

In OC take any point P*, and draw PD, PE, perpendicular to OA and OB respectively; draw EH, EK, perpendicular to PD and OA respectively.

Then the angle $EPH = 90° - HEP = BEH = AOB = x$.

$$\sin(x - y) = \frac{DP}{OP} = \frac{EK - HP}{OP} = \frac{EK}{OP} - \frac{HP}{OP}$$

$$= \frac{EK}{OE} \cdot \frac{OE}{OP} - \frac{HP}{PE} \cdot \frac{PE}{OP}$$

$$= \sin x \cos y - \cos x \sin y.$$

$$\cos(x - y) = \frac{OD}{OP} = \frac{OK + EH}{OP} = \frac{OK}{OP} + \frac{EH}{OP}$$

$$= \frac{OK}{OE} \cdot \frac{OE}{OP} + \frac{EH}{PE} \cdot \frac{PE}{OP}$$

$$= \cos x \cos y + \sin x \sin y.$$

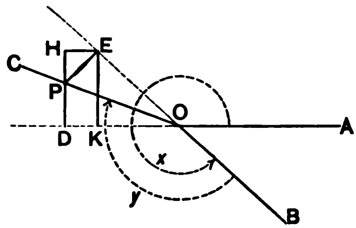
NOTE 1.—The sign in the expression of the sine is the *same* as it is in the angle expanded; in the cosine it is the *opposite*.

* P is taken in the line bounding the angle under consideration; *i.e.*, AOC.

NOTE 2. — In this proof the angle $x - y$ is acute; but the proof, like the one given in Art. 43, applies to angles of any magnitude whatever. For example,

Let AOB, measured in the positive direction, $= x$, and BOC $= y$. Then AOC $= x - y$.

In OC take any point P, and draw PD, PE, perpendicular to OA and OB produced: draw EH, EK, perpendicular to DP and AO produced.



Then,

angle EPH = EOK = AOB = $360° - x$,

and POE = $180° - y$.

$$\sin(x-y) = \frac{PD}{OP} = \frac{EK - HP}{OP}$$

$$= \frac{EK}{OE} \cdot \frac{OE}{OP} - \frac{HP}{PE} \cdot \frac{PE}{OP}$$

$$= \sin(360° - x)\cos(180° - y) - \cos(360° - x)\sin(180° - y)$$

$$= (-\sin x)(-\cos y) - \cos x \sin y$$

$$= \sin x \cos y - \cos x \sin y.$$

$$\cos(x-y) = -\frac{OD}{OP} = -\frac{OK + HE}{OP}$$

$$= -\frac{OK}{OE} \cdot \frac{OE}{OP} - \frac{HE}{PE} \cdot \frac{PE}{OP}$$

$$= -\cos(360° - x)\cos(180° - y) - \sin(360° - x)\sin(180° - y)$$

$$= (-\cos x)(-\cos y) - (-\sin x)\sin y$$

$$= \cos x \cos y + \sin x \sin y.$$

NOTE 3. — The four fundamental formulæ just proved are very important, *and must be committed to memory*. It will be convenient to refer to them as the 'x, y' formulæ. From any one of them, all the others can be deduced in the following manner:

Thus, from $\cos(x - y)$ to deduce $\sin(x + y)$. We have

$$\cos(x - y) = \cos x \cos y + \sin x \sin y \quad . \quad . \quad . \quad . \quad (1)$$

Substitute $90° - x$ for x in (1), and it becomes

$$\cos\{90° - (x + y)\} = \cos(90° - x)\cos y + \sin(90° - x)\sin y.$$

$$\therefore \sin(x + y) = \sin x \cos y + \cos x \sin y. \qquad \text{(Art. 29)}$$

The student should make the substitutions indicated below, and satisfy himself that the corresponding results follow:

From

$\sin(x+y)$	to deduce	$\cos(x+y)$	substitute	$(90°+x)$ for x.
"	" "	$\cos(x-y)$	"	$(90°-x)$ for x.
$\cos(x+y)$	" "	$\sin(x+y)$	"	$(90°+x)$ for x.
"	" "	$\sin(x-y)$	"	$(90°-x)$ for x.
"	" "	$\cos(x-y)$	"	$-y$ for y.
etc.		etc.		etc.

EXAMPLES.

1. To find the value of $\sin 15°$.

$$\sin 15° = \sin(45° - 30°)$$
$$= \sin 45° \cos 30° - \cos 45° \sin 30°$$
$$= \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{3}}{2} - \frac{1}{\sqrt{2}} \cdot \frac{1}{2}$$
$$= \frac{\sqrt{3}-1}{2\sqrt{2}}.$$

2. Show that $\sin 75° = \dfrac{\sqrt{3}+1}{2\sqrt{2}}$.

3. Show that $\cos 75° = \dfrac{\sqrt{3}-1}{2\sqrt{2}}$.

4. Show that $\cos 15° = \dfrac{\sqrt{3}+1}{2\sqrt{2}}$.

5. If $\sin x = \frac{3}{5}$, and $\cos y = \frac{5}{13}$, find $\sin(x+y)$ and $\cos(x-y)$.

Ans. $\frac{63}{65}$, and $\frac{56}{65}$.

6. If $\sin x = \frac{1}{2}$, and $\cos y = \frac{1}{2}$, find $\sin(x+y)$ and $\cos(x-y)$.

Ans. 1, and $\frac{\sqrt{3}}{2}$.

45. Formulæ for the Transformation of Sums into Products. — From the four fundamental formulæ of Arts. 43 and 44 we have, by addition and subtraction, the following:

$$\sin(x+y)+\sin(x-y)=2\sin x\cos y \quad . \quad . \quad . \quad (1)$$

$$\sin(x+y)-\sin(x-y)=2\cos x\sin y \quad . \quad . \quad . \quad (2)$$

$$\cos(x+y)+\cos(x-y)=2\cos x\cos y \quad . \quad . \quad . \quad (3)$$

$$\cos(x-y)-\cos(x+y)=2\sin x\sin y \quad . \quad . \quad . \quad (4)$$

These formulæ are useful in proving identities by transforming products into terms of first degree. They enable us, when read from right to left, to replace the *product* of a *sine or a cosine into a sine or a cosine* by half the *sum* or half the *difference* of two such ratios.

Let $x+y=\text{A}$, and $x-y=\text{B}$.

$$\therefore \; x=\tfrac{1}{2}(\text{A}+\text{B}), \text{ and } y=\tfrac{1}{2}(\text{A}-\text{B}).$$

Substituting these values in the above formulæ, and putting, for the sake of uniformity of notation, x, y instead of A, B, we get

$$\sin x+\sin y=2\sin\tfrac{1}{2}(x+y)\cos\tfrac{1}{2}(x-y) \quad . \quad . \quad (5)$$

$$\sin x-\sin y=2\cos\tfrac{1}{2}(x+y)\sin\tfrac{1}{2}(x-y) \quad . \quad . \quad (6)$$

$$\cos x+\cos y=2\cos\tfrac{1}{2}(x+y)\cos\tfrac{1}{2}(x-y) \quad . \quad . \quad (7)$$

$$\cos y-\cos x=2\sin\tfrac{1}{2}(x+y)\sin\tfrac{1}{2}(x-y) \quad . \quad . \quad (8)$$

The formulæ are of great importance in mathematical investigations (especially in computations by logarithms); they enable us to express *the sum or the difference* of two sines or two cosines in the form of a *product*. The student is recommended to become familiar with them, and to commit the following enunciations to memory:

Of any two angles, the

Sum of the sines $= 2\sin\frac{1}{2}\text{sum}\cdot\cos\frac{1}{2}\text{diff.}$

Diff. " " " $= 2\cos\frac{1}{2}\text{sum}\cdot\sin\frac{1}{2}\text{diff.}$

Sum of the cosines $= 2\cos\frac{1}{2}\text{sum}\cdot\cos\frac{1}{2}\text{diff.}$

Diff. " " " $= 2\sin\frac{1}{2}\text{sum}\cdot\sin\frac{1}{2}\text{diff.}$

EXAMPLES.

1. $\sin 5x\cos 3x = \frac{1}{2}(\sin 8x + \sin 2x).$

For, $\sin 5x\cos 3x = \frac{1}{2}\{\sin(5x+3x) + \sin(5x-3x)\}$
$= \frac{1}{2}(\sin 8x + \sin 2x).$

2. Prove $\sin\theta\sin 3\theta = \frac{1}{2}(\cos 2\theta - \cos 4\theta).$
3. " $2\sin\theta\cos\phi = \sin(\theta+\phi) + \sin(\theta-\phi).$
4. " $2\sin 2\theta\cos 3\phi = \sin(2\theta+3\phi) + \sin(2\theta-3\phi).$
5. " $\sin 60° + \sin 30° = 2\sin 45°\cos 15°.$
6. " $\sin 40° - \sin 10° = 2\cos 25°\sin 15°.$
7. " $\sin 10\theta + \sin 6\theta = 2\sin 8\theta\cos 2\theta.$
8. " $\sin 8\alpha - \sin 4\alpha = 2\cos 6\alpha\sin 2\alpha.$
9. " $\sin 3x + \sin x = 2\sin 2x\cos x.$
10. " $\sin 3x - \sin x = 2\cos 2x\sin x.$
11. " $\sin 4x + \sin 2x = 2\sin 3x\cos x.$

46. Useful Formulæ. — The following formulæ, which are of frequent use, may be deduced by taking the quotient of each pair of the formulæ (5) to (8) of Art. 45 as follows:

1. $$\frac{\sin x + \sin y}{\sin x - \sin y} = \frac{2\sin\frac{1}{2}(x+y)\cos\frac{1}{2}(x-y)}{2\cos\frac{1}{2}(x+y)\sin\frac{1}{2}(x-y)}$$
$$= \tan\tfrac{1}{2}(x+y)\cot\tfrac{1}{2}(x-y)$$
$$= \frac{\tan\frac{1}{2}(x+y)}{\tan\frac{1}{2}(x-y)}. \qquad \text{(Art. 24)}$$

The following may be proved by the student in a similar manner:

2. $$\frac{\sin x + \sin y}{\cos x + \cos y} = \tan\tfrac{1}{2}(x+y),$$

3. $\dfrac{\sin x + \sin y}{\cos y - \cos x} = \cot \frac{1}{2}(x - y),$

4. $\dfrac{\sin x - \sin y}{\cos x + \cos y} = \tan \frac{1}{2}(x - y),$

5. $\dfrac{\sin x - \sin y}{\cos y - \cos x} = \cot \frac{1}{2}(x + y),$

6. $\dfrac{\cos x + \cos y}{\cos y - \cos x} = \cot \frac{1}{2}(x + y) \cot \frac{1}{2}(x - y).$

47. The Tangent of the Sum and Difference of Two Angles. — Expressions for the value of $\tan(x+y)$, $\tan(x-y)$, etc., may be established *geometrically*. It is simpler, however, to deduce them from the formulæ already established, as follows:

Dividing the first of the '*x, y*' formulæ by the second, we have, by Art. 23,

$$\tan(x+y) = \frac{\sin(x+y)}{\cos(x+y)} = \frac{\sin x \cos y + \cos x \sin y}{\cos x \cos y - \sin x \sin y}.$$

Dividing both terms of the fraction by $\cos x \cos y$,

$$\tan(x+y) = \frac{\dfrac{\sin x \cos y}{\cos x \cos y} + \dfrac{\cos x \sin y}{\cos x \cos y}}{\dfrac{\cos x \cos y}{\cos x \cos y} - \dfrac{\sin x \sin y}{\cos x \cos x}}$$

$$= \frac{\tan x + \tan y}{1 - \tan x \tan y} \text{ (Art. 23)} \quad . \; . \; . \quad (1)$$

In the same manner may be derived

$$\tan(x-y) = \frac{\tan x - \tan y}{1 + \tan x \tan y} \quad . \; . \; . \; . \; . \; . \; . \quad (2)$$

Also, $$\cot(x+y) = \frac{\cot x \cot y - 1}{\cot x + \cot y} \quad . \; . \; . \; . \; . \; . \; . \quad (3)$$

and $$\cot(x-y) = \frac{\cot x \cot y + 1}{\cot y - \cot x} \quad . \; . \; . \; . \; . \; . \; . \quad (4)$$

EXERCISES.

Prove the following:

1. $$\tan(x+45^\circ)=\frac{\tan x+1}{1-\tan x}.$$

2. $$\tan(x-45^\circ)=\frac{\tan x-1}{1+\tan x}.$$

3. $$\frac{\sin(x+y)}{\sin(x-y)}=\frac{\tan x+\tan y}{\tan x-\tan y}.$$

4. $$\frac{\cos(x-y)}{\cos(x+y)}=\frac{\tan x\tan y+1}{1-\tan x\tan y}.$$

5. $$\sin(x+y)\sin(x-y)=\sin^2 x-\sin^2 y$$
$$=\cos^2 y-\cos^2 x.$$

6. $$\cos(x+y)\cos(x-y)=\cos^2 x-\sin^2 y$$
$$=\cos^2 y-\sin^2 x.$$

7. $$\tan x\pm\tan y=\frac{\sin(x\pm y)}{\cos x\cos y}.$$

8. $$\cot x\pm\cot y=\frac{\sin(y\pm x)}{\sin x\sin y}.$$

9. $$\frac{\sin 2x}{\sin x}-\frac{\cos 2x}{\cos x}=\sec x.$$

10. If $\tan x=\frac{1}{2}$ and $\tan y=\frac{1}{4}$, prove that $\tan(x+y)=\frac{6}{7}$, and $\tan(x-y)=\frac{2}{9}$.

11. Prove that $\tan 15^\circ=2-\sqrt{3}$.

12. If $\tan x=\frac{5}{6}$, and $\tan y=\frac{1}{11}$, prove that $\tan(x+y)=1$. What is $(x+y)$ in this case?

48. Formulæ for the Sum of Three or More Angles. — Let x, y, z be any three angles; we have by Art. 43,

$$\sin(x+y+z) = \sin(x+y)\cos z + \cos(x+y)\sin z$$
$$= \sin x \cos y \cos z + \cos x \sin y \cos z$$
$$+ \cos x \cos y \sin z - \sin x \sin y \sin z \quad . \quad . \quad (1)$$

In like manner,

$$\cos(x+y+z) = \cos x \cos y \cos z - \sin x \sin y \cos z$$
$$- \sin x \cos y \sin z - \cos x \sin y \sin z \quad . \quad . \quad (2)$$

Dividing (1) by (2), and reducing by dividing both terms of the fraction by $\cos x \cos y \cos z$, we get

$$\tan(x+y+z) = \frac{\tan x + \tan y + \tan z - \tan x \tan y \tan z}{1 - \tan x \tan y - \tan y \tan z - \tan z \tan x} \quad (3)$$

EXAMPLES.

1. Prove that $\sin x + \sin y + \sin z - \sin(x+y+z)$

$$= 4 \sin \tfrac{1}{2}(x+y) \sin \tfrac{1}{2}(y+z) \sin \tfrac{1}{2}(z+x).$$

By (6) of Art. 44 we have

$$\sin x - \sin(x+y+z) = -2\cos \tfrac{1}{2}(2x+y+z) \sin \tfrac{1}{2}(y+z),$$

and $\qquad \sin y + \sin z = 2 \sin \tfrac{1}{2}(y+z) \cos \tfrac{1}{2}(y-z).$

$\therefore\ \sin x + \sin y + \sin z - \sin(x+y+z)$

$$= 2\sin \tfrac{1}{2}(y+z) \cos \tfrac{1}{2}(y-z) - 2\cos \tfrac{1}{2}(2x+y+z) \sin \tfrac{1}{2}(y+z)$$
$$= 2\sin \tfrac{1}{2}(y+z) \{\cos \tfrac{1}{2}(y-z) - \cos \tfrac{1}{2}(2x+y+z)\}$$
$$= 2\sin \tfrac{1}{2}(y+z)\, 2\sin \tfrac{1}{2}(x+y) \sin \tfrac{1}{2}(x+z)$$
$$= 4\sin \tfrac{1}{2}(x+y) \sin \tfrac{1}{2}(y+z) \sin \tfrac{1}{2}(z+x).$$

Prove the following:

2. $\cos x + \cos y + \cos z + \cos(x+y+z)$

$$= 4\cos \tfrac{1}{2}(y+z) \cos \tfrac{1}{2}(z+x) \cos \tfrac{1}{2}(x+y).$$

3. $\sin(x+y-z) = \sin x \cos y \cos z + \cos x \sin y \cos z$
$$- \cos x \cos y \sin z + \sin x \sin y \sin z.$$

4. $\sin x + \sin y - \sin z - \sin(x+y-z)$
$$= 4 \sin \tfrac{1}{2}(x-z) \sin \tfrac{1}{2}(y-z) \sin \tfrac{1}{2}(x+y).$$

5. $\sin(y-z) + \sin(z-x) + \sin(x-y)$
$$+ 4 \sin \tfrac{1}{2}(y-z) \sin \tfrac{1}{2}(z-x) \sin \tfrac{1}{2}(x-y) = 0.$$

49. Functions of Double Angles. — To express the trigonometric functions of the angle $2x$ in terms of those of the angle x.

Put $y = x$ in (1) of Art. 42, and it becomes

$$\sin 2x = \sin x \cos x + \cos x \sin x,$$

or
$$\sin 2x = 2 \sin x \cos x \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Put $y = x$ in (2) of Art. 42, and it becomes

$$\cos 2x = \cos^2 x - \sin^2 x \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$
$$= 1 - 2 \sin^2 x \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

or
$$= 2 \cos^2 x - 1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Put $y = x$ in (1) and (3) of Art. 47, and they become

$$\tan 2x = \frac{2 \tan x}{1 - \tan^2 x} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

$$\cot 2x = \frac{\cot^2 x - 1}{2 \cot x} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

Transposing 1 in (4), and dividing it into (1), we have

$$\frac{\sin 2x}{1 + \cos 2x} = \tan x \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

NOTE. — These seven formulæ are very important. The student must notice that x is *any* angle, and therefore these formulæ will be true whatever we put for x. Thus, if we write $\frac{x}{2}$ for x, we get

$$\sin x = 2\sin\frac{x}{2}\cos\frac{x}{2} \quad \dots \dots \dots \dots \quad (8)$$

$$\cos x = \cos^2\frac{x}{2} - \sin^2\frac{x}{2} \quad \dots \dots \dots \dots \quad (9)$$

or

$$= 1 - 2\sin^2\frac{x}{2} = 2\cos^2\frac{x}{2} - 1 \quad \dots \dots \dots \quad (10)$$

and so on.

EXAMPLES.

Prove the following:

1. $2\operatorname{cosec} 2x = \sec x \operatorname{cosec} x.$

2. $\dfrac{\operatorname{cosec}^2 x}{\operatorname{cosec} x - 2} = \sec 2x.$

3. $\dfrac{2\tan x}{1 + \tan^2 x} = \sin 2x.$

4. $\dfrac{1 - \tan^2 x}{1 + \tan^2 x} = \cos 2x.$

5. $\tan x + \cot x = 2\operatorname{cosec} 2x.$

6. $\cot x - \tan x = 2\cot 2x.$

7. $\dfrac{\sin x}{1 + \cos x} = \tan\dfrac{x}{2}.$

8. $\dfrac{\sin x}{1 - \cos x} = \cot\dfrac{x}{2}.$

9. Given $\sin 45° = \dfrac{1}{\sqrt{2}}$; find $\tan 22\frac{1}{2}°$. *Ans.* $\sqrt{2} - 1.$

10. Given $\tan x = \dfrac{3}{4}$; find $\tan 2x$, and $\sin 2x$. $\dfrac{24}{7}, \dfrac{24}{25}.$

50. To Express the Functions of $3x$ in Terms of the Functions of x.

Put $y = 2x$ in (1) of Art. 42, and it becomes

$$\begin{aligned}\sin 3x &= \sin(2x + x)\\ &= \sin 2x \cos x + \cos 2x \sin x\\ &= 2\sin x \cos^2 x + (1 - 2\sin^2 x)\sin x \qquad \text{(Art. 49)}\\ &= 2\sin x(1 - \sin^2 x) + \sin x - 2\sin^3 x\\ &= 3\sin x - 4\sin^3 x.\end{aligned}$$

$$\begin{aligned}\cos 3x &= \cos(2x + x)\\ &= \cos 2x \cos x - \sin 2x \sin x\\ &= (2\cos^2 x - 1)\cos x - 2\sin^2 x \cos x \qquad \text{(Art. 49)}\\ &= 4\cos^3 x - 3\cos x.\end{aligned}$$

$$\begin{aligned}\tan 3x &= \tan 2x + x\\ &= \frac{\tan 2x + \tan x}{1 - \tan 2x \tan x}\\ &= \frac{\dfrac{2\tan x}{1 - \tan^2 x} + \tan x}{1 - \dfrac{2\tan^2 x}{1 - \tan^2 x}}\\ &= \frac{3\tan x - \tan^3 x}{1 - 3\tan^2 x}.\end{aligned}$$

EXAMPLES.

Prove the following:

1. $\dfrac{\sin 3x}{\sin x} = 2\cos 2x + 1.$

2. $\dfrac{\sin 3x - \sin x}{\cos 3x + \cos x} = \tan x.$

3. $\dfrac{\sin 3x + \cos 3x}{\cos x - \sin x} = 2\sin 2x - 1.$

4. $\dfrac{1}{\tan 3x - \tan x} + \dfrac{1}{\cot x - \cot 3x} = \cot 2x.$

5. $\dfrac{1 - \cos 3x}{1 - \cos x} = (1 + 2\cos x)^2.$

51. Functions of Half an Angle. — To express the functions of $\frac{x}{2}$ in terms of the functions of x.

Since $$\cos x = 1 - 2\sin^2\frac{x}{2},$$

or $$= 2\cos^2\frac{x}{2} - 1 \quad . \quad . \quad . \quad [\text{Art. } 49, (10)]$$

$$\therefore \sin^2\frac{x}{2} = \frac{1 - \cos x}{2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

and $$\cos^2\frac{x}{2} = \frac{1 + \cos x}{2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Or $$\sin\frac{x}{2} = \pm\sqrt{\frac{1 - \cos x}{2}} \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

and $$\cos\frac{x}{2} = \pm\sqrt{\frac{1 + \cos x}{2}} \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

By division, $$\tan\frac{x}{2} = \pm\sqrt{\frac{1 - \cos x}{1 + \cos x}} = \pm\frac{1 - \cos x}{\sin x} \quad . \quad . \quad (5)$$

By formulæ (3), (4), and (5) the functions of half an angle may be found when the cosine of the whole angle is given.

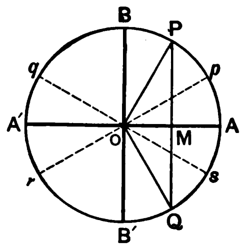
52. If the Cosine of an Angle be given, the Sine and the Cosine of its Half are each Two-Valued.

By Art. 51, each value of $\cos x$ (nothing else being known about the angle x) gives *two* values each for $\sin\frac{x}{2}$ and $\cos\frac{x}{2}$, one positive and one negative. But if the value of x be given, we know the quadrant in which $\frac{x}{2}$ lies, and hence we know which sign is to be taken.

Thus, if x lies between 0° and 360°, $\frac{x}{2}$ lies between 0° and 180°, and therefore $\sin\frac{x}{2}$ is *positive;* but if x lies between 360° and 720°, $\frac{x}{2}$ lies between 180° and 360°, and hence $\sin\frac{x}{2}$ is *negative.* Also, if x lie between 0° and 180°, $\cos\frac{x}{2}$ is *positive;* but if x lie between 180° and 360°, $\cos\frac{x}{2}$ is *negative.*

The case may be investigated geometrically thus:

Let OM = the given cosine (radius being unity, Art. 16), $= \cos x$. Through M draw PQ perpendicular to OA; and draw OP, OQ. Then all angles whose cosines are equal to $\cos x$ are terminated either by OP or OQ, and the halves of these angles are terminated by the dotted lines Op, Oq, Or, or Os. The *sines* of angles ending at Op and Oq are the same, and equal numerically to those of angles ending at Or and Os; but in the former case they are positive, and in the latter, negative; hence we obtain *two,* and *only two,* values of $\sin\frac{x}{2}$ from a given value of $\cos x$.



Also, the *cosines* of angles ending at Op and Os are the same, and have the positive sign. They are equal numerically to the cosines of the angles ending at Oq and Or, but the latter are negative; hence we obtain *two,* and *only two,* values of $\cos\frac{x}{2}$ from a given value of $\cos x$.

Also, the *tangent* of half the angle whose cosine is given is two-valued. This follows immediately from (5) of Art. 51.

53. If the Sine of an Angle be given, the Sine and the Cosine of its Half are each Four-Valued.

We have $$2\sin\frac{x}{2}\cos\frac{x}{2}=\sin x \quad . \quad . \quad . \quad \text{(Art. 49)}$$

and $$\sin^2\frac{x}{2}+\cos^2\frac{x}{2}=1 \quad . \quad . \quad . \quad . \quad \text{(Art. 23)}$$

By addition, $$\left(\sin\frac{x}{2}+\cos\frac{x}{2}\right)^2=1+\sin x.$$

By subtraction, $$\left(\sin\frac{x}{2}-\cos\frac{x}{2}\right)^2=1-\sin x.$$

$$\therefore \sin\frac{x}{2}+\cos\frac{x}{2}=\pm\sqrt{1+\sin x} \quad . \quad . \quad (1)$$

and $$\sin\frac{x}{2}-\cos\frac{x}{2}=\pm\sqrt{1-\sin x} \quad . \quad . \quad (2)$$

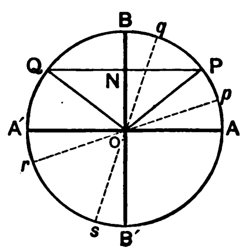
$$\therefore 2\sin\frac{x}{2}=\pm\sqrt{1+\sin x}\pm\sqrt{1-\sin x} \quad . \quad . \quad (3)$$

and $$2\cos\frac{x}{2}=\pm\sqrt{1+\sin x}\mp\sqrt{1-\sin x} \quad . \quad . \quad (4)$$

Thus, if we are given the value of $\sin x$ (nothing else being known about the angle x), it follows from (3) and (4) that $\sin\frac{x}{2}$ and $\cos\frac{x}{2}$ have each *four* values equal, two by two, in absolute value, but of contrary signs.

The case may be investigated geometrically thus:

Let ON = the given sine (radius being unity) $=\sin x$. Through N draw PQ parallel to OA; and draw OP, OQ. Then all angles whose sines are equal to $\sin x$ are terminated either by OP or OQ, and the halves of these angles are terminated by the dotted lines Op, Oq, Or, or Os. The sines of angles ending at Op, Oq, Or, and Os are all different



in value; and so are their cosines. Hence we obtain *four* values for $\sin\frac{x}{2}$, and four also for $\cos\frac{x}{2}$, in terms of x.

When the angle x is given, there is no ambiguity in the calculations; for $\frac{x}{2}$ is then known, and therefore the signs and relative magnitudes of $\sin\frac{x}{2}$ and $\cos\frac{x}{2}$ are known. Then equations (1) and (2), which should always be used, immediately determine the signs to be taken in equations (3) and (4).

Thus, when $\frac{x}{2}$ lies between $-45°$ and $+45°$, $\cos\frac{x}{2} > \sin\frac{x}{2}$, and is *positive.*

Therefore (1) is *positive*, and (2) is *negative* · and hence (3) and (4) become

$$2\sin\frac{x}{2} = \sqrt{1+\sin x} - \sqrt{1-\sin x},$$

$$2\cos\frac{x}{2} = \sqrt{1+\sin x} + \sqrt{1-\sin x}.$$

When $\frac{x}{2}$ lies between 45° and 135°, $\sin\frac{x}{2} > \cos\frac{x}{2}$, and is *positive.*

Therefore (1) and (2) are both positive; and hence (3) and (4) become

$$2\sin\frac{x}{2} = \sqrt{1+\sin x} + \sqrt{1-\sin x},$$

$$2\cos\frac{x}{2} = \sqrt{1+\sin x} - \sqrt{1-\sin x}.$$

And so on.

54. If the Tangent of an Angle be given, the Tangent of its Half is Two-Valued.

We have $$\tan\theta = \frac{2\tan\frac{\theta}{2}}{1-\tan^2\frac{\theta}{2}} \quad . \quad . \quad . \quad . \quad . \quad \text{(Art. 49)}$$

Put $\tan\frac{\theta}{2} = x$; thus

$$(1 - x^2)\tan\theta = 2x,$$

$$x^2 + \frac{2}{\tan\theta}x = 1.$$

$$\therefore\ \tan\frac{\theta}{2} = x = \frac{-1 \pm \sqrt{1 + \tan^2\theta}}{\tan\theta}.$$

Thus, given $\tan\theta$, we find *two* unequal values for $\tan\frac{\theta}{2}$, one positive and one negative.

This result may be proved geometrically, an exercise which we leave for the student.

55. If the Sine of an Angle be given, the Sine of One-Third of the Angle is Three-Valued.

We have $\qquad \sin 3x = 3\sin x - 4\sin^3 x \quad . \quad . \quad$ (Art. 50)

Put $\qquad x = \frac{\theta}{3}$, and we get

$$\sin\theta = 3\sin\frac{\theta}{3} - 4\sin^3\frac{\theta}{3},$$

a cubic equation, which therefore has three roots.

EXAMPLES.

1. Determine the limits between which A must lie to satisfy the equation

$$2\sin A = -\sqrt{1 + \sin 2A} - \sqrt{1 - \sin 2A}.$$

By (1) and (2) of Art. 53, $2\sin A$ can have this value only when

$$\sin A + \cos A = -\sqrt{1 + \sin 2A},$$

and $\qquad \sin A - \cos A = -\sqrt{1 - \sin 2A}$;

i.e., when $\sin A > \cos A$ and *negative*.

Therefore A lies between 225° and 315°, or between the angles formed by adding or subtracting any multiple of four right angles to each of these; *i.e.*, A lies between

$$2n\pi + \frac{5\pi}{4} \text{ and } 2n\pi + \frac{7\pi}{4},$$

where n is zero or any positive or negative integer.

2. Determine the limits between which A must lie to satisfy the equation

$$2\cos A = \sqrt{1+\sin 2A} - \sqrt{1-\sin 2A}.$$

By (1) and (2) of Art. 53, $2\cos A$ can have this value only when

$$\cos A + \sin A = \sqrt{1+\sin 2A},$$

and

$$\cos A - \sin A = -\sqrt{1-\sin 2A};$$

i.e., when $\sin A > \cos A$ and *positive*.

Therefore A lies between

$$2n\pi + \frac{\pi}{4} \text{ and } 2n\pi + \frac{3\pi}{4},$$

where n is any positive or negative integer.

3. State the signs of $(\sin\theta + \cos\theta)$ and $(\sin\theta - \cos\theta)$ when θ has the following values: (1) 22°; (2) 191°; (3) 290°; (4) 345°; (5) $-22°$; (6) $-275°$; (7) $-470°$; (8) 1000°.

Ans. (1) +, −; (2) −, +; (3) −, −; (4) +, −; (5) +, −; (6) +, +; (7) −, −; (8) −, −.

4. Prove that the formulæ which give the values of $\sin\frac{x}{2}$ and of $\cos\frac{x}{2}$ in terms of $\sin x$ are unaltered when x has the values

(1) 92°, 268°, 900°, $4n\pi + \frac{3}{4}\pi$, or $(4n+2)\pi - \frac{3}{4}\pi$;

(2) 88°, $-88°$, 770°, $-770°$, or $4n\pi \pm \frac{\pi}{8}$.

5. Find the limits between which A must lie when

$$2\sin A = \sqrt{1+\sin 2A} - \sqrt{1-\sin 2A}.$$

56. Find the Values of the Functions of $22\frac{1}{2}°$. — In (3), (4), and (5) of Art. 51, put $x = 45°$. Then

$$\sin 22\tfrac{1}{2}° = \sqrt{\frac{1-\cos 45°}{2}} = \frac{\sqrt{2-\sqrt{2}}}{2},$$

$$\cos 22\tfrac{1}{2}° = \sqrt{\frac{1+\cos 45°}{2}} = \frac{\sqrt{2+\sqrt{2}}}{2},$$

$$\tan 22\tfrac{1}{2}° = \frac{1-\cos 45°}{\sin 45°} = \sqrt{2} - 1.$$

Since $22\frac{1}{2}°$ is an acute angle, its functions are all positive. The above results are also the cosine, sine, and cotangent respectively of $67\frac{1}{2}°$, since the latter is the complement of $22\frac{1}{2}°$ (Art. 15).

57. Find the Sine and Cosine of 18°.

Let $x = 18°$; then $2x = 36°$, and $3x = 54°$.

$$\therefore\ 2x + 3x = 90°.$$

$$\therefore\ \sin 2x = \cos 3x \quad . \ . \ . \ . \ . \ . \quad \text{(Art. 15)}$$

$$\therefore\ 2\sin x\cos x = 4\cos^3 x - 3\cos x \quad . \ . \quad \text{(Art. 50)}$$

or

$$2\sin x = 4\cos^2 x - 3$$

$$= 1 - 4\sin^2 x.$$

Solving the quadratic, and taking the upper sign, since $\sin 18°$ must be positive, we get

$$\sin 18° = \frac{\sqrt{5}-1}{4}.$$

Also, $\cos 18° = \sqrt{1-\sin^2 18°} = \dfrac{\sqrt{10+2\sqrt{5}}}{4}.$

Hence we have also the sine and cosine of 72° (Art. 15).

58. Find the Sine and Cosine of 36°.

$$\cos 36° = 1 - 2\sin^2 18° \quad . \quad . \quad . \quad [(3) \text{ of Art. } 49]$$

$$= 1 - \frac{6 - 2\sqrt{5}}{8} = \frac{\sqrt{5}+1}{4}.$$

$$\therefore \sin 36° = \sqrt{1 - \cos^2 36°} = \frac{\sqrt{10 - 2\sqrt{5}}}{4}.$$

The above results are also the sine and cosine, respectively, of 54° (Art. 15).

Otherwise thus: Let $x = 36°$; then $2x = 72°$, and $3x = 108°$.

$$\therefore 2x + 3x = 180°.$$

$$\therefore \sin 2x = \sin 3x. \quad . \quad . \quad . \quad . \quad . \quad (\text{Art. } 29)$$

$$2\sin x \cos x = 3\sin x - 4\sin^3 x,$$

$$2\cos x = 3 - 4\sin^2 x$$

$$= 4\cos^2 x - 1.$$

Solving, $$\cos x = \frac{\pm\sqrt{5}+1}{4}.$$

But 36° is an acute angle, and therefore its cosine is positive.

$$\therefore \cos 36° = \frac{\sqrt{5}+1}{4}.$$

59. If $A + B + C = 180°$, or if A, B, C are the Angles of a Triangle, prove the Following Identities:

(1) $\sin A + \sin B + \sin C = 4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}.$

(2) $\cos A + \cos B + \cos C = 1 + 4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}.$

(3) $\tan A + \tan B + \tan C = \tan A \tan B \tan C.$

We have $A + B + C = 180°$.

$\therefore\ \sin(A+B) = \sin C$, and $\sin\frac{A+B}{2} = \cos\frac{C}{2}$. (Arts. 15 and 30)

Now $\quad \sin A + \sin B = 2\sin\frac{A+B}{2}\cos\frac{A-B}{2}$. (Art. 45)

$$= 2\cos\frac{C}{2}\cos\frac{A-B}{2} \quad . \ . \quad \text{(Art. 15)}$$

and $\quad \sin C = 2\sin\frac{C}{2}\cos\frac{C}{2}$ . . . . (Art. 49)

$$= 2\cos\frac{A+B}{2}\cos\frac{C}{2} \quad . \ . \quad \text{(Art. 15)}$$

$$\therefore\ \sin A + \sin B + \sin C = 2\cos\frac{C}{2}\cos\frac{A-B}{2} + 2\cos\frac{C}{2}\cos\frac{A+B}{2}$$

$$= 2\cos\frac{C}{2}\left(\cos\frac{A-B}{2} + \cos\frac{A+B}{2}\right)$$

$$= 2\cos\frac{C}{2}\left(2\cos\frac{A}{2}\cos\frac{B}{2}\right). \quad \text{(Art. 45)}$$

$$= 4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2} \quad . \ . \ . \ . \quad (1)$$

Again, $\quad \cos A + \cos B = 2\cos\frac{A+B}{2}\cos\frac{A-B}{2}$. (Art. 45)

$$= 2\sin\frac{C}{2}\cos\frac{A-B}{2};$$

and $\quad \cos C = 1 - 2\sin^2\frac{C}{2}$ . . . . (Art. 49)

$$\therefore\ \cos A + \cos B + \cos C = 1 + 2\sin\frac{C}{2}\left(\cos\frac{A-B}{2} - \sin\frac{C}{2}\right)$$

$$= 1 + 2\sin\frac{C}{2}\left(\cos\frac{A-B}{2} - \cos\frac{A+B}{2}\right)$$

$$= 1 + 4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} \quad . \ . \quad (2)$$

Again, $\tan(A+B) = -\tan C$ . . . . . . (Art. 30)

$$= \frac{\tan A + \tan B}{1 - \tan A \tan B}. \quad . \quad . \qquad \text{(Art. 47)}$$

$$\therefore \tan A + \tan B = -\tan C\,(1 - \tan A \tan B).$$

$$\therefore \tan A + \tan B + \tan C = \tan A \tan B \tan C \quad . \quad . \quad . \quad . \quad (3)$$

NOTE. — The student will observe that (1), (2), and (3) follow directly from Examples 1 and 2, and formula (3), respectively, of Art. 48, by putting

$$A + B + C = 180°.$$

EXAMPLES.

Prove the following statements if $A + B + C = 180°$:

1. $\cos(A + B - C) = -\cos 2C.$
2. $\sin A + \sin B - \sin C = 4 \sin\frac{A}{2} \sin\frac{B}{2} \cos\frac{C}{2}.$
3. $\sin 2A + \sin 2B + \sin 2C = 4 \sin A \sin B \sin C.$
4. $\sin 2A + \sin 2B - \sin 2C = 4 \sin C \cos A \cos B.$
5. $\tan 7A - \tan 4A - \tan 3A = \tan 7A \tan 4A \tan 3A.$
6. $\sin A - \sin B + \sin C = 4 \sin\frac{A}{2} \cos\frac{B}{2} \sin\frac{C}{2}.$
7. $\cot\frac{A}{2} + \cot\frac{B}{2} + \cot\frac{C}{2} = \cot\frac{A}{2} \cot\frac{B}{2} \cot\frac{C}{2}.$
8. $\tan A - \cot B = \sec A \operatorname{cosec} B \cos C.$

60. Inverse Trigonometric Functions. — The equation $\sin\theta = x$ means that θ is the angle whose sine is x; this may be written $\theta = \sin^{-1}x$, where $\sin^{-1}x$ is an abbreviation for *the angle (or arc) whose sine is x.*

So the symbols $\cos^{-1}x$, $\tan^{-1}x$, and $\sec^{-1}y$, are read "the angle (or arc) whose cosine is x," "the angle (or arc) whose tangent is x," and "the angle (or arc) whose secant is y." These angles are spoken of as being *the inverse sine of x, the* *inverse cosine of x, the inverse tangent of x*, and *the inverse secant of y*, respectively. Such expressions are called *inverse trigonometric functions.*

NOTE. — The student must be careful to notice that -1 is not an exponent, $\sin^{-1}x$ is not $(\sin x)^{-1}$, which $=\dfrac{1}{\sin x}$.

Notice also that $\sin^{-1}\dfrac{\sqrt{3}}{2}=\cos^{-1}\dfrac{1}{2}$ is not an *identity*, but is true only for the particular angle 60°.

This notation is only *analogous* to the use of exponents in multiplication, where we have $a^{-1}a=a^0=1$. Thus, $\cos^{-1}(\cos x)=x$, and $\sin(\sin^{-1}x)=x$; that is, $\cos^{-1}$ is inverse to cos, and applied to it annuls it; and so for other functions.

The French method of writing inverse functions is *arc sin x, arc cos x, arc tan x*, and so on.

EXAMPLES.

1. Show that 30° is one value of $\sin^{-1}\frac{1}{2}$.

We know that $\sin 30°=\frac{1}{2}$. $\therefore$ 30° is an angle whose sine is $\frac{1}{2}$; or $30°=\sin^{-1}\frac{1}{2}$.

2. Prove that $\tan^{-1}\frac{1}{2}+\tan^{-1}\frac{1}{3}=45°$.

$\tan^{-1}\frac{1}{2}$ is one of the angles whose tangent is $\frac{1}{2}$, and $\tan^{-1}\frac{1}{3}$ is one of the angles whose tangent is $\frac{1}{3}$.

Let $$\alpha=\tan^{-1}\tfrac{1}{2},\quad\text{and}\quad\beta=\tan^{-1}\tfrac{1}{3};$$

then $$\tan\alpha=\tfrac{1}{2}\quad\text{and}\quad\tan\beta=\tfrac{1}{3}.$$

Now $$\tan(\alpha+\beta)=\frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}\quad . \quad . \quad . \quad \text{(Art. 47)}$$

$$=\frac{\frac{1}{2}+\frac{1}{3}}{1-\frac{1}{2}\times\frac{1}{3}}=1.$$

But $\tan 45°=1$, $\quad\therefore\ \alpha+\beta=45°$;

that is, $$\tan^{-1}\tfrac{1}{2}+\tan^{-1}\tfrac{1}{3}=45°.$$

Therefore 45° is one value of $\tan^{-1}\frac{1}{2}+\tan^{-1}\frac{1}{3}$.

3. Prove that $\tan^{-1}x + \tan^{-1}y = \tan^{-1}\dfrac{x+y}{1-xy}$.

Let $\tan^{-1}x = A$. $\therefore \tan A = x$.

$\tan^{-1}y = B$. $\therefore \tan B = y$.

Now $$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$$

$$= \frac{x+y}{1-xy}.$$

$$\therefore A + B = \tan^{-1}\frac{x+y}{1-xy}.$$

$$\therefore \tan^{-1}x + \tan^{-1}y = \tan^{-1}\frac{x+y}{1-xy}.$$

Any relations which have been established among the trigonometric functions may be expressed by means of the inverse notation. Thus, we know that

4. $$\cos x = \sqrt{1 - \sin^2 x}.$$

This may be written $x = \cos^{-1}\sqrt{1-\sin^2 x}$ . . (1)

Put $\sin x = \theta$; then $x = \sin^{-1}\theta$.

Thus (1) becomes $\sin^{-1}\theta = \cos^{-1}\sqrt{1-\theta^2}$.

5. By Art. 49, $\cos 2\theta = 2\cos^2\theta - 1$,

which may be written $2\theta = \cos^{-1}(2\cos^2\theta - 1)$.

Put $\cos\theta = x$. $\therefore 2\cos^{-1}x = \cos^{-1}(2x^2 - 1)$.

6. By Art. 49, $\sin 2\theta = 2\sin\theta\cos\theta$,

which may be written $2\theta = \sin^{-1}(2\sin\theta\cos\theta)$.

Put $\sin\theta = x$. $\therefore 2\sin^{-1}x = \sin^{-1}(2x\sqrt{1-x^2})$.

7. Prove $\sin^{-1}x = \cos^{-1}\sqrt{1-x^2} = \tan^{-1}\dfrac{x}{\sqrt{1-x^2}}.$

8. " $\tan^{-1}x = \sin^{-1}\dfrac{x}{\sqrt{1+x^2}} = \cos^{-1}\dfrac{1}{\sqrt{1+x^2}}.$

9. " $2\tan^{-1}x = \tan^{-1}\dfrac{2x}{1-x^2}.$

10. " $\sin(2\sin^{-1}x) = 2x\sqrt{1-x^2}.$

11. " $\tan^{-1}\dfrac{5}{7} + \tan^{-1}\dfrac{1}{6} = \dfrac{\pi}{4}.$

12. " $\cos^{-1}\dfrac{1}{2} + 2\sin^{-1}\dfrac{1}{2} = 120°.$

13. " $\cot^{-1}3 + \operatorname{cosec}^{-1}\sqrt{5} = \dfrac{\pi}{4}.$

14. " $3\sin^{-1}x = \sin^{-1}(3x - 4x^3)$

61. Table of Useful Formulæ. — The following is a list of important formulæ proved in this chapter, and summed up for the convenience of the student:

1. $\sin(x+y) = \sin x \cos y + \cos x \sin y$ . . (Art. 43)
2. $\cos(x+y) = \cos x \cos y - \sin x \sin y.$
3. $\sin(x-y) = \sin x \cos y - \cos x \sin y$ . . (Art. 44)
4. $\cos(x-y) = \cos x \cos y + \sin x \sin y.$
5. $2\sin x \cos y = \sin(x+y) + \sin(x-y)$. (Art. 45)
6. $2\cos x \sin y = \sin(x+y) - \sin(x-y).$
7. $2\cos x \cos y = \cos(x+y) + \cos(x-y).$
8. $2\sin x \sin y = \cos(x-y) - \cos(x+y).$
9. $\sin x + \sin y = 2\sin\frac{1}{2}(x+y)\cos\frac{1}{2}(x-y).$
10. $\sin x - \sin y = 2\cos\frac{1}{2}(x+y)\sin\frac{1}{2}(x-y).$

11. $\cos x + \cos y = 2\cos\frac{1}{2}(x+y)\cos\frac{1}{2}(x-y)$.

12. $\cos y - \cos x = 2\sin\frac{1}{2}(x+y)\sin\frac{1}{2}(x-y)$.

13. $\dfrac{\sin x + \sin y}{\sin x - \sin y} = \dfrac{\tan\frac{1}{2}(x+y)}{\tan\frac{1}{2}(x-y)}$ . . . . . . (Art. 46)

14. $\tan(x+y) = \dfrac{\tan x + \tan y}{1 - \tan x \tan y}$ . . . . . . (Art. 47)

15. $\tan(x-y) = \dfrac{\tan x - \tan y}{1 + \tan x \tan y}$.

16. $\cot(x+y) = \dfrac{\cot x \cot y - 1}{\cot x + \cot y}$.

17. $\cot(x-y) = \dfrac{\cot x \cot y + 1}{\cot x - \cot y}$.

18. $\tan(x \pm 45°) = \dfrac{\tan x \mp 1}{\tan x \pm 1}$ . . . . (Art. 47)

19. $\sin(x+y)\sin(x-y) = \sin^2 x - \sin^2 y = \cos^2 y - \cos^2 x$.

20. $\cos(x+y)\cos(x-y) = \cos^2 x - \sin^2 y = \cos^2 y - \sin^2 x$.

21. $\tan x \pm \tan y = \dfrac{\sin(x \pm y)}{\cos x \cos y}$.

22. $\cot x \pm \cot y = \dfrac{\sin(y \pm x)}{\sin x \sin y}$.

23. $\sin 2x = 2\sin x\cos x = \dfrac{2\tan x}{1+\tan^2 x}$ . . . (Art. 49)

24. $\cos 2x = \cos^2 x - \sin^2 x = 1 - 2\sin^2 x = 2\cos^2 x - 1$

$$= \frac{1-\tan^2 x}{1+\tan^2 x}.$$

25. $\dfrac{1-\cos 2x}{1+\cos 2x} = \dfrac{2\sin^2 x}{2\cos^2 x} = \tan^2 x$.

26. $\tan 2x = \dfrac{2\tan x}{1-\tan^2 x}$.

27. $\cot 2x = \dfrac{\cot^2 x - 1}{2\cot x}$.

28. $\sin 3x = 3\sin x - 4\sin^3 x$ . . . . (Art. 50)

29. $\cos 3x = 4\cos^3 x - 3\cos x$.

30. $\tan 3x = \dfrac{3\tan x - \tan^3 x}{1 - 3\tan^2 x}$.

31. $\sin^2 \dfrac{x}{2} = \dfrac{1-\cos x}{2}$ . . . . (Art. 51)

32. $\cos^2 \dfrac{x}{2} = \dfrac{1+\cos x}{2}$.

33. $\tan^{-1} x + \tan^{-1} y = \tan^{-1}\dfrac{x+y}{1-xy}$ . . . . (Art. 60)

EXAMPLES.

1. If $\sin\alpha = \frac{1}{3}$, and $\sin\beta = \frac{2}{3}$, find a value for $\sin(\alpha+\beta)$, and $\sin(\alpha-\beta)$. *Ans.* $\dfrac{\sqrt{5}+4\sqrt{2}}{9}$; $\dfrac{\sqrt{5}-4\sqrt{2}}{9}$.

2. If $\cos\alpha = \frac{4}{5}$, and $\cos\beta = \frac{40}{41}$, find a value for $\sin(\alpha+\beta)$, and $\cos(\alpha+\beta)$. *Ans.* $\dfrac{156}{205}$, $\dfrac{133}{205}$.

3. If $\cos\alpha = \frac{3}{4}$, and $\cos\beta = \frac{2}{5}$, find a value for $\sin(\alpha+\beta)$, and $\sin(\alpha-\beta)$. *Ans.* $\dfrac{2\sqrt{7}+3\sqrt{21}}{20}$, $\dfrac{2\sqrt{7}-3\sqrt{21}}{20}$.

4. If $\sin\alpha = \frac{4}{5}$, and $\sin\beta = \frac{3}{5}$, find a value for $\sin(\alpha+\beta)$, and $\cos(\alpha+\beta)$. *Ans.* 1, $\dfrac{24}{25}$.

5. If $\sin\alpha = .6$, and $\sin\beta = \frac{5}{13}$, find a value for $\sin(\alpha - \beta)$, and $\cos(\alpha + \beta)$.

Ans. $\frac{16}{65}$, $\frac{33}{65}$.

6. If $\sin\alpha = \frac{1}{\sqrt{5}}$, and $\sin\beta = \frac{1}{\sqrt{10}}$, show that one value of $\alpha + \beta$ is 45°.

7. Prove $\cos\theta + \cos 3\theta = 2\cos 2\theta \cos\theta$.

8. " $2\cos\alpha\cos\beta = \cos(\alpha - \beta) + \cos(\alpha + \beta)$.

9. " $2\sin 3\theta \cos 5\theta = \sin 8\theta - \sin 2\theta$.

10. " $2\cos\frac{3}{2}\theta\cos\frac{\theta}{2} = \cos\theta + \cos 2\theta$.

11. " $\sin 4\theta \sin\theta = \frac{1}{2}(\cos 3\theta - \cos 5\theta)$.

12. " $2\cos 10^\circ \sin 50^\circ = \sin 60^\circ + \sin 40^\circ$.

13. Simplify $2\cos 2\theta\cos\theta - 2\sin 4\theta \sin\theta$.

Ans. $2\cos 3\theta \cos 2\theta$.

14. Simplify $\sin\frac{5\theta}{2}\cos\frac{\theta}{2} - \sin\frac{9\theta}{2}\cos\frac{3\theta}{2}$. $-\cos 4\theta \sin 2\theta$.

Prove the following statements:

15. $\cos 3\alpha - \cos 7\alpha = 2\sin 5\alpha \sin 2\alpha$.

16. $\sin 60^\circ + \sin 20^\circ = 2\sin 40^\circ \cos 20^\circ$.

17. $\sin 3\theta + \sin 5\theta = 2\sin 4\theta\cos\theta$.

18. $\sin 7\theta - \sin 5\theta = 2\cos 6\theta \sin\theta$.

19. $\cos 5\theta + \cos 9\theta = 2\cos 7\theta\cos 2\theta$.

20. $\dfrac{\sin 2\theta + \sin\theta}{\cos\theta + \cos 2\theta} = \tan\dfrac{3\theta}{2}$.

21. $\cos(60^\circ + A) + \cos(60^\circ - A) = \cos A$.

22. $\cos(45^\circ + A) + \cos(45^\circ - A) = \sqrt{2}\cos A.$

23. $\sin(45^\circ + A) - \sin(45^\circ - A) = \sqrt{2}\sin A.$

24. $\cos 2\theta + \cos 4\theta = 2\cos 3\theta \cos\theta.$

25. $\cos 4\theta - \cos 6\theta = 2\sin 5\theta \sin\theta.$

26. $\cos\theta + \cos 3\theta + \cos 5\theta + \cos 7\theta = 4\cos\theta\cos 2\theta\cos 4\theta.$

27. $\cot\alpha + \tan\beta = \dfrac{\cos(\alpha - \beta)}{\sin\alpha\cos\beta}.$

28. $\cot\alpha - \tan\beta = \dfrac{\cos(\alpha + \beta)}{\sin\alpha\cos\beta}.$

29. $\sin(A - 45^\circ) = \dfrac{\sin A - \cos A}{\sqrt{2}}.$

30. $\sqrt{2}\sin(A + 45^\circ) = \sin A + \cos A.$

31. $\cos(A + 45^\circ) + \sin(A - 45^\circ) = 0.$

32. $\dfrac{\tan(\theta - \phi) + \tan\phi}{1 - \tan(\theta - \phi)\tan\phi} = \tan\theta.$

33. $\dfrac{\tan(\theta + \phi) + \tan\phi}{1 + \tan(\theta + \phi)\tan\theta} = \tan\phi.$

34. $\cos(\theta + \phi) - \sin(\theta - \phi) = 2\sin\left(\frac{\pi}{4} - \theta\right)\cos\left(\frac{\pi}{4} - \phi\right).$

35. $\sin n\theta\cos\theta + \cos n\theta\sin\theta = \sin(n + 1)\theta.$

36. $\cot\left(\theta - \frac{\pi}{4}\right) = \dfrac{\cot\theta + 1}{1 - \cot\theta}.$

37. $\tan\left(\theta - \frac{\pi}{4}\right) + \cot\left(\theta + \frac{\pi}{4}\right) = 0.$

38. $\cot\left(\theta - \frac{\pi}{4}\right) + \tan\left(\theta + \frac{\pi}{4}\right) = 0.$

39. $\dfrac{\tan(n + 1)\phi - \tan n\phi}{1 + \tan(n + 1)\phi\tan n\phi} = \tan\phi.$

40. If $\tan x = 1$, and $\tan y = \dfrac{1}{\sqrt{3}}$, prove that

$$\tan(x+y) = 2 + \sqrt{3}.$$

41. If $\tan\alpha = \dfrac{m}{m+1}$, and $\tan\beta = \dfrac{1}{2m+1}$, prove that

$$\tan(\alpha+\beta) = 1.$$

42. If $\tan\alpha = m$, and $\tan\beta = n$, prove that

$$\cos(\alpha+\beta) = \frac{1-mn}{\sqrt{(1+m^2)(1+n^2)}}.$$

43. If $\tan\theta = (a+1)$, and $\tan\phi = (a-1)$, prove that

$$2\cot(\theta-\phi) = a^2.$$

Prove the following statements:

44. $\cos(x-y+z) = \cos x\cos y\cos z + \cos x\sin y\sin z$
$\qquad - \sin x\cos y\sin z + \sin x\sin y\cos z.$

45. $\sin(x-y-z) = \sin x + \sin y + \sin z$
$\qquad + 4\sin\frac{1}{2}(x-y)\sin\frac{1}{2}(x-z)\sin\frac{1}{2}(y+z).$

46. $\sin(x+y-z) + \sin(x+z-y) + \sin(y+z-x)$
$\qquad = \sin(x+y+z) + 4\sin x\sin y\sin z.$

47. $\sin 2x + \sin 2y + \sin 2z - \sin 2(x+y+z)$
$\qquad = 4\sin(x+y)\sin(y+z)\sin(z+x).$

48. $\cos 2x + \cos 2y + \cos 2z + \cos 2(x+y+z)$
$\qquad = 4\cos(x+y)\cos(y+z)\cos(z+x).$

49. $\cos(x+y-z) + \cos(y+z-x) + \cos(z+x-y)$
$\qquad + \cos(x+y+z) = 4\cos x\cos y\cos z.$

50. $\sin^2 x + \sin^2 y + \sin^2 z + \sin^2 (x + y + z)$

$$= 2\{1 - \cos (x + y) \cos (y+z) \cos (z+x)\}.$$

51. $\cos^2 x + \cos^2 y + \cos^2 z + \cos^2 (x + y - z)$

$$= 2\{1 + \cos (x + y) \cos (x-z) \cos (y-z)\}.$$

52. $\cos x \sin(y-z) + \cos y \sin(z-x) + \cos z \sin(x-y) = 0.$

53. $\sin x \sin(y-z) + \sin y \sin(z-x) + \sin z \sin(x-y) = 0.$

54. $\cos (x + y) \cos (x - y) + \sin (y + z) \sin (y - z)$

$$- \cos (x + z) \cos (x - z) = 0.$$

55. $\dfrac{2 - \sec^2\theta}{\sec^2\theta} = \cos 2\theta.$

56. $\cos^2\theta (1 - \tan^2\theta) = \cos 2\theta.$

57. $\cot 2\theta = \dfrac{\cot^2\theta - 1}{2\cot\theta}.$

58. $\sec 2\theta = \dfrac{\cot^2\theta + 1}{\cot^2\theta - 1}.$

59. $\left(\sin\dfrac{\theta}{2} + \cos\dfrac{\theta}{2}\right)^2 = 1 + \sin\theta.$

60. $\left(\sin\dfrac{\theta}{2} - \cos\dfrac{\theta}{2}\right)^2 = 1 - \sin\theta.$

61. $\dfrac{1 + \sec\theta}{\sec\theta} = 2\cos^2\dfrac{\theta}{2}.$

62. $\dfrac{\cos 2\theta}{1 + \sin 2\theta} = \dfrac{1 - \tan\theta}{1 + \tan\theta}.$

63. $\dfrac{\cos\theta}{1 - \sin\theta} = \dfrac{1 + \tan\dfrac{\theta}{2}}{1 - \tan\dfrac{\theta}{2}}.$

64. $\dfrac{1+\sin x+\cos x}{1+\sin x-\cos x}=\cot\dfrac{x}{2}.$

65. $\dfrac{\cos^3 x+\sin^3 x}{\cos x+\sin x}=\dfrac{2-\sin 2x}{2}.$

66. $\dfrac{\cos^3 x-\sin^3 x}{\cos x-\sin x}=\dfrac{2+\sin 2x}{2}.$

67. $\cos^4\theta-\sin^4\theta=\cos 2\theta.$

68. $\dfrac{\sin 3\theta}{\sin\theta}-\dfrac{\cos 3\theta}{\cos\theta}=2.$

69. $\dfrac{\cos 3\theta}{\sin\theta}+\dfrac{\sin 3\theta}{\cos\theta}=2\cot 2\theta.$

70. $\dfrac{\sin 4\theta}{\sin 2\theta}=2\cos 2\theta.$

71. $\dfrac{\sin\dfrac{5\pi}{12}}{\sin\dfrac{\pi}{12}}-\dfrac{\cos\dfrac{5\pi}{12}}{\cos\dfrac{\pi}{12}}=2\sqrt{3}.$

72. $\tan(45^\circ+x)-\tan(45^\circ-x)=2\tan 2x.$

73. $\tan(45^\circ-x)+\cot(45^\circ-x)=2\sec 2x.$

74. $\dfrac{\tan^2(45^\circ+x)-1}{\tan^2(45^\circ+x)+1}=\sin 2x.$

75. $\dfrac{\cos(x+45^\circ)}{\cos(x-45^\circ)}=\sec 2x-\tan 2x.$

76. $\tan x=\dfrac{\sin x+\sin 2x}{1+\cos x+\cos 2x}.$

77. $\tan x=\dfrac{\sin 2x-\sin x}{1-\cos x+\cos 2x}.$

78. $\dfrac{\cos 3x}{\cos x}=2\cos 2x-1.$

79. $\dfrac{3\sin x - \sin 3x}{\cos 3x + 3\cos x} = \tan^3 x.$

80. $\cot 3x = \dfrac{\cot^3 x - 3\cot x}{3\cot^2 x - 1}.$

81. $\dfrac{1 - \cos 3x}{1 - \cos x} = (1 + 2\cos x)^2.$

82. $\dfrac{\sin x + \cos x}{\cos x - \sin x} = \tan 2x + \sec 2x.$

83. $\dfrac{\cos 2x + \cos 12x}{\cos 6x + \cos 8x} + \dfrac{\cos 7x - \cos 3x}{\cos x - \cos 3x} + \dfrac{2\sin 4x}{\sin 2x} = 0.$

84. $\sin 2x \sin 2y = \sin^2(x + y) - \sin^2(x - y).$

85. $\tan 50° + \cot 50° = 2\sec 10°.$

86. $\sin 3x = 4\sin x \sin (60° + x) \sin (60° - x).$

87. $\cot\frac{\pi}{8} - \tan\frac{\pi}{8} = 2.$

88. $\tan 4\theta = \dfrac{4\tan\theta\,(1 - \tan^2\theta)}{1 - 6\tan^2\theta + \tan^4\theta}.$

89. $2\cos\frac{\pi}{8} = \sqrt{2 + \sqrt{2}}.$

90. $(3\sin\theta - 4\sin^3\theta)^2 + (4\cos^3\theta - 3\cos\theta)^2 = 1.$

91. $\dfrac{\sin 2\theta \cos\theta}{(1 + \cos 2\theta)(1 + \cos\theta)} = \tan\frac{\theta}{2}.$

92. If $\tan\theta = \frac{1}{7}$, and $\tan\phi = \frac{2}{11}$, prove $\tan(2\theta + \phi) = \frac{1}{2}.$

93. Prove that $\tan\frac{\theta}{2}$ and $\cot\frac{\theta}{2}$ are the roots of the equation

$$x^2 - 2x \operatorname{cosec}\theta + 1 = 0.$$

94. If $\tan\theta = \frac{b}{a}$, prove that

$$\sqrt{\frac{a+b}{a-b}} + \sqrt{\frac{a-b}{a+b}} = \frac{2\cos\theta}{\sqrt{\cos 2\theta}}.$$

95. Find the values of (1) $\sin 9°$, (2) $\cos 9°$, (3) $\sin 81°$, (4) $\cos 189°$, (5) $\tan 202\frac{1}{2}°$, (6) $\tan 97\frac{1}{2}°$.

Ans. (1) $\frac{1}{4}(\sqrt{3+\sqrt{5}} - \sqrt{5-\sqrt{5}})$,

(2) $\frac{1}{4}(\sqrt{3+\sqrt{5}} + \sqrt{5-\sqrt{5}})$,

(3) $\sin 81° = \cos 9°$,

(4) $\cos 189° = -\cos 9°$,

(5) $\sqrt{2} - 1$,

(6) $-(\sqrt{3}+\sqrt{2})(\sqrt{2}+1)$.

96. If $A = 200°$, prove that

(1) $2\sin\frac{A}{2} = +\sqrt{1+\sin A} + \sqrt{1-\sin A}$.

(2) $\tan\frac{A}{2} = \frac{-(1+\sqrt{1+\tan^2 A})}{\tan A}$.

97. If A lies between 270° and 360°, prove that

(1) $2\sin\frac{A}{2} = +\sqrt{1-\sin A} - \sqrt{1+\sin A}$.

(2) $\tan\frac{A}{2} = -\cot A + \operatorname{cosec} A$.

98. If A lies between 450° and 630°, prove that

(1) $2\sin\frac{A}{2} = -\sqrt{1+\sin A} - \sqrt{1-\sin A}$.

(2) $2\cos\frac{A}{2} = -\sqrt{1+\sin A} + \sqrt{1-\sin A}$.

Prove the following statements, A, B, C being the angles of a triangle.

99. $\dfrac{\sin A - \sin B}{\sin A + \sin B} = \tan\dfrac{C}{2}\tan\dfrac{A - B}{2}.$

100. $\dfrac{\sin 3B - \sin 3C}{\cos 3C - \cos 3B} = \dfrac{\tan 3A}{2}.$

101. $\sin\dfrac{A}{2}\cos\dfrac{A}{2} + \sin\dfrac{B}{2}\cos\dfrac{B}{2} + \sin\dfrac{C}{2}\cos\dfrac{C}{2}$

$$= 2\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}.$$

102. $\cos^2\dfrac{A}{2} + \cos^2\dfrac{B}{2} - \cos^2\dfrac{C}{2} = 2\cos\dfrac{A}{2}\cos\dfrac{B}{2}\sin\dfrac{C}{2}.$

103. $\sin A\cos A - \sin B\cos B + \sin C\cos C$

$$= 2\cos A\sin B\cos C.$$

104. $\cos 2A + \cos 2B + \cos 2C = -1 - 4\cos A\cos B\cos C.$

105. $\sin^2 A - \sin^2 B + \sin^2 C = 2\sin A\cos B\sin C.$

106. $\tan\dfrac{B}{2}\tan\dfrac{C}{2} + \tan\dfrac{C}{2}\tan\dfrac{A}{2} + \tan\dfrac{A}{2}\tan\dfrac{B}{2} = 1.$

Prove the following statements when we take for $\sin^{-1}$, $\cos^{-1}$, etc., their least *positive* value.

107. $\sin^{-1}\dfrac{1}{2} = \cos^{-1}\dfrac{\sqrt{3}}{2} = \cot^{-1}\sqrt{3}.$

108. $2\tan^{-1}(\cos 2\theta) = \tan^{-1}\left(\dfrac{\cot^2\theta - \tan^2\theta}{2}\right).$

109. $4\tan^{-1}\dfrac{1}{5} - \tan^{-1}\dfrac{1}{239} = \dfrac{\pi}{4}.$

110. $\sin^{-1}\dfrac{4}{5} + \sin^{-1}\dfrac{8}{17} + \sin^{-1}\dfrac{13}{85} = \dfrac{\pi}{2}.$

111. $\tan^{-1}\sqrt{5}\,(2-\sqrt{3})-\cot^{-1}\sqrt{5}\,(2+\sqrt{3})=\cot^{-1}\sqrt{5}.$

112. $\sec^{-1}\sqrt{3}=2\cot^{-1}\sqrt{2}.$

113. $2\cot^{-1}x=\operatorname{cosec}^{-1}\dfrac{1+x^2}{2x}.$

114. $\tan^{-1}\dfrac{\sqrt{3}+\sqrt{2}}{\sqrt{3}-\sqrt{2}}+\tan^{-1}\sqrt{\dfrac{3}{2}}=\dfrac{3\pi}{4}.$

115. $\sin^{-1}\dfrac{1}{\sqrt{5}}+\cot^{-1}3=\dfrac{\pi}{4}.$

116. $\cos^{-1}\dfrac{63}{65}+2\tan^{-1}\dfrac{1}{5}=\sin^{-1}\dfrac{3}{5}.$

117. If $\theta=\sin^{-1}\dfrac{3}{5}$, and $\phi=\cos^{-1}\dfrac{3}{5}$, then $\theta+\phi=90°$.

118. Prove that $\cos(2\tan^{-1}x)=\dfrac{1-x^2}{1+x^2}.$

119. " " $\tan^{-1}\dfrac{1}{2}+\operatorname{cosec}^{-1}\sqrt{10}=\dfrac{\pi}{4}.$

120. " " $2\tan^{-1}\dfrac{2}{3}-\operatorname{cosec}^{-1}\dfrac{5}{3}=\sin^{-1}\dfrac{33}{65}.$

121. " " $2\tan^{-1}\dfrac{1}{2}+\cos^{-1}\dfrac{4}{5}=\dfrac{\pi}{2}.$

122. " " $\sin^{-1}(\cos x)+\cos^{-1}(\sin y)+x+y=\pi.$

123. " " $\tan^{-1}\dfrac{1}{1+x}+\tan^{-1}\dfrac{1}{1-x}+\tan^{-1}\dfrac{2}{x^2}=n\pi.$

124. " " $\tan^{-1}\dfrac{x-1}{x}+\tan^{-1}\dfrac{1}{2x-1}=n\pi+\dfrac{\pi}{4}.$

125. " " $\sin^{-1}x-\sin^{-1}y$

$$=\cos^{-1}(xy\pm\sqrt{1-x^2-y^2+x^2y^2}).$$

CHAPTER IV.

LOGARITHMS AND LOGARITHMIC TABLES. — TRIGONOMETRIC TABLES.

62. Nature and Use of Logarithms. — The numerical calculations which occur in Trigonometry are very much abbreviated by the aid of logarithms; and thus it is necessary to explain the nature and use of logarithms, and the manner of calculating them.

The *logarithm* of a number to a given *base* is the exponent of the power to which the base must be raised to give the number.

Thus, if $a^x = m$, x is called the "logarithm of m to the base a," and is usually written $x = \log_a m$, the base being put as a suffix.*

The relation between the base, logarithm, and number is expressed by the equation,

$$(\text{base})^{\log} = \text{number}.$$

Thus, if the base of a system of logarithms is 2, then 3 is the logarithm of the number 8, because $2^3 = 8$.

If the base be 5, then 3 is the logarithm of 125, because $5^3 = 125$.

63. Properties of Logarithms. — The use of logarithms depends on the following *properties* which are true for all logarithms, whatever may be the base.

* From the definition it follows that (1) $\log_a a^x = x$, and conversely (2) $a^{\log_a m} = m$. Taking the logarithms of both sides of the equation $a^x = m$, we have $\log_a a^x = x = \log m$. Conversely, taking the exponentials of both sides of $x = \log_a m$ to base a, we have $a^x = a^{\log_a m} = m$. $a^x = m$ and $x = \log_a m$ are thus seen to be equivalent, and to express the same relation between a number, m, and its logarithm, x, to base a.

(1) *The logarithm of* 1 *is zero.*

For $a^0 = 1$, whatever a may be; therefore $\log 1 = 0$.

(2) *The logarithm of the base of any system is unity.*

For $a^1 = a$, whatever a may be; therefore $\log_a a = 1$.

(3) *The logarithm of zero in any system whose base is greater than* 1 *is minus infinity.*

For $a^{-\infty} = \frac{1}{a^\infty} = \frac{1}{\infty} = 0$; therefore $\log 0 = -\infty$.

(4) *The logarithm of a product is equal to the sum of the logarithms of its factors.*

For let $$x = \log_a m, \text{ and } y = \log_a n.$$

$$\therefore\ m = a^x, \quad \text{and } n = a^y.$$

$$\therefore\ mn = a^{x+y}.$$

$$\therefore\ \log_a mn = x + y = \log_a m + \log_a n.$$

Similarly, $$\log_a mnp = \log_a m + \log_a n + \log_a p,$$

and so on for any number of factors.

Thus, $$\log 60 = \log(3 \times 4 \times 5),$$
$$= \log 3 + \log 4 + \log 5.$$

(5) *The logarithm of a quotient is equal to the logarithm of the dividend minus the logarithm of the divisor.*

For let $$x = \log_a m, \text{ and } y = \log_a n.$$

$$\therefore\ m = a^x, \quad \text{and } n = a^y.$$

$$\therefore\ \frac{m}{n} = a^{x-y}.$$

$$\therefore\ \log_a \frac{m}{n} = x - y = \log_a m - \log_a n.$$

Thus, $$\log \frac{17}{5} = \log 17 - \log 5.$$

(6) *The logarithm of any power of a number is equal to the logarithm of the number multiplied by the exponent of the power.*

For let $x = \log_a m. \quad \therefore m = a^x.$

$$\therefore m^p = a^{px}.$$

$$\therefore \log_a m^p = px = p \log_a m.$$

(7) *The logarithm of any root of a number is equal to the logarithm of the number divided by the index of the root.*

For let $x = \log_a m. \quad \therefore m = a^x.$

$$\therefore m^{\frac{1}{r}} = a^{\frac{x}{r}}.$$

$$\therefore \log (m^{\frac{1}{r}}) = \frac{x}{r} = \frac{1}{r} \log_a m.$$

It follows from these propositions that by means of logarithms, the operations of *multiplication* and *division* are changed into those of *addition* and *subtraction;* and the operations of *involution* and *evolution* are changed into those of *multiplication* and *division.*

1. Suppose, for instance, it is required to find the product of 246 and 357; we add the logarithms of the factors, and the sum is the logarithm of the product: thus,

$$\begin{aligned} \log_{10} 246 &= 2.39093 \\ \log_{10} 357 &= \underline{2.55267} \\ & 4.94360 \end{aligned}$$

which is the logarithm of 87822, the product required.

2. If we are required to divide 371.49 by 52.376, we proceed thus:

$$\begin{aligned} \log_{10} 371.49 &= 2.56995 \\ \log_{10} 52.376 &= \underline{1.71913} \\ & 0.85082 \end{aligned}$$

which is the logarithm of 7.092752, the quotient required.

3. If we have to find the fourth power of 13, we proceed thus:

$$\begin{array}{r} \log_{10} 13 = 1.11394 \\ 4 \\ \hline 4.45576 \end{array}$$

which is the logarithm of 28561, the number required.

4. If we are to find the fifth root of 16807, we proceed thus:

$$\begin{array}{l} 5\underline{)4.22549} = \log_{10} 16807, \\ \quad 0.845098 \end{array}$$

which is the logarithm of 7, the root required.

5. Given $\log_{10} 2 = 0.30103$; find $\log_{10} 128$, $\log_{10} 512$.

Ans. 2.10721, 2.70927.

6. Given $\log_{10} 3 = 0.47712$; find $\log_{10} 81$, $\log_{10} 2187$.

Ans. 1.90849, 3.33985.

7. Given $\log_{10} 3$; find $\log_{10} \sqrt[5]{3^3}$. 0.28627.

8. Find the logarithms to the base a of

$$a^3,\ a^{\frac{10}{3}},\ \sqrt[4]{a},\ \sqrt[3]{a^2},\ a^{-\frac{5}{2}}.$$

9. Find the logarithms to the base 2 of 8, 64, $\frac{1}{2}$, .125, .015625, $\sqrt[3]{64}$. *Ans.* 3, 6, -1, -3, -6, 2.

10. Find the logarithms to base 4 of 8, $\sqrt[3]{16}$, $\sqrt[4]{.5}$, $\sqrt[3]{.015625}$. *Ans.* $\frac{3}{2}$, $\frac{2}{3}$, $-\frac{1}{4}$, -1.

Express the following logarithms in terms of $\log a$, $\log b$, and $\log c$:

11. $\log \sqrt{(a^2 b^3 c)^6}$. *Ans.* $6 \log a + 9 \log b + 3 \log c$.

12. $\log \sqrt[4]{a^3 b^5 c^7}$. $\frac{3}{4} \log a + \frac{5}{4} \log b + \frac{7}{4} \log c$.

13. $\log \dfrac{\sqrt[3]{ab^{-1} c^{-2}}}{(a^{-1} b^{-2} c^{-4})^{\frac{1}{6}}}$. $\frac{1}{2} \log a$.

64. Common System of Logarithms. — There are two systems of logarithms in use, viz., the *Naperian** system and the *common* system.

The *Naperian* system is used for purely *theoretic investigations;* its base is $e = 2.7182818$.

The *common* system† of logarithms is the system that is used in all practical calculations; its base is 10.

By a *system of logarithms* to the base 10, is meant a succession of values of x which satisfy the equation

$$m = 10^x,$$

for all *positive* values of m, *integral* or *fractional.* Thus, if we suppose m to assume in succession every value from 0 to ∞, the corresponding values of x will form a *system of logarithms,* to the base 10.

Such a system is formed by means of the series of logarithms of the natural numbers from 1 to 100000, which constitute the logarithms registered in our ordinary tables.

Now $\quad 10^0 = 1, \qquad \therefore \log 1 = 0;$

$\quad 10^1 = 10, \qquad \therefore \log 10 = 1;$

$\quad 10^2 = 100, \qquad \therefore \log 100 = 2;$

$\quad 10^3 = 1000, \qquad \therefore \log 1000 = 3.$

and so on.

Also, $\quad 10^{-1} = \frac{1}{10} = .1, \qquad \therefore \log .1 = -1;$

$\quad 10^{-2} = \frac{1}{100} = .01, \qquad \therefore \log .01 = -2;$

$\quad 10^{-3} = \frac{1}{1000} = .001, \qquad \therefore \log .001 = -3.$

and so on.

Hence, in the common system, the logarithm of any number between

1 and 10 is some number between 0 and 1; *i.e.*, 0 + a decimal;

* So called from its inventor, *Baron Napier*, a Scotch mathematician.
† First introduced in 1615 by *Briggs*, a contemporary of Napier.

10 and 100 is some number between 1 and 2; *i.e.*, 1 + a decimal;

100 and 1000 is some number between 2 and 3; *i.e.*, 2 + a decimal;

1 and .1 is some number between 0 and −1; *i.e.*, −1 + a decimal;

.1 and .01 is some number between −1 and −2; *i.e.*, −2 + a decimal;

.01 and .001 is some number between −2 and −3; *i.e.*, −3 + a decimal;

and so on.

It thus appears that

(1) The (common) logarithm of any number greater than 1 is *positive.*

(2) The logarithm of any positive number less than 1 is *negative.*

(3) In general, the common logarithm of a number consists of two parts, an integral part and a decimal part.

The *integral* part of a logarithm is called the *characteristic* of the logarithm, and may be *either positive or negative.*

The *decimal* part of a logarithm is called the *mantissa* of the logarithm, and is *always kept positive.*

NOTE. — It is *convenient* to keep the *decimal part* of the logarithms always *positive,* in order that numbers consisting of the same digits in the same order may correspond to the same mantissa.

It is evident from the above examples that the characteristic of a logarithm can always be obtained by the following rule:

RULE. — The characteristic of the logarithm of a number greater than unity is *one less* than the number of digits in the whole number.

The characteristic of the logarithm of a number less than unity is *negative,* and is *one more* than the number of ciphers immediately after the decimal point.

Thus, the characteristics of the logarithms of 1234, 123.4, 1.234, .1234, .00001234, 12340, are respectively, 3, 2, 0, -1, -5, 4.

NOTE. — When the characteristic is negative, the minus sign is written *over* it to indicate that *the characteristic* alone is negative, the mantissa being always positive.

Write down the characteristics of the common logarithms of the following numbers:

1. 17601, 361.1, 4.01, 723000, 29. *Ans.* 4, 2, 0, 5, 1.

2. .04, .0000612, .7963, .001201, .1.
Ans. -2, -5, -1, -3, -1.

3. How many digits are there in the integral part of the numbers whose common logarithms are respectively 3.461, 0.30203, 5.47123, 2.67101?

4. Given $\log 2 = 0.30103$; find the number of digits in the integral part of 8^{10}, 2^{12}, 16^{20}, 2^{100}. *Ans.* 10, 4, 25, 31.

65. Comparison of Two Systems of Logarithms. — Given the logarithm of a number to base a; to find the logarithm of the same number to base b.

Let m be any number whose logarithm to base b is required.

Let $$x = \log_b m; \text{ then } b^x = m.$$

$$\therefore \log_a(b^x) = \log_a m; \text{ or } x \log_a b = \log_a m.$$

$$\therefore x = \frac{1}{\log_a b} \times \log_a m,$$

or $$\log_b m = \frac{\log_a m}{\log_a b} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Hence, to transform the logarithm of a number from base a to base b, we multiply it by $\frac{1}{\log_a b}$.

This constant multiplier $\frac{1}{\log_a b}$ is called the *modulus of the system of which the base is b* with reference to the system of which the base is a.

If, then, a list of logarithms to some base e can be made, we can deduce from it a list of common logarithms by multiplying each logarithm in the given list by the *modulus of the common system* $\frac{1}{\log_e 10}$.

Putting a for m in (1), we have

$$\log_b a = \frac{\log_a a}{\log_a b} = \frac{1}{\log_a b}, \text{ by (2) of Art. 63.}$$

$$\therefore \log_b a \times \log_a b = 1.$$

EXAMPLES.

1. Show how to transform logarithms with base 5 to logarithms with base 125.

Let m be any number, and let x be its logarithm to base 125.

Then $$m = 125^x = (5^3)^x = 5^{3x}. \quad \therefore 3x = \log_5 m.$$

$$\therefore x = \log_{125} m = \tfrac{1}{3} \log_5 m.$$

Thus, the logarithm of any number to base 5, divided by 3 (*i.e.*, by $\log_5 125$), is the logarithm of the same number to the base 125.

Otherwise by the rule given in (1). Thus,

$$\log_{125} m = \frac{\log_5 m}{\log_5 125} = \frac{\log_5 m}{3}.$$

Show how to transform

2. Logarithms with base 2 to logarithms with base 8.

Ans. Divide each logarithm by 3.

3. Logarithms with base 9 to logarithms with base 3.

Ans. Multiply each logarithm by 2.

4. Find $\log_2 8$, $\log_5 1$, $\log_8 2$, $\log_7 1$, $\log_{32} 128$.

Ans. 3, 0, $\frac{1}{3}$, 0, $\frac{7}{5}$.

66. Tables of Logarithms. — The common logarithms of all integers from 1 to 100000 have been found and registered in tables, which are therefore called *tabular logarithms.* In most tables they are given to six places of decimals, though they may be calculated to various degrees of approximation, such as five, six, seven, or a higher number of decimal places. Tables of logarithms to seven places of decimals are in common use for astronomical and mathematical calculations. The common system to base 10 is the one in practical use, and it has two great advantages:

(1) From the rule (Art. 64) the characteristics can be written down at once, so that only the mantissæ have to be given in the tables.

(2) The mantissæ are the same for the logarithms of all numbers which have the same significant digits, in the same order, so that it is sufficient to tabulate the mantissæ of the logarithms of *integers.*

For, since altering the position of the decimal point without changing the sequence of figures merely multiplies or divides the number by an integral power of 10, it follows that its logarithm will be increased or diminished by an integer; *i.e.*, that the mantissa of the logarithm remains unaltered.

In General. — If N be any number, and p and q any integers, it follows that $N \times 10^p$ and $N \div 10^q$ are numbers whose significant digits are the same as those of N.

Then $\log(N \times 10^p) = \log N + p \log 10 = \log N + p.$ (1)

Also, $\log(N \div 10^q) = \log N - q \log 10 = \log N - q.$ (2)

In (1) the logarithm of N is increased by an integer, and in (2) it is diminished by an integer.

That is, the same mantissa serves for the logarithms of all numbers, *whether greater or less than unity*, which have the same significant digits, and differ only in the position of the decimal point.

This will perhaps be better understood if we take a particular case.

From a table of logarithms we find the mantissa of the logarithm of 787 to be 895975; therefore, prefixing the characteristic with its appropriate sign according to the rule, we have

$$\log 787 = 2.895975.$$

Now $$\log 7.87 = \log\frac{787}{100} = \log 787 - 2 = 0.895975.$$

Also, $$\log .0787 = \log\left(\frac{787}{10000}\right) = \log 787 - 4 = \bar{2}.895975.$$

Also, $$\log 78700 = \log(787 \times 100) = \log 787 + 2 = 4.895975.$$

NOTE 1. — We do not write $\log_{10} 787$; for so long as we are treating of logarithms to the particular base 10, we may omit the suffix.

NOTE 2. — Sometimes in working with negative logarithms, an arithmetic artifice will be necessary to make the mantissa positive. For example, a result such as -2.69897, in which the whole expression is negative, may be transformed by subtracting 1 from the characteristic, and adding 1 to the mantissa. Thus,

$$-2.69897 = -3 + (1 - .69897) = \bar{3}.30103.$$

NOTE 3. — When the characteristic of a logarithm is negative, it is often, especially in Astronomy and Geodesy, for convenience, made positive by the addition of 10, which can lead to no error, if we are careful to subtract 10.

Thus, instead of the logarithm $\bar{3}.603582$, we may write $7.603582 - 10$.

In calculations with negative characteristics we follow the rules of Algebra.

EXAMPLES.

1. Add together

$$\begin{array}{r} \bar{2}.2143 \\ \bar{1}.3142 \\ \bar{5}.9068 \\ \hline \bar{7}.4353 \end{array} \quad \textit{Ans.}$$

2. From $\bar{3}.24569$

take $\bar{5}.62493$

$$\begin{array}{r} \bar{3}.24569 \\ \bar{5}.62493 \\ \hline 1.62076 \end{array}$$

the 1 carried from the last subtraction in decimal places changes -5 into -4, and then -4 subtracted from -3 gives 1 as a result.

3. Multiply $\bar{2}.1528$ by 7.

$$\begin{array}{r} \bar{2}.1528 \\ 7 \\ \hline \overline{13}.0696 \end{array}$$

the 1 carried from the last multiplication of the decimal places being added to -14, and thus giving -13 as a result.

NOTE 4. — When a logarithm with negative characteristic has to be divided by a number which is not an exact divisor of the characteristic, we proceed as follows in order to keep the characteristic integral. *Increase the characteristic numerically by a number which will make it exactly divisible, and prefix an equal positive number to the mantissa.*

4. Divide $\bar{3}.7268$ by 5.

Increase the negative characteristic so that it may be exactly divisible by 5; thus

$$\frac{\bar{3}.7268}{5} = \frac{\bar{5} + 2.7268}{5} = \bar{1}.5453.$$

Given that $\log 2 = .30103$, $\log 3 = .47712$, and $\log 7 = .84510$; find the values of

5. $\log 6$, $\log 42$, $\log 16$. *Ans.* .77815, 1.62325, 1.20412.

6. log 49, log 36, log 63. *Ans.* 1.69020, 1.55630, 1.79934.

7. log 200, log 600, log 70. 2.30103, 2.77815, 1.84510.

8. log 60, log .03, log 1.05, log .0000432.

NOTE. — The logarithm of 5 and its powers can always be obtained from log 2.

Ans. 1.77815, $\bar{2}.47712$, .02119, $\bar{5}.63548$.

9. Given log 2 = .30103; find log 128, log 125, and log 2500.
Ans. 2.10721, 2.09691, 3.39794.

Given the logarithms of 2, 3, and 7, as above; find the logarithms of the following:

10. 20736, 432, 98, 686, 1.728, .336.

Ans. 4.31672, 2.63548, 1.99122, 2.83632, .23754, $\bar{1}.52634$.

11. $\sqrt{.2}$, $(.03)^{\frac{1}{4}}$, $(.0021)^{\frac{1}{5}}$, $(.098)^3$, $(.00042)^5$, $(.0336)^{\frac{1}{2}}$.
Ans. $\bar{1}.65052$, $\bar{1}.61928$, $\bar{1}.46444$, $\bar{4}.97368$, $\overline{17}.11625$, $\bar{1}.26317$.

67. Use of Tables of Logarithms* of Numbers. — In our explanations of the use of tables of common logarithms we shall use tables of seven places of decimals.† These tables are arranged so as to give the mantissæ of the logarithms of the natural members from 1 to 100000; *i.e.*, of numbers containing from one to five digits.

A table of logarithms of numbers correct to seven decimal places is *exact* for all the practical purposes of Astronomy and Geodesy. For an actual measurement of any kind must be made with the greatest care, with the most accurate instruments, by the most skilful observers, if it is to attain to anything like the accuracy represented by seven significant figures.

* The methods by which these tables are formed will be given in Chap. VIII.

† The student should here provide himself with logarithmic and trigonometric tables of seven decimal places. The most convenient seven-figure tables used in this country are Stanley's, Vega's, Bruhns', etc. In the appendix to the Elementary Trigonometry are given five-figured tables, which are sufficiently near for most practical applications.

If the measure of any *length* is known accurately to seven figures, it is practically exact; *i.e.*, it is known to within the limits of observation.

If the measure of any *angle* is known to within the tenth part of a second, the greatest accuracy possible, at present, in the measurement of angles is reached. The tenth part of a second is about the two-millionth part of a radian. This degree of accuracy is attainable only with the largest and best instruments, and under the most favorable conditions.

On page 101 is a specimen page of Logarithmic Tables. It consists of the mantissæ of the logarithms, correct to seven places of decimals, of all numbers between 62500 and 63009. The figures of the *number* are those in the left column headed N, followed by one in larger type at the top of the page. The first three figures of the mantissæ 795, 796, 797 ⋯, and the remaining four are in the same horizontal line with the first four figures of the number, and in the vertical column under the last.

Logarithms are in general *incommensurable* numbers. Their values can therefore only be given *approximately*. Throughout all *approximate* calculations it is usual to take for the last figure which we retain, that figure which gives the nearest approach to the true value. When only a certain number of decimal places is required, the general rule is this: *Strike out the rest of the figures, and increase the last figure retained by 1 if the first figure struck off is 5 or greater than 5.*

68. To find the Logarithm of a Given Number. — When the given number has not more than five digits, we have merely to take the mantissa immediately from the table, and prefix the *characteristic* by the rule (Art. 64).

Thus, suppose we require the logarithm of 62541. The table gives .7961648 as the mantissa, and the characteristic is 4, by the rule; therefore

$$\log\ 62541 = 4.7961648.$$

Similarly, $\log .006\dot{2}81 = \bar{3}.7980288$ . . (Art. 64)

Suppose, however, that the given number has more than five digits. For example:

Suppose we require to find log 62761.6.

We find from the table

$$\log 62761 = 4.7976899$$
$$\log 62762 = 4.7976968$$

and
$$\text{diff. for } 1 = 0.0000069$$

Thus for an increase of 1 in the number there is an increase of .0000069 in the logarithm.

Hence, *assuming* that the increase of the logarithm is proportional to the increase of the number, then an increase in the number of .6 will correspond to an increase in the logarithm of $.6 \times .0000069 = .0000041$, to the nearest seventh decimal place.

Hence,
$$\log 62761 = 4.7976899$$
$$\text{diff. for } .6 = \qquad 41$$
$$\therefore \log 62761.6 = 4.7976940$$

This explains the use of the column of proportional parts on the extreme right of the page. It will be seen that the difference between the logarithms of two consecutive numbers is not always the same; for instance, those in the upper part of the page before us differ by .0000070, while those in the middle and the lower parts differ by .0000069 and .0000068. Under the column with the heading 69 we see the difference 41 corresponding to the figure 6, which implies that when the difference between the logarithms of two consecutive members is .0000069, the increase in the *logarithm* corresponding to an increase of .6 in the *number* is .0000041; for .06 it is evidently .0000004, and so on.

NOTE. — We assume in this method that the increase in a logarithm is *proportional* to the increase in the number. Although this is not *strictly* true, yet it is in most cases sufficiently exact for practical purposes.

Had we taken a *whole* number or a *decimal*, the process would have been the same.

N.	0	1	2	3	4	5	6	7	8	9	P.P.	
6250	795 8800	8870	8939	9009	9078	9148	9217	9287	9356	9426		
51	9495	9564	9634	9703	9773	9842	9912	9981	0̄051	0̄120		
52	796 0190	0259	0329	0398	0468	0537	0606	0676	0745	0815		
53	0884	0954	1023	1093	1162	1232	1301	1370	1440	1509		
54	1579	1648	1718	1787	1857	1926	1995	2065	2134	2204		
55	2273	2343	2412	2481	2551	2620	2690	2759	2829	2898		
56	2967	3037	3106	3176	3245	3314	3384	3453	3523	3592		70
57	3662	3731	3800	3870	3939	4009	4078	4147	4217	4286	1	7.0
58	4356	4425	4494	4564	4633	4703	4772	4841	4911	4980	2	14.0
59	5050	5119	5188	5258	5327	5396	5466	5535	5605	5674	3	21.0
6260	796 5743	5813	5882	5951	6021	6090	6160	6229	6298	6368	4	28.0
61	6437	6506	6576	6645	6714	6784	6853	6923	6992	7061	5	35.0
62	7131	7200	7269	7339	7408	7477	7547	7616	7685	7755	6	42.0
63	7824	7893	7963	8032	8101	8171	8240	8309	8379	8448	7	49.0
64	8517	8587	8656	8725	8795	8864	8933	9003	9072	9141	8	56.0
65	9211	9280	9349	9419	9488	9557	9627	9696	9765	9835	9	63.0
66	9904	9973	0̄043	0̄112	0̄181	0̄250	0̄320	0̄389	0̄458	0̄528		
67	797 0597	0666	0736	0805	0874	0943	1013	1082	1151	1221		
68	1290	1359	1428	1498	1567	1636	1706	1775	1844	1913		
69	1983	2052	2121	2191	2260	2329	2398	2468	2537	2606		
6270	797 2675	2745	2814	2883	2952	3022	3091	3160	3229	3299		
71	3368	3437	3507	3576	3645	3714	3784	3853	3922	3991		69
72	4060	4130	4199	4268	4337	4407	4476	4545	4614	4684	1	6.9
73	4753	4822	4891	4961	5030	5099	5168	5237	5307	5376	2	13.8
74	5445	5514	5584	5653	5722	5791	5860	5930	5999	6068	3	20.7
75	6137	6207	6276	6345	6414	6483	6553	6622	6691	6760	4	27.6
76	6829	6899	6968	7037	7106	7175	7245	7314	7383	7452	5	34.5
77	7521	7590	7660	7729	7798	7867	7936	8006	8075	8144	6	41.4
78	8213	8282	8351	8421	8490	8559	8628	8697	8766	8836	7	48.3
79	8905	8974	9043	9112	9181	9251	9320	9389	9458	9527	8	55.2
6280	797 9596	9666	9735	9804	9873	9942	0̄011	0̄080	0̄150	0̄219	9	62.1
81	798 0288	0357	0426	0495	0565	0634	0703	0772	0841	0910		
82	0979	1048	1118	1187	1256	1325	1394	1463	1532	1601		
83	1671	1740	1809	1878	1947	2016	2085	2154	2224	2293		
84	2362	2431	2500	2569	2638	2707	2776	2846	2915	2984		
85	3053	3122	3191	3260	3329	3398	3467	3536	3606	3675		
86	3744	3813	3882	3951	4020	4089	4158	4227	4296	4366		68
87	4435	4504	4573	4642	4711	4780	4849	4918	4987	5056	1	6.8
88	5125	5194	5263	5333	5402	5471	5540	5609	5678	5747	2	13.6
89	5816	5885	5954	6023	6092	6161	6230	6299	6368	6437	3	20.4
6290	798 6506	6575	6645	6714	6783	6852	6921	6990	7059	7128	4	27.2
91	7197	7266	7335	7404	7473	7542	7611	7680	7749	7818	5	34.0
92	7887	7956	8025	8094	8163	8232	8301	8370	8439	8508	6	40.8
93	8577	8646	8715	8784	8853	8922	8991	9060	9129	9198	7	47.6
94	9267	9336	9405	9474	9543	9612	9681	9750	9819	9888	8	54.4
95	9957	0̄026	0̄095	0̄164	0̄233	0̄302	0̄371	0̄440	0̄509	0̄578	9	61.2
96	799 0647	0716	0785	0854	0923	0992	1061	1130	1199	1268		
97	1337	1406	1475	1544	1613	1682	1751	1820	1889	1958		
98	2027	2096	2164	2233	2302	2371	2440	2509	2578	2647		
99	2716	2785	2854	2923	2992	3061	3130	3199	3268	3337		
6300	799 3405	3474	3543	3612	3681	3750	3819	3888	3957	4026		
N.	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**P.P.**	

Thus, suppose we require to find log 627616 and log .627616. The mantissa is exactly the same as before (Art. 66), and the only difference to be made in the final result is to change the characteristic according to rule (Art. 64).

Thus $\log 627616 = 5.7976942$,

and $\log .627616 = \bar{1}.7976942$.

69. To find the Number corresponding to a Given Logarithm. — If the decimal part of the logarithm is found exactly in the table, we can take out the corresponding number, and put the decimal point in the number, in the place indicated by the characteristic.

Thus if we have to find the number whose logarithm is $\bar{2}.7982915$, we look in the table for the mantissa .7982915, and we find it set down opposite the number 62848: and as the characteristic is $\bar{2}$, there must be one cipher before the first significant figure (Art. 64).

Hence $\bar{2}.7982915$ is the logarithm of .062848.

Next, suppose that the decimal part of the logarithm is not found exactly in the table. For example, suppose we have to find the number whose logarithm is 2.7974453.

We find from the table

$$\begin{aligned} \log 62726 &= 4.7974476 \\ \log 62725 &= 4.7974407 \\ \hline \text{diff. for } 1 &= .0000069 \end{aligned}$$

Thus for a difference of 1 in the numbers there is a difference of .0000069 in the logarithms. The excess of the *given* mantissa above .7974407 is (.7974453 − .7974407) or .0000046.

Hence, assuming that the increase of the number is proportional to the increase of the logarithm, we have

$$.0000069 : .0000046 :: 1 : \text{number to be added to } 627.25.$$

$$\therefore \text{ number to be added} = \frac{.0000046}{.0000069} = \frac{46}{69} = .667$$

$$\begin{array}{r} 69)46.0(.666 \\ \underline{41\,4} \\ 4\,60 \\ \underline{4\,14} \\ 460 \end{array}$$

$\therefore \log 62725.667 = 4.7974453$,

and $\therefore \log 627.25667 = 2.7974453$;

therefore number required is 627.25667.

We might have saved the labor of dividing 46 by 69, by using the table of proportional parts as follows:

given mantissa	=	.7974453
mantissa of 62725	=	.7974407
diff. of mantissæ	=	46
proportional part for .6	=	41.4
		4.6
" " " .06	=	4.14
		.46
" " " .006	=	.414
		and so on.

$\therefore$ number $= 627.256666 \cdots$.

69 a. Arithmetic Complement. — By the *arithmetic complement* of the logarithm of a number, or, briefly, the *cologarithm* of the number, is meant the remainder found by subtracting the logarithm from 10. To subtract one logarithm from another is the same as to add the cologarithm and then subtract 10 from the result.

Thus, $$a - b = a + (10 - b) - 10,$$

where a and b are logarithms, and $10 - b$ is the arithmetic complement of b.

When one logarithm is to be subtracted from the sum of several others, it is more convenient to *add* its *cologarithm* to the sum, and reject 10. The advantage of using the cologarithm is that it enables us to exhibit the work in a more compact form.

The cologarithm is easily taken from the table mentally by subtracting the last significant figure on the right from 10, and all the others from 9.

EXAMPLES.

1. Given $\log 52502 = 4.7201758,$

$\log 52503 = 4.7201841;$

find $\log 52502.5.$ *Ans.* 4.7201799.

2. Given $\log 3.0042 = 0.4777288,$

$\log 3.0043 = 0.4777433;$

find $\log 300.425.$ 2.4777360.

3. Given $\log 7.6543 = 0.8839055,$

$\log 7.6544 = 0.8839112;$

find $\log 7.65432.$ .8839066.

4. Given $\log 6.4371 = 0.8086903,$

$\log 6.4372 = 0.8086970;$

find $\log 6437125.$ 6.8086920.

5. Given $\log 12954 = 4.1124039,$

$\log 12955 = 4.1124374;$

find the number whose logarithm is 4.1124307. 12954.8.

6. Given $\log 60195 = 4.7795532,$

$\log 60196 = 4.7795604;$

find the number whose logarithm is 2.7795561. 601.95403.

7. Given $\log 3.7040 = .5686710,$

$\log 3.7041 = .5686827;$

find the number whose logarithm is .5686760. 3.70404.

8. Given $\log 2.4928 = .3966874,$

$\log 2.4929 = .3967049;$

find the number whose logarithm is 6.3966938. 2492837.

9. Given $\log 32642 = 4.5137768,$
$\log 32643 = 4.5137901;$

find $\log 32642.5.$ *Ans.* 4.5137835.

10. Find the logarithm of 62654326. 7.7969510.

Use specimen page.

11. Find the number whose logarithm is 4.7989672.
Ans. 62945.876.

70. Use of Trigonometric Tables. — Trigonometric Tables are of two kinds, — *Tables of Natural Trigonometric Functions* and *Tables of Logarithmic Trigonometric Functions.* As the greater part of the computations of Trigonometry is carried on by logarithms, the latter tables are by far the most useful.

We have explained in Art. 27 how to find the *actual numerical values* of certain trigonometric functions, exactly or approximately.

Thus, $\sin 30° = \frac{1}{2}$; that is, .5 exactly.

Also, $\tan 60° = \sqrt{3}$; that is, 1.73205 approximately.

A table of natural trigonometric functions gives their approximate numerical values for angles at regular intervals in the first quadrant. In some tables the angles succeed each other at intervals of 1″, in others, at intervals of 10″, but in ordinary tables at intervals of 1′: and the values of the functions are given correct to *five, six,* and *seven* places. The functions of intermediate angles can be found by the principle of *proportional parts* as applied in the table of logarithms of numbers (Arts. 68 and 69).

It is sufficient to have tables which give the functions of angles only in the first quadrant, since the functions of all angles of whatever size can be reduced to functions of angles less than 90° (Art. 35).

71. Use of Tables of Natural Trigonometric Functions. — These tables, which consist of the *actual numerical values* of the trigonometric functions, are commonly called tables of *natural sines, cosines,* etc., so as to distinguish them from the tables of the *logarithms* of the sines, cosines, etc.

We shall now explain, first, how to determine the value of a function that lies between the functions of two consecutive angles given in the tables; and secondly, how to determine the angle to which a given ratio corresponds.

72. To find the Sine of a Given Angle.

Find the sine of 25° 14′ 20″, having given from the table

$$\begin{aligned} \sin 25° \, 15' &= .4265687 \\ \sin 25° \, 14' &= \underline{.4263056} \\ \text{diff. for } 1' &= .0002631 \end{aligned}$$

Let $d =$ diff. for 20″; and assuming that an increase in the angle is proportional to an increase in the sine, we have

$$60 : 20 :: .0002631 : d.$$

$$\therefore \; d = \frac{20 \times .0002631}{60} = .0000877.$$

$$\begin{aligned} \therefore \; \sin 25° \, 14' \, 20'' &= .4263056 + .0000877 \\ &= .4263933. \end{aligned}$$

NOTE. — We assumed here that an increase in the angle is proportional to the increase in the corresponding sine, which is sufficiently exact for practical purposes, with certain exceptions.

73. To find the Cosine of a Given Angle.

Find the cosine of 44° 35′ 25″, having given from the table

$$\begin{aligned} \cos 44° \, 35' &= .7122303 \\ \cos 44° \, 36' &= \underline{.7120260} \\ \text{diff. for } 1' &= .0002043 \end{aligned}$$

observing that the cosine *decreases* as the angle increases from 0° to 90°.

Let $d =$ decrease of cosine for 25″; then

$$60 : 25 :: .0002043 : d.$$

$$\therefore\ d = \frac{25}{60} \times .0002043 = .0000851.$$

$$\therefore\ \cos 44^\circ\, 35'\, 25'' = .7122303 - .0000851$$
$$= 7121452.$$

Similarly, we may find the values of the other trigonometric functions, remembering that, in the first quadrant, the tangent and secant *increase* and the cotangent and cosecant *decrease*, as the angle increases.

EXAMPLES.

1. Given $\sin 44^\circ\, 35' = .7019459$,
$\sin 44^\circ\, 36' = .7021531$;
find $\sin 44^\circ\, 35'\, 25''$. *Ans.* .7020322.

2. Given $\sin 42^\circ\, 15' = .6723668$,
$\sin 42^\circ\, 16' = .6725821$;
find $\sin 42^\circ\, 15'\, 16''$. .6724242.

3. Given $\sin 43^\circ\, 23' = .6868761$,
$\sin 43^\circ\, 22' = .6866647$;
find $\sin 43^\circ\, 22'\, 50''$. .6868408.

4. Given $\sin 31^\circ\, 6' = .5165333$,
$\sin 31^\circ\, 7' = .5167824$;
find $\sin 31^\circ\, 6'\, 25''$. .5166371.

5. Given $\cos 74^\circ\, 45' = .4265687$,
$\cos 74^\circ\, 46' = .4263056$;
find $\cos 74^\circ\, 45'\, 40''$. .4263933.

6. Given $\cos 41° 13' = .7522233$,

$\cos 41° 14' = .7520316$;

find $\cos 41° 13' 26''$. *Ans.* .7521403.

7. Given $\cos 47° 38' = .6738727$,

$\cos 47° 39' = .6736577$;

find $\cos 47° 38' 30''$. .6737652.

74. To find the Angle whose Sine is Given.

Find the angle whose sine is .5082784, having given from the table

$$\begin{aligned} \sin 30° 33' &= .5082901 \\ \sin 30° 32' &= .5080396 \\ \hline \text{diff. for } 1' &= .0002505 \end{aligned}$$

$$\begin{aligned} \text{given sine} &= .5082784 \\ \sin 30° 32' &= .5080396 \\ \hline \text{diff.} &= .0002388 \end{aligned}$$

Let d = diff. between $30° 32'$ and required angle; then $.0002505 : .0002388 :: 60 : d$.

$$\therefore d = \frac{2388 \times 60}{2505} = \frac{6552}{167}$$

$$= 57.2 \text{ nearly.}$$

$$\therefore \text{required angle} = 30° 32' 57''.2.$$

75. To find the Angle whose Cosine is Given.

Find the angle whose cosine is .4043281, having given from the table

$$\begin{aligned} \cos 66° \ 9' &= .4043436 \\ \cos 66° 10' &= .4040775 \\ \hline \text{diff. for } 1' &= .0002661 \end{aligned}$$

$$\begin{aligned} \cos 66° \ 9' &= .4043436 \\ \text{given cosine} &= .4043281 \\ \hline \text{diff.} &= .0000155 \end{aligned}$$

Let $d =$ diff. between $66° 9'$ and required angle; then $.0002661 : .0000155 :: 60 : d$.

$$\therefore\ d = \frac{155 \times 60}{2661} = 3.5.$$

Required angle is greater than $66° 9'$ because its cosine is *less* than $\cos 66° 9'$.

$$\therefore\ \text{required angle} = 66° 9' 3''.5.$$

EXAMPLES.

1. Given $\sin 44° 12' = .6971651,$
$\sin 44° 11' = .6969565;$
find the angle whose sine is .6970886. *Ans.* $44° 11' 38''$.

2. Given $\sin 48° 47' = .7522233,$
$\sin 48° 46' = .7520316;$
find the angle whose sine is .752140. $48° 46' 34''$.

3. Given $\sin 24° 11' = .4096577,$
$\sin 24° 12' = .4099230;$
find the angle whose sine is .4097559. $24° 11' 22''.2$.

4. Given $\cos 32° 31' = .8432351,$
$\cos 32° 32' = .8430787;$
find the angle whose cosine is .8432. $32° 31' 13''.5$.

5. Given $\cos 44° 11' = .7171134,$
$\cos 44° 12' = .7169106;$
find the angle whose cosine is .7169848. $44° 11' 38''$.

6. Given $\cos 70° 32' = .3332584,$
$\cos 70° 31' = .3335326;$
find the angle whose cosine is .3333333. $70° 31' 43''.6$.

76. Use of Tables of Logarithmic Trigonometric Functions. — Since the sines, cosines, tangents, etc., of angles are numbers, we may use the logarithms of these numbers in numerical calculations in which trigonometric functions are involved; and these logarithms are in practice much more useful than the numbers themselves, as with their assistance we are able to abbreviate greatly our calculations; this is especially the case, as we shall see hereafter, in the *solution of triangles*. In order to avoid the trouble of referring *twice* to tables — first to the table of natural functions for the value of the function, and then to a table of logarithms for the logarithm of that function — the logarithms of the trigonometric functions have been calculated and arranged in tables, forming tables of the *logarithms of the sines, logarithms of the cosines*, etc.; these tables are called *tables of logarithmic sines, logarithmic cosines*, etc.

Since the sines and cosines of all angles and the tangents of angles less than 45° are *less than unity*, the logarithms of these functions are *negative*. To avoid the inconvenience of using negative characteristics, 10 is added to the logarithms of all the functions before they are entered in the table. The logarithms so increased are called the *tabular logarithms* of the sine, cosine, etc. Thus, the tabular logarithmic sine of 30° is

$$10 + \log \sin 30° = 10 + \log \frac{1}{2} = 10 - \log 2 = 9.6989700.$$

In calculations we have to remember and allow for this increase of the true logarithms. When the value of any one of the *tabular* logarithms is given, we must take away 10 from it to obtain the *true* value of the logarithm.

Thus in the tables we find

$$\log \sin 31° 15' = 9.7149776.$$

Therefore the *true* value of the logarithm of the sine of $31° 15'$ is $9.7149776 - 10 = \bar{1}.7149776$.

Similarly with the logarithms of other functions.

NOTE. — English authors usually denote these tabular logarithms by the letter L. Thus, L sin A denotes the tabular logarithm of the sine of A.

French authors use the logarithms of the tables diminished by 10. Thus,

$$\log \sin A = \bar{1}.8598213, \text{ instead of } 9.8598213.$$

The *Tables* contain the tabular logs of the functions of all angles in the first quadrant at intervals of 1′; and from these the logarithmic functions of all other angles can be found.*

Since every angle between 45° and 90° is the complement of another angle between 45° and 0°, every sine, tangent, etc., of an angle less than 45° is the cosine, cotangent, etc., of another angle greater than 45° (Art. 16). Hence the degrees at the top of the tables are generally marked from 0° to 45°, and those at the bottom from 45° to 90°, while the minutes are marked both in the first column at the *left*, and in the last column at the *right*. Every number therefore in each column, except those marked diff., stands for two functions — the one named at the top of the column, and the complemental function named at the bottom of the column. In looking for a function of an angle, if it be less than 45°, the degrees are found at the top, and the minutes at the left-hand side. If greater than 45°, the degrees are found at the foot, and the minutes at the right-hand side.

On page 113 is a specimen page of Mathematical Tables. It gives the *tabular logarithmic functions* of all angles between 38° and 39°, and also of those between 51° and 52°, both inclusive, at intervals of 1′. The names of the functions for 38° are printed at the top of the page, and those for 51° at the foot. The column of minutes for 38° is on the left, that for 51° is on the right.

Thus we find

$$\log \sin 38° 29' = 9.7939907.$$

$$\log \cos 38° 45' = 9.8920303.$$

$$\log \tan 51° 18' = 10.0962856.$$

* Many tables are calculated for angles at intervals of 10″.

77. To find the Logarithmic Sine of a Given Angle.

Find log sin 38° 52′ 46″.

We have from page 113

$$\log \sin 38° 53' = 9.7977775$$
$$\log \sin 38° 52' = 9.7976208$$
$$\text{diff. for } 1' = .0001567$$

Let $d =$ diff. for 46″, and assuming that the change in the log sine is proportional to the change in the angle, we have

$$60 : 46 :: .0001567 : d.$$

$$\therefore\ d = \frac{46 \times .0001567}{60} = .0001201.$$

$$\therefore\ \log \sin 38° 52' 46'' = 9.7976208 + .0001201$$
$$= 9.7977409.$$

78. To find the Logarithmic Cosine of a Given Angle.

Find log cos 83° 27′ 23″, having given from the table

$$\log \cos 83° 27' = 9.0571723$$
$$\log \cos 83° 28' = 9.0560706$$
$$\text{diff. for } 1' = .0011017$$

Let $d =$ decrease of log cosine for 23″; then

$$60 : 23 :: .0011017 : d.$$

$$\therefore\ d = \frac{23 \times .0011017}{60} = .0004223, \text{ nearly.}$$

$$\therefore\ \log \cos 83° 27' 23'' = 9.0571723 - .0004223$$
$$= 9.0567500.$$

EXAMPLES.

1. Given log sin 6° 33′ = 9.0571723,
log sin 6° 32′ = 9.0560706;
find log sin 6° 32′ 37″. *Ans.* 9.05675.

′	Sine.	Diff.	Tang.	Diff.	Cotang.	Diff.	Cosine.	′
0	9.7893420	1616	9.8928098	2604	10.1071902	987	9.8965321	60
1	9.7895036	1616	9.8930702	2604	10.1069298	988	9.8964334	59
2	9.7896652	1614	9.8933306	2603	10.1066694	988	9.8963346	58
3	9.7898266	1614	9.8935909	2602	10.1064091	989	9.8962358	57
4	9.7899880	1613	9.8938511	2603	10.1061489	990	9.8961369	56
5	9.7901493	1611	9.8941114	2601	10.1058886	990	9.8960379	55
6	9.7903104	1611	9.8943715	2602	10.1056285	991	9.8959389	54
7	9.7904715	1610	9.8946317	2601	10.1053683	992	9.8958398	53
8	9.7906325	1608	9.8948918	2601	10.1051082	992	9.8957406	52
9	9.7907933	1608	9.8951519	2600	10.1048481	992	9.8956414	51
10	9.7909541	1607	9.8954119	2600	10.1045881	993	9.8955422	50
11	9.7911148	1606	9.8956719	2600	10.1043281	994	9.8954429	49
12	9.7912754	1605	9.8959319	2599	10.1040681	995	9.8953435	48
13	9.7914359	1604	9.8961918	2599	10.1038082	995	9.8952440	47
14	9.7915963	1603	9.8964517	2599	10.1035483	995	9.8951445	46
15	9.7917566	1602	9.8967116	2598	10.1032884	997	9.8950450	45
16	9.7919168	1601	9.8969714	2598	10.1030286	996	9.8949453	44
17	9.7920769	1600	9.8972312	2598	10.1027688	998	9.8948457	43
18	9.7922369	1599	9.8974910	2597	10.1025090	998	9.8947459	42
19	9.7923968	1598	9.8977507	2597	10.1022493	998	9.8946461	41
20	9.7925566	1597	9.8980104	2596	10.1019896	1000	9.8945463	40
21	9.7927163	1597	9.8982700	2596	10.1017300	999	9.8944463	39
22	9.7928760	1595	9.8985296	2596	10.1014704	1001	9.8943464	38
23	9.7930355	1594	9.8987892	2595	10.1012108	1001	9.8942463	37
24	9.7931949	1594	9.8990487	2595	10.1009513	1001	9.8941462	36
25	9.7933543	1592	9.8993082	2595	10.1006918	1003	9.8940461	35
26	9.7935135	1592	9.8995677	2594	10.1004323	1002	9.8939458	34
27	9.7936727	1590	9.8998271	2594	10.1001729	1004	9.8938456	33
28	9.7938317	1590	9.9000865	2594	10.0999135	1004	9.8937452	32
29	9.7939907	1589	9.9003459	2593	10.0996541	1004	9.8936448	31
30	9.7941496	1587	9.9006052	2593	10.0993948	1005	9.8935444	30
31	9.7943083	1587	9.9008645	2592	10.0991355	1006	9.8934439	29
32	9.7944670	1586	9.9011237	2593	10.0988763	1007	9.8933433	28
33	9.7946256	1585	9.9013830	2592	10.0986170	1007	9.8932426	27
34	9.7947841	1584	9.9016422	2591	10.0983578	1007	9.8931419	26
35	9.7949425	1583	9.9019013	2591	10.0980987	1008	9.8930412	25
36	9.7951008	1582	9.9021604	2591	10.0978396	1009	9.8929404	24
37	9.7952590	1581	9.9024195	2591	10.0975805	1010	9.8928395	23
38	9.7954171	1580	9.9026786	2590	10.0973214	1010	9.8927385	22
39	9.7955751	1579	9.9029376	2590	10.0970624	1010	9.8926375	21
40	9.7957330	1579	9.9031966	2589	10.0968034	1011	9.8925365	20
41	9.7958909	1577	9.9034555	2589	10.0965445	1012	9.8924354	19
42	9.7960486	1576	9.9037144	2589	10.0962856	1013	9.8923342	18
43	9.7962062	1576	9.9039733	2588	10.0960267	1013	9.8922329	17
44	9.7963638	1574	9.9042321	2589	10.0957679	1013	9.8921316	16
45	9.7965212	1574	9.9044910	2587	10.0955090	1014	9.8920303	15
46	9.7966786	1573	9.9047497	2588	10.0952503	1015	9.8919289	14
47	9.7968359	1571	9.9050085	2587	10.0949915	1016	9.8918274	13
48	9.7969930	1571	9.9052672	2587	10.0947328	1016	9.8917258	12
49	9.7971501	1570	9.9055259	2586	10.0944741	1016	9.8916242	11
50	9.7973071	1569	9.9057845	2586	10.0942155	1018	9.8915226	10
51	9.7974640	1568	9.9060431	2586	10.0939569	1017	9.8914208	9
52	9.7976208	1567	9.9063017	2586	10.0936983	1019	9.8913191	8
53	9.7977775	1566	9.9065603	2585	10.0934397	1019	9.8912172	7
54	9.7979341	1565	9.9068188	2585	10.0931812	1020	9.8911153	6
55	9.7980906	1564	9.9070773	2584	10.0929227	1020	9.8910133	5
56	9.7982470	1564	9.9073357	2584	10.0926643	1021	9.8909113	4
57	9.7984034	1562	9.9075941	2584	10.0924059	1021	9.8908092	3
58	9.7985596	1562	9.9078525	2584	10.0921475	1022	9.8907071	2
59	9.7987158	1560	9.9081109	2583	10.0918891	1023	9.8906049	1
60	9.7988718		9.9083692		10.0916308		9.8905026	0
′	Cosine.	Diff.	Cotang.	Diff.	Tang.	Diff.	Sine.	′

2. Given $\log\sin 55° 33' = 9.9162539$,
$\log\sin 55° 34' = 9.9163406$;
find $\log\sin 55° 33' 54''$. *Ans.* 9.9163319.

3. Given $\log\cos 37° 28' = 9.8996604$,
$\log\cos 37° 29' = 9.8995636$;
find $\log\cos 37° 28' 36''$. 9.8996023.

4. Given $\log\cos 44° 35' 20'' = 9.8525789$,
$\log\cos 44° 35' 30'' = 9.8525582$;
find $\log\cos 44° 35' 25''.7$. 9.8525671.

See foot-note of Art. 76.

5. Given $\log\cos 55° 11' = 9.7565999$,
$\log\cos 55° 12' = 9.7564182$;
find $\log\cos 55° 11' 12''$. 9.7565636.

6. Given $\log\tan 27° 13' = 9.7112148$,
$\log\tan 27° 14' = 9.7115254$;
find $\log\tan 27° 13' 45''$. 9.7114477.

79. To find the Angle whose Logarithmic Sine is Given.

Find the angle whose log sine is 8.8785940, having given from the table

$$\log\sin 4° 21' = 8.8799493$$
$$\log\sin 4° 20' = 8.8782854$$
$$\text{diff. for } 1' = .0016639$$

$$\text{given log sine} = 8.8785940$$
$$\log\sin 4° 20' = 8.8782854$$
$$\text{diff.} = .0003086$$

Let $d =$ diff. between $4° 20'$ and required angle; then

$$.0016639 : .0003086 :: 60 : d.$$

$$\therefore\ d = \frac{3086 \times 60}{16639} = 24, \text{ nearly.}$$

$$\therefore\ \text{required angle} = 4° 20' 24''.$$

80. To find the Angle whose Logarithmic Cosine is Given.

Find the angle whose log cosine is 9.8934342.

We have from page 113

$$\begin{aligned} \log\cos 38^\circ\, 31' &= 9.8934439 \\ \log\cos 38^\circ\, 32' &= 9.8933433 \\ \text{diff. for } 1' &= .0001006 \end{aligned}$$

$$\begin{aligned} \log\cos 38^\circ\, 31' &= 9.8934439 \\ \text{given log cosine} &= 9.8934342 \\ \text{diff.} &= .0000097 \end{aligned}$$

Let $d =$ diff. between $38^\circ\, 31'$ and required angle; then

$$.0001006 : .0000097 :: 60 : d.$$

$$\therefore\ d = \frac{.0000097}{.0001006} \times 60 = \frac{97 \times 60}{1006} = 5''.8.$$

$$\therefore\ \text{required angle} = 38^\circ\, 31'\, 5''.8.$$

NOTE. — **In using both the tables of the *natural* sines, cosines, etc., and the tables of the *logarithmic* sines, cosines, etc., the student will remember that, in the first quadrant, as the angle *increases*, the sine, tangent, and secant *increase*, but the cosine, cotangent, and cosecant *decrease*.**

EXAMPLES.

1. Given $\log\sin 14^\circ\, 24' = 9.3956581,$
$\log\sin 14^\circ\, 25' = 9.3961499;$
find the angle whose log sine is 9.3959449. *Ans.* $14^\circ\, 24'\, 35''$.

2. Given $\log\sin 71^\circ\, 40' = 9.9773772,$
$\log\sin 71^\circ\, 41' = 9.9774191;$
find the angle whose log sine is 9.9773897. $71^\circ\, 40'\, 18''$.

3. Given $\log\cos 28^\circ\, 17' = 9.9447862,$
$\log\cos 28^\circ\, 16' = 9.9448541;$
find the angle whose log cosine is 9.9448230. $28^\circ\, 16'\, 27''.5$.

4. Given $\log\cos 80° 53' = 9.1998793$,
$\log\cos 80° 52' 50'' = 9.2000105$;
find the angle whose log cosine is 9.2000000.

Ans. $80° 52' 51''$.

5. Given $\log\tan 35° 4' = 9.8463018$,
$\log\tan 35° 5' = 9.8465705$;
find the angle whose log tangent is 9.8464028. $35° 4' 23''$.

6. Given $\log\sin 44° 35' 30'' = 9.8463678$,
$\log\sin 44° 35' 20'' = 9.8463464$;
find the angle whose log sine is 9.8463586. $44° 35' 25''.7$.

7. Find the angle by page 113 whose log tangent is 10.1018542. *Ans.* $51° 39' 28''.7$.

81. Angles near the Limits of the Quadrant. — It was assumed in Arts. 72–80 that, in general, the differences of the trigonometric functions, both natural and logarithmic, are approximately proportional to the differences of their corresponding angles, *with certain exceptions.* The exceptional cases are as follows:

(1) *Natural functions.* — For the *sine* the differences are insensible for angles near 90°; for the *cosine* they are insensible for angles near 0°. For the *tangent* the differences are irregular for angles near 90°; for the *cotangent* they are irregular for angles near 0°.

(2) *Logarithmic functions.* — The principle of proportional parts fails both for angles near 0° and angles near 90°. For the *log sine* and the *log cosecant* the differences are *irregular* for angles near 0°, and *insensible* for angles near 90°. For the *log cosine* and the *log secant* the differences are *insensible* for angles near 0°, and *irregular* for angles near 90°. For the *log tangent* and the *log cotangent* the differences are *irregular* for angles near 0° and angles near 90°.

It follows, therefore, that angles near 0° and angles near 90° cannot be found with exactness from their log trigonometric functions. These difficulties may be met in three ways.

(1) For an angle near 0° use the principle that *the sines and tangents of small angles are approximately proportional to the angles themselves.* (See Art. 130.)

(2) For an angle near 90° use the half angle (Art. 99).

(3) In using the proportional parts, find two, three, or more orders of differences (Alg., Art. 197).

Special tables are employed for angles near the limits of the quadrant.

EXAMPLES.

1. Given $\log_{10} 7 = .8450980$, find $\log_{10} 343$, $\log_{10} 2401$, and $\log_{10} 16.807$. *Ans.* 2.5352940, 3.3803920, 1.2254900.

2. Find the logarithms to the base 3 of 9, 81, $\frac{1}{3}$, $\frac{1}{27}$, .1, $\frac{1}{81}$.
Ans. 2, 4, -1, -3, -2, -4.

3. Find the value of $\log_2 8$, $\log_2 .5$, $\log_3 243$, $\log_5 (.04)$, $\log_{10} 1000$, $\log_{10} .001$. *Ans.* 3, -1, 5, -2, 3, -3.

4. Find the value of $\log_a a^{\frac{4}{3}}$, $\log_b \sqrt[3]{b^2}$, $\log_8 2$, $\log_{27} 3$, $\log_{100} 10$.
Ans. $\frac{4}{3}$, $\frac{2}{3}$, $\frac{1}{3}$, $\frac{1}{3}$, $\frac{1}{2}$.

Given $\log_{10} 2 = .3010300$, $\log_{10} 3 = .4771213$, and $\log_{10} 7 = .8450980$, find the values of the following:

5. $\log_{10} 35$, $\log_{10} 150$, $\log_{10} .2$.
Ans. 1.544068, 2.1760913, $\bar{1}.30103$.

6. $\log_{10} 3.5$, $\log_{10} 7.29$, $\log_{10} .081$.
Ans. .5440680, .8627278, $\bar{2}.9084852$.

7. $\log_{10} \frac{7}{3}$, $\log_{10} 3^5$, $\log_{10} \sqrt[3]{\frac{12}{7}}$.
Ans. .3679767, 2.3856065, .0780278.

8. Write down the integral part of the common logarithms of 7963, .1, 2.61, 79.6341, 1.0006, .00000079.

Ans. 3, -1, 0, 1, 0, -7.

9. Give the position of the first *significant* figure in the numbers whose logarithms are

$$\bar{2}.4612310,\ \bar{1}.2793400,\ \bar{6}.1763241.$$

10. Give the position of the first *significant* figure in the numbers whose logarithms are 4.2990713, .3040595, 2.5860244, $\bar{3}.1760913$, $\bar{1}.3180633$, .4980347.

Ans. ten thousands, units, hundreds, 3rd dec. pl., 1st dec. pl., units.

11. Given $\log 7 = .8450980$, find the number of digits in the integral part of 7^{10}, 49^6, $343^{\frac{100}{3}}$, $(\frac{10}{7})^{20}$, $(4.9)^{12}$, $(3.43)^{10}$.

Ans. 9, 11, 85, 4, 9, 6.

12. Find the position of the first significant figure in the numerical value of 20^7, $(.02)^7$, $(.007)^2$, $(3.43)^{\frac{1}{10}}$, $(.0343)^8$, $(.0343)^{\frac{1}{10}}$.

Ans. tenth integral pl., 12th dec. pl., 5th dec. pl., units, 12th dec. pl., 1st dec. pl.

Show how to transform

13. Common logarithms to logarithms with base 2.

Ans. Divide each logarithm by .30103.

14. Logarithms with base 3 to common logarithms.

Ans. Multiply each log by .4771213.

15. Given $\log_{10} 2 = .3010300$, find $\log_2 10$. 3.32190.

16. Given $\log_{10} 7 = .8450980$, find $\log_7 10$. 1.183.

17. Given $\log_{10} 2 = .3010300$, find $\log_8 10$. 1.10730.

18. The mantissa of the log of 85762 is 9332949; find (1) the log of $\sqrt[11]{.0085762}$, and (2) the number of figures in $(85762)^{11}$, when it is multiplied out.

Ans. (1) $\bar{1}.8121177$, (2) 55.

19. What are the characteristics of the logarithms of 3742 to the bases 3, 6, 10, and 12 respectively?

Ans. 7, 4, 3, 3.

20. Prove that $7 \log \frac{15}{16} + 6 \log \frac{8}{3} + 5 \log \frac{2}{5} + \log \frac{32}{25} = \log 3$.

21. Given $\log_{10} 7$, find $\log_7 490$. *Ans.* $2 + \dfrac{1}{\log_{10} 7}$.

22. From $\bar{5}.3429$ take $\bar{3}.6284$. $\bar{3}.7145$.

23. Divide $\overline{13}.2615$ by 8. $\bar{2}.4076$.

24. Prove that $6 \log \frac{2}{3} + 4 \log \frac{9}{10} + 2 \log \frac{25}{6} = 0$.

25. Find $\log \{297\sqrt{11}\}^{\frac{3}{5}}$ to the base $3\sqrt{11}$. 1.8.

Given $\log 2 = .3010300$, $\log 3 = .4771213$.

26. Find log 216, 6480, 5400, $\frac{4}{9}$.

Ans. 2.3344539, 3.8115752, 3.7323939, $\bar{1}.6478174$.

27. Find $\log .03$, $6^{-\frac{1}{3}}$, $(5\frac{1}{3})^{-\frac{1}{2}}$.

Ans. $\bar{2}.4771213$, $\bar{1}.7406162$, $\bar{1}.6365006$.

28. Find $\log .18$, $\log 2.4$, $\log \frac{3}{16}$.

Ans. $\bar{1}.2552726$, .3802113, $\bar{1}.2730013$.

29. Find $\log (6.25)^{\frac{1}{7}}$, $\log 4\sqrt{.005}$. .1136971, $\bar{1}.45154$.

30. Given $\log 56321 = 4.7506704$,
$\log 56322 = 4.7506781$;
find $\log 5632147$. 6.7506740.

31. Given $\log 53403 = 4.7275657$,
$\log 53402 = 4.7275575$;
find $\log 5340234$. 6.7275603.

32. Given $\log 56412 = 4.7513715$,
$\log 56413 = 4.7513792$;
find $\log 564.123$. 2.7513738.

33. Given $\log 87364 = 4.9413325$,
$\log 87365 = 4.9413375$;
find $\log .0008736416$. *Ans.* $\bar{4}.9413333$.

34. Given $\log 37245 = 4.5710680$,
$\log 37246 = 4.5710796$;
find $\log 3.72456$. $.5710750$.

35. Given $\log 32025 = 4.5054891$,
$\log 32026 = 4.5055027$;
find $\log 32.025613$. 1.5054974.

36. Given $\log 65931 = 4.8190897$,
$\log 65932 = 4.8190962$;
find $\log .000006593171$. $\bar{6}.8190943$.

37. Given $\log 25819 = 4.4119394$,
$\log 25820 = 4.4119562$;
find $\log 2.581926$. $.4119438$.

38. Given $\log 23454 = 4.3702169$,
$\log 23453 = 4.3701984$;
find $\log 23453487$. 7.3702074.

39. Given $\log 45740 = 4.6602962$,
$\log 45741 = 4.6603057$;
find the number whose logarithm is 4.6602987. 45740.26.

40. Given $\log 43965 = 4.6431071$,
$\log 43966 = 4.6431170$;
find the number whose logarithm is $\bar{4}.6431150$. $.000439658$.

41. Given $\log 56891 = 4.7550436$,
$\log 56892 = 4.7550512$;
find the number whose logarithm is $.7550480$. 5.689158.

42. Given $\log 34572 = 4.5387245,$

$\log 34573 = 4.5387371;$

find the number whose logarithm is 2.5387359.

Ans. 345.7291.

43. Given $\log 10905 = 4.0376257,$

$\log 10906 = 4.0376655;$

find the number whose logarithm is 3.0376371. 1090.5286.

44. Given $\log 25725 = 4.4103554,$

$\log 25726 = 4.4103723;$

find the number whose logarithm is $\bar{7}.4103720$.

Ans. .00000025725982.

In the following six examples the student must take his logarithms from the tables.

45. Required the product of 3670.257 and 12.61158, by logarithms. *Ans.* 46287.74.

46. Required the quotient of .1234567 by 54.87645, by logarithms. *Ans.* .002249721.

47. Required the cube of .3180236, by logarithms.

Ans. .03216458.

48. Required the cube root of .3663265, by logarithms.

Ans. .7155216.

49. Required the eleventh root of 63.742. 1.45894.

50. Required the fifth root of .07. .58752.

51. Given $\sin 42° 21' = .6736577,$

$\sin 42° 22' = .6738727;$

find $\sin 42° 21' 30''$. .6737652.

52. Given $\sin 67° 22' = .9229865,$

$\sin 67° 23' = .9230984;$

find $\sin 67° 22' 48''.5$. .9230769.

53. Given $\sin 7^\circ 17' = .1267761,$
$\sin 7^\circ 18' = .1270646;$
find $\sin 7^\circ 17' 25''.$ *Ans.* .1268963.

54. Given $\cos 21^\circ 27' = .9307370,$
$\cos 21^\circ 28' = .9306306;$
find $\cos 21^\circ 27' 45''.$ .9306572.

55. Given $\cos 34^\circ 12' = .8270806,$
$\cos 34^\circ 13' = .8269170;$
find $\cos 34^\circ 12' 19''.6.$ .8270272.

56. Given $\sin 41^\circ 48' = .6665325,$
$\sin 41^\circ 49' = .6667493;$
find the angle whose sine is .6666666. $41^\circ 48' 37''.$

57. Given $\sin 73^\circ 44' = .9599684,$
$\sin 73^\circ 45' = .9600499;$
find the angle whose sine is .96. $73^\circ 44' 23''.2.$

58. Given $\cos 75^\circ 32' = .2498167,$
$\cos 75^\circ 31' = .2500984;$
find the angle whose cosine is .25. $75^\circ 31' 21''.$

59. Given $\cos 53^\circ 7' = .6001876,$
$\cos 53^\circ 8' = .5999549;$
find the angle whose cosine is .6. $53^\circ 7' 48''.4.$

60. Given $\log \sin 45^\circ 16' = 9.8514969,$
$\log \sin 45^\circ 17' = 9.8516220;$
find $\log \sin 45^\circ 16' 30''.$ 9.8515594.

61. Given $\log \sin 38^\circ 24' = 9.7931949,$
$\log \sin 38^\circ 25' = 9.7933543;$
find $\log \sin 38^\circ 24' 27''.$ 9.7932666.

62. Given	log sin 32° 28′ = 9.7298197,		
	log sin 32° 29′ = 9.7300182;		
find	log sin 32° 28′ 36″.	*Ans.*	9.7299388.
.63. Given	log sin 17° 1′ = 9.4663483.		
	log sin 17° 0′ = 9.4659353;		
find	log sin 17° 0′ 12″.		9.4660179.
64. Given	log sin 26° 24′ = 9.6480038.		
	log sin 26° 25′ = 9.6482582;		
find	log sin 26° 24′ 12″.		9.6480547.
65. Given	log cos 17° 31′ = 9.9793796,		
	log cos 17° 32′ = 9.9793398;		
find	log cos 17° 31′ 25″.2.		9.9793629.
66. Given	log tan 21° 17′ = 9.5905617,		
	log tan 21° 18′ = 9.5909351;		
find	log tan 21° 17′ 12″.		9.5906364.
67. Given	log tan 27° 26′ = 9.7152419,		
	log tan 27° 27′ = 9.7155508;		
find	log tan 27° 26′ 42″.		9.7154581.
68. Given	log cot 72° 15′ = 9.5052891,		
	log cot 72° 16′ = 9.5048538;		
find	log cot 72° 15′ 35″.		9.5050352.
69. Given	log cot 36° 18′ = 10.1339650,		
	log cot 36° 19′ = 10.1337003;		
find	log cot 36° 18′ 20″.		10.1338768.
70. Given	log cot 51° 17′ = 9.9039733,		
	log cot 51° 18′ = 9.9037144;		
find	log cot 51° 17′ 32″.		9.9038352.

71. Given $\log\sin 16^\circ\, 19' = 9.4486227,$

$\log\sin 16^\circ\, 20' = 9.4490540;$

find the angle whose log sine is 9.4488105.

Ans. 16° 19′ 26″.

72. Given $\log\sin\ 6^\circ\, 53' = 9.0786310,$

$\log\sin\ 6^\circ\, 53'\, 10'' = 9.0788054;$

find the angle whose log sine is 9.0787743. 6° 53′ 8″.

73. Given $\log\cos 22^\circ\, 28'\, 20'' = 9.9657025,$

$\log\cos 22^\circ\, 28'\, 10'' = 9.9657112;$

find the angle whose log cosine is 9.9657056. 22° 28′ 16″.

In the following examples the tables are to be used:

74. Find log tan 55° 37′ 53″. *Ans.* 10.1650011.

75. Find log sin 73° 20′ 15″.7. 9.9813707.

76. Find log cos 55° 11′ 12″. 9.7565636.

77. Find log tan 16° 0′ 27″. 9.4577109.

78. Find log sec 16° 0′ 27″. 10.0171747.

79. Find the angle whose log cosine is 9.9713383.

Ans. 20° 35′ 16″.

80. Find the angle whose log cosine is 9.9165646.

Ans. 34° 23′ 25″.

81. Find log cos 34° 24′ 26″. 9.9164762.

82. Find log cos 37° 19′ 47″. 9.9004540.

83. Find log sin 37° 19′ 47″. 9.7827599.

84. Find log tan 37° 19′ 47″. 9.8823059.

85. Find log sin 32° 18′ 24″.6. 9.7279096.

86. Find log cos 32° 18′ 24″.6. 9.9269585.

87. Find log tan 32° 18′ 24″.6. 9.8009511.

Prove the following by the use of logarithms:

88. $\dfrac{(7.014)^3 - 1}{(7.014)^3 + 1} = .9942207.$

89. $\dfrac{\sqrt[3]{5.12} \times \sqrt[5]{.00003075}}{\sqrt[3]{80} \div \sqrt[3]{.0000001}} = .000232432.$

90. $\sqrt[900]{\dfrac{(2002)^{1001} \times (1001)^{2002}}{1001 \times 2002}} = 21840300000.$

CHAPTER V.

SOLUTION OF TRIGONOMETRIC EQUATIONS.

82. A Trigonometric Equation is an equation in which the unknown quantities involve trigonometric functions.

The *solution* of a trigonometric equation is the process of finding the values of the unknown quantity which satisfy the equation. As in Algebra, we may have two or more simultaneous equations, the number of angles involved being equal to the number of equations.

EXAMPLES.

1. Solve $\sin\theta = \frac{1}{2}$.

This is a trigonometric equation. To solve it we must find some angle whose sine is $\frac{1}{2}$. We know that $\sin 30° = \frac{1}{2}$.

Therefore, if 30° be put for θ, the equation is satisfied.

$$\therefore\ \theta = 30° \text{ is a solution of the equation.}$$

$$\therefore\ \theta = n\pi + (-1)^n\frac{\pi}{6} \quad . \ . \ . \ . \ . \ . \ . \quad \text{(Art. 38)}$$

2. Solve $\cos\theta + \sec\theta = \frac{5}{2}$.

The usual method of solution is to express all the functions in terms of one of them.

Thus, we put $\frac{1}{\cos\theta}$ for $\sec\theta$, and get

$$\cos\theta + \frac{1}{\cos\theta} = \frac{5}{2}.$$

This is an equation in which θ, and therefore $\cos\theta$, is unknown. We proceed to solve the equation algebraically just as we should if x occupied the place of $\cos\theta$, thus:

$$\cos^2\theta - \frac{5}{2}\cos\theta = -1.$$

$$\therefore\ \cos\theta = \frac{5}{4} \pm \frac{3}{4}$$

$$= 2 \text{ or } \frac{1}{2}.$$

The value 2 is inadmissible, for there is no angle whose cosine is numerically greater than 1 (Art. 21).

$$\therefore\ \cos\theta = \frac{1}{2}.$$

But $$\cos 60° = \frac{1}{2}.$$

$$\therefore\ \cos\theta = \cos 60°.$$

Therefore *one* value of θ which satisfies the equation is 60°.

3. Solve $\operatorname{cosec}\theta - \cot^2\theta + 1 = 0.$

We have $\operatorname{cosec}\theta - (\operatorname{cosec}^2\theta - 1) + 1 = 0$. . (Art. 23)

$$\operatorname{cosec}^2\theta - \operatorname{cosec}\theta = 2.$$

$$\therefore\ \operatorname{cosec}\theta = \frac{1}{2} \pm \frac{3}{2}$$

$$= 2 \text{ or } -1.$$

But $$\operatorname{cosec} 30° = 2.$$

$$\therefore\ \operatorname{cosec}\theta = \operatorname{cosec} 30°.$$

Therefore 30° is one value of θ which satisfies the equation.

Find a value of θ which will satisfy the following equations:

4. $\cos\theta = \cos 2\theta.$ *Ans.* $\frac{2}{3}\pi$.

5. $2\cos\theta = \sec\theta.$ 45°.

6. $4 \sin \theta - 3 \operatorname{cosec} \theta = 0.$ *Ans.* $60°$.

7. $4 \cos \theta = 3 \sec \theta.$ $30°$.

8. $3 \sin \theta - 2 \cos^2\theta = 0.$ $30°$.

9. $\sqrt{2} \sin \theta = \tan \theta.$ $0°$ or $45°$.

10. $\tan \theta = 3 \cot \theta.$ $60°$.

11. $\tan \theta + 3 \cot \theta = 4.$ $45°$.

12. $\cos \theta + \cos 3\theta + \cos 5\theta = 0.$ $\frac{\pi}{2}, \frac{2\pi}{3}$.

13. $\sin (\theta - \phi) = \frac{1}{2}, \cos (\theta + \phi) = 0.$ $\theta = 60°, \phi = 30°$.

83. *Solve the equations*

$$m \sin \phi = a \quad \dots\dots\dots \quad (1)$$

$$m \cos \phi = b \quad \dots\dots\dots \quad (2)$$

where a and b are given, and the values of m and ϕ are required.

Dividing (1) by (2), we get

$$\tan \phi = \frac{a}{b},$$

which gives two values of ϕ, differing by 180°, and therefore two values of m also from either of the equations

$$m = \frac{a}{\sin \phi} = \frac{b}{\cos \phi}.$$

The two values of m will be equal numerically with opposite signs.

In practice, m is almost always *positive* by the conditions of the problem. Accordingly, $\sin \phi$ has the sign of a, and $\cos \phi$ the sign of b, and hence ϕ must be taken in the quadrant denoted by these signs. These cases may be considered as follows:

(1) Sin ϕ and cos ϕ *both positive.* This requires that the angle ϕ be taken in the *first* quadrant, because $\sin \phi$ and $\cos \phi$ are *both* positive in no other quadrant.

(2) Sin ϕ *positive* and cos ϕ *negative.* This requires that ϕ be taken in the *second* quadrant, because only in this quadrant is sin ϕ positive and cos ϕ negative for the same angle.

(3) Sin ϕ and cos ϕ *both negative.* This requires that ϕ be taken in the *third* quadrant, because only in this quadrant are sin ϕ and cos ϕ both negative for the same angle.

(4) Sin ϕ *negative* and cos ϕ *positive.* This requires that ϕ be taken in the *fourth* quadrant, because only in this quadrant is sin ϕ negative and cos ϕ positive for the same angle.

Ex. 1. Solve the equations $m \sin \phi = 332.76$, and $m \cos \phi = 290.08$, for m and ϕ.

$$\begin{aligned} \log m \sin \phi &= 2.52213 \\ \log m \cos \phi &= \underline{2.46252} \\ \log \tan \phi &= 0.05961 \end{aligned}$$

$\therefore \ \phi = 48° \ 55'.2.$

$$\begin{aligned} \log m \sin \phi &= 2.52213 \\ \log \sin \phi &= \underline{9.87725} \\ \log m &= 2.64488 \end{aligned}$$

$\therefore \ m = 441.45.$

Ex. 2. Solve $m \sin \phi = -72.631$, and $m \cos \phi = 38.412$.

Ans. $\phi = 117° \ 52'.3$, $m = -82.164$.

84. *Solve the equation*

$$a \sin \phi + b \cos \phi = c \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

a, b, and c being given, and ϕ required.

Find in the tables the angle whose tangent is $\frac{b}{a}$; let it be β.

Then $\frac{b}{a} = \tan \beta$, and (1) becomes

$$a(\sin \phi + \tan \beta \cos \phi) = c;$$

or
$$a\left(\frac{\sin \phi \cos \beta + \cos \phi \sin \beta}{\cos \beta}\right) = c;$$

or
$$\sin(\phi + \beta) = \frac{c}{a} \cos \beta = \frac{c}{b} \sin \beta \quad . \quad . \quad . \quad . \quad (2)$$

There will be two solutions from the two values of $\phi + \beta$ given in (2).

Find from the tables the value of $\cos\beta$. Next find from the tables the magnitude of the angle α whose sine $= \frac{c}{a}\cos\beta$, and we get

$$\sin(\phi + \beta) = \sin\alpha,$$

$$\therefore\ \phi + \beta = n\pi + (-1)^n\alpha \quad . \quad . \quad \text{(Art. 38)}$$

$$\therefore\ \phi = -\beta + n\pi + (-1)^n\alpha,$$

where n is zero or any positive or negative integer.

In order that the solution may be possible, it is necessary to have $\frac{c}{a}\cos\beta =$, or < 1.

NOTE.—This example might have been solved by squaring both sides of the equation; but in solving trigonometric equations, it is important, if possible, to avoid squaring both sides of the equation.

Thus, solve $\cos\theta = k\sin\theta$ . . . . (3)

If we square both sides we get

$$\cos^2\theta = k^2\sin^2\theta = k^2(1 - \cos^2\theta).$$

$$\therefore\ \cos^2\theta = \frac{k^2}{1+k^2};\ \text{or}\ \cos\theta = \pm\frac{k}{\sqrt{1+k^2}} \quad . \quad . \quad . \quad (4)$$

Now if α be the least angle whose cosine $= \frac{k}{\sqrt{1+k^2}}$, we get from (4)

$$\theta = n\pi \pm \alpha \quad . \quad . \quad . \quad (5)$$

But (3) may be written $\cot\theta = k$.

$$\therefore\ \theta = n\pi + \alpha \quad . \quad . \quad . \quad (6)$$

(6) is the complete solution of the given equation (3), while (5) is the solution of both $\cos\theta = k\sin\theta$, and also of $\cos\theta = -k\sin\theta$. Therefore by squaring both members of an equation we obtain solutions which do not belong to the given equation.

EXAMPLES.

1. Solve $0.7466898\sin\phi - 1.0498\cos\phi = -0.431689$, when $\phi < 180°$.

$$\log b = 0.02112-{*}$$

$$\log a = \bar{1}.87314$$

$$\log\tan\beta = 0.14798-$$

$\therefore\ \beta = 125°\ 25'\ 20''$.

* The minus sign is written thus to denote that it belongs to the *natural number* and does not affect the *logarithm*. Sometimes the letter n is written instead of the minus sign, to denote the same thing.

$$\begin{aligned} \log \sin \beta &= 9.91111 \\ \log c &= \bar{1}.63517- \\ \text{colog}\, b &= 9.97888- \\ \hline \log \sin(\phi + \beta) &= 9.52516+ \end{aligned}$$

$\therefore\ \phi + \beta = 19°\ 34'\ 40''$ or $160°\ 25'\ 20''$.

$\therefore\ \phi = -105°\ 50'\ 40''$ or $35°\ 0'\ 0''$.

2. Solve $-23.8 \sin \phi + 19.3 \cos \phi = 17.5 (\phi < 180°)$.
Ans. $\phi = 4°\ 12'.7$ or $-106°\ 7'.9$.

3. Solve $2 \sin \theta + 2 \cos \theta = \sqrt{2}$. *Ans.* $-\frac{\pi}{4} + n\pi + (-1)^n \frac{\pi}{6}$.

4. " $\sin \theta + \sqrt{3} \cos \theta = 1$. $-\frac{\pi}{3} + n\pi + (-1)^n \frac{\pi}{6}$.

5. " $\sin \theta - \cos \theta = 1$. $\frac{\pi}{4} + n\pi + (-1)^n \frac{1}{4}\pi$.

6. " $\sqrt{3} \sin \theta - \cos \theta = \sqrt{2}$. $\frac{1}{6}\pi + n\pi + (-1)^n \frac{1}{4}\pi$.

85. *Solve the equation*

$$\sin(\alpha + x) = m \sin x \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

in which α and m are given.

From (1) we have

$$\frac{\sin(\alpha + x) + \sin x}{\sin(\alpha + x) - \sin x} = \frac{m+1}{m-1} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

$$= \frac{\tan\left(x + \frac{\alpha}{2}\right)}{\tan \frac{\alpha}{2}} \quad . \quad . \quad \text{(Art. 46)}$$

$$\therefore\ \tan(x + \tfrac{1}{2}\alpha) = \frac{m+1}{m-1} \tan \frac{\alpha}{2} \quad . \quad . \quad . \quad . \quad . \quad (3)$$

which determines $x + \frac{1}{2}\alpha$, and therefore x.

If we introduce an *auxiliary* angle, the calculation of equation (3) is facilitated.

Thus, let $m = \tan\phi$; then we have by [(14) of Art. 61]

$$\frac{m+1}{m-1} = \frac{\tan\phi + 1}{\tan\phi - 1} = \cot(\phi - 45°),$$

which in (3) gives

$$\tan\left(x + \frac{\alpha}{2}\right) = \cot(\phi - 45°)\tan\tfrac{1}{2}\alpha \quad . \quad (4)$$

This, with $\tan\phi = m$,

gives the logarithmic solution.

The logarithmic solution of the equation

$$\sin(\alpha - x) = m\sin x$$

is found in the same manner to be

$$\tan\phi = m,$$

and $$\tan\left(x - \frac{\alpha}{2}\right) = \cot(\phi + 45°)\tan\frac{\alpha}{2},$$

which the student may show.

Example. — Solve

$$\sin(106° + x) = -1.263\sin x (x < 180°).$$

$$\log\tan\phi = \log m = \log(-1.263) = 0.10140 -.$$

$$\therefore \phi = 128°\ 22'.3.$$

$$\phi - 45° = 83°\ 22'.3; \quad \log\cot(\phi - 45°) = 9.06523$$

$$\tfrac{1}{2}\alpha = 53°\ 0'.0, \quad \log\tan\tfrac{1}{2}\alpha = \underline{10.12289}$$

$$\log\tan\left(x + \frac{\alpha}{2}\right) = 9.18812$$

$$x + \frac{\alpha}{2} = 8°\ 46'.0 \text{ or } 188°\ 46'.$$

$$\therefore x = -44°\ 14' \text{ or } 135°\ 46'.$$

86. *Solve the equation*

$$\tan(\alpha + x) = m\tan x \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

in which α and m are given.

From (1) we have

$$\frac{\tan(\alpha+x)+\tan x}{\tan(\alpha+x)-\tan x}=\frac{m+1}{m-1}$$

$$=\frac{\sin(\alpha+2x)}{\sin\alpha} \quad [(21) \text{ of Art. } 61]$$

$$\therefore\ \sin(\alpha+2x)=\frac{m+1}{m-1}\sin\alpha \quad . \quad . \quad . \quad . \quad . \quad (2)$$

$$=\cot(\phi-45°)\sin\alpha \quad . \quad (\text{Art. } 85)$$

where $\tan\phi = m.$

Example. — Solve $\tan(23° \, 16' + x) = .296 \tan x.$

$\log\tan\phi = \log m = \log(.296) = \bar{1}.47129.$

$\therefore\ \phi = 16° \, 29'.3.$

$\phi - 45° = -28° \, 30'.7$; $\quad \log\cot(\phi-45°) = 10.26502-$

$\alpha = 23° \, 16'.1,$ $\quad \log\sin\alpha = \ \ 9.59661$

$\log\sin(\alpha+2x) = \ \ 9.86163-$

$\alpha + 2x = 226° \, 38'.9$ or $313° \, 21'.1.$

$\therefore\ x = 101° \, 41'.5$ or $145° \, 2'.6.$

87. *Solve the equation*

$$\tan(\alpha+x)\cdot\tan x = m \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

in which α and m are given.

From (1) we have

$$\frac{1+\tan(\alpha+x)\tan x}{1-\tan(\alpha+x)\tan x}=\frac{1+m}{1-m}$$

$$=\frac{\cos\alpha}{\cos(\alpha+2x)} \quad . \quad . \quad (\text{Ex. 4 of Art. } 47)$$

$$\therefore\ \cos(\alpha+2x)=\frac{1-m}{1+m}\cos\alpha$$

$$=\tan(45°-\phi)\cos\alpha \quad [(16) \text{ of Art. } 61]$$

where $\tan\phi = m.$

Example. — Solve $\tan(65° + x)\tan x = 1.5196 (\alpha < 180°)$.

$\log\tan\phi = \log m = 0.18173.$

$\therefore\ \phi = 50°\,39'\,9'';$

$45° - \phi = -11°\,39'\,9'';\qquad \log\tan(45° - \phi) = 9.31434-$

$\alpha = 65°\,0'\,0'';\qquad \log\cos\alpha = 9.62595$

$\log\cos(\alpha + 2x) = 8.94029-$

$\alpha + 2x = 95° \text{ or } 265°.$

$\therefore\ 2x = 30° \text{ or } 200°.$

$\therefore\ x = 15° \text{ or } 100°.$

88. *Solve the equations*

$$m\sin(\theta + x) = a \quad \ldots\ldots \quad (1)$$

$$m\sin(\phi + x) = b \quad \ldots\ldots \quad (2)$$

for m and x, the other four quantities, θ, ϕ, a, b, being known.

Expanding (1) and (2) by (Art. 44), we get

$$m\sin\theta\cos x + m\cos\theta\sin x = a \quad \ldots\ldots \quad (3)$$

$$m\sin\phi\cos x + m\cos\phi\sin x = b \quad \ldots\ldots \quad (4)$$

Multiplying (3) by $\sin\phi$ and (4) by $\sin\theta$, and subtracting the latter from the former, we have

$$m\sin x(\sin\phi\cos\theta - \cos\phi\sin\theta) = a\sin\phi - b\sin\theta.$$

$$\therefore\ m\sin x = \frac{a\sin\phi - b\sin\theta}{\sin(\phi - \theta)} \quad \ldots\ldots \quad (5)$$

To find the value of $m\cos x$, multiply (3) and (4) by $\cos\phi$ and $\cos\theta$, respectively, and subtract the former from the latter. Thus

$$m\cos x(\sin\phi\cos\theta - \cos\phi\sin\theta) = b\cos\theta - a\cos\phi.$$

$$\therefore\ m\cos x = \frac{b\cos\theta - a\cos\phi}{\sin(\phi - \theta)} \quad \ldots\ldots \quad (6)$$

Having obtained the values of $m\sin x$ and $m\cos x$ from (5) and (6), m and x can be calculated by Art. 83.

EXAMPLES.

1. Solve $m\cos(\theta + x) = a$, and $m\sin(\phi + x) = b$, for $m\sin x$ and $m\cos x$.

Ans. $m\sin x = \dfrac{b\cos\theta - a\sin\phi}{\cos(\theta - \phi)}$,

$$m\cos x = \frac{b\sin\theta + a\cos\phi}{\cos(\theta - \phi)}.$$

2. Solve $m\cos(\theta + x) = a$, and $m\cos(\phi - x) = b$, for $m\sin x$ and $m\cos x$.

Ans. $m\sin x = \dfrac{b\cos\theta - a\cos\phi}{\sin(\theta + \phi)}$,

$$m\cos x = \frac{b\sin\theta + a\sin\phi}{\sin(\theta + \phi)}.$$

89. *Solve the equation*

$$x\cos\alpha + y\sin\alpha = m \quad \dots\dots\dots \quad (1)$$

$$x\sin\alpha - y\cos\alpha = n \quad \dots\dots\dots \quad (2)$$

for x and y.

Multiplying (1) by $\cos\alpha$ and (2) by $\sin\alpha$, and adding, we get

$$x = m\cos\alpha + n\sin\alpha.$$

To find the value of y, multiply (1) by $\sin\alpha$ and (2) by $\cos\alpha$, and subtract the latter from the former. Thus

$$y = m\sin\alpha - n\cos\alpha.$$

Example. — Solve

$$x\sin\alpha + y\cos\alpha = a,$$

$$x\cos\alpha - y\sin\alpha = b.$$

90. To adapt Formulæ to Logarithmic Computation. — As calculations are performed principally by means of logarithms, and as we are not able by logarithms directly* to add and subtract quantities, it becomes necessary to know how to transform sums and differences into products

* Addition and Subtraction Tables are published, by means of which the logarithm of the sum or difference of two numbers may be obtained. (See Tafeln der Additions, und Subtractions, Logarithmen für sieben Stellen, von J. Zech, Berlin.)

and quotients. An expression in the form of a product or quotient is said to be *adapted to logarithmic computation.*

An angle, introduced into an expression in order to adapt it to logarithmic computation, is called a *Subsidiary Angle.* Such an angle was introduced into each of the Arts. 84, 85, 86, and 87.

The following are further examples of the use of subsidiary angles:

1. Transform $a\cos\theta \pm b\sin\theta$ into a product, so as to adapt it to logarithmic computation.

Put $\frac{b}{a} = \tan\phi$;* thus

$$a\cos\theta \pm b\sin\theta = a\left(\cos\theta \pm \frac{b}{a}\sin\theta\right)$$
$$= a(\cos\theta \pm \tan\phi\sin\theta)$$
$$= \frac{a}{\cos\phi}\cos(\theta \mp \phi).$$

2. Similarly,

$$a\sin\theta \pm b\cos\theta = \frac{a}{\cos\phi}\sin(\theta \pm \phi).$$

3. Transform $a \pm b$ into a product,

$$a + b = a\left(1 + \frac{b}{a}\right) = a(1 + \tan^2\phi) = a\sec^2\phi,$$

if $$\frac{b}{a} = \tan^2\phi.$$

$$a - b = a\left(1 - \frac{b}{a}\right) = a\cos^2\phi,$$

if $$\frac{b}{a} = \sin^2\phi.$$

* The fundamental formulæ $\cos(x \pm y)$ and $\sin(x \pm y)$ (Art. 42) afford examples of one term equal to the sum or difference of two terms; hence we may transform an expression $a\cos\theta \pm b\sin\theta$ into an equivalent product, by conforming it to the formulæ just mentioned.

Thus, comparing the identity, $m\cos\phi\cos\theta \pm m\sin\phi\sin\theta = m\cos(\phi \mp \theta)$ or $m\cos(\theta \mp \phi)$, with $a\cos\theta \pm b\sin\theta$, we will have $a\cos\theta \pm b\sin\theta = m\cos(\theta \mp \phi)$ if we assume $a = m\cos\phi$ and $b = m\sin\phi$; *i.e.* (Art. 83), if $\tan\phi = \frac{b}{a}$ and $m = \frac{a}{\cos\phi} = \frac{b}{\sin\phi}$ as above. See Art. 84.

$$\therefore\ \log(a+b) = \log a + 2\log\sec\phi;$$

and
$$\log(a-b) = \log a + 2\log\cos\phi.$$

4. Transform $1239.3 \sin\theta - 724.6 \cos\theta$ to a product.

$$\log b = \log(-724.6) = 2.86010-$$
$$\log a = \log(1239.3) = \underline{3.09318}$$
$$\log\tan\phi = 9.76692-$$
$$\therefore\ \phi = -30^\circ\ 18'.8.$$

$$\log a = 3.09318$$
$$\log\cos\phi = \underline{9.93615}$$
$$\log\frac{a}{\cos\phi} = 3.15703$$

$$\therefore\ \frac{a}{\cos\phi} = 1435.6.$$

$$\therefore\ 1239.3\sin\theta - 724.6\cos\theta = 1435.6\sin(\theta - 30^\circ\ 18'.8).$$

91. *Solve the equations*

$$r\cos\phi\cos\theta = a \quad \ldots\ldots\ldots \quad (1)$$
$$r\cos\phi\sin\theta = b \quad \ldots\ldots\ldots \quad (2)$$
$$r\sin\phi = c \quad \ldots\ldots\ldots \quad (3)$$

for r, ϕ, and θ.

Dividing (2) by (1), we have

$$\tan\theta = \frac{b}{a},$$

from which we obtain θ.

From (1) and (2) we have

$$r\cos\phi = \frac{a}{\cos\theta} = \frac{b}{\sin\theta}. \quad \ldots\ldots \quad (4)$$

from which we obtain $r\cos\phi$.

From (3) and (4) we obtain r and ϕ (Art. 83).

EXAMPLES.

1. Solve
$$r\cos\phi\cos\theta = -53.953,$$
$$r\cos\phi\sin\theta = 197.207,$$
$$r\sin\phi = -39.062,$$
for r, ϕ, θ.

$$\begin{aligned}\log b &= 2.29493\\ \log a &= \underline{1.73201}-\\ \log\tan\theta &= 0.56292-\\ \therefore\ \theta &= 105^\circ\,18'.0.\\ \log\sin\theta &= \underline{9.98433}\\ \log r\cos\phi &= 2.31060\\ \log r\sin\phi &= \underline{1.59175}-\\ \log\tan\phi &= 9.28115-\end{aligned}
\qquad
\begin{aligned}\therefore\ \phi &= -10^\circ\,49'.\\ \log r\cos\phi &= 2.31060\\ \log\cos\phi &= \underline{9.99221}\\ \log r &= 2.31839\\ \therefore\ r &= 208.16.\end{aligned}$$

92. Trigonometric Elimination. — Several simultaneous equations may be given, as in Algebra, by the combination of which certain quantities may be eliminated, and a result obtained involving the remaining quantities.

Trigonometric elimination occurs chiefly in the application of Trigonometry to the higher branches of Mathematics, as, for example, in Physical Astronomy, Mechanics, Analytic Geometry, etc. As no special rules can be given, we illustrate the process by a few examples.

EXAMPLES.

1. Eliminate ϕ from the equations
$$x = a\cos\phi,\quad y = b\sin\phi.$$

From the given equations we have
$$\frac{x}{a} = \cos\phi,\quad \frac{y}{b} = \sin\phi,$$
which in
$$\cos^2\phi + \sin^2\phi = 1,$$
gives
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

2. Eliminate ϕ from the equations

$$a\cos\phi + b\sin\phi = c,$$
$$b\cos\phi + c\sin\phi = a.$$

Solving these equations for $\sin\phi$ and $\cos\phi$, we have

$$\sin\phi = \frac{bc - a^2}{b^2 - ac},$$
$$\cos\phi = \frac{c^2 - ab}{ac - b^2};$$

which in $\cos^2\phi + \sin^2\phi = 1,$

gives $(bc - a^2)^2 + (c^2 - ab)^2 = (ac - b^2)^2.$

3. Eliminate ϕ from the equations

$$y\cos\phi - x\sin\phi = a\cos 2\phi,$$
$$y\sin\phi + x\cos\phi = 2a\sin 2\phi.$$

Solve for x and y, then add and subtract, and we get

$$x + y = a(\sin\phi + \cos\phi)(1 + \sin 2\phi),$$
$$x - y = a(\sin\phi - \cos\phi)(1 - \sin 2\phi).$$
$$\therefore\ (x + y)^2 = a^2(1 + \sin 2\phi)^3,$$
$$(x - y)^2 = a^2(1 - \sin 2\phi)^3.$$
$$\therefore\ (x + y)^{\frac{2}{3}} + (x - y)^{\frac{2}{3}} = 2a^{\frac{2}{3}}.$$

4. Eliminate α and β from the equations

$$a = \sin\alpha\cos\beta\sin\theta + \cos\alpha\cos\theta \quad . \quad . \quad . \quad (1)$$
$$b = \sin\alpha\cos\beta\cos\theta - \cos\alpha\sin\theta \quad . \quad . \quad . \quad (2)$$
$$c = \sin\alpha\sin\beta\sin\theta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Squaring (1) and (2), and adding, we get

$$a^2 + b^2 = \sin^2\alpha\cos^2\beta + \cos^2\alpha \quad . \quad . \quad . \quad . \quad . \quad (4)$$
$$\frac{c^2}{\sin^2\theta} = \sin^2\alpha\sin^2\beta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Adding (4) and (5), we have

$$a^2 + b^2 + \frac{c^2}{\sin^2\theta} = 1.$$

5. Eliminate ϕ from the equations

$$a\sin\phi + b\cos\phi = c,$$
$$a\cos\phi - b\sin\phi = d. \qquad \textit{Ans. } a^2 + b^2 = c^2 + d^2.$$

6. Eliminate θ from the equations

$$m = \operatorname{cosec}\theta - \sin\theta,$$
$$n = \sec\theta - \cos\theta. \qquad m^{\frac{2}{3}}n^{\frac{2}{3}}(m^{\frac{2}{3}} + n^{\frac{2}{3}}) = 1.$$

7. Eliminate θ and ϕ from the equations

$$\sin\theta + \sin\phi = a,$$
$$\cos\theta + \cos\phi = b,$$
$$\cos(\theta - \phi) = c. \qquad a^2 + b^2 - 2c = 2.$$

8. Eliminate x and y from the equations

$$\tan x + \tan y = a,$$
$$\cot x + \cot y = b,$$
$$x + y = c. \qquad \cot c = \frac{1}{a} - \frac{1}{b}.$$

9. Eliminate ϕ from the equations

$$x = \cos 2\phi + \cos\phi,$$
$$y = \sin 2\phi + \sin\phi.$$
$$\textit{Ans. } 2x = (x^2 + y^2)^2 - 3(x^2 + y^2).$$

EXAMPLES.

Solve the following equations:

1. $\tan\theta + \cot\theta = 2.$ *Ans.* 45°.

2. $2\sin^2\theta + \sqrt{2}\cos\theta = 2.$ 90°, or 45°.

3. $3\tan^2\theta - 4\sin^2\theta = 1.$ 45°.

4. $2\sin^2\theta + \sqrt{2}\sin\theta = 2.$ 45°.

5. $\cos^2\theta - \sqrt{3}\cos\theta + \frac{3}{4} = 0.$ *Ans.* $30°.$

6. $\sin 5\theta = 16\sin^5\theta.$ $n\pi$, or $n\pi \pm \frac{\pi}{6}.$

7. $\sin 9\theta - \sin\theta = \sin 4\theta.$ $\frac{1}{4}n\pi$, or $\frac{2}{5}n\pi \pm \frac{\pi}{15}.$

8. $2\sin\theta = \tan\theta.$ $n\pi$, or $2n\pi \pm \frac{\pi}{3}.$

9. $6\cot^2\theta - 4\cos^2\theta = 1.$ $n\pi \pm \frac{\pi}{3}.$

10. $\tan\theta + \tan(\theta - 45°) = 2.$ $n\pi \pm \frac{\pi}{3}.$

11. $\cos\theta + \sqrt{3}\sin\theta = \sqrt{2}.$ $2n\pi \pm \frac{\pi}{3}.$

12. $\tan(\theta + 45°) = 1 + \sin 2\theta.$ $n\pi - \frac{\pi}{4}$, or $n\pi.$

13. $(\cot\theta - \tan\theta)^2(2 + \sqrt{3}) = 4(2 - \sqrt{3}).$ $\frac{1}{2}n\pi \pm \frac{\pi}{24}.$

14. $\operatorname{cosec}\theta\cot\theta = 2\sqrt{3}.$ $2n\pi \pm \frac{\pi}{6}.$

15. $\operatorname{cosec}\theta + \cot\theta = \sqrt{3}.$ $2n\pi + \frac{1}{3}\pi.$

16. $\sin\frac{\theta}{2} = \operatorname{cosec}\theta - \cot\theta.$ $2n\pi.$

17. $\sin 5\theta\cos 3\theta = \sin 9\theta\cos 7\theta.$ $\frac{1}{16}n\pi + (-1)\frac{\theta}{2}.$

18. $\sin^2\theta + \cos^2 2\theta = \frac{3}{4}.$ $n\pi \pm \frac{\pi}{10}$, or $n\pi \pm \frac{3}{10}\pi.$

19. $\sqrt{3}\sin\theta - \cos\theta = \sqrt{2}.$ $n\pi + \frac{\pi}{6} + (-1)^n\frac{\pi}{4}.$

20. $\tan\theta + \cot\theta = 4.$ $n\pi + \frac{5}{12}\pi.$

21. $\sin(\theta + \phi) = \frac{\sqrt{3}}{2}$, $\cos(\theta - \phi) = \frac{\sqrt{3}}{2}.$ $\theta = \frac{\pi}{4}$, $\phi = \frac{\pi}{12}.$

22. Solve $m\sin\phi = 1.29743$, and $m\cos\phi = 6.0024.$

Ans. $(\phi < 180°).$

23. Solve $m \sin \phi = -0.3076258$, and $m \cos \phi = 0.4278735$. (m positive.) *Ans.* $\phi = 324° 17' 6''.6$, $m = 0.52698$.

24. Solve $m \sin \phi = 0.08219$, and $m \cos \phi = 0.1288$.

25. Solve $m \sin \phi = 194.683$, and $m \cos \phi = 8460.7$.

26. If $a \sin \theta + b \cos \theta = c$, and $a \cos \theta + b \sin \theta = c \sin \theta \cos \theta$, show that $\sin 2\theta (c^2 - a^2 - b^2) = 2ab$.

Solve the following equations:

27. $\sqrt{2} \sin \theta + \sqrt{2} \cos \theta = \sqrt{3}$. *Ans.* $-\frac{\pi}{4} + n\pi + (-1)^n \frac{\pi}{3}$.

28. $2 \sin x + 5 \cos x = 2$. Sug. [$2.5 = \tan 68° 12'$].
Ans. $x = -68° 12' + n\,180° + (-1)^n (21° 48')$.

29. $3 \cos x - 8 \sin x = 3$. Sug. [$2.6 = \tan 69° 26' 30''$].
Ans. $x = -69° 26' 30'' + 2n\,180° \pm (69° 26' 30'')$.

30. $4 \sin x - 15 \cos x = 4$. Sug. [$3.75 = \tan 75° 4'$].
Ans. $x = 75° 4' + n\,180° + (-1)^n (14° 56')$.

31. $\cos(\alpha + x) = \sin(\alpha + x) + \sqrt{2} \cos \beta$.
Ans. $x = -\alpha - \frac{\pi}{4} + 2n\pi \pm \beta$.

32. $\cos \theta + \cos 3\theta + \cos 5\theta = 0$.
Ans. $\frac{1}{6}(2n + 1)\pi$, or $\frac{1}{3}(3n \pm 1)\pi$.

33. $\sin 5\theta = \sin 3\theta + \sin \theta = 3 - 4 \sin^2 \theta$.
Ans. $n\pi \pm \frac{\pi}{3}$, or $\frac{1}{6}(2n + 1)\pi$.

34. $2 \sin^2 3\theta + \sin^2 6\theta = 2$.
Ans. $\frac{1}{6}(2n + 1)\pi$, or $\frac{n\pi}{3} + (-1)^n \frac{\pi}{12}$.

35. $a(\cos 2\theta - 1) + 2b(\cos \theta + 1) = 0$.
Ans. $(2n + 1)\pi$, or $\cos^{-1} \frac{a - b}{a}$.

36. Solve

$$m\sin(\theta+x)=a\cos\beta,\ \text{and}\ m\cos(\theta-x)=a\sin\beta,$$

for $m\sin x$ and $m\cos x$. (Art. 67.)

Ans. $m\sin x=\dfrac{a\cos(\beta+\theta)}{\cos 2\theta}$,

$m\cos x=\dfrac{a\sin(\beta-\theta)}{\cos 2\theta}$.

37. Solve $m\cos(\theta+\phi)=3.79$, and $m\cos(\theta-\phi)=2.06$, for m and θ, when $\phi=31^\circ 27'.4$. (Art. 67.)

38. Solve

$$r\cos\phi\cos\theta=1.271,$$
$$r\cos\phi\sin\theta=-0.981,$$
$$r\sin\phi=0.890,$$

for r, ϕ, θ. (Art. 70.)

39. Solve

$$r\cos\phi\cos\theta=-2,$$
$$r\cos\phi\sin\theta=+3,$$
$$r\sin\phi=-4,$$

for r, ϕ, θ.

40. Solve

$$r\sin\phi\sin\theta=19.765,$$
$$r\sin\phi\cos\theta=-7.192,$$
$$r\cos\phi=12.124,$$

for r, ϕ, θ.

41. Solve $\cos(2x+3y)=\frac{1}{2}$, $\cos(3x+2y)=\frac{1}{2}\sqrt{3}$.

Ans. $x=\frac{2}{5}n\pi\pm\frac{2}{15}\pi\pm\frac{1}{2}\pi$, $y=\frac{2}{5}n\pi\pm\frac{1}{5}\pi\pm\frac{1}{15}\pi$.

42. Solve $\cos 3\theta+\cos 5\theta+\sqrt{2}(\cos\theta+\sin\theta)\cos\theta=0$.

Ans. $4\theta\pm\theta=2n\pi\pm\frac{3}{4}\pi$, or $\frac{1}{2}(2n+1)\pi$.

43. Solve $\cos 3\theta+\sin 3\theta=\cos\theta+\sin\theta$.

Ans. $\sin\theta=0$, or $\tan\theta=-1\pm\sqrt{2}$.

44. Solve $3\sin\theta+\cos\theta=2x$, $\sin\theta+2\cos\theta=x$.

Ans. $\theta=71^\circ 34'$, $x=\frac{1}{2}\sqrt{10}$.

45. Solve $1.268 \sin\phi = 0.948 + m \sin(25° 27'.2)$,

$$1.268 \cos\phi = 0.281 + m\cos(25° 27'.2).$$

Ans. $\phi = 60° 53'.8$, $m = 0.372$.

46. Transform $x^4 + y^4 + z^4 - 2y^2z^2 - 2z^2x^2 - 2x^2y^2$ into a product. *Ans.* $-(x+y+z)(y+z-x)(z+x-y)(x+y-z)$.

47. Eliminate θ from the equations

$$m\sin 2\theta = n\sin\theta,\ p\cos 2\theta = q\cos\theta.$$

Ans. $m^2 + p^2 = n^2 + q^2$.

48. Eliminate θ and ϕ from the equations

$$x = a\cos^m\theta\cos^m\phi,\ y = b\cos^m\theta\sin^m\phi,\ z = c\sin^m\theta.$$

Ans. $\left(\frac{x}{a}\right)^{\frac{2}{m}} + \left(\frac{y}{b}\right)^{\frac{2}{m}} + \left(\frac{z}{c}\right)^{\frac{2}{m}} = 1.$

49. Eliminate θ from the equations

$$a\sin\theta + b\cos\theta = h,\ a\cos\theta - b\sin\theta = k.$$

Ans. $a^2 + b^2 = h^2 + k^2$.

50. Eliminate θ from the equations

$$a\tan\theta + b\sec\theta = c,\ a'\cot\theta + b'\operatorname{cosec}\theta = c'.$$

Ans. $(a'b + cb')^2 + (ab' + c'b)^2 = (cc' - aa')^2$.

51. Eliminate θ from the equations

$$x = 2a\cos\theta\cos 2\theta - a\cos\theta,$$

$$y = 2a\cos\theta\sin 2\theta - a\sin\theta.$$

Ans. $x^2 + y^2 = a^2$.

52. Eliminate θ from the equations

$$x = a\cos\theta + b\cos 2\theta, \text{ and } y = a\sin\theta + b\sin 2\theta.$$

Ans. $a^2[(x+b)^2 + y^2] = [x^2 + y^2 - b^2]^2$.

53. Eliminate α and β from the equations

$$b + c\cos\alpha = u\cos(\alpha - \theta),$$

$$b + c\cos\beta = u\cos(\beta - \theta),\ \alpha - \beta = 2\phi;$$

and show that

$$u^2 - 2uc\cos\theta + c^2 = b^2\sec^2\phi.$$

54. Eliminate θ and ϕ from the equations

$$x\cos\theta + y\sin\theta = a,\ \ b\sin(\theta+\phi) = a\sin\phi,$$

$$x\cos(\theta+2\phi) - y\sin(\theta+2\phi) = a.$$

Ans. $x^2 + y^2 = a^2 + \dfrac{a^2y^2}{b^2}.$

55. Eliminate θ from the equations

$$\frac{x}{a} = \frac{\sec^2\theta - \cos^2\theta}{\sec^2\theta + \cos^2\theta},$$

$$\frac{25}{y} = \sec^2\theta + \cos^2\theta.$$

$\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1.$

56. Eliminate θ from the equations

$$(a+b)\tan(\theta-\phi) = (a-b)\tan(\theta+\phi),$$

$a\cos 2\phi + b\cos 2\theta = c.$ *Ans.* $b^2 = c^2 + a^2 - 2ac\cos 2\phi.$

57. Eliminate θ from the equations

$$x\sin\theta - y\cos\theta = \sqrt{x^2+y^2},\ \ \frac{\cos^2\theta}{a^2} + \frac{\sin^2\theta}{b^2} = \frac{1}{x^2+y^2}.$$

Ans. $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1.$

58. Eliminate θ and ϕ from the equations

$$a^2\cos^2\theta - b^2\cos^2\phi = c^2,\ \ a\cos\theta + b\cos\phi = r,$$

$$a\tan\theta = b\tan\phi.$$

Ans. $a^2\left[\dfrac{4r^2a^2}{(r^2+c^2)^2} - 1\right] = b^2\left[\dfrac{4r^2b^2}{(r^2-c^2)^2} - 1\right].$

59. Eliminate ϕ from the equations

$$n\sin\theta - m\cos\theta = 2m\sin\phi,$$

$$n\sin 2\theta - m\cos 2\phi = n.$$

Ans. $(n\sin\theta + m\cos\theta)^2 = 2m(m+n).$

60. Eliminate α from the equations

$$x\tan(\alpha-\beta) = y\tan(\alpha+\beta),$$

$$(x-y)\cos 2\alpha + (x+y)\cos 2\beta = z.$$

Ans. $z^2 + 4xy = 2z(x+y)\cos 2\beta.$

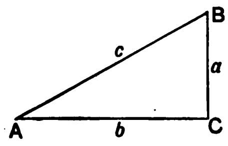
CHAPTER VI.

RELATIONS BETWEEN THE SIDES OF A TRIANGLE AND THE FUNCTIONS OF ITS ANGLES.

93. Formulæ. — In this chapter we shall deduce formulæ which express certain relations between the sides of a triangle and the functions of its angles. These relations will be applied in the next chapter to the *solution of triangles.* One of the principal objects of Trigonometry, as its name implies (Art. 1), is to establish certain relations between the sides and angles of triangles, so that when some of these are known the rest may be determined.

RIGHT TRIANGLES.

94. Let ABC be a triangle, right-angled at C. Denote the angles of the triangle by the letters A, B, C, and the lengths of the sides respectively opposite these angles, by the letters a, b, c.* Then we have (Art. 14) the following relations:



$$a = c \sin A = c \cos B = b \tan A = b \cot B \quad . \quad . \quad (1)$$

$$b = c \sin B = c \cos A = a \tan B = a \cot A \quad . \quad . \quad (2)$$

$$c = b \sec A = a \sec B = b \operatorname{cosec} B = a \operatorname{cosec} A \quad . \quad (3)$$

which may be expressed in the following general theorems:

* The student must remember that a, b, c, are *numbers* expressing the lengths of the sides in terms of some unit of length, such as a foot or a mile. The unit may be whatever we please, but must be the same for all the sides.

I. *In a right triangle each side is equal to the product of the hypotenuse into the sine of the opposite angle or the cosine of the adjacent angle.*

II. *In a right triangle each side is equal to the product of the other side into the tangent of the angle adjacent to that other side, or the cotangent of the angle adjacent to itself.*

III. *In a right triangle the hypotenuse is equal to the product of a side into the secant of its adjacent angle, or the cosecant of its opposite angle.*

EXAMPLES.

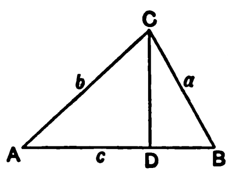
In a right triangle ABC, in which C is a right angle, prove the following:

1. $\tan B = \cot A + \cos C$.
2. $\sin 2A = \sin 2B$.
3. $\cos 2A + \cos 2B = 0$.
4. $\sin 2A = \frac{2ab}{c^2}$.
5. $\operatorname{cosec} 2B = \frac{a}{2b} + \frac{b}{2a}$.
6. $\cos 2A = \frac{b^2 - a^2}{c^2}$.
7. $\tan 2A = \frac{2ab}{b^2 - a^2}$.
8. $\sin 3A = \frac{3ab^2 - a^3}{c^3}$.

OBLIQUE TRIANGLES.

95. Law of Sines. — *In any triangle the sides are proportional to the sines of the opposite angles.*

Let ABC be any triangle. Draw CD perpendicular to AB.

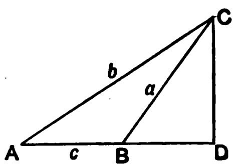


We have, then, in both figures

$$CD = a \sin B = b \sin A. \quad \text{(Art. 94)}$$

$$\therefore\ a \sin B = b \sin A.$$

$$\therefore\ \frac{a}{\sin A} = \frac{b}{\sin B}.$$



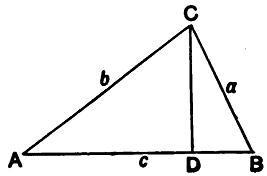
Similarly, by drawing a perpendicular from A or B to the opposite side, we may prove that

$$\frac{b}{\sin B}=\frac{c}{\sin C}, \text{ and } \frac{c}{\sin C}=\frac{a}{\sin A}.$$

$$\therefore \frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C},$$

or $$a : b : c = \sin A : \sin B : \sin C.$$

96. Law of Cosines. — *In any triangle the square of any side is equal to the sum of the squares of the other two sides minus twice the product of these sides and the cosine of the included angle.*



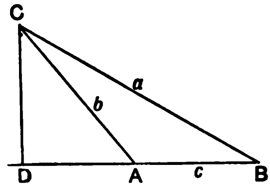
In an *acute-angled* triangle (see first figure) we have (Geom., Book III., Prop. 26)

$$\overline{BC}^2 = \overline{AC}^2 + \overline{AB}^2 - 2\,AB \times AD,$$

or $$a^2 = b^2 + c^2 - 2c \cdot AD.$$

But $$AD = b \cos A.$$

$$\therefore a^2 = b^2 + c^2 - 2bc \cos A.$$



In an *obtuse-angled* triangle (see second figure) we have (Geom., Book III., Prop. 27)

$$\overline{BC}^2 = \overline{AC}^2 + \overline{AB}^2 + 2\,AB \times AD,$$

or $$a^2 = b^2 + c^2 + 2c \cdot AD.$$

But $$AD = b \cos CAD = -b \cos A.$$

$$\therefore a^2 = b^2 + c^2 - 2bc \cos A.$$

Similarly, $$b^2 = c^2 + a^2 - 2ca \cos B,$$

$$c^2 = a^2 + b^2 - 2ab \cos C.$$

NOTE. — When one equation in the solution of triangles has been obtained, the other two may generally be obtained by advancing the letters so that a becomes b, b becomes c, and c becomes a; the order is abc, bca, cab. It is obvious that the formulæ thus obtained are true, since the naming of the sides makes no difference, provided the right order is maintained.

97. Law of Tangents. — *In any triangle the sum of any two sides is to their difference as the tangent of half the sum of the opposite angles is to the tangent of half their difference.*

By Art. 95, $a : b = \sin A : \sin B$.

By composition and division,

$$\frac{a+b}{a-b} = \frac{\sin A + \sin B}{\sin A - \sin B}$$

$$= \frac{\tan \frac{1}{2}(A+B)}{\tan \frac{1}{2}(A-B)}, \text{ by (13) of Art. 61} \quad (1)$$

Similarly.

$$\frac{b+c}{b-c} = \frac{\tan \frac{1}{2}(B+C)}{\tan \frac{1}{2}(B-C)} \quad . \; . \; . \; . \; . \; . \; . \quad (2)$$

$$\frac{c+a}{c-a} = \frac{\tan \frac{1}{2}(C+A)}{\tan \frac{1}{2}(C-A)} \quad . \; . \; . \; . \; . \; . \; . \quad (3)$$

Since $\tan \frac{1}{2}(A+B) = \tan(90^\circ - \frac{1}{2}C) = \cot \frac{1}{2}C$,

the result in (1) may be written

$$\frac{a+b}{a-b} = \frac{\cot \frac{1}{2}C}{\tan \frac{1}{2}(A-B)} \quad . \; . \; . \; . \; . \; . \; . \quad (4)$$

and similar expressions for (2) and (3).

98. To show that in any triangle $c = a \cos B + b \cos A$.

In an *acute-angled* triangle (first figure of Art. 96) we have

$$c = DB + DA$$
$$= a \cos B + b \cos A.$$

In an *obtuse-angled* triangle (second figure of Art. 96) we have

$$c = DB - DA$$
$$= a \cos B - b \cos CAD.$$
$$\therefore c = a \cos B + b \cos A.$$

Similarly,

$$b = c \cos A + a \cos C,$$
$$a = b \cos C + c \cos B.$$

EXAMPLES.

1. In the triangle ABC prove (1)

$$a+b:c=\cos\tfrac{1}{2}(A-B):\sin\tfrac{1}{2}C,$$

and (2)
$$a-b:c=\sin\tfrac{1}{2}(A-B):\cos\tfrac{1}{2}C.$$

2. If AD bisects the angle A of the triangle ABC, prove

$$BD:DC=\sin C:\sin B.$$

3. If AD′ bisects the external vertical angle A, prove

$$BD':CD'=\sin C:\sin B.$$

4. Hence prove
$$\frac{1}{DC}=\frac{2\cos\frac{1}{2}A\cos\frac{1}{2}(B-C)}{a\sin B};$$

and also
$$\frac{1}{D'C}=\frac{2\sin\frac{1}{2}A\sin\frac{1}{2}(C-B)}{a\sin B}.$$

99. To express the Sine, the Cosine, and the Tangent of Half an Angle of a Triangle in Terms of the Sides.

I. By Art. 96 we have

$$\cos A=\frac{b^2+c^2-a^2}{2bc}=1-2\sin^2\frac{A}{2} \quad . \quad . \quad . \quad . \quad \text{(Art. 49)}$$

$$\therefore\ 2\sin^2\frac{A}{2}=1-\frac{b^2+c^2-a^2}{2bc}$$

$$=\frac{a^2-(b-c)^2}{2bc}$$

$$=\frac{(a+b-c)(a-b+c)}{2bc}.$$

Let
$$a+b+c=2s;$$

then
$$a+b-c=2(s-c),\text{ and }a-b+c=2(s-b).$$

$$\therefore\ 2\sin^2\frac{A}{2}=\frac{2(s-c)2(s-b)}{2bc}.$$

$$\therefore\ \sin\frac{A}{2}=\sqrt{\frac{(s-b)(s-c)}{bc}} \quad . \quad . \quad . \quad . \quad (1)$$

Similarly, $\sin\frac{B}{2} = \sqrt{\frac{(s-c)(s-a)}{ac}}$ . . . . (2)

$\sin\frac{C}{2} = \sqrt{\frac{(s-a)(s-b)}{ab}}$ . . . . (3)

II. $\cos A = 2\cos^2\frac{A}{2} - 1$. . . . . . . (Art. 49)

$$\therefore\ 2\cos^2\frac{A}{2} = 1 + \frac{b^2+c^2-a^2}{2bc}$$

$$= \frac{(b+c)^2 - a^2}{2bc}$$

$$= \frac{(a+b+c)(b+c-a)}{2bc}$$

$$= \frac{2s \cdot 2(s-a)}{2bc}.$$

$\therefore\ \cos\frac{A}{2} = \sqrt{\frac{s(s-a)}{bc}}$ . . . . (4)

Similarly, $\cos\frac{B}{2} = \sqrt{\frac{s(s-b)}{ac}}$ . . . . (5)

$\cos\frac{C}{2} = \sqrt{\frac{s(s-c)}{ab}}$ . . . . (6)

III. Dividing (1) by (4), we get

$\tan\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}$ . . . . (7)

Similarly, $\tan\frac{B}{2} = \sqrt{\frac{(s-c)(s-a)}{s(s-b)}}$ . . . . (8)

$\tan\frac{C}{2} = \sqrt{\frac{(s-a)(s-b)}{s(s-c)}}$ . . . . (9)

Since any angle of a triangle is $< 180°$, the half angle is $< 90°$; therefore the *positive* sign must be given to the radicals which occur in this article.

100. To express the Sine of an Angle in Terms of the Sides.

$$\sin A = 2\sin\frac{A}{2}\cos\frac{A}{2} \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(Art. 49)}$$

$$= 2\sqrt{\frac{(s-b)(s-c)}{bc}}\cdot\sqrt{\frac{s(s-a)}{bc}}. \quad \text{(Art. 99)}$$

$$\therefore\ \sin A = \frac{2}{bc}\sqrt{s(s-a)(s-b)(s-c)}.$$

Similarly, $\sin B = \dfrac{2}{ac}\sqrt{s(s-a)(s-b)(s-c)},$

$$\sin C = \frac{2}{ab}\sqrt{s(s-a)(s-b)(s-c)}.$$

Cor. $\sin A = \dfrac{1}{2bc}\sqrt{2b^2c^2 + 2c^2a^2 + 2a^2b^2 - a^4 - b^4 - c^4},$

and similar expressions for $\sin B$, $\sin C$.

EXAMPLES.

In any triangle ABC prove the following statements:

1. $a(b\cos C - c\cos B) = b^2 - c^2.$

2. $(b+c)\cos A + (c+a)\cos B + (a+b)\cos C = a + b + c.$

3. $\dfrac{\sin A + 2\sin B}{a + 2b} = \dfrac{\sin C}{c}.$

4. $\dfrac{\sin^2 A - m\sin^2 B}{a^2 - mb^2} = \dfrac{\sin^2 C}{c^2}.$

5. $a\cos A + b\cos B - c\cos C = 2c\cos A\cos B.$

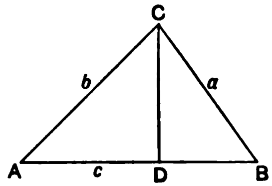
6. $\dfrac{\cos A}{\sin B\sin C} + \dfrac{\cos B}{\sin C\sin A} + \dfrac{\cos C}{\sin A\sin B} = 2.$

7. $a\sin(B - C) + b\sin(C - A) + c\sin(A - B) = 0.$

8. $\tan\frac{1}{2}A\tan\frac{1}{2}B = \dfrac{s - c}{s}.$

9. $\tan\frac{1}{2}A \div \tan\frac{1}{2}B = (s - b) \div (s - c).$

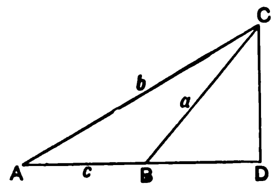
101. Expressions for the Area of a Triangle.



(1) *Given two sides and their included angle.*

Let S denote the area of the triangle ABC. Then by Geometry,

$$2S = c \times CD.$$



But in either figure, by Art. 94,

$$CD = b \sin A.$$

$$\therefore \; S = \tfrac{1}{2} bc \sin A.$$

Similarly, $\quad S = \tfrac{1}{2} ac \sin B,$

$$S = \tfrac{1}{2} ab \sin C.$$

(2) *Given one side and the angles.*

Since $\quad a : b = \sin A : \sin B \quad . \; . \; . \; . \; . \; .$ (Art. 95)

$$\therefore \; b = \frac{a \sin B}{\sin A},$$

which is $\quad S = \tfrac{1}{2} ab \sin C$, gives

$$S = \frac{a^2 \sin B \sin C}{2 \sin A}.$$

Similarly, $\quad S = \dfrac{b^2 \sin A \sin C}{2 \sin B} = \dfrac{c^2 \sin A \sin B}{2 \sin C}.$

(3) *Given the three sides.*

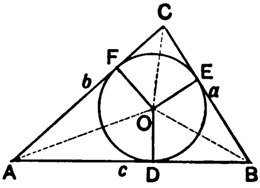
$$\sin A = \frac{2}{bc}\sqrt{s(s-a)(s-b)(s-c)} \quad \text{(Art. 100)}$$

Substituting in

$$S = \tfrac{1}{2} bc \sin A,$$

we get $\quad S = \sqrt{s(s-a)(s-b)(s-c)}.$

102. Inscribed Circle. — *To find the radius of the inscribed circle of a triangle.*



Let ABC be a triangle, O the centre of the inscribed circle, and r its radius. Draw radii to the points of contact D, E, F; and join OA, OB, OC. Then

$$S = \text{area of ABC}$$

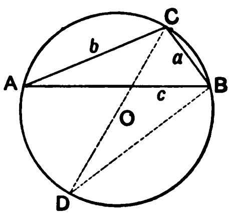
$$= \triangle AOB + \triangle BOC + \triangle COA$$

$$= \tfrac{1}{2}rc + \tfrac{1}{2}ra + \tfrac{1}{2}rb$$

$$= r\frac{a+b+c}{2} = rs \quad . \quad . \quad . \quad . \quad \text{(Art. 99)}$$

$$\therefore\ r = \frac{S}{s} = \sqrt{\frac{(s-a)(s-b)(s-c)}{s}} \quad . \quad \text{(Art. 101)}$$

103. Circumscribed Circle. — *To find the radius of the circumscribed circle of a triangle in terms of the sides of the triangle.*



Let O be the centre of the circle described about the triangle ABC, and R its radius.

Through O draw the diameter CD and join BD.

Then $\angle BDC = \angle BAC = \angle A.$

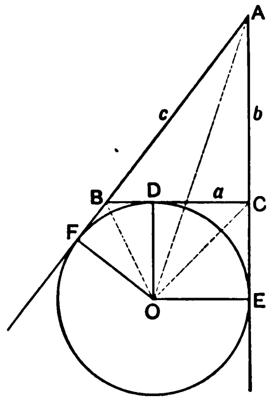
$$\therefore\ BC = 2R\sin A, \text{ or } a = 2R\sin A.$$

$$\therefore\ R = \frac{a}{2\sin A} = \frac{b}{2\sin B} = \frac{c}{2\sin C} \quad . \quad . \quad . \quad (1)$$

But $$\sin A = \frac{2S}{bc} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(Art. 101)}$$

$$\therefore\ R = \frac{abc}{4S} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

104. Escribed Circle. — *To find the radii of the escribed circles of a triangle.*



A circle, which touches one side of a triangle and the other two sides produced, is called an *escribed circle* of the triangle.

Let O be the centre of the escribed circle which touches the side BC and the other sides produced, at the points D, E, and F, respectively, and let the radius of this circle be r_1.

We then have from the figure

$$\triangle ABC = \triangle AOB + \triangle AOC - \triangle BOC.$$

$$\therefore\ S = \frac{cr_1}{2} + \frac{br_1}{2} - \frac{ar_1}{2}$$

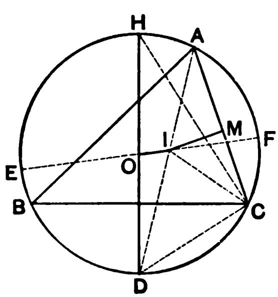
$$= \tfrac{1}{2} r_1(b + c - a) = r_1(s - a)\ . \quad \text{(Art. 99)}$$

$$\therefore\ r_1 = \frac{S}{s - a} \quad . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \quad (1)$$

Similarly it may be proved that if r_2, r_3 are the radii of the circles touching AC and AB respectively,

$$r_2 = \frac{S}{s - b};\ r_3 = \frac{S}{s - c}.$$

105. To find the Distance between the Centres of the Inscribed and Circumscribed Circles* of a Triangle.



Let I and O be the incentre and circumcentre, respectively, of the triangle ABC, IA and IC bisect the angles BAC and BCA;

* Often called the *incentre* and *circumcentre* of a triangle.

therefore the arc BD is equal to the arc DC, and DOH bisects BC at right angles.

Draw IM perpendicular to AC. Then

$$\angle \text{DIC} = \frac{\text{A} + \text{C}}{2} = \text{BCD} + \text{BCI} = \text{DCI}.$$

$$\therefore \text{DI} = \text{DC} = 2\text{R} \sin \frac{\text{A}}{2}.$$

Also, $$\text{AI} = \text{IM} \operatorname{cosec} \frac{\text{A}}{2} = r \operatorname{cosec} \frac{\text{A}}{2}.$$

$$\therefore \text{DI} \cdot \text{AI} = 2\text{R}r = \text{EI} \cdot \text{IF};$$

that is, $$(\text{R} + \text{OI})(\text{R} - \text{OI}) = 2\text{R}r.$$

$$\therefore \overline{\text{OI}}^2 = \text{R}^2 - 2\text{R}r.$$

EXAMPLES.

1. The sides of a triangle are 18, 24, 30; find the radii of its inscribed, escribed, and circumscribed circles.

Ans. 6, 12, 18, 36, 15.

2. Prove that the area of the triangle ABC is

$$\frac{1}{2} \frac{c^2}{\cot \text{A} + \cot \text{B}}.$$

3. Find the area of the triangle ABC when

(1) $a = 4$, $b = 10$ ft., $\text{C} = 30°$. *Ans.* 10 sq. ft.

(2) $b = 5$, $c = 20$ inches, $\text{A} = 60°$. 43.3 sq. in.

(3) $a = 13$, $b = 14$, $c = 15$ chains. 84 sq. chains.

4. Prove $\frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3}$.

5. Prove $r = \frac{a \sin \frac{1}{2} \text{B} \sin \frac{1}{2} \text{C}}{\cos \frac{1}{2} \text{A}}$.

6. Prove that the area of the triangle ABC is represented by each of the three expressions:

$$2\,R^2 \sin A \sin B \sin C,$$
$$rs, \text{ and}$$
$$Rr(\sin A + \sin B + \sin C).$$

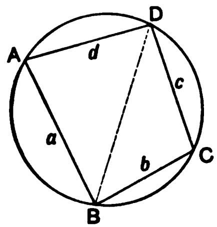
7. If $A = 60°$, $a = \sqrt{3}$, $b = \sqrt{2}$, prove that the area $= \frac{1}{4}(3 + \sqrt{3})$.

8. Prove $R(\sin A + \sin B + \sin C) = s$.

9. Prove that the bisectors of the angles A, B, C, of a triangle are, respectively, equal to

$$\frac{2\,bc\cos\frac{A}{2}}{b+c},\quad \frac{2\,ca\cos\frac{B}{2}}{c+a},\quad \frac{2\,ab\cos\frac{C}{2}}{a+b}.$$

106. To find the Area of a Cyclic* Quadrilateral.



Let ABCD be the quadrilateral, and a, b, c, and d its sides. Join BD.

Then, area of figure = S

$$= \tfrac{1}{2}\,ad \sin A + \tfrac{1}{2}\,bc \sin C$$
$$= \tfrac{1}{2}(ad + bc)\sin A \quad . \quad . \quad . \quad (1)$$

Now in $\triangle$ ABD, $\overline{BD}^2 = a^2 + d^2 - 2\,ad\cos A$,

and in $\triangle$ CBD, $\overline{BD}^2 = b^2 + c^2 - 2\,bc\cos C$
$$= b^2 + c^2 - 2\,bc\cos A.$$

$$\therefore \cos A = \frac{a^2 - b^2 - c^2 + d^2}{2(ad + bc)}.$$

$$\therefore \sin A = \sqrt{1 - \left[\frac{a^2 - b^2 - c^2 + d^2}{2(ad + bc)}\right]^2}$$

$$= \frac{\sqrt{(2\,ad + 2\,bc)^2 - (a^2 - b^2 - c^2 + d^2)^2}}{2(ad + bc)}$$

* See Geometry, Art. 251.

$$=\frac{\sqrt{[(a+d)^2-(b-c)^2][(b+c)^2-(a-d)^2]}}{2(ad+bc)}$$

$$=\frac{\sqrt{(a+d+b-c)(a+d-b+c)(b+c+a-d)(b+c-a+d)}}{2(ad+bc)}$$

$$=\frac{2\sqrt{(s-a)(s-b)(s-c)(s-d)}}{ad+bc}$$

(where $2s=a+b+c+d$).

Substituting in (1), we have

$$S=\sqrt{(s-a)(s-b)(s-c)(s-d)}.$$

The more important formulæ proved in this chapter are summed up as follows:

1. $\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}$ . . . . (Art. 95)

2. $a^2=b^2+c^2-2bc\cos A$ . . . . (Art. 96)

3. $\frac{a+b}{a-b}=\frac{\tan\frac{1}{2}(A+B)}{\tan\frac{1}{2}(A-B)}$ . . . . (Art. 97)

4. $\sin\frac{1}{2}A=\sqrt{\frac{(s-b)(s-c)}{bc}}$ . . . . (Art. 99)

5. $\cos\frac{1}{2}A=\sqrt{\frac{s(s-a)}{bc}}$.

6. $\tan\frac{1}{2}A=\sqrt{\frac{(s-b)(s-c)}{s(s-a)}}$.

7. $\sin A=\frac{2}{bc}\sqrt{s(s-a)(s-b)(s-c)}$ . . (Art. 100)

$$=\frac{1}{2bc}\sqrt{2b^2c^2+2c^2a^2+2a^2b^2-a^4-b^4-c^4}.$$

8. Area of $\triangle=\sqrt{s(s-a)(s-b)(s-c)}$. . (Art. 101)

9. Area of $\Delta = \frac{r}{2}(a+b+c) = rs$ . . . . . (Art. 102)

10. $$r = \sqrt{\frac{(s-a)(s-b)(s-c)}{s}}.$$

11. $$R = \frac{abc}{4S}$$ . . . . (Art. 103)

EXAMPLES.

In a right triangle ABC, in which C is the right angle, prove the following:

1. $\cos 2B = \frac{\sin^2 A - \sin^2 B}{\sin^2 A + \sin^2 B}.$

2. $\sin^2 \frac{B}{2} = \frac{c-a}{2c}.$

3. $\left(\cos \frac{A}{2} + \sin \frac{A}{2}\right)^2 = \frac{a+c}{c}.$

4. $\cos^2 \frac{A}{2} = \frac{b+c}{2c}.$

5. $\sin (A - B) + \cos 2A = 0.$

6. $\frac{a-b}{a+b} = \tan \frac{A-B}{2}.$

7. $\sin (A - B) + \sin (2A + C) = 0.$

8. $\tan \frac{1}{2} A = \frac{a}{b+c}.$

9. $(\sin A - \sin B)^2 + (\cos A + \cos B)^2 = 2.$

10. $\sqrt{\frac{a+b}{a-b}} + \sqrt{\frac{a-b}{a+b}} = \frac{2 \sin A}{\sqrt{\cos 2B}}.$

In any triangle ABC, prove the following statements:

11. $(a+b) \sin \frac{C}{2} = c \cos \frac{A-B}{2}.$

12. $(b-c)\cos\frac{A}{2} = a\sin\frac{B-C}{2}.$

13. $a(b^2+c^2)\cos A + b(c^2+a^2)\cos B + c(a^2+b^2)\cos C = 3abc.$

14. $\frac{a-b}{c} = \frac{\cos B - \cos A}{1+\cos C}.$

15. $\frac{b+c}{a} = \frac{\cos B + \cos C}{1-\cos A}.$

16. $\sqrt{bc\sin B\sin C} = \frac{b^2\sin C + c^2\sin B}{b+c}.$

17. $a+b+c = (b+c)\cos A + (c+a)\cos B + (a+b)\cos C.$

18. $b+c-a = (b+c)\cos A - (c-a)\cos B + (a-b)\cos C.$

19. $a\cos(A+B+C) - b\cos(B+A) - c\cos(A+C) = 0.$

20. $\frac{\cos A}{a} + \frac{\cos B}{b} + \frac{\cos C}{c} = \frac{a^2+b^2+c^2}{2abc}.$

21. $\tan A = \frac{a\sin C}{b - a\cos C}.$

22. $b\cos^2\frac{C}{2} + c\cos^2\frac{B}{2} = s.$

23. $\tan\frac{B}{2}\tan\frac{C}{2} = \frac{b+c-a}{b+c+a}.$

24. $\tan\frac{A}{2}(b+c-a) = \tan\frac{B}{2}(c+a-b).$

25. $c^2 = (a+b)^2\sin^2\frac{C}{2} + (a-b)^2\cos^2\frac{C}{2}.$

26. $c(\cos A + \cos B) = 2(a+b)\sin^2\frac{C}{2}.$

27. $c(\cos A - \cos B) = 2(b-a)\cos^2\frac{C}{2}.$

28. $\tan B \div \tan C = (a^2+b^2-c^2) \div (a^2-b^2+c^2).$

29. $a^2 + b^2 + c^2 = 2(ab\cos C + bc\cos A + ca\cos B)$.

30. $\cos^2\frac{A}{2} \div \cos^2\frac{B}{2} = (s-a) \div b(s-b)$.

31. $b\sin^2\frac{C}{2} + c\sin^2\frac{B}{2} = s - a$.

32. If p is the length of the perpendicular from A on BC,

$$\sin A = \frac{ap}{bc}.$$

33. If $A = 3B$, then $\sin B = \frac{1}{2}\sqrt{\frac{3b-a}{b}}$.

34. If $\sqrt{bc\sin B\sin C} = \frac{b^2\sin B + c^2\sin C}{b+c}$, then $B = C$.

35. $a\cos\frac{B}{2}\cos\frac{C}{2}\operatorname{cosec}\frac{A}{2} = s$.

36. If $\cos A = \frac{3}{5}$, and $\cos B = \frac{12}{13}$, then $\cos C = -\frac{16}{65}$.

37. If $\sin^2 B + \sin^2 C = \sin^2 A$, then $A = 90°$.

38. If D is the middle point of BC, prove that

$$4\,\overline{AD}^2 = 2b^2 + 2c^2 - a^2.$$

39. If $a = 2b$, and $A = 3B$, prove that $C = 60°$.

40. If D, E, F, are the middle points of the sides, BC, CA, AB, prove

$$4(\overline{AD}^2 + \overline{BE}^2 + \overline{CF}^2) = 3(a^2 + b^2 + c^2).$$

41. If a, b, c, the sides of a triangle, are in arithmetic progression, prove

$$\tan\frac{A}{2}\tan\frac{C}{2} = \frac{1}{3}.$$

42. If $\frac{\tan A - \tan B}{\tan A + \tan B} = \frac{c-b}{c}$, prove that $A = 60°$.

43. If $\cos B = \frac{\sin A}{2\sin C}$, prove that $B = C$.

44. If $a^2 = b^2 - bc + c^2$, prove that $A = 60°$.

45. If the sides of a triangle are a, b, and $\sqrt{a^2 + ab + b^2}$, prove that its greatest angle is 120°.

46. Prove that the vertical angle of any triangle is divided by the median which bisects the base, into segments whose sines are inversely proportional to the adjacent sides.

47. If AD be the median that bisects BC, prove (1)

$$(b^2 - c^2) \tan \text{ADB} = 2\,bc \sin \text{A},$$

and (2) $\cot \text{BAD} + \cot \text{DAC} = 4 \cot \text{A} + \cot \text{B} + \cot \text{C}.$

48. Find the area of the triangle ABC when $a = 625$, $b = 505$, $c = 904$ yards. *Ans.* 151872 sq. yards.

49. Find the radii of the inscribed and each of the escribed circles of the triangle ABC when $a = 13$, $b = 14$, $c = 15$. *Ans.* 4; 10.5; 12; 14.

50. Prove the area $S = \frac{1}{2} a^2 \sin B \sin C \operatorname{cosec} A$.

51. " " " " $= \sqrt{rr_1r_2r_3}$.

52. " " " " $= \dfrac{2\,abc}{a+b+c}\left(\cos\dfrac{A}{2}\cos\dfrac{B}{2}\cos\dfrac{C}{2}\right)$.

53. Prove that the lengths of the sides of the pedal triangle, that is, the triangle formed by joining the feet of the perpendiculars, are $a \cos A$, $b \cos B$, $c \cos C$, respectively.

54. Prove that the angles of the pedal triangle are, respectively, $\pi - 2A$, $\pi - 2B$, $\pi - 2C$.

55. Prove $r_1r_2r_3 = r^3 \cot^2\dfrac{A}{2} \cot^2\dfrac{B}{2} \cot^2\dfrac{C}{2}$.

56. Prove $r_1 \cos\dfrac{A}{2} = a \cos\dfrac{B}{2} \cos\dfrac{C}{2}$.

57. Prove that the area of the incircle : area of the triangle : : $\pi : \cot\dfrac{A}{2} \cot\dfrac{B}{2} \cot\dfrac{C}{2}$.

Prove the following statements:

58. If a, b, c, are in A. P., then $ac = 6rR$.

59. If the altitude of an isosceles triangle is equal to the base, R is five-eighths of the base.

60. $bc = 4R^2(\cos A + \cos B \cos C)$.

61. If C is a right angle, $2r + 2R = a + b$.

62. $r_2r_3 + r_3r_1 + r_1r_2 = s^2$.

63. $\frac{1}{bc} + \frac{1}{ca} + \frac{1}{ab} = \frac{1}{2rR}$.

64. $r_1 + r_2 = c \cot \frac{C}{2}$.

65. $r \cos \frac{A}{2} = a \sin \frac{B}{2} \sin \frac{C}{2}$.

66. If p_1, p_2, p_3 be the distances to the sides from the circumcentre, then

$$\frac{a}{p_1} + \frac{b}{p_2} + \frac{c}{p_3} = \frac{abc}{4p_1p_2p_3}.$$

67. The radius R of the circumcircle

$$= \frac{1}{2}\sqrt[3]{\frac{abc}{\sin A \sin B \sin C}}.$$

68. $S = \frac{a^2}{4} \sin 2B + \frac{b^2}{4} \sin 2A$.

69. $\frac{1}{s-a} + \frac{1}{s-b} + \frac{1}{s-c} - \frac{1}{s} = \frac{4R}{S}$.

70. $abcr = 4R(s-a)(s-b)(s-c)$.

71. The distances between the centres of the inscribed and escribed circles of the triangle ABC are $4R \sin \frac{A}{2}$, $4R \sin \frac{B}{2}$, $4R \sin \frac{C}{2}$.

72. If A is a right angle, $r_2 + r_3 = a$.

73. In an equilateral triangle $3R = 6r = 2r_1$.

74. If r, r_1, r_2, r_3 denote the radii of the inscribed and escribed circles of a triangle,

$$\tan^2\frac{A}{2} = \frac{rr_1}{r_2r_3}.$$

75. The sides of a triangle are in arithmetic progression, and its area is to that of an equilateral triangle of the same perimeter as 3 is to 5. Find the ratio of the sides and the value of the largest angle. *Ans.* As 7, 5, 3; 120°.

76. If an equilateral triangle be described with its angular points on the sides of a given right isosceles triangle, and one side parallel to the hypotenuse, its area will be $2a^2\sin^2 15°\sin 60°$, where a is a side of the given triangle.

77. If h be the difference between the sides containing the right angle of a right triangle, and S its area, the diameter of the circumscribing circle $= \sqrt{h^2 + 4S}$.

78. Three circles touch one another externally: prove that the square of the area of the triangle formed by joining their centres is equal to the product of the sum and product of their radii.

79. On the sides of any triangle equilateral triangles are described externally, and their centres are joined: prove that the triangle thus formed is equilateral.

80. If O_1, O_2, O_3 are the centres of the escribed circles of a triangle, then the area of the triangle $O_1O_2O_3 =$ area of triangle ABC$\left[1 + \frac{a}{b+c-a} + \frac{b}{a+c-b} + \frac{c}{a+b-c}\right]$.

81. If the centres of the three escribed circles of a triangle are joined, then the area of the triangle thus formed is $\frac{abc}{2r}$, where r is the radius of the inscribed circle of the original triangle.

CHAPTER VII.

SOLUTION OF TRIANGLES.

107. Triangles. — In every triangle there are *six elements*, the three sides and the three angles. When any three elements are given, one at least of the three being a side, the other three can be calculated. The process of determining the unknown elements from the known is called the *solution of triangles.*

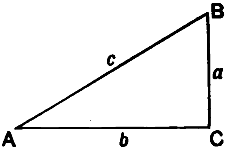
NOTE. — If the three *angles* only of a triangle are given, it is impossible to determine the sides, for there is an infinite number of triangles that are equiangular to one another.

Triangles are divided in Trigonometry into *right* and *oblique.* We shall commence with right triangles, and shall suppose C the right angle.

RIGHT TRIANGLES.

108. There are Four Cases of Right Triangles.

I. *Given one side and the hypotenuse.*
II. *Given an acute angle and the hypotenuse.*
III. *Given one side and an acute angle.*
IV. *Given the two sides.*



Let ABC be a triangle, right-angled at C, and let a, b, and c, as before, be the sides opposite the angles A, B, and C, respectively.

The formulæ for the solution of right triangles are (1), (2), (3) of Art. 94.

109. Case I. — *Given a side and the hypotenuse, as a and c; to find A, B, b.*

We have $$\sin A = \frac{a}{c}.$$

$$\therefore \log \sin A = \log a - \log c,$$

from which A is determined; then $B = 90° - A$.

Lastly, $$b = c \cos A.$$

$$\therefore \log b = \log c + \log \cos A.$$

Thus A, B, and b are determined.

Ex. 1. Given $a = 536$, $c = 941$; find A, B, b.

Solution by Natural Functions.

We have $$\sin A = \frac{a}{c} = \frac{536}{941} = .569607.$$

From a table of natural sines we find that

$$A = 34° 43' 22''. \quad \therefore B = 55° 16' 38''.$$

Lastly, $$b = c \cos A = 941 \times .821918$$
$$= 773.425.$$

Logarithmic Solution.

$$\log \sin A = \log a - \log c.$$

$$\log a = 2.7291648$$
$$\log c = 2.9735896$$
$$\log \sin A^* = 9.7555752$$
$$\therefore A = 34° 43' 22''.$$
$$\therefore B = 55° 16' 38''.$$

$$\log b = \log c + \log \cos A.$$

$$\log c = 2.9735896$$
$$\log \cos A = 9.9148283$$
$$\log b = 2.8884179$$
$$\therefore b = 773.424.$$

Our two methods of calculation give results which do not quite agree. The discrepancies arise from the defects of the tables.

* *Ten* is *added* so as to get the tabular logarithms (Art. 76).

The process of solution by natural sines, cosines, etc., can be used to advantage only in cases in which the measures of the sides are small numbers.

We might have determined b thus:

$$b = \sqrt{(c-a)(c+a)};$$

or thus: $$b = a \tan B.$$

NOTE. — It is generally better to compute all the required parts from the given ones, so that if an error is made in determining one part, that error will not affect the computation of the other parts.

To test the accuracy of the work, compute the same parts by different formulæ.

Ex. 2. Given $a = 21$, $c = 29$; find A, B, b.

Ans. $A = 46° 23' 50''$, $B = 43° 36' 10''$, $b = 20$.

NOTE. — In these examples the student must find the necessary logarithms from the tables.

110. Case II. — *Given an acute angle and the hypotenuse, as A and c; to find B, a, b.*

We have $$B = 90° - A.$$

Also $$a = c \sin A, \text{ and } b = c \cos A.$$

$$\therefore \log a = \log c + \log \sin A;$$

and $$\log b = \log c + \log \cos A.$$

Thus B, a, and b are determined.

Ex. 1. Given $A = 54° 28'$, $c = 125$; find B, a, b.

$$B = 90° - A = 35° 32'.$$

Solution by Natural Functions.

We have $$a = c \sin A, \text{ and } b = c \cos A.$$

Using a table of natural sines, we have

$$a = 125 \times .813778 = 101.722,$$

and $$b = 125 \times .581177 = 72.647.$$

Logarithmic Solution.

$\log a = \log c + \log \sin A.$	$\log b = \log c + \log \cos A.$
$\log c = 2.0969100$	$\log c = 2.0969100$
$\log \sin A = 9.9105057$	$\log \cos A = 9.7643080$
$\log a\,^* = 2.0074157$	$\log b\,^* = 1.8612180$
$\therefore\ a = 101.722.$	$\therefore\ b = 72.647.$

Ex. 2. Given $A = 37° 10'$, $c = 8762$; find a and b.
Ans. 5293.4; 6982.3.

111. Case III. — *Given a side and an acute angle, as A and a; to find B, b, c.*

We have $B = 90° - A.$

Also $b = \dfrac{a}{\tan A}$, and $c = \dfrac{a}{\sin A}.$

$$\therefore \log b = \log a - \log \tan A,$$

and $$\log c = \log a - \log \sin A.$$

Ex. 1. Given $A = 32° 15' 24''$, $a = 5472.5$; find B, b, c.

Solution.

$$B = 90° - A = 57° 44' 36''.$$

$\log b = \log a - \log \tan A.$	$\log c = \log a - \log \sin A.$
$\log a = 3.7381858$	$\log a = 3.7381858$
$\log \tan A = 9.8001090$	$\log \sin A = 9.7273076$
$\log b = 3.9380768$	$\log c = 4.0108782$
$\therefore\ b = 8671.152.$	$\therefore\ c = 10253.64.$

Ex. 2. Given $A = 34° 18'$, $a = 237.6$; find B, b, c.
Ans. $B = 55° 42'$; $b = 348.31$; $c = 421.63$.

* *Ten* is *rejected* because the tabular logarithmic functions are too large by ten (Art. 76).

112. Case IV. — *Given the two sides, as a and b; to find A, B, c.*

We have $\tan A = \frac{a}{b}$; then $B = 90° - A.$

Also $c = a \operatorname{cosec} A = \frac{a}{\sin A}.$

$$\therefore \log \tan A = \log a - \log b,$$

and $\log c = \log a - \log \sin A.$

Ex. Given $a = 2266.35$, $b = 5439.24$; find A, B, c.

Solution.

$\log \tan A = \log a - \log b.$	$\log c = \log a - \log \sin A.$
$\log a = 3.3553270$	$\log a = 3.3553270$
$\log b = 3.7355382$	$\log \sin A = 9.5850266$
$\log \tan A = 9.6197888$	$\log c = 3.7703004$
$\therefore A = 22° 37' 12''.$	$\therefore c = 5892.51.$
$\therefore B = 67° 22' 48''.$	

NOTE. — In this example we might have found c by means of the formula $c = \sqrt{a^2 + b^2}$; but we would have had to go through the process of *squaring* the values of a and b. If these values are simple numbers, it is often easier to find c in this way; but this value of c is not adapted to logarithms. A formula which consists entirely of *factors* is always preferred to one which consists of *terms*, when any of those terms contain any power of the quantities involved.

113. When a Side and the Hypotenuse are nearly Equal. — When a side and the hypotenuse are given, as a and c in Case I., and are nearly equal in value, the angle A is very near 90°, and cannot be determined with much accuracy from the tables, because the sines of angles near 90° differ very little from one another (Art. 81). It is therefore desirable, in this case, to find B first, by either of the following formulæ:

$$\sin \frac{B}{2} = \sqrt{\frac{1 - \cos B}{2}} \quad . \; . \; . \; . \; . \; . \quad \text{(Art. 50)}$$

$$= \sqrt{\frac{c - a}{2c}} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (1)$$

$$\tan\frac{B}{2}=\sqrt{\frac{1-\cos B}{1+\cos B}} \quad . \quad . \quad . \quad . \quad . \quad \text{(Art. 50)}$$

$$=\sqrt{\frac{c-a}{c+a}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Then $$b=c\cos A \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

or $$=\sqrt{(c+a)(c-a)} \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Ex. 1. Given $a=4602.21059$, $c=4602.836$; find B.

$c-a=0.62541$, $\qquad \log(c-a)=\bar{1}.7961648$

$2c=9205.672$, $\qquad \log 2c=3.9640555$

$$2)\overline{\bar{5}.8321093}$$

$$\log\sin\frac{B}{2}=7.9160547$$

$B=56'\,40''.36.$ $\qquad \therefore\ \frac{B}{2}=28'\,20''.18.$

NOTE.—The characteristic $\bar{5}$ is increased numerically to $\bar{6}$ to make it divisible by 2 (see Note 4 of Art. 66). Ten is then added to the characteristic $\bar{3}$, making it 7, so as to agree with the Tables (Art. 76).

There is a slight error in the above value of B on account of the irregular differences of the log sines for angles near 0° (Art. 81). A more accurate value may be found by the principle that the sines of small angles are approximately proportional to the angles (Art. 130).

EXAMPLES.

The following right triangles must be solved by logarithms.

1. Given $a=60$, $c=100$; find A, B, b.

Ans. $A=36°\,52'$; $B=53°\,8'$; $b=80$.

2. Given $a=137.66$, $c=240$; find A, B, b.

Ans. $A=35°$; $B=55°$; $b=196.59$.

3. Given $a = 147$, $c = 184$; find A, B, b.
Ans. $A = 53^\circ\, 1'\, 35''$; $B = 36^\circ\, 58'\, 25''$; $b = 110.67$.

4. Given $a = 100$, $c = 200$; find A, B, b.
Ans. $A = 30^\circ$; $B = 60^\circ$; $b = 100\sqrt{3}$.

5. Given $A = 40^\circ$, $c = 100$; find B, a, b.
Ans. $B = 50^\circ$; $a = 64.279$; $b = 76.604$.

6. Given $A = 30^\circ$, $c = 150$; find B, a, b.
Ans. $B = 60^\circ$; $a = 75$; $b = 75\sqrt{3}$.

7. Given $A = 32^\circ$, $c = 1760$; find B, a, b.
Ans. $B = 58^\circ$; $a = 932.66$; $b = 1492.57$.

8. Given $A = 35^\circ\, 16'\, 25''$, $c = 672.3412$; find B, a, b.
Ans. $B = 54^\circ\, 43'\, 35''$; $a = 388.26$; $b = 548.9$.

9. Given $A = 75^\circ$, $a = 80$; find B, b, c.
Ans. $B = 15^\circ$; $b = 80(2 - \sqrt{3})$; $c = 80(\sqrt{6} - \sqrt{2})$.

10. Given $A = 36^\circ$, $a = 520$; find B, b, c.
Ans. $B = 54^\circ$; $b = 715.72$; $c = 884.68$.

11. Given $A = 34^\circ\, 15'$, $a = 843.2$; find B, b, c.
Ans. $B = 55^\circ\, 45'$; $c = 1498.2$.

12. Given $A = 67^\circ\, 37'\, 15''$, $b = 254.73$; find B, a, c.
Ans. $B = 22^\circ\, 22'\, 45''$; $a = 618.66$; $c = 669.05$.

13. Given $a = 75$, $b = 75$; find A, B, c.
Ans. $A = 45^\circ = B$; $c = 75\sqrt{2}$.

14. Given $a = 21$, $b = 20$; find A, B, c.
Ans. $A = 46^\circ\, 23'\, 50''$; $c = 29$.

15. Given $a = 300.43$, $b = 500$; find A, B, c.
Ans. $A = 31^\circ$; $B = 59^\circ$; $c = 583.31$.

16. Given $a = 4845$, $b = 4742$; find A, B, c.
Ans. $A = 45^\circ\, 36'\, 56''$.

OBLIQUE TRIANGLES.

114. There are Four Cases of Oblique Triangles.

I. *Given a side and two angles.*
II. *Given two sides and the angle opposite one of them.*
III. *Given two sides and the included angle.*
IV. *Given the three sides.*

The formulæ for the solution of oblique triangles will be taken from Chap. VI. Special attention must be given to the following three, proved in Arts. 95, 96, 97.

(1) The Sine-rule, $$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}.$$

(2) The Cosine-rule, $$\cos A = \frac{b^2 + c^2 - a^2}{2bc}.$$

(3) The Tangent-rule, $$\tan\frac{A - B}{2} = \frac{a - b}{a + b}\cot\frac{C}{2}.$$

115. Case I. *Given a side and two angles, as a, B, C; find A, b, c.*

(1) $A = 180° - (B + C)$. $\therefore$ A is found.

(2) $\frac{b}{\sin B} = \frac{a}{\sin A}$. $\therefore b = \frac{a \sin B}{\sin A}$.

(3) $\frac{c}{\sin C} = \frac{a}{\sin A}$. $\therefore c = \frac{a \sin C}{\sin A}$.

These determine b and c.

Ex. 1. Given $a = 7012.6$, $B = 38° 12' 48''$, $C = 60°$; find A, b, c.

Solution.

$$A = 180° - (B + C) = 81° 47' 12''.$$

$\log a = 3.8458729$	$\log a = 3.8458729$
$\log \sin B = 9.9714038$	$\log \sin C = 9.9375306$
colog* $\sin A = 0.0044775$	colog $\sin A = 0.0044775$
$\log b = 3.8217542$	$\log c = 3.7878810$
$\therefore b = 6633.67.$	$\therefore c = 6135.94.$

* See Art. 69.

Ex. 2. Given $a=1000$, $B=45°$, $C=127° 19'$; find A, b, c.

Ans. $A=7° 41'$; $b=5288.8$; $c=5948.5$.

116. Case II. *Given two sides and the angle opposite one of them, as a, b, A; find B, C, c.*

(1) $\sin B = \frac{b \sin A}{a}$; thus B is found.

(2) $C = 180° - (A + B)$; thus C is found.

(3) $c = \frac{a \sin C}{\sin A}$; thus c is found.

This is usually known as the *ambiguous* case, as shown in geometry (B. II., Prop. 31). The ambiguity is found in the equation

$$\sin B = \frac{b \sin A}{a}.$$

Since the angle is determined by its sine, it admits of two values, which are supplements of each other (Art. 29). Therefore, either value of B may be taken, unless excluded by the conditions of the problem.

I. If $a < b \sin A$, $\sin B > 1$, which is impossible; and therefore there is no triangle with the given parts.

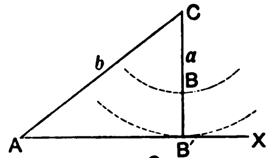
II. If $a = b \sin A$, $\sin B = 1$, and $B = 90°$; therefore there is one triangle — a right triangle — with the given parts.

III. If $a > b \sin A$, and $< b$, $\sin B < 1$; hence there are two values of B, one being the supplement of the other, *i.e.*, one acute, the other obtuse, and both are admissible; therefore there are two triangles with the given parts.

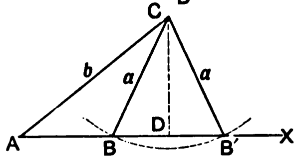
IV. If $a > b$, then $A > B$, and since A is given, B must be acute; thus there is only one triangle with the given parts.

These four cases may be illustrated geometrically.

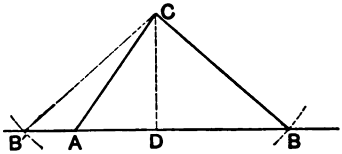
Draw A, the given angle. Make AC $= b$; draw the perpendicular CD, which $= b \sin$ A. With centre C and radius a, describe a circle.



I. If $a < b \sin A$, the circle will not meet AX, and therefore no triangle can be formed with the given parts.



II. If $a = b \sin A$, the circle touches AX in B′; therefore there is one triangle, right-angled at B.



III. If $a > b \sin A$, and $< b$, the circle cuts AX in two points B and B′, on the same side of A; thus there are two triangles ABC and AB′C, each having the given parts, the angles ABC, AB′C being supplementary.

IV. If $a > b$, the circle cuts AX on *opposite* sides of A, and only the triangle ABC has the given parts, because the angle B′AC of the triangle AB′C is not the given angle A, but its supplement.

These results may be stated as follows:

$a < b \sin A$,	no solution.
$a = b \sin A$,	one solution (right triangle).
$a > b \sin A$ and $< b$,	two solutions.
$a > b$,	one solution.

These results may be obtained algebraically thus:

We have $$a^2 = b^2 + c^2 - 2bc \cos A \quad . \quad . \quad . \quad \text{(Art. 96)}$$

$$\therefore \; c = b \cos A \pm \sqrt{a^2 - b^2 \sin^2 A},$$

giving two roots, *real* and *unequal*, *equal* or *imaginary*, according as $a >, =,$ or $< b \sin A$.

A discussion of these two values of c gives the same results as are found in the above four cases. We leave the discussion as an exercise for the student.

Note. — When two sides and the angle opposite the greater are given, there can be no ambiguity, for the angle opposite the less must be acute.

When the given angle is a right angle or obtuse, the other two angles are both acute, and there can be no ambiguity.

In the solution of triangles there can be no ambiguity, except when an angle is determined by the sine or cosecant, and in no case whatever when the triangle has a right angle.

Ex. 1. Given $a = 7$, $b = 8$, $A = 27° 47' 45''$; find B, C, c.

Solution.

$$\begin{array}{r|r}
\log b = 0.9030900 & \log a = 0.8450980 \\
\log \sin A = 9.6686860 & \log \sin C = 9.9375306 \\
\text{colog } a = 9.1549020 & \text{colog} \sin A = 0.3313140 \\
\hline
\log \sin B = 9.7266780 & \log c = 1.1139426
\end{array}$$

$$\therefore\ B = 32° 12' 15'', \text{ or } 147° 47' 45''. \qquad \therefore\ c = 13.$$

$$\therefore\ C = 120°, \text{ or } 4° 24' 30''.$$

Taking the second value of C as follows:

$$\begin{aligned}
\log a &= 0.8450980 \\
\log \sin C &= 8.8857232 \\
\text{colog} \sin A &= 0.3313140 \\
\hline
\log c &= 0.0621352 \\
\therefore\ c &= 1.1538.
\end{aligned}$$

Thus, there are two solutions. See Case III.

Ex. 2. Given $a = 31.239$, $b = 49.5053$, $A = 32° 18'$; find B, C, c.

Ans. $B = 56° 56' 56''.3$, or $123° 3' 3''.7$;
$C = 90° 45' 3''.7$, or $24° 38' 56''.3$;
$c = 58.456$, or 24.382.

117. Case III. — *Given two sides and the included angle, as a, b, C; find A, B, c.*

(1) $$\tan\frac{A-B}{2}=\frac{a-b}{a+b}\cot\frac{C}{2} \quad . \quad . \quad \text{(Art. 114)}$$

Hence $\frac{A-B}{2}$ is known, and $\frac{A+B}{2}=90°-\frac{C}{2}$.

∴ A and B are found.

(2) $$c=\frac{a\sin C}{\sin A}, \text{ or } \frac{b\sin C}{\sin B},$$

and thus c is found and the triangle solved.

In simple cases the third side c may be found directly by the formula

$$c=\sqrt{a^2+b^2-2ab\cos C} \quad . \quad . \quad . \quad \text{(Art. 96)}$$

or the formula may be adapted to logarithmic calculation by the use of a subsidiary angle (Art. 90).

Ex. 1. Given $a=234.7$, $b=185.4$, $C=84°\,36'$; find A, B, c.

Solution.

$$\begin{aligned} a &= 234.7 \\ b &= 185.4 \\ \hline a-b &= 49.3 \\ a+b &= 420.1 \\ \therefore \frac{C}{2} &= 42°\,18'. \end{aligned}$$

$$\begin{aligned} \log(a-b) &= 1.6928469 \\ \text{colog}(a+b) &= 7.3766473 \\ \log\cot\frac{C}{2} &= 10.0409920 \\ \hline \log\tan\frac{A-B}{2} &= 9.1104862 \\ \therefore \frac{A-B}{2} &= 7°\,20'\,56''. \end{aligned}$$

$$\begin{aligned} \frac{A+B}{2} &= 47°\,42'. \\ \therefore A &= 55°\ \ 2'\,56'', \\ B &= 40°\,21'\ \ 4'', \\ C &= 84°\,36', \\ c &= 285.0745. \end{aligned}$$

$$\begin{aligned} \log b &= 2.2681097 \\ \log\sin C &= 9.9980683 \\ \text{colog}\sin B &= 0.1887804 \\ \hline \log c &= 2.4549584 \end{aligned}$$

Ex. 2. Given $a = .062387$, $b = .023475$, $C = 110° 32'$; find A, B, c.

Ans. $A = 52° 10' 33''$; $B = 17° 17' 27''$; $c = .0739635$.

118. Case IV. *Given the three sides, as a, b, c; find A, B, C.*

The solution in this case may be performed by the formulæ of Art. 99. By means of these formulæ we may compute two of the angles, and find the third by subtracting their sum from 180°. But in practice it is better to compute the three angles independently, and check the accuracy of the work by taking their sum.

If only *one* angle is to be found, the formulæ for the *sines* or *cosines* may be used. If *all* the angles are to be found, the *tangent* formulæ are the most convenient, because then we require only the logarithms of the same four quantities, s, $s - a$, $s - b$, $s - c$, to find all the angles; whereas the sine and cosine formulæ require in addition the logs of a, b, c.

The tangent formulæ (Art. 99) may be reduced as follows:

$$\tan\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}$$

$$= \frac{1}{s-a}\sqrt{\frac{(s-a)(s-b)(s-c)}{s}}.$$

$$\therefore \tan\frac{A}{2} = \frac{r}{s-a} \quad . \quad . \quad . \quad . \quad . \quad . \quad (\text{Art. } 102)$$

Similarly,
$$\tan\frac{B}{2} = \frac{r}{s-b},$$

$$\tan\frac{C}{2} = \frac{r}{s-c}.$$

NOTE. — The quantity r is the radius of the inscribed circle (Art. 102).

Ex. 1. Given $a = 13$, $b = 14$, $c = 15$; find A, B, C.

Solution.

$$\begin{array}{ll}
a = 13 & \log(s-a) = .9030900 \\
b = 14 & \log(s-b) = .8450980 \\
c = 15 & \log(s-c) = .7781513 \\
\hline
2s = 42 & \text{colog } s = 8.6777807 \\
s = 21. & \log r^2 = 1.2041200 \\
s - a = 8, & \log r = .6020600. \\
s - b = 7, & \therefore \log\tan\frac{A}{2} = 9.6989700. \\
s - c = 6. & \therefore \log\tan\frac{B}{2} = 9.7569620. \\
 & \therefore \log\tan\frac{C}{2} = 9.8239087.
\end{array}$$

$$\therefore A = 53^\circ\, 7'\, 48''.38;$$

$$B = 59^\circ\, 29'\, 23''.18;$$

$$C = 67^\circ\, 22'\, 48''.44.$$

Without the use of logarithms, the angles may be found by the cosine formulæ (Art. 96). These may sometimes be used with advantage, when the given lengths of a, b, c each contain less than three digits.

Ex. 2. Find the greatest angle in the triangle whose sides are 13, 14, 15.

Let $a = 15$, $b = 14$, $c = 13$. Then A is the greatest angle.

Then $$\cos A = \frac{b^2 + c^2 - a^2}{2bc} = \frac{14^2 + 13^2 - 15^2}{2 \times 14 \times 13}$$

$$= \tfrac{5}{13} = .384615 = \cos 67^\circ\, 23', \text{ nearly}$$

(by the table of natural sines).

$$\therefore \text{the greatest angle is } 67^\circ\, 23'.$$

EXAMPLES.

1. Given $a = 254$, $B = 16°$, $C = 64°$; find $b = 71.0919$.

2. Given $c = 338.65$, $A = 53° 24'$, $B = 66° 27'$; find $a = 313.46$.

3. Given $c = 38$, $A = 48°$, $B = 54°$; find $a = 28.87$, $b = 31.43$.

4. Given $a = 7012.5$, $B = 38° 12' 48''$, $C = 60°$; find b and c.
Ans. $b = 4382.82$; $c = 6135.94$.

5. Given $a = 528$, $b = 252$, $A = 124° 34'$; find B and C.
Ans. $B = 23° 8' 33''$; $C = 32° 17' 27''$.

6. Given $a = 170.6$, $b = 140.5$, $B = 40°$; find A and C.
Ans. $A = 51° 18' 21''$, or $128° 41' 39''$;
$C = 88° 41' 39''$, or $11° 18' 21''$.

7. Given $a = 97$, $b = 119$, $A = 50°$; find B and C.
Ans. $B = 70° 0' 56''$, or $109° 59' 4''$;
$C = 59° 59' 4''$, or $20° 0' 56''$.

8. Given $a = 7$, $b = 8$, $A = 27° 47' 45''$; find B, C, c.
Ans. $B = 32° 12' 15''$, or $147° 47' 45''$;
$C = 120°$, or $4° 24' 30''$;
$c = 13$, or 1.15385.

9. Given $b = 55$, $c = 45$, $A = 6°$; find B and C.
Ans. $B = 149° 20' 31''$; $C = 24° 39' 29''$.

10. Given $b = 131$, $c = 72$, $A = 40°$; find B and C.
Ans. $B = 108° 36' 30''$; $C = 31° 23' 30''$.

11. Given $a = 35$, $b = 21$, $C = 50°$; find A and B.
Ans. $A = 93° 11' 49''$; $B = 36° 48' 11''$.

12. Given $a = 601$, $b = 289$, $C = 100° 19' 6''$; find A and B.
Ans. $A = 56° 8' 42''$; $B = 23° 32' 12''$.

13. Given $a = 222$, $b = 318$, $c = 406$; find $A = 32° 57' 8''$.

14. Given $a = 275.35$, $b = 189.28$, $c = 301.47$; find A, B, C.
Ans. $A = 63° 30' 57''$; $B = 37° 58' 20''$; $C = 78° 30' 43''$.

15. Given $a = 5238$, $b = 5662$, $c = 9384$; find A and B.
Ans. $A = 29° 17' 16''$; $B = 31° 55' 31''$.

16. Given $a = 317$, $b = 533$, $c = 510$; find A, B, C.
Ans. $A = 35° 18' 0''$; $B = 76° 18' 52''$; $C = 68° 23' 8''$.

119. Area of a Triangle (Art. 101).

EXAMPLES.

Find the area:

1. Given $a = 116.082$, $b = 100$, $C = 118° 15' 41''$. *Ans.* 5112.25.
2. Given $a = 8$, $b = 5$, $C = 60°$. 17.3205.
3. Given $b = 21.5$, $c = 30.456$, $A = 41° 22'$. 216.372.
4. Given $a = 72.3$, $A = 52° 35'$, $B = 63° 17'$. 2644.94.
5. Given $b = 100$, $A = 76° 38' 13''$, $C = 40° 5'$. 3506.815.
6. Given $a = 31.325$, $B = 13° 57' 2''$, $A = 53° 11' 18''$. *Ans.* 135.3545.
7. Given $a = .582$, $b = .601$, $c = .427$. .117655.
8. Given $a = 408$, $b = 41$, $c = 401$. 8160.
9. Given $a = .9$, $b = 1.2$, $c = 1.5$. .54.
10. Given $a = 21$, $b = 20$, $c = 29$. 210.
11. Given $a = 24$, $b = 30$, $c = 18$. 216.
12. Given $a = 63.89$, $b = 138.24$, $c = 121.15$. 3869.2.

MEASUREMENT OF HEIGHTS AND DISTANCES.

120. Definitions. — One of the most important applications of Trigonometry is the determination of the heights and distances of objects which cannot be actually measured.

The actual measurement, with scientific accuracy, of a *line* of any considerable length, is a very long and difficult operation. But the accurate measurement of an *angle*, with proper instruments, can be made with comparative ease and rapidity.

By the aid of the Solution of Triangles we can determine:

(1) The distance between points which are inaccessible.

(2) The magnitude of angles which cannot be practically observed.

(3) The relative heights of distant and inaccessible points.

A *vertical line* is the line assumed by a plummet when freely suspended by a cord, and allowed to come to rest.

A *vertical plane* is any plane containing a vertical line.

A *horizontal plane* is a plane perpendicular to a vertical line.

A *vertical angle* is one lying in a vertical plane.

A *horizontal angle* is one lying in a horizontal plane.

An *angle of elevation* is a vertical angle having one side horizontal and the other ascending.

An *angle of depression* is a vertical angle having one side horizontal and the other descending.

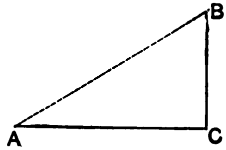
By *distance* is meant the *horizontal* distance, unless otherwise named.

By *height* is meant the *vertical* height above or below the horizontal plane of the observer.

For a description of the requisite instruments, and the method of using them, the student is referred to books on practical surveying.*

* See Johnson's Surveying, Gillespie's Surveying, Clarke's Geodesy, Gore's Geodesy, etc.

121. To find the Height of an Object standing on a Horizontal Plane, the Base of the Object being Accessible.



Let BC be a vertical object, such as a church spire or a tower.

From the base C measure a horizontal line CA.

At the point A measure the angle of elevation CAB.

We can then determine the height of the object BC; for

$$BC = AC \tan CAB.$$

EXAMPLES.

1. If $AC = 100$ feet and $CAB = 60°$, find BC.

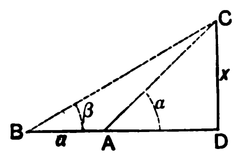
Ans. 173.2 feet.

2. If $AC = 125$ feet and $CAB = 52° 34'$, find BC.

Ans. 163.3 feet.

3. AC, the breadth of a river, is 100 feet. At the point A, on one bank, the angle of elevation of B, the top of a tree on the other bank directly opposite, is $25° 37'$; find the height of the tree. *Ans.* 47.9 feet.

122. To find the Height and Distance of an Inaccessible Object on a Horizontal Plane.



Let CD be the object, whose base D is inaccessible; and let it be required to find the height CD, and its horizontal distance from A, the nearest accessible point.

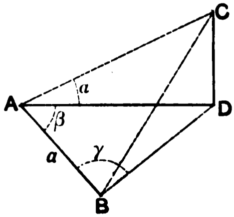
(1) At A in the horizontal line BAD observe the $\angle DAC = \alpha$; measure $AB = a$, and at B observe the $\angle DBC = \beta$.

Then $$CA = \frac{a \sin \beta}{\sin (\alpha - \beta)} \quad . \; . \; . \; . \; . \quad \text{(Art. 95)}$$

$$\therefore \ CD = CA \sin \alpha = \frac{a \sin \alpha \sin \beta}{\sin(\alpha - \beta)},$$

and $$AD = AC \cos \alpha = \frac{a \sin \beta \cos \alpha}{\sin(\alpha - \beta)}.$$

(2) *When the line BA cannot be measured directly toward the object.*



At A observe the vertical $\angle$ CAD $= \alpha$, and the horizontal $\angle$ DAB $= \beta$; measure AB $= a$, and at B observe the $\angle$ DBA $= \gamma$.

Then $$AD = \frac{a \sin \gamma}{\sin(\beta + \gamma)}.$$

$$\therefore \ CD = AD \tan \alpha$$

$$= \frac{a \sin \gamma \tan \alpha}{\sin(\beta + \gamma)}.$$

EXAMPLES.

1. A river 300 feet wide runs at the foot of a tower, which subtends an angle of 22° 30′ at the edge of the remote bank; find the height of the tower. *Ans.* 124.26 feet.

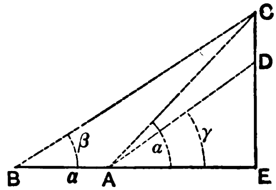
2. At 360 feet from the foot of a steeple the elevation is half what it is at 135 feet; find its height. *Ans.* 180 feet.

3. A person standing on the bank of a river observes the angle subtended by a tree on the opposite bank to be 60°, and when he retires 40 feet from the river's bank he finds the angle to be 30°; find the height of the tree and the breadth of the river. *Ans.* $20\sqrt{3}$; 20.

4. What is the height of a hill whose angle of elevation, taken at the bottom, was 46°, and 100 yards farther off, on a level with the bottom, the angle was 31°?

Ans. 143.14 yards.

123. To find the Height of an Inaccessible Object situated above a Horizontal Plane, and its Height above the Plane.



Let CD be the object, and let A and B be two points in the horizontal plane, and in the same vertical plane with CD.

At A, in the horizontal line BAE, observe the $\angle$CAE $= \alpha$, and DAE $= \gamma$; measure AB $= a$, and at B observe the $\angle$CBE $= \beta$.

Then $$\text{CE} = \frac{a \sin \alpha \sin \beta}{\sin(\alpha - \beta)} \quad \ldots\ldots \quad \text{(Art. 122)}$$

Also, $$\text{AE} = \frac{a \cos \alpha \sin \beta}{\sin(\alpha - \beta)} \quad \ldots\ldots \quad \text{(Art. 122)}$$

$$\therefore \text{DE} = \frac{a \cos \alpha \sin \beta \tan \gamma}{\sin(\alpha - \beta)}.$$

$$\therefore \text{CD} = \frac{a \sin \beta}{\sin(\alpha - \beta)} \{\sin \alpha - \cos \alpha \tan \gamma\}$$

$$= \frac{a \sin \beta \sin(\alpha - \gamma)}{\cos \gamma \sin(\alpha - \beta)}.$$

EXAMPLES.

1. A man 6 feet high stands at a distance of 4 feet 9 inches from a lamp-post, and it is observed that his shadow is 19 feet long: find the height of the lamp. *Ans.* $7\frac{1}{2}$ feet.

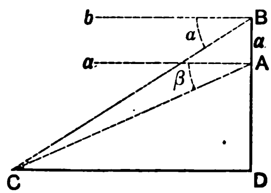
2. A flagstaff, 25 feet high, stands on the top of a cliff, and from a point on the seashore the angles of elevation of the highest and lowest points of the flagstaff are observed to be 47° 12′ and 45° 13′ respectively: find the height of the cliff. *Ans.* 348 feet.

3. A castle standing on the top of a cliff is observed from two stations at sea, which are in line with it; their

distance is a quarter of a mile: the elevation of the top of the castle, seen from the remote station, is 16° 28′; the elevations of the top and bottom, seen from the near station, are 52° 24′ and 48° 38′ respectively: (1) what is its height, and (2) what its elevation above the sea?

Ans. (1) 60.82 feet; (2) 445.23 feet.

124. To find the Distance of an Object on a Horizontal Plane, from Observations made at Two Points in the Same Vertical Line, above the Plane.



Let the points of observation A and B be in the same vertical line, and at a given distance from each other; let C be the point observed, whose horizontal distance CD and vertical distance AD are required.

Measure the angles of depression, *b*BC, *a*AC, equal to α and β respectively, and denote AB by a.

Then $$BD = CD \tan\alpha,\ AD = CD \tan\beta.$$

$$\therefore\ a = CD(\tan\alpha - \tan\beta).$$

$$\therefore\ CD = \frac{a\cos\alpha\cos\beta}{\sin(\alpha-\beta)},$$

and $$AD = \frac{a\cos\alpha\sin\beta}{\sin(\alpha-\beta)}.$$

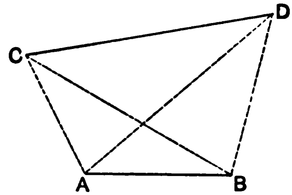
EXAMPLES.

1. From the top of a house, and from a window 30 feet below the top, the angles of depression of an object on the ground are 15° 40′ and 10°: find (1) the horizontal distance of the object, and (2) the height of the house.

Ans. (1) 288.1 feet; (2) 80.8 feet.

2. From the top and bottom of a castle, which is 68 feet high, the depressions of a ship at sea are observed to be 16° 28′ and 14°: find its distance. *Ans.* 570.2 yards.

125. To find the Distance between Two Inaccessible Objects on a Horizontal Plane.



Let C and D be the two inaccessible objects.

Measure a base line AB, from whose extremities C and D are visible. At A observe the angles CAD, DAB; and at B observe the angles CBA and CBD.

Then, in the triangle ABC, we know two angles and the side AB. ∴ AC may be found. In the triangle ABD we know two angles and the side AB. ∴ AD may be found.

Lastly, in the triangle ACD, AC and AD have been determined, and the included angle CAD has been measured; and thus CD can be found.

EXAMPLES.

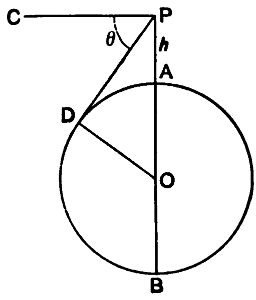
1. Let $AB = 1000$ yards, the angles BAC, $BAD = 76° 30'$ and $44° 10'$, respectively; and the angles ABD, $ABC = 81° 12'$ and $46° 5'$, respectively: find the distance between C and D. *Ans.* 669.8 yards.

2. A and B are two trees on one side of a river; at two stations P and Q on the other side observations are taken, and it is found that the angles APB, BPQ, AQP are each equal to 30°, and that the angle AQB is equal to 60°. If $PQ = a$, show that

$$AB = \frac{a}{6}\sqrt{21}.$$

126. The Dip of the Horizon. — Since the surface of the earth is spherical, it is obvious that an object on it will be visible only for a certain distance depending on its height; and, conversely, that at a certain height above the ground the visible horizon will be limited.

Let O be the centre of the earth, P a point above the surface, PD a tangent to the surface at D. Then D is a point on the terrestrial horizon; and CPD, which is the angle of depression of the most distant point on the horizon seen from P, is called *the dip of the horizon* at P. The angle DOP is equal to it.



Denote the angle CPD by θ, the height AP by h, and the radius OD by r. Then

$$h = \text{OP} - \text{OA} = r\sec\theta - r$$

$$= \frac{r(1-\cos\theta)}{\cos\theta}.$$

$$\therefore\ r = \frac{h\cos\theta}{1-\cos\theta}.$$

$$\text{PD} = r\tan\theta = \frac{h\sin\theta}{1-\cos\theta}$$

$$= h\cot\frac{\theta}{2} \quad . \quad . \quad \text{(Art. 48, Ex. 8)}$$

Also $\overline{\text{PD}}^2 = \text{PA}\times\text{PB} = h(h+2r)$ . . . (Geom.)

Since, in all cases which can occur in practice, h is very small compared with $2r$, we have approximately

$$\overline{\text{PD}}^2 = 2hr.$$

Let n = the number * of miles in PD, h = the feet in PA, and r = 4000 miles nearly. Then

$$h = \frac{\overline{\text{PD}}^2}{2r} = \frac{(5280n)^2}{8000\times 5280}$$

$$= \frac{5280n^2}{8000} = \frac{5.28}{8}n^2 = \frac{2}{3}n^2.$$

* It will be noticed that n is a *number* merely, and that the result will be in feet, since the miles have been reduced to feet.

That is, the height at which objects can be seen varies as the square of the distance.

Thus, if $n = 1$ mile, we have

$$h = \tfrac{2}{3} \text{ feet} = 8 \text{ inches};$$

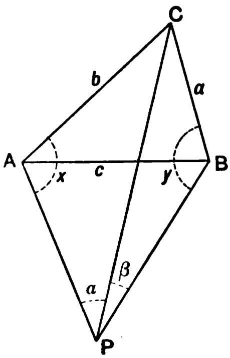
if $n = 2$ miles, $\qquad h = \tfrac{2}{3} \cdot 2^2 = \tfrac{8}{3}$ feet, etc., etc.

Thus it appears that an object less than 8 inches above the surface of still water will be invisible to an eye on the surface at the distance of a mile.

Example. From a balloon, at an elevation of 4 miles, the dip of the sea-horizon is observed to be 2° 33′ 40″: find (1) the diameter of the earth, and (2) the distance of the horizon from the balloon.

Ans. (1) 8001.24 miles; (2) 178.944 miles.

127. Problem of Pothenot or of Snellius. — *To determine a point in the plane of a given triangle, at which the sides of the triangle subtend given angles.*



Let ABC be the given triangle, and P the required point. Join P with A, B, C.

Let the given angles APC, BPC be denoted by α, β, and the unknown angles PAC, PBC by x, y respectively; then α and β are known; and when x and y are found, the position of P can be determined, for the distances PA and PB can be found by solving the triangles PAC, PBC.

We have $\quad x + y = 2\pi - \alpha - \beta - C \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$

Also $\qquad \dfrac{b \sin x}{\sin \alpha} = \dfrac{a \sin y}{\sin \beta} = \text{PC}.$

Assume an auxiliary angle ϕ such that

$$\tan \phi = \frac{a \sin \alpha}{b \sin \beta};$$

then the value of ϕ can be found from the tables.

Thus, $\dfrac{\sin x}{\sin y} = \tan \phi.$

$$\therefore \frac{\sin x - \sin y}{\sin x + \sin y} = \frac{\tan \phi - 1}{\tan \phi + 1} = \tan (\phi - 45^\circ)$$

[(14) of Art. 61].

$$\therefore \tan \tfrac{1}{2}(x - y) = \tan \tfrac{1}{2}(x + y) \tan (\phi - 45^\circ)$$

[(13) of Art. 61].

$$= \tan (45^\circ - \phi) \tan \tfrac{1}{2}(\alpha + \beta + C) \quad . \quad (2)$$

thus from (1) and (2) x and y are found.

EXAMPLES.

Solve the following right triangles:

1. Given $a = 51.303$, $c = 150$;
   find $A = 20^\circ$, $B = 70^\circ$, $b = 140.95$.

2. Given $a = 157.33$, $c = 250$;
   find $A = 39^\circ$, $B = 51^\circ$, $b = 194.28$.

3. Given $a = 104$, $c = 185$;
   find $A = 34^\circ 12' 19''.6$, $B = 55^\circ 47' 40''.4$, $b = 153$.

4. Given $a = 304$, $c = 425$;
   find $A = 45^\circ 40' \; 2''.3$, $B = 44^\circ 19' 57''.7$, $b = 297$.

5. Given $b = 3$, $c = 5$;
   find $A = 53^\circ \; 7' 48''.4$, $B = 36^\circ 52' 11''.6$, $a = 4$.

6. Given $b = 15$, $c = 17$;
   find $A = 28^\circ \; 4' 20''.9$, $B = 61^\circ 55' 39''.1$, $a = 8$.

7. Given $b = 21$, $c = 29$;
   find $A = 43^\circ 36' 10''.1$, $B = 46^\circ 23' 49''.9$, $a = 20$.

8. Given $b = 7$, $c = 25$;
   find $A = 73^\circ 44' 23''.3$, $B = 16^\circ 15' 36''.7$, $a = 24$.

9. Given $b=33$, $c=65$;
find $A=59^\circ 29' 23''.2$, $B=30^\circ 30' 36''.8$, $a=56$.

10. Given $c=625$, $A=44^\circ$;
find $a=434.161$, $b=449.587$.

11. Given $c=300$, $A=52^\circ$;
find $a=236.403$, $b=184.698$.

12. Given $c=13$, $A=67^\circ 22' 48''.5$;
find $B=22^\circ 37' 11''.5$, $a=12$ $b=5$.

13. Given $A=77^\circ 19' 10''.6$, $c=41$;
find $B=12^\circ 40' 49''.4$, $a=40$, $b=9$.

14. Given $B=48^\circ 53' 16''.5$, $c=73$;
find $A=41^\circ 6' 43''.5$, $a=48$, $b=55$.

15. Given $B=64^\circ 0' 38''.8$, $c=89$;
find $A=25^\circ 59' 21''.2$, $a=39$, $b=80$.

16. Given $A=77^\circ 19' 10''.6$, $a=40$;
find $B=12^\circ 40' 49''.4$, $b=9$, $c=41$.

17. Given $A=87^\circ 12' 20''.3$, $a=840$;
find $B=2^\circ 47' 39''.7$, $b=41$, $c=841$.

18. Given $A=32^\circ 31' 13''.5$, $a=336$;
find $B=57^\circ 28' 46''.5$, $b=527$, $c=625$.

19. Given $A=82^\circ 41' 44''$, $a=1100$;
find $B=7^\circ 18' 16''$, $b=141$, $c=1109$.

20. Given $A=75^\circ 23' 18''.5$, $b=195$;
find $B=14^\circ 36' 41''.5$, $a=748$, $c=773$.

21. Given $B=87^\circ 49' 10''$, $b=42536.37$;
find $A=2^\circ 10' 50''$, $a=1619.626$, $c=42567.2$.

22. Given	A = 88° 59′	b = 2.234875;	
find	B = 1° 1′,	a = 125.9365,	c = 125.9563.
23. Given	A = 35° 16′ 25″,	a = 388.2647;	
find	B = 54° 43′ 35,″	b = 548.9018,	c = 672.3412.
24. Given	a = 7694.5,	b = 8471;	
find	A = 42° 15′,	B = 47° 45′,	c = 11444.
25. Given	a = 736,	b = 273;	
find	A = 69° 38′ 56″.3,	B = 20° 21′ 3″.7,	c = 785.
26. Given	a = 200,	b = 609;	
find	A = 18° 10′ 50″,	B = 71° 49′ 10″,	c = 641.
27. Given	a = 276,	b = 493;	
find	A = 29° 14′ 30″.3,	B = 60° 45′ 29.″7,	c = 565.
28. Given	a = 396,	b = 403;	
find	A = 44° 29′ 53″,	B = 45° 30′ 7″,	c = 565.
29. Given	a = 278.3,	b = 314.6;	
find	A = 41° 30′,	B = 48° 30′,	c = 420.
30. Given	a = 372,	b = 423.924;	
find	A = 41° 16′ 2″.7,	B = 48° 43′ 57″.3,	c = 564.
31. Given	a = 526.2,	b = 414.745;	
find	A = 51° 45′ 18″.7,	B = 38° 14′ 41″.3,	c = 670.

Solve the following oblique triangles:

32. Given	B = 50° 30′,	C = 122° 9′,	a = 90;
find	A = 7° 21′,	b = 542.850,	c = 595.638.
33. Given	A = 82° 20′,	B = 43° 20′,	a = 479;
find	C = 54° 20′,	b = 331.657,	c = 392.473.
34. Given	A = 79° 59′,	B = 44° 41′,	a = 795;
find	C = 51° 20′,	b = 567.888,	c = 663.986.

35.	Given	$B = 37°58'$,	$C = 65°2'$,	$a = 999$;
	find	$A = 77°\ 0'$,	$b = 630.771$,	$c = 829.480$.
36.	Given	$A = 70°55'$,	$C = 52°\ 9'$,	$a = 6412$;
	find	$B = 56°56'$,	$b = 5686.00$,	$c = 5357.50$.
37.	Given	$A = 48°20'$,	$B = 81°\ 2'16''$,	$b = 5.75$;
	find	$C = 50°37'44''$,	$a = 4.3485$,	$c = 4.5$.
38.	Given	$A = 72°\ 4'$,	$B = 41°56'18''$,	$c = 24$;
	find	$C = 65°59'42''$,	$a = 24.995$,	$b = 17.559$.
39.	Given	$A = 43°36'10''.1$,	$C = 124°58'33''.6$,	$b = 29$;
	find	$B = 11°25'16''.3$,	$a = 101$,	$c = 120$.
40.	Given	$A = 69°59'\ 2''.5$,	$C = 70°42'30''$,	$b = 149$;
	find	$B = 39°18'27''.5$,	$a = 221$,	$c = 222$.
41.	Given	$A = 21°14'25''$,	$a = 345$,	$b = 695$;
	find	$B = 46°52'10''$,	$C = 111°53'25''$,	$c = 883.65$.
	or	$B' = 133°\ 7'50''$,	$C' = 25°37'45''$,	$c' = 411.92$.
42.	Given	$A = 41°13'\ 0''$,	$a = 77.04$,	$b = 91.06$;
	find	$B = 51°\ 9'\ 6''$,	$C = 87°37'54''$,	$c = 116.82$,
	or	$B' = 128°50'54''$,	$C' = 9°56'\ 6''$,	$c' = 20.172$.
43.	Given	$A = 21°14'25''$,	$a = 309$,	$b = 360$;
	find	$B = 24°51'54''$,	$C = 133°47'41''$,	$c = 615.67$,
	or	$B' = 155°\ 2'\ 6''$,	$C' = 3°43'29''$,	$c' = 55.41$.
44.	Given	$B = 68°10'24''$,	$a = 83.856$,	$b = 83.153$;
	find	$A = 65°\ 5'10''$,	$C = 45°44'26''$,	$c = 65.696$.
45.	Given	$B = 60°\ 0'32''$,	$a = 27.548$,	$b = 35.055$;
	find	$A = 42°53'34''$,	$C = 77°\ 5'54''$,	$c = 39.453$.
46.	Given	$A = 60°$,	$a = 120$,	$b = 80$;
	find	$B = 35°15'52''$,	$C = 84°44'\ 8''$,	$c = 137.9796$.

47. Given A = 50°, $a=119$, $b=97$;
find B = 38° 38′ 24″, C = 91° 21′ 36″, $c=155.3$.

48. Given C = 65° 59′, $a=25$, $c=24$;
find A = 72° 4′ 48″, B = 41° 56′ 12″, $b=17.56$,
or A′ = 107° 55′ 12″, B′ = 6° 5′ 48″.

49. Given A = 18° 55′ 28″.7, $a=13$, $b=37$;
find B = 67° 22′ 48″.1, or B′ = 112° 37′ 11″.9.

50. Given C = 15° 11′ 21″, $a=232$, $b=229$;
find A = 85° 11′ 58″, B = 79° 36′ 40″, $c=61$.

51. Given C = 126° 12′ 14″, $a=5132$, $b=3476$;
find A = 32° 28′ 19″, B = 21° 19′ 27″, $c=7713.3$.

52. Given C = 55° 12′ 3″, $a=20.71$, $b=18.87$;
find A = 67° 28′ 51″.5, B = 57° 19′ 5″.5, $c=18.41$.

53. Given C = 12° 35′ 8″, $a=8.54$, $b=6.39$;
find A = 136° 15′ 48″, B = 31° 9′ 4″, $c=2.69$.

54. Given C = 34° 9′ 16″, $a=3184$, $b=917$;
find A = 133° 51′ 34″, B = 11° 59′ 10″, $c=2479.2$.

55. Given C = 32° 10′ 53″.8, $a=101$, $b=29$;
find A = 136° 23′ 49″.9, B = 11° 25′ 16″.3, $c=78$.

56. Given C = 96° 57′ 20″.1, $a=401$, $b=41$;
find A = 77° 19′ 10″.6, B = 5° 43′ 29″.2, $c=408$.

57. Given C = 30° 40′ 35″, $a=221$, $b=149$;
find A = 110° 0′ 57″.5, B = 39° 18′ 27″.5, $c=120$.

58. Given C = 66° 59′ 25″.4, $a=109$, $b=61$;
find A = 79° 36′ 40″, B = 33° 23′ 54″.6, $c=102$.

59. Given C = 131° 24′ 44″, $a=229$, $b=109$;
find A = 33° 23′ 54″.6, B = 15° 11′ 21″.4, $c=312$.

60. Given $C=104°3'51''$, $a=241$, $b=169$;
find $A=45°46'16''.5$, $B=30°\ 9'52''.5$, $c=332.97$.

61. Given $a=289$, $b=601$, $c=712$;
find $A=23°32'12''$, $B=56°\ 8'42''$, $C=100°19'\ 6''$.

62. Given $a=17$, $b=113$, $c=120$;
find $A=\ 7°37'42''$, $B=61°55'38''$, $C=110°26'40''$.

63. Given $a=15.47$, $b=17.39$, $c=22.88$;
find $A=42°30'44''$, $B=49°25'49''$, $C=88°3'27''$.

64. Given $a=5134$, $b=7268$, $c=9313$;
find $A=33°15'39''$, $B=50°56'\ 0''$, $C=95°48'21''$.

65. Given $a=99$, $b=101$, $c=158$;
find $A=37°22'19''$, $B=38°15'41''$, $C=104°22'0''$.

66. Given $a=11$, $b=13$, $c=16$;
find $A=43°\ 2'56''$, $B=53°46'44''$, $C=83°10'20''$.

67. Given $a=25$, $b=26$, $c=27$;
find $A=56°15'\ 4''$, $B=59°51'10''$, $C=63°53'46''$.

68. Given $a=197$, $b=53$, $c=240$;
find $A=31°53'26''.8$, $B=\ 8°10'16''.4$, $C=139°56'16''.8$.

69. Given $a=509$, $b=221$, $c=480$;
find $A=84°32'50''.5$, $B=25°36'30''.7$, $C=69°50'38''.8$.

70. Given $a=533$, $b=317$, $c=510$;
find $A=76°18'52''$, $B=35°18'\ 0''.9$, $C=68°23'7''.1$.

71. Given $a=565$, $b=445$, $c=606$;
find $A=62°51'32''.9$, $B=44°29'53''$, $C=72°38'34''.1$.

72. Given $a=10$, $b=12$, $c=14$;
find $A=44°24'55''.2$, $B=57°\ 7'18''$, $C=78°27'47''$.

73. Given $a=.8706$, $b=.0916$, $c=.7902$;
find $A=149°49'0''.4$, $B=\ 3°\ 1'56''.2$, $C=27°9'3''.4$.

Find the area:

74. Given $a=10$, $b=12$, $C=60°$. *Ans.* $30\sqrt{3}$.

75. " $a=40$, $b=60$, $C=30°$. 600.

76. " $b=7$, $c=5\sqrt{2}$, $A=135°$. $17\frac{1}{2}$.

77. " $a=32.5$, $b=56.3$, $C=47° 5' 30''$. 670.

78. " $b=149$, $A=70° 42' 30''$, $B=39° 18' 28''$. 15540.

79. " $c=8.025$, $B=100° 5' 23''$, $C=31° 6' 12''$. 46.177.

80. " $a=5$, $b=6$, $c=7$. 12.

81. " $a=625$, $b=505$, $c=904$. 151872.

82. " $a=409$, $b=169$, $c=510$. 30600.

83. " $a=577$, $b=73$, $c=520$. 12480.

84. " $a=52.53$, $b=48.76$, $c=44.98$. 1016.9487.

85. " $a=13$, $b=14$, $c=15$. 84.

86. " $a=242$ yards, $b=1212$ yards, $c=1450$ yards. *Ans.* 6 acres.

87. " $a=7.152$, $b=8.263$, $c=9.375$. 28.47717.

88. The sides of a triangle are as $2:3:4$: show that the radii of the escribed circles are as $\frac{1}{5}:\frac{1}{3}:1$.

89. The area of a triangle is an acre; two of its sides are 127 yards and 150 yards: find the angle between them. *Ans.* $30° 32' 23''$.

90. The adjacent sides of a parallelogram are 5 and 8, and they include an angle of 60°: find (1) the two diagonals, and (2) the area. *Ans.* (1) 7, $\sqrt{129}$; (2) $20\sqrt{3}$.

91. Two angles of a triangular field are $22\frac{1}{2}°$ and 45°, and the length of the side opposite the latter is a furlong. Show that the field contains $2\frac{1}{2}$ acres.

HEIGHTS AND DISTANCES.

92. At a point 200 feet in a horizontal line from the foot of a tower, the angle of elevation of the top of the tower is observed to be 60°: find the height of the tower.

Ans. 346 feet.

93. From the top of a vertical cliff, the angle of depression of a point on the shore 150 feet from the base of the cliff, is observed to be 30°: find the height of the cliff.

Ans. 86.6 feet.

94. From the top of a tower 117 feet high, the angle of depression of the top of a house 37 feet high is observed to be 30°: how far is the top of the house from the tower?

Ans. 138.5 feet.

95. The shadow of a tower in the sunlight is observed to be 100 feet long, and at the same time the shadow of a lamp-post 9 feet high is observed to be $3\sqrt{3}$ feet long: find the angle of elevation of the sun, and height of the tower.

Ans. 60°; 173.2 feet.

96. A flagstaff 25 feet high stands on the top of a house; from a point on the plain on which the house stands, the angles of elevation of the top and bottom of the flagstaff are observed to be 60° and 45° respectively: find the height of the house above the point of observation.

Ans. 34.15 feet.

97. From the top of a cliff 100 feet high, the angles of depression of two ships at sea are observed to be 45° and 30° respectively; if the line joining the ships points directly to the foot of the cliff, find the distance between the ships.

Ans. 73.2.

98. A tower 100 feet high stands on the top of a cliff; from a point on the sand at the foot of the cliff the angles

of elevation of the top and bottom of the tower are observed to be 75° and 60° respectively: find the height of the cliff.

Ans. 86.6 feet.

99. A man walking a straight road observes at one milestone a house in a direction making an angle of 30° with the road, and at the next milestone the angle is 60°: how far is the house from the road? *Ans.* 1524 yds.

100. A man stands at a point A on the bank AB of a straight river and observes that the line joining A to a post C on the opposite bank makes with AB an angle of 30°. He then goes 400 yards along the bank to B and finds that BC makes with BA an angle of 60°: find the breadth of the river. *Ans.* 173.2 yards.

101. From the top of a hill the angles of depression of the top and bottom of a flagstaff 25 feet high at the foot of the hill are observed to be 45° 13′ and 47° 12′ respectively: find the height of the hill. *Ans.* 373 feet.

102. From each of two stations, east and west of each other, the altitude of a balloon is observed to be 45°, and its bearings to be respectively N.W. and N.E.; if the stations be 1 mile apart, find the height of the balloon.

Ans. 3733 feet.

103. The angle of elevation of a balloon from a station due south of it is 60°, and from another station due west of the former and distant a mile from it is 45°: find the height of the balloon. *Ans.* 6468 feet.

104. Find the height of a hill, the angle of elevation at its foot being 60°, and at a point 500 yards from the foot along a horizontal plane 30°. *Ans.* $250\sqrt{3}$ yards.

105. A tower 51 feet high has a mark at a height of 25 feet from the ground: find at what distance from the foot the two parts subtend equal angles.

106. The angles of a triangle are as $1:2:3$, and the perpendicular from the greatest angle on the opposite side is 30 yards: find the sides. *Ans.* $20\sqrt{3}$, 60, $40\sqrt{3}$.

107. At two points A, B, an object DE, situated in the same vertical line CE, subtends the same angle α; if AC, BC be in the same right line, and equal to a and b, respectively, prove

$$\text{DE} = (a + b)\tan\alpha.$$

108. From a station B at the foot of an inclined plane BC the angle of elevation of the summit A of a mountain is 60°, the inclination of BC is 30°, the angle BCA 135°, and the length of BC is 1000 yards: find the height of A over B. *Ans.* $500(3+\sqrt{3})$ yards.

109. A right triangle rests on its hypotenuse, the length of which is 100 feet; one of the angles is 36°, and the inclination of the plane of the triangle to the horizon is 60°: find the height of the vertex above the ground. *Ans.* $25\sqrt{3}\cos 18°$.

110. A station at A is due west of a railway train at B; after traveling N.W. 6 miles, the bearing of A from the train is S. $22\frac{1}{2}°$ W.: find the distance AB. *Ans.* 6 miles.

111. The angles of depression of the top and bottom of a column observed from a tower 108 feet high are 30° and 60° respectively: find the height of the column. *Ans.* 72 feet.

112. At the foot of a mountain the elevation of its summit is found to be 45°. After ascending for one mile, at a slope of 15°, towards the summit, its elevation is found to be 60°: find the height of the mountain.

Ans. $\dfrac{\sqrt{3}+1}{\sqrt{2}}$ miles.

113. A and B are two stations on a hillside. The inclination of the hill to the horizon is 30°. The distance between A and B is 500 yards. C is the summit of another hill in

the same vertical plane as A and B, on a level with A, but at B its elevation above the horizon is 15°: find the distance between A and C. *Ans.* $500(\sqrt{3}+1)$.

114. From the top of a cliff the angles of depression of the top and bottom of a lighthouse 97.25 feet high are observed to be 23° 17′ and 24° 19′ respectively: how much higher is the cliff than the lighthouse? *Ans.* 1942 feet.

115. The angle of elevation of a balloon from a station due south of it is 47° 18′ 30″, and from another station due west of the former, and distant 671.38 feet from it, the elevation is 41° 14′: find the height of the balloon.

Ans. 1000 feet.

116. A person standing on the bank of a river observes the elevation of the top of a tree on the opposite bank to be 51°; and when he retires 30 feet from the river's bank he observes the elevation to be 46°: find the breadth of the river. *Ans.* 155.823 feet.

117. From the top of a hill I observe two milestones on the level ground in a straight line before me, and I find their angles of depression to be respectively 5° and 15°: find the height of the hill. *Ans.* 228.6307 yards.

118. A tower is situated on the top of a hill whose angle of inclination to the horizon is 30°. The angle subtended by the tower at the foot of the hill is found by an observer to be 15°; and on ascending 485 feet up the hill the tower is found to subtend an angle of 30°: find (1) the height of the tower, and (2) the distance of its base from the foot of the hill. *Ans.* (1) 280.015; (2) 765.015 feet.

119. The angle of elevation of a tower at a place A due south of it is 30°; and at a place B, due west of A, and at a distance a from it, the elevation is 18°: show that the height of the tower is $\dfrac{a}{\sqrt{2+2\sqrt{5}}}$.

120. On the bank of a river there is a column 200 feet high supporting a statue 30 feet high. The statue to an observer on the opposite bank subtends an equal angle with a man 6 feet high standing at the base of the column: find the breadth of the river. *Ans.* $10\sqrt{115}$ feet.

121. A man walking along a straight road at the rate of 3 miles an hour, sees in front of him, at an elevation of 60°, a balloon which is travelling horizontally in the same direction at the rate of 6 miles an hour; ten minutes after he observes that the elevation is 30°: prove that the height of the balloon above the road is $440\sqrt{3}$ yards.

122. An observer in a balloon observes the angle of depression of an object on the ground, due south, to be 35° 30′. The balloon drifts due east, at the same elevation, for $2\frac{1}{2}$ miles, when the angle of depression of the same object is observed to be 23° 14′: find the height of the balloon. *Ans.* 1.34394 miles.

123. A column, on a pedestal 20 feet high, subtends an angle 45° to a person on the ground; on approaching 20 feet, it again subtends an angle 45°: show that the height of the column is 100 feet.

124. A tower 51 feet high has a mark 25 feet from the ground: find at what distance the two parts subtend equal angles to an eye 5 feet from the ground. *Ans.* 160 feet.

125. From the extremities of a sea-wall, 300 feet long, the bearings of a boat at sea were observed to be N. 23° 30′ E., and N. 35° 15′ W.: find the distance of the boat from the sea-wall. *Ans.* 262.82 feet.

126. ABC is a triangle on a horizontal plane, on which stands a tower CD, whose elevation at A is 50° 3′ 2″; AB is 100.62 feet, and BC and AC make with AB angles 40° 35′ 17″ and 9° 59′ 50″ respectively: find CD. *Ans.* 101.166 feet.

127. The angle of elevation of a tower at a distance of 20 yards from its foot is three times as great as the angle of elevation 100 yards from the same point: show that the height of the tower is $\dfrac{300}{\sqrt{7}}$ feet.

128. A man standing at a point A, due south of a tower built on a horizontal plain, observes the altitude of the tower to be 60°. He then walks to a point B due west from A and observes the altitude to be 45°, and then at the point C in AB produced he observes the altitude to be 30°: prove that AB = BC.

129. The angle of elevation of a balloon, which is ascending uniformly and vertically, when it is one mile high is observed to be 35° 20′; 20 minutes later the elevation is observed to be 55° 40′: how fast is the balloon moving?

Ans. 3 (sin 20° 20′) (sec 55° 40′) (cosec 35° 20′) miles per hour.

130. The angle of elevation of the top of a steeple at a place due south of it is 45°, and at another place due west of the former station and distant 100 feet from it the elevation is 15°: show that the height of the steeple is $50(3^{\frac{1}{4}} - 3^{-\frac{1}{4}})$ feet.

131. A tower stands at the foot of an inclined plane whose inclination to the horizon is 9°; a line is measured up the incline from the foot of the tower, of 100 feet in length. At the upper extremity of this line the tower subtends an angle of 54°: find the height of the tower.

Ans. 114.4 feet.

132. The altitude of a certain rock is observed to be 47°, and after walking 1000 feet towards the rock, up a slope inclined at an angle of 32° to the horizon, the observer finds that the altitude is 77°: prove that the vertical height of the rock above the first point of observation is 1034 feet.

133. From a window it is observed that the angle of elevation of the top of a house on the opposite side of the street is 29°, and the angle of depression of the bottom of the house is 56°: find the height of the house, supposing the breadth of the street to be 80 feet. *Ans.* 162.95 feet.

134. A and B are two positions on opposite sides of a mountain; C is a point visible from A and B; AC and BC are 10 miles and 8 miles respectively, and the angle BCA is 60°: prove that the distance between A and B is 9.165 miles.

135. P and Q are two inaccessible objects; a straight line AB, in the same plane as P and Q, is measured, and found to be 280 yards; the angle PAB is 95°, the angle QAB is $47\frac{1}{2}°$, the angle QBA is 110°, and the angle PBA is 52° 20′: find the length of PQ. *Ans.* 509.77 yards.

136. Two hills each 264 feet high are just visible from each other over the sea: how far are they apart? (Take the radius of the earth = 4000 miles.) *Ans.* 40 miles.

137. A ship sailing out of harbor is watched by an observer from the shore; and at the instant she disappears below the horizon he ascends to a height of 20 feet, and thus keeps her in sight 40 minutes longer: find the rate at which the ship is sailing, assuming the earth's radius to be 4000 miles, and neglecting the height of the observer.

Ans. $40\sqrt{330}$ feet per minute.

138. From the top of the mast of a ship 64 feet above the level of the sea the light of a distant lighthouse is just seen in the horizon; and after the ship has sailed directly towards the light for 30 minutes it is seen from the deck of the ship, which is 16 feet above the sea: find the rate at which the ship is sailing. (Take radius = 4000 miles.)

Ans. $8\sqrt{\frac{50}{33}}$ miles per hour.

139. A, B, C, are three objects at known distances apart; namely, AB = 1056 yards, AC = 924 yards, BC = 1716 yards. An observer places himself at a station P from which C appears directly in front of A, and observes the angle CPB to be 14° 24′: find the distance CP.

Ans. 2109.824 yards.

140. A, B, C, are three objects such that AB = 320 yards, AC = 600 yards, and BC = 435 yards. From a station P it is observed that APB = 15°, and BPC = 30°: find the distances of P from A, B, and C; the point B being nearest to P, and the angle APC being the sum of the angles APB and BPC. *Ans.* PA = 777, PB = 502, PC = 790.

CHAPTER VIII.

CONSTRUCTION OF LOGARITHMIC AND TRIGONOMETRIC TABLES.

128. Logarithmic and Trigonometric Tables. — In Chapters IV., V., and VII., it was shown how to *use* logarithmic and trigonometric tables; it will now be shown how to *calculate* such tables. Although the trigonometric functions are seldom capable of being expressed *exactly*, yet they can be found *approximately* for any angle; and the calculations may be carried to any assigned degree of accuracy. We shall first show how to calculate logarithmic tables, and shall repeat here substantially Arts. 208, 209, 210, from the College Algebra.

129. Exponential Series. — *To expand e^x in a series of ascending powers of x.*

By the Binomial Theorem,

$$\left(1+\frac{1}{n}\right)^{nx} = 1 + nx\frac{1}{n} + \frac{nx(nx-1)}{\underline{|2}}\frac{1}{n^2} + \frac{nx(nx-1)(nx-2)}{\underline{|3}}\frac{1}{n^3} + \cdots$$

$$= 1 + x + \frac{x\left(x-\frac{1}{n}\right)}{\underline{|2}} + \frac{x\left(x-\frac{1}{n}\right)\left(x-\frac{2}{n}\right)}{\underline{|3}} + \cdots \quad (1)$$

Similarly,

$$\left[\left(1+\frac{1}{n}\right)^{n}\right]^x = \left[1 + 1 + \frac{1-\frac{1}{n}}{\underline{|2}} + \frac{\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)}{\underline{|3}} + \cdots\right]^x \quad (2)$$

and therefore series (1) is equal to series (2) however great n may be. Hence if n be indefinitely increased, we have from (1) and (2)

$$1+x+\frac{x^2}{\lfloor 2}+\frac{x^3}{\lfloor 3}+\frac{x^4}{\lfloor 4}+\cdots=\left(1+1+\frac{1}{\lfloor 2}+\frac{1}{\lfloor 3}+\frac{1}{\lfloor 4}+\cdots\right)^x.$$

The series in the parenthesis is usually denoted by e; hence

$$e^x=1+x+\frac{x^2}{\lfloor 2}+\frac{x^3}{\lfloor 3}+\frac{x^4}{\lfloor 4}+\cdots \quad (3)$$

which is the expansion of e^x in powers of x.

This result is called the *Exponential Theorem.*

If we put $x=1$, we have from (3)

$$e=1+1+\frac{1}{\lfloor 2}+\frac{1}{\lfloor 3}+\frac{1}{\lfloor 4}+\frac{1}{\lfloor 5}+\frac{1}{\lfloor 6}+\cdots$$

From this series we may readily compute the approximate value of e to any required degree of accuracy. This constant value e is called the *Napierian base* (Art. 64). To ten places of decimals it is found to be 2.7182818284.

Cor. Let $a=e^c$; then $c=\log_e a$, and $a^x=e^{cx}$. Substituting in (3), we have

$$e^{cx}=1+cx+\frac{c^2x^2}{\lfloor 2}+\frac{c^3x^3}{\lfloor 3}+\cdots$$

or

$$a^x=1+x\log_e a+\frac{x^2(\log_e a)^2}{\lfloor 2}+\frac{x^3(\log_e a)^3}{\lfloor 3}+\cdots \quad (4)$$

which is the expansion of a^x in powers of x.

130. Logarithmic Series. — *To expand* $\log_e(1+x)$ *in a series of ascending powers of* x.

By the Binomial Theorem,

$$a^x=(1+a-1)^x=1+x(a-1)+\frac{x(x-1)}{\lfloor 2}(a-1)^2$$
$$+\frac{x(x-1)(x-2)}{\lfloor 3}(a-1)^3+\cdots$$

$$= 1 + x[a - 1 - \tfrac{1}{2}(a-1)^2 + \tfrac{1}{3}(a-1)^3 - \tfrac{1}{4}(a-1)^4 + \cdots]$$
$$+ \text{ terms involving } x^2,\ x^3, \text{ etc.}$$

Comparing this value of a^x with that given in (4) of Art. 129, and equating the coefficients of x, we have

$$\log_e a = a - 1 - \tfrac{1}{2}(a-1)^2 + \tfrac{1}{3}(a-1)^3 - \tfrac{1}{4}(a-1)^4 + \cdots$$

Put $a = 1 + x$; then

$$\log_e (1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

This is the *Logarithmic Series;* but unless x be very small, the terms diminish so slowly that a large number of them will have to be taken; and hence the series is of little practical use for numerical calculation. If $x > 1$, the series is altogether unsuitable. We shall therefore deduce some more convenient formulæ.

Changing x into $-x$, (1) becomes

$$\log_e(1 - x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \cdots \quad . \quad . \quad . \quad (2)$$

Subtracting (2) from (1), we have

$$\log_e \frac{1+x}{1-x} = 2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} + \cdots\right) \quad . \quad . \quad . \quad (3)$$

Put $\dfrac{1+x}{1-x} = \dfrac{n+1}{n}$. $\quad \therefore\ x = \dfrac{1}{2n+1}$,

and (3) becomes

$$\log_e \frac{n+1}{n} = 2\left[\frac{1}{2n+1} + \frac{1}{3(2n+1)^3} + \frac{1}{5(2n+1)^5} + \cdots\right],$$

or $\log_e (n + 1)$

$$= \log_e n + 2\left[\frac{1}{2n+1} + \frac{1}{3(2n+1)^3} + \frac{1}{5(2n+1)^5} + \cdots\right] \quad . \quad (4)$$

This series is rapidly convergent, and gives the logarithm of either of two consecutive numbers to any extent when the logarithm of the other number is known.

131. Computation of Logarithms. — Logarithms to the base e are called *Napierian Logarithms* (Art. 64). They are also called *natural logarithms*, because they are the first logarithms which occur in the investigation of a method of calculating logarithms. Logarithms to the base 10 are called *common logarithms.* When logarithms are used in theoretical investigations, the base e is always understood, just as in all practical calculations the base 10 is invariably employed. It is only necessary to compute the logarithms of *prime* numbers from the series, since the logarithm of a *composite* number may be obtained by adding together the logarithms of its component factors. The logarithm of $1 = 0$. Putting $n = 1, 2, 4, 6$, etc., successively, in (4) of Art. 130, we obtain the following

Napierian Logarithms:

$$\log_e 2 = 2\left[\frac{1}{3}+\frac{1}{3\cdot 3^3}+\frac{1}{5\cdot 3^5}+\frac{1}{7\cdot 3^7}+\frac{1}{9\cdot 3^9}+\cdots\right] = 0.69314718.$$

$$\log_e 3 = \log_e 2 + 2\left[\frac{1}{5}+\frac{1}{3\cdot 5^3}+\frac{1}{5\cdot 5^5}+\frac{1}{7\cdot 5^7}+\cdots\right] = 1.09861228.$$

$$\log_e 4 = 2\log_e 2 = 1.38629436.$$

$$\log_e 5 = \log_e 4 + 2\left[\frac{1}{9}+\frac{1}{3\cdot 9^3}+\frac{1}{5\cdot 9^5}+\frac{1}{7\cdot 9^7}+\cdots\right] = 1.60943790.$$

$$\log_e 6 = \log_e 2 + \log_e 3 = 1.79175946.$$

$$\log_e 7 = \log_e 6 + 2\left[\frac{1}{13}+\frac{1}{3\cdot 13^3}+\frac{1}{5\cdot 13^5}+\cdots\right] = 1.94590996.$$

$$\log_e 8 = 3\log_e 2 = 2.07944154.$$

$$\log_e 9 = 2\log_e 3 = 2.19722456.$$

$$\log_e 10 = \log_e 5 + \log_e 2 = 2.30258509.$$

And so on.

The number of terms of the series which it is necessary to include diminishes as n increases. Thus, in computing the logarithm of 101, the first term of the series gives the result true to seven decimal places.

By changing b to 10 and a to e in (1) of Art. 65, we have

$$\log_{10} m = \frac{\log_e m}{\log_e 10} = \frac{\log_e m}{2.30258509} = .43429448 \log_e m,$$

or $\quad$ common $\log m =$ Napierian $\log m \times .43429448.$

The number .43429448 is called *the modulus of the common system.* It is usually denoted by μ.

Hence, the common logarithm of any number is equal to the Napierian logarithm of the same number multiplied by the modulus of the common system, .43429448.

Multiplying (4) of Art. 130 by μ, we obtain a series by which *common logarithms* may be computed; thus,

$$\log_{10}(n+1) = \log_{10} n + 2\mu\left[\frac{1}{2n+1} + \frac{1}{3(2n+1)^3} + \frac{1}{5(2n+1)^5} + \cdots\right] \quad (1)$$

Common Logarithms.

$\log_{10} 2 = \mu \log_e 2 = .43429448 \times .69314718 = .3010300.$

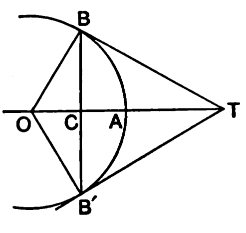
$\log_{10} 3 = \mu \log_e 3 = .43429448 \times 1.09861228 = .4771213.$

$\log_{10} 4 = 2 \log_{10} 2 = .6020600.$

$\log_{10} 5 = \mu \log_e 5 = .43429448 \times 1.60943790 = .6989700.$

And so on.

132. If θ be the Circular Measure of an Acute Angle, sin θ, θ, and tan θ are in Ascending Order of Magnitude.



With centre O, and any radius, describe an arc BAB′. Bisect the angle BOB′ by OA; join BB′, and draw the tangents BT, B′T.

Let AOB = AOB′ = θ. Then

$$BB' < \text{arc } BAB' < BT + B'T$$

(Geom., Art. 246)

$\therefore\ BC < \text{arc } BA < BT.$

$$\therefore \frac{BC}{OB} < \frac{BA}{OB} < \frac{BT}{OB}.$$

$$\therefore \quad \sin\theta < \theta < \tan\theta.$$

133. The Limit of $\frac{\sin\theta}{\theta}$, when θ is Indefinitely Diminished, is Unity.

We have $\quad \sin\theta < \theta < \tan\theta$ . . . . (Art. 132)

$$\therefore 1 < \frac{\theta}{\sin\theta} < \sec\theta.$$

Now as θ is diminished indefinitely, $\sec\theta$ approaches the limit unity; then when $\theta = 0$, we have $\sec\theta = 1$.

$\therefore$ the limit of $\frac{\theta}{\sin\theta}$, which lies between $\sec\theta$ and unity, is unity.

$\therefore$ also $\frac{\sin\theta}{\theta}$ approaches the limit unity.

As $\frac{\tan\theta}{\theta} = \frac{\sin\theta}{\theta} \times \sec\theta$, the limit of $\frac{\tan\theta}{\theta}$, when θ is indefinitely diminished, is also unity.

This is often stated briefly thus:

$$\frac{\sin\theta}{\theta} = 1, \text{ and } \frac{\tan\theta}{\theta} = 1, \text{ when } \theta = 0.$$

NOTE.—From this it follows that the *sines* and the *tangents* of very small angles are proportional to the angles themselves.

134. If θ is the Circular Measure of an Acute Angle, $\sin\theta$ lies between θ and $\theta - \frac{\theta^3}{4}$; and $\cos\theta$ lies between $1 - \frac{\theta^2}{2}$ and $1 - \frac{\theta^2}{2} + \frac{\theta^4}{16}$.

(1) We have $\quad \tan\frac{\theta}{2} > \frac{\theta}{2}$ . . . . . . (Art. 132)

$$\therefore \sin\frac{\theta}{2} > \frac{\theta}{2}\cos\frac{\theta}{2}.$$

$$\therefore\ 2\sin\frac{\theta}{2}\cos\frac{\theta}{2} > \theta\cos^2\frac{\theta}{2}.$$

$$\therefore\ \sin\theta > \theta\left(1-\sin^2\frac{\theta}{2}\right)$$

$$> \theta\left(1-\frac{\theta^2}{4}\right) \quad . \quad . \quad \text{(Art. 132)}$$

$$\therefore\ \sin\theta < \theta \text{ and } > \theta - \frac{\theta^3}{4}.$$

(2) $$\cos\theta = 1 - 2\sin^2\frac{\theta}{2}.$$

$$> 1 - 2\left(\frac{\theta^2}{4}\right).$$

$$\therefore\ \cos\theta > 1 - \frac{\theta^2}{2}.$$

Also, $$\sin\frac{\theta}{2} > \frac{\theta}{2} - \frac{1}{4}\left(\frac{\theta}{2}\right)^3 \text{ by (1).}$$

$$\therefore\ \cos\theta < 1 - 2\left[\frac{\theta}{2} - \frac{\theta^3}{32}\right]^2$$

$$< 1 - \frac{\theta^2}{2} + \frac{\theta^4}{16} - \frac{\theta^6}{512}.$$

$$\therefore\ \cos\theta > 1 - \frac{\theta^2}{2} \text{ and } < 1 - \frac{\theta^2}{2} + \frac{\theta^4}{16}.$$

Note.—It may be proved that $\sin\theta > \theta - \frac{\theta^3}{6}$, as follows:

We have $$3\sin\frac{\theta}{3} - \sin\theta = 4\sin^3\frac{\theta}{3} \text{ (Art. 50)} \quad . \quad . \quad . \quad (1)$$

$$\therefore\ 3\sin\frac{\theta}{3^2} - \sin\frac{\theta}{3} = 4\sin^3\frac{\theta}{3^2} \text{ (by putting } \frac{\theta}{3} \text{ for } \theta) \quad . \quad . \quad . \quad (2)$$

. . . .

$$3\sin\frac{\theta}{3^n} - \sin\frac{\theta}{3^{n-1}} = 4\sin^3\frac{\theta}{3^n} \quad . \quad . \quad . \quad (n)$$

Multiply (1), (2), ⋯ (n) by 1, 3, ⋯ 3^{n-1}, respectively, and add them,

$$3^n\sin\frac{\theta}{3^n} - \sin\theta = 4\left(\sin^3\frac{\theta}{3} + 3\sin^3\frac{\theta}{3^2} + \cdots 3^{n-1}\sin^3\frac{\theta}{3^n}\right).$$

$$\therefore\ \theta\cdot\frac{\sin\frac{\theta}{3^n}}{\frac{\theta}{3^n}}-\sin\theta<4\left(\frac{\theta^3}{3^3}+\frac{\theta^3}{3^5}+\cdots\frac{\theta^3}{3^{2n+1}}\right)\ \ldots\ldots\ \text{(Art. 132)}$$

$$<\frac{4}{3^3}\theta^3\left(1+\frac{1}{3^2}+\cdots\frac{1}{3^{2n-2}}\right).$$

If $n=\infty$, then
$$\frac{\sin\frac{\theta}{3^n}}{\frac{\theta}{3^n}}=1\ \ldots\ldots\ \text{(Art. 133)}$$

and
$$\frac{4}{3^3}\theta^3\left(1+\frac{1}{3^2}+\cdots\frac{1}{3^{2n-2}}\right)=\frac{4\theta^3}{3^3}\cdot\frac{1}{1-\frac{1}{3^2}}=\frac{\theta^3}{6}.$$

$$\therefore\ \theta-\sin\theta<\frac{\theta^3}{6},\ \text{and}\ \therefore\ \sin\theta>\theta-\frac{\theta^3}{6}.$$

This makes the limits for $\sin\theta$ closer than in (1) of this Art.

135. To calculate the Sine and Cosine of 10″ and of 1′.

(1) Let θ be the circular measure of 10″.

Then
$$\theta=\frac{10\pi}{180\times 60\times 60}=\frac{3.141592653589793\cdots}{64800},$$

or $\theta=.000048481368110\cdots$, correct to 15 decimal places.

$$\therefore\ \frac{\theta^3}{4}=.000000000000032\cdots,\quad \text{“ “ “ “}$$

$$\therefore\ \theta-\frac{\theta^3}{4}=.000048481368078\cdots,\quad \text{“ “ “ “}$$

Hence the two quantities θ and $\theta-\frac{\theta^3}{4}$ agree to 12 decimal places; and since $\sin\theta<\theta$ and $>\theta-\frac{\theta^3}{4}$ (Art. 134),

$$\therefore\ \sin 10''=.000048481368, \text{ to 12 decimal places.}$$

We have
$$\cos 10''=\sqrt{1-\sin^2 10''}=1-\tfrac{1}{2}\sin^2 10''$$
$$=.9999999988248\cdots, \text{ to 13 decimal places.}$$

Or we may use the results established in (2) of Art. 134, and obtain the same value.

(2) Let θ be the circular measure of $1'$.

Then

$$\theta = \frac{\pi}{180 \times 60} = .000290888208665, \text{ to 15 decimal places.}$$

$$\therefore \frac{\theta^3}{4} = .000000000006 \quad \text{to 12} \quad \text{"} \quad \text{"}$$

$$\therefore \theta - \frac{\theta^3}{4} = .00029088820 \quad \text{to 11} \quad \text{"} \quad \text{"}$$

Hence θ and $\theta - \frac{\theta^3}{4}$ differ only in the twelfth decimal.

$$\therefore \sin 1' = .00029088820 \text{ to 11 decimal places.}$$

$$\cos 1' = \sqrt{1 - \sin^2 1'} = .999999957692025 \text{ to 15 decimal places,}$$

Otherwise thus:

$$1 - \frac{\theta^2}{2} = .999999957692025029 \text{ to 18 decimal places.}$$

and $$\frac{\theta^4}{16} = .00000000000000044 \text{ to 17 decimal places.}$$

But $\cos 1' > 1 - \frac{\theta^2}{2}$ and $< 1 - \frac{\theta^2}{2} + \frac{\theta^4}{16}$ . . . (Art. 134)

$$\therefore \cos 1' = .999999957692025 \text{ to 15 decimal places, as before.}$$

Cor. 1. *The sine of* 10″ *equals the circular measure of* 10″, *to* 12 *decimal places; and the sine of* 1′ *equals the circular measure of* 1′ *to* 11 *decimal places.*

Cor. 2. *If n denote any number of seconds less than* 60, *we shall have approximately*

$$\sin n'' = n \sin 1'',$$

for the sine of n'' = the circular measure of n'', approximately, $= n$ times the circular measure of $1''$.

Cor. 3. $n = \frac{\text{circular measure of } n''}{\sin 1''}$, approximately; that is, the number of seconds in any small angle is found approximately by dividing the circular measure of that angle by the sine of one second.

136. To construct a Table of Natural Sines and Cosines at Intervals of 1′.

We have, by Art. 45,

$$\sin(x+y)=2\sin x\cos y-\sin(x-y),$$
$$\cos(x+y)=2\cos x\cos y-\cos(x-y).$$

Suppose the angles to increase by 1′; putting $y=1'$, we have,

$$\sin(x+1')=2\sin x\cos 1'-\sin(x-1') \quad . \quad . \quad . \quad . \quad (1)$$
$$\cos(x+1')=2\cos x\cos 1'-\cos(x-1') \quad . \quad . \quad . \quad . \quad (2)$$

Putting $x=1', 2', 3', 4'$, etc., in (1) and (2), we get for the sines

$$\sin 2'=2\sin 1'\cos 1'-\sin 0'=.0005817764,$$
$$\sin 3'=2\sin 2'\cos 1'-\sin 1'=.0008726646,$$
$$\sin 4'=2\sin 3'\cos 1'-\sin 2'=.0011635526;$$

and for the cosines

$$\cos 2'=2\cos 1'\cos 1'-\cos 0'=.9999998308,$$
$$\cos 3'=2\cos 2'\cos 1'-\cos 1'=.9999996193,$$
$$\cos 4'=2\cos 3'\cos 1'-\cos 2'=.9999993223.$$

We can proceed in this manner* until we find the values of the sines and cosines of all angles at intervals of 1′ from 0° to 30°.

137. Another Method.

Let α denote any angle. Then, in the identity,

$$\sin(n+1)\alpha=2\sin n\alpha\cos\alpha-\sin(n-1)\alpha,$$

put $\quad 2(1-\cos\alpha)=k,$

and we get

$$\sin(n+1)\alpha-\sin n\alpha=\sin n\alpha-\sin(n-1)\alpha-k\sin n\alpha \quad . \quad (1)$$

* This method is due to Thomas Simpson, an English geometrician.

This formula enables us to construct a table of *sines* of angles whose common difference is α.

Thus, suppose $\alpha = 10''$, and let $n = 1, 2, 3, 4$, etc.

Then

$$\sin 20'' - \sin 10'' = \sin 10'' - k \sin 10'',$$

$$\sin 30'' - \sin 20'' = \sin 20'' - \sin 10'' - k \sin 20'',$$

$$\sin 40'' - \sin 30'' = \sin 30'' - \sin 20'' - k \sin 30'', \text{ etc.}$$

These equations give in succession $\sin 20''$, $\sin 30''$, etc. It will be seen that the most laborious part of this work is the multiplication of k by the sines of $10''$, $20''$, etc., as they are successively found. But from the value of $\cos 10''$, we have

$$k = 2(1 - \cos 10'') = .0000000023504,$$

the smallness of which facilitates the process.

In the same manner a table of *cosines* can be constructed by means of the formula,

$$\cos(n+1)\alpha - \cos n\alpha = \cos n\alpha - \cos(n-1)\alpha - k \cos n\alpha,$$

which is obtained from the identity,

$$\cos(n+1)\alpha = 2 \cos n\alpha \cos \alpha - \cos(n-1)\alpha,$$

by putting $2(1 - \cos \alpha) = k$, as before.

138. The Sines and Cosines from 30° to 60°. — It is not necessary to calculate in this way the sines and cosines of angles beyond 30°, as we can obtain their values for angles from 30° to 60° more easily by means of the formulæ (Art. 45):

$$\sin(30° + \alpha) = \cos \alpha - \sin(30° - \alpha),$$

$$\cos(30° + \alpha) = \cos(30° - \alpha) - \sin \alpha,$$

by giving α all values up to 30°. Thus,

$$\sin 30° 1' = \cos 1' - \sin 29° 59',$$

$$\cos 30° 1' = \cos 29° 59' - \sin 1', \text{ and so on.}$$

139. Sines of Angles greater than 45°. — When the sines of angles up to 45° have been calculated, those of angles between 45° and 90° may be deduced by the formula

$$\sin(45° + \alpha) - \sin(45° - \alpha) = \sqrt{2}\sin\alpha \quad . \quad \text{(Art. 45)}$$

Also, when the sines of angles up to 60° have been found, the remainder up to 90° can be found still more easily from the formula

$$\sin(60° + \alpha) - \sin(60° - \alpha) = \sin\alpha.$$

Having completed a table of sines, the cosines are known, since

$$\cos\alpha = \sin(90° - \alpha).$$

Otherwise thus: When the sines and cosines of the angles up to 45° have been obtained, those of angles between 45° and 90° are obtained from the fact that the sine of an angle is equal to the cosine of its complement, so that it is not necessary to proceed in the calculation beyond 45°.

NOTE. — A more modern method of calculating the sines and cosines of angles is to use series (3) and (4) of Art. 156.

140. Tables of Tangents and Secants. — To form a table of tangents, we find the tangents of angles up to 45°, from the tables of sines and cosines, by means of the formula

$$\tan\alpha = \frac{\sin\alpha}{\cos\alpha}.$$

Then the tangents of angles from 45° to 90° may be obtained by means of the identity *

$$\tan(45° + \alpha) = \tan(45° - \alpha) + 2\tan 2\alpha.$$

When the tangents have been found, the cotangents are known, since the cotangent of any angle is equal to the tangent of its complement.

A table of cosecants may be obtained by calculating the reciprocals of the sines; or they may be obtained more

* Called Cagnoli's formula.

easily from the tables of the tangents by means of the formula

$$\operatorname{cosec} \alpha = \tan \frac{\alpha}{2} + \cot \alpha.$$

The secants are then known, since the secant of any angle is equal to the cosecant of its complement.

141. Formulæ of Verification. — *Formulæ used to test the accuracy of the calculated sines or cosines of angles are called Formulæ of Verification.*

It is necessary to have methods of verifying from time to time the correctness of the values of the sines and cosines of angles calculated by the preceding method, since any error made in obtaining the value of one of the functions would be repeated to the end of the work. For this purpose we may compare the value of the sine of any angle obtained by the preceding method with its value obtained independently.

Thus, for example, we know that $\sin 18° = \frac{\sqrt{5}-1}{4}$ (Art. 57); hence the sine of 18° may be calculated to any degree of approximation, and by comparison with the value obtained in the tables, we can judge how far we can rely upon the tables.

Similarly, we may compare our results for the angles $22\frac{1}{2}°$, 30°, 36°, 45°, etc., calculated by the preceding method with the sines and cosines of the same angles as obtained in Arts. 26, 27, 56, 57, 58, etc.

There are, however, certain well-known formulæ of verification which can be used to verify any part of the calculated tables; these are

Euler's Formulæ:

$$\sin(36° + A) - \sin(36° - A) + \sin(72° - A) - \sin(72° + A) = \sin A.$$

$$\cos(36° + A) + \cos(36° - A) - \cos(72° + A) - \cos(72° - A) = \cos A.$$

Legendre's Formula:

$$\sin(54° + A) + \sin(54° - A) - \sin(18° + A) - \sin(18° - A) = \cos A.$$

The verification consists in giving to A any value, and taking from the tables the sines and cosines of the angles involved: these values must satisfy the above equations.

To prove Euler's Formulæ:

$$\sin(36° + A) - \sin(36° - A) = 2\cos 36° \sin A \quad . \quad \text{(Art. 45)}$$
$$= \frac{\sqrt{5}+1}{2}\sin A \quad . \quad \text{(Art. 58)}$$
$$\sin(72° + A) - \sin(72° - A) = 2\cos 72° \sin A \quad . \quad \text{(Art. 45)}$$
$$= \frac{\sqrt{5}-1}{2}\sin A \quad . \quad \text{(Art. 57)}$$

Subtracting the latter from the former, we get sin A.

Similarly, Euler's second formula may be proved.

By substituting 90° – A for A in this formula we obtain Legendre's Formula.

142. Tables of Logarithmic Trigonometric Functions. — To save the trouble of referring twice to tables — first to the table of *natural functions* for the value of the function, and then to a table of *logarithms* for the logarithm of that function — it is convenient to calculate the logarithms of trigonometric functions, and arrange them in tables, called tables of logarithmic sines, cosines, etc.

When tables of natural sines and cosines have been constructed, tables of logarithmic sines and cosines may be made by means of tables of ordinary logarithms, which will give the logarithm of the calculated numerical value of the sine or cosine of any angle; adding 10 to the logarithm so found we have the corresponding tabular logarithm. The logarithmic tangents may be found by the relation

$$\log\tan A = 10 + \log\sin A - \log\cos A;$$

and thus a table of logarithmic tangents may be constructed.

PROPORTIONAL PARTS.

143. The Principle of Proportional Parts. — It is often necessary to find from a table of logarithms, the logarithm of a number containing more digits than are given in the table. In order to do this, we assumed, in Chapter IV., *the principle of proportional parts,* which is as follows:

The differences between three numbers are proportional to the corresponding differences between their logarithms, provided the differences between the numbers are small compared with the numbers.

By means of this principle, we are enabled to use tables of a more moderate size than would otherwise be necessary.

We shall now investigate how far, and with what exceptions, the principle or *rule* of proportional increase is true.

144. To prove the Rule for the Table of Common Logarithms.

We have

$$\log(n+d) - \log n = \log\frac{n+d}{n} = \log\left(1+\frac{d}{n}\right)$$

$$= \mu\left(\frac{d}{n} - \frac{d^2}{2n^2} + \frac{d^3}{3n^3} - \cdots\right) \quad . \quad \text{(Art. 130)}$$

where $\mu = .43429448 \cdots$, a quantity $< \frac{1}{2}$.

Now let n be an integer not < 10000, and d not > 1;

then $\frac{d}{n}$ is not greater than .0001.

$\therefore \frac{\mu d^2}{2n^2}$ is not $> \frac{1}{4}(.0001)^2$, *i.e.*, not $> .0000000025$;

and $\frac{\mu d^3}{3n^3}$ is much less than this.

$\therefore \log(n+d) - \log n = \mu\frac{d}{n}$, correct *at least as far as seven decimal places.*

Hence if the number be changed from n to $n + d$, the corresponding change in the logarithm is approximately $\frac{\mu d}{n}$.

Therefore, *the change of the logarithm is approximately proportional to the change of the number.*

145. To prove the Rule for the Table of Natural Sines.

$$\sin(\theta + h) - \sin\theta = \sin h \cos\theta - \sin\theta(1 - \cos h)$$
$$= \sin h \cos\theta\left(1 - \tan\theta \tan\frac{h}{2}\right) \quad . \quad \text{(Art. 51)}$$

If h is the circular measure of a very small angle, $\sin h = h$ nearly, and $\tan\frac{h}{2} = \frac{h}{2}$ nearly.

$$\therefore \sin(\theta + h) - \sin\theta = h\cos\theta\left(1 - \tan\theta\tan\frac{h}{2}\right)$$
$$= h\cos\theta - \frac{h^2}{2}\sin\theta.$$

If h is the circular measure of an angle not $> 1'$, then h is not $> .0003$ (Art. 135). $\therefore \frac{h^2}{2}$ is not $> .00000005$; and $\sin\theta$ is not > 1.

$\therefore \sin(\theta+h) - \sin\theta = h\cos\theta$, as far as seven decimal places, which proves the proposition.

Similarly, $\sin(\theta - h) - \sin\theta = -h\cos\theta$, approximately.

146. To prove the Rule for a Table of Natural Cosines.

$$\cos(\theta - h) - \cos\theta = \sin h \sin\theta - \cos\theta(1 - \cos h)$$
$$= \sin h \sin\theta\left(1 - \cot\theta\tan\frac{h}{2}\right).$$

If h is the circular measure of a very small angle, $\sin h = h$ nearly, and $\tan\frac{h}{2} = \frac{h}{2}$ nearly.

$$\therefore \cos(\theta - h) - \cos\theta = h\sin\theta\left(1 - \cot\theta\tan\frac{h}{2}\right)$$
$$= h\sin\theta - \frac{h^2}{2}\cos\theta.$$

We may prove, as in Art. 145, that

$$\frac{h^2}{2}\cos\theta \text{ is not } > .00000005.$$

$\therefore \cos(\theta - h) - \cos\theta = h\sin\theta$, as far as seven decimal places, which proves the proposition.

Similarly, $\cos(\theta + h) - \cos\theta = -h\sin\theta$, approximately.

147. To prove the Rule for a Table of Natural Tangents.

$$\tan(\theta + h) - \tan\theta = \frac{\sin(\theta + h)}{\cos(\theta + h)} - \frac{\sin\theta}{\cos\theta} = \frac{\sin h}{\cos(\theta + h)\cos\theta}$$

$$= \frac{\tan h}{\cos^2\theta\,(1 - \tan\theta\tan h)}.$$

If h is the circular measure of a very small angle, $\tan h = h$ nearly.

$$\therefore \tan(\theta + h) - \tan\theta = \frac{h\sec^2\theta}{1 - h\tan\theta}$$

$$= h\sec^2\theta + h^2\sin\theta\sec^3\theta.$$

$$\therefore \tan(\theta + h) - \tan\theta = h\sec^2\theta, \text{ approximately,}$$

unless $\sin\theta\sec^3\theta$ is large, which proves the proposition.

Similarly, $\cot(\theta - h) - \cot\theta = h\operatorname{cosec}^2\theta$, approximately.

Sch. 1. If h is the circular measure of an angle not $> 1'$, then h is not $> .0003$. Hence the greatest value of $h^2\sin\theta\sec^3\theta$ is not $> .00000009\sin\theta\sec^3\theta$. Therefore, when $\theta > \frac{\pi}{4}$, we are liable to an error in the seventh place of decimals. Hence the rule is not true for tables of tangents calculated for every minute, when the angle is between 45° and 90°.

Sch. 2. Since the cotangent of an angle is equal to the tangent of its complement, it follows immediately that the rule must not be used for a table of cotangents, calculated for every minute, when the angle lies between 0° and 45°.

148. To prove the Rule for a Table of Logarithmic Sines.

$$\sin(\theta+h)-\sin\theta = h\cos\theta - \frac{h^2}{2}\sin\theta \quad . \quad . \quad \text{(Art. 145)}$$

$$\therefore \ \frac{\sin(\theta+h)}{\sin\theta} = 1 + h\cot\theta - \frac{h^2}{2}.$$

$\therefore$ $\log\sin(\theta+h) - \log\sin\theta$

$$= \mu\log\left(1 + h\cot\theta - \frac{h^2}{2}\right)$$

$$= \mu\left[h\cot\theta - \frac{h^2}{2} - \frac{1}{2}\left(h\cot\theta - \frac{h^2}{2}\right)^2 + \cdots\right] \quad \text{(Art. 130)}$$

$$= \mu h\cot\theta - \frac{\mu h^2}{2}(1+\cot^2\theta) + \cdots$$

$$= \mu h\cot\theta - \frac{\mu h^2}{2}\operatorname{cosec}^2\theta + \cdots$$

If h is the circular measure of an angle not $> 10''$, then h is not $> .00005$, and therefore, unless $\cot\theta$ is small or $\operatorname{cosec}^2\theta$ large, we have

$$\log\sin(\theta+h) - \log\sin\theta = \mu h\cot\theta,$$

as far as seven decimal places, which proves the rule to be *generally* true.

Sch. 1. When θ is small, $\operatorname{cosec}\theta$ is large. If the log sines are calculated to every $10''$, then h is not $>.00005$, and μ is not $> .5$.

$$\therefore \ \tfrac{1}{2}\mu h^2\operatorname{cosec}^2\theta \text{ is not } > \frac{6\operatorname{cosec}^2\theta}{10^{10}}.$$

In order that this error may not affect the *seventh* decimal place, $6\operatorname{cosec}^2\theta$ must not be $> 10^3$, that is, θ must not be less than about $5°$.

When θ is small, $\cot\theta$ is large. Hence, when the angles are *small*, the differences of consecutive log sines are *irregular*, and they are *not* insensible. Therefore the rule does not apply to the log sine when the angle is less than 5°.

Sch. 2. When θ is nearly a right angle, $\cot\theta$ is small, and $\operatorname{cosec}\theta$ approaches unity.

Hence, when the angles are nearly right angles, the differences of consecutive log sines are irregular and nearly insensible.

149. To prove the Rule for a Table of Logarithmic Cosines.

$$\cos(\theta-h)-\cos\theta = h\sin\theta - \frac{h^2}{2}\cos\theta. \quad \text{(Art. 146)}$$

$$\therefore\ \frac{\cos(\theta-h)}{\cos\theta} = 1 + h\tan\theta - \frac{h^2}{2}.$$

$$\therefore\ \log\cos(\theta-h) - \log\cos\theta$$

$$= \mu\log\left(1 + h\tan\theta - \frac{h^2}{2}\right)$$

$$= \mu\left[h\tan\theta - \frac{h^2}{2} - \frac{1}{2}\left(h\tan\theta - \frac{h^2}{2}\right)^2\right]$$

$$= \mu h\tan\theta - \frac{\mu h^2}{2}\sec^2\theta + \cdots$$

In this case the differences will be irregular and large when θ is nearly a right angle, and irregular and insensible when θ is nearly zero. This is also clear because the sine of an angle is the cosine of its complement.

150. To prove the Rule for a Table of Logarithmic Tangents.

$$\tan(\theta+h) - \tan\theta = h\sec^2\theta + h^2\sin\theta\sec^3\theta\ . \quad \text{(Art. 147)}$$

$$\therefore\ \frac{\tan(\theta+h)}{\tan\theta} = 1 + \frac{h}{\sin\theta\cos\theta} + h^2\sec^2\theta.$$

$\therefore$ $\log\tan(\theta+h)-\log\tan\theta$

$$=\mu\left[\frac{h}{\sin\theta\cos\theta}+h^2\sec^2\theta-\frac{1}{2}\left(\frac{h}{\sin\theta\cos\theta}+h^2\sec^2\theta\right)^2+\cdots\right]$$

$$=\frac{\mu h}{\sin\theta\cos\theta}+\mu h^2\left(\sec^2\theta-\frac{1}{2\sin^2\theta\cos^2\theta}\right)\cdots$$

$\therefore$ $\log\tan(\theta+h)-\log\tan\theta$

$$=\frac{\mu h}{\sin\theta\cos\theta}-2\mu h^2\frac{\cos 2\theta}{\sin^2 2\theta}.$$

151. Cases where the Principle of Proportional Parts is Inapplicable.

It appears from the last six Articles that if h is small enough, the differences are proportional to h, *for values of θ which are neither very small nor nearly equal to a right angle.*

The following exceptional cases arise:

(1) The difference $\sin(\theta+h)-\sin\theta$ is *insensible* when θ is *nearly* 90°, for in that case $h\cos\theta$ is *very small;* it is then also *irregular,* for $\frac{1}{2}h^2\sin\theta$ may become comparable with $h\cos\theta$.

(2) The difference $\cos(\theta+h)-\cos\theta$ is both *insensible* and *irregular* when θ is *small.*

(3) The difference $\tan(\theta+h)-\tan\theta$ is *irregular* when θ is *nearly* 90°, for $h^2\sin\theta\sec^3\theta$ may then become comparable with $h\sec^2\theta$; it is never *insensible,* since $\sec\theta$ is not <1.

(4) The difference $\log\sin(\theta+h)-\log\sin\theta$ is *irregular* when θ is *small,* and both *irregular* and *insensible* when θ is *nearly* 90°.

(5) The difference $\log\cos(\theta+h)-\log\cos\theta$ is *insensible* and *irregular* when θ is *small,* and *irregular* when θ is *nearly* 90°.

(6) The difference $\log\tan(\theta+h)-\log\tan\theta$ is *irregular* when θ is either *small* or *nearly* 90°.

A difference which is *insensible* is also *irregular;* but the converse does not hold.

When the differences for a function are *insensible* to the number of decimal places of the tables, the tables will give the *functions* when the *angle* is known, but we cannot use the tables to find any intermediate *angle* by means of this *function;* thus, we cannot determine θ from the value log cos θ, for small angles, or from the value log sin θ, for angles nearly 90°.

When the differences for a function are *irregular without being insensible*, the approximate method of proportional parts is not sufficient for the determination of the angle by means of the function, nor the function by means of the angle; thus, the approximation is inadmissible for log sin θ, when θ is small, for log cos θ, when θ is nearly 90°, and for log tan θ in either case. (Compare Art. 81.)

In these cases of *irregularity without insensibility*, the following three means may be used to effect the purpose of finding the angle corresponding to a given value of the function, or of the function corresponding to a given angle.*

152. Three Methods to replace the Rule of Proportional Parts.

(1) The simplest plan is to have tables of log sines and log tangents, for *each second*, for the first few degrees of the quadrant, and of log cosines and log cotangents, for each second, for the few degrees near 90°. Such tables are generally given in trigonometric tables of seven places; we can then use the principle of proportional parts for all angles which are not extremely near 0° or 90°.

(2) *Delambre's Method.* In this method a table is constructed which gives the value of $\log\dfrac{\sin\theta}{\theta} + \log\sin 1''$ for every second for the first few degrees of the quadrant.

* This article has been taken substantially from Hobson's Trigonometry.

Let θ be the circular measure of n seconds. Then, when θ is small, we have $\theta = n \sin 1''$, approximately.

$$\therefore \log \frac{\sin \theta}{\theta} = \log \frac{\sin n''}{n \sin 1''} = \log \sin n'' - \log n - \log \sin 1''.$$

$$\therefore \log \sin n'' = \log n + \left(\log \frac{\sin \theta}{\theta} + \log \sin 1''\right).$$

Hence, if the angle is known, the table gives the value of the expression in parenthesis, and $\log n$ can be found from the ordinary table of the logs of numbers; thus log $\sin n''$ can be found.

If $\log \sin n''$ is given, we can find *approximately* the value of n, and then from the table we have the value of the expression in parenthesis; thus we can find $\log n$, and then n from an ordinary table of logs of numbers.

Rem. When θ is small (less than 5°),

$$\frac{\sin \theta}{\theta} = 1 - \frac{\theta^2}{6}, \text{ approximately} \quad . \quad . \quad \text{(Art. 134, Note)}$$

Hence a small error in θ will not produce a sensible error in the result, since $\log \dfrac{\sin \theta}{\theta}$ will vary much less rapidly than θ.

(3) *Maskelyne's Method.* The principle of this method is the same as that of Delambre's. If θ is a small angle, we have

$$\sin \theta = \theta - \frac{\theta^3}{6}, \text{ approximately,}$$

and

$$\cos \theta = 1 - \frac{\theta^2}{2}, \quad \text{"} \quad . \quad . \quad \text{(Art. 134)}$$

$$\therefore \frac{\sin \theta}{\theta} = \left(1 - \frac{\theta^2}{2}\right)^{\frac{1}{3}}, \quad \text{"}$$

$$= (\cos \theta)^{\frac{1}{3}}, \quad \text{"}$$

$$\therefore \log \sin \theta = \log \theta + \tfrac{1}{3} \log \cos \theta, \text{ approximately.}$$

When θ is a small angle, the differences of $\log\cos\theta$ are insensible (Art. 149); hence, if θ be given, we can find $\log\theta$ accurately from the table of natural logarithms, and also an approximate value of $\log\cos\theta$; the formula then gives $\log\sin\theta$ at once.

If $\log\sin\theta$ be given, we must first find an approximate value of θ from the table, and use that for finding $\log\cos\theta$, approximately; θ is then obtained from the formula.

EXAMPLES.

1. Prove $1 + 2 + \frac{3}{\lfloor 2} + \frac{4}{\lfloor 3} + \cdots = 2e.$

2. Prove $\log\frac{3}{2} = \frac{1}{2} - \left(\frac{2}{1\cdot 3\cdot 2^3} + \frac{3}{2\cdot 5\cdot 2^5} + \cdots\right).$

3. Prove $$\frac{e}{2} = \frac{1}{\lfloor 2} + \frac{1+2}{\lfloor 3} + \frac{1+2+3}{\lfloor 4} + \cdots$$

4. Prove $$\frac{1}{e} = \frac{2}{\lfloor 3} + \frac{4}{\lfloor 5} + \frac{6}{\lfloor 7} + \cdots.$$

5. Prove $\tan\theta + \frac{1}{3}\tan^3\theta + \frac{1}{5}\tan^5\theta + \cdots$
$$= \tfrac{1}{2}\log\left(\frac{\cos\left(\theta - \frac{\pi}{4}\right)}{\cos\left(\theta + \frac{\pi}{4}\right)}\right).$$

6. Prove $\log_e 11 = 2.39789527\cdots$, by (4) of Art. 130.

7. Prove $\log_e 13 = 2.56494935\cdots$, " "

8. Prove $\log_e 17 = 2.83321334\cdots$, " "

9. Prove $\log_e 19 = 2.9444394\ \cdots$, " "

10. Find, by means of the table of common logarithms and the modulus, the Napierian logarithms of 1325.07, 52.9381, and .085623. *Ans.* 7.18923, 3.96913, − 2.4578.

11. Prove that the limit of $m \sin \frac{\theta}{m}$ is θ, when $m = \infty$.

12. " " " $m \tan \frac{\theta}{m}$ is θ, " $m = \infty$.

13. " " " $\frac{nr^2}{2} \sin \frac{2\pi}{n}$ is πr^2, " $n = \infty$.

14. " " " $\pi r^2 \tan \frac{\pi}{n}$ is πr^2, " $n = \infty$.

15. " " " $\frac{\text{vers}\, a\theta}{\text{vers}\, b\theta}$ is $\frac{a^2}{b^2}$, " $\theta = 0$.

16. " " " $\left(\cos \frac{\theta}{n}\right)^r$ is 1, " $n = \infty$.

17. " " " $\left(\sin \frac{\theta}{n}\right)^r$ is 1, " $n = \infty$.

18. " " " $\left(\cos \frac{\theta}{n}\right)^n$ is 1, " $n = \infty$.

19. " " " $\left(\frac{\sin \frac{\theta}{n}}{\frac{\theta}{n}}\right)^n$ is 1, " $n = \infty$.

20. " " " $\left(\cos \frac{\theta}{n}\right)^{n^2}$ is $e^{-\frac{\theta^2}{2}}$, " $n = \infty$.

21. " " " $\left(\cos \frac{\theta}{n}\right)^{n^3}$ is zero, when $n = \infty$.

22. If θ is the circular measure of an acute angle, prove (1) $\cos \theta < 1 - \frac{\theta^2}{2} + \frac{\theta^4}{24}$, and (2) $\tan \theta > \theta + \frac{\theta^3}{3}$.

23. Given $\frac{\sin \theta}{\theta} = \frac{1013}{1014}$: prove that $\theta = 4° 24'$, nearly.

24. Given $\frac{\sin \theta}{\theta} = \frac{2165}{2166}$: prove that $\theta = 3°$, nearly.

25. Given $\sin\phi = n\sin\theta$, $\tan\phi = 2\tan\theta$: find the limiting values of n that these equations may coexist.

Ans. n must lie between 1 and 2, or between -1 and -2.

26. Find the limit of $(\cos ax)^{\operatorname{cosec}^2 bx}$, when $x = 0$.

Ans. $e^{-\frac{a^2}{2b^2}}$.

27. From a table of natural tangents of seven decimal places, show that when an angle is near 60° it may be determined within about $\frac{1}{200}$ of a second.

28. When an angle is very near 64° 36′, show that the angle can be determined from its log sine within about $\frac{1}{10}$ of a second; having given $(\log_e 10)\tan 64° 36' = 4.8492$, and the tables reading to seven decimal places.

CHAPTER IX.

DE MOIVRE'S THEOREM.* — APPLICATIONS.

153. De Moivre's Theorem. — *For any value of n, positive or negative, integral or fractional.*

$$(\cos\theta + \sqrt{-1}\sin\theta)^n = \cos n\theta + \sqrt{-1}\sin n\theta \quad . \quad . \quad (1)$$

I. *When n is a positive integer.*

We have the product

$$(\cos\alpha + \sqrt{-1}\sin\alpha)(\cos\beta + \sqrt{-1}\sin\beta)$$
$$= (\cos\alpha\cos\beta - \sin\alpha\sin\beta) + \sqrt{-1}(\cos\alpha\sin\beta + \sin\alpha\cos\beta)$$
$$= \cos(\alpha+\beta) + \sqrt{-1}\sin(\alpha+\beta).$$

Similarly, the product

$$[\cos(\alpha+\beta) + \sqrt{-1}\sin(\alpha+\beta)][\cos\gamma + \sqrt{-1}\sin\gamma]$$
$$= \cos(\alpha+\beta+\gamma) + \sqrt{-1}\sin(\alpha+\beta+\gamma).$$

Proceeding in this way we find that the product of any number n of factors, each of the form

$$\cos\alpha + \sqrt{-1}\sin\alpha = \cos(\alpha+\beta+\gamma+\cdots n \text{ terms})$$
$$+ \sqrt{-1}\sin(\alpha+\beta+\gamma+\cdots n \text{ terms}).$$

Suppose now that $\alpha = \beta = \gamma = \text{etc.} = \theta$, then we have

$$(\cos\theta + \sqrt{-1}\sin\theta)^n = \cos n\theta + \sqrt{-1}\sin n\theta,$$

which proves the theorem when n is a positive integer.

* From the name of the French geometer who discovered it.

II. *When n is a negative integer.*

Let $n=-m$; then m is a positive integer. Then

$$(\cos\theta+\sqrt{-1}\sin\theta)^n=(\cos\theta+\sqrt{-1}\sin\theta)^{-m}$$

$$=\frac{1}{(\cos\theta+\sqrt{-1}\sin\theta)^m}=\frac{1}{\cos m\theta+\sqrt{-1}\sin m\theta}\ \text{(by I.)}$$

$$=\frac{1}{\cos m\theta+\sqrt{-1}\sin m\theta}\times\frac{\cos m\theta-\sqrt{-1}\sin m\theta}{\cos m\theta-\sqrt{-1}\sin m\theta}$$

$$=\frac{\cos m\theta-\sqrt{-1}\sin m\theta}{\cos^2 m\theta+\sin^2 m\theta}=\cos m\theta-\sqrt{-1}\sin m\theta$$

$$=\cos(-m\theta)+\sqrt{-1}\sin(-m\theta).$$

$$\therefore\ (\cos\theta+\sqrt{-1}\sin\theta)^n=\cos n\theta+\sqrt{-1}\sin n\theta,$$

which proves the theorem when n is a negative integer.

III. *When n is a fraction, positive or negative.*

Let $n=\frac{p}{q}$, where p and q are integers. Then

$$(\cos\theta+\sqrt{-1}\sin\theta)^p=\cos p\theta+\sqrt{-1}\sin p\theta\ \text{(by I. and II.)}.$$

But $\left(\cos\frac{p}{q}\theta+\sqrt{-1}\sin\frac{p}{q}\theta\right)^q=\cos p\theta+\sqrt{-1}\sin p\theta.$

$$\therefore\ (\cos\theta+\sqrt{-1}\sin\theta)^p=\left(\cos\frac{p}{q}\theta+\sqrt{-1}\sin\frac{p}{q}\theta\right)^q.$$

$$\therefore\ (\cos\theta+\sqrt{-1}\sin\theta)^{\frac{p}{q}}=\cos\frac{p}{q}\theta+\sqrt{-1}\sin\frac{p}{q}\theta;$$

that is, one of the values of $(\cos\theta+\sqrt{-1}\sin\theta)^{\frac{p}{q}}$

is $$\cos\frac{p\theta}{q}+\sqrt{-1}\sin\frac{p\theta}{q}.$$

In like manner,

$$(\cos\theta-\sqrt{-1}\sin\theta)^n=\cos n\theta-\sqrt{-1}\sin n\theta.$$

Thus, De Moivre's Theorem is completely established. It shows that to raise the binomial $\cos\theta+\sqrt{-1}\sin\theta$ to any power, we have only to multiply the arc θ by the exponent of the power. This theorem is a fundamental one in Analytic Mathematics.

154. To find All the Values of $(\cos\theta+\sqrt{-1}\sin\theta)^{\frac{p}{q}}$. — When n is an integer, the expression $(\cos\theta+\sqrt{-1}\sin\theta)^n$ can have only one value. But if n is a fraction $=\frac{p}{q}$, the expression becomes

$$(\cos\theta+\sqrt{-1}\sin\theta)^{\frac{p}{q}}=\sqrt[q]{(\cos\theta+\sqrt{-1}\sin\theta)^p},$$

which has q different values, from the principle of Algebra (Art. 235). In III. of Art. 153, we found *one* of the values of $(\cos\theta+\sqrt{-1}\sin\theta)^{\frac{p}{q}}$; we shall now find an expression which will give *all* the q values of $(\cos\theta+\sqrt{-1}\sin\theta)^{\frac{p}{q}}$.

Now both $\cos\theta$ and $\sin\theta$ remain unchanged when θ is increased by any multiple of 2π; that is, the expression $\cos\theta+\sqrt{-1}\sin\theta$ is unaltered if for θ we put $(\theta+2r\pi)$, where r is an integer (Art. 36).

$$\therefore\ (\cos\theta+\sqrt{-1}\sin\theta)^{\frac{p}{q}}$$
$$=[\cos(\theta+2r\pi)+\sqrt{-1}\sin(\theta+2r\pi)]^{\frac{p}{q}}$$
$$=\cos\frac{p(\theta+2r\pi)}{q}+\sqrt{-1}\sin\frac{p(\theta+2r\pi)}{q}\quad \text{(Art. 153)} \quad (1)$$

The second member of (1) has q different values, and no more; these q values are found by putting $r=0, 1, 2, \cdots q-1$, successively, by which we obtain the following series of angles.

When $r=0$, $\quad \cos\frac{p(\theta+2r\pi)}{q}=\cos\frac{p\theta}{q}$.

" $\quad r=1$, $\quad$ " $\quad =\cos\frac{p(\theta+2\pi)}{q}$.

" $\quad r=2$, $\quad$ " $\quad =\cos\frac{p(\theta+4\pi)}{q}$,

etc. etc.

When $r=q-1$, $\cos\frac{p(\theta+2r\pi)}{q}=\cos\frac{p[\theta+2\pi(q-1)]}{q}$

$$=\cos\frac{p(\theta+2q\pi-2\pi)}{q}.$$

All these q values are different.

When $r=q$, $\cos\frac{p(\theta+2r\pi)}{q}=\cos\frac{p(\theta+2q\pi)}{q}$

$$=\cos\left(\frac{p\theta}{q}+2p\pi\right)=\cos\frac{p\theta}{q},$$

the same value as when $r=0$.

When $r=q+1$, $\cos\frac{p(\theta+2r\pi)}{q}=\cos\frac{[\theta+(2q+2)\pi]}{q}$

$$=\cos\frac{p(\theta+2\pi)}{q},$$

the same as when $r=1$,

etc.,

from which it appears that there are q and *only* q different values of $\cos\frac{p(\theta+2r\pi)}{q}$, since the same values afterwards recur in the same order.

Similarly for $\sin\frac{p(\theta+2r\pi)}{q}$.

Therefore the expression

$$\cos\frac{p(\theta+2r\pi)}{q}+\sqrt{-1}\sin\frac{p(\theta+2r\pi)}{q}$$

gives *all* the q values of $(\cos\theta+\sqrt{-1}\sin\theta)^{\frac{p}{q}}$ and no more. And this agrees with the Theory of Equations that there must be q values of x, and no more, which satisfy the equation $x^q=c$, where c is either real or of the form $a+b\sqrt{-1}$.

APPLICATIONS OF DE MOIVRE'S THEOREM.

155. To develop $\cos n\theta$ and $\sin n\theta$ in Powers of $\sin\theta$ and $\cos\theta$.

We shall generally in this chapter write i for $\sqrt{-1}$ in accordance with the usual notation.

By De Moivre's Theorem (Art. 153) we have

$$\cos n\theta + i\sin n\theta = (\cos\theta + i\sin\theta)^n \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Let n be a positive integer. Expand the second member of (1) by the binomial theorem, remembering that $i^2 = -1$, $i^3 = -i$, and that $i^4 = +1$ (Algebra, Art. 219). Equate the real and imaginary parts of the two members. Thus,

$$\cos n\theta = \cos^n\theta - \frac{n(n-1)}{\underline{|2}}\cos^{n-2}\theta\sin^2\theta + \frac{n(n-1)(n-2)(n-3)}{\underline{|4}}\cos^{n-4}\theta\sin^4\theta - \text{etc.} \quad . \quad (2)$$

$$\sin n\theta = n\cos^{n-1}\theta\sin\theta - \frac{n(n-1)(n-2)}{\underline{|3}}\cos^{n-3}\theta\sin^3\theta + \frac{n(n-1)(n-2)(n-3)(n-4)}{\underline{|5}}\cos^{n-5}\theta\sin^5\theta - \text{etc.} \quad (3)$$

The last terms in the series for $\cos n\theta$ and for $\sin n\theta$ will be different according as n is even or odd.

The last term in the expansion of $(\cos\theta + i\sin\theta)^n$ is $i^n\sin^n\theta$; and the last term but one is $ni^{n-1}\cos\theta\sin^{n-1}\theta$. Therefore:

When *n is even*, the last term of $\cos n\theta$ is $i^n\sin^n\theta$ or $(-1)^{\frac{n}{2}}\sin^n\theta$, and the last term of $\sin n\theta$ is $ni^{n-2}\cos\theta\sin^{n-1}\theta$ or $n(-1)^{\frac{n-2}{2}}\cos\theta\sin^{n-1}\theta$.

When *n is odd*, the last term of $\cos n\theta$ is $ni^{n-1}\cos\theta\sin^{n-1}\theta$ or $n(-1)^{\frac{n-1}{2}}\cos\theta\sin^{n-1}\theta$, and the last term of $\sin n\theta$ is $i^{n-1}\sin^n\theta$ or $(-1)^{\frac{n-1}{2}}\sin^n\theta$.

EXAMPLES.

Prove the following statements:

1. $\sin 4\theta = 4\cos^3\theta \sin\theta - 4\cos\theta \sin^3\theta.$

2. $\cos 4\theta = \cos^4\theta - 6\cos^2\theta \sin^2\theta + \sin^4\theta.$

156. To develop $\sin\theta$ and $\cos\theta$ in Series of Powers of θ.

Put $n\theta = \alpha$ in (2) and (3) of Art. 155; and let n be increased without limit while α remains unchanged. Then since $\theta = \frac{\alpha}{n}$, θ must diminish without limit. Therefore the above formulæ may be written

$$\cos\alpha = \cos^n\theta - \frac{\alpha(\alpha-\theta)}{|\underline{2}}\cos^{n-2}\theta\left(\frac{\sin\theta}{\theta}\right)^2 + \frac{\alpha(\alpha-\theta)(\alpha-2\theta)(\alpha-3\theta)}{|\underline{4}}\cos^{n-4}\theta\left(\frac{\sin\theta}{\theta}\right)^4 - \cdots \quad (1)$$

and
$$\sin\alpha = \alpha\cos^{n-1}\theta\left(\frac{\sin\theta}{\theta}\right) - \frac{\alpha(\alpha-\theta)(\alpha-2\theta)}{|\underline{3}}\cos^{n-3}\theta\left(\frac{\sin\theta}{\theta}\right)^3 + \cdots \quad (2)$$

If $n = \infty$, then $\theta = 0$, and the limit of $\cos\theta$ and its powers is 1; also the limit of $\left(\frac{\sin\theta}{\theta}\right)$ and its powers is 1. Hence (1) and (2) become

$$\cos\alpha = 1 - \frac{\alpha^2}{2} + \frac{\alpha^4}{|\underline{4}} - \frac{\alpha^6}{|\underline{6}} + \cdots \quad (3)$$

$$\sin\alpha = \alpha - \frac{\alpha^3}{|\underline{3}} + \frac{\alpha^5}{|\underline{5}} - \cdots \quad (4)$$

Sch. In the series for $\sin\alpha$ and $\cos\alpha$, just found, α is *the circular* measure of the angle considered.

Cor. 1. If α be an angle so small that α^2 and higher powers of α may be neglected when compared with unity, (3) becomes $\cos\alpha = 1$, and (4), $\sin\alpha = \alpha$.

If α^2, α^3 be retained, but higher powers of α be neglected, (3) and (4) give

$$\sin\alpha = \alpha - \frac{\alpha^3}{6}; \quad \cos\alpha = 1 - \frac{\alpha^2}{2} \quad \text{(Compare Art. 134)}$$

Cor. 2. By dividing (3) by (4), we obtain

$$\tan\alpha = \alpha + \frac{\alpha^3}{3} + \frac{2\alpha^5}{3\cdot 5} + \frac{17\alpha^7}{3\cdot 3\cdot 5\cdot 7} + \text{etc.} \quad . \quad . \quad (5)$$

157. Convergence of the Series. — The series (3) and (4) of Art. 156 may be proved to be convergent, as follows:

The numerical value of the ratio of the successive pairs of consecutive terms in the series for $\sin\alpha$ are

$$\frac{\alpha^2}{2\cdot 3},\ \frac{\alpha^2}{4\cdot 5},\ \frac{\alpha^2}{6\cdot 7},\ \frac{\alpha^2}{8\cdot 9},\ \text{etc.}$$

Hence the ratio of the $(n+1)$th term to the nth term is $\frac{\alpha^2}{2n(2n+1)}$; and whatever be the value of α, we can take n so large that for such value of n and all greater values, this fraction can be made less than any assignable quantity; hence the series is convergent.

Similarly, it may be shown that the series for $\cos\alpha$ is always convergent.

158. Expansion of $\cos^n\theta$ in Terms of Cosines of Multiples of θ, when n is a Positive Integer.

Let $x = \cos\theta + i\sin\theta$;

then $$\frac{1}{x} = \frac{1}{\cos\theta + i\sin\theta} = \cos\theta - i\sin\theta.$$

$$\therefore\ x + \frac{1}{x} = 2\cos\theta;\ \text{and}\ x - \frac{1}{x} = 2i\sin\theta \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Also $x^n = (\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$ (Art. 153) (2)

and $\frac{1}{x^n} = (\cos\theta - i\sin\theta)^n = \cos n\theta - i\sin n\theta$ . . . (3)

$\therefore\ 2\cos n\theta = x^n + \frac{1}{x^n}$, and $2i\sin n\theta = x^n - \frac{1}{x^n}$. (4)

Hence $(2\cos\theta)^n = (x + x^{-1})^n$, by (1),

$$= x^n + nx^{n-2} + \frac{n(n-1)}{\lfloor 2}x^{n-4} + \text{etc.} + nx^{-(n-2)} + x^{-n}$$

$$= \left(x^n + \frac{1}{x^n}\right) + n\left(x^{n-2} + \frac{1}{x^{n-2}}\right) + \frac{n(n-1)}{\lfloor 2}\left(x^{n-4} + \frac{1}{x^{n-4}}\right) + \text{etc.}$$

$$= 2\cos n\theta + n\,2\cos(n-2)\theta + \frac{n(n-1)}{\lfloor 2}2\cos(n-4)\theta + \text{etc.}$$

$$\therefore\ 2^{n-1}\cos^n\theta = \cos n\theta + n\cos(n-2)\theta + \frac{n(n-1)}{\lfloor 2}\cos(n-4)\theta + \text{etc.} \quad (5)$$

NOTE.—In the expansion of $(x + x^{-1})^n$ there are $n+1$ terms; thus when n is even there is a *middle* term, the $\left(\frac{n}{2}+1\right)$th, which is independent of θ, and which is

$$\frac{n(n-1)\dots(n-\frac{1}{2}n+1)}{\lfloor \frac{1}{2}n}, \text{ i.e., } \frac{n(n-1)\dots\left(\frac{n}{2}+1\right)}{\lfloor \frac{1}{2}n}.$$

Hence when n is *even* the last term in the expansion of $2^{n-1}\cos^n\theta$ is

$$\frac{n(n-1)\dots\left(\frac{n}{2}+1\right)}{\lfloor \frac{1}{2}n}.$$

When n is *odd* the last term in the expansion of $2^{n-1}\cos^n\theta$ is

$$\frac{n(n-1)\dots\frac{1}{2}(n+3)}{\lfloor \frac{1}{2}(n-1)}\cos\theta.$$

159. Expansion of $\sin^n\theta$ in Terms of Cosines of Multiples of θ, when n is an Even Positive Integer.

$(2i\sin\theta)^n = \left(x - \frac{1}{x}\right)^n$ by (1) of Art. 158

$$= x^n - nx^{n-2} + \frac{n(n-1)}{\lfloor 2}x^{n-4} + \cdots + \frac{n(n-1)}{\lfloor 2}x^{-(n-4)} - nx^{-(n-2)} + x^{-n}$$

$$=\left(x^n+\frac{1}{x^n}\right)-n\left(x^{n-2}-\frac{1}{x^{n-2}}\right)+\frac{n(n-1)}{\lfloor 2}\left(x^{n-4}+\frac{1}{x^{n-4}}\right)$$

$$+\cdots+\frac{(-1)^{\frac{n}{2}}n(n-1)\cdots\left(\frac{n}{2}+1\right)}{\lfloor \frac{1}{2}n}.$$

$\therefore\ 2^{n-1}(-1)^{\frac{n}{2}}\sin^n\theta$

$$=\cos n\theta-n\cos(n-2)\theta+\frac{n(n-1)}{\lfloor 2}\cos(n-4)\theta$$

$$-\cdots+\frac{(-1)^{\frac{n}{2}}n(n-1)\cdots\left(\frac{n}{2}+1\right)}{2\lfloor \frac{1}{2}n}.$$

160. Expansion of $\sin^n\theta$ in Terms of Sines of Multiples of θ, when n is an Odd Positive Integer.

$$(2i\sin\theta)^n=\left(x-\frac{1}{x}\right)^n \text{by (1) of Art. 158}$$

$$=x^n-nx^{n-2}+\frac{n(n-1)}{\lfloor 2}x^{n-4}-\cdots$$

$$-\frac{n(n-1)}{\lfloor 2}x^{-(n-4)}+nx^{-(n-2)}-x^{-n}$$

$$=x^n-\frac{1}{x^n}-n\left(x^{n-2}-\frac{1}{x^{n-2}}\right)+\frac{n(n-1)}{\lfloor 2}\left(x^{n-4}-\frac{1}{x^{n-4}}\right)-\cdots$$

$$+\frac{(-1)^{\frac{n-1}{2}}n(n-1)\cdots\frac{1}{2}(n+3)}{\lfloor \frac{1}{2}(n-1)}\left(x-\frac{1}{x}\right).$$

$\therefore\ (2i\sin\theta)^n$

$$=2i\sin n\theta-n\,2i\sin(n-2)\theta$$

$$+\frac{n(n-1)}{\lfloor 2}2i\sin(n-4)\theta-\cdots$$

$$+\frac{(-1)^{\frac{n-1}{2}}n(n-1)\cdots\frac{1}{2}(n+3)}{\lfloor \frac{1}{2}(n-1)}2i\sin\theta \quad [(4) \text{ of Art. } 158]$$

Whence dividing by $2i$, we have

$$2^{n-1}(-1)^{\frac{n-1}{2}}\sin^n\theta$$
$$= \sin n\theta - n\sin(n-2)\theta + \frac{n(n-1)}{\lfloor 2}\sin(n-4)\theta + \cdots$$
$$+ \frac{(-1)^{\frac{n-1}{2}} n(n-1)\cdots\frac{1}{2}(n+3)}{\lfloor\frac{1}{2}(n-1)}\sin\theta.$$

EXAMPLES.

Prove that

1. $128\cos^8\theta = \cos 8\theta + 8\cos 6\theta + 28\cos 4\theta + 56\cos 2\theta + 35.$

2. $64\cos^7\theta = \cos 7\theta + 7\cos 5\theta + 21\cos 3\theta + 35\cos\theta.$

161. Exponential Values of Sine and Cosine.

Since $e^x = 1 + x + \frac{x^2}{\lfloor 2} + \frac{x^3}{\lfloor 3} + \frac{x^4}{\lfloor 4} + \cdots$ . . . (Art. 129)

$$\therefore\ e^{i\theta} = 1 - \frac{\theta^2}{\lfloor 2} + \frac{\theta^4}{\lfloor 4} - \text{etc.} + i\left(\theta - \frac{\theta^3}{\lfloor 3} + \frac{\theta^5}{\lfloor 5} - \text{etc.}\right)$$
$$= \cos\theta + i\sin\theta \quad . \ . \ . \ . \ . \ . \ . \ . \quad \text{(Art. 156)}$$

and
$$e^{-i\theta} = 1 - \frac{\theta^2}{\lfloor 2} + \frac{\theta^4}{\lfloor 4} - \text{etc.} - i\left(\theta - \frac{\theta^3}{\lfloor 3} + \frac{\theta^5}{\lfloor 5} - \text{etc.}\right)$$
$$= \cos\theta - i\sin\theta.$$

$$\therefore\ 2\cos\theta = e^{i\theta} + e^{-i\theta}, \text{ and } 2i\sin\theta = e^{i\theta} - e^{-i\theta} \quad . \ . \quad (1)$$

$$\therefore\ \cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}, \text{ and } \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i} \quad . \ . \ . \quad (2)$$

which are called the *exponential values** of the cosine and sine.

Cor. From these exponential values we may deduce similar values for the other trigonometric functions. Thus,

$$\tan\theta = \frac{e^{i\theta} - e^{-i\theta}}{i(e^{i\theta} + e^{-i\theta})} \quad . \ . \ . \ . \ . \quad (3)$$

* Called also *Euler's* equations, after Euler, their discoverer.

Sch. These results may be applied to prove any *general* formula in elementary Trigonometry, and are of great importance in the Higher Mathematics.

EXAMPLES.

1. Prove $\frac{\sin 2\theta}{1+\cos 2\theta} = \tan\theta.$

We have $$\frac{2i\sin 2\theta}{2+2\cos 2\theta} = \frac{e^{2i\theta} - e^{-2i\theta}}{2 + e^{2i\theta} + e^{-2i\theta}} \text{ by (1)}$$

$$= \frac{e^{i\theta} - e^{-i\theta}}{e^{i\theta} + e^{-i\theta}} = i\tan\theta \text{ by (3).} \quad \therefore \text{ etc.}$$

Prove the following, by the exponential values of the sine and cosine.

2. $\cos 2\alpha = \cos^2\alpha - \sin^2\alpha.$

3. $\sin\theta = -\sin(-\theta).$

4. $\cos 3\theta = 4\cos^3\theta - 3\cos\theta.$

Rem.—If we omit the i from the exponential values of the sine, cosine, and tangent of θ, the results are called respectively the hyperbolic sine, cosine, and tangent of θ, and are written $\sinh\theta$, $\cosh\theta$, and $\tanh\theta$, respectively. Thus we have

$$\sinh\theta = -i\sin i\theta,\ \cosh\theta = \cos i\theta,\ \tanh\theta = -i\tan i\theta.$$

Hyperbolic functions are so called, because they have geometric relations with the equilateral hyperbola analogous to those between the circular functions and the circle. A consideration of hyperbolic functions is clearly beyond the limits of this treatise.

For an excellent discussion of such functions, the student is referred to such works as Casey's Trigonometry, Hobson's Trigonometry, Lock's Higher Trigonometry, etc.

162. Gregory's Series. — *To expand θ in powers of $\tan\theta$ where θ lies between $-\frac{\pi}{2}$ and $+\frac{\pi}{2}$.*

By (3) of Art. 161, we have

$$i\tan\theta = \frac{e^{i\theta} - e^{-i\theta}}{e^{i\theta} + e^{-i\theta}}.$$

$$\therefore \frac{1 + i\tan\theta}{1 - i\tan\theta} = \frac{2e^{i\theta}}{2e^{-i\theta}} = e^{2i\theta}.$$

$\therefore \log e^{2i\theta} = \log(1 + i\tan\theta) - \log(1 - i\tan\theta).$

$\therefore 2i\theta = 2i(\tan\theta - \frac{1}{3}\tan^3\theta + \frac{1}{5}\tan^5\theta - \text{etc.})$ (Art. 130)

$$\therefore \theta = \tan\theta - \tfrac{1}{3}\tan^3\theta + \tfrac{1}{5}\tan^5\theta - \text{etc.} \quad . \quad . \quad . \quad . \quad (1)$$

which is *Gregory's Series.*

This series is convergent if $\tan\theta =$ or < 1, *i.e.*, if θ lies between $-\frac{\pi}{4}$ and $\frac{\pi}{4}$, or between $\frac{3}{4}\pi$ and $\frac{5}{4}\pi$.

Sch. This series may also be obtained by reverting (5) in Cor. 2, Art. 156.

Cor. 1. If $\tan\theta = x$, we have from (1)

$$\tan^{-1}x = x - \frac{x^3}{3} + \frac{x^5}{5} - \text{etc.} \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Cor. 2. If $\theta = \frac{\pi}{4}$, we have from (1)

$$\frac{\pi}{4} = 1 - \tfrac{1}{3} + \tfrac{1}{5} - \tfrac{1}{7} + \text{etc.} \quad . \quad . \quad . \quad . \quad . \quad (3)$$

a series which is very slowly convergent, so that a large number of terms would have to be taken to calculate π to a close approximation. We shall therefore show how series, which are more rapidly convergent, may be obtained from Gregory's series.

163. Euler's Series.

$$\tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{3} = \frac{\pi}{4} \quad . \quad . \quad . \quad \text{(by Ex. 2, Art. 60)}$$

Put $\theta = \tan^{-1}\frac{1}{2}$. $\therefore \tan\theta = \frac{1}{2}$, which in (1) of Art. 162 gives

$$\tan^{-1}\frac{1}{2} = \frac{1}{2} - \frac{1}{3\cdot 2^3} + \frac{1}{5\cdot 2^5} - \frac{1}{7\cdot 2^7} + \text{etc.} \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Put $\theta = \tan^{-1}\frac{1}{3}$. $\therefore \tan\theta = \frac{1}{3}$, and (1) becomes

$$\tan^{-1}\frac{1}{3} = \frac{1}{3} - \frac{1}{3\cdot 3^3} + \frac{1}{5\cdot 3^5} - \frac{1}{7\cdot 3^7} + \text{etc.} \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Adding (1) and (2) we have

$$\frac{\pi}{4}=\left(\frac{1}{2}-\frac{1}{3\cdot 2^3}+\frac{1}{5\cdot 2^5}-\cdots\right)+\left(\frac{1}{3}-\frac{1}{3\,.\,3^3}+\frac{1}{5\,.\,3^5}-\cdots\right) \quad (3)$$

a series which converges much more rapidly than (3) of Art. 162.

164. Machin's Series.

Since $2\tan^{-1}\frac{1}{5}=\tan^{-1}\frac{\frac{2}{5}}{1-\frac{1}{25}}$ (by Ex. 3, Art. 60) $=\tan^{-1}\frac{5}{12}$,

$$\therefore\ 4\tan^{-1}\frac{1}{5}=2\tan^{-1}\frac{5}{12}=\tan^{-1}\frac{\frac{5}{6}}{1-\frac{25}{144}}=\tan^{-1}\frac{120}{119}.$$

Also, $\tan^{-1}\frac{120}{119}-\tan^{-1}1=\tan^{-1}\frac{\frac{120}{119}-1}{1+\frac{120}{119}}=\tan^{-1}\frac{1}{239}.$

$$\therefore\ 4\tan^{-1}\frac{1}{5}-\frac{\pi}{4}=\tan^{-1}\frac{1}{239}.$$

$$\therefore\ \frac{\pi}{4}=4\tan^{-1}\frac{1}{5}-\tan^{-1}\frac{1}{239}.$$

$$\therefore\ \frac{\pi}{4}=4\left(\frac{1}{5}-\frac{1}{3\cdot 5^3}-\frac{1}{5\cdot 5^5}-\cdots\right)$$
$$-\left(\frac{1}{239}-\frac{1}{3\cdot(239)^3}+\frac{1}{5\,(239)^5}-\cdots\right).$$

In this way it is found that $\pi=3.141592653589793\cdots$.

Cor. Since $\tan^{-1}\frac{1}{99}+\tan^{-1}\frac{1}{239}=\tan^{-1}\frac{\frac{1}{99}+\frac{1}{239}}{1-\frac{1}{99}\times\frac{1}{239}}$

$$=\tan^{-1}\frac{1}{70},$$

$$\therefore\ \frac{\pi}{4}=4\tan^{-1}\frac{1}{5}-\tan^{-1}\frac{1}{70}+\tan^{-1}\frac{1}{99}.$$

NOTE. — The series for $\tan^{-1}\frac{1}{70}$ and $\tan^{-1}\frac{1}{99}$ are much more convenient for purposes of numerical calculation than the series for $\tan^{-1}\frac{1}{239}$.

Example. — Find the numerical value of π to 6 figures by Machin's series.

165. Given $\sin\theta = x\sin(\theta+\alpha)$; expand θ in a Series of Ascending Powers of x.

We have $e^{i\theta} - e^{-i\theta} = x\left[e^{i\theta+i\alpha} - e^{-i\theta-i\alpha}\right]$ . . (Art. 161)

$$\therefore\ e^{2i\theta} - 1 = x\left[e^{2i\theta}\cdot e^{i\alpha} - e^{-i\alpha}\right]$$

$$\therefore\ e^{2i\theta} = \frac{1 - xe^{-i\alpha}}{1 - xe^{i\alpha}}.$$

$$\therefore\ 2i\theta = \log(1 - xe^{-i\alpha}) - \log(1 - xe^{i\alpha})$$

$$= x(e^{i\alpha} - e^{-i\alpha}) + \frac{x^2}{2}(e^{2i\alpha} - e^{-2i\alpha}) + \frac{x^3}{3}(e^{3i\alpha} - e^{-3i\alpha})\cdots \quad \text{(Art. 130)}$$

$$\therefore\ \theta = x\sin\alpha + \frac{x^2}{2}\sin 2\alpha + \frac{x^3}{3}\sin 3\alpha + \cdots \quad \text{(Art. 161) (1)}$$

Example. If $\alpha = \pi - 2\theta$, then $x = 1$. $\therefore$ (1) becomes

$$\theta = \sin 2\theta - \tfrac{1}{2}\sin 4\theta + \tfrac{1}{3}\sin 6\theta - \tfrac{1}{4}\sin 8\theta + \cdots.$$

166. Given $\tan x = n\tan\theta$; expand x in Powers of n.

$$\frac{e^{ix} - e^{-ix}}{e^{ix} + e^{-ix}} = n\frac{e^{i\theta} - e^{-i\theta}}{e^{i\theta} + e^{-i\theta}} \quad . \ . \ . \ . \quad \text{(Art. 161)}$$

$$\therefore\ \frac{e^{2ix} - 1}{e^{2ix} + 1} = n\frac{e^{2i\theta} - 1}{e^{2i\theta} + 1}.$$

$$\therefore\ e^{2ix} = \frac{(e^{2i\theta} + 1) + n(e^{2i\theta} - 1)}{(e^{2i\theta} + 1) - n(e^{2i\theta} - 1)}$$

$$= \frac{(1+n)e^{2i\theta} + 1 - n}{(1-n)e^{2i\theta} + 1 + n}$$

$$= \frac{e^{2i\theta} + m}{me^{2i\theta} + 1} \left(\text{where } m = \frac{1-n}{1+n}\right)$$

$$= e^{2i\theta}\left(\frac{1 + me^{-2i\theta}}{1 + me^{2i\theta}}\right).$$

$$\therefore\ 2ix = 2i\theta + \log(1 + me^{-2i\theta}) - \log(1 + me^{2i\theta})$$

$$= 2i\theta - m(e^{2i\theta} - e^{-2i\theta}) + \frac{m^2}{2}(e^{4i\theta} - e^{-4i\theta}) - \cdots.$$

$$\therefore\ x = \theta - m\sin 2\theta + \frac{m^2}{2}\sin 4\theta - \cdots \quad . \ . \quad \text{(Art. 161)}$$

RESOLUTION OF EXPRESSIONS INTO FACTORS.

167. Resolve $x^n - 1$ into Factors.

Since $\cos 2r\pi \pm \sqrt{-1}\sin 2r\pi = 1,$

where r is any integer, and $x^n = 1,$

$$\therefore\ x^n = \cos 2r\pi \pm \sqrt{-1}\sin 2r\pi.$$

$$\therefore\ x = (\cos 2r\pi \pm \sqrt{-1}\sin 2r\pi)^{\frac{1}{n}}$$

$$= \cos\frac{2r\pi}{n} \pm \sqrt{-1}\sin\frac{2r\pi}{n}^{*} \quad . \quad . \quad \text{(Art. 153)}\ (1)$$

(1) *When n is even.* If $r = 0$, we obtain from (1) a real root 1; if $r = \frac{n}{2}$, we obtain a real root -1, and the two corresponding factors are $x - 1$ and $x + 1$. If we put

$$r = 1, 2, 3 \cdots \frac{n}{2} - 1,$$

in succession in (1), we obtain $n - 2$ additional roots, since each value of r gives two roots.

The product of the two factors, which are

$$\left(x - \cos\frac{2r\pi}{n} - \sqrt{-1}\sin\frac{2r\pi}{n}\right)$$

and

$$\left(x - \cos\frac{2r\pi}{n} + \sqrt{-1}\sin\frac{2r\pi}{n}\right)$$

$$= \left(x - \cos\frac{2r\pi}{n}\right)^2 + \sin^2\frac{2r\pi}{n}$$

$$= x^2 - 2x\cos\frac{2r\pi}{n} + 1 \quad . \quad . \quad . \quad . \quad . \quad (2)$$

which is a real quadratic factor.

$$\therefore\ x^n - 1 = (x^2 - 1)\left(x^2 - 2x\cos\frac{2\pi}{n} + 1\right)\left(x^2 - 2x\cos\frac{4\pi}{n} + 1\right)\cdots$$

$$\cdots\left(x^2 - 2x\cos\frac{n-4}{n}\pi + 1\right)\left(x^2 - 2x\cos\frac{n-2}{n}\pi + 1\right)\cdots\ (3)$$

* This expression gives the n nth roots of unity.

(2) *When n is odd.* The only real root is 1, found by putting $r=0$ in (1); the other $n-1$ roots are found by putting $r=1, 2, 3, \cdots \frac{n-1}{2}$ in (1) or (2) in succession.

$$\therefore\ x^n-1=(x-1)\left(x^2-2x\cos\frac{2\pi}{n}+1\right)\left(x^2-2x\cos\frac{4\pi}{n}+1\right)\cdots$$

$$\cdots\left(x^2-2x\cos\frac{n-3}{n}\pi+1\right)\left(x^2-2x\cos\frac{n-1}{n}\pi+1\right)\cdots \quad (4)$$

168. Resolve x^n+1 into Factors.

Since $\cos(2r+1)\pi \pm \sqrt{-1}\sin(2r+1)\pi = -1$,

where r is any integer, and $x^n=-1$,

$$\therefore\ x^n = \cos(2r+1)\pi \pm \sqrt{-1}\sin(2r+1)\pi.$$

$$\therefore\ x = [\cos(2r+1)\pi \pm \sqrt{-1}\sin(2r+1)\pi]^{\frac{1}{n}}$$

$$= \cos\frac{2r+1}{n}\pi \pm \sqrt{-1}\sin\frac{2r+1}{n}\pi \quad . \quad . \quad . \quad (1)$$

which is a root of the equation $x^n=-1$; *i.e.*, -1 is a root.

(1) *When n is even.* There is no real root; the n roots are all imaginary, and are found by putting

$$r=0, 1, 2, \cdots \frac{n}{2}-1,$$

successively, in (1).

The product of the two factors,

$$\left(x-\cos\frac{2r+1}{n}\pi-\sqrt{-1}\sin\frac{2r+1}{n}\pi\right)$$

and $$\left(x-\cos\frac{2r+1}{n}\pi+\sqrt{-1}\sin\frac{2r+1}{n}\pi\right)$$

$$= x^2-2x\cos\frac{2r+1}{n}\pi+1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

which is a real quadratic factor.

$$\therefore\ x^n+1=\left(x^2-2x\cos\frac{\pi}{n}+1\right)\left(x^2-2x\cos\frac{3\pi}{n}+1\right)\cdots$$

$$\cdots\left(x^2-2x\cos\frac{n-3}{n}\pi+1\right)\left(x^2-2x\cos\frac{n-1}{n}\pi+1\right)\cdots \quad (3)$$

(2) *When n is odd.* The only real root is -1; the other $n-1$ roots are found by putting $r=0, 1, 2, \cdots \frac{n-3}{2}$ in (1), in succession.

$$\therefore\ x^n+1=(x+1)\left(x^2-2x\cos\frac{\pi}{n}+1\right)\left(x^2-2x\cos\frac{3\pi}{n}+1\right)\cdots$$

$$\cdots\left(x^2-2x\cos\frac{n-4}{n}\pi+1\right)\left(x^2-2x\cos\frac{n-2}{n}\pi+1\right) \quad (4)$$

EXAMPLES.

1. Find the roots of the equation $x^5-1=0$.

Ans. 1, $\cos\frac{1}{5}(2r\pi)+i\sin\frac{1}{5}(2r\pi)$, where $r=1, 2, 3, 4$.

2. Find the quadratic factors of x^8-1.

Ans. $(x^2-1)(x^2-\sqrt{2}\,x+1)(x^2+1)(x^2+\sqrt{2}\,x+1)$.

3. Find the roots of the equation $x^4+1=0$, and write down the quadratic factors of x^4+1.

Ans. $\pm\frac{1}{\sqrt{2}}\pm\sqrt{-1}\frac{1}{\sqrt{2}}$; $(x^2-x\sqrt{2}+1)(x^2+x\sqrt{2}+1)$.

169. Resolve $x^{2n}-2x^n\cos\theta+1$ into Factors.

Let $x^{2n}-2x^n\cos\theta+1=0$.

$\therefore\ x^{2n}-2x^n\cos\theta+\cos^2\theta=-\sin^2\theta$.

$$\therefore\ x^n-\cos\theta=\pm\sqrt{-1}\sin\theta=\pm i\sin\theta.$$

$$\therefore\ x=(\cos\theta\pm i\sin\theta)^{\frac{1}{n}}=\cos\frac{2r\pi+\theta}{n}\pm i\sin\frac{2r\pi+\theta}{n} \quad (1)$$

since $\cos\theta$ is unaltered if for θ we put $\theta+2r\pi$. If we put $r=0, 1, 2, \cdots n-1$, successively in (1), we find $2n$ different roots, since each value of r gives two roots.

The product of the two factors in (1)

$$=\left(x-\cos\frac{2r\pi+\theta}{n}-i\sin\frac{2r\pi+\theta}{n}\right)$$

$$\times\left(x-\cos\frac{2r\pi+\theta}{n}+i\sin\frac{2r\pi+\theta}{n}\right)$$

$$=x^2-2x\cos\frac{2r\pi+\theta}{n}+1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (2)$$

$\therefore\ x^{2n}-2x^n\cos\theta+1$

$$=\left(x^2-2x\cos\frac{\theta}{n}+1\right)\left(x^2-2x\cos\frac{2\pi+\theta}{n}+1\right)\cdots$$

$$\cdots\left(x^2-2x\cos\frac{(2n-4)\pi+\theta}{n}+1\right)\cdots$$

$$\cdots\left(x^2-2x\cos\frac{(2n-2)\pi+\theta}{n}+1\right)\cdots \quad . \quad . \quad . \quad . \quad . \qquad (3)$$

Cor. Change x into $\frac{x}{a}$ in (3) and clear of fractions, and we get

$$x^{2n}-2a^nx^n\cos\theta+a^{2n}=\left(x^2-2ax\cos\frac{\theta}{n}+a^2\right)\cdots$$

$$\cdots\left(x^2-2ax\cos\frac{2\pi+\theta}{n}+a^2\right)\left(x^2-2ax\cos\frac{4\pi+\theta}{n}+a^2\right)\cdots$$

$$\cdots\text{to } n \text{ factors} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (4)$$

EXAMPLES.

Find the quadratic factors of the following:

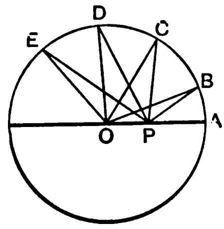
1. $x^8-2x^4\cos 60°+1=0.$

   Ans. $(x^2-2x\cos 15°+1)(x^2-2x\cos 105°+1)$
   $\times(x^2-2x\cos 195°+1)(x^2-2x\cos 285°+1)=0.$

2. $x^{10}-2x^5\cos 10°+1=0.$

   Ans. $(x^2-2x\cos 2°+1)(x^2-2x\cos 74°+1)$
   $\times(x^2-2x\cos 146°+1)(x^2-2x\cos 218°+1)$
   $(x^2-2x\cos 290°+1)=0.$

170. De Moivre's Property of the Circle. — Let O be the centre of a circle, P any point in its plane. Divide the circumference into n equal parts BC, CD, DE, ⋯, beginning at any point B; and join O and P with the points of division B, C, D, ⋯. Let POB $= \theta$; then will



$$\overline{OB}^{2n} - 2\,\overline{OB}^n \cdot \overline{OP}^n \cos n\theta + \overline{OP}^{2n}$$
$$= \overline{PB}^2 \cdot \overline{PC}^2 \cdot \overline{PD}^2 \cdots \text{ to } n \text{ terms.}$$

For, put OB $= a$, OP $= x$, and $\theta = \frac{\alpha}{n}$; then

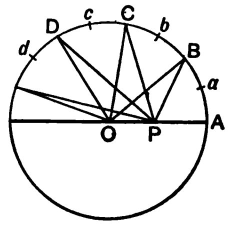
$$\overline{PB}^2 = \overline{OP}^2 + \overline{OB}^2 - 2\,OP \cdot OB \cos\theta$$
$$= x^2 + a^2 - 2\,ax \cos\frac{\alpha}{n} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$
$$\overline{PC}^2 = \overline{OP}^2 + \overline{OC}^2 - 2\,OP \cdot OC \cos\frac{\alpha + 2\pi}{n}$$
$$= x^2 + a^2 - 2\,ax \cos\frac{\alpha + 2\pi}{n}; \text{ and so on} \quad . \quad . \quad (2)$$

Multiplying (1), (2), (3), ⋯ together, we have

$$\overline{PB}^2 \cdot \overline{PC}^2 \cdot \overline{PD}^2 \cdots \text{ to } n \text{ terms}$$
$$= \left(x^2 - 2\,ax \cos\frac{\alpha}{n} + a^2\right)\left(x^2 - 2\,ax\cos\frac{\alpha + 2\pi}{n} + a^2\right)$$
$$\times\left(x^2 - 2\,ax \cos\frac{\alpha + 4\pi}{n} + a^2\right)\cdots$$
$$= x^{2n} - 2\,a^n x^n \cos\alpha + a^{2n} \quad . \quad \text{[by (4) of Art. 169]}$$
$$= \overline{OP}^{2n} - 2\,\overline{OP}^n \cdot \overline{OB}^n \cos n\theta + \overline{OB}^{2n} \quad . \quad . \quad . \quad (3)$$

which proves the proposition.

171. Cote's Properties of the Circle. — These are particular cases of De Moivre's property of the circle.



(1) Let OP, produced if necessary, meet the circle at A, and let

$$AB = BC = CD, \text{ etc.}, = \frac{2\pi}{n};$$

then $n\theta$ is a multiple of 2π. Hence we have from (3) of Art. 170, after taking the square root of both members,

$$\overline{OB}^n - \overline{OP}^n = PB \cdot PC \cdot PD \cdots \text{ to } n \text{ factors} \quad . \quad . \quad . \quad \text{I.}$$

(2) Let the arcs AB, BC, $\cdots$ be bisected in the points a, b, $\cdots$; then we have, by (1),

$$\overline{OB}^{2n} - \overline{OP}^{2n} = Pa \cdot PB \cdot Pb \cdot PC \cdot Pc \cdots \text{ to } 2n \text{ factors.}$$

Hence, by division,

$$\overline{OB}^n + \overline{OP}^n = Pa \cdot Pb \cdot Pc \cdots \text{ to } n \text{ factors} \quad . \quad . \quad . \quad \text{II.}$$

Cor. If the arcs AB, BC, $\cdots$ be trisected in the points a_1, a_2, b_1, b_2, $\cdots$, then we have

$$\overline{OB}^{2n} + \overline{OB}^n \cdot \overline{OP}^n + \overline{OP}^{2n} = Pa_1 \cdot Pa_2 \cdot Pb_1 \cdot Pb_2 \cdots \text{ to } 2n \text{ factors.}$$

172. Resolve sin θ into Factors.

(1) Put $x = 1$; then we get from (3) of Art. 169

$$2(1 - \cos\theta) = 2^n\left(1 - \cos\frac{\theta}{n}\right)\left(1 - \cos\frac{2\pi + \theta}{n}\right)\left(1 - \cos\frac{4\pi + \theta}{n}\right)\cdots$$
$$\cdots\left(1 - \cos\frac{(2n-2)\pi + \theta}{n}\right) \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Put $\theta = 2n\phi$ in (1), and let $2n\alpha = \pi$.

$$1 - \cos\theta = 1 - \cos 2n\phi = 2\sin^2 n\phi;$$

then extracting the square root, we have

$$\sin n\phi = 2^{n-1}\sin\phi \cdot \sin(\phi + 2\alpha)\sin(\phi + 4\alpha) \times$$
$$\cdots \times \sin(\phi + 2n\alpha - 2\alpha) \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

But $\sin(\phi + 2n\alpha - 2\alpha) = \sin(\phi + \pi - 2\alpha) = \sin(2\alpha - \phi)$,

$\sin(\phi + 2n\alpha - 4\alpha) = \sin(4\alpha - \phi)$, and so on.

Hence, when n is *odd*, multiplying together the second factor and the last, the third and the last but one, and so on, we have

$$\sin n\phi$$
$$= 2^{n-1}\sin\phi\sin(2\alpha+\phi)\sin(2\alpha-\phi)\sin(4\alpha+\phi)\sin(4\alpha-\phi)$$
$$\cdots \times \sin[(n-1)\alpha + \phi]\sin[(n-1)\alpha - \phi].$$

But $\sin(2\alpha+\phi)\sin(2\alpha-\phi) = \sin^2 2\alpha - \sin^2\phi$, and so on.

$$\therefore \sin n\phi = 2^{n-1}\sin\phi(\sin^2 2\alpha - \sin^2\phi)(\sin^2 4\alpha - \sin^2\phi) \times \cdots$$
$$\cdots \times [\sin^2(n-1)\alpha - \sin^2\phi] \quad . \quad . \quad . \quad . \quad (3)$$

Divide both members of (3) by $\sin\phi$, and then diminish ϕ indefinitely. Since the limit of $\sin n\phi \div \sin\phi$ is n, we get

$$n = 2^{n-1}\sin^2 2\alpha\sin^2 4\alpha\sin^2 6\alpha \times \cdots \times \sin^2(n-1)\alpha \quad (4)$$

Divide (3) by (4); thus

$$\sin n\phi = n\sin\phi\left(1 - \frac{\sin^2\phi}{\sin^2 2\alpha}\right)\left(1 - \frac{\sin^2\phi}{\sin^2 4\alpha}\right) \times \cdots \quad (5)$$

Put $n\phi = \theta$, and let n be increased while ϕ is diminished without limit, θ remaining unchanged; then since $2n\alpha = \pi$, the limit of

$$\frac{\sin^2\phi}{\sin^2 2\alpha} = \frac{\sin^2\frac{\theta}{n}}{\sin^2\frac{\pi}{n}} = \frac{\theta^2}{\pi^2} \times \frac{\sin^2\frac{\theta}{n}}{\left(\frac{\theta}{n}\right)^2} \times \frac{\left(\frac{\pi}{n}\right)^2}{\sin^2\frac{\pi}{n}} = \frac{\theta^2}{\pi^2} \text{ (Art. 133)}$$

and the limit of $n\sin\phi =$ that of $n\sin\frac{\theta}{n} = \theta$; and so on.

Hence (5) becomes

$$\sin\theta = \theta\left(1 - \frac{\theta^2}{\pi^2}\right)\left(1 - \frac{\theta^2}{2^2\pi^2}\right)\left(1 - \frac{\theta^2}{3^2\pi^2}\right)\cdots \quad . \quad (6)$$

NOTE. — The same result will be obtained if we suppose n *even*.

Rem. — When $\theta > 0$ and $< \pi$, $\sin\theta$ is +, and every factor in the second member of (6) is positive; when $\theta > \pi$ and $< 2\pi$, $\sin\theta$ is −, and only the second factor is negative; when $\theta > 2\pi$ and $< 3\pi$, both members are positive, since only the second and third factors are negative; and so on. Hence the + sign was taken in extracting the square root of (1).

Cor. Let $\theta = \frac{\pi}{2}$, then $\sin\frac{\pi}{2} = 1$, and $\frac{\theta}{\pi} = \frac{1}{2}$. Hence (6) becomes

$$1 = \frac{\pi}{2}\left(1 - \frac{1}{2^2}\right)\left(1 - \frac{1}{2^2 \cdot 2^2}\right)\left(1 - \frac{1}{3^2 \cdot 2^2}\right)\cdots$$

$$= \frac{\pi}{2} \cdot \frac{1 \cdot 3}{2^2} \cdot \frac{3 \cdot 5}{4^2} \cdot \frac{5 \cdot 7}{6^2} \cdots$$

$$\therefore \frac{\pi}{2} = \frac{2^2}{1 \cdot 3} \cdot \frac{4^2}{3 \cdot 5} \cdot \frac{6^2}{5 \cdot 7} \cdot \frac{8^2}{7 \cdot 9} \cdots$$

which is *Wallis's* expression for π.

173. Resolve $\cos\theta$ into Factors. — In (2) of Art. 172, change ϕ into $\phi + \alpha$, then $n\phi$ becomes $n\phi + n\alpha$, *i.e.*, $n\phi + \frac{\pi}{2}$. Hence (2) becomes

$$\cos n\phi = 2^{n-1} \sin(\phi + \alpha)\sin(\phi + 3\alpha)\sin(\phi + 5\alpha) \times \cdots \sin[\phi + (2n-1)\alpha] \quad . \; . \; . \; . \; . \quad (1)$$

But $\sin(\phi + 2n\alpha - \alpha) = \sin(\phi + \pi - \alpha) = \sin(\alpha - \phi)$,
$\sin(\phi + 2n\alpha - 3\alpha) = \sin(3\alpha - \phi)$, and so on.

Hence when n is *even* we have from (1)

$$\cos n\phi = 2^{n-1} \sin(\alpha + \phi)\sin(\alpha - \phi)\sin(3\alpha + \phi)\sin(3\alpha - \phi) \times \cdots \times \sin[(n-1)\alpha + \phi]\sin[(n-1)\alpha - \phi]$$

$$= 2^{n-1}(\sin^2\alpha - \sin^2\phi)(\sin^2 3\alpha - \sin^2\phi) \times \cdots \times [\sin^2(n-1)\alpha - \sin^2\phi] \quad . \; . \; . \; . \; . \quad (2)$$

Therefore, putting $n\phi = \theta$, as in Art. 172, we obtain

$$\cos\theta = \left(1 - \frac{4\theta^2}{\pi^2}\right)\left(1 - \frac{4\theta^2}{3^2\pi^2}\right)\left(1 - \frac{4\theta^2}{5^2\pi^2}\right) \quad . \; . \; . \; . \; . \quad (3)$$

NOTE. — For an alternative proof of the propositions of Arts. 172 and 173, see Lock's Higher Trigonometry, pp. 92–95.

EXAMPLES.

1. If $\alpha = \dfrac{\pi}{4n}$, prove that

$$\sin\alpha \sin 5\alpha \sin 9\alpha \cdots \sin(4n-3)\alpha = 2^{-n+\frac{1}{2}}.$$

2. Show that

$$16\cos\theta\cos(72^\circ-\theta)\cos(72^\circ+\theta)\cos(144^\circ-\theta)\cos(144^\circ+\theta) = \cos 5\theta.$$

SUMMATION OF TRIGONOMETRIC SERIES.

174. Sum the Series

$$\sin\alpha + \sin(\alpha+\beta) + \sin(\alpha+2\beta) + \cdots + \sin[\alpha+(n-1)\beta].$$

We have

$$2\sin\alpha\sin\tfrac{1}{2}\beta = \cos\left(\alpha-\frac{\beta}{2}\right) - \cos\left(\alpha+\frac{\beta}{2}\right) \quad \text{(Art. 45)}$$

$$2\sin(\alpha+\beta)\sin\tfrac{1}{2}\beta = \cos\left(\alpha+\frac{\beta}{2}\right) - \cos(\alpha+\tfrac{3}{2}\beta),$$

$$2\sin(\alpha+2\beta)\sin\tfrac{1}{2}\beta = \cos(\alpha+\tfrac{3}{2}\beta) - \cos(\alpha+\tfrac{5}{2}\beta),$$

$$\text{etc.} = \text{etc.}$$

$$2\sin[\alpha+(n-1)\beta]\sin\tfrac{1}{2}\beta = \cos\left[\alpha+\frac{2n-3}{2}\beta\right] - \cos\left[\alpha+\frac{2n-1}{2}\beta\right].$$

Therefore, if S_n denote the sum of n terms, we have, by addition,

$$2S_n\sin\tfrac{1}{2}\beta = \cos(\alpha-\tfrac{1}{2}\beta) - \cos\left[\alpha+\frac{2n-1}{2}\beta\right]$$

$$= 2\sin\left[\alpha+\frac{n-1}{2}\beta\right]\sin\tfrac{1}{2}n\beta \quad . \quad . \quad \text{(Art. 45)}$$

$$\therefore\ S_n = \frac{\sin\left[\alpha+\frac{n-1}{2}\beta\right]\sin\tfrac{1}{2}n\beta}{\sin\tfrac{1}{2}\beta}.$$

175. Sum the Series

$\cos\alpha + \cos(\alpha+\beta) + \cos(\alpha+2\beta) + \cdots + \cos[\alpha+(n-1)\beta]$.

We have $2\cos\alpha\sin\frac{1}{2}\beta = \sin(\alpha+\frac{1}{2}\beta) - \sin(\alpha-\frac{1}{2}\beta)$,

$$2\cos(\alpha+\beta)\sin\tfrac{1}{2}\beta = \sin(\alpha+\tfrac{3}{2}\beta) - \sin(\alpha+\tfrac{1}{2}\beta),$$

$$\text{etc.} = \text{etc.}$$

$$2\cos[\alpha+(n-1)\beta]\sin\tfrac{1}{2}\beta$$
$$= \sin\left[\alpha+\frac{2n-1}{2}\beta\right] - \sin\left[\alpha+\frac{2n-3}{2}\beta\right].$$

Denoting the sum of n terms by S_n, and adding, we get

$$2S_n\sin\tfrac{1}{2}\beta = \sin\left[\alpha+\frac{2n-1}{2}\beta\right] - \sin(\alpha-\tfrac{1}{2}\beta).$$

$$\therefore\ S_n = \frac{\cos\left[\alpha+\frac{n-1}{2}\beta\right]\sin\frac{1}{2}n\beta}{\sin\frac{1}{2}\beta}.$$

Rem. — **The sum of the series in this article may be deduced from that in Art. 174 by putting $\alpha+\frac{\pi}{2}$ for α. The sums of these two series are often useful;* and the student is advised to commit them to memory.**

Cor. If we put $\beta = \frac{2\pi}{n}$, then $\sin\frac{1}{2}n\beta = \sin\pi = 0$. Hence we have from Arts. 174 and 175

$$\sin\alpha + \sin\left(\alpha+\frac{2\pi}{n}\right) + \sin\left(\alpha+\frac{4\pi}{n}\right) + \cdots \sin\left[\alpha+\frac{2(n-1)}{n}\pi\right] = 0.$$

$$\cos\alpha + \cos\left(\frac{\alpha+2\pi}{n}\right) + \cos\left(\alpha+\frac{4\pi}{n}\right) + \cdots \cos\left[\alpha+\frac{2(n-1)}{n}\pi\right] = 0.$$

Note. — These two results are very important, and the student should carefully notice them.

176. Sum the Series

$\sin^m\alpha + \sin^m(\alpha+\beta) + \sin^m(\alpha+2\beta) + \cdots + \sin^m[\alpha+(n-1)\beta]$.

This may be done by the aid of Art. 159 or Art. 160.

* See Thompson's Dynamo-Electric Machinery. 3d ed., pp. 345, 346.

Thus, if m is even, we have from Art. 159

$$2^{m-1}\sin^m \alpha = (-1)^{\frac{m}{2}}[\cos m\alpha - m\cos(m-2)\alpha + \cdots] \quad . \quad (1)$$

$$2^{m-1}\sin^m(\alpha+\beta)$$
$$= (-1)^{\frac{m}{2}}[\cos m(\alpha+\beta) - m\cos(m-2)(\alpha+\beta) + \cdots] \quad (2)$$

and so on; and the required sum may be obtained from the known sum of the series

$$[\cos m\alpha + \cos m(\alpha+\beta) + \cos m(\alpha+2\beta) + \cdots]$$

and $\{\cos(m-2)\alpha + \cos[(m-2)(\alpha+\beta)]$
$$+ \cos[(m-2)(\alpha+2\beta)] + \cdots\}, \text{ etc.}$$

We may find the sum of the series

$$\cos^m \alpha + \cos^m(\alpha+\beta) + \cos^m(\alpha+2\beta) + \text{etc.}$$

to n terms in a similar manner by the aid of Art. 158.

EXAMPLES.

1. Sum to n terms the series

$$\sin^2\alpha + \sin^2(\alpha+\beta) + \sin^2(\alpha+2\beta) + \cdots$$

We have

$$2\sin^2\alpha = -(\cos 2\alpha - 1) \text{ by } (1),$$

$$2\sin^2(\alpha+\beta) = -[\cos 2(\alpha+\beta) - 1] \text{ by } (2),$$

$$2\sin^2(\alpha+2\beta) = -[\cos 2(\alpha+2\beta) - 1], \text{ and so on.}$$

Hence

$$2S_n = n - [\cos 2\alpha + \cos 2(\alpha+\beta) + \cos 2(\alpha+2\beta) + \cdots]$$

$$= n - \frac{\cos[2\alpha + (n-1)\beta]\sin n\beta}{\sin\beta} \quad . \quad (\text{Art. } 175)$$

$$\therefore S_n = \frac{n}{2} - \frac{\cos[2\alpha + (n-1)\beta]\sin n\beta}{2\sin\beta}.$$

2. Sum to n terms the series

$$\cos^3 \alpha + \cos^3 2\alpha + \cos^3 3\alpha + \cdots$$

Ans. $$\frac{2\cos\left[3\alpha + \frac{1}{2}(n-1)3\alpha\right]\sin\frac{3}{2}n\alpha}{8\sin\frac{3}{2}\alpha} + \frac{6\cos\left[\alpha + \frac{n-1}{2}\alpha\right]}{8\sin\frac{1}{2}\alpha}\sin\frac{n\alpha}{2}.$$

177. Sum the Series

$$\sin\alpha - \sin(\alpha+\beta) + \sin(\alpha + 2\beta) - \cdots \text{ to } n \text{ terms} \quad . \quad (1)$$

Change β into $\beta + \pi$, and (1) becomes

$$\sin\alpha + \sin(\alpha + \pi + \beta) + \sin(\alpha + 2\pi + 2\beta) + \cdots \quad . \quad . \quad (2)$$

Therefore we have from Art. 174

$$S_n = \frac{\sin\left[\alpha + \frac{(n-1)(\pi+\beta)}{2}\right]\sin\frac{n(\pi+\beta)}{2}}{\sin\frac{\pi+\beta}{2}} \quad . \quad . \quad . \quad (3)$$

Similarly,

$$\cos\alpha - \cos(\alpha+\beta) + \cos(\alpha+2\beta) - \cdots \text{ to } n \text{ terms}$$

$$= \frac{\cos\left[\alpha + \frac{(n-1)(\pi+\beta)}{2}\right]\sin\frac{n(\pi+\beta)}{2}}{\sin\frac{\pi+\beta}{2}} \quad . \quad . \quad . \quad (4)$$

178. Sum the Series

$$\operatorname{cosec}\theta + \operatorname{cosec}2\theta + \operatorname{cosec}4\theta + \cdots \text{ to } n \text{ terms.}$$

We have $$\operatorname{cosec}\theta = \cot\frac{\theta}{2} - \cot\theta,$$

$$\operatorname{cosec}2\theta = \cot\theta - \cot 2\theta,$$

$$\text{etc.} = \text{etc.}$$

$$\operatorname{cosec}2^{n-1}\theta = \cot 2^{n-2}\theta - \cot 2^{2n-1}\theta.$$

Therefore, by addition, as in Art. 174,

$$S_n = \cot\tfrac{1}{2}\theta - \cot 2^{2n-1}\theta.$$

NOTE. — The artifice employed in this Art., of resolving each term into the difference of two others, is extensively used in the summation of series.

Practice alone will give the student readiness in effecting such transformations. If he cannot discover the mode of resolution in any example, he will often easily recognize it when he sees the *result* of summation.

The student, however, is advised to resort to this method of solution only as a last resource.

179. Sum the Series

$$\tan\theta + \tfrac{1}{2}\tan\frac{\theta}{2} + \tfrac{1}{4}\tan\frac{\theta}{4} + \cdots \text{ to } n \text{ terms.}$$

We have
$$\tan\theta = \cot\theta - 2\cot 2\theta,$$
$$\tfrac{1}{2}\tan\frac{\theta}{2} = \tfrac{1}{2}\cot\frac{\theta}{2} - \cot\theta,$$
$$\tfrac{1}{4}\tan\frac{\theta}{4} = \tfrac{1}{4}\cot\frac{\theta}{4} - \tfrac{1}{2}\cot\frac{\theta}{2},$$
$$\text{etc} = \text{etc.}$$
$$\frac{1}{2^{n-1}}\tan\frac{\theta}{2^{n-1}} = \frac{1}{2^{n-1}}\cot\frac{\theta}{2^{n-1}} - \frac{1}{2^{n-2}}\cot\frac{\theta}{2^{n-2}}.$$
$$\therefore\ S_n = \frac{1}{2^{n-1}}\cot\frac{\theta}{2^{n-1}} - 2\cot 2\theta.$$

180. Sum the Series

$$\sin\alpha + x\sin(\alpha+\beta) + x^2\sin(\alpha+2\beta) + \cdots x^{n-1}\sin[\alpha+(n-1)\beta].$$

Denote the sum by S_n, and substitute for the sines their exponential values (Art. 161). Thus,

$$2iS_n = (e^{i\alpha} - e^{-i\alpha}) + x(e^{i(\alpha+\beta)} - e^{-i(\alpha+\beta)})$$
$$+ x^2(e^{i(\alpha+2\beta)} - e^{-i(\alpha+2\beta)}) + x^{n-1}\left[e^{i(\alpha+n\beta-\beta)} - e^{-i(\alpha+n\beta-\beta)}\right]$$

$$= \frac{e^{i\alpha} - x^n e^{i(\alpha+n\beta)}}{1 - xe^{i\beta}} - \frac{e^{-i\alpha} - x^n e^{-i(\alpha+n\beta)}}{1 - xe^{-i\beta}} \quad \text{[Alg. (3) Art. 163]}$$

$$= \frac{e^{i\alpha} - e^{-i\alpha} - x\left[e^{i(\alpha-\beta)} - e^{-i(\alpha-\beta)}\right] - x^n\left[e^{i(\alpha+n\beta)} - e^{-i(\alpha+n\beta)}\right]}{1 - x(e^{i\beta} + e^{-i\beta}) + x^2}$$
$$\qquad + \frac{x^{n+1}\left[e^{i(\alpha+n\beta-\beta)} - e^{-i(\alpha+n\beta-\beta)}\right]}{1 - x(e^{i\beta} + e^{-i\beta}) + x^2}.$$

$\therefore S_n =$

$$\frac{\sin\alpha - x\sin(\alpha-\beta) - x^n \sin(\alpha+n\beta) + x^{n+1}\sin[\alpha+(n-1)\beta]}{1-2x\cos\beta+x^2}. \quad (1)$$

Cor. If $x<1$, and n be indefinitely increased,

$$S_\infty = \frac{\sin\alpha - x\sin(\alpha-\beta)}{1-2x\cos\beta+x^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Sch. Similarly,

$\cos\alpha + x\cos(\alpha+\beta) + x^2\cos(\alpha+2\beta) + \cdots$ to n terms $=$

$$\frac{\cos\alpha - x\cos(\alpha-\beta) - x^n\cos(\alpha+n\beta) + x^{n+1}\cos[\alpha+(n-1)\beta]}{1-2x\cos\beta+x^2} \quad (3)$$

We may obtain (3) from (1) by changing α to $\alpha+\frac{\pi}{2}$.

Also $$S_\infty = \frac{\cos\alpha - x\cos(\alpha-\beta)}{1-2x\cos\beta+x^2} \quad . \quad . \quad . \quad . \quad (4)$$

181. Sum the Infinite Series

$$x\sin(\alpha+\beta) + \frac{x^2}{\lfloor 2}\sin(\alpha+2\beta) + \frac{x^3}{\lfloor 3}\sin(\alpha+3\beta) + \cdots,$$

and $x\cos(\alpha+\beta) + \frac{x^2}{\lfloor 2}\cos(\alpha+2\beta) + \frac{x^3}{\lfloor 3}\cos(\alpha+3\beta) + \cdots.$

Let S denote the former series, and C the latter.

Then $C + iS = xe^{i(\alpha+\beta)} + \frac{x^2}{\lfloor 2}e^{i(\alpha+2\beta)} + \frac{x^3}{\lfloor 3}e^{i(\alpha+3\beta)} + \cdots$

$$= e^{i\alpha}\left(xe^{i\beta} + \frac{x^2}{\lfloor 2}e^{i2\beta} + \frac{x^3}{\lfloor 3}e^{i3\beta} + \cdots\right)$$

$$= e^{i\alpha}(e^{xe^{i\beta}} - 1) \quad . \quad . \quad \text{[by (3) of Art. 129]}$$

$$= e^{i\alpha}(e^{x(\cos\beta + i\sin\beta)} - 1) \quad . \quad . \quad . \quad \text{(Art 161)}$$

$$= e^{x\cos\beta}e^{i(\alpha + x\sin\beta)} - e^{i\alpha}$$

$$= e^{x\cos\beta}[\cos(\alpha + x\sin\beta) + i\sin(\alpha + x\sin\beta)]$$

$$-(\cos\alpha + i\sin\alpha) \quad . \quad . \quad . \quad . \quad \text{(Art. 161)}$$

Equating real and imaginary parts, we have

$$C = e^{x\cos\beta}\cos(\alpha + x\sin\beta) - \cos\alpha,$$
$$S = e^{x\cos\beta}\sin(\alpha + x\sin\beta) - \sin\alpha.$$

EXAMPLES.

Prove the following statements:

1. The two values of $(\cos 4\theta + \sqrt{-1}\sin 4\theta)^{\frac{1}{2}}$ are

$$\pm(\cos 2\theta + \sqrt{-1}\sin 2\theta) \quad . \quad . \quad . \quad \text{(Art. 154)}$$

2. The three values of $(\cos\theta + \sqrt{-1}\sin\theta)^{\frac{1}{3}}$ are

$$\cos\frac{\theta}{3} + \sqrt{-1}\sin\frac{\theta}{3},\ \cos\frac{2\pi+\theta}{3} + \sqrt{-1}\sin\frac{2\pi+\theta}{3},$$
$$\cos\frac{4\pi+\theta}{3} + \sqrt{-1}\sin\frac{4\pi+\theta}{3}.$$

3. The three values of $(-1)^{\frac{1}{3}}$ are

$$\frac{1+\sqrt{-3}}{2},\ -1,\ \frac{1-\sqrt{-3}}{2} \quad . \quad . \quad . \quad . \quad . \quad \text{(Art. 154)}$$

4. The six values of $(-1)^{\frac{1}{6}}$ are contained in

$$\cos\frac{(2r+1)\pi}{6} \pm \sqrt{-1}\sin\frac{(2r+1)\pi}{6},\ \text{where } r = 0,\ 1,\ \text{or } 2.$$

5. The three values of $(1+\sqrt{-1})^{\frac{1}{3}}$ are contained in

$$2^{\frac{1}{6}}\left[\cos\frac{\theta}{3} + \sqrt{-1}\sin\frac{\theta}{3}\right],\ \text{where } \theta = \frac{\pi}{4},\ \tfrac{9}{4}\pi,\ \text{or } \tfrac{17}{4}\pi.$$

6. The three values of $(3+4\sqrt{-1})^{\frac{1}{3}}$ are contained in

$$\sqrt[3]{5}\left[\cos\frac{2r\pi+\theta}{3} + \sqrt{-1}\sin\frac{2r\pi+\theta}{3}\right],\ \text{where } r = 0,\ 1,\ \text{or } 2.$$

7. $\cos 6\theta = \cos^6\theta - 15\cos^4\theta\sin^2\theta + 15\cos^2\theta\sin^4\theta - \sin^6\theta.$

8. $\sin 9\theta = 9\cos^8\theta\sin\theta - 84\cos^6\theta\sin^3\theta + 126\cos^4\theta\sin^5\theta$
$\qquad - 36\cos^2\theta\sin^7\theta + \sin^9\theta.$

9. $\tan n\theta$

$$= \frac{n\tan\theta - \frac{n(n-1)(n-2)}{\lfloor 3}\tan^3\theta + \cdots}{1 - \frac{n(n-1)}{\lfloor 2}\tan^2\theta + \frac{n(n-1)(n-2)(n-3)}{\lfloor 4}\tan^4\theta - \cdots}.$$

10. Given $\frac{\sin\theta}{\theta} = \frac{2165}{2166}$: show that θ is nearly the circular measure of 3°.

Prove the following:

11. $-64\sin^7\theta = \sin 7\theta - 7\sin 5\theta + 21\sin 3\theta - 35\sin\theta.$

12. $-2^9\sin^{10}\theta = \cos 10\theta - 10\cos 8\theta + 45\cos 6\theta - 120\cos 4\theta + 210\cos 2\theta - 126.$

13. $2^6(\cos^8\theta + \sin^8\theta) = \cos 8\theta + 28\cos 4\theta + 35.$

14. $\cos^6\theta + \sin^6\theta = \frac{1}{8}(5 + 3\cos 4\theta).$

15. Expand $(\sin\theta)^{4n+2}$ in terms of cosines of multiples of θ.

16. Expand $(\sin\theta)^{4n+1}$ in terms of sines of multiples of θ.

17. Expand $(\cos\theta)^{2n}$ in terms of cosines of multiples of θ.

Use the exponential values of the sine and cosine to prove the following:

18. $\frac{\sin\theta}{1-\cos\theta} = \cot\frac{\theta}{2}.$

19. If $\log(x + y\sqrt{-1}) = \alpha + \beta\sqrt{-1}$, prove that

$$x^2 + y^2 = e^{2\alpha}, \quad \text{and} \quad y = x\tan\beta.$$

20. If $\sin(\alpha + \beta\sqrt{-1}) = x + y\sqrt{-1}$, prove that

$$x^2\operatorname{cosec}^2\alpha - y^2\sec^2\alpha = 1.$$

21. $2\cos(n\cos^{-1}x) = (x + \sqrt{-1}\sqrt{1-x^2})^n + (x - \sqrt{-1}\sqrt{1-x^2})^n.$

22. $(\sqrt{-1})^{\sqrt{-1}} = e^{-\frac{\pi}{2}}$.

23. $e^{\theta}(\cos\theta + \sqrt{-1}\sin\theta) = e^{\theta\sqrt{2}}\left(\cos\frac{\pi}{4} + \sqrt{-1}\sin\frac{\pi}{4}\right)$.

24. The coefficients of x^n in the expansion, (1) of $e^{ax}\cos bx$, and (2) of $e^{ax}\sin bx$, in powers of x, are

$$\frac{(a^2+b^2)^{\frac{n}{2}}}{\lfloor n}\cos n\theta \text{ and } \frac{(a^2+b^2)^{\frac{n}{2}}}{\lfloor n}\sin n\theta.$$

25. The coefficient of x^n in the expansion of $e^x\cos x$ in powers of x is $\frac{2^{\frac{n}{2}}}{\lfloor n}\cos\frac{n\pi}{4}$.

26. If the sides of a right triangle are 49 and 51, then the angles opposite them are 43° 51′ 15″ and 46° 8′ 45″ nearly.

27. If a and b be the sides of a triangle, A and B the opposite angles, then will $\log b - \log a$

$$= \cos 2\mathrm{A} - \cos 2\mathrm{B} + \tfrac{1}{2}(\cos 4\mathrm{A} - \cos 4\mathrm{B}) + \tfrac{1}{3}(\cos 6\mathrm{A} - \cos 6\mathrm{B}) + \cdots.$$

28. If $\mathrm{A} + i\mathrm{B} = \log(m + in)$, then

$$\tan\mathrm{B} = \frac{n}{m}, \text{ and } 2\mathrm{A} = \log(n^2 + m^2).$$

29. $\cos(\theta + i\phi) = \cos\theta\left(\frac{e^{-\phi} + e^{\phi}}{2}\right) + i\sin\theta\left(\frac{e^{-\phi} - e^{\phi}}{2}\right)$.

30. $\sin(\theta + i\phi) = \sin\theta\left(\frac{e^{-\phi} + e^{\phi}}{2}\right) - i\cos\theta\left(\frac{e^{-\phi} - e^{\phi}}{2}\right)$.

31. $2\cos(\alpha + i\beta) = \cos\alpha(e^{\beta} + e^{-\beta}) - i\sin\alpha(e^{\beta} - e^{-\beta})$.

32. $(a + ib)^{(\alpha + i\beta)}$

$$= r^{\alpha}e^{-\beta r}[\cos(\beta\log r + \alpha r) + i\sin(\beta\log r + \alpha r)],$$

where $a + ib = r(\cos r + i\sin r)$.

33. $\log(a + ib) = \frac{1}{2}\log(a^2 + b^2) + i\tan^{-1}\frac{b}{a}$.

34. $[\sin(\alpha-\theta)+e^{\pm i\alpha}\sin\theta]^n$
$$=\sin^{n-1}\alpha[\sin(\alpha-n\theta)+e^{\pm i\alpha}\sin n\theta].$$

35. $\frac{\pi}{8}=\frac{1}{1\cdot 3}+\frac{1}{5\cdot 7}+\frac{1}{9\cdot 11}+\cdots.$

36. Write down the quadratic factors of $x^{13}-1$.

Ans. $(x-1)[x^2-2x\cos\frac{1}{13}(2r\pi)+1]$, six factors, putting $r=1, 2, 3, 4, 5, 6$.

37. Solve the equation $x^6-1=0$.

Ans. $(x^2-1)(x^2-x+1)(x^2+x+1)=0.$

38. Give the general quadratic factor of $x^{20}-a^{20}$.

Ans. $x^2-2ax\cos\frac{1}{10}(r\pi)+a^2.$

39. Find all the values of $\sqrt[12]{1}$.

Ans. $\cos\frac{1}{6}(r\pi)+i\sin\frac{1}{6}(r\pi)$, r having each integral value from 0 to 11.

40. Write down the quadratic factors of x^6+1.

Ans. $(x^2-\sqrt{3}x+1)(x^2+1)(x^2+\sqrt{3}x+1).$

41. Write down the general quadratic factor of $x^{20}+1$.

Ans. $x^2-2x\cos(1+2r)9°+1.$

42. Find the factors of $x^{13}+1=0$.

Ans. $(x+1)[x^2-2x\cos\frac{1}{13}(\pi+2r\pi)+1]$, seven factors in all.

43. Find a general expression for all the values of $\sqrt[n]{-1}$.

Ans. $\cos\frac{\pi+2r\pi}{n}+i\sin\frac{\pi+2r\pi}{n}$, where r may have any integral value.

44. Solve $x^{12}-2x^6\cos\frac{2}{3}\pi+1=0$.

Ans. $x^2-2x\cos\frac{1}{9}(3r\pi+\pi)+1=0$, six quadratics.

45. Solve $x^{10} + \sqrt{3}\,x^5 + 1 = 0$.

Ans. $x^2 + 2x\cos(r \times 72° + 6°) + 1 = 0$, five quadratics.

46. Write down the quadratic factors of

$$x^{2n} - 2x^n y^n \cos\alpha + y^{2n}.$$

Ans. $x^2 - 2xy\cos\dfrac{\alpha + 2r\pi}{n} + y^2$, n factors.

Prove the following:

47. $\tan\phi\tan\left(\phi + \dfrac{\pi}{n}\right)\tan\left(\phi + \dfrac{2\pi}{n}\right)\cdots\tan\left(\phi + \dfrac{n-1}{n}\pi\right) = (-1)^{\frac{n}{2}}$, where n is even. [Use (2) of Art. 172.]

48. $\sin 5\theta - \cos 5\theta = 16\cos(\theta - 27°)\cos(\theta + 9°) \times \sin(\theta + 27°)\sin(\theta - 9°)(\cos\theta - \sin\theta)$.

49. $e^{\theta} + e^{-\theta} = 2\left(1 + \dfrac{4\theta^2}{\pi^2}\right)\left(1 + \dfrac{4\theta^2}{3^2\pi^2}\right)\cdots$

50. $e^{\theta} - e^{-\theta} = 2\theta\left(1 + \dfrac{\theta^2}{\pi^2}\right)\left(1 + \dfrac{\theta^2}{2^2\pi^2}\right)\cdots$

51. $\dfrac{1}{1^2} + \dfrac{1}{2^2} + \dfrac{1}{3^2} + \dfrac{1}{4^2} + \cdots = \dfrac{\pi^2}{6}$.

52. $\dfrac{1}{1^2} + \dfrac{1}{3^2} + \dfrac{1}{5^2} + \dfrac{1}{7^2} + \cdots = \dfrac{\pi^2}{8}$.

53. $\pi = 3 \cdot \dfrac{36}{35} \cdot \dfrac{144}{143} \cdot \dfrac{324}{323} \cdot \dfrac{576}{575} \cdots$.

54. $\dfrac{\pi}{2} = \dfrac{2 \cdot 2 \cdot 4 \cdot 4 \cdot 6 \cdot 6 \cdot 8 \cdot 8 \cdots}{1 \cdot 3 \cdot 3 \cdot 5 \cdot 5 \cdot 7 \cdot 7 \cdot 9 \cdots}$.

55. $\sqrt{2} = \dfrac{4 \cdot 36 \cdot 100 \cdot 196 \cdot 324 \cdots}{3 \cdot 35 \cdot 99 \cdot 195 \cdot 323 \cdots}$.

56. $\frac{1}{2}\sqrt{3} = \dfrac{8 \cdot 80 \cdot 224 \cdot 440 \cdots}{9 \cdot 81 \cdot 225 \cdot 441 \cdots}$.

57. $\cos x + \tan\frac{y}{2}\sin x =$

$$\left(1+\frac{2x}{\pi-y}\right)\left(1-\frac{2x}{\pi+y}\right)\left(1+\frac{2x}{3\pi-y}\right)\left(1-\frac{2x}{3\pi+y}\right)\left(1+\frac{2x}{5\pi-y}\right)\cdots$$

58. $\cos x - \cot\frac{y}{2}\sin x =$

$$\left(1-\frac{2x}{y}\right)\left(1+\frac{2x}{2\pi-y}\right)\left(1-\frac{2x}{2\pi+y}\right)\left(1+\frac{2x}{4\pi-y}\right)\left(1-\frac{2x}{4\pi+y}\right)\cdots$$

59. By aid of the formula $\cos\theta = \frac{\sin 2\theta}{2\sin\theta}$ and Art. 172, deduce the value for $\cos\theta$ obtained in Art. 173.

60. By expanding both sides of Ex. 57 in powers of x and equating the coefficients of x, prove that

$$\tan\frac{y}{2} = \frac{2}{\pi-y} - \frac{2}{\pi+y} + \frac{2}{3\pi-y} - \frac{2}{3\pi+y} + \frac{2}{5\pi-y} - \frac{2}{5\pi+y} + \cdots$$

61. Prove in like manner from Ex. 58 that

$$\cot\frac{y}{2} = \frac{2}{y} - \frac{2}{2\pi-y} + \frac{2}{2\pi+y} - \frac{2}{4\pi-y} + \frac{2}{4\pi+y} - \cdots$$

62. Prove $\frac{\pi}{3\sqrt{3}} = 1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{5} + \frac{1}{7} - \frac{1}{8} + \frac{1}{10} - \cdots$

63. Prove $\frac{\pi}{2\sqrt{3}} = 1 - \frac{1}{5} + \frac{1}{7} - \frac{1}{11} + \frac{1}{13} - \frac{1}{17} + \frac{1}{19} - \cdots$

64. Prove that $\frac{1}{\sin y} =$

$$\frac{1}{y} + \frac{1}{\pi-y} - \frac{1}{2\pi-y} - \frac{1}{\pi+y} + \frac{1}{2\pi+y} + \frac{1}{3\pi-y} - \frac{1}{4\pi-y} - \frac{1}{3\pi+y} + \cdots$$

Sum the following series to n terms:

65. $\sin\alpha + \sin 2\alpha + \sin 3\alpha + \cdots + \sin n\alpha = \dfrac{\sin\frac{n+1}{2}\alpha \sin\frac{n\alpha}{2}}{\sin\frac{\alpha}{2}}.$

66. $\cos\alpha+\cos 2\alpha+\cos 3\alpha+\cdots+\cos n\alpha=\dfrac{\cos\dfrac{n+1}{2}\alpha\sin\dfrac{n\alpha}{2}}{\sin\dfrac{\alpha}{2}}.$

67. $\sin\alpha+\sin 3\alpha+\sin 5\alpha+\cdots=\dfrac{\sin^2 n\alpha}{\sin\alpha}.$

68. $\cos\alpha+\cos 3\alpha+\cos 5\alpha+\cdots=\dfrac{\sin 2n\alpha}{2\sin\alpha}.$

69. $\sin^2\alpha+\sin^2 2\alpha+\sin^2 3\alpha+\cdots=\dfrac{n\sin\alpha-\sin n\alpha\cos(n+1)\alpha}{2\sin\alpha}.$

70. $\cos^2\alpha+\cos^2 2\alpha+\cos^2 3\alpha+\cdots=\dfrac{n\sin\alpha+\cos(n+1)\alpha\sin n\alpha}{2\sin\alpha}.$

71. $\sin^3\alpha+\sin^3(\alpha+\beta)+\sin^3(\alpha+2\beta)+\cdots$

$$=\frac{3}{4}\frac{\sin\left(\alpha+\dfrac{n-1}{2}\beta\right)\sin\dfrac{n\beta}{2}}{\sin\frac{1}{2}\beta}-\frac{1}{4}\frac{\sin\left(3\alpha+\dfrac{n-1}{2}3\beta\right)\sin\dfrac{3n\beta}{2}}{\sin\frac{3}{2}\beta}.$$

72. $\sin^3\alpha+\sin^3 2\alpha+\sin^3 3\alpha+\cdots$

$$=\frac{3}{4}\frac{\sin\dfrac{n\alpha}{2}\sin\left(\dfrac{n+1}{2}\right)\alpha}{\sin\dfrac{\alpha}{2}}-\frac{\sin\dfrac{3n\alpha}{2}\sin\dfrac{3(n+1)\alpha}{2}}{4\sin\dfrac{3\alpha}{2}}.$$

73. $\sin\alpha\sin 2\alpha+\sin 2\alpha\sin 3\alpha+\sin 3\alpha\sin 4\alpha+\cdots$

$$=\frac{n\sin\alpha\cos\alpha-\sin n\alpha\cos(n+2)\alpha}{2\sin\alpha}.$$

74. $\tan\alpha+2\tan 2\alpha+2^2\tan 2^2\alpha+\cdots=\cot\alpha-2^n\cot 2^n\alpha.$

75. $(\tan\alpha+\cot\alpha)+(\tan 2\alpha+\cot 2\alpha)+(\tan 2^2\alpha+\cot 2^2\alpha)$
$+\cdots=2\cot\alpha-2\cot 2^n\alpha.$

76. $\sec\alpha\sec 2\alpha+\sec 2\alpha\sec 3\alpha+\cdots$
$=\operatorname{cosec}\alpha[\tan(n+1)\alpha-\tan\alpha].$

77. $\operatorname{cosec}\alpha\operatorname{cosec}2\alpha+\operatorname{cosec}2\alpha\operatorname{cosec}3\alpha+\cdots$

$$=\operatorname{cosec}\alpha\,[\cot\alpha-\cot(n+1)\alpha].$$

78. $\dfrac{\sin 2\theta}{\cos\theta\cos 3\theta}+\dfrac{\sin 4\theta}{\cos 3\theta\cos 5\theta}+\cdots=\dfrac{\sec(2n+1)\theta-\sec\theta}{2\sin\theta}.$

79. $\cos^4\alpha+\cos^4(\alpha+\beta)+\cos^4(\alpha+2\beta)+\cdots=\frac{3}{8}n+$

$$\frac{\cos[2\alpha+(n-1)\beta]\sin n\beta}{2\sin\beta}+\frac{\cos[4\alpha+(n-1)2\beta]\sin 2n\beta}{8\sin 2\beta}.$$

80. $\tan n\theta=\dfrac{\sin\theta+\sin 3\theta+\sin 5\theta+\cdots\text{to } n \text{ terms}}{\cos\theta+\cos 3\theta+\cos 5\theta+\cdots\text{to } n \text{ terms}}.$

81. $\cos\theta\cos(\theta+\alpha)+\cos(\theta+\alpha)\cos(\theta+2\alpha)$

$$+\cos(\theta+2\alpha)\cos(\theta+3\alpha)+\cdots$$

$$=\frac{n}{2}\cos\alpha+\frac{\cos(2\theta+n\alpha)\sin n\alpha}{2\sin\alpha}.$$

82. $\dfrac{\sin\theta-\sin 2\theta+\sin 3\theta-\cdots\text{to } n \text{ terms}}{\cos\theta-\cos 2\theta+\cos 3\theta-\cdots\text{to } n \text{ terms}}=\tan\dfrac{n+1}{2}(\pi+\theta).$

83. $\sin(p+1)\theta\cos\theta+\sin(p+2)\theta\cos 2\theta+\cdots$

$$=\frac{n\sin p\theta}{2}+\frac{\sin(p+1+n)\theta\sin n\theta}{2\sin\theta}.$$

84. $\sin 3\theta\sin\theta+\sin 6\theta\sin 2\theta+\sin 12\theta\sin 4\theta+\cdots$

$$=\tfrac{1}{2}(\cos 2\theta-\cos 2^{n+1}\theta).$$

85. $\sin\theta\left(\sin\dfrac{\theta}{2}\right)^2+2\sin\dfrac{\theta}{2}\left(\sin\dfrac{\theta}{4}\right)^2+4\sin\dfrac{\theta}{4}\left(\sin\dfrac{\theta}{8}\right)^2+\cdots$

$$=2^{n-2}\sin\frac{\theta}{2^{n-1}}-\tfrac{1}{4}\sin 2\theta.$$

86. $\tan\dfrac{\theta}{2}\sec\theta+\tan\dfrac{\theta}{4}\sec\dfrac{\theta}{2}+\tan\dfrac{\theta}{8}\sec\dfrac{\theta}{4}+\cdots=\tan\theta-\tan\dfrac{\theta}{2^n}.$

87. $\cot\theta\operatorname{cosec}\theta+2\cot 2\theta\operatorname{cosec}2\theta+2^2\cot 2^2\theta\operatorname{cosec}2^2\theta+\cdots$

$$=\frac{1}{2\sin^2\dfrac{\theta}{2}}-\frac{2^{n-1}}{\sin^2 2^{n-1}\theta}.$$

88. $\dfrac{1}{\sin\theta\sin 2\theta}+\dfrac{1}{\sin 2\theta\sin 3\theta}+\dfrac{1}{\sin 3\theta\sin 4\theta}+\cdots$

$$=\frac{1}{\sin\theta}(\cot 3\theta-\cot 4\theta).$$

89. $\dfrac{1}{\sin\theta\cos 2\theta}-\dfrac{1}{\cos 2\theta\sin 3\theta}+\dfrac{1}{\sin 3\theta\cos 4\theta}-\cdots$

$$=\operatorname{cosec}\left(\theta+\frac{\pi}{2}\right)\left[\tan(n+1)\left(\theta+\frac{\pi}{2}\right)-\tan\left(\theta+\frac{\pi}{2}\right)\right].$$

90. $\tan^{-1}\dfrac{1}{1+1+1^2}+\tan^{-1}\dfrac{1}{1+2+2^2}+\tan^{-1}\dfrac{1}{1+3+3^2}$

$$+\cdots=\frac{\pi}{4}-\tan^{-1}\frac{1}{n+1}.$$

91. $\tan^{-1}x+\tan^{-1}\dfrac{x}{1+1\cdot 2\cdot x^2}+\tan^{-1}\dfrac{x}{1+2\cdot 3\cdot x^2}+\cdots$

$$=\tan^{-1}nx.$$

92. $\sin\alpha\sin 3\alpha+\sin\dfrac{\alpha}{2}\sin\dfrac{3\alpha}{2}+\sin\dfrac{\alpha}{2^2}\sin\dfrac{3\alpha}{2^2}+\cdots$

$$=\frac{1}{2}\left(\cos\frac{\alpha}{2^{n-2}}-\cos 4\alpha\right).$$

93. $\dfrac{1}{\cos\theta+\cos 3\theta}+\dfrac{1}{\cos\theta+\cos 5\theta}+\dfrac{1}{\cos\theta+\cos 7\theta}+\cdots$

$$=\tfrac{1}{2}\operatorname{cosec}\theta[\tan(n+1)\theta-\tan\theta].$$

94. $\dfrac{1}{2}\sec\theta+\dfrac{1}{2^2}\sec\theta\sec 2\theta+\dfrac{1}{2^3}\sec\theta\sec 2\theta\sec 2^2\theta+\cdots$

$$=\sin\theta(\cot\theta-\cot 2^n\theta).$$

95. $\dfrac{1}{2}\log\tan 2\theta+\dfrac{1}{2^2}\log\tan 2^2\theta+\dfrac{1}{2^3}\log\tan 2^3\theta+\cdots$

$$=\log 2\sin 2\theta-\frac{1}{2^n}\log 2\sin 2^{n+1}\theta.$$

Sum the following series to infinity:

96. $\cos\theta+\dfrac{\cos\theta}{1}\cos 2\theta+\dfrac{\cos^2\theta}{\lfloor 2}\cos 3\theta+\dfrac{\cos^2\theta}{\lfloor 3}\cos 4\theta+\cdots$

$$=e^{\cos^2\theta}\cos(\theta+\sin\theta\cos\theta).$$

97. $\sin\theta - \frac{\sin 2\theta}{\lfloor 2} + \frac{\sin 3\theta}{\lfloor 3} - \cdots = e^{-\cos\theta}\sin(\sin\theta).$

98. $1 - \frac{\cos 2\theta}{\lfloor 2} + \frac{\cos 4\theta}{\lfloor 4} - \cdots = \frac{1}{2}\cos(\cos\theta)(e^{\sin\theta} + e^{-\sin\theta}).$

99. $2\cos\theta + \frac{3}{2}\cos^2\theta + \frac{4}{3}\cos^3\theta + \frac{5}{4}\cos^4\theta + \cdots$

$$= \frac{\cos\theta}{1-\cos\theta} - \log(1-\cos\theta).$$

100. $\sin\theta\cos\theta + \frac{\sin 2\theta\cos^2\theta}{\lfloor 2} + \frac{\sin 3\theta\cos^3\theta}{\lfloor 3} + \cdots$

$$= e^{\cos^2\theta}\sin(\sin\theta\cos\theta).$$

101. $\cos\theta + \frac{\sin\theta}{1}\cos 2\theta + \frac{\sin^2\theta}{\lfloor 2}\cos 3\theta + \cdots$

$$= e^{\sin\theta\cos\theta}\cos(\theta + \sin^2\theta).$$

102. $\sin\theta + \frac{\sin\theta}{1}\sin 2\theta + \frac{\sin^2\theta}{\lfloor 2}\sin 3\theta +$

$$= e^{\sin\theta\cos\theta}\sin(\theta + \sin^2\theta).$$

103. $\cos\theta - \frac{1}{2}\cos 2\theta + \frac{1}{3}\cos 3\theta - \cdots = \log\left(2\cos\frac{\theta}{2}\right).$

104. $\cos 2\theta + \frac{1}{3}\cos 6\theta + \frac{1}{5}\cos 10\theta + \cdots = \frac{1}{2}\log\cot\frac{\theta}{2}.$

105. $x\sin\theta - \frac{x^2\sin 2\theta}{2} + \frac{x^3\sin 3\theta}{3} - \cdots = \cot^{-1}\left(\frac{\operatorname{cosec}\theta}{x} + \cot\theta\right).$

106. $x\cos\theta - \frac{x^2}{2}\cos 2\theta + \frac{x^3}{3}\cos 3\theta - \frac{x^4}{4}\cos 4\theta + \cdots$

$$= \log(1 + 2x\cos\theta + x^2)^{\frac{1}{2}}.$$

107. $\sin\theta\frac{\sin\theta}{1} - \sin 2\theta\frac{\sin^2\theta}{2} + \sin 3\theta\frac{\sin^3\theta}{3} - \cdots$

$$= \cot^{-1}(1 + \cot^2\theta + \cot\theta).$$

108. $\frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \cdots = \frac{\pi^4}{90}.$

109. $\frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \frac{1}{7^4} + \cdots = \frac{\pi^4}{96}.$

PART II.

SPHERICAL TRIGONOMETRY.

CHAPTER X.

FORMULÆ RELATIVE TO SPHERICAL TRIANGLES.

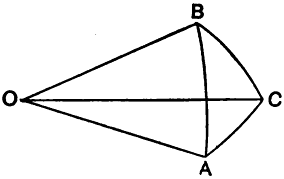
182. *Spherical Trigonometry* has for its object the solution of spherical triangles.

A *spherical triangle* is the figure formed by joining any three points on the surface of a sphere by arcs of great circles. The three points are called the *vertices* of the triangle; the three arcs are called the *sides* of the triangle.

Any two points on the surface of a sphere can be joined by two distinct arcs, which together make up a great circle passing through the points. Hence, when the points are not diametrically opposite, these arcs are unequal, one of them being less, the other greater, than 180°. It is not necessary to consider triangles in which a side is greater than 180°, since we may always replace such a side by the remaining arc of the great circle to which it belongs.

183. Geometric Principles. — It is shown in geometry (Art. 702), that if the vertex of a triedral angle is made the centre of a sphere, then the planes which form the triedral angle will cut the surface of the sphere in three arcs of great circles, forming a spherical triangle.

Thus, let O be the vertex of a triedral angle, and AOB, BOC, COA its face-angles. We may construct a sphere with its centre at O, and with any radius OA. Let AB, BC, CA be the arcs of great circles in which the planes of the face-angles AOB, BOC, COA cut the surface of this sphere; then ABC is a spherical triangle, and the arcs AB, BC, CA are its sides.



Now it is shown in geometry that the three face-angles AOB, BOC, COA are measured by the sides AB, BC, CA, respectively, of the spherical triangle, and that the diedral angles OA, OB, OC are equal to the angles A, B, C, respectively, of the spherical triangle ABC, and also that a diedral angle is measured by its plane angle.

There is then a correspondence between the triedral angle O–ABC and the spherical triangle ABC: the six parts of the triedral angle are represented by the corresponding six parts of the spherical triangle, and all the relations among the parts of the former are the same as the relations among the corresponding parts of the latter.

184. Fundamental Definitions and Properties. — The following definitions and properties are from Geometry, Book VIII.:

In every spherical triangle

Each side is less than the sum of the other two.

The sum of the three sides lies between 0° and 360°.

The sum of the three angles lies between 180° and 540°.

Each angle is greater than the difference between 180° and the sum of the other two.

If two sides are equal, the angles opposite them are equal; and conversely.

If two sides are unequal, the greater side lies opposite the greater angle; and conversely.

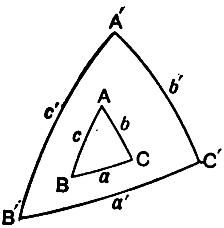
The perpendicular from the vertex to the base of an isosceles triangle bisects both the vertical angle and the base.

The *axis* of a circle is the diameter of the sphere perpendicular to the plane of the circle. The *poles* of a circle are the two points in which its axis meets the surface of the sphere.

One spherical triangle is called the *polar triangle* of a second spherical triangle when the sides of the first triangle have their poles at the vertices of the second.

If the first of two spherical triangles is the polar triangle of the second, then the second is the polar triangle of the first.

Two such triangles are said to be *polar* with respect to each other. Thus:



If A′B′C′ is the polar triangle of ABC, then ABC is the polar triangle of A′B′C′.

In two polar triangles, each angle of one is measured by the supplement of the corresponding side of the other. Thus:

$$A = 180^\circ - a', \qquad B = 180^\circ - b', \qquad C = 180^\circ - c',$$
$$a = 180^\circ - A', \qquad b = 180^\circ - B', \qquad c = 180^\circ - C'.$$

This result is of great importance; for if any general equation be established between the sides and angles of a spherical triangle, it holds of course for the polar triangle also. Hence, *by means of the above formulæ any theorem of a spherical triangle may be at once transformed into another theorem by substituting for each side and angle respectively the supplements of its opposite angle and side.*

If a spherical triangle has one right angle, it is called a *right* triangle; if it has two right angles, it is called a *bi-rectangular* triangle; and if it has three right angles, it is called a *tri-rectangular* triangle. If it has one side equal to a quadrant, it is called a *quadrantal* triangle; and if it has two sides equal to a quadrant, it is called a *bi-quadrantal* triangle.

Note. — It is shown in geometry that a spherical triangle may, in general, be *constructed* when any three of its six parts are given (not excepting the case in which the given parts are the three angles). In spherical trigonometry we investigate the methods by which the unknown parts of a spherical triangle may be *computed* from the above data.

EXAMPLES.

1. In the spherical triangle whose angles are A, B, C, prove

$$B + C - A < \pi \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$C + A - B < \pi \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

$$A + B - C < \pi \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

2. If C is a right angle, prove

$$A + B < \tfrac{3}{2}\pi \ (1), \text{ and } A - B < \frac{\pi}{2} \ (2).$$

3. The angles of a triangle are A, 45°, and 120°; find the maximum value of A. *Ans.* $A < 105°$.

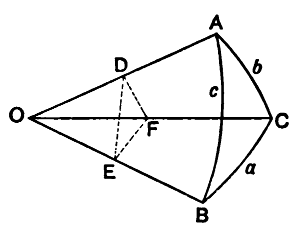
4. The angles of a triangle are A, 30°, and 150°; find the maximum value of A. *Ans.* $A < 60°$.

5. The angles of a triangle are A, 20°, and 110°; find the maximum value of A. *Ans.* $A < 90°$.

6. Any side of a triangle is greater than the difference between the other two.

RIGHT SPHERICAL TRIANGLES.

185. Formulæ for Right Triangles. — Let ABC be a spherical triangle in which C is a right angle, and let O be the centre of the sphere; then will OA, OB, OC be radii: let a, b, c denote the sides of the triangle opposite the angles A, B, C, respectively; then a, b, and c are the measures of the angles BOC, COA, and AOB.



From any point D in OA draw DE ⊥ to OC, and from E draw EF ⊥ to OB, and join DF. Then DE is ⊥ to EF (Geom. Art. 537). Hence (Geom. Art. 507),

DF is ⊥ to OB; ∴ ∠ DFE = ∠ B. (Art. 183)

Now $\frac{OF}{OD} = \frac{OF}{OE} \cdot \frac{OE}{OD}$; that is, $\cos c = \cos a \cos b$ . . (1)

$\frac{DE}{OD} = \frac{DE}{DF} \cdot \frac{DF}{OD}$; that is, $\sin b = \sin B \sin c$ . . (2)

Interchanging a's and b's, $\sin a = \sin A \sin c$ . . (3)

$\frac{EF}{OF} = \frac{EF}{DF} \cdot \frac{DF}{OF}$; that is, $\tan a = \cos B \tan c$ . . (4)

Interchanging a's and b's, $\tan b = \cos A \tan c$ . . (5)

$\frac{DE}{OE} = \frac{DE}{EF} \cdot \frac{EF}{OE}$; that is, $\tan b = \tan B \sin a$ . . (6)

Interchanging a's and b's, $\tan a = \tan A \sin b$ . . (7)

Multiply (6) and (7) together, and we get

$$\tan A \tan B = \frac{1}{\cos a \cos b} = \frac{1}{\cos c}, \text{ by (1)}$$

$$\therefore \cos c = \cot A \cot B \quad . . . . . . (8)$$

Multiply crosswise (3) and (4), and we get

$$\sin a \cos B \tan c = \tan a \sin A \sin c.$$

$$\therefore \cos B = \frac{\sin A \cos c}{\cos a} = \sin A \cos b, \text{ by (1)} \quad . . . (9)$$

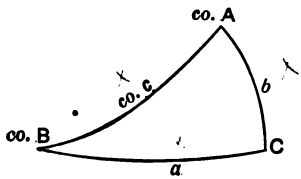
Interchanging a's and b's,

$$\cos A = \sin B \cos a \quad . . . . (10)$$

Sch. By these ten formulæ, every case of right triangles can be solved; for every one of these ten formulæ is a distinct combination, involving three out of the five quantities, a, b, c, A, B, and there can be but ten combinations in all. Hence, any two of the five quantities being given and a third required, that third quantity may be determined by some one of the above ten formulæ.

186. Napier's Rules.—The ten preceding formulæ, which may be found difficult to remember, have been included under two simple rules, called after their inventor, *Napier's Rules of the Circular Parts.*

Let ABC be a right spherical triangle. Omit the right angle C. Then the two sides a and b, which include the right angle, the complement of the hypotenuse c, and the complements of the oblique angles A and B, are called the *circular parts* of the triangle. Thus, there are *five* circular parts, arranged in the figure in the following order: a, b, *co.* A, *co.* c, *co.* B.



Any one of these five parts may be selected and called the *middle part;* then the two parts next to it are called *adjacent parts,* and the remaining two parts are called *opposite parts.* Thus, if *co.* A is selected as the middle part, then b and *co.* c are the adjacent parts, and a and *co.* B are the opposite parts.

Then Napier's Rules are:

(1) *The sine of the middle part equals the product of the tangents of the adjacent parts.*

(2) *The sine of the middle part equals the product of the cosines of the opposite parts.*

NOTE 1.—It will assist the student in remembering these rules to notice the occurrence of the vowel *i* in *sine* and *middle,* of the vowel *a* in *tangent* and *adjacent,* and of the vowel *o* in *cosine* and *opposite.*

Napier's Rules* may be made evident by taking in detail each of the five parts as middle part, and comparing the equations thus found with the formulæ of Art. 185.

Thus, let *co. c be the middle part.* The rules give

$\sin(co.\,c) = \tan(co.\,A)\tan(co.\,B)$; $\therefore \cos c = \cot A \cot B$ . . . . (8)

$\sin(co.\,c) = \cos a \cos b$; $\therefore \cos c = \cos a \cos b$ . . . . (1)

co. B *the middle part.*

$\sin(co.\,B) = \tan a \tan(co.\,c)$; $\therefore \cos B = \tan a \cot c$ . . . . (4)

$\sin(co.\,B) = \cos b \cos(co.\,A)$; $\therefore \cos B = \cos b \sin A$ . . . . (9)

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* While some find these rules to be useful aids to the memory, others question their utility.

a the middle part.

$\sin a = \tan b \tan(co.\,B)$; $\quad \therefore \sin a = \tan b \cot B$ . . . . (6)

$\sin a = \cos(co.\,A)\cos(co.\,c)$; $\quad \therefore \sin a = \sin A \sin c$ . . . . (3)

b the middle part.

$\sin b = \tan a \tan(co.\,A)$; $\quad \therefore \sin b = \tan a \cot A$ . . . . (7)

$\sin b = \cos(co.\,c)\cos(co.\,B)$; $\quad \therefore \sin b = \sin c \sin B$ . . . . (2)

co. A *the middle part.*

$\sin(co.\,A) = \tan b \tan(co.\,c)$; $\quad \therefore \cos A = \tan b \cot c$ . . . . (5)

$\sin(co.\,A) = \cos a \cos(co.\,B)$; $\quad \therefore \cos A = \cos a \sin B$ . . . . (10)

NOTE 2. — In applying these rules it is not necessary to use the notation *co. c*, *co.* A, *co.* B, since we may write at once cos *c* for sin (*co. c*), etc.

187. The Species of the Parts. — If two parts of a spherical triangle are either both less than 90° or both greater than 90°, they are said to be of the *same species*. But if one part is less than 90° and the other part is greater than 90°, they are of *different species*.

In order to determine whether the required parts are less or greater than 90°, it will be necessary carefully to observe their algebraic signs. If the required part is determined by means of its cosine, tangent, or cotangent, the algebraic sign of the result will show whether it is less or greater than 90°. But when a required part is found in terms of its sine, it will be ambiguous, since the sines are positive in both the first and second quadrants. This ambiguity, however, may generally be removed by either of the following principles:

(1) *In a right spherical triangle, either of the sides containing the right triangle is of the same species as the opposite angle.*

(2) *The three sides of a right spherical triangle* (omitting bi-rectangular or tri-rectangular triangles) *are either all acute, or else one is acute and the other two obtuse.*

The first follows from the equation

$$\cos A = \cos a \sin B,$$

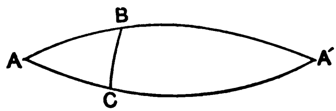
in which, since $\sin B$ is always positive $(B < 180°)$, $\cos A$ and $\cos a$ must have the same sign; *i.e.*, A and a must be either both $<$ or both $>$ 90°.

The second follows from the equation

$$\cos c = \cos a \cos b.$$

188. Ambiguous Solution. — When the given parts of a right triangle are a side and its opposite angle, the triangle cannot be determined.

For two right spherical triangles ABC, A′BC, right angled at C, may always be found, having the angles A and A′ equal, and BC, the side opposite these angles, the same in both triangles,



but the remaining sides, AB, AC, and the remaining angle ABC of the one triangle are the supplements of the remaining sides A′B, A′C, and the remaining angle A′BC of the other triangle. It is therefore ambiguous whether ABC or A′BC be the triangle required.

This ambiguity will also be found to exist, if it be attempted to determine the triangle by the equation

$$\sin b = \tan a \cot A,$$

since it cannot be determined from this equation whether the side AC is to be taken or its supplement A′C.

189. Quadrantal Triangles. — The polar triangle of a right triangle has one side a quadrant, and is therefore a *quadrantal triangle* (Art. 184). The formulæ for quadrantal triangles may be obtained by applying the ten formulæ of Art. 185 to the polar triangle. They are as follows, c being the quadrantal side:

$$\cos C = -\cos A \cos B \quad . \quad . \quad . \quad (1)$$

$$\sin B = \sin b \sin C \quad . \quad . \quad . \quad (2)$$

$\sin A = \sin a \sin C$ . . . . (3)

$\cos b = -\tan A \cot C$ . . . . (4)

$\cos a = -\tan B \cot C$ . . . . (5)

$\sin A = \tan B \cot b$ . . . . (6)

$\sin B = \tan A \cot a$ . . . . (7)

$\cos C = -\cot a \cot b$ . . . . (8)

$\cos b = \cos B \sin a$ . . . . (9)

$\cos a = \cos A \sin b$ . . . . (10)

EXAMPLES.

In the right triangle ABC in which the angle C is the right angle, prove the following relations:

1. $\sin^2 a + \sin^2 b - \sin^2 c = \sin^2 a \sin^2 b.$

2. $\cos^2 A \sin^2 c = \sin^2 c - \sin^2 a.$

3. $\sin^2 A \cos^2 c = \sin^2 A - \sin^2 a.$

4. $\sin^2 A \cos^2 b \sin^2 c = \sin^2 c - \sin^2 b.$

5. $2 \cos c = \cos(a + b) + \cos(a - b).$

6. $\tan \frac{1}{2}(c + a) \tan \frac{1}{2}(c - a) = \tan^2 \frac{1}{2} b.$

7. $\sin^2 \frac{c}{2} = \sin^2 \frac{a}{2} \cos^2 \frac{b}{2} + \cos^2 \frac{a}{2} \sin^2 \frac{b}{2}.$

8. $\sin(c - b) = \tan^2 \frac{A}{2} \sin(c + b).$

9. If $b = c = \frac{\pi}{2}$, prove $\cos a = \cos A.$

10. If $a = b = c$, prove $\sec A = 1 + \sec a.$

11. If $c < 90°$, show that a and b are of the same species.

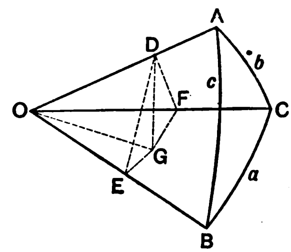
12. If $c > 90°$, a and b are of different species.

13. A side and the hypotenuse are of the same or opposite species, according as the included angle $<$, or $> \frac{\pi}{2}$.

OBLIQUE SPHERICAL TRIANGLES.

190. Law of Sines.—*In any spherical triangle the sines of the sides are proportional to the sines of the opposite angles.*

Let ABC be a spherical triangle, O the centre of the sphere; and let a, b, c denote the sides of the triangle opposite the angles A, B, C, respectively. Then a, b, and c are the measures of the angles BOC, COA, and AOB.



From any point D in OA draw DG ⊥ to the plane BOC, and from G draw GE, GF ⊥ to OB, OC. Join DE, DF, and GO. Then DG is ⊥ to GE, GF, and GO (Geom. Art. 487). Hence, DE is ⊥ to OB, and DF ⊥ to OC (Geom. Art. 507).

$$\therefore \angle \text{DEG} = \angle \text{B, and } \angle \text{DFG} = \angle \text{C} \quad . \quad . \quad \text{(Art. 183)}$$

In the right plane triangles DGE, DGF, ODE, ODF,

$$\text{DG} = \text{DE} \sin \text{B} = \text{OD} \sin \text{DOE} \sin \text{B} = \text{OD} \sin c \sin \text{B},$$

$$\text{DG} = \text{DF} \sin \text{C} = \text{OD} \sin \text{DOF} \sin \text{C} = \text{OD} \sin b \sin \text{C}.$$

$$\therefore \sin c \sin \text{B} = \sin b \sin \text{C};$$

or
$$\sin b : \sin c :: \sin \text{B} : \sin \text{C}.$$

Similarly, it may be shown that

$$\sin a : \sin c :: \sin \text{A} : \sin \text{C}.$$

$$\therefore \frac{\sin a}{\sin \text{A}} = \frac{\sin b}{\sin \text{B}} = \frac{\sin c}{\sin \text{C}}.$$

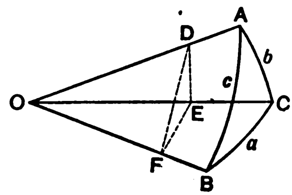
NOTE.—The common value of these three ratios is called the *modulus* of the spherical triangle.

Sch. In the figure, B, C, b, c are each less than a right angle; but it will be found on examination that the proof will hold when the figure is modified to meet any case which can occur. For example, if B alone is greater than

90°, the point G will fall outside of OB instead of between OB and OC. Then DEG will be the *supplement* of B, and thus we shall still have $\sin \text{DEG} = \sin \text{B}$.

191. Law of Cosines. — *In any spherical triangle, the cosine of each side is equal to the product of the cosines of the other two sides, plus the product of the sines of those sides into the cosine of their included angle.*

Let ABC be a spherical triangle, O the centre of the sphere, and a, b, c the sides of the triangle opposite the angles A, B, C, respectively. Then



$$a = \angle \text{BOC},$$
$$b = \angle \text{COA},$$
$$c = \angle \text{AOB}.$$

From any point D in OA draw, in the planes AOB, AOC, respectively, the lines DE, DF $\perp$ to OA. Then

$$\angle \text{EDF} = \angle \text{A} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\text{Art. } 183)$$

Join EF; then in the plane triangles EOF, EDF, we have

$$\overline{\text{EF}}^2 = \overline{\text{OE}}^2 + \overline{\text{OF}}^2 - 2\,\text{OE} \cdot \text{OF} \cos \text{EOF} \quad . \quad . \quad (1)$$

$$\overline{\text{EF}}^2 = \overline{\text{DE}}^2 + \overline{\text{DF}}^2 - 2\,\text{DE} \cdot \text{DF} \cos \text{EDF} \quad . \quad . \quad (2)$$

also in the right triangles EOD, FOD, we have

$$\overline{\text{OE}}^2 - \overline{\text{DE}}^2 = \overline{\text{OD}}^2, \text{ and } \overline{\text{OF}}^2 - \overline{\text{DF}}^2 = \overline{\text{OD}}^2 \quad . \quad (3)$$

Subtracting (2) from (1), and reducing by (3), and transposing, we get

$$2\,\text{OE} \cdot \text{OF} \cos \text{EOF} = 2\,\overline{\text{OD}}^2 + 2\,\text{DE} \cdot \text{DF} \cos \text{EDF}.$$

$$\therefore \cos \text{EOF} = \frac{\text{OD}}{\text{OF}} \cdot \frac{\text{OD}}{\text{OE}} + \frac{\text{DF}}{\text{OF}} \cdot \frac{\text{DE}}{\text{OE}} \cos \text{EDF},$$

or

$$\cos a = \cos b \cos c + \sin b \sin c \cos \text{A} \quad (4)$$

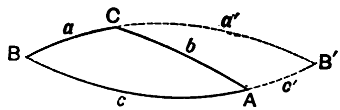
By treating the other edges in order in the same way, or by advancing letters (see Note, Art. 96) we get

$$\cos b = \cos c \cos a + \sin c \sin a \cos B \quad . \quad . \quad (5)$$

$$\cos c = \cos a \cos b + \sin a \sin b \cos C \quad . \quad . \quad (6)$$

Sch. Formula (4) has been proved only for the case in which the sides b and c are less than quadrants; but it may be shown to be true when these sides are not less than quadrants, as follows:

(1) Suppose c is greater than 90°. Produce BA, BC to meet in B′, and put $AB' = c'$, $CB' = a'$.



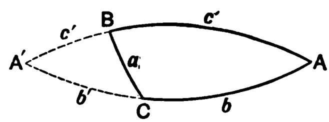
Then, from the triangle AB′C, we have by (4)

$$\cos a' = \cos b \cos c' + \sin b \sin c' \cos B'AC,$$

or $\cos(\pi - a) = \cos b \cos(\pi - c) + \sin b \sin(\pi - c) \cos(\pi - A)$.

$$\therefore \cos a = \cos b \cos c + \sin b \sin c \cos A.$$

(2) Suppose both b and c to be greater than 90°. Produce AB, AC to meet in A′, and put $A'B = c'$, $A'C = b'$.



Then, from the triangle A′BC, we have by (4)

$$\cos a = \cos b' \cos c' + \sin b' \sin c' \cos A';$$

but $\quad b' = \pi - b, \; c' = \pi - c, \; A' = A.$

$$\therefore \cos a = \cos b \cos c + \sin b \sin c \cos A.$$

The triangle AB′C is called the *colunar* triangle of ABC.

192. Relation between a Side and the Three Angles. — *In any spherical triangle* ABC,

$$\cos A = -\cos B \cos C + \sin B \sin C \cos a.$$

Let A′B′C′ be the polar triangle of ABC, and denote its angles and sides by A′, B′, C′, a', b', c'; then we have by (4) of Art. 191

$$\cos a' = \cos b' \cos c' + \sin b' \sin c' \cos A';$$

but $a' = \pi - A,\ b' = \pi - B,\ c' = \pi - C$, etc. . (Art. 184)

Hence, substituting, we get

$$\cos A = -\cos B \cos C + \sin B \sin C \cos a \quad . \quad . \quad . \quad . \quad (1)$$

Similarly,

$$\cos B = -\cos C \cos A + \sin C \sin A \cos b \quad . \quad . \quad . \quad . \quad (2)$$

$$\cos C = -\cos A \cos B + \sin A \sin B \cos c \quad . \quad . \quad . \quad . \quad (3)$$

Rem. — This process is called "applying the formula to the polar triangle." By means of the polar triangle, any formula of a spherical triangle may be immediately transformed into another, in which angles take the place of sides, and sides of angles.

193. To show that in a spherical triangle ABC,

$$\cot a \sin b = \cot A \sin C + \cos C \cos b.$$

Multiply (6) of Art. 191 by $\cos b$, and substitute the result in (4) of Art. 191, and we get

$$\cos a = \cos a \cos^2 b + \sin a \sin b \cos b \cos C + \sin b \sin c \cos A.$$

Transpose $\cos a \cos^2 b$, and divide by $\sin a \sin b$; thus,

$$\cot a \sin b = \cos b \cos C + \frac{\sin c \cos A}{\sin a}$$

$$= \cos b \cos C + \cot A \sin C \quad . \quad \text{(by Art. 190)}$$

By interchanging the letters, we obtain five other formulæ like the preceding one. The six formulæ are as follows:

$$\cot a \sin b = \cot A \sin C + \cos C \cos b \quad . \quad . \quad . \quad . \quad (1)$$

$$\cot a \sin c = \cot A \sin B + \cos B \cos c \quad . \quad . \quad . \quad . \quad (2)$$

$$\cot b \sin a = \cot B \sin C + \cos C \cos a \quad . \quad . \quad . \quad . \quad (3)$$

$$\cot b \sin c = \cot B \sin A + \cos A \cos c \quad . \quad . \quad . \quad . \quad (4)$$

$$\cot c \sin a = \cot C \sin B + \cos B \cos a \quad . \quad . \quad . \quad . \quad (5)$$

$$\cot c \sin b = \cot C \sin A + \cos A \cos b \quad . \quad . \quad . \quad . \quad (6)$$

EXAMPLES.

1. If a, b, c be the sides of a spherical triangle, a', b', c' the sides of its polar triangle, prove

$$\sin a : \sin b : \sin c = \sin a' : \sin b' : \sin c'.$$

2. If the bisector AD of the angle A of a spherical triangle divide the side BC into the segments $\text{CD} = b'$, $\text{BD} = c'$, prove

$$\sin b : \sin c = \sin b' : \sin c'.$$

3. If D be any point of the side BC, prove that

$$\cot \text{AB} \sin \text{DAC} + \cot \text{AC} \sin \text{DAB} = \cot \text{AD} \sin \text{BAC}.$$

$$\cot \text{ABC} \sin \text{DC} + \cot \text{ACB} \sin \text{BD} = \cot \text{ADB} \sin \text{BC}.$$

4. If α, β, γ be the perpendiculars of a triangle, prove that

$$\sin a \sin \alpha = \sin b \sin \beta = \sin c \sin \gamma.$$

5. In Ex. 4 prove that

$$\sin a \cos \alpha = \sqrt{\cos^2 b + \cos^2 c - 2 \cos a \cos b \cos c}.$$

194. Useful Formulæ. — Several other groups of useful formulæ are easily obtained from those of Art. 191; the following are left as exercises for the student:

$$\sin a \cos \text{B} = \cos b \sin c - \sin b \cos c \cos \text{A} \quad . \quad . \quad (1)$$

$$\sin a \cos \text{C} = \sin b \cos c - \cos b \sin c \cos \text{A} \quad . \quad . \quad (2)$$

$$\sin b \cos \text{A} = \cos a \sin c - \sin a \cos c \cos \text{B} \quad . \quad . \quad (3)$$

$$\sin b \cos \text{C} = \sin a \cos c - \cos a \sin c \cos \text{B} \quad . \quad . \quad (4)$$

$$\sin c \cos \text{A} = \cos a \sin b - \sin a \cos b \cos \text{C} \quad . \quad . \quad (5)$$

$$\sin c \cos \text{B} = \sin a \cos b - \cos a \sin b \cos \text{C} \quad . \quad . \quad (6)$$

Applying these six formulæ to the polar triangle, we obtain the following six:

$$\sin A \cos b = \cos B \sin C + \sin B \cos C \cos a \quad . \quad . \quad (7)$$

$$\sin A \cos c = \sin B \cos C + \cos B \sin C \cos a \quad . \quad . \quad (8)$$

$$\sin B \cos a = \cos A \sin C + \sin A \cos C \cos b \quad . \quad . \quad (9)$$

$$\sin B \cos c = \sin A \cos C + \cos A \sin C \cos b \quad . \quad . \quad (10)$$

$$\sin C \cos a = \cos A \sin B + \sin A \cos B \cos c \quad . \quad . \quad (11)$$

$$\sin C \cos b = \sin A \cos B + \cos A \sin B \cos c \quad . \quad . \quad (12)$$

195. Formulæ for the Half Angles. — *To express the sine, cosine, and tangent of half an angle of a spherical triangle in terms of the sides.*

I. By (4) of Art. 191 we have

$$\cos A = \frac{\cos a - \cos b \cos c}{\sin b \sin c} = 1 - 2\sin^2\frac{A}{2} \quad \text{(Art. 49)}$$

$$\therefore\ 2\sin^2\frac{A}{2} = 1 - \frac{\cos a - \cos b \cos c}{\sin b \sin c}$$

$$= \frac{\cos(b-c) - \cos a}{\sin b \sin c}.$$

$$\therefore\ \sin^2\frac{A}{2} = \frac{\sin\frac{1}{2}(a+b-c)\sin\frac{1}{2}(a-b+c)}{\sin b \sin c} \quad \text{(Art. 45)}$$

Let $2s = a + b + c$; so that s is half the sum of the sides of the triangle; then

$$a + b - c = 2(s - c), \text{ and } a - b + c = 2(s - b).$$

$$\therefore\ \sin^2\frac{A}{2} = \frac{\sin(s-b)\sin(s-c)}{\sin b \sin c}.$$

$$\therefore\ \sin\frac{A}{2} = \sqrt{\frac{\sin(s-b)\sin(s-c)}{\sin b \sin c}} \quad . \quad . \quad . \quad (1)$$

Advancing letters,

$$\sin\frac{B}{2}=\sqrt{\frac{\sin(s-c)\sin(s-a)}{\sin c\sin a}} \quad . \quad . \quad . \quad (2)$$

$$\sin\frac{C}{2}=\sqrt{\frac{\sin(s-a)\sin(s-b)}{\sin a\sin b}} \quad . \quad . \quad . \quad (3)$$

II. $$2\cos^2\frac{A}{2}=1+\cos A \quad . \quad . \quad . \quad . \quad . \quad . \quad (\text{Art. } 49)$$

$$=1+\frac{\cos a-\cos b\cos c}{\sin b\sin c}$$

$$=\frac{\cos a-\cos(b+c)}{\sin b\sin c}.$$

$$\therefore\ \cos^2\frac{A}{2}=\frac{\sin\frac{1}{2}(a+b+c)\sin\frac{1}{2}(b+c-a)}{\sin b\sin c}$$

$$=\frac{\sin s\sin(s-a)}{\sin b\sin c}.$$

$$\therefore\ \cos\frac{A}{2}=\sqrt{\frac{\sin s\sin(s-a)}{\sin b\sin c}} \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Advancing letters,

$$\cos\frac{B}{2}=\sqrt{\frac{\sin s\sin(s-b)}{\sin c\sin a}} \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

$$\cos\frac{C}{2}=\sqrt{\frac{\sin s\sin(s-c)}{\sin a\sin b}} \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

III. By division, we obtain

$$\tan\frac{A}{2}=\sqrt{\frac{\sin(s-b)\sin(s-c)}{\sin s\sin(s-a)}} \quad . \quad . \quad . \quad (7)$$

$$\tan\frac{B}{2}=\sqrt{\frac{\sin(s-c)\sin(s-a)}{\sin s\sin(s-b)}} \quad . \quad . \quad . \quad (8)$$

$$\tan\frac{C}{2}=\sqrt{\frac{\sin(s-a)\sin(s-b)}{\sin s\sin(s-c)}} \quad . \quad . \quad . \quad (9)$$

Sch. The positive sign must be given to the radicals in each case in this article, because $\frac{1}{2}A$, $\frac{1}{2}B$, $\frac{1}{2}C$ are each less than 90°.

Cor. 1.
$$\tan\frac{A}{2}\tan\frac{B}{2}=\frac{\sin(s-c)}{\sin s} \quad . \quad . \quad . \quad . \quad . \quad (10)$$

$$\tan\frac{B}{2}\tan\frac{C}{2}=\frac{\sin(s-a)}{\sin s} \quad . \quad . \quad . \quad . \quad . \quad (11)$$

$$\tan\frac{C}{2}\tan\frac{A}{2}=\frac{\sin(s-b)}{\sin s} \quad . \quad . \quad . \quad . \quad . \quad (12)$$

Cor. 2. Since $\sin A = 2\sin\frac{A}{2}\cos\frac{A}{2}$,

$$\therefore \sin A = \frac{2\sqrt{\sin s\sin(s-a)\sin(s-b)\sin(s-c)}}{\sin b\sin c} \quad . \quad (13)$$

$$= \frac{2n}{\sin b\sin c} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

where $n^2=\sin s\sin(s-a)\sin(s-b)\sin(s-c)$.

EXAMPLES.

1. Prove $\sin^2 A=\frac{1-\cos^2 a-\cos^2 b-\cos^2 c+2\cos a\cos b\cos c}{\sin^2 b\sin^2 c}$

$$=\frac{4n^2}{\sin^2 b\sin^2 c},$$

where $4n^2=1-\cos^2 a-\cos^2 b-\cos^2 c+2\cos a\cos b\cos c$.

2. Prove $\cos c=\cos(a+b)\sin^2\frac{C}{2}+\cos(a-b)\cos^2\frac{C}{2}$.

3. Prove $\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}=\frac{\sin(s-a)\sin(s-b)\sin(s-c)}{\sin a\sin b\sin c}$.

4. Prove $\frac{\cos A+\cos B}{1-\cos C}=\frac{\sin(a+b)}{\sin c}$.

5. Prove $\Sigma\frac{\cos A+\cos B}{1-\cos C}\sin(a-b)\sin c=0$.

6. Prove $\frac{\cos A-\cos B}{1+\cos C}=\frac{\sin(a\sim b)}{\sin c}$.

196. Formulæ for the Half Sides. — *To express the sine, cosine, and tangent of half a side of a spherical triangle in terms of the angles.*

By (1) of Art. 192, we have

$$\cos a = \frac{\cos A + \cos B \cos C}{\sin B \sin C} = 1 - 2\sin^2\frac{a}{2} \quad . \quad \text{(Art. 49)}$$

$$\therefore\ 2\sin^2\frac{a}{2} = -\frac{\cos A + \cos(B + C)}{\sin B \sin C}.$$

$$\therefore\ \sin^2\frac{a}{2} = -\frac{\cos\frac{1}{2}(A+B+C)\cos\frac{1}{2}(B+C-A)}{\sin B \sin C} \quad \text{(Art. 45)}$$

Let $2S = A + B + C$; then $B + C - A = 2(S - A)$.

Proceeding in the same way as in Art. 195, we find the following expressions for the sides, in terms of the three angles:

$$\sin\frac{a}{2} = \sqrt{-\frac{\cos S \cos(S - A)}{\sin B \sin C}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$\sin\frac{b}{2} = \sqrt{-\frac{\cos S \cos(S - B)}{\sin C \sin A}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

$$\sin\frac{c}{2} = \sqrt{-\frac{\cos S \cos(S - C)}{\sin A \sin B}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

$$\cos\frac{a}{2} = \sqrt{\frac{\cos(S - B)\cos(S - C)}{\sin B \sin C}} \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

$$\cos\frac{b}{2} = \sqrt{\frac{\cos(S - C)\cos(S - A)}{\sin C \sin A}} \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

$$\cos\frac{c}{2} = \sqrt{\frac{\cos(S - A)\cos(S - B)}{\sin A \sin B}} \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

$$\tan\frac{a}{2} = \sqrt{-\frac{\cos S \cos(S - A)}{\cos(S - B)\cos(S - C)}} \quad . \quad . \quad . \quad . \quad . \quad (7)$$

$$\tan\frac{b}{2}=\sqrt{-\frac{\cos S\cos(S-B)}{\cos(S-C)\cos(S-A)}} \quad . \quad . \quad . \quad . \quad . \quad (8)$$

$$\tan\frac{c}{2}=\sqrt{-\frac{\cos S\cos(S-C)}{\cos(S-A)\cos(S-B)}} \quad . \quad . \quad . \quad . \quad . \quad (9)$$

Sch. 1. These formulæ may also be obtained immediately from those of Art. 195 by means of the polar triangle.

Sch. 2. The positive sign must be given to the above radicals, because $\frac{a}{2}, \frac{b}{2}, \frac{c}{2}$, are each less than 90°.

Sch. 3. These values of the sines, cosines, and tangents of the half sides are always *real.*

For S is $> 90°$ and $< 270°$ (Art. 184), so that cos S is always *negative*.

Also, in the polar triangle, any side is less than the sum of the other two (Art. 184).

$$\therefore\ \pi - A < \pi - B + \pi - C.$$

$$\therefore\ B + C - A < \pi.$$

$$\therefore\ \cos(S-A) \text{ is } \textit{positive}.$$

Similarly, cos (S − B) and cos (S − C) are positive.

Cor. Since $\qquad \sin a = 2\sin\frac{a}{2}\cos\frac{a}{2},$

$$\therefore\ \sin a = \frac{2\sqrt{-\cos S\cos(S-A)\cos(S-B)\cos(S-C)}}{\sin B\sin C} \quad (10)$$

$$= \frac{2N}{\sin B\sin C},$$

where $N = \sqrt{-\cos S\cos(S-A)\cos(S-B)\cos(S-C)}$.

EXAMPLES.

1. Prove $\cos C = -\cos(A+B)\cos^2\frac{c}{2} - \cos(A-B)\sin^2\frac{c}{2}$.

2. Prove $\sin\frac{a}{2}\sin\frac{b}{2}\sin\frac{c}{2} = \frac{-N\cos S}{\sin A\sin B\sin C}$,

where $N = \sqrt{-\cos S\cos(S-A)\cos(S-B)\cos(S-C)}$.

197. Napier's Analogies.

Let $$m = \frac{\sin A}{\sin a} = \frac{\sin B}{\sin b} \text{ (Art. 190)} \quad (1)$$

$$= \frac{\sin A + \sin B}{\sin a + \sin b} \text{ (Algebra)} \quad (2)$$

or $$= \frac{\sin A - \sin B}{\sin a - \sin b} \quad . \quad . \quad . \quad (3)$$

$$\cos A + \cos B\cos C = \sin B\sin C\cos a \quad \text{(Art. 192)}$$

$$= m\sin C\sin b\cos a, \text{ by } (1) (4)$$

and $$\cos B + \cos C\cos A = \sin C\sin A\cos b$$

$$= m\sin C\sin a\cos b \quad . \quad . \quad (5)$$

$$\therefore (\cos A + \cos B)(1 + \cos C) = m\sin C\sin(a+b), \quad (6)$$

from (4) and (5)

Dividing (2) by (6),

$$\frac{\sin A + \sin B}{\cos A + \cos B} = \frac{\sin a + \sin b}{\sin(a+b)} \cdot \frac{1 + \cos C}{\sin C}.$$

$$\therefore \tan\tfrac{1}{2}(A+B) = \frac{\cos\frac{1}{2}(a-b)}{\cos\frac{1}{2}(a+b)}\cot\frac{C}{2}, \quad (7)$$

(Arts. 45, 46, and 49)

Similarly, $$\tan\tfrac{1}{2}(A-B) = \frac{\sin\frac{1}{2}(a-b)}{\sin\frac{1}{2}(a+b)}\cot\frac{C}{2} \quad . \quad . \quad (8)$$

Writing $\pi - A$ for a, etc., by Art. 184, we obtain from (7) and (8)

$$\tan \tfrac{1}{2}(a+b) = \frac{\cos \frac{1}{2}(A-B)}{\cos \frac{1}{2}(A+B)} \tan \frac{c}{2} \quad . \quad . \quad . \quad . \quad (9)$$

$$\tan \tfrac{1}{2}(a-b) = \frac{\sin \frac{1}{2}(A-B)}{\sin \frac{1}{2}(A+B)} \tan \frac{c}{2} \quad . \quad . \quad . \quad . \quad (10)$$

Sch. The formulæ (7), (8), (9), (10) are known as *Napier's Analogies*, after their discoverer. The last two may be proved without the polar triangle by starting with the formulæ of Art. 191.

Cor. In any spherical triangle whose parts are positive, and less than 180°, *the half-sum of any two sides and the half-sum of their opposite angles are of the same species.*

For, since $\cos \frac{1}{2}(a-b)$ and $\cot \frac{C}{2}$ are necessarily positive, therefore by (7) $\tan \frac{1}{2}(A+B)$ and $\cos \frac{1}{2}(a+b)$ are both positive or both negative.

$\therefore \frac{1}{2}(A+B)$ and $\frac{1}{2}(a+b)$ are both $>$ or both $<$ or both $= 90°$.

198. Delambre's (or Gauss's) Analogies.

$$\sin \tfrac{1}{2}(A+B)$$

$$= \sin \frac{A}{2} \cos \frac{B}{2} + \cos \frac{A}{2} \sin \frac{B}{2}$$

$$= \sqrt{\frac{\sin(s-b)\sin(s-c)}{\sin b \sin c}} \sqrt{\frac{\sin s \sin(s-b)}{\sin c \sin a}}$$

$$+ \sqrt{\frac{\sin s \sin(s-a)}{\sin b \sin c}} \sqrt{\frac{\sin(s-c)\sin(s-a)}{\sin c \sin a}} \quad \text{(Art. 195)}$$

$$= \frac{\sin(s-b) + \sin(s-a)}{\sin c} \sqrt{\frac{\sin s \sin(s-c)}{\sin a \sin b}}$$

$$= \frac{\cos \frac{1}{2}(a-b)}{\cos \frac{c}{2}} \cos \frac{C}{2} \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(Arts. 45 and 195)}$$

$$\therefore \sin \tfrac{1}{2}(A + B) \cos \frac{c}{2} = \cos \tfrac{1}{2}(a - b) \cos \frac{C}{2} \quad . \quad . \quad . \quad (1)$$

Similarly, we obtain the following three equations:

$$\sin \tfrac{1}{2}(A - B) \sin \frac{c}{2} = \sin \tfrac{1}{2}(a - b) \cos \frac{C}{2} \quad . \quad . \quad . \quad (2)$$

$$\cos \tfrac{1}{2}(A + B) \cos \frac{c}{2} = \cos \tfrac{1}{2}(a + b) \sin \frac{C}{2} \quad . \quad . \quad . \quad (3)$$

$$\cos \tfrac{1}{2}(A - B) \sin \frac{c}{2} = \sin \tfrac{1}{2}(a + b) \sin \frac{C}{2} \quad . \quad . \quad . \quad (4)$$

Sch. 1. When the sides and angles are all less than 180°, both members of these equations are positive.

Sch. 2. Napier's analogies may be obtained from Delambre's by division.

Note.—Delambre's analogies were discovered by him in 1807, and published in the Connaissance des Temps for 1809, p. 443. They were subsequently discovered independently by Gauss, and published by him, and are sometimes improperly called Gauss's equations. Both systems may be proved geometrically. The geometric proof is the one originally given by Delambre. It was rediscovered by Professor Crofton in 1869, and published in the Proceedings of the London Mathematical Society, Vol. III. [Casey's Trigonometry, p. 41].

EXAMPLES.

In the right triangle ABC, in which C is the right angle, prove the following relations in Exs. 1–45:

1. $\sin^2 a \cos^2 b = \sin(c + b) \sin(c - b)$.

2. $\tan^2 a : \tan^2 b = \sin^2 c - \sin^2 b : \sin^2 c - \sin^2 a$.

3. $\cos^2 a \cos^2 B = \sin^2 A - \sin^2 a$.

4. $\cos^2 A + \cos^2 c = \cos^2 A \cos^2 c + \cos^2 a$.

5. $\sin^2 A - \cos^2 B = \sin^2 a \sin^2 B$.

6. If one of the sides of a right triangle be equal to the opposite angle, the remaining parts are each equal to 90°.

7. If the angle A of a right triangle be acute, show that the difference of the sides which contain it is less than 90°.

8. Prove $$\tan\frac{B}{2}=\frac{\sin(s-a)}{\sin s}.$$

9. Prove (1) $2n=\sin a\sin b$; (2) $2N=\sin a\sin B$.

10. Prove $\sin^2 a\sin^2 b=\sin^2 a+\sin^2 b-\sin^2 c$.

11. Prove $$\tan^2\frac{A}{2}=\frac{\sin(c-b)}{\sin(c+b)}.$$

12. Prove $$2\sin^2\frac{c}{2}=\sin^2\tfrac{1}{2}(a+b)+\sin^2\tfrac{1}{2}(a-b).$$

13. In a spherical triangle, if $c=90°$, prove that
$$\tan a\tan b+\sec C=0.$$

14. In a spherical triangle, if $c=90°$, prove that
$$\sin^2 p=\cot\theta\cot\phi,$$
where p is the perpendicular on c, and θ and ϕ are the segments of the vertical angle.

15. Show that the ratio of the cosines of the segments of the base made by the perpendicular from the vertex is equal to the ratio of the cosines of the sides.

16. If B be the bisector of the hypotenuse, show that
$$\sin^2 B=\frac{\sin^2 a+\sin^2 b}{4\cos^2\frac{c}{2}}.$$

17. Prove $$\tan S=\cot\frac{a}{2}\cot\frac{b}{2}.$$

18. Construct a triangle, being given the hypotenuse and (1) the sum of the base angles, and (2) the difference of the base angles.

19. Given the hypotenuse and the sum or difference of the sides: construct the triangle.

20. Given the sum of the sides a and b, and the sum of the base angles: solve the triangle.

21. Show that $\sin\frac{A}{2}=\frac{\sqrt{\sin c+\sin a}+\sqrt{\sin c-\sin a}}{2\sqrt{\sin c}}$.

22. $\sin\frac{1}{2}A=\sqrt{\frac{\sin(c-b)}{2\cos b\sin c}}$.

23. $\cos\frac{1}{2}A=\sqrt{\frac{\sin(c+b)}{2\cos b\sin c}}$.

24. $\sin(a+b)\tan\frac{1}{2}(A+B)=\sin(a-b)\cot\frac{1}{2}(A-B)$.

25. $\sin(A+B)=\frac{\cos a+\cos b}{1+\cos a\cos b}$.

26. $\sin(A-B)=\frac{\cos b-\cos a}{1-\cos a\cos b}$.

27. $\cos(A+B)=-\frac{\sin a\sin b}{1+\cos a\cos b}$.

28. $\cos(A-B)=\frac{\sin a\sin b}{1-\cos a\cos b}$.

29. $\sin^2\frac{c}{2}=\sin^2\frac{a}{2}\cos^2\frac{b}{2}+\cos^2\frac{a}{2}\sin^2\frac{b}{2}$.

30. $\sin(c-b)=\sin(c+b)\tan^2\frac{A}{2}$.

31. $\sin(a-b)=\sin a\tan\frac{A}{2}-\sin b\tan\frac{B}{2}$.

32. $\sin(c-a)=\cos a\sin b\tan\frac{B}{2}$.

33. If ABC is a spherical triangle, right-angled at C, and $\cos A=\cos^2 a$, show that if A be not a right angle, $b+c=\frac{1}{2}\pi$ or $\frac{3}{2}\pi$, according as b and c are both $<$ or both $>\frac{\pi}{2}$.

34. If α, β be the arcs drawn from the right angle respectively perpendicular to and bisecting the hypotenuse c, show that

$$\sin^2\frac{c}{2}(1+\sin^2\alpha)=\sin^2\beta.$$

35. In a triangle, if C be a right angle and D the middle point of AB, show that

$$4\cos^2\frac{c}{2}\sin^2 \mathrm{CD}=\sin^2 a+\sin^2 b.$$

In a right triangle, if p be the length of the arc drawn from the right angle C perpendicular to the hypotenuse AB, prove:

36. $\cot^2 p=\cot^2 a+\cot^2 b.$

37. $\cos^2 p=\cos^2 \mathrm{A}+\cos^2 \mathrm{B}.$

38. $\tan^2 a=\mp\tan a'\tan c.$

39. $\tan^2 b=\pm\tan b'\tan c.$

40. $\tan^2 a:\tan^2 b=\tan a':\tan b'.$

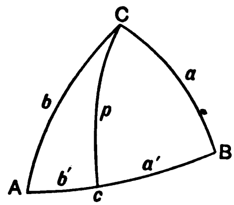
41. $\sin^2 p=\sin a'\sin b'.$

42. $\sin p\sin c=\sin a\sin b.$

43. $\tan a\tan b=\tan c\sin p.$

44. $\tan^2 a+\tan^2 b=\tan^2 c\cos^2 p.$

45. $\cot \mathrm{A}:\cot \mathrm{B}=\sin a':\sin b'.$



In the oblique triangle ABC, prove the following:

46. If the difference between any two angles of a triangle is 90°, the remaining angle is less than 90°.

47. If a triangle is equilateral or isosceles, its polar triangle is equilateral or isosceles.

48. If the sides of a triangle are each $\frac{\pi}{3}$, find the sides of the polar triangle.

49. If in a triangle the side $a = 90°$, show that

$$\cos A + \cos B \cos C = 0.$$

50. If θ and θ' are the angles which the internal and external bisectors of the vertical angle of a triangle make with the base, show that

$$\cos\theta = \frac{\cos A \sim \cos B}{2\cos\frac{C}{2}}, \text{ and } \cos\theta' = \frac{\cos A + \cos B}{2\sin\frac{C}{2}}.$$

51. Given the base c and $\frac{\cos A}{\cos B} = -\cos C$: find the locus of the vertex.

52. Prove $4N^2 = 1 - \cos^2 A - \cos^2 B - \cos^2 C$

$$- 2\cos A \cos B \cos C.$$

53. If p, q, r be the perpendiculars from the vertices on the opposite sides, show that

(1) $\sin a \sin p = \sin b \sin q = \sin c \sin r = 2n.$

(2) $\sin A \sin p = \sin B \sin q = \sin C \sin r = 2N.$

54. Prove $8n^3 = \sin^2 a \sin^2 b \sin^2 c \sin A \sin B \sin C.$

55. Prove $\dfrac{\sin^2 A + \sin^2 B + \sin^2 C}{\sin^2 a + \sin^2 b + \sin^2 c} = \dfrac{1 + \cos A \cos B \cos C}{1 - \cos a \cos b \cos c}.$

56. If l be the length of the arc joining the middle point of the base to the vertex, find an expression for its length in terms of the sides.

Ans. $\cos l = \dfrac{\cos a + \cos b}{2\cos\frac{c}{2}}.$

57. If CD, CD′ are the internal and external bisectors of the angle C of a triangle, prove that

$$\cot CD = \frac{\cot a + \cot b}{2\cos\frac{C}{2}}, \text{ and } \cot CD' = \frac{\cot a \sim \cot b}{2\sin\frac{C}{2}}.$$

58. Show that the angles θ and θ', made by the bisectors of the angle C in Ex. 55 with the opposite side c, are thus given:

$$\cot\theta = \frac{\cot a - \cot b}{2\sin\frac{C}{2}}\sin CD,$$

$$\cot\theta' = \frac{\cot a + \cot b}{2\cos\frac{C}{2}}\sin CD'.$$

59. Show that the arc intercepted on the base by the bisectors in Ex. 55 is thus given:

$$\cot DD' = \frac{\sin^2 A - \sin^2 B}{2\sin A\sin B\sin C}.$$

60. Prove that

$$\frac{\cos^2 b - \cos^2 c}{\cos b\cot B - \cos c\cot C} = \frac{\cos^2 c - \cos^2 a}{\cos c\cot C - \cos a\cot A}$$
$$= \frac{\cos^2 a - \cos^2 b}{\cos a\cot A - \cos b\cos B}.$$

61. If s and s' are the segments of the base made by the perpendicular from the vertex, and m and m' those made by the bisector of the vertical angle, show that

$$\tan\frac{s-s'}{2}\tan\frac{m-m'}{2} = \tan^2\frac{a-b}{2}.$$

62. Prove

$$\sin b\sin c + \cos b\cos c\cos A = \sin B\sin C - \cos B\cos C\cos a.$$

63. Show that the arc l joining the middle points of the two sides a and b of a triangle is thus given:

$$\cos l = \frac{1 + \cos a + \cos b + \cos c}{4\cos\frac{a}{2}\cos\frac{b}{2}}.$$

64. If the side c of a triangle be 90°, and δ the arc drawn at right angles to it from the opposite vertex, show that

$$\cot^2 \delta = \cot^2 A + \cot^2 B.$$

65. Prove that the angle ϕ between the perpendicular from the vertex on the base and the bisector of the vertical angle is thus given:

$$\tan \phi = \frac{\cos \frac{1}{2}(a-b)}{\cos \frac{1}{2}(a+b)} \tan \tfrac{1}{2}(A-B).$$

66. In an isosceles triangle, if each of the base angles be double the vertical angle, prove that

$$\cos a \cos \frac{a}{2} = \cos\left(c + \frac{a}{2}\right).$$

67. If a side c of a triangle be 90°, show that

(1) $\cot a \cot b + \cos C = 0.$

(2) $\cos S \cos(S - C) + \cos(S - A)\cos(S - B) = 0.$

68. In any triangle prove

$$\frac{\cos a - \cos b}{1 - \cos c} + \frac{\sin(A-B)}{\sin C} = 0.$$

69. $\tan \frac{1}{2}(A+B) : \tan \frac{1}{2}(A-B)$

$$= \tan \tfrac{1}{2}(a+b) : \tan \tfrac{1}{2}(a-b).$$

70. $\tan \frac{1}{2}(A+a) : \tan \frac{1}{2}(A-a)$

$$= \tan \tfrac{1}{2}(B+b) : \tan \tfrac{1}{2}(B-b).$$

71. If the bisector of the exterior angle, formed by producing BA through A, meet the base BC in D′, and if BD $= c''$, CD′ $= b''$, prove

$$\sin b : \sin c = \sin b'' : \sin c''.$$

72. If D be any point in the side BC of a triangle, prove

$$\frac{\sin BD}{\sin CD} = \frac{\sin BAD}{\sin CAD} \cdot \frac{\sin C}{\sin B}.$$

73. If $A = a$, show that B and b are either equal or supplemental, as also C and c.

74. If $A = B + C$, and D be the middle point of a, show that $a = 2\,AD$.

75. When does the polar triangle coincide with the primitive triangle?

76. If D be the middle point of c, show that

$$\cos a + \cos b = 2 \cos \frac{c}{2} \cos CD.$$

77. In an equilateral triangle show that

$$(1)\quad 2 \cos \frac{a}{2} \sin \frac{A}{2} = 1.$$

$$(2)\quad \tan^2 \frac{a}{2} + 2 \cos A = 1.$$

78. If $b + c = \pi$, show that $\sin 2B + \sin 2C = 0$.

79. Show that

$$\sin b \sin c + \cos b \cos c \cos A = \sin B \sin C - \cos B \cos C \cos a.$$

80. If D be any point in the side BC of a triangle, show that

$$\cos AD \sin a = \cos c \sin DC + \cos b \sin BD.$$

81. Prove $\cos^2 \frac{c}{2} = \cos^2 \frac{1}{2}(a+b) \sin^2 \frac{C}{2} + \cos^2 \frac{1}{2}(a-b) \cos^2 \frac{C}{2}$.

82. " $\sin^2 \frac{C}{2} = \sin^2 \frac{1}{2}(a+b) \sin^2 \frac{C}{2} + \sin^2 \frac{1}{2}(a-b) \cos^2 \frac{C}{2}$.

83. " $\sin s \sin (s-a) \sin (s-b) \sin (s-c)$

$$= \frac{1}{4}(1 - \cos^2 a - \cos^2 b - \cos^2 c + 2 \cos a \cos b \cos c).$$

84. If AD be the bisector of the angle A, prove that

$$(1)\quad \cos B + \cos C = 2 \sin \frac{A}{2} \sin ADB \cos AD.$$

$$(2)\quad \cos C - \cos B = 2 \cos \frac{A}{2} \cos ADB.$$

85. Prove $\cos a \sin b = \sin a \cos b \cos C + \cos A \sin c$.

86. " $\sin C \cos a = \cos A \sin B + \sin A \cos B \cos C$.

87. In a triangle if $A = \frac{\pi}{5}$, $B = \frac{\pi}{3}$, $C = \frac{\pi}{2}$, show that $a + b + c = \frac{\pi}{2}$.

88. Prove $\sin(S - A) = \dfrac{1 + \cos a - \cos b - \cos c}{4 \cos \frac{a}{2} \sin \frac{b}{2} \sin \frac{c}{2}}$.

89. If δ be the length of the arc from the vertex of an isosceles triangle, dividing the base into segments α and β, prove that

$$\tan \frac{\alpha}{2} \tan \frac{\beta}{2} = \tan \frac{a + \delta}{2} \tan \frac{a - \delta}{2}.$$

90. If $b = c$, show that

$$\sin b = \frac{\sin \frac{a}{2}}{\sin \frac{A}{2}}, \text{ and } \sin B = \frac{\cos \frac{A}{2}}{\cos \frac{a}{2}}.$$

91. If AB, AC be produced to B′, C′, so that BB′, CC′ shall be the semi-supplements of AB, AC respectively, prove that the arc B′C′ will subtend an angle at the centre of the sphere equal to the angle between the chords of AB, AC.

CHAPTER XI.

SOLUTION OF SPHERICAL TRIANGLES.

199. Preliminary Observations.—In every spherical triangle there are *six elements*, the three sides and the three angles, besides the radius of the sphere, which is supposed constant. *The solution of spherical triangles* is the process by which, when the values of *any three* elements are given, we calculate the values of the remaining three (Art. 184, Note).

In making the calculations, attention must be paid to the algebraic signs of the functions. When angles greater than 90° occur in calculation, we replace them by their supplements; and if the functions of such angles be either *cosine*, *tangent*, *cotangent*, or *secant*, we take account of the change of sign.

It is necessary to avoid the calculation of very small angles by their cosines, or of angles near 90° by their sines, for their tabular differences vary too slowly (Art. 81). It is better to determine such angles, for example, by means of their tangents.

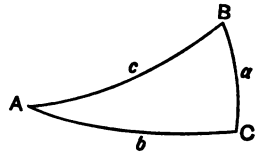
We shall begin with the right triangle; here two elements, in addition to the right angle, will be supposed known.

SOLUTION OF RIGHT SPHERICAL TRIANGLES.

200. The Solution of Right Spherical Triangles presents Six Cases, which may be solved by the formulæ of Art. 185. If the formula required for any case be not remembered, it is always easy to find it by Napier's Rules (Art.

186). In applying these rules, we must choose the middle part as follows:

When the three parts considered are all adjacent, the one between is, of course, the middle part. When only two are adjacent, the other one is the middle part.



Let ABC be a spherical triangle, right-angled at C, and let a, b, c denote the sides opposite the angles A, B, C, respectively.

We shall assume that the parts are all positive and less than 180° (Art. 182).

201. Case I. — Given the hypotenuse c and an angle A; to find a, b, B.

By (3), (5), and (8) of Art. 185, or by Napier's Rules, we have

$$\sin a = \sin c \sin A,$$

$$\tan b = \tan c \cos A,$$

$$\cot B = \cos c \tan A.$$

Since a is found by its sine, it would be ambiguous, but the ambiguity is removed because a and A are of the same species [Art. 187, (1)]. B and b are determined immediately without ambiguity.

If a be very near 90°, we commence by calculating the values of b and B, and then determine a by either of the formulæ

$$\tan a = \sin b \tan A, \tan a = \tan c \cos B.$$

Check. — As a final step, in order to guard against numerical errors, it is often expedient to check the logarithmic work, which may be done in every case without the necessity of new logarithms. To check the work, we make up a formula between the three required parts, and see whether it is satisfied by the results. In the present case, when the three parts a, b, B have been found, the *check formula* is

$$\sin a = \tan b \cot B \quad . \quad . \quad . \quad . \quad [(6) \text{ of Art. } 185]$$

Ex. 1. Given $c = 81° 29' 32''$, $A = 32° 18' 17''$; find a, b, B.

Solution.

$\log \sin c = 9.9951945$	$\log \tan c = 10.8250982$
$\log \sin A = 9.7278843$	$\log \cos A = 9.9269687$
$\log \sin a = 9.7230788$	$\log \tan b = 10.7520669$
$\therefore\ a = 31° 54' 25''$.	$\therefore\ b = 79° 51' 48''.65$.
$\log \cos c = 9.1700960$	*Check.*
$\log \tan A = 9.8009157$	$\log \tan b = 10.7520669$
$\log \cot B = 8.9710117$	$\log \cot B = 8.9710117$
$\therefore\ B = 84° 39' 21''.33$.	$\log \sin a = 9.7230786$

Ex. 2. Given $c = 110° 46' 20''$, $A = 80° 10' 30''$; find a, b, B.

Ans. $a = 67° 5' 52''.7$, $b = 155° 46' 42''.7$, $B = 153° 58' 24''.5$.

202. Case II. — Given the hypotenuse c and a side a; to find b, A, B.

By (1), (3), (4) of Art. 185, or by Napier's Rules, we have

$$\cos b = \frac{\cos c}{\cos a}, \ \sin A = \frac{\sin a}{\sin c}, \ \cos B = \frac{\tan a}{\tan c}.$$

The *check formula* involves b, A, B; therefore, from (9) of Art. 185 we have

$$\cos B = \sin A \cos b.$$

In this case there is an apparent ambiguity in the value of A, but this is removed by considering that A and a are always of the same species (Art. 187).

Ex. 1. Given $c = 140°$, $a = 20°$; find b, A, B.

Solution.

$\log\cos c \quad = 9.8842540-$	$\log\sin a \quad = 9.5340517$
$\text{colog}\cos a = 0.0270142$	$\text{colog}\sin c = 0.1919325$
$\log\cos b \quad = 9.9112682-$	$\log\sin A \quad = 9.7259842$
$\therefore\ b \quad = 144°\ 36'\ 28''.4.$	$\therefore\ A \quad = 32°\ 8'\ 48''.1.$
$\log\tan a \quad = 9.5610659$	*Check.*
$\text{colog}\tan c = 0.0761865-$	$\log\sin A \quad = 9.7259842$
	$\log\cos b \quad = 9.9112682$
$\log\cos B \quad = 9.6372524-$	
$\therefore\ B \quad = 115°\ 42'\ 23''.8.$	$\log\cos B \quad = 9.6372524$

Ex. 2. Given $c = 72°\ 30'$, $a = 45°\ 15'$; find b, A, B.

Ans. $b = 64°\ 42'\ 52''$, $A = 48°\ 7'\ 44''.5$, $B = 71°\ 27'\ 15''$.

203. Case III.—Given a side a and the adjacent angle B; to find A, b, c.

By (10), (6), (4) of Art. 185, we have

$$\cos A = \cos a \sin B,\ \tan b = \sin a \tan B,\ \cot c = \cot a \cos B.$$

Check formula, $\cos A = \tan b \cot c.$

In this case there is evidently no ambiguity.

Ex. 1. Given $a = 31°\ 20'\ 45''$, $B = 55°\ 30'\ 30''$; find A, b, c.

Solution.

$\log\cos a = 9.9314797$	$\log\sin a = 9.7161724$
$\log\sin B = 9.9160371$	$\log\tan B = 0.1630010$
$\log\cos A = 9.8475168$	$\log\tan b = 9.8791734$
$\therefore\ A = 45°\ 15'\ 30''.6.$	$\therefore\ b = 37°\ 7'\ 50''.$
$\log\cot a = 0.2153073$	*Check.*
$\log\cos B = 9.7530361$	$\log\tan b = 9.8791734$
$\log\cot c = 9.9683434$	$\log\cot c = 9.9683434$
$\therefore\ c = 47°\ 5'\ 11''.$	$\log\cos A = 9.8475168$

Ex. 2. Given $a=112° 0' 0''$, $B=152° 23' 1''.3$; find A, b, c.

Ans. $A=100°$, $b=154° 7' 26''.5$, $c=70° 18' 10''.2$.

204. Case IV. — Given a side a and the opposite angle A; to find b, c, B.

By (7), (3), (10) of Art. 185, we have

$$\sin b = \tan a \cot A, \quad \sin c = \frac{\sin a}{\sin A}, \quad \sin B = \frac{\cos A}{\cos a}.$$

Check formula, $\sin b = \sin c \sin B.$

In this case there is an ambiguity, as the parts are determined by their sines, and two values for each are in general admissible. But for each value of b there will, *in general*, be only one value for c, since c and b are connected by the relation $\cos c = \cos a \cos b$ (Art. 185); and at the same time only one admissible value for B, since $\cos c = \cot A \cot B$. Hence there will be, *in general*, only *two* triangles having the given parts, except when the side a is a quadrant and the angle A is also 90°, in which case the solution becomes *indeterminate.*

It is also easily seen from a figure that the ambiguity must occur in general (Art. 188).

When $a = A$, the formulæ, and also the figure, show that b, c, and B are each 90°.

Ex. 1. Given $a = 46° 45'$, $A = 59° 12'$; find b, c, B.

Solution.

$\log \tan a = 0.0265461$	$\log \sin a = 9.8623526$
$\log \cot A = 9.7753334$	$\log \sin A = 9.9339729$
$\log \sin b = 9.8018795$	$\log \sin c = 9.9283797$
$\therefore b = 39° 19' 23''.5$,	$\therefore c = 57° 59' 29''$,
or $140° 40' 36''.5$.	or $122° 0' 31''$.

	Check.
$\log \cos A = 9.7093063$	
$\log \cos a = 9.8358066$	$\log \sin c = 9.9283797$
$\log \sin B = 9.8734997$	$\log \sin B = 9.8734997$
$\therefore B = 48° 21' 28''$,	$\log \sin b = 9.8018794$
or $131° 38' 32''$.	

Ex. 2. Given $a = 112°$, $A = 100°$; find b, c, B.

Ans. $b = 154° \; 7' 26''.5$, $c = 70° 18' 10''.2$, $B = 152° 23' \; 1''.3$, or $25° 52' 33''.5$, or $109° 41' 49''.8$, or $27° 36' 58''.7$.

205. Case V.—**Given the two sides** a **and** b; **to find** A, B, c.

By (7), (6), (1) of Art. 185, we have

$\cot A = \cot a \sin b$, $\cot B = \cot b \sin a$, $\cos c = \cos a \cos b$.

Check formula, $\qquad \cos c = \cot A \cot B$.

In this case there is no ambiguity.

Ex. 1. Given $a = 54° 16'$, $b = 33° 12'$; find A, B, c.

Ans. $A = 68° 29' 53''$, $B = 38° 52' 26''$, $c = 60° 44' 46''$.

Ex. 2. Given $a = 56° 34'$, $b = 27° 18'$; find A, B, c.

Ans. $A = 73° 9' 13''$, $B = 31° 44' 9''$, $c = 60° 41' 9''$.

206. Case VI.—**Given the two angles** A **and** B; **to find** a, b, **and** c.

By (10), (9), (8)

$$\cos a = \frac{\cos A}{\sin B}, \; \cos b = \frac{\cos B}{\sin A}, \; \cos c = \cot A \cot B.$$

Check formula, $\qquad \cos c = \cos a \cos b$.

There is no ambiguity in this case.

Ex. 1. Given $A = 74° 15'$, $B = 32° 10'$; find a, b, c.

Ans. $a = 59° 20' 44''$, $b = 28° 24' 54''$, $c = 63° 21' 24''.5$.

Ex. 2. Given $A = 91° 11'$, $B = 111° 11'$; find a, b, c.

Ans. $a = 91° 16' 8''$, $b = 111° 11' 16''$, $c = 89° 32' 29''$.

207. Quadrantal and Isosceles Triangles. — Since the polar triangle of a quadrantal triangle is a right triangle (Art. 184), we have only to solve the polar triangle by the formulæ of Art. 185, and take the supplements of the parts thus found for the required parts of the given triangle; or we can solve the quadrantal triangle immediately by the formulæ of Art. 189.*

A biquadrantal triangle is indeterminate unless either the base or the vertical angle be given.

An *isosceles triangle* is easily solved by dividing it into two equal right triangles by drawing an arc from the vertex to the middle of the base.

The solution of triangles in which $a + b = \pi$, or $A + B = \pi$, can be made to depend on the solution of right triangles. Thus (see the second figure of Art. 191) the triangle B′AC has the two equal sides, a' and b, given, or the two equal angles, A and B′, given, according as $a + b = \pi$ or $A + B = \pi$ in the triangle ABC.

EXAMPLES.

Solve the following right triangles :

1.	Given $c=32°34'$,	$a=44°44'$;	
	find $a=22°15'43''$,	$b=24°24'19''$,	$B=50°8'21''$.
2.	Given $c=69°25'11''$,	$A=54°54'42''$;	
	find $a=50°\ 0'\ 0''$,	$b=56°50'49''$,	$B=63°25'4''$.
3.	Given $c=55°\ 9'32''$,	$a=22°15'\ 7''$;	
	find $b=51°53'$,	$A=27°28'37'.5$,	$B=73°27'11''.1$.
4.	Given $c=127°12'$,	$a=141°11'$;	
	find $b=39°\ 6'25''$,	$A=128°5'54''$,	$B=52°21'49''$.
5.	Given $a=118°54'$,	$B=12°19'$;	
	find $A=95°55'\ 2''$,	$b=10°49'17''$,	$c=118°20'20''$.

* Quadrantal triangles are generally avoided in practice, but when unavoidable, they are readily solved by either of these methods.

6.	Given $a=29°46'\ 8''$,	$B=137°24'21''$;	
	find $A=54°\ 1'16''$,	$b=155°27'54''$,	$c=142°9'13''$.
7.	Given $a=77°21'50''$,	$A=83°56'40''$;	
	find $b=28°14'31''.1$,	$c=78°53'20''$,	$B=28°49'57''.4$,
	or $b'=151°45'28''.9$,	$c'=101°6'40''$,	$B'=151°10'2''.6$.
8.	Given $a=68°$,	$A=80°$;	
	find $b=25°52'33''.5$,	$c=70°18'10''.2$,	$B=27°36'58''.7$,
	or $b'=154°7'26''.5$,	$c'=109°41'49''.8$,	$B'=152°23'1''.3$.
9.	Given $a=144°27'3''$,	$b=32°8'56''$;	
	find $A=126°40'24''$,	$B=47°13'43''$,	$c=133°32'26''$.
10.	Given $a=36°27'$,	$b=43°32'31''$;	
	find $A=46°59'43''.3$,	$B=57°59'19''.2$,	$c=54°20'$.
11.	Given $A=63°15'12''$,	$B=135°33'39''$;	
	find $a=49°59'56''$,	$b=143°5'12''$,	$c=120°55'34''$.
12.	Given $A=67°54'47''$,	$B=99°57'35''$;	
	find $a=67°33'27''$,	$b=100°45'$,	$c=94°5'$.

13. Solve the quadrantal triangle in which

$$c = 90°,\ A = 42°1',\ B = 121°20'.$$

Ans. $C = 67°16'22''$, $b = 112°10'20''$, $a = 46°31'36''$.

14. Solve the quadrantal triangle in which

$$a = 174°12'49''.1,\ b = 94°8'20'',\ c = 90°.$$

Ans. $A = 175°57'10''$, $B = 135°42'55''$, $C = 135°34'8''$.

SOLUTION OF OBLIQUE SPHERICAL TRIANGLES.

208. The Solution of Oblique Spherical Triangles presents Six Cases; as follows:

I. *Given two sides and the included angle,* a, b, C.

II. *Given two angles and the included side,* A, B, c.

III. *Given two sides and an angle opposite one of them,* a, b, A.

IV. *Given two angles and a side opposite one of them,* A, B, a.

V. *Given the three sides,* a, b, c.

VI. *Given the three angles,* A, B, C.

These six cases are immediately resolved into *three pairs* of cases by the aid of the polar triangle (Art. 184).

For when two sides and the included angle are given, and the remaining parts are required, the application of the data to the polar triangle transforms the problem into the supplemental problem: given two angles and the included side, to find the remaining parts.

Similarly, cases III. and IV. are supplemental, also V. and VI.

The parts are all positive and less than 180° (Art. 182). The attention of the student is called to Art. 199.

209. Case I. — Given two sides, a, b, and the included angle C; to find A, B, c.

By Napier's Analogies, (7) and (8) of Art. 197,

$$\tan \tfrac{1}{2}(A+B) = \frac{\cos \frac{1}{2}(a-b)}{\cos \frac{1}{2}(a+b)} \cot \frac{C}{2}.$$

$$\tan \tfrac{1}{2}(A-B) = \frac{\sin \frac{1}{2}(a-b)}{\sin \frac{1}{2}(a+b)} \cot \frac{C}{2}.$$

These determine $\frac{1}{2}(A+B)$ and $\frac{1}{2}(A-B)$, and therefore A and B; then c can be found by Art. 190, or by one of Gauss's equations (Art. 198). Since c is found from its *sine* in Art. 190, it may be uncertain which of two values is to be given to it: if we determine c from one of Gauss's equations, it is free from ambiguity. We may therefore find c from (3) of Art. 198. Thus

$$\cos \tfrac{1}{2}(A+B) \cos \frac{c}{2} = \cos \tfrac{1}{2}(a+b) \sin \frac{C}{2}.$$

Check, $\tan \frac{1}{2}(a+b) \cos \frac{1}{2}(A+B) = \cos \frac{1}{2}(A-B) \tan \frac{c}{2}.$

There is no ambiguity in this case.

Ex. 1. Given $a = 43° 18'$, $b = 19° 24'$, $C = 74° 22'$; find A, B, c.

Solution.

$$\tfrac{1}{2}(a+b) = 31° 21', \ \tfrac{1}{2}(a-b) = 11° 57', \ \tfrac{1}{2}C = 37° 11'.$$

$$\begin{aligned} \log \cos \tfrac{1}{2}(a-b) &= 9.9904848 \\ \log \sec \tfrac{1}{2}(a+b) &= 0.0685395 \\ \log \cot \frac{C}{2} &= 10.1199969 \\ \hline \log \tan \tfrac{1}{2}(A+B) &= 10.1790212 \end{aligned}$$

$$\therefore \ \tfrac{1}{2}(A+B) = 56° 29' 17''$$
$$\tfrac{1}{2}(A-B) = 27° 41' \ 0''.5$$
$$\therefore \ A = 84° 10' 17''.5$$
$$B = 28° 48' 16''.5.$$
$$c = 41° 35' 48''.5.$$

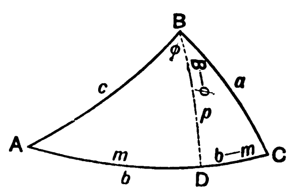
$$\begin{aligned} \log \sin \tfrac{1}{2}(a-b) &= 9.3160921 \\ \log \operatorname{cosec} \tfrac{1}{2}(a+b) &= 0.2837757 \\ \log \cot \frac{C}{2} &= 10.1199969 \\ \hline \log \tan \tfrac{1}{2}(A-B) &= 9.7198647 \end{aligned}$$

$$\therefore \ \tfrac{1}{2}(A-B) = 27° 41' 0''.5.$$

$$\begin{aligned} \log \cos \tfrac{1}{2}(a+b) &= 9.9314605 \\ \log \sec \tfrac{1}{2}(A+B) &= 0.2579737 \\ \log \sin \frac{C}{2} &= 9.7813010 \\ \hline \log \cos \frac{c}{2} &= 9.9707352 \end{aligned}$$

$$\therefore \ \frac{c}{2} = 20° 47' 54''.25.$$

Otherwise thus: Let fall the perpendicular BD, dividing the triangle ABC into two right triangles, BDA, BDC. Denote AD by m, the angle ABD by ϕ, and BD by p. Then by Napier's Rules, we have



$$\cos C = \tan(b-m) \cot a;$$

$$\sin(b-m) = \cot C \tan p; \ \sin m = \cot A \tan p.$$

$$\therefore \ \tan(b-m) = \tan a \cos C \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

and $$\tan A \sin m = \tan C \sin(b-m) \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

From (1) m is determined, and from (2) A is determined. In a similar manner B may be found.

Also, from the same triangles, we have by Napier's Rules

$$\cos a = \cos(b-m)\cos p; \quad \cos c = \cos m \cos p.$$

$$\therefore \cos c \cos(b-m) = \cos m \cos a,$$

from which c is found.

NOTE. — This method has the advantage that, in using it, nothing need be remembered except Napier's Rules.

If only the side c is wanted, it may be found from (4) of Art. 191, without previously determining A and B. This formula may be adapted to logarithms by the use of a *subsidiary angle* (Art. 90).

Ex. 2. Given $b = 120° 30' 30''$, $c = 70° 20' 20''$, $A = 50° 10' 10''$; find B, C, a.

Ans. $B = 135° 5' 28''.8$, $C = 50° 30' 8''.4$, $a = 69° 34' 56''$.

210. Case II. — Given two angles, A, B, and the included side c; to find a, b, C.

By Napier's Analogies (9) and (10) of Art. 197,

$$\tan\tfrac{1}{2}(a+b) = \frac{\cos\frac{1}{2}(A-B)}{\cos\frac{1}{2}(A+B)}\tan\frac{c}{2},$$

$$\tan\tfrac{1}{2}(a-b) = \frac{\sin\frac{1}{2}(A-B)}{\sin\frac{1}{2}(A+B)}\tan\frac{c}{2},$$

from which a and b are found.

The remaining part C may be found by (2) of Art. 198.

$$\sin\tfrac{1}{2}(a-b)\cos\frac{C}{2} = \sin\tfrac{1}{2}(A-B)\sin\frac{c}{2}.$$

Check, $\cos\tfrac{1}{2}(a-b)\cot\frac{C}{2} = \cos\tfrac{1}{2}(a+b)\tan\tfrac{1}{2}(A+B).$

There is no ambiguity in this case.

Ex. 1. Given A=68° 40′, B=56° 20′, c=84° 30′; find a, b, C.

Solution.

$$\tfrac{1}{2}(A+B)=62°\,30',\ \tfrac{1}{2}(A-B)=6°\,10',\ \frac{c}{2}=42°\,15'.$$

$$\begin{aligned}
\log\cos\tfrac{1}{2}(A-B) &= 9.9974797\\
\log\sec\tfrac{1}{2}(A+B) &= 0.3355944\\
\log\tan\frac{c}{2} &= 9.9582465\\
\hline
\log\tan\tfrac{1}{2}(a+b) &= 10.2913206\\
\therefore\ \tfrac{1}{2}(a+b) &= 62°\,55'\,9''\\
\tfrac{1}{2}(a-b) &= 6°\,16'\,39''\\
\hline
a &= 69°\,11'\,48''\\
b &= 56°\,38'\,30''.\\
C &= 97°\,19'\,3''.5.
\end{aligned}$$

$$\begin{aligned}
\log\sin\tfrac{1}{2}(A-B) &= 9.0310890\\
\log\operatorname{cosec}\tfrac{1}{2}(A+B) &= 0.0520711\\
\log\tan\frac{c}{2} &= 9.9582465\\
\hline
\log\tan\tfrac{1}{2}(a-b) &= 9.0414066\\
\therefore\ \tfrac{1}{2}(a-b) &= 6°\,16'\,39''.\\
\log\sin\tfrac{1}{2}(A-B) &= 9.0310890\\
\log\operatorname{cosec}\tfrac{1}{2}(a-b) &= 0.9612050\\
\log\sin\frac{c}{2} &= 9.8276063\\
\hline
\log\cos\frac{C}{2} &= 9.8199003\\
\therefore\ \frac{C}{2} &= 48°\,39'\,31\tfrac{3}{4}''.
\end{aligned}$$

Otherwise thus: Let fall the perpendicular BD (see last figure). Denote, as before, AD by m, the angle ABD by ϕ, and BD by p. Then by Napier's Rules, we have

$$\cos c=\cot\phi\cot A;$$

$$\cos\phi=\cot c\tan p;\ \cos(B-\phi)=\cot a\tan p.$$

$$\therefore\ \cot\phi=\tan A\cos c \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$\tan a\cos(B-\phi)=\cos\phi\tan c \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

From (1) ϕ is determined, and from (2) a is found. Similarly b may be found.

Also, from the same triangles, we have

$$\cos C\sin\phi=\cos A\sin(B-\phi),$$

from which C is found.

Ex. 2. Given

$$A = 135° \, 5' \, 28''.6, \quad C = 50° \, 30' \, 8''.6, \quad b = 69° \, 34' \, 56''.2;$$

find a, c, B.

Ans. $a = 120° \, 30' \, 30''$, $c = 70° \, 20' \, 20''$, $B = 50° \, 10' \, 10''$.

211. Case III. — Given two sides, a, b, **and the angle** A **opposite one of them**; **to find** B, C, c.

The angle B is found from the formula,

$$\sin B = \frac{\sin b}{\sin a} \sin A \quad . \quad . \quad . \quad . \quad . \quad (\text{Art. } 190) \quad (1)$$

Then C and c are found from Napier's Analogies,

$$\tan \frac{C}{2} = \frac{\sin \frac{1}{2}(a-b)}{\sin \frac{1}{2}(a+b)} \cot \frac{1}{2}(A-B) \quad . \quad . \quad . \quad . \quad (2)$$

$$\tan \frac{c}{2} = \frac{\sin \frac{1}{2}(A+B)}{\sin \frac{1}{2}(A-B)} \tan \frac{1}{2}(a-b) \quad . \quad . \quad . \quad . \quad (3)$$

Check, $$\frac{\sin A}{\sin a} = \frac{\sin B}{\sin b} = \frac{\sin C}{\sin c}.$$

Since B is found from its sine in (1), it will have two values, if $\sin A \sin b < \sin a$, and the triangle, in general, will admit of two solutions.

When $\sin A \sin b > \sin a$, there will be no solution, for then $\sin B > 1$.

In order that either of these values found for B may be admissible, it is necessary and sufficient that, when substituted in (2) and (3), they give positive values for $\tan \frac{C}{2}$ and $\tan \frac{c}{2}$, or which is the same thing, that $A - B$ and $a - b$ have the same sign. Hence we have the following

Rule. — *If both values of* B *obtained from* (1) *be such as that* $A - B$ *and* $a - b$ *have like signs, there are two complete solutions. If only one of the values of* B *satisfies this condition, there is only one triangle that satisfies the problem, since* *in this case* C, *or* $c > 180°$. *If neither of the values of* B *makes* $A - B$ *and* $a - b$ *of the same signs, the problem is impossible.*

This case is known as the *ambiguous case,* and is like the analogous ambiguity in Plane Trigonometry (Art. 116), though it is somewhat more complex. For a complete discussion of the *Ambiguous Case,* the student is referred to Todhunter's Spherical Trigonometry, pp. 53–58; McCollend and Preston's Spherical Trigonometry, pp. 137–143; Serret's Trigonometry, pp. 191–195, etc.

Ex. 1. Given $a = 42°45'$, $b = 47°15'$, $A = 56°30'$; find B, C, c.

Solution.

$\log\sin b = 9.8658868$	$\frac{1}{2}(a+b) = 45°\ 0'\ 0''$.
$\text{colog}\sin a = 0.1682577$	$\frac{1}{2}(a-b) = -\ 2°\ 15'\ 0''$.
$\log\sin A = 9.9211066$	$\frac{1}{2}(A+B) = 60°\ 28'\ 2''$.
$\log\sin B = 9.9552511$	$\frac{1}{2}(A-B) = -\ 3°\ 58'\ 2''$.
$\therefore\ B = 64°\ 26'\ 4''$,	$\frac{1}{2}(A+B') = 86°\ 1'\ 58''$.
$B' = 115°\ 33'\ 56''$.	$\frac{1}{2}(A-B') = -\ 29°\ 31'\ 58''$.

Since both values of B are such that $A - B$, $A - B'$, and $a - b$, are all negative, there are *two* solutions, by the above Rule.

(1) *When* $B = 64°\ 26'\ 4''$.

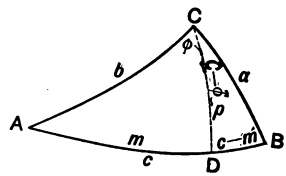
$\log\sin\frac{1}{2}(a-b) = 8.5939483-$	$\log\sin\frac{1}{2}(A+B) = 9.9395560$
$\text{colog}\sin\frac{1}{2}(a+b) = 0.1505150$	$\text{colog}\sin\frac{1}{2}(A-B) = 1.1599832-$
$\log\cot\frac{1}{2}(A-B) = 1.1589413-$	$\log\tan\frac{1}{2}(a-b) = 8.5942832-$
$\log\tan\frac{C}{2} = 9.9034046$	$\log\tan\frac{c}{2} = 9.6938224$
$\therefore\ \frac{C}{2} = 38°\ 40'\ 48''$.	$\therefore\ \frac{c}{2} = 26°\ 17'\ 40''$.
$\therefore\ C = 77°\ 21'\ 36''$.	$\therefore\ c = 53°\ 35'\ 20''$.

(2) *When* $B' = 115° 33' 56''$.

$\log\sin\frac{1}{2}(a-b) = 8.5939483-$	$\log\sin\frac{1}{2}(A+B') = 9.9989581$
$\text{colog}\sin\frac{1}{2}(a+b) = 0.1505150$	$\text{colog}\sin\frac{1}{2}(A-B') = 0.3072223-$
$\log\cot\frac{1}{2}(A-B') = 0.2467784-$	$\log\tan\frac{1}{2}(a-b) = 8.5942832-$
$\log\tan\frac{C'}{2} = 8.9912417$	$\log\tan\frac{c'}{2} = 8.9004636$
$\therefore \frac{C'}{2} = 5° 35' 50\frac{1}{4}''$.	$\therefore \frac{c'}{2} = 4° 32' 47\frac{1}{4}''$.
$\therefore C' = 11° 11' 40\frac{1}{2}''$.	$\therefore c' = 9° \ 5' 34\frac{1}{2}''$.

Ans. $B = 64° 26' \ 4''$, $C = 77° 21' 36''$, $c = 53° 35' 20''$;
$B' = 115° 35' 56''$, $C' = 11° 11' 40\frac{1}{2}''$, $c' = 9° \ 5' 34\frac{1}{2}''$.

Otherwise thus: Let fall the perpendicular CD; denote AD by m, the angle ACD by ϕ, and CD by p. Then we have



$$\cos A = \tan m \cot b;$$

$$\therefore \tan m = \cos A \tan b \qquad (1)$$

$$\cos b = \cot A \cot \phi. \quad \therefore \cot \phi = \cos b \tan A \quad . \quad (2)$$

Again,

$$\cos a = \cos (c - m) \cos p; \ \cos b = \cos m \cos p.$$

$$\therefore \cos (c - m) = \cos a \cos m \div b \quad . \quad . \quad . \quad (3)$$

Also, $\cos (C - \phi) = \cot a \tan p$; $\cos \phi = \cot b \tan p$.

$$\therefore \cos (C - \phi) = \cot a \tan b \cos \phi \quad . \quad . \quad . \quad (4)$$

Lastly, $$\sin B = \frac{\sin b}{\sin a} \sin A \quad . \quad . \quad . \quad . \quad . \quad (5)$$

The required parts are given by (1), (2), (3), (4), (5).

Ex. 2. Given $a = 73° 49' 38''$, $b = 120° 53' 35''$, $A = 88° 52' 42''$: find B, C, c.

Ans. $B = 116° 44' 48''$, $C = 116° 44' 48''$, $c = 120° 55' 35''$.

212. Case IV. — Given two angles, A, B, **and the side** a **opposite one of them; to find** b, c, C.

This case reduces, by aid of the polar triangle, to the preceding case, and gives rise to the same ambiguities. Hence the same remarks made in Art. 211 apply in this case also, and the direct solution may be obtained in the same way as in Case III. Thus,

The side b is found from the formula

$$\sin b = \frac{\sin B}{\sin A} \sin a \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Then c and C are found from Napier's Analogies.

$$\tan \frac{c}{2} = \frac{\cos \frac{1}{2}(A+B)}{\cos \frac{1}{2}(A-B)} \tan \tfrac{1}{2}(a+b) \quad . \quad . \quad . \quad . \quad (2)$$

$$\tan \frac{C}{2} = \frac{\cos \frac{1}{2}(a-b)}{\cos \frac{1}{2}(a+b)} \cot \tfrac{1}{2}(A+B) \quad . \quad . \quad . \quad (3)$$

Check, $\dfrac{\sin A}{\sin a} = \dfrac{\sin B}{\sin b} = \dfrac{\sin C}{\sin c}$.

Ex. 1. Given $A = 66° 7' 20''$, $B = 52° 50' 20''$, $a = 59° 28' 27''$; find b, c, C.

Solution.

By (1) we find $b = 48° 39' 16''$, or $131° 20' 44''$.

$\frac{1}{2}(A+B) = 59° 28' 50''$.	$\frac{1}{2}(a+b) = 54° \ 3' 51\frac{1}{2}''$.
$\frac{1}{2}(A-B) = 6° 38' 30''$.	$\frac{1}{2}(a-b) = 5° 24' 35\frac{1}{2}''$.
$\log \cos \frac{1}{2}(A+B) = 9.7057190$	$\log \cos \frac{1}{2}(a-b) = 9.9980612$
$\operatorname{colog} \cos \frac{1}{2}(A-B) = 0.0029244$	$\operatorname{colog} \cos \frac{1}{2}(a+b) = 0.2314530$
$\log \tan \frac{1}{2}(a+b) = 10.1397643$	$\log \cot \frac{1}{2}(A+B) = 9.7704854$
$\log \tan \frac{c}{2} = 9.8484077$	$\log \tan \frac{C}{2} = 9.9999996$
$\therefore \frac{c}{2} = 35° 11' 50\frac{3}{4}''$.	$\therefore \frac{C}{2} = 45°$.

The second value of b is inadmissible (see Rule of Art. 211), and therefore there is only one solution.

Ans. $b = 48° 39' 16''$, $c = 70° 23' 41\frac{1}{2}''$, $C = 90°$.

Ex. 2. Given $A = 110° 10'$, $B = 133° 18'$, $a = 147° 5' 32''$; find b, c, C.

Ans. $b = 155° 5' 18''$, $c = 33° 1' 45''$, $C = 70° 20' 50''$.

213. Case V. — Given the three sides, a, b, c; to find the angles.

The angles may be computed by any of the formulæ of Art. 195; but since an angle near 90° cannot be accurately determined by its sine, nor one near 0° by its cosine (Art. 151), neither of the first six formulæ can be used with advantage in all cases. The formulæ for the tangents however are accurate in all parts of the quadrant, and are therefore to be preferred for the solution of a triangle in which all three sides or all three angles are given.

By (7) of Art. 195 we have

$$\tan\frac{A}{2} = \sqrt{\frac{\sin(s-b)\sin(s-c)}{\sin s\sin(s-a)}}$$

$$= \frac{1}{\sin(s-a)}\sqrt{\frac{\sin(s-a)\sin(s-b)\sin(s-c)}{\sin s}}.$$

Since the part under the radical is a symmetric function of the sides, it is in the formulæ for determining all three angles A, B, C, and when once calculated, it may be utilized in the calculation of each angle. For convenience in computation, denote this term by $\tan r$. Then

$$\tan r = \sqrt{\frac{\sin(s-a)\sin(s-b)\sin(s-c)}{\sin s}};$$

and (7), (8), (9) of Art. 195 become

$$\tan\frac{A}{2} = \frac{\tan r}{\sin(s-a)} \quad \ldots \ldots \ldots \quad (1)$$

$$\tan\frac{B}{2} = \frac{\tan r}{\sin(s-b)} \quad \ldots \ldots \ldots \quad (2)$$

$$\tan\frac{C}{2} = \frac{\tan r}{\sin(s-c)} \quad \ldots \ldots \ldots \quad (3)$$

Check, $\dfrac{\sin A}{\sin a}=\dfrac{\sin B}{\sin b}=\dfrac{\sin C}{\sin c}.$

Ex. 1. Given $a=46^\circ\,24'$, $b=67^\circ\,14'$, $c=81^\circ\,12'$; find A, B, C.

Solution.

$a=46^\circ\,24'$	$\log\sin(s-a)=9.8906049$
$b=67^\circ\,14'$	$\log\sin(s-b)=9.7013681$
$c=81^\circ\,12'$	$\log\sin(s-c)=9.4460251$
$2s=194^\circ\,50'$	$\text{colog}\sin s=0.0036487$
$s=97^\circ\,25'$,	$\log\tan^2 r=9.0416468$
$s-a=51^\circ\;1'$,	$\log\tan r=9.5208234.$
$s-b=30^\circ\,11'$,	
$s-c=16^\circ\,13'.$	

$\tan r=9.5208234$	$\tan r=9.5208234$	$\tan r=9.5208234$
$\sin(s-a)=9.8906049$	$\sin(s-b)=9.7013681$	$\sin(s-c)=9.4460251$
$\tan\dfrac{A}{2}=9.6302185$	$\tan\dfrac{B}{2}=9.8194553$	$\tan\dfrac{C}{2}=10.0747983$
$\therefore\ \dfrac{A}{2}=23^\circ\;6'\,45''.$	$\therefore\ \dfrac{B}{2}=33^\circ\,25'\,10''.$	$\therefore\ \dfrac{C}{2}=49^\circ\,54'\,35''.$
$A=46^\circ\,13'\,30''.$	$B=66^\circ\,50'\,20''.$	$C=99^\circ\,49'\,10''.$

Ex. 2. Given $a=100^\circ$, $b=37^\circ\,18'$, $c=62^\circ\,46'$; find A, B, C.

Ans. $A=176^\circ\,15'\,46''.56$, $B=2^\circ\,17'\,55''.08$, $C=3^\circ\,22'\,25''.46$.

214. Case VI. — Given the three angles, A, B, C; **to find the sides.**

As in Art. 213, the formulæ for the tangents are to be preferred.

Putting $\tan R=\sqrt{\dfrac{-\cos S}{\cos(S-A)\cos(S-B)\cos(S-C)}},$

we have, from (7), (8), (9) of Art. 196,

$$\tan\frac{a}{2}=\tan R\cos(S-A),$$

$$\tan\frac{b}{2} = \tan R \cos(S - B),$$

$$\tan\frac{c}{2} = \tan R \cos(S - C),$$

by which the three sides may be found.

Check, $$\frac{\sin A}{\sin a} = \frac{\sin B}{\sin b} = \frac{\sin C}{\sin c}.$$

Ex. 1. Given $A = 68° 30'$, $B = 74° 20'$, $C = 83° 10'$; find a, b, c.

Solution.

$$\begin{aligned} A &= 68° 30' \\ B &= 74° 20' \\ C &= 83° 10' \\ \hline 2S &= 226° \ 0' \end{aligned} \qquad \begin{aligned} S &= 113° \ 0' \\ S - A &= 44° 30' \\ S - B &= 38° 40' \\ S - C &= 29° 50' \end{aligned}$$

$$\begin{aligned} \log(-\cos S) &= 9.5918780 \\ \log\cos(S - A) &= 9.8532421 \\ \log\cos(S - B) &= 9.8925365 \\ \log\cos(S - C) &= 9.9382576 \\ \hline \log\tan^2 R &= 9.9078418 \\ \log\tan R &= 9.9539209.* \end{aligned}$$

$$\begin{aligned} \log\tan\frac{a}{2} &= 9.8071630 \\ \log\tan\frac{b}{2} &= 9.8464574 \\ \log\tan\frac{c}{2} &= 9.8921785 \\ \hline a &= 65° 21' 22\tfrac{1}{2}'' \\ b &= 70° \ 9' \ 9\tfrac{1}{2}'', \\ c &= 75° 55' \ 9''. \end{aligned}$$

Check, $$\frac{\sin a}{\sin A} = \frac{\sin b}{\sin B} = \frac{\sin c}{\sin C}.$$

Ex. 2. Given $A = 59° 55' 10''$, $B = 85° 36' 50°$, $C = 59° 55' 10''$; find a, b, c.

Ans. $a = 129° 11' 40''$, $b = 63° 15' 12''$, $c = 129° 11' 40''$.

* The necessary additions may be conveniently performed by writing log tan R on a slip of paper, and holding it successively over log cos (S − A), log cos (S − B), and log cos (S − C).

EXAMPLES.

Solve the following right triangles:

1.	Given $c = 84°20'$,	$A = 35°25'$;	
	find $a = 35°13'\ 4''$,	$b = 83°\ 3'29''$,	$B = 85°59'\ 1''$.
2.	Given $c = 67°54'$,	$A = 43°28'$;	
	find $a = 39°35'51''$,	$b = 60°46'25\frac{1}{2}''$,	$B = 70°22'21''$.
3.	Given $c = 22°18'30''$,	$A = 47°39'36''$;	
	find $a = 16°17'41''$,	$b = 15°26'53''$,	$B = 44°33'53''.4$.
4.	Given $c = 145°$,	$A = 23°28'$;	
	find $a = 13°12'12''$,	$b = 147°17'15''$,	$B = 109°34'33''$.
5.	Given $c = 98°\ 6'43''$,	$A = 138°27'18''$;	
	find $a = 137°\ 6'$,	$b = 77°51'$,	$B = 80°55'27''$.
6.	Given $c = 46°40'12''$,	$A = 37°46'\ 9''$;	
	find $a = 26°27'23''.8$,	$b = 39°57'41''.4$,	$B = 62°\ 0'\ 4''$.
7.	Given $c = 76°42'$,	$a = 47°18'$;	
	find $b = 70°10'13''$,	$A = 49°\ 2'24''.5$,	$B = 75°9'24''.75$.
8.	Given $c = 91°18'$,	$a = 72°27'$;	
	find $b = 94°18'53''.8$,	$A = 72°29'48''$,	$B = 94°\ 6'53''.3$.
9.	Given $c = 86°51'$,	$b = 18°\ 1'50''$;	
	find $a = 86°41'14''$,	$A = 88°58'25''$,	$B = 18°\ 3'32''$.
10.	Given $c = 23°49'51''$,	$a = 14°16'35''$;	
	find $b = 19°17'$,	$A = 37°36'49''.3$,	$B = 54°49'23''.3$.
11.	Given $c = 97°13'\ 4''$,	$a = 132°14'12''$;	
	find $b = 79°13'38''.2$,	$A = 131°43'50''$,	$B = 81°58'53''.3$.
12.	Given $c = 37°40'20''$,	$a = 37°40'12''$;	
	find $b = 0°26'37''.2$,	$A = 89°25'37''$,	$B = 0°43'33''$.

13.	Given $a= 82° \; 6'$,	$B= 43°28'$;	
	find $A= 84°34'28''$,	$b= 43°11'38''$,	$c= 84°14'57''$.
14.	Given $a= 42°30'30''$,	$B= 53°10'30''$;	
	find $A= 53°50'12''$,	$b= 42° \; 3'47''$,	$c= 56°49' \; 8''$.
15.	Given $a= 20°20'20''$,	$B= 38°10'10''$;	
	find $A= 54°35'16''.7$,	$b= 15°16'50''.4$,	$c= 25°14'38''.2$.
16.	Given $a= 92°47'32''$;	$B= 50° \; 2' \; 1''$;	
	find $A= 92° \; 8'23''$,	$b= 50°$,	$c= 91°47'40''$.
17.	Given $b= 54°30'$,	$A= 35°30'$;	
	find $B= 70°17'35''$,	$a= 30° \; 8'39''.2$,	$c= 59°51'20''$.
18.	Given $b=155°46'42''.7$,	$A= 80°10'30''$;	
	find $B=153°58'24''.5$,	$a= 67° \; 6'52''.6$,	$c=110°46'20''$.
19.	Given $a= 35°44'$,	$A= 37°28'$;	
	find $b= 69°50'24''$,	$c= 73°45'15''$,	$B= 77°54'$,
	or $b'=110° \; 9'36''$,	$c'=106°14'45''$,	$B=102° \; 6'$.
20.	Given $a=129°33'$,	$A=104°59'$;	
	find $b= 18°54'38''$,	$c=127° \; 2'27''$,	$B= 23°57'19''$,
	or $b'=161° \; 5'22''$,	$c'= 52°57'33''$,	$B'=156° \; 2'41''$.
21.	Given $a= 21°39'$,	$A= 42°10'10''$;	
	find $b= 25°59'27''.8$,	$c= 33°20'13''.4$,	$B= 52°23' \; 2''.8$,
	or $b'=154° \; 0'32''.2$,	$c'=146°39'46''.6$,	$B'=127°36'57''.2$.
22.	Given $a= 42°18'45''$,	$A= 46°15'25''$;	
	find $b= 60°36'10''$,	$c= 68°42'59''$,	$B= 69°13'47''$,
	or $b'=119°23'50''$,	$c'=111°17' \; 1''$,	$B'=110°46'13''$.
23.	Given $b=160°$,	$B=150°$;	
	find $a= 39° \; 4'50''.7$,	$c=136°50'23''.3$,	$A= 67° \; 9'42''.7$,
	or $a'=140°55' \; 9''.3$,	$c'= 43° \; 9'36''.7$,	$A'=112°50'17''.3$.

24.	Given $a=$ 25° 18′ 45″,	A= 15° 58′ 15″.	
	Ans. Impossible; why?		
25.	Given $a=$ 32° 9′ 17″,	$b=$ 32° 41′;	
	find A= 49° 20′ 17″,	B= 50° 19′ 16″,	$c=$ 44° 33′ 17″.
26.	Given $a=$ 55° 18′,	$b=$ 39° 27′;	
	find A= 66° 15′ 6″,	B= 45° 1′ 31″,	$c=$ 63° 55′ 21″.
27.	Given $a=$ 56° 20′,	$b=$ 78° 40′;	
	find A= 56° 51′ 7″,	B= 80° 31′ 48″,	$c=$ 83° 44′ 44½″.
28.	Given $a=$ 86° 40′,	$b=$ 32° 40′;	
	find A= 88° 11′ 57″.8,	B= 32° 42′ 37″.8,	$c=$ 87° 11′ 39″.8.
29.	Given $a=$ 37° 48′ 12″,	$b=$ 59° 44′ 16″;	
	find A= 41° 55′ 45″,	B= 70° 19′ 15″,	$c=$ 66° 32′ 6″.
30.	Given $a=$116°,	$b=$ 16°;	
	find A= 97° 39′ 24″.4,	B= 17° 41′ 39″.9,	$c=$114° 55′ 20″.4.
31.	Given A= 52° 26′,	B= 49° 15′;	
	find $a=$ 36° 24′ 34″.5,	$b=$ 34° 33′ 40″,	$c=$ 48° 29′ 20″.
32.	Given A= 64° 15′,	B= 48° 24′;	
	find $a=$ 54° 28′ 53″,	$b=$ 42° 30′ 47″,	$c=$ 64° 38′ 38″.
33.	Given A= 54° 1′ 15″,	B=137° 24′ 21″;	
	find $a=$ 29° 46′ 8″,	$b=$155° 27′ 55″,	$c=$142° 9′ 12″.
34.	Given A= 46° 59′ 42″,	B= 57° 59′ 17″;	
	find $a=$ 36° 27′,	$b=$ 43° 32′ 37″,	$c=$ 54° 20′ 3″.
35.	Given A= 55° 32′ 45″,	B=101° 47′ 56″;	
	find $a=$ 54° 41′ 35″,	$b=$104° 21′ 28″,	$c=$ 98° 14′ 24″.
36.	Given A= 60° 27′ 24″.3,	B= 57° 16′ 20″.2;	
	find $a=$ 54° 32′ 32″.1,	$b=$ 51° 43′ 36″.1,	$c=$ 68° 56′ 28″.9.

Solve the following quadrantal triangles:

37.	Given B = 74°45′,	a = 18°12′,	c = 90°;
	find b = 85°17′15″.5,	A = 17°34′ 2″,	C = 104°31′13″.
38.	Given A = 110°47′50″,	B = 135°35′34″.5,	c = 90°;
	find a = 104°53′ 0″.8,	b = 133°39′47″.7,	C = 104°41′37″.2.

Solve the following oblique triangles:

39.	Given a = 73°58′,	b = 38°45′,	C = 46°33′39″;
	find A = 116° 8′28″,	B = 35°46′39″,	c = 51° 1′11″.
40.	Given a = 96°24′30″,	b = 68°27′26″,	C = 84°46′40″;
	find A = 97°53′0¼″,	B = 67°59′39¼″,	c = 87°31′37″.
41.	Given a = 76°24′40″,	b = 58°18′36″,	C = 116°30′28″;
	find A = 63°48′35¼″,	B = 51°46′12¼″,	c = 104°13′27″.
42.	Given a = 86°18′40″,	b = 45°36′20″,	C = 120°46′30″;
	find A = 64°48′53¾″,	B = 40°23′15¾″,	c = 108°39′11½″.
43.	Given a = 88°24′,	b = 56°48′,	C = 128°16′;
	find A = 65°13′ 3½″,	B = 49°27′51″,	c = 120°10′52″.
44.	Given a = 68°20′25″,	b = 52°18′15″,	C = 117°12′20″;
	find A = 56°16′15″,	B = 45° 4′41″,	c = 96°20′44″.
45.	Given a = 88°12′20″,	b = 124° 7′17″,	C = 50° 2′ 1″;
	find A = 63°15′12″,	B = 132°17′59″,	c = 59° 4′25″.
46.	Given a = 32°23′57″,	b = 32°23′57″,	C = 66°49′17″;
	find A = 60°53′ 2″,	B = 60°53′ 2″,	c = 34°19′11″.
47.	Given b = 99°40′48″,	c = 100°49′30″,	A = 65°33′10″;
	find B = 95°38′ 4″,	C = 97°26′29″.1,	a = 64°23′15″.1.
48.	Given A = 31°34′26″,	B = 30°28′12″,	c = 70° 2′ 3″;
	find a = 40° 1′ 5¼″,	b = 38°31′ 3¼″,	C = 130° 3′50″.

49.	Given A = 130° 5′ 22″.4,	B = 32° 26′ 6″.41,	c = 51° 6′ 11″.6;
	find a = 84° 14′ 29″,	b = 44° 13′ 45″,	C = 36° 45′ 26″.
50.	Given A = 96° 46′ 30″,	B = 84° 30′ 20″,	c = 126° 46′;
	find a = 102° 21′ 42″,	b = 78° 17′ 2″,	C = 125° 28′ 13¼″.
51.	Given A = 84° 30′ 20″,	B = 76° 20′ 40″,	c = 130° 46′;
	find a = 94° 34′ 52¼″,	b = 76° 40′ 48¼″,	C = 130° 51′ 33½″.
52.	Given A = 107° 47′ 7″,	B = 38° 58′ 27″,	c = 51° 41′ 14″;
	find a = 70° 20′ 50″,	b = 38° 27′ 59″,	C = 52° 29′ 45″.
53.	Given A = 128° 41′ 49″,	B = 107° 33′ 20″,	c = 124° 12′ 31″;
	find a = 125° 44′ 44″,	b = 82° 47′ 35″,	C = 127° 22′ 7″.
54.	Given A = 129° 58′ 30″,	B = 34° 29′ 30″,	c = 50° 6′ 20″;
	find a = 85° 59′,	b = 47° 29′ 20″,	C = 36° 6′ 50″.
55.	Given A = 95° 38′ 4″,	C = 97° 26′ 29″,	b = 64° 23′ 15″;
	find a = 99° 40′ 48″,	c = 100° 49′ 30″,	B = 65° 33′ 10″.
56.	Given A = 70°,	B = 131° 18′,	c = 116°;
	find a = 57° 56′ 53″,	b = 137° 20′ 33″,	C = 94° 48′ 12″.
57.	Given a = 62° 15′ 24″,	b = 103° 18′ 47″,	A = 53° 42′ 38″;
	find B = 62° 24′ 24″.8,	C = 155° 43′ 11″.3,	c = 153° 9′ 35½″,
	or B′ = 117° 35′ 35″.2,	C′ = 59° 6′ 10″.6,	c′ = 70° 25′ 26″.
58.	Given a = 52° 45′ 20″,	b = 71° 12′ 40″,	A = 46° 22′ 10″;
	find B = 59° 24′ 15¾″,	C = 115° 39′ 55½″,	c = 97° 33′ 18″.8,
	or B′ = 120° 35′ 44¼″,	C′ = 26° 59′ 55″.2,	c′ = 29° 57′ 10″.5.
59.	Given a = 48° 45′ 40″,	b = 67° 12′ 20″,	A = 42° 20′ 30″;
	find B = 55° 39′ 57″,	C = 116° 34′ 18″,	c = 93° 8′ 9″.6,
	or B′ = 124° 20′ 3″,	C′ = 24° 32′ 15″,	c′ = 27° 37′ 20″.
60.	Given a = 46° 20′ 45″,	b = 65° 18′ 15″,	A = 40° 10′ 30″;
	find B = 54° 6′ 19″,	C = 116° 55′ 26″,	c = 90° 31′ 46″,
	or B′ = 125° 53′ 41″,	C′ = 24° 12′ 53″.3,	c′ = 27° 23′ 14″.

61.	Given *a* = 150° 57′ 5″,	*b* = 134° 15′ 54″,	A = 144° 22′ 42″;
	find B = 120° 47′ 44″,	C = 97° 42′ 55″,	*c* = 55° 42′ 8″,
	or B′ = 59° 12′ 16″,	C′ = 29° 9′ 9″,	*c*′ = 23° 57′ 29″.
62.	Given *a* = 50° 45′ 20″,	*b* = 69° 12′ 40″,	A = 44° 22′ 10″;
	find B = 57° 34′ 51″.4,	C = 115° 57′ 50″.6,	*c* = 95° 18′ 16″.4,
	or B′ = 122° 25′ 8″.6,	C′ = 25° 44′ 31″.6,	*c*′ = 28° 45′ 5″.2.
63.	Given *a* = 40° 5′ 25″.6,	*b* = 118° 22′ 7″.3,	A = 29° 42′ 33″.8;
	find B = 42° 37′ 17″.5,	C = 160° 1′ 24″.4,	*c* = 153° 38′ 42″.4,
	or B′ = 137° 22′ 42″.5,	C′ = 50° 18′ 55″.2,	*c*′ = 90° 5′ 41″.0.
64.	Given *a* = 99° 40′ 48″,	*b* = 64° 23′ 15″,	A = 95° 38′ 4″;
	find B = 65° 33′ 10″,	C = 97° 26′ 29″,	*c* = 100° 49′ 30″.
	(No ambiguity; why?)		
65.	Given A = 79° 30′ 45″,	B = 46° 15′ 15″,	*a* = 53° 18′ 20″;
	find *b* = 36° 5′ 34¾″,	*c* = 50° 24′ 57″,	C = 70° 55′ 35″.
	(No ambiguity; why?)		
66.	Given A = 73° 11′ 18″,	B = 61° 18′ 12″,	*a* = 46° 45′ 30″;
	find *b* = 41° 52′ 34¾″,	*c* = 41° 35′ 4″,	C = 60° 42′ 46″.5.
	(Only one solution; why?)		
67.	Given A = 46° 30′ 40″,	B = 36° 20′ 20″,	*a* = 42° 15′ 20″;
	find *b* = 33° 18′ 47½″,	*c* = 60° 32′ 6″,	C = 110° 3′ 14″.6.
	(Only one solution; why?)		
68.	Given A = 61° 29′ 30″,	B = 24° 30′ 30″,	*a* = 34° 30″;
	find *b* = 15° 30′ 30″.5,	*c* = 39° 33′ 52″,	C = 98° 48′ 58″.5.
	(Only one solution; why?)		
69.	Given A = 36° 20′ 20″,	B = 46° 30′ 40″,	*a* = 42° 15′ 20″;
	find *b* = 55° 25′ 2½″,	*c* = 81° 27′ 26¼″,	C = 119° 22′ 27½″,
	or *b*′ = 124° 34′ 57½″,	*c*′ = 162° 34′ 27″,	C′ = 164° 41′ 55″.

70.	Given A = 52° 50′ 20″,	B = 66° 7′ 20″,	*a* = 59° 28′ 27″;
	find *b* = 81° 15′ 15″,	*c* = 110° 10′ 50½″,	C = 119° 43′ 48″,
	or *b*′ = 98° 44′ 45″,	*c*′ = 138° 45′ 26″,	C′ = 142° 24′ 59″.
71.	Given A = 115° 36′ 45″,	B = 80° 19′ 12″,	*b* = 84° 21′ 56″;
	find *a* = 114° 26′ 50″,	*c* = 82° 33′ 31″,	C = 79° 10′ 30″.
72.	Given A = 61° 37′ 52″.7,	B = 139° 54′ 34″.4,	*b* = 150° 17′ 26″.2;
	find *a* = 42° 37′ 17″.5,	*c* = 129° 41′ 4″.8,	C = 89° 54′ 19″.0,
	or *a*′ = 137° 22′ 42″.5,	*c*′ = 19° 58′ 35″.6,	C′ = 26° 21′ 17″.6.
73.	Given A = 70°,	B = 120°,	*b* = 80°.
	Ans. Impossible; why?		
74.	Given *a* = 108° 14′,	*b* = 75° 29′,	*c* = 56° 37′;
	find A = 123° 53′ 47″,	B = 57° 46′ 56″,	C = 46° 51′ 51″.5.
75.	Given *a* = 57° 17′,	*b* = 20° 39′,	*c* = 76° 22′;
	find A = 21° 1′ 2″,	B = 8° 38′ 46″,	C = 155° 31′ 36″.5.
76.	Given *a* = 68° 45′,	*b* = 53° 15′,	*c* = 46° 30′;
	find A = 94° 52′ 40″,	B = 58° 5′ 10″,	C = 50° 50′ 52½″.
77.	Given *a* = 63° 54′,	*b* = 47° 18′,	*c* = 53° 26′;
	find A = 86° 30′ 40″,	B = 54° 46′ 14″,	C = 63° 12′ 55½″.
78.	Given *a* = 70° 14′ 20″,	*b* = 49° 24′ 10″,	*c* = 38° 46′ 10″;
	find A = 110° 51′ 16″,	B = 48° 56′ 4″,	C = 38° 26′ 48″.
79.	Given *a* = 124° 12′ 31″,	*b* = 54° 18′ 16″,	*c* = 97° 12′ 25″;
	find A = 127° 22′ 7″,	B = 51° 18′ 11″,	C = 72° 26′ 40″.
80.	Given *a* = 50° 12′ 4″,	*b* = 116° 44′ 48″,	*c* = 129° 11′ 42″;
	find A = 59° 4′ 25″,	B = 94° 23′ 10″,	C = 120° 4′ 50″.
81.	Given *a* = 100°,	*b* = 50°,	*c* = 60°;
	find A = 138° 15′ 45″.4,	B = 31° 11′ 14″.0,	C = 35° 49′ 58″.2.
82.	Given A = 86° 20′,	B = 76° 30′,	C = 94° 40′;
	find *a* = 87° 20′ 28″,	*b* = 76° 44′ 2½″,	*c* = 93° 55′ 31″.

83.	Given $A = 96°45'$,	$B = 108°30'$,	$C = 116°15'$;
	find $a = 88°27'49''$,	$b = 107°19'52''$,	$c = 115°28'13\frac{1}{4}''$.
84.	Given $A = 78°30'$,	$B = 118°40'$,	$C = 93°20'$;
	find $a = 74°57'46''$,	$b = 120°\ 8'49''$,	$c = 100°18'11\frac{3}{4}''$.
85.	Given $A = 57°50'$,	$B = 98°20'$,	$C = 63°40'$;
	find $a = 58°\ 8'19''$,	$b = 83°\ 5'36''$,	$c = 64°\ 3'20''$.
86.	Given $A = 129°\ 5'28''$,	$B = 142°12'42''$,	$C = 105°\ 8'10''$;
	find $a = 135°49'20''$,	$b = 144°37'15''$,	$c = 60°\ 4'54''$.
87.	Given $A = 138°15'50''$,	$B = 31°11'10''$,	$C = 35°50'$;
	find $a = 100°\ 0'\ 8''.4$,	$b = 49°59'56''.4$,	$c = 60°\ 0'11''.2$.
88.	Given $A = 102°14'12''$,	$B = 54°32'24''$,	$C = 89°\ 5'46''$;
	find $a = 104°25'\ 8''$,	$b = 53°49'25''$,	$c = 97°44'18''$.
89.	Given $A = 20°\ 9'56''$,	$B = 55°52'32''$,	$C = 114°20'14''$;
	find $a = 20°16'38''$,	$b = 56°19'41''$,	$c = 66°20'43''$.

90. If a, b, c are each $< \frac{\pi}{2}$, show that the greater angle *may* exceed $\frac{\pi}{2}$.

91. If a alone $> \frac{1}{2}\pi$, show that A *must* exceed $\frac{\pi}{2}$.

92. If a and b are each $> \frac{1}{2}\pi$, and $c < \frac{1}{2}\pi$, prove that:

(1) The greatest angle A *must* be $> \frac{1}{2}\pi$;

(2) B *may* be $> \frac{1}{2}\pi$;

(3) C may or may not be $< \frac{1}{2}\pi$.

93. If $\cos a$, $\cos b$, $\cos c$ are all negative, prove that $\cos A$, $\cos B$, $\cos C$ are all necessarily negative.

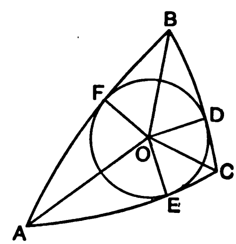
94. In a spherical triangle, of the five products, $\cos a \cos A$, $\cos b \cos B$, $\cos c \cos C$, $\cos a \cos b \cos c$, $-\cos A \cos B \cos C$, show that one is negative, the other four being positive.

CHAPTER XII.

THE IN-CIRCLES AND EX-CIRCLES. — AREAS.

215. The In-Circle (Inscribed Circle). — *To find the angular radius of the in-circle of a triangle.*

Let ABC be the triangle; bisect the angles A and B by the arcs AO, BO; from O draw OD, OE, OF perpendicular to the sides. Then it may be shown that O is the in-centre, and that the perpendiculars OD, OE, OF are each equal to the required angular radius.



Let $2s =$ the sum of the sides of the triangle ABC. The right triangles OAE, OAF are equal.

$$\therefore \text{AF} = \text{AE}.$$

Similarly, $\text{BD} = \text{BF}$, and $\text{CD} = \text{CE}$.

$$\therefore \text{BC} + \text{AF} = \text{AC} + \text{BF} = s.$$

$$\therefore \text{AF} = s - \text{BC} = s - a.$$

Now $\tan \text{OF} = \tan \text{OAF} \sin \text{AF}$. (Art. 186)

or, denoting the radius OF by r, we have

$$\tan r = \tan \frac{A}{2} \sin (s - a) \quad \ldots \ldots \ldots \quad (1)$$

or $$\tan r = \sqrt{\frac{\sin (s-a) \sin (s-b) \sin (s-c)}{\sin s}}$$

$$= \frac{n}{\sin s} \quad \ldots \ldots \ldots \quad \text{(Art. 195) (2)}$$

Also, $\sin(s-a)$

$$=\sin\left[\tfrac{1}{2}(b+c)-\frac{a}{2}\right]$$

$$=\sin\tfrac{1}{2}(b+c)\cos\tfrac{1}{2}a-\cos\tfrac{1}{2}(b+c)\sin\tfrac{1}{2}a$$

$$=\frac{\sin\frac{1}{2}a\cos\frac{1}{2}a}{\sin\frac{A}{2}}\left[\cos\tfrac{1}{2}(B-C)-\cos\tfrac{1}{2}(B+C)\right] \quad \text{(Art. 198)}$$

$$=\frac{\sin a\sin\frac{1}{2}B\sin\frac{1}{2}C}{\sin\frac{1}{2}A},$$

which in (1) gives

$$\tan r=\frac{\sin\frac{B}{2}\sin\frac{C}{2}}{\cos\frac{1}{2}A}\sin a \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (3)$$

$$=\frac{N}{2\cos\frac{1}{2}A\cos\frac{1}{2}B\cos\frac{1}{2}C} \quad . \; . \; . \quad \text{(Art. 196)} \quad (4)$$

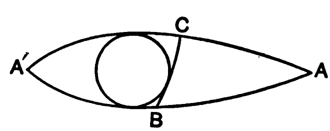
an equation which is equivalent to the following:

$$\cot r=\frac{1}{2N}\left[\cos S+\cos(S-A)+\cos(S-B)+\cos(S-C)\right] \quad (5)$$

216. The Ex-Circles. — *To find the angular radii of the ex-circles of a triangle.*

A circle which touches one side of a triangle and the other two sides produced, is called an *escribed circle,* or *ex-circle,* of the triangle. It is clear that the three ex-circles of any triangle are the in-circles of its colunar triangles (Art. 191, Sch.).

Since the circle escribed to the side a of the triangle ABC is the in-circle of the colunar triangle A′BC, the parts of which are a, $\pi-b$, $\pi-c$, A, $\pi-$ B, $\pi-$ C, the problem becomes identical with that of Art. 215; and we obtain the value for the in-radius of the colunar triangle A′BC, by substituting for b, c, B, C, their supplements in the five equations of that article.



Hence, denoting the radius by r_a, we get

$$\tan r_a = \tan \tfrac{1}{2} \mathrm{A} \sin s \quad \text{. . . . .} \quad (1)$$

$$= \frac{n}{\sin(s-a)} \quad \text{. . . . .} \quad (2)$$

$$= \frac{\cos \frac{1}{2}\mathrm{B} \cos \frac{1}{2}\mathrm{C}}{\cos \frac{1}{2}\mathrm{A}} \sin a \quad \text{. . . . . .} \quad (3)$$

$$= \frac{\mathrm{N}}{2 \cos \frac{1}{2}\mathrm{A} \sin \frac{1}{2}\mathrm{B} \sin \frac{1}{2}\mathrm{C}} \quad \text{. . . . .} \quad (4)$$

$$\cot r_a = \frac{1}{2\mathrm{N}}[-\cos \mathrm{S} - \cos(\mathrm{S}-\mathrm{A}) + \cos(\mathrm{S}-\mathrm{B}) + \cos(\mathrm{S}-\mathrm{C})] \quad (5)$$

These formulæ may also be found independently by methods similar to those employed in Art. 215, for the in-circle, as the student may show.

Sch. Similarly, another triangle may be formed by producing BC, BA to meet again, and another by producing CA, CB to meet again. The colunar triangles on the sides b and c have each two parts, b and B, c and C, equal to parts of the primitive triangle, while their remaining parts are the supplements in the former case of a, c, A, C, and in the latter, of a, b, A, B.

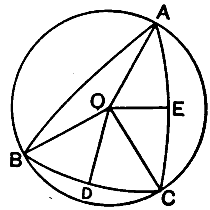
The values for the radii r_b and r_c are therefore found in the same way as the above values for r_a; or they may be obtained from the values of r_a by advancing the letters.

Thus, $\tan r_b = \tan \frac{1}{2} \mathrm{B} \sin s = \dfrac{n}{\sin(s-b)}$, etc.,

and $\tan r_c = \tan \frac{1}{2} \mathrm{C} \sin s = \dfrac{n}{\sin(s-c)}$, etc.

217. The Circumcircle. — *To find the angular radius of the circumcircle of a triangle.*

The small circle passing through the vertices of a spherical triangle is called the *circumscribing circle*, or *circumcircle*, of the triangle.



Let ABC be the triangle; bisect the sides CB, CA at D, E, and let O be the intersection of perpendiculars to CB, CA, at D, E; then O is the circumcentre.

For, join OA, OB, OC; then (Art. 186)

$$\cos OB = \cos BD \cos OD,$$

$$\cos OC = \cos DC \cos OD.$$

$$\therefore\ OB = OC. \quad \text{Similarly, } OC = OA.$$

Now the angle

$$OAB = OBA,\ OBC = OCB,\ OCA = OAC.$$

$$\therefore\ OCB + A = \tfrac{1}{2}(A + B + C) = S.$$

$$\therefore\ OCB = S - A.$$

Let $OC = R$; then, in the triangle ODC, we have

$$\cos OCD = \tan CD \cot CO = \tan \tfrac{1}{2} a \cot R \quad . \quad \text{(Art. 186)}$$

$$\therefore\ \tan R = \frac{\tan \frac{1}{2} a}{\cos(S - A)} \quad . \ . \ . \ . \ . \ . \ . \ . \ . \quad (1)$$

or

$$\tan R = -\frac{\cos S}{N} \quad . \ . \ . \ . \ . \ . \quad \text{(Art. 196)} \quad (2)$$

Also $\cos(S - A) = \cos \tfrac{1}{2}[(B + C) - A]$

$$= \cos \tfrac{1}{2}(B + C) \cos \tfrac{1}{2} A + \sin \tfrac{1}{2}(B + C) \sin \tfrac{1}{2} A$$

$$= \frac{\sin \frac{1}{2} A \cos \frac{1}{2} A}{\cos \frac{1}{2} a}[\cos \tfrac{1}{2}(b + c) + \cos \tfrac{1}{2}(b - c)] \quad \text{(Art. 198)}$$

$$= \frac{\sin A}{\cos \frac{1}{2} a} \cos \tfrac{1}{2} b \cos \tfrac{1}{2} c,$$

which in (1) gives

$$\tan R = \frac{\sin \frac{1}{2} a}{\sin A \cos \frac{1}{2} b \cos \frac{1}{2} c} \quad . \ . \ . \ . \ . \ . \quad (3)$$

$$= \frac{2 \sin \frac{1}{2} a \sin \frac{1}{2} b \sin \frac{1}{2} c}{n} \quad . \quad \text{(Art. 195)} \quad (4)$$

which may be reduced to the following:

$$\tan R = \frac{1}{2n}[\sin(s-a) + \sin(s-b) + \sin(s-c) - \sin s] \quad (5)$$

218. Circumcircles of Colunar Triangles. — *To find the angular radii of the circumcircles of the three colunar triangles.*

Let R_1, R_2, R_3 be the angular radii of the circumcircles of the colunar triangles on the sides a, b, c, respectively. Then, since R_1 is the circumradius of the triangle A'BC whose parts are a, $\pi - b$, $\pi - c$, A, $\pi - B$, $\pi - C$, we have, from Art. 217,

$$\tan R_1 = -\frac{\tan \frac{1}{2} a}{\cos S} \quad \ldots \quad (1)$$

$$\tan R_1 = \frac{\cos(S - A)}{N} \quad \ldots \quad (2)$$

$$\tan R_1 = \frac{\sin \frac{1}{2} a}{\sin A \sin \frac{1}{2} b \sin \frac{1}{2} c} \quad \ldots \quad (3)$$

$$\tan R_1 = \frac{2 \sin \frac{1}{2} a \cos \frac{1}{2} b \cos \frac{1}{2} c}{n} \quad \ldots \quad (4)$$

$$\tan R_1 = \frac{1}{2n}[\sin s - \sin(s-a) + \sin(s-b) + \sin(s-c)] \quad (5)$$

Similarly,

$$\tan R_2 = -\frac{\tan \frac{1}{2} b}{\cos S} = \frac{\cos(S - B)}{N} = \text{etc.},$$

and $$\tan R_3 = -\frac{\tan \frac{1}{2} c}{\cos S} = \frac{\cos(S - C)}{N} = \text{etc.}$$

EXAMPLES.

Prove the following:

1. $\cos s + \cos(s-a) + \cos(s-b) + \cos(s-c)$
$$= 4 \cos \tfrac{1}{2} a \cos \tfrac{1}{2} b \cos \tfrac{1}{2} c.$$

2. $\cos(s-b) + \cos(s-c) - \cos(s-a) - \cos s$
$$= 4 \cos \tfrac{1}{2} a \sin \tfrac{1}{2} b \sin \tfrac{1}{2} c.$$

3. $\tan r_b = \dfrac{\cos\frac{1}{2}C\cos\frac{1}{2}A}{\cos\frac{1}{2}B}\sin b = \dfrac{N}{2\cos\frac{1}{2}B\sin\frac{1}{2}C\sin\frac{1}{2}A}.$

4. $\tan r_c = \dfrac{\cos\frac{1}{2}A\cos\frac{1}{2}B}{\cos\frac{1}{2}C}\sin c = \dfrac{N}{2\cos\frac{1}{2}C\sin\frac{1}{2}A\sin\frac{1}{2}B}.$

5. $\cot r : \cot r_1 : \cot r_2 : \cot r_3$
$$= \sin s : \sin(s-a) : \sin(s-b) : \sin(s-c).$$

6. $\tan r \tan r_1 \tan r_2 \tan r_3 = n^2.$

7. $\cot r \tan r_1 \tan r_2 \tan r_3 = \sin^2 s.$

8. $\tan R_2 = \dfrac{2\cos\frac{1}{2}a\sin\frac{1}{2}b\cos\frac{1}{2}c}{n}.$

9. $\tan R_3 = \dfrac{2\cos\frac{1}{2}a\cos\frac{1}{2}b\sin\frac{1}{2}c}{n}.$

10. $\tan R_1 : \tan R_2 : \tan R_3 = \cos(S-A) : \cos(S-B) : \cos(S-C).$

11. $\cot R \cot R_1 \cot R_2 \cot R_3 = N^2.$

12. $\tan R \cot R_1 \cot R_2 \cot R_3 = \cos^2 S.$

AREAS OF TRIANGLES.

219. Problem. — *To find the area of a spherical triangle, having given the three angles.*

Let r = the radius of the sphere.

E = the spherical excess $= A + B + C - 180°$.

K = area of triangle ABC.

It is shown in Geometry (Art. 738) that *the absolute area of a spherical triangle is to that of the surface of the sphere as its spherical excess, in degrees, is to 720°.*

$$\therefore\ K : 4\pi r^2 = E : 720°.$$

$$\therefore\ K = \frac{E}{180°}\pi r^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Cor. The areas of the colunar triangles are

$$\frac{(2A-E)}{180°}\pi r^2,\quad \frac{(2B-E)}{180°}\pi r^2,\quad \frac{(2C-E)}{180°}\pi r^2.$$

220. Problem. — *To find the area of a triangle, having given the three sides.*

Here the object is to express E in terms of the sides.

I. *Cagnoli's Theorem.*

$$\sin\tfrac{1}{2}E = \sin\tfrac{1}{2}(A+B+C-\pi)$$

$$= \sin\tfrac{1}{2}(A+B)\sin\tfrac{1}{2}C - \cos\tfrac{1}{2}(A+B)\cos\tfrac{1}{2}C$$

$$= \frac{\sin\frac{1}{2}C\cos\frac{1}{2}C}{\cos\frac{1}{2}c}\left[\cos\tfrac{1}{2}(a-b) - \cos\tfrac{1}{2}(a+b)\right] \quad . \text{ (Art. 198)}$$

$$= \frac{\sin\frac{1}{2}a\sin\frac{1}{2}b\sin C}{\cos\frac{1}{2}c} = \frac{\sin\frac{1}{2}a\sin\frac{1}{2}b}{\cos\frac{1}{2}c}\cdot\frac{2n}{\sin a\sin b} \quad \text{(Art. 195) (1)}$$

$$\therefore\ \sin\tfrac{1}{2}E = \frac{n}{2\cos\frac{1}{2}a\cos\frac{1}{2}b\cos\frac{1}{2}c} \quad . \ . \ . \ . \ . \ . \ . \ . \quad (2)$$

II. *Lhuilier's Theorem.*

$$\tan\tfrac{1}{4}E = \frac{\sin\frac{1}{4}(A+B+C-\pi)}{\cos\frac{1}{4}(A+B+C-\pi)}$$

$$= \frac{\sin\frac{1}{2}(A+B) - \sin\frac{1}{2}(\pi-C)}{\cos\frac{1}{2}(A+B) + \cos\frac{1}{2}(\pi-C)} \quad . \ . \ . \ . \ . \ . \quad \text{(Art. 45)}$$

$$= \frac{\sin\frac{1}{2}(A+B) - \cos\frac{1}{2}C}{\cos\frac{1}{2}(A+B) + \sin\frac{1}{2}C}$$

$$= \frac{\cos\frac{1}{2}(a-b) - \cos\frac{1}{2}c}{\cos\frac{1}{2}(a+b) + \cos\frac{1}{2}c}\cdot\frac{\cos\frac{1}{2}C}{\sin\frac{1}{2}C} \quad . \ . \ . \ . \ . \ . \ . \quad \text{(Art. 198)}$$

$$= \frac{\sin\frac{1}{2}(s-b)\sin\frac{1}{2}(s-a)}{\cos\frac{1}{2}s\cos\frac{1}{2}(s-c)}\cot\tfrac{1}{2}C \quad . \ . \ . \ . \ . \ . \quad \text{(Art. 45)}$$

$$= \sqrt{\tan\tfrac{1}{2}s\tan\tfrac{1}{2}(s-a)\tan\tfrac{1}{2}(s-b)\tan\tfrac{1}{2}(s-c)} \quad \text{(Art. 195) (3)}$$

221. Problem. — *To find the area of a triangle, having given two sides and the included angle.*

$$\cos\tfrac{1}{2}E = \cos\left[\tfrac{1}{2}(A+B) - (\tfrac{1}{2}\pi - \tfrac{1}{2}C)\right]$$

$$= \cos\tfrac{1}{2}(A+B)\sin\tfrac{1}{2}C + \sin\tfrac{1}{2}(A+B)\cos\tfrac{1}{2}C$$

$$= \cos\tfrac{1}{2}(a+b)\sin^2\tfrac{1}{2}C + \cos\tfrac{1}{2}(a-b)\cos^2\tfrac{1}{2}C \quad \text{(Art. 198)}$$

$$= \left[\cos\tfrac{1}{2}a\cos\tfrac{1}{2}b + \sin\tfrac{1}{2}a\sin\tfrac{1}{2}b\cos C\right]\sec\tfrac{1}{2}c \quad . \quad . \quad (1)$$

Dividing (1) of Art. 220 by this equation, and reducing, we have

$$\tan\tfrac{1}{2}E = \frac{\tan\tfrac{1}{2}a\tan\tfrac{1}{2}b\sin C}{1 + \tan\tfrac{1}{2}a\tan\tfrac{1}{2}b\cos C} \quad . \quad . \quad . \quad . \quad . \quad (2)$$

EXAMPLES.

1. Given $a = 113^\circ\,2'\,56''.64$, $b = 82^\circ\,39'\,28''.4$, $c = 74^\circ\,54'\,31''.06$; find the area of the triangle, the radius of the sphere being r.

By formula (3) of Art. 220,

$$a = 113^\circ\ 2'56''.64$$
$$b = 82^\circ 39'28''.40$$
$$c = 74^\circ 54'31''.06$$
$$2s = 270^\circ 36'56''.10$$
$$s = 135^\circ 18'28''.05$$
$$s-a = 22^\circ 15'31''.41,$$
$$s-b = 52^\circ 38'59''.65,$$
$$s-c = 60^\circ 23'56''.99.$$

$$\tfrac{1}{2}s = 67^\circ 39'14''.025$$
$$\tfrac{1}{2}(s-a) = 11^\circ\ 7'45''.705$$
$$\tfrac{1}{2}(s-b) = 26^\circ 19'29''.825$$
$$\tfrac{1}{2}(s-c) = 30^\circ 11'58''.495$$
$$\log\tan\tfrac{1}{2}s = 0.3860840$$
$$\log\tan\tfrac{1}{2}(s-a) = 9.2938583$$
$$\log\tan\tfrac{1}{2}(s-b) = 9.6944058$$
$$\log\tan\tfrac{1}{2}(s-c) = 9.7649261$$
$$\log\tan^2\tfrac{1}{4}E = 9.1392742$$
$$\log\tan\tfrac{1}{4}E = 9.5696371.$$
$$\tfrac{1}{4}E = 20^\circ\,21'\,58''.25.$$
$$E = 81^\circ 27'53''$$
$$= 293273''.$$

$$\therefore\ K = \tfrac{293273}{648000} \times \pi r^2 \quad . \quad . \quad . \quad . \quad [(1) \text{ of Art. } 219]$$

$$= \tfrac{293273}{206265} r^2 = \text{area}.$$

2. Given $A = 84°20'19''$, $B = 27°22'40''$, $C = 75°33'$;
find $E = 7°15'59''$.

3. Given $a = 46°24'$, $b = 67°14'$, $c = 81°12'$;
find $K = \frac{118880}{206265} r^2$.

4. Given $a = 108°14'$, $b = 75°29'$, $c = 56°37'$;
find $E = 48°32'34''.5$.

5. Prove $\cos\frac{1}{2}E = \dfrac{1 + \cos a + \cos b + \cos c}{4\cos\frac{1}{2}a\cos\frac{1}{2}b\cos\frac{1}{2}c}$

$$= \frac{\cos^2\frac{1}{2}a + \cos^2\frac{1}{2}b + \cos^2\frac{1}{2}c - 1}{2\cos\frac{1}{2}a\cos\frac{1}{2}b\cos\frac{1}{2}c}.$$

6. " $\sin\frac{1}{4}E = \sqrt{\dfrac{\sin\frac{1}{2}s\sin\frac{1}{2}(s-a)\sin\frac{1}{2}(s-b)\sin\frac{1}{2}(s-c)}{\cos\frac{1}{2}a\cos\frac{1}{2}b\cos\frac{1}{2}c.}}.$

7. " $\cos\frac{1}{4}E = \sqrt{\dfrac{\cos\frac{1}{2}s\cos\frac{1}{2}(s-a)\cos\frac{1}{2}(s-b)\cos\frac{1}{2}(s-c)}{\cos\frac{1}{2}a\cos\frac{1}{2}b\cos\frac{1}{2}c}}.$

8. " $\cot\frac{1}{2}E = \dfrac{\cot\frac{1}{2}a\cot\frac{1}{2}b + \cos C}{\sin C}$

$$= \frac{\cot\frac{1}{2}b\cot\frac{1}{2}c + \cos A}{\sin A}$$

$$= \frac{\cot\frac{1}{2}c\cot\frac{1}{2}a + \cos B}{\sin B}.$$

EXAMPLES.

Prove the following:

1. $\sin(s-a) + \sin(s-b) + \sin(s-c) - \sin s$
$$= 4\sin\tfrac{1}{2}a\sin\tfrac{1}{2}b\sin\tfrac{1}{2}c.$$

2. $\sin s + \sin(s-b) + \sin(s-c) - \sin(s-a)$
$$= 4\sin\tfrac{1}{2}a\cos\tfrac{1}{2}b\cos\tfrac{1}{2}c.$$

3. $\sin(s-b)\sin(s-c)+\sin(s-c)\sin(s-a)$
$$+\sin(s-a)\sin(s-b)+\sin s\sin(s-a)$$
$$+\sin s\sin(s-b)+\sin s\sin(s-c)$$
$$=\sin b\sin c+\sin c\sin a+\sin a\sin b.$$

4. $\sin(s-b)\sin(s-c)+\sin(s-c)\sin(s-a)$
$$-\sin(s-a)\sin(s-b)+\sin s\sin(s-a)$$
$$+\sin s\sin(s-b)-\sin s\sin(s-c)$$
$$=\sin b\sin c+\sin c\sin a-\sin a\sin b.$$

5. $\sin^2 s+\sin^2(s-a)+\sin^2(s-b)+\sin^2(s-c)$
$$=2(1-\cos a\cos b\cos c).$$

6. $\sin^2 s+\sin^2(s-a)-\sin^2(s-b)-\sin^2(s-c)$
$$=2\cos a\sin b\sin c.$$

7. $\cos^2 s+\cos^2(s-a)+\cos^2(s-b)+\cos^2(s-c)$
$$=2(1+\cos a\cos b\cos c).$$

8. $\cos^2 s+\cos^2(s-a)-\cos^2(s-b)-\cos^2(s-c)$
$$=-2\cos a\sin b\sin c.$$

9. $\tan r\cot r_1\tan r_2\tan r_3=\sin^2(s-a).$

10. $\tan r\tan r_1\cot r_2\tan r_3=\sin^2(s-b).$

11. $\tan r\tan r_1\tan r_2\cot r_3=\sin^2(s-c).$

12. $\cot r\sin s=\cot\frac{1}{2}A\cot\frac{1}{2}B\cot\frac{1}{2}C.$

13. $\tan r_1+\tan r_2+\tan r_3-\tan r=\dfrac{4N\sin S}{\sin A\sin B\sin C}.$

14. $\cot r_1+\cot r_2+\cot r_3-\cot r=\dfrac{4\sin\frac{1}{2}a\sin\frac{1}{2}b\sin\frac{1}{2}c}{n}.$

15. $\tan r_1:\tan r_2:\tan r_3=\dfrac{\sin a}{1+\cos A}:\dfrac{\sin b}{1+\cos B}:\dfrac{\sin c}{1+\cos C}.$

16. $\dfrac{\tan r_1+\tan r_2+\tan r_3-\tan r}{\cot r_1+\cot r_2+\cot r_3-\cot r}=\frac{1}{2}(1+\cos a+\cos b+\cos c).$

17. $\cot^2 r_1 + \cot^2 r_2 + \cot^2 r_3 + \cot^2 r = \dfrac{2(1-\cos a \cos b \cos c)}{n^2}.$

18. $\dfrac{1}{\sin^2 r} + \dfrac{1}{\sin^2 r_1} - \dfrac{1}{\sin^2 r_2} - \dfrac{1}{\sin^2 r_3} = \dfrac{2 \cos a \sin b \sin c}{n^2}.$

19. $\cot r_2 \cot r_3 + \cot r_3 \cot r_1 + \cot r_1 \cot r_2$
$$+ \cot r(\cot r_1 + \cot r_2 + \cot r_3)$$
$$= \frac{\sin b \sin c + \sin c \sin a + \sin a \sin b}{n^2}.$$

20. $\tan r_2 \tan r_3 + \tan r_3 \tan r_1 + \tan r_1 \tan r_2$
$$+ \tan r(\tan r_1 + \tan r_2 + \tan r_3)$$
$$= \sin b \sin c + \sin c \sin a + \sin a \sin b.$$

21. $\cot R \tan R_1 \cot R_2 \cot R_3 = \cos^2 (S - A).$

22. $\cot R \cot R_1 \tan R_2 \cot R_3 = \cos^2 (S - B).$

23. $\cot R \cot R_1 \cot R_2 \tan R_3 = \cos^2 (S - C).$

24. $\tan R_1 + \tan R_2 = \cot r + \cot r_3.$

25. $\tan R_1 + \tan R_2 + \tan R_3 - \tan R = 2 \cot r.$

26. $\tan R - \tan R_1 + \tan R_2 + \tan R_3 = 2 \cot r_1.$

27. $\tan R + \tan R_1 - \tan R_2 + \tan R_3 = 2 \cot r_2.$

28. $\tan R + \tan R_1 + \tan R_2 - \tan R_3 = 2 \cot r_3.$

29. $\cot r_1 + \cot r_2 + \cot r_3 - \cot r = 2 \tan R.$

30. $\cot r - \cot r_1 + \cot r_2 + \cot r_3 = 2 \tan R_1.$

31. $\cot r + \cot r_1 - \cot r_2 + \cot r_3 = 2 \tan R_2.$

32. $\cot r + \cot r_1 + \cot r_2 - \cot r_3 = 2 \tan R_3.$

33. $\tan R + \cot r = \tan R_1 + \cot r_1 = \text{etc.},$
$$= \tfrac{1}{2}(\cot r + \cot r_1 + \cot r_2 + \cot r_3).$$

34. $\tan^2 R + \tan^2 R_1 + \tan^2 R_2 + \tan^2 R_3$
$$= \frac{2(1 + \cos A \cos B \cos C)}{N^2}.$$

35. $\dfrac{\tan^2 R + \tan^2 R_1 + \tan^2 R_2 + \tan^2 R_3}{\cot^2 r + \cot^2 r_1 + \cot^2 r_2 + \cot^2 r_3} = 1.$

36. $\tan^2 R + \tan^2 R_1 - \tan^2 R_2 - \tan^2 R_3$

$$= -\frac{2(\cos A \sin B \sin C)}{N^2}.$$

37. $\cot^2 r + \cot^2 r_1 - \cot^2 r_2 - \cot^2 r_3 = \dfrac{2 \cos a \sin b \sin c}{n^2}.$

38. $\dfrac{\tan^2 R + \tan^2 R_1 - \tan^2 R_2 - \tan^2 R_3}{\cot^2 r + \cot^2 r_1 - \cot^2 r_2 - \cot^2 r_3} = -\dfrac{\cos A}{\cos a}.$

39. $\tan R \cot R_1 = \tan \tfrac{1}{2} b \tan \tfrac{1}{2} c.$

40. $(\cot r + \tan R)^2 + 1 = \left(\dfrac{\sin a + \sin b + \sin c}{2n}\right)^2.$

41. $(\cot r_1 - \tan R)^2 + 1 = \left(\dfrac{\sin b + \sin c - \sin a}{2n}\right)^2.$

42. $\tan \tfrac{1}{2} A \sin (s - a) = \dfrac{N}{2 \cos \frac{1}{2} A \cos \frac{1}{2} B \cos \frac{1}{2} C}.$

43. $\dfrac{\tan r}{\tan R} = \dfrac{\cos (S - A) \cos (S - B) \cos (S - C)}{2 \cos \frac{1}{2} A \cos \frac{1}{2} B \cos \frac{1}{2} C}.$

44. $\cot (s - b) \cot (s - c) + \cot (s - c) \cot (s - a)$

$$+ \cot (s - a) \cot (s - b) = \operatorname{cosec}^2 r.$$

45. $\cot (s-b) \cot (s-c) - \cot s \cot (s-b) - \cot s \cot (s-c)$

$$= \operatorname{cosec}^2 r_1.$$

46. $\cot (s-c) \cot (s-a) - \cot s \cot (s-c) - \cot s \cot (s-a)$

$$= \operatorname{cosec}^2 r_2.$$

47. $\cot (s-a) \cot (s-b) - \cot s \cot (s-a) - \cot s \cot (s-b)$

$$= \operatorname{cosec}^2 r_3.$$

48. $\dfrac{\cot (s-a)}{\sin^2 r_1} + \dfrac{\cot (s-b)}{\sin^2 r_2} + \dfrac{\cot (s-c)}{\sin^2 r_3} + \dfrac{2 \cot s}{\sin^2 r}$

$$= 3 \cot (s-a) \cot (s-b) \cot (s-c).$$

49. $\operatorname{cosec}^2 r_1 + \operatorname{cosec}^2 r_2 + \operatorname{cosec}^2 r_3 - \operatorname{cosec}^2 r$
$$= -2 \cot s[\cot(s-a) + \cot(s-b) + \cot(s-c)].$$

50. $\dfrac{1}{\sin^2 r} + \dfrac{1}{\sin^2 r_1} + \dfrac{1}{\sin^2 r_2} + \dfrac{1}{\sin^2 r_3}$
$$= \frac{-2\Sigma \tan(s-a)\tan(s-b)}{\tan s \tan(s-a)\tan(s-b)\tan(s-c)}.$$

51. $\cot R - \cot R_1 - \cot R_2 - \cot R_3 = \dfrac{2 N \cos s}{n}.$

52. $\sin(A - \tfrac{1}{2}E) = \dfrac{n}{2 \cos\frac{1}{2}a \sin\frac{1}{2}b \sin\frac{1}{2}c}.$

53. $\sin(B - \tfrac{1}{2}E) = \dfrac{n}{2 \sin\frac{1}{2}a \cos\frac{1}{2}b \sin\frac{1}{2}c}.$

54. $\sin(C - \tfrac{1}{2}E) = \dfrac{n}{2 \sin\frac{1}{2}a \sin\frac{1}{2}b \cos\frac{1}{2}c}.$

55. $\cos(A - \tfrac{1}{2}E) = \dfrac{\sin^2\frac{1}{2}b + \sin^2\frac{1}{2}c - \sin^2\frac{1}{2}a}{2 \cos\frac{1}{2}a \sin\frac{1}{2}b \sin\frac{1}{2}c}.$

56. $\cos(B - \tfrac{1}{2}E) = \dfrac{\sin^2\frac{1}{2}c + \sin^2\frac{1}{2}a - \sin^2\frac{1}{2}b}{2 \sin\frac{1}{2}a \cos\frac{1}{2}b \sin\frac{1}{2}c}.$

57. $\cos(C - \tfrac{1}{2}E) = \dfrac{\sin^2\frac{1}{2}a + \sin^2\frac{1}{2}b - \sin^2\frac{1}{2}c}{2 \sin\frac{1}{2}a \sin\frac{1}{2}b \cos\frac{1}{2}c}.$

58. $\cot(A - \tfrac{1}{2}E) = \dfrac{\cot\frac{1}{2}a \tan\frac{1}{2}b - \cos C}{\sin C}$
$$= \frac{\tan\frac{1}{2}b \tan\frac{1}{2}c + \cos A}{\sin A} = \frac{\tan\frac{1}{2}a \cot\frac{1}{2}b - \cos B}{\sin B}.$$

59. $\tan\tfrac{1}{2}(A - \tfrac{1}{2}E) = \sqrt{\cot\tfrac{1}{2}s \cot\tfrac{1}{2}(s-a) \tan\tfrac{1}{2}(s-b) \tan\tfrac{1}{2}(s-c)}.$

60. $\tan\tfrac{1}{2}(B - \tfrac{1}{2}E) = \sqrt{\cot\tfrac{1}{2}s \tan\tfrac{1}{2}(s-a) \cot\tfrac{1}{2}(s-b) \tan\tfrac{1}{2}(s-c)}.$

61. $\tan\tfrac{1}{2}(C - \tfrac{1}{2}E) = \sqrt{\cot\tfrac{1}{2}s \tan\tfrac{1}{2}(s-a) \tan\tfrac{1}{2}(s-b) \cot\tfrac{1}{2}(s-c)}.$

62. If S, S_1, S_2, S_3 denote the sums of the angles of a triangle and its three colunars, prove that $S + S_1 + S_2 + S_3 = 3\pi$.

63. In an equilateral triangle, $\tan R = 2 \tan r$.

64. If E_1, E_2, E_3 denote the spherical excesses of the colunars on a, b, c, respectively, show that $E + E_1 + E_2 + E_3 = 2\pi$; and therefore the sum of the areas of any triangle and its colunars is half the area of the sphere.

65. Given $a = 108° 14'$, $b = 75° 29'$, $c = 56° 37'$;
find $E = 48° 32' 34''.5$.

66. Given $a = 63° 54'$, $b = 47° 18'$, $c = 53° 26'$;
find $E = 24° 29' 49\frac{1}{2}''$.

67. Given $a = 69° 15' 6''$, $b = 120° 42' 47''$, $c = 159° 18' 33''$;
find $E = 216° 40' 23''$.

68. Given $a = 33° 1' 45''$, $b = 155° 5' 18''$, $C = 110° 10'$;
find $E = 133° 48' 55''$.

69. Given $a = b = c = 1°$, on the earth's surface;
find $E = 27''.21$.

70. Given $a = b = c = 60°$, on a sphere of 6 inches radius; find the area of the triangle. *Ans.* 19.845 square inches.

71. If $a = b = \frac{\pi}{3}$, and $c = \frac{\pi}{2}$, prove $\sin\frac{1}{2}E = \frac{1}{3}$, and $\cos E = \frac{7}{9}$.

72. If $C = \frac{\pi}{2}$, prove $\sin\frac{1}{2}E = \sin\frac{1}{2}a \sin\frac{1}{2}b \sec\frac{1}{2}c$, and $\cos\frac{1}{2}E = \cos\frac{1}{2}a \cos\frac{1}{2}b \sec\frac{1}{2}c$.

73. If $a = b$, and $C = \frac{\pi}{2}$, prove $\tan E = \frac{1}{2}\tan a \sec a$.

74. If $A + B + C = 2\pi$, prove $\cos^2\frac{1}{2}a + \cos^2\frac{1}{2}b + \cos^2\frac{1}{2}c = 1$.

75. If $a + b = \pi$, prove that $E = C$; and if E' denote the spherical excess of the polar triangle, prove that

$$\sin\tfrac{1}{2}E' = \sin a \cos\tfrac{1}{2}C.$$

76. Prove $\sin^2\frac{1}{2}E = \dfrac{\sqrt{\sin\frac{1}{2}E \sin\frac{1}{2}E_1 \sin\frac{1}{2}E_2 \sin\frac{1}{2}E_3}}{\cot\frac{1}{2}a \cot\frac{1}{2}b \cot\frac{1}{2}c}$.

CHAPTER XIII.

APPLICATIONS OF SPHERICAL TRIGONOMETRY.

SPHERICAL ASTRONOMY.

222. Astronomical Definitions.

The *celestial sphere* is the imaginary concave surface of the visible heavens in which all the heavenly bodies appear to be situated.

The *sensible horizon* of a place is the circle in which a plane tangent to the earth's surface at the place meets the celestial sphere.

The *rational horizon* is the great circle in which a plane through the centre of the earth parallel to the sensible horizon meets the celestial sphere. Because the radius of the celestial sphere is so great, in comparison with the radius of the earth, these two horizons will sensibly coincide, and form a great circle called the *celestial horizon.*

The *zenith* of a place is that pole of the horizon which is exactly overhead; the other pole of the horizon directly underneath is called the *nadir.*

Vertical circles are great circles passing through the zenith and nadir. The two principal vertical circles are the celestial meridian and the prime vertical.

The *celestial meridian* of a place is the great circle in which the plane of the terrestrial meridian meets the celestial sphere; the points in which it cuts the horizon are called the *north* and *south* points.

The *prime vertical* is the vertical circle which is perpendicular to the meridian; the points in which it cuts the horizon are called the *east* and *west* points.

The *axis* of the earth or of the celestial sphere is the imaginary line about which the earth rotates.

The *celestial equator*, or *equinoctial*, is the great circle in which the plane of the earth's equator intersects the celestial sphere.

The *poles* of the equinoctial are the points in which the axis pierces the celestial sphere.

Hour circles, or *circles of declination*, are great circles passing through the poles of the equinoctial.

The *ecliptic* is a great circle of the celestial sphere, and the apparent path of the sun due to the real motion of the earth round the sun.

The *equinoxes* are the points in which the ecliptic cuts the equinoctial. There are two, called the *vernal* and the *autumnal* equinox, which the sun passes on March 20 and September 22.

The *obliquity* of the ecliptic is the angle between the planes of the ecliptic and equator, and is about 23° 27′.

Circles of latitude are great circles passing through the poles of the ecliptic.

223. Spherical Coördinates. — The position of a point on the celestial sphere may be denoted by any one of three systems. In each system two great circles are taken as standards of reference, and the point is determined by means of these circles, which are called its *spherical coördinates*, as follows:

I. *The horizon and the celestial meridian of the place.*

The *azimuth* of a star is the arc of the horizon intercepted between the south point and the vertical circle passing through the star; it is generally reckoned from the south point of the horizon round by the west, from 0° to 360°.

The *altitude* of a star is its angular distance above the horizon, measured on a vertical circle. The complement of the altitude is called the *zenith distance*.

II. *The equinoctial and the hour circle through the vernal equinox.*

The *right ascension* of a star is the arc of the equinoctial included between the vernal equinox and the hour circle passing through the star; it is reckoned eastward from 0° to 360°, or from 0^h to 24^h.

The angle at the pole between the hour circle of the star and the meridian of the place is called the *hour angle* of the star.

The *declination* of a star is its distance from the equinoctial, measured on its hour circle; it may be north or south, and is usually reckoned from 0° to 90°. It corresponds to terrestrial latitude.

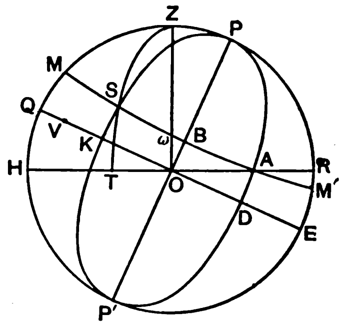
The *polar distance* of a star is its distance from the pole, and is the complement of its declination. The right ascension and declination of celestial bodies are given in *nautical almanacs*.

III. *The ecliptic and the circle of latitude through the vernal equinox.*

The *latitude* of a star is its angular distance from the ecliptic measured on a circle of latitude; it may be north or south, and is reckoned from 0° to 90°.

The *longitude* of a star is the arc of the ecliptic intercepted between the vernal equinox and the circle of latitude passing through the star.

224. Graphic Representation of the Spherical Coördinates. — The figure will serve to illustrate the preceding definitions. O is the earth, PHP′R is the meridian, P the north pole, HR the horizon, EQ the equinoctial, Z the zenith. Then, of a place whose zenith is Z, QZ is the terrestrial latitude; and since



$$QZ = PR,$$

$$\therefore \; PR = \text{the latitude.}$$

But PR is the elevation of the pole above the horizon.

Hence *the elevation of the pole above the horizon is equal to the latitude.*

Let V be the vernal equinox, and let S be any heavenly body, such as the sun or a star; then its position is denoted as follows:

VK = right ascension	of the body				$= \alpha$,
KS = declination	"	"	"		$= \delta$,
ZPS or QK = hour angle	"	"	"		$= t$,
PS = north polar distance	"	"	"		$= p$,
HT = azimuth	"	"	"		$= a$,
TS = altitude	"	"	"		$= h$,
ZS = zenith distance	"	"	"		$= z$,
QZ = PR = latitude of the observer					$= \phi$.

The triangle ZPS is called the *astronomical triangle;* $ZP = 90° - \phi$ = co-latitude of the observer,

$$PS = 90° - \delta, \; SZ = 90° - h.$$

Let the small circle MM′, passing through S, and parallel to the equinoctial, represent the apparent diurnal motion of the heavenly body S (the declination being supposed constant); then the body S will appear to *rise* at A (if we suppose the Eastern hemisphere is represented in the diagram). It will be at B at 6 o'clock in the morning, at M at noon, at M′ at midnight, and at ω it will be east.

225. Problems. — By means of the foregoing definitions and diagram we may solve several astronomical problems of an elementary character as follows:

(1) *Given the latitude of a place and the declination of a star; to find the time of its rising.*

Let A be the position of the star in the horizon. Then in the triangle APR, right angled at R, we have

$$\cos \text{RPA} = -\cos \text{ZPA} = \tan \text{RP} \cot \text{AP}.$$

$$\therefore \cos t = -\tan \phi \tan \delta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

from which the hour angle is found.

Since the hourly rate at which a heavenly body appears to move from east to west is 15°, if the hour angle be divided by 15 the time will be found. In the case of the sun, formula (1) gives the time from sunrise to noon, and hence the length of the day.

Ex. Required the apparent time of sunrise at a place whose latitude is 40° 36′ 23″.9, on July 4, 1881, when the sun's declination is 22° 52′ 1″.

$\phi = 40° \, 36' \, 23''.9,$	$\log \tan \phi = 9.9331352$
$\delta = 22° \, 52' \, 1''.$	$\log \tan \delta = 9.6250362$
	$\log \cos t = 9.5581714-$
	$\therefore t = 111° \, 11' \, 44''$
	$= 7^h \, 24^m \, 47^s$, nearly,

which taken from 12^h, the time of apparent noon, gives

$4^h\ 35^m\ 13^s$, the time of apparent sunrise.*

(2) *Given the latitude of a place and the declination of a star; to find its azimuth from the north at rising.*

Let A = the azimuth = AR. Then in the triangle APR we have

$$\sin AP = \cos AR \cos PR,$$

or

$$\sin \delta = \cos A \cos \phi.$$

$$\therefore \cos A = \sin \theta \sec \phi \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Ex. Required the hour angle and azimuth of Arcturus when it rises to an observer in New York, lat. 40° 42′ N., the declination being 19° 57′ N.

Ans. $7^h\ 12^m\ 46^s.3$; N. 63° 15′ 11″ E.

(3) *Given the latitude of the observer and the hour angle and declination of a star; to find its azimuth and altitude.*

Here we have given, in the triangle ZPS, two sides and the included angle; that is, $PZ = 90° - \phi$, $PS = 90° - \delta$, and $ZPS = t$. Let A = the azimuth from the north = RT, p = the angle ZSP, and z = ZS. Then by Delambre's Analogies (Art. 198),

$$\sin \tfrac{1}{2}(p + A) \cos \tfrac{1}{2} z = \cos \tfrac{1}{2}(\delta - \phi) \cos \tfrac{1}{2} t,$$

$$\sin \tfrac{1}{2}(p - A) \sin \tfrac{1}{2} z = \sin \tfrac{1}{2}(\delta - \phi) \cos \tfrac{1}{2} t,$$

$$\cos \tfrac{1}{2}(p + A) \cos \tfrac{1}{2} z = \sin \tfrac{1}{2}(\delta + \phi) \sin \tfrac{1}{2} t,$$

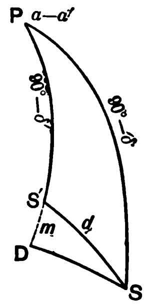
$$\cos \tfrac{1}{2}(p - A) \sin \tfrac{1}{2} z = \cos \tfrac{1}{2}(\delta + \phi) \sin \tfrac{1}{2} t.$$

Hence, when ϕ, t, and δ are given, that is, the latitude of the place, and the hour angle and the declination of a heavenly body, A, z, and p can be found.

In a similar manner may be solved the converse problem: *Given the latitude of the observer and the azimuth and altitude of a star; to find its hour angle and declination.*

* In these examples no corrections are applied for refraction, semi-diameter of the sun, change in declination from noon, etc.

(4) *Given the right ascensions and declinations of two stars; to find the distance between them.*



Let P be the pole, S and S′ the two stars. Let α and α' be the right ascensions of the stars;

δ and δ' their declinations; and

d the required distance.

Then we have given, in the triangle PSS′, two sides and the included angle; that is, $PS = 90^\circ - \delta = p$, $PS' = 90^\circ - \delta' = p'$, and $P = \alpha - \alpha'$.

This may be solved by Art. 198, or by the second method of Art. 209, as follows:

Draw SD perpendicular to PS′ produced; let $PD = m$. Then

$$\cos P = \tan PD \cot PS.$$

$$\therefore\ \tan m = \cos P \tan p.$$

Also $$\cos SS' = \cos PS \cos S'D \sec PD \quad . \quad \text{(Art. 209)}$$

$$\therefore\ \cos d = \cos p \cos S'D \sec m.$$

Ex. Required the distance between Sirius and Aldebaran, the right ascensions being $6^h\,38^m\,37^s.6$ and $4^h\,27^m\,25^s.9$, and the declinations $16^\circ\,31'\,2''$ S. and $16^\circ\,12'\,27''$ N., respectively.

Here $P = 2^h\,11^m\,11^s.7$
$= 32^\circ\,47'\,55''$
$p = 106^\circ\,31'\,2''$
$m = 109^\circ\,25'\,55''$
$p' = 73^\circ\,47'\,33''$
$S'D = 35^\circ\,38'\,22''$

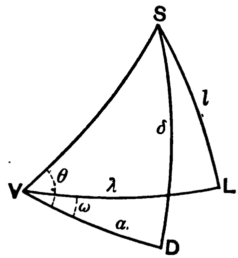
$\log\cos P = 9.9245789$
$\log\tan p = 0.5279161-$
$\log\tan m = 0.4524950-$
$\therefore\ m = 109^\circ\,25'\,55''.$

$p = 106^\circ\,31'\,2''$,
$m = 109^\circ\,25'\,55''$,
$SS' = d = 46^\circ\,0'\,44''$.

$\log\cos S'D = 9.9099302$
$\log\cos p = 9.4537823-$
$\text{colog}\cos m = 0.4779643-$
$\log\cos d = 9.8416768$

(5) *Given the right ascension and declination of a star; to find its latitude and longitude.*

Let V be the vernal equinox, S the star, VD, VL the the equator and the ecliptic, SD, SL perpendicular to VD, VL. Then VD = right ascension = α, SD = declination = δ, VL = longitude = λ, SL = latitude = l. Denote the obliquity of the ecliptic DVL by ω, and the angle DVS by θ.



From the right triangles SVD, SVL we get

$$\cot\theta = \sin\alpha\cot\delta \quad \ldots\ldots\ldots \quad (1)$$

$$\tan\lambda = \cos(\theta-\omega)\tan\alpha\sec\theta \quad \ldots\ldots \quad (2)$$

$$\sin l = \sin(\theta-\omega)\sin\delta\,\mathrm{cosec}\,\theta \quad \ldots\ldots \quad (3)$$

From (1), θ is determined; and from (2) and (3), λ and l are determined.

Ex. Given the right ascension of a star $5^h\,6^m\,42^s.01$, and its declination $45^\circ\,51'\,20''.1$ N.; to find its longitude and latitude, the obliquity of the ecliptic being $23^\circ\,27'\,19''.45$.

$\alpha = 76^\circ\,40'\,30''.15$	$\log\sin\alpha = 9.9881479$
$\delta = 45^\circ\,51'\,20''.1$	$\log\cot\delta = 9.9870277$
$\theta = 46^\circ\,38'\,11''.8$	$\log\cot\theta = 9.9751756$
$\omega = 23^\circ\,27'\,19''.45$	
$\theta-\omega = 23^\circ\,10'\,52''.35$	$\log\cos(\theta-\omega) = 9.9634401$
	$\log\tan\alpha = 10.6255266$
	$\operatorname{colog}\cos\theta = 0.1632816$
$\lambda = 79^\circ\,58'\,3''.44.$	$\log\tan\lambda = 10.7522483$
	$\log\sin(\theta-\omega) = 9.5950996$
	$\log\sin\delta = 9.8558743$
	$\operatorname{colog}\sin\theta = 0.1384575$
$l = 22^\circ\,51'\,48''.4.$	$\log\sin l = 9.5894314$

EXAMPLES.

1. Find the apparent time of sunrise at a place whose latitude is 40° 42′, when the sun's declination is 17° 49′ N.

Ans. $4^h\ 56^m$.

2. Given the latitude of a place = 40° 36′ 23″.9, the hour angle of a star = 46° 40′ 4″.5, and its declination = 23° 4′ 24″.3; to find its azimuth and altitude.

Ans. Azimuth = 80° 23′ 4″.47, altitude = 47° 15′ 18″.3.

3. Find the altitude and azimuth of a star to an observer in latitude 38° 53′ N., when the hour angle of the star is $3^h\ 15^m\ 20^s$ W., and the declination is 12° 42′ N.

Ans. Altitude = 39° 38′ 0″; azimuth = S. 72° 28′ 14″ W.

4. Given the latitudes of New York City and Liverpool 40° 42′ 44″ N. and 53° 25′ N., respectively, and their longitudes 74° 0′ 24″ W. and 3° W., respectively; to find the shortest distance on the earth's surface between them in miles, considering the earth as a perfect sphere whose radius is 3956 miles.

NOTE. — This is evidently a case of (4) where two sides and the included angle are given, to find the third side.

Ans. 3305 miles.

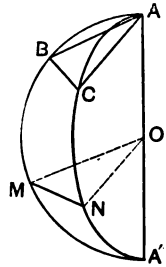
5. The latitudes of Paris and Pekin are 48° 50′ 14″ N. and 39° 54′ 13″ N., and their difference of longitude is 114° 7′ 30″; find the distance between them in degrees.

Ans. 73° 56′ 40″.

GEODESY.

226. The Chordal Triangle. — *Given two sides and the included angle of a spherical triangle; to find the corresponding angle of the chordal triangle.*

The *chordal triangle* is the triangle formed by the chords of the sides of a spherical triangle.



Let ABC be a spherical triangle, O the centre of the sphere, A′BC the colunar triangle, and M, N the middle points of the arcs A′B, A′C. Then the chord AB is parallel to the radius OM, since they are both perpendicular to the chord A′B. Similarly, AC is parallel to ON.

In the spherical triangle A′MN, we have

$$\cos MN = \cos A'N \cos A'M + \sin A'N \sin A'M \cos A' \quad \text{(Art. 191)}$$

Denote the angle BAC of the chordal triangle by A_1.

Then arc MN or angle MON $= A_1$, $A'N = \frac{1}{2}(\pi - b)$, $A'M = \frac{1}{2}(\pi - c)$, and $A' = A$.

$$\therefore \cos A_1 = \sin \tfrac{1}{2}b \sin \tfrac{1}{2}c + \cos \tfrac{1}{2}b \cos \tfrac{1}{2}c \cos A \quad . \quad (1)$$

with similar values for $\cos B_1$ and $\cos C_1$.

Cor. 1. If the sides b and c are small compared with the radius of the sphere, A_1 will not differ much from A.

Let $A_1 = A - \theta$; then

$$\cos A_1 = \cos A + \theta \sin A, \text{ nearly.}$$

But $\quad \sin \frac{1}{2}b \sin \frac{1}{2}c = \sin^2 \frac{1}{4}(b + c) - \sin^2 \frac{1}{4}(b - c)$,

and $\quad \cos \frac{1}{2}b \cos \frac{1}{2}c = \cos^2 \frac{1}{4}(b + c) - \sin^2 \frac{1}{4}(b - c)$.

Substituting in (1) and reducing, we get

$$\theta = \tan \tfrac{1}{2}A \sin^2 \tfrac{1}{4}(b + c) - \cot \tfrac{1}{2}A \sin^2 \tfrac{1}{4}(b - c) \quad . \quad (2)$$

which is *the circular measure of the excess of an angle of the spherical triangle over the corresponding angle of the chordal triangle.*

The value in seconds is obtained by dividing the circular measure by the circular measure of one second, or, approximately, by the *sine* of one second.

Cor. 2. *The angles of the chordal triangle are, respectively, equal to the arcs joining the middle points of the sides of the colunar triangles.*

227. Legendre's Theorem. — *If the sides of a spherical triangle be small compared with the radius of the sphere, then each angle of the spherical triangle exceeds by one-third of the spherical excess the corresponding angle of the plane triangle, the sides of which are of the same length as the arcs of the spherical triangle.*

Let a, b, c be the lengths of the sides of the spherical triangle, and r the radius of the sphere; then the circular measures* of the sides are respectively $\frac{a}{r}, \frac{b}{r}, \frac{c}{r}$. Hence, neglecting powers of $\frac{1}{r}$ above the *fourth*,

$$\cos A = \frac{\cos\frac{a}{r} - \cos\frac{b}{r}\cos\frac{c}{r}}{\sin\frac{b}{r}\sin\frac{c}{r}} \quad . \ . \ . \ . \ . \ . \ . \ . \ . \quad \text{(Art. 191)}$$

$$= \frac{\left(1-\frac{a^2}{2r^2}+\frac{a^4}{24r^4}\right)-\left(1-\frac{b^2}{2r^2}+\frac{b^4}{24r^4}\right)\left(1-\frac{c^2}{2r^2}+\frac{c^4}{24r^4}\right)}{\frac{bc}{r^2}\left(1-\frac{b^2}{6r^2}\right)\left(1-\frac{c^2}{6r^2}\right)} \quad \text{(Art. 156)}$$

$$= \frac{r^2}{bc}\left(\frac{b^2+c^2-a^2}{2r^2}+\frac{a^4-b^4-c^4-6b^2c^2}{24r^4}\right)\left(1-\frac{b^2+c^2}{6r^2}\right)^{-1}$$

$$= \left(\frac{b^2+c^2-a^2}{2bc}+\frac{a^4-b^4-c^4-6b^2c^2}{24bcr^2}\right)\left(1+\frac{b^2+c^2}{6r^2}\right)$$

$$= \frac{b^2+c^2-a^2}{2bc} - \frac{2b^2c^2+2c^2a^2+2a^2b^2-a^4-b^4-c^4}{24bcr^2} \quad . \ . \quad (1)$$

* The term $\frac{a}{r}$ is the circular measure of the angle which the arc a subtends at the centre of the sphere; and similarly for $\frac{b}{r}$ and $\frac{c}{r}$.

Now if A′, B′, C′ denote the angles of the plane triangle whose sides are a, b, c, respectively, we have

$$\cos A' = \frac{b^2 + c^2 - a^2}{2bc} \quad . \quad . \quad . \quad . \quad \text{(Art. 96)}$$

and $$\sin^2 A' = \frac{2b^2c^2 + 2c^2a^2 + 2a^2b^2 - a^4 - b^4 - c^4}{4b^2c^2} \quad \text{(Art. 100)}$$

Therefore (1) becomes

$$\cos A = \cos A' - \frac{bc \sin^2 A'}{6r^2} \quad . \quad . \quad . \quad . \quad (2)$$

Let $A = A' + \theta$, where θ is a very small quantity; then

$$\cos A = \cos A' - \theta \sin A', \text{ nearly.}$$

$$\therefore \ \theta = \frac{bc \sin A'}{6r^2} = \frac{\Delta}{3r^2} \quad . \quad . \quad . \quad . \quad . \quad \text{(Art. 101)}$$

where Δ denotes the area of the plane triangle whose sides are a, b, c.

We have therefore $$A = A' + \frac{\Delta}{3r^2}.$$

Similarly $$B = B' + \frac{\Delta}{3r^2}, \quad C = C' + \frac{\Delta}{3r^2}.$$

$$\therefore \ A + B + C - A' - B' - C' = \frac{\Delta}{r^2};$$

or $$A + B + C - \pi = \frac{\Delta}{r^2} = \text{spherical excess (Art. 219)}$$

$$\therefore \ A - A' = B - B' = C - C' = \frac{\Delta}{3r^2} = \tfrac{1}{3} \text{ spherical excess.}$$

Cor. 1. *If the sides of a spherical triangle be very small compared with the radius of the sphere, the area of the spherical triangle is approximately equal to*

$$\Delta\left(1 + \frac{a^2 + b^2 + c^2}{24r^2}\right).$$

For, $\tan\frac{1}{4}E = \sqrt{\tan\frac{s}{2r}\tan\frac{s-a}{2r}\tan\frac{s-b}{2r}\tan\frac{s-c}{2r}}$ (Art. 220)

and

$$\tan\frac{s}{2r} = \frac{s}{2r}\left[1+\frac{s^2}{12r^2}\right];\ \tan\frac{s-a}{2r} = \frac{s-a}{2r}\left[1+\frac{(s-a)^2}{12r^2}\right], \text{ etc.}$$

(Art. 156)

$$\therefore \tan\tfrac{1}{4}E = \sqrt{\frac{s}{2r}\cdot\frac{s-a}{2r}\cdot\frac{s-b}{2r}\cdot\frac{s-c}{2r}\left[1+\frac{s^2}{12r^2}\right]\left[1+\frac{(s-a)^2}{12r^2}\right]\cdots}$$

$$\therefore \tfrac{1}{4}E = \frac{\Delta}{4r^2}\sqrt{1+\frac{s^2+(s-a)^2+(s-b)^2+(s-c)^2}{12r^2}}$$

(Arts. 101 and 156)

$$= \frac{\Delta}{4r^2}\left(1+\frac{a^2+b^2+c^2}{12r^2}\right)^{\frac{1}{2}} = \frac{\Delta}{4r^2}\left(1+\frac{a^2+b^2+c^2}{24r^2}\right).$$

$$\therefore\ Er^2 = \Delta\left(1+\frac{a^2+b^2+c^2}{24r^2}\right).$$

That is: *the area of the spherical triangle exceeds the area of the plane triangle by* $\frac{a^2+b^2+c^2}{24r^2}$ *part of the latter.*

Cor. 2. If we omit terms of the second degree in $\frac{1}{r}$, we have

$$Er^2 = \Delta.$$

Hence, *if the sides of a spherical triangle be very small compared with the radius of the sphere, its area is approximately equal to the area of the plane triangle having sides of the same length.*

228. Roy's Rule. — *The area of a spherical triangle on the Earth's surface being known, to establish a formula for computing the spherical excess in seconds.*

Let A be the area of the triangle in square feet, and n the number of seconds in the spherical excess. Then we have

$$A = \frac{E \times 60 \times 60 \pi r^2}{180° \times 60 \times 60} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\text{Art. } 219)$$

$$= \frac{n\pi r^2}{180 \times 60 \times 60} = \frac{nr^2}{206265} \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Now, the length of a degree on the Earth's surface is found by actual measurement to be 365155 feet.

$$\therefore \ \frac{\pi r}{180°} = 365155. \ \therefore \ r = \frac{180 \times 365155}{\pi}.$$

Substituting this value of r in (1), and reducing, we get

$$\log n = \log A - 9.3267737 \quad . \quad . \quad . \quad . \quad . \quad (2)$$

This formula is called General Roy's rule, as it was used by him in the Trigonometric Survey of the British Isles. He gave it in the following form: *From the logarithm of the area of the triangle, taken as a plane triangle, in square feet, subtract the constant logarithm* 9.3267737; *and the remainder is the logarithm of the excess above* 180°, *in seconds, nearly.*

Ex. If the observed angles of a spherical triangle are 42° 2′ 32″, 67° 55′ 39″, 70° 1′ 48″, and the side opposite the angle A is 27404.2 feet, required the number of seconds in the sum of the errors made in observing the three angles.

Here the *apparent* spherical excess is

$$A + B + C - 180° = -1''.$$

The area of the triangle is calculated from the expression

$$\frac{a^2 \sin B \sin C}{2 \sin A} \quad . \quad . \quad . \quad . \quad . \quad . \quad (\text{Art. } 101)$$

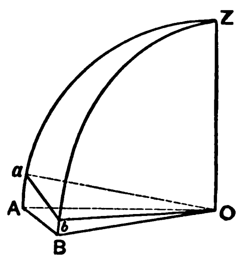
and by Roy's Rule the *computed* spherical excess is found to be .23″.

Now since the *computed* spherical excess may be supposed to be the *real* spherical excess, the sum of the observed angles ought to have been 180° + .23″.

Hence it appears that the sum of the errors of the observations is $.23'' - (-1'') = 1''.23$, which the observer must add to the three observed angles, in such proportions as his judgment may direct. One way is to increase each of the observed angles by one-third of $1''.23$, and take the angles thus corrected for the true angles.

229. Reduction of an Angle to the Horizon. — *Given the angles of elevation or depression of two objects, which are at a small angular distance from the horizon, and the angle which the objects subtend, to find the horizontal angle between them.*

Let a, b be the two objects, the angular distance between which is measured by an observer at O; let OZ be the direction at right angles to the observer's horizon. Describe a sphere round O as a centre, and let vertical planes through Oa, Ob, meet the horizon at OA, OB, respectively; then the horizontal angle AOB, or AB, is required.



Let $ab = \theta$, $AB = \theta + x$, $Aa = h$, $Bb = k$. Then in the triangle aZb we have

$$\cos AB = \cos aZb = \frac{\cos ab - \cos aZ \cos bZ}{\sin aZ \sin bZ},$$

or $$\cos(\theta + x) = \frac{\cos\theta - \sin h \sin k}{\cos h \cos k}.$$

This gives the exact value of AB; by approximation we obtain, where x is essentially small,

$$\cos\theta - x\sin\theta = \frac{\cos\theta - hk}{1 - \frac{1}{2}(h^2 + k^2)}.$$

$$\therefore\ x\sin\theta = hk - \tfrac{1}{2}(h^2 + k^2)\cos\theta, \text{ nearly.}$$

$$\therefore\ x = \frac{2hk - (h^2 + k^2)\left(\cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2}\right)}{2\sin\theta}$$

$$= \tfrac{1}{4}\left[(h + k)^2 \tan\tfrac{1}{2}\theta - (h - k)^2 \cot\tfrac{1}{2}\theta\right].$$

EXAMPLES.

1. Prove that the angles subtended by the sides of a spherical triangle at the pole of its circumcircle are respectively double the corresponding angles of its chordal triangle.

2. If A_1, B_1, C_1; A_2, B_2, C_2; A_3, B_3, C_3; be the angles of the chordal triangles of the colunars, prove that

$$\cos A_1 = \cos \tfrac{1}{2} a \sin S, \quad \cos B_1 = \sin \tfrac{1}{2} b \sin (S - C), \quad \cos C_1 = \sin \tfrac{1}{2} c \sin (S - B),$$
$$\cos A_2 = \sin \tfrac{1}{2} a \sin (S - C), \quad \cos B_2 = \cos \tfrac{1}{2} b \sin S, \quad \cos C_2 = \sin \tfrac{1}{2} c \sin (S - A),$$
$$\cos A_3 = \sin \tfrac{1}{2} a \sin (S - B), \quad \cos B_3 = \sin \tfrac{1}{2} b \sin (S - A), \quad \cos C_3 = \cos \tfrac{1}{2} c \sin S.$$

3. Prove Legendre's Theorem from either of the formulæ for $\sin \frac{1}{2} A$, $\cos \frac{1}{2} A$, $\tan \frac{1}{2} A$, respectively, in terms of the sides.

4. If $C = A + B$, prove $\cos C = - \tan \frac{1}{2} a \tan \frac{1}{2} b$.

230. Small Variations in the Parts of a Spherical Triangle.

It is sometimes important in Geodesy and Astronomy to determine the *error* introduced into one of the *computed* parts of a triangle from any *small error* in the *given* parts.

If two parts of a spherical triangle remain constant, to determine the relation between the small variations of any other two parts.

Suppose C and c to remain constant.

(1) Required the relation between the small variations of a side and the opposite angle (a, A).

Take the equation

$$\sin A \sin c = \sin C \sin a \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

We suppose a and A to receive very small increments da and dA; then we require the ratio of da and dA when both are extremely small. Thus

$$\sin (A + dA) \sin c = \sin C \sin (a + da),$$

or $(\sin A \cos dA + \cos A \sin dA) \sin c$

$$= \sin C (\sin a \cos da + \cos a \sin da) \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Because the arcs dA and da are extremely small, their sines are equal to the arcs themselves and their cosines equal 1: therefore (2) may be written

$$\sin A \sin c + \cos A \sin c dA = \sin C \sin a + \sin C \cos a da \quad (3)$$

Subtracting (1) from (3), we have

$$\cos A \sin c dA = \sin C \cos a da.$$

$$\therefore \frac{da}{dA} = \frac{\cos A \sin c}{\sin C \cos a} = \frac{\tan a}{\tan A}.$$

(2) Required the relation between the small variations of the other sides (a, b). We have

$$\cos c = \cos a \cos b + \sin a \sin b \cos C \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$\therefore \cos c = \cos(a+da)\cos(b+db) + \sin(a+da)\sin(b+db)\cos C,$$

or $$= (\cos a - \sin a da)(\cos b - \sin b db)$$
$$+ (\sin a + \cos a da)(\sin b + \cos b db) \cos C \quad . \quad (2)$$

Subtracting (2) from (1) and neglecting the product $da\, db$, we have

$$0 = (\sin a \cos b - \cos a \sin b \cos C)\, da$$
$$+ (\cos a \sin b - \sin a \cos b \cos C)\, db,$$

or $$0 = \frac{(\cot b \sin a - \cos a \cos C)}{\sin a} da + \frac{(\cot a \sin b - \cos b \cos C)}{\sin b} db.$$

$$0 = \frac{\cot B \sin C\, da}{\sin a} + \frac{\cot A \sin C\, db}{\sin b} \quad . \quad . \quad . \quad . \quad (\text{Art. } 193)$$

$$\therefore \frac{da}{db} = -\frac{\cos A}{\cos B}.$$

(3) Required the relation between the small variations of the other angles (A, B).

By means of the polar triangle, we may deduce from the result just found, that

$$\frac{dA}{dB} = -\frac{\cos a}{\cos b}.$$

(4) Required the relation between the small variations of a side and the adjacent angle (b, A). We have

$$\cot c \sin b = \cot C \sin A + \cos b \cos A \quad . \quad . \quad . \quad \text{(Art. 193)}$$

Giving to b and A very small increments, and subtracting, as before, we get

$$\cot c \cos b db = \cot C \cos A dA - \sin b \cos A db - \cos b \sin A dA.$$

$$(\cot c \cos b + \sin b \cos A) db = (\cot C \cos A - \cos b \sin A) dA.$$

$$\therefore \frac{\cos a}{\sin c} db = -\frac{\cos B}{\sin C} da \quad . \quad . \quad \text{(Arts. 191 and 192)}$$

$$\therefore \quad \frac{db}{dA} = -\frac{\cos B \sin b}{\cos a \sin B} = -\frac{\sin b \cot B}{\cos a}.$$

EXAMPLES.

1. If A and c are constant, prove the following relations between the small variations of any two parts of the other elements:

$$\frac{da}{dC} = -\frac{\tan a}{\tan C}; \qquad \frac{db}{dB} = \frac{\sin a}{\sin C}.$$

$$\frac{db}{dC} = -\frac{\tan a}{\sin C}; \qquad \frac{dC}{dB} = -\cos a.$$

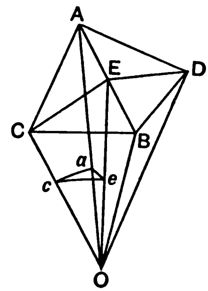
2. If B and C remain constant, prove the following:

$$\frac{db}{dc} = \frac{\tan b}{\tan c}; \qquad \frac{dA}{da} = \sin B \sin C.$$

$$\frac{dA}{dc} = \sin A \tan b; \qquad \frac{da}{dc} = \frac{\sin a}{\sin c \cos b}.$$

POLYEDRONS.

231. To find the Inclination of Two Adjacent Faces of a Regular Polyedron.



Let C and D be the centres of the circles inscribed in the two adjacent faces whose common edge is AB; bisect AB in E, and join CE and DE; CE and DE will be perpendicular to AB. ∴ ∠CED is the inclination of the two adjacent faces, which denote by I.

In the plane CED draw CO and DO at right angles to CE and DE, respectively, and meeting in O. Join OA, OE, OB, and from O as centre describe a sphere, cutting OA, OC, OE at *a*, *c*, *e*, respectively; then *ace* is a spherical triangle. Since AB is perpendicular to CE and DE, it is perpendicular to the plane CED; therefore the plane AOB, in which AB lies, is perpendicular to the plane CED. ∴ ∠ *aec* is a right angle.

Let *m* be the number of sides in each face, and *n* the number of plane angles in each solid angle. Then

$$\angle ace = \angle \mathrm{ACE} = \frac{2\pi}{2m} = \frac{\pi}{m},$$

and $$\angle cae = \tfrac{1}{2}\angle \text{ of the planes OAC and OAD.}$$

$$\therefore\ \angle cae = \frac{2\pi}{2n} = \frac{\pi}{n}.$$

In the right triangle *cae* we have

$$\cos cae = \cos ce \sin ace.$$

But $$\cos ce = \cos \mathrm{COE} = \cos\left(\frac{\pi}{2} - \frac{\mathrm{I}}{2}\right) = \sin\frac{\mathrm{I}}{2}.$$

$$\therefore\ \cos\frac{\pi}{n} = \sin\frac{\mathrm{I}}{2}\sin\frac{\pi}{m}.$$

$$\therefore\ \sin\frac{\mathrm{I}}{2} = \cos\frac{\pi}{n}\operatorname{cosec}\frac{\pi}{m}.$$

Cor. 1. *If r be the radius of the inscribed sphere, and a be a side of one of the faces, then*

$$r = \frac{a}{2}\cot\frac{\pi}{m}\tan\frac{I}{2}.$$

For, $$r = OC = CE \tan CEO = AE \cot ACE \tan CEO$$

$$= \frac{a}{2}\cot\frac{\pi}{m}\tan\frac{I}{2}.$$

Cor. 2. *If* R *be the circumradius of the polyedron, then*

$$R = \frac{a}{2}\tan\frac{\pi}{n}\tan\frac{I}{2}.$$

For, $r = OA \cos aoc = R \cot eca \cot eac = R \cot\frac{\pi}{m}\cot\frac{\pi}{n}.$

$$\therefore\ R = \frac{a}{2}\tan\frac{\pi}{n}\tan\frac{I}{2}.$$

Cor. 3. *The surface of a regular polyedron,* F *being the number of faces,* $= \frac{ma^2F}{4}\cot\frac{\pi}{m}.$

For, the area of one face $= \frac{a^2}{4} m \cot\frac{\pi}{m}.$ $\therefore$ etc.

Cor. 4. *The volume of a regular polyedron*

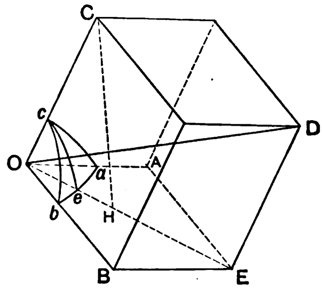
$$= \frac{ma^2rF}{12}\cot\frac{\pi}{m}.$$

For, the volume of the pyramid which has one face of the polyedron for base and O for vertex

$$= \frac{r}{3}\cdot\frac{ma^2}{4}\cot\frac{\pi}{m}. \quad \therefore \text{ etc.}$$

232. Volume of a Parallelopiped. — *To find the volume of a parallelopiped in terms of its edges and their inclinations to one another.*

Let the edges be OA $= a$, OB $= b$, OC $= c$, and let the inclinations be BOC $= \alpha$, COA $= \beta$, AOB $= \gamma$. Draw CH perpendicular to the face AOBE. Describe a sphere round O as centre, meeting OA, OB, OC, OE, in a, b, c, e, respectively.



The volume of the parallelopiped is equal to the area of the base OAEB multiplied by the altitude CH; that is,

$$\text{volume} = ab \sin\gamma \cdot \text{CH} = abc \sin\gamma \sin ce$$

where ce is the perpendicular arc from c on ab.

$$\therefore \text{ volume} = abc \sin\gamma \sin ac \sin bac \quad . \quad . \quad . \quad \text{(Art. 186)}$$

$$= abc \sin\gamma \sin\beta \frac{2n}{\sin\beta \sin\gamma} \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(Art. 195)}$$

$$= abc\sqrt{1 - \cos^2\alpha - \cos^2\beta - \cos^2\gamma + 2\cos\alpha\cos\beta\cos\gamma}.$$

Cor. 1. *The surface of a parallelopiped*

$$= 2\,(bc \sin\alpha + ca \sin\beta + ab \sin\gamma).$$

Cor. 2. *The volume of a tetraedron*

$$= \tfrac{1}{6}\,abc\sqrt{1 - \cos^2\alpha - \cos^2\beta - \cos^2\gamma + 2\cos\alpha\cos\beta\cos\gamma}.$$

For, a tetraedron is one-sixth of a parallelopiped which has the same altitude and its base double that of the tetraedron.

233. Diagonal of a Parallelopiped. — *To find the diagonal of a parallelopiped in terms of its edges, and their mutual inclinations.*

Let OD (figure of Art. 232) be a parallelopiped, whose edges OA $= a$, OB $= b$, OC $= c$, and their inclinations BOC $= \alpha$, COA $= \beta$, AOB $= \gamma$; let OD be the diagonal required, and OE the diagonal of the face OAB. Then the triangle OED gives

$$\overline{OD}^2 = \overline{OE}^2 + \overline{ED}^2 + 2\,OE \cdot ED \cos COE$$
$$= a^2 + b^2 + 2\,ab \cos \gamma + c^2 + 2\,c \cdot OE \cos COE \qquad (1)$$

Now, it is clear that OE cos COE is the projection of OE on the line OC, and therefore it must be equal to the sum of the projections of OB and BE (or of OB and OA), on the same line.*

$$\therefore\ OE \cos COE = b \cos \alpha + a \cos \beta,$$

which in (1) gives

$$\overline{OD}^2 = a^2 + b^2 + c^2 + 2\,bc \cos \alpha + 2\,ca \cos\ \beta + 2\,ab \cos \gamma\,. \qquad (2)$$

234. Table of Formulæ in Spherical Trigonometry. — For the convenience of the student, many of the preceding formulæ are summed up in the following table:

1. $\cos c\ = \cos a \cos b$ . . . . (Art. 185)
2. $\sin b\ = \sin B \sin c.$
3. $\sin a\ = \sin A \sin c.$
4. $\cos C = -\cos A \cos B$ . . . . (Art. 189)
5. $\sin B = \sin b \sin C.$
6. $\sin A = \sin a \sin C.$
7. $\dfrac{\sin a}{\sin A} = \dfrac{\sin b}{\sin B} = \dfrac{\sin c}{\sin C}$ . . . . (Art. 190)
8. $\cos a = \cos b \cos c + \sin b \sin c \cos A$ . . (Art. 191)
9. $\cos b\ = \cos c \cos a + \sin c \sin a \cos B.$
10. $\cos c\ = \cos a \cos b + \sin a \sin b \cos C.$
11. $\cos A = -\cos B \cos C + \sin B \sin C \cos a$ (Art. 192)
12. $\cos B = -\cos C \cos A + \sin C \sin A \cos b.$

* From the nature of projections (Plane and Solid Geom., Art. 326).

13. $\cos C = -\cos A \cos B + \sin A \sin B \cos c.$

14. $\cot a \sin b = \cot A \sin C + \cos C \cos b$ . . (Art. 193)

15. $\cot a \sin c = \cot A \sin B + \cos B \cos c.$

16. $\cot b \sin a = \cot B \sin C + \cos C \cos a.$

17. $\cot b \sin c = \cot B \sin A + \cos A \cos c.$

18. $\cot c \sin a = \cot C \sin B + \cos B \cos a.$

19. $\cot c \sin b = \cot C \sin A + \cos A \cos b.$

20. $\sin a \cos B = \cos b \sin c - \sin b \cos c \cos A$ (Art. 194)

21. $\sin a \cos C = \sin b \cos c - \cos b \sin c \cos A.$

22. $\sin b \cos A = \cos a \sin c - \sin a \cos c \cos B.$

23. $\sin b \cos C = \sin a \cos c - \cos a \sin c \cos B.$

24. $\sin c \cos A = \cos a \sin b - \sin a \cos b \cos C.$

25. $\sin c \cos B = \sin a \cos b - \cos a \sin b \cos C.$

26. $\sin \frac{1}{2} A = \sqrt{\frac{\sin(s-b)\sin(s-c)}{\sin b \sin c}}$ . . . (Art. 195)

27. $\cos \frac{1}{2} A = \sqrt{\frac{\sin s \sin(s-a)}{\sin b \sin c}}.$

28. $\tan \frac{1}{2} A = \sqrt{\frac{\sin(s-b)\sin(s-c)}{\sin s \sin(s-a)}}.$

29. $$\sin A = \frac{2\sqrt{\sin s \sin(s-a)\sin(s-b)\sin(s-c)}}{\sin b \sin c} = \frac{2n}{\sin b \sin c},$$

where $n = \sqrt{\sin s \sin(s-a)\sin(s-b)\sin(s-c)}.$

30. $\sin \frac{1}{2} a = \sqrt{\frac{-\cos S \cos(S-A)}{\sin B \sin C}}$ . . . (Art. 196)

31. $\cos \frac{1}{2} a = \sqrt{\frac{\cos(S-B)\cos(S-C)}{\sin B \sin C}}.$

32. $\tan \frac{1}{2} a = \sqrt{-\frac{\cos S \cos (S-A)}{\cos (S-B) \cos (S-C)}}.$

33. $\sin a = \frac{2\sqrt{-\cos S \cos(S-A) \cos(S-B) \cos(S-C)}}{\sin B \sin C}.$

34. $\tan \frac{1}{2}(A+B) = \frac{\cos \frac{1}{2}(a-b)}{\cos \frac{1}{2}(a+b)} \cot \frac{1}{2} C$ . . (Art. 197)

35. $\tan \frac{1}{2}(A-B) = \frac{\sin \frac{1}{2}(a-b)}{\sin \frac{1}{2}(a+b)} \cot \frac{1}{2} C.$

36. $\tan \frac{1}{2}(a+b) = \frac{\cos \frac{1}{2}(A-B)}{\cos \frac{1}{2}(A+B)} \tan \frac{1}{2} c.$

37. $\tan \frac{1}{2}(a-b) = \frac{\sin \frac{1}{2}(A-B)}{\sin \frac{1}{2}(A+B)} \tan \frac{1}{2} c.$

38. $\sin \frac{1}{2}(A+B) \cos \frac{1}{2} c = \cos \frac{1}{2}(a-b) \cos \frac{1}{2} C$ (Art. 198)

39. $\sin \frac{1}{2}(A-B) \sin \frac{1}{2} c = \sin \frac{1}{2}(a-b) \cos \frac{1}{2} C.$

40. $\cos \frac{1}{2}(A+B) \cos \frac{1}{2} c = \cos \frac{1}{2}(a+b) \sin \frac{1}{2} C.$

41. $\cos \frac{1}{2}(A-B) \sin \frac{1}{2} c = \sin \frac{1}{2}(a+b) \sin \frac{1}{2} C.$

42. $\tan r = \sqrt{\frac{\sin (s-a) \sin (s-b) \sin (s-c)}{\sin s}}$

$= \frac{n}{\sin s}$ . . . . (Art. 215)

43. $\tan R = \frac{1}{2n} [\sin(s-a) + \sin(s-b) + \sin(s-c) - \sin s]$

(Art. 217)

44. $K = \text{area of } \triangle = \frac{E}{180} \pi r^2$ . . . . (Art. 219)

45. $\sin \frac{1}{2} E = \frac{n}{2 \cos \frac{1}{2} a \cos \frac{1}{2} b \cos \frac{1}{2} c}$ . . . (Art. 220)

46. $\tan \frac{1}{4} E = \sqrt{\tan \frac{1}{2} s \tan \frac{1}{2}(s-a) \tan \frac{1}{2}(s-b) \tan \frac{1}{2}(s-c)}.$

EXAMPLES.

1. Find the time of sunrise at a place whose latitude is 42° 33′ N., when the sun's declination is 13° 28′ N.

Ans. $5^h\ 9^m\ 13^s$.

2. Find the time of sunset at Cincinnati, lat. 39° 6′ N., when the sun's declination is 15° 56′ S. *Ans.* $5^h\ 6^m$.

3. Find the time of sunrise at lat. 40° 43′ 48″ N., in the longest day in the year, the sun's greatest declination being 23° 27′ N. *Ans.* $4^h\ 32^m\ 16^s.4$.

4. Find the time of sunrise at Boston, lat. 42° 21′ N., when the sun's declination is 8° 47′ S. *Ans.* $6^h\ 14^m$.

5. Find the length of the longest day at lat. 42° 16′ 48″.3 N., the sun's greatest declination being 23° 27′ N.

Ans. $15^h\ 5^m\ 50^s$.

6. Find the length of the shortest day at New Brunswick, N.J., lat. 40° 29′ 52″.7 N., the sun's greatest declination being 23° 27′ S.

7. Find the hour angle and azimuth of Antares, declination 26° 6′ S., when it sets to an observer at Philadelphia, lat. 39° 57′ N. *Ans.* $4^h\ 23^m\ 5^s.7$; S. 54° 58′ 44″ W.

8. Find the hour angle and azimuth of the Nebula of Andromeda, declination 40° 35′ N., when it rises to an observer at New Brunswick, N.J., lat. 40° 29′ 52″.7 N.

9. Find the azimuth and altitude of Regulus, declination 16° 13′ N., to an observer at New York, lat. 40° 42′ N., when the star is three hours east of the meridian.

Ans. Azimuth = S. 71° 12′ 30″ E.; Altitude = 44° 10′ 33″.

10. Find the azimuth and altitude of Fomalhaut, declination 30° 25′ S., to an observer in lat. 42° 22′ N., when the star is $2^h\ 5^m\ 36^s$ east of the meridian.

Ans. Azimuth = S. 27° 18′ 40″ E.; Altitude = 11° 41′ 37″.

11. Find the azimuth and altitude of a star to an observer in lat. 39° 57′ N., when the hour angle of the star is $5^h\ 17^m\ 40^s$ east, and the declination is 62° 33′ N.

Ans. Azimuth = N. 35° 54′ E.; Altitude = 39° 24′.

12. Find the hour angle (t) and declination (δ) of a star to an observer in lat. 40° 36′ 23″.9 N., when the azimuth of the star is 80° 23′ 4″.47, and the altitude is 47° 15′ 18″.3.

Ans. $t = 46° 40' 4''.53$; $\delta = 23° 4' 24''.33$.

13. Find the distance between Regulus and Antares, the right ascensions being $10^h\ 0^m\ 29^s.11$ and $16^h\ 20^m\ 20^s.35$, and the polar distances 77° 18′ 41″.4 and 116° 5′ 55″.5.

Ans. 99° 55′ 44″.9.

14. Find the distance between the sun and moon when the right ascensions are $12^h\ 39^m\ 3^s.22$ and $6^h\ 55^m\ 32^s.73$, and the declinations 9° 23′ 16″.7 S. and 22° 50′ 21″.9 N.

Ans. 89° 52′ 55″.5.

15. Find the shortest distance on the earth's surface, in miles, from New York, lat. 40° 42′ 44″ N., long. 74° 0′ 24″ W., to San Francisco, lat. 37° 48′ N., long. 122° 23′ W.

Ans. 2562 miles.

16. Find the shortest distance on the earth's surface from San Francisco, lat. 37° 48′ N., long. 122° 23′ W., to Port Jackson, lat. 33° 51′ S., long. 151° 19′ E.

Ans. 6444 nautical miles.

17. Given the right ascension of a star $10^h\ 1^m\ 9^s.34$, and its declination 12° 37′ 36″.8 N.; to find its latitude and longitude, the obliquity of the ecliptic being 23° 27′ 19″.45.

Ans. Latitude = ; Longitude = .

18. Given the obliquity of the ecliptic ω, and the sun's longitude λ; to find his right ascension α and declination δ.

Ans. $\tan\alpha = \cos\omega\tan\lambda$; $\sin\delta = \sin\omega\sin\lambda$.

19. Given the obliquity of the ecliptic $23° 27' 18''.5$, and the sun's longitude $59° 40' 1''.6$; to find his right ascension (α), and declination (δ).

Ans. $\alpha = 3^h 49^m 52^s.62$; $\delta = 20° 5' 33''.9$ N.

20. Given the sun's declination $16° 0' 56''.4$ N., and the obliquity of the ecliptic $23° 27' 18''.2$; to find his right ascension (α), and longitude (λ).

Ans. $\alpha = 9^h 14^m 19^s.2$; $\lambda = 136° 7' 6''.5$.

21. Given the sun's right ascension $14^h 8^m 19^s.06$, and the obliquity of the ecliptic $23° 27' 17''.8$; to find his longitude (λ), and declination (δ).

Ans. $\lambda = 214° 20' 34''.7$; $\delta = 12° 58' 34''.4$ S.

22. Given the sun's longitude $280° 23' 52''.3$, and his declination $23° 2' 52''.2$ S.; to find his right ascension (α).

Ans. $\alpha = 18^h 45^m 14^s.7$.

23. In latitude 45° N., prove that the shadow at noon of a vertical object is three times as long when the sun's declination is 15° S. as when it is 15° N.

24. Given the azimuth of the sun at setting, and also at 6 o'clock; find the sun's declination, and the latitude.

25. If the sun's declination be 15° N., and length of day four hours, prove $\tan \phi = \sin 60° \tan 75°$.

26. Given the sun's declination and the latitude; show how to find the time when he is due east.

27. If the sun rise northeast in latitude ϕ, prove that cot hour angle at sunrise $= -\sin \phi$.

28. Given the latitudes and longitudes of two places; find the sun's declination when he is on the horizon of both at the same instant.

29. Given the sun's declination δ, his altitude h at 6 o'clock, and his altitude h' when due east; prove $\sin^2 \delta = \sin h \sin h'$.

30. Given the declination of a star 30°; find at what latitude its azimuth is 45° at the time of rising.

31. Given the sun's declination δ, and the latitude of the place ϕ; find his altitude when due east.

32. Given the declinations of two stars, and the difference of their altitudes when they are on the prime vertical; find the latitude of the place.

33. If the difference between the lengths of the longest and shortest day at a given place be six hours, find the latitude.

34. If the radius of the earth be 4000 miles, what is the area of a spherical triangle whose spherical excess is 1°?

35. If A″, B″, C″ be the chordal angles of the polar triangle of ABC, prove

$$\cos A'' = \sin \tfrac{1}{2} A \cos(s - a), \text{ etc.}$$

36. If the area of a spherical triangle be one-fourth the area of the sphere, show that the bisector of a side is the supplement of half that side.

37. If the area of a spherical triangle be one-fourth the area of the sphere, show that the arcs joining the middle points of its sides are quadrants.

38. Given the base and area; show that the arc joining the middle points of the sides is constant; and if it is a quadrant, then the area of the triangle is πr^2.

39. Two circles of angular radii, α and β, intersect orthogonally on a sphere of radius r; find in any manner the area common to the two.

40. If E be the spherical excess of a triangle, prove that

$$\tfrac{1}{2}E = \tan\tfrac{1}{2}a \tan\tfrac{1}{2}b \sin C - \tfrac{1}{2}(\tan\tfrac{1}{2}a \tan\tfrac{1}{2}b)^2 \sin 2C + \text{etc.}$$

41. Show that the sum of the three arcs joining the middle points of the sides of the colunars is equal to two

right angles, the sides of the original triangle being regarded as the bases of the colunars.

42. Prove that

$$\cos^2\tfrac{1}{2}a\sin^2 S+\sin^2\tfrac{1}{2}b\sin^2(S-C)+\sin^2\tfrac{1}{2}c\sin^2(S-B)$$
$$+2\cos\tfrac{1}{2}a\sin\tfrac{1}{2}b\sin\tfrac{1}{2}c\sin S\sin(S-B)\sin(S-C)=1.$$

43. Having given the base and the arc joining the middle points of the colunar on the base, the circumcircle is fixed.

44. Prove $\sin\tfrac{1}{2}b\sin\tfrac{1}{2}c\sin(S-A)+\cos\tfrac{1}{2}b\cos\tfrac{1}{2}c\sin S$
$=\cos\tfrac{1}{2}a.$

45. If $A+B+C=2\pi$, prove that

$$\cos^2\tfrac{1}{2}a+\cos^2\tfrac{1}{2}b+\cos^2\tfrac{1}{2}c=1,$$

and

$$\cos C=-\cot\tfrac{1}{2}a\cot\tfrac{1}{2}b.$$

46. Solve the equations,

$$\sin b\cos c\sin Z+\sin c\cos b\sin Y=\sin a,$$
$$\sin c\cos a\sin X+\sin a\cos c\sin Z=\sin b,$$
$$\sin a\cos b\sin Y+\sin b\cos a\sin X=\sin c,$$

for $\sin X$, $\sin Y$, and $\sin Z$.

47. If b and c are constant, prove the following relations between the small variations of any two parts of the other elements of the spherical triangle ABC:

$$\frac{dB}{dC}=\frac{\tan B}{\tan C};\qquad \frac{da}{dB}=-\sin a\tan C;$$
$$\frac{da}{dA}=\sin B\sin c;\qquad \frac{dA}{dB}=-\frac{\sin A}{\sin B\cos C};$$
$$\frac{da}{dC}=-\sin a\tan B;\qquad \frac{dA}{dC}=-\frac{\sin A}{\cos B\sin C}.$$

48. If A and c remain constant, prove the following:

$$\frac{da}{dB}=\frac{\sin a}{\tan C};\qquad \frac{da}{db}=\cos C.$$

49. If B and C remain constant, prove the following:

$$\frac{dA}{db} = \sin A \tan c; \qquad \frac{da}{db} = \frac{\sin a}{\sin b \cos c}.$$

50. If A and a remain constant, prove the following:

$$\frac{db}{dB} = \frac{\tan b}{\tan B}; \qquad \frac{dc}{dC} = \frac{\tan c}{\tan C};$$

$$\frac{db}{dc} = -\frac{\cos B}{\cos C}; \qquad \frac{db}{dC} = -\frac{\sin b}{\tan B \cos c}.$$

51. Two equal small circles are drawn touching each other; show that the angle between their planes is twice the complement of their spherical radius.

52. On a sphere whose radius is r a small circle of spherical radius θ is described, and a great circle is described having its pole on the small circle; show that the length of their common chord is $\frac{2r}{\sin\theta}\sqrt{-\cos 2\theta}$.

53. Given the base c of a triangle, and that

$$\tan \tfrac{1}{2} a \tan \tfrac{1}{2} b = \tan^2 \tfrac{1}{2} B,$$

B being the bisector of the base, find $a - b$ in terms of c.

54. If $C = A + B$, show that

$$1 - \cos a - \cos b + \cos c = 0.$$

55. If A denote one of the angles of an equilateral triangle, and A′ an angle of its polar triangle, show that

$$\cos A \cos A' = \cos A + \cos A'.$$

56. Show that

$$\frac{\cos a \cos B - \cos b \cos A}{\sin a - \sin b} = \frac{\cos C + \cos c}{\sin c}.$$

57. Prove $\quad \cos A = \dfrac{\cos a \sin b - \sin a \cos b \cos C}{\sin c},$

and $\quad \cos A + \cos B = \dfrac{2 \sin (a + b) \sin^2 \frac{1}{2} C}{\sin c}.$

58. Prove Legendre's Theorem by means of the relations

$$\frac{\sin A}{\sin a} = \frac{\sin B}{\sin b} = \frac{\sin C}{\sin c}.$$

59. Two places are situated on the same parallel of latitude ϕ; find the difference of the distances sailed over by two ships passing between them, one keeping to the great circle course, the other to the parallel; the difference of longitude of the places being 2λ.

Ans. $2r[\lambda\cos\phi - \sin^{-1}(\cos\phi\sin\lambda)]$.

60. If the sides of a triangle be each 60°, show that the circles described, each having a vertex for pole, and passing through the middle points of the sides which meet at it, have the sides of the supplemental triangle for common tangents.

61. Find the volume and also the inclination of two adjacent faces (1) of a regular tetraedron, (2) of a regular octaedron, (3) of a regular dodecaedron, and (4) of a regular icosaedron, the edge being one inch.

Ans. (1) 117.85 cu. in., 70° 31′ 43″.4;
(2) .4714 cu. in., 109° 28′ 16″;
(3) 7.663 cu. in., 116° 33′ 54″;
(4) 2.1817 cu. in., 138° 11′ 22″.6.

62. In the tetraedron, prove (1) that the circumradius is equal to three times its in-radius, and (2) that the radius of the sphere touching its six edges is a mean proportional between the in-radius and circumradius.

63. Prove that the ratio of the in-radius to the circumradius is the same in the cube and the octaedron, and also in the dodecaedron and icosaedron.